



This is to certify that the

thesis entitled

"Generalised Sylvester-Gallai
Configurations"

presented by

Sonde Ndubeze Nwankpa

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

L. M. Kelly
Major professor

Date November 20, 1970

ABSTRACT

GENERALISED SYLVESTER GALLAI CONFIGURATIONS

By

Sonde Ndubeze Nwankpa

In linear spaces it is of interest to study cutting hyperplanes of a family of, say, disjoint compact sets as opposed to studying supporting and separating hyperplanes.

Kelly and Edelstein have shown that "if $\{S_i\}$ is a finite collection of disjoint compact sets in a Hausdorff linear topological space Σ , spanning a space of dimension $d > 1$ and if $|\cup S_i| = \infty$ then there is a line in Σ , intersecting precisely two of the sets of $\{S_i\}$." Furthermore, they have shown that if $d > 3$ the condition that $|\cup S_i| = \infty$ can be removed. For $d = 2$ in real linear space there exists a large number of examples of what we now call generalised Sylvester-Gallai configurations (i.e. a finite collection of disjoint finite sets spanning a space of dimension 2 such that no line cuts precisely two of the sets).

For $d = 3$ only one example is known, the classically studied desmic configuration consisting of 3 sets of 4 points each.

This thesis represents an effort to understand these generalised Sylvester-Gallai configurations better, both

in the ordered linear setting, and in some general projective spaces.

It is easily shown that a necessary condition for the existence of a GSG configuration of 3 sets in dimension $d \geq 3$ is the existence of a completely self perspective arrangement (CSPA) in dimension $d-1$. Accordingly a considerable portion of the work is devoted to the study of CSPA particularly in the ordered projective plane. Our conjecture was (and is) that such an arrangement fails to exist in ordered projective space for sets of more than four points. We were only able to verify this for sets with fewer than nine points. This is the main result of Chapter 3.

Generalised S.G. Configurations in which each S_i consists of a single point are called simply S.G. configurations and have received considerable attention from several prominent sources. For example, J. P. Serre [AMM 73, p. 89 1966] asked if an S.G. configuration spanning complex projective 3-space exists. In Chapter 2 we show that such a set must contain at least 40 points. We also continue the program initiated by T. Motzkin [TAMS 70 (1951) 451-464] of characterizing abstract S.G. configurations of low orders or with other restrictions. For example, we completely analyze those which are subsets of 3 lines.

Typical theorems proved in the thesis

Theorem "If G is a GSG whose point sets $\{S_i\}$ span an ordered projective 3-space then no line intersects more than three of the sets."

Theorem "If S is an S.G. configuration spanning
complex projective 3-space then $|S| \geq 40$."

Theorem "If $\{S_1, S_2, \Gamma\}$ is a multiply perspective
arrangement of class $[k, k, d]$ then $k \geq 2^{d-1}$."

GENERALISED SYLVESTER GALLAI CONFIGURATIONS

By

Sonde Ndubeze Nwankpa

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1970

TO

Hanna Neumann my mentor,
Onyehuruchi my wife,
and the children of Biafra.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Professor L. M. Kelly for his helpful suggestions and guidance during the research. He would also like to thank M. Edelstein for the use of his unpublished results, and the guidance committee for their suggestions in the final preparation of the manuscript. He would like to thank the U. S. Government for their support through the Fulbright-Hayes Act Scholarship program administered by the Institute for International Education (I.I.E.).

TABLE OF CONTENTS

List of Figures.	Page iv
List of Symbols.	vii
CHAPTER 1. Introduction	1
CHAPTER 2. Sylvester-Gallai Configurations.	8
1. Configurations.	8
2. Complete characterizations of S.G. configura- tions for low orders not exceeding 12	16
3. S. G. Configurations as subsets of 3-lines.	36
4. On the existence of S. G. configurations in complex 3-space	41
CHAPTER 3. The Structure of a GSG Spanning Ordered 3-Space.	49
1. Introduction.	49
2. Completely self perspective sets of low orders.	59
3. Completely self perspective sets of order 8	65
CHAPTER 4. Desmic Triads in General Spaces.	83
1. Introduction.	83
2. Multiply perspective arrangements	85
3. Multiply perspective arrangements of class $[k, k, d]$, $k \leq 7$	93
Bibliography	118

LIST OF FIGURES

Figure	Page
1.1, 1.2.	3
1.3	4
2.1, 2.2.	.11
2.3	.18
2.4, 2.5.	.20
2.6	.21
2.7	.24
2.8, 2.9.	.25
2.10.	.26
2.11.	.27
2.12.	.28
2.13.	.29
2.14.	.30
2.15.	.31
2.16.	.32
2.17.	.33
2.18.	.34
2.19.	.35
2.20.	.37
2.21.	.39
2.22.	.40

Figure	Page
2.23.	.47
3.1	.51
3.2	.53
3.3	.54
3.4	.59
3.5	.58
3.6	.60
3.7, 3.8.	.61
3.9, 3.10, 3.11	.62
3.12, 3.13.	.63
3.14.	.64
3.15, 3.16, 3.17.	.65
For $ S = 8$, con'd diagrams: Case 1.	.66, 67
Case 2.	.67
Case 3, 4	.68
Case 5a,b	.69
Case 5c	.70
Case 6a	.72
Case 6b,c	.73
Case 7b	.74
Case 7a	.75
Case 8a,b	.76
Case 8c	.77
Case 9a	.78
Case 9b, 10	.79
Case 11	.81

Figures	Page
Case 12	.82
4.1	.92
4.2	.96
4.3	100
4.4, 4.5.	101
4.6	102
4.7	103
4.8	104
4.9	106
4.10.	108
4.11.	110
4.12, 4.13.	111
4.14.	112
4.15, 4.16.	114

LIST OF SYMBOLS

Symbol	Page
k-secant	.1
GSG.	.3
GDC.	.6
SG	10
[1x,23] points 1 and x separate points 2 and 3 . .	11
(a_α, b_β) A configuration with 'a' points and 'b' lines such that there are β points on one line and α lines on one point. . . .	12
$(n_1, n_2, \dots, n_k)$	15
[P] , S^*	60
θ_{ij}	61
$A_1 \Psi B_1(46)$	70
$\{A, B, \Gamma\}$, $[k, p, d]$, $\{A, \Gamma\}$	85
$[k, p, d]$ SPA.	86
(k, k, d) SPA.	91
$[k \ k \ d]$ MPA.	94
	95

INTRODUCTION

In the study of convex sets in linear spaces the concepts of supporting and separating hyperplanes is of central importance. While not as strong a case can be made for the study of cutting hyperplanes of a family of, say, disjoint compact sets in a linear space it does seem natural and possibly useful to probe such matters.

For example, we may ask if corresponding to any finite family of bounded, closed disjoint sets in a Banach space there is at least one hyperplane cutting exactly one set of the family. The answer is no. It is possible to construct two disjoint bounded closed sets in c_0 such that any hyperplane cutting one of the sets intersects the other set. However, if the sets, instead of being merely bounded and closed are also compact, then the corresponding question can be answered in the affirmative

Definition 1.1: k -secant

A line intersecting precisely k -members of a family of point sets is a k -secant of that family.

Theorem 1.1: (Kelly and Edelstein)

A finite family of disjoint compact sets in a topological linear space has a 1-secant.

How about 2-secants? The answer has turned out to be rather surprising and leads us out of the linear topological setting into the linear combinatorial setting with which this thesis will be concerned.

The question about 2-secants was originally posed by B. Grünbaum [4] who proved that a finite family of disjoint continua in E_2 , not all subsets of the same line, must have a 2-secant. He thought of this as a generalization of the Sylvester phenomenon, namely that a finite non-linear set of points of E_n , or more generally in any ordered projective space, is cut by some line in exactly two points.

Herzog and Kelly (9) extended the Grünbaum result to a finite family of disjoint compact sets in E_n at least one of which is infinite.

Edelstein, Herzog and Kelly (2) strengthened the above result to the following:

Theorem 1.2:

A finite family of disjoint compact sets in a topological linear space at least one of which is infinite and not all of which are subsets of the same line is cut by some hyperplane in exactly two sets. That the condition of infinite number of points in one set cannot be dropped follows from the following example. Consider the Pappus configuration in E_2 : A, B, C on a line ℓ , $A'B'C'$ on $\ell' \neq \ell$ with

$$AB' \cap A'B = C''$$

$$A'C \cap AC' = B''$$

$$BC' \cap B'C = A''$$

The sets $\{A, A', A''\}$

$\{B, B', B''\}, \{C, C', C''\}$

constitute a finite family of

disjoint compact sets in E_2

having no 2-secant.

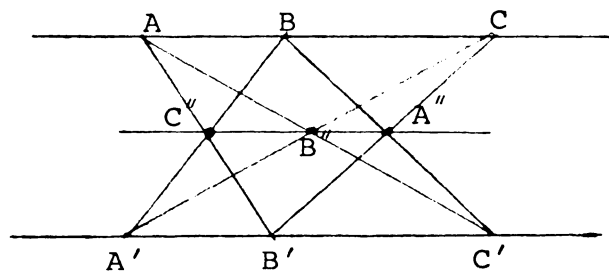


Figure 1.1

The hope was that this and possibly a few other configurations would be the only one standing in the way of improved version of theorem 1.2. Unfortunately, this is not the case. Consider as a further example, the vertices of a regular hexagon alternately carrying the number 1 and 2, together with the three

points at infinity on the side

lines carrying the number 3. The

points carrying the same number con-

stitute a set and the three such sets

have no 2-secants. The interesting

thing about this example is that it is

extendable to any regular $2n$ -gon. So

in E_2 we have an infinite class of examples of what we

now call generalized Sylvester-Gallai configurations.

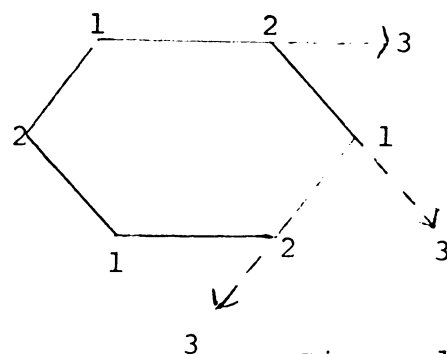


Figure 1.2

Definition 1.2:

A generalized Sylvester-Gallai configuration (GSG) is a finite family of disjoint finite sets not all on one line which has no 2-secant.

Definition 1.3:

If in the above each set of the family consists of a single point, then the configuration is called simply a Sylvester-Gallai configuration.

At this stage Kelly and Edelman [3] anticipated a flood of further counterexamples in E_3 and E_n . For example they found that numbering the 8 vertices of a cube in E_3 alternately with the integers 1 and 2, the center and three ideal points on the edges with the integer 3, yielded a counterexample. This is the classically studied desmic configuration discovered by Stepanos in 1890. However, they could find no other example spanning E_3 and this example failed to generalize to higher dimensions.

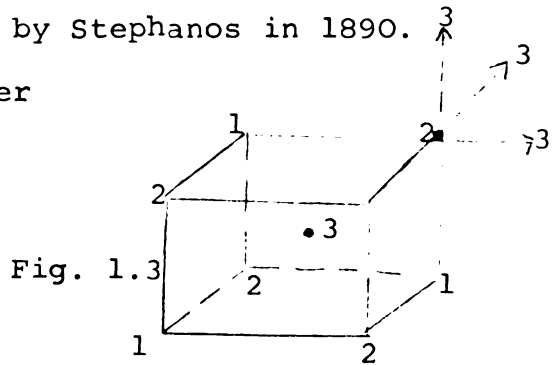


Fig. 1.3

Note incidentally that the topology has completely vanished at this stage and one is concerned only with incidence structure and order. That the flood, alluded to above, would not materialize became evident from the discovery that in ordered projective spaces there are no generalized Sylvester-Gallai configurations spanning a space of more than three dimensions, specifically:

Theorem 1.3:

If $\{S_i\}$ is a finite collection of two or more non-empty disjoint finite sets in an ordered projective space such that $\cup S_i$ spans a subspace of dimension at least 4,

then there exists a line (and therefore also a hyperplane) cutting precisely two of the sets.

This theorem is proved in Chapter 3.

In his Transactions' paper, Motzkin [8] initiated this study, showing among other things, that Sylvester-Gallai configurations exist in a wide variety of non-ordered projective spaces. It is, of course, clear that any finite projective space is a Sylvester-Gallai configuration but it is not so obvious that such configurations exist in projective spaces of infinite cardinality. Motzkin showed that such configurations always exist in projective spaces over a field which contains roots of unity other than ± 1 .

We continue this program in Chapter 2, characterizing such configurations of low orders as well as those which are subsets of three or four lines. The latter part of the chapter is concerned specifically with complex projective space. Specifically we try to cast some light on a question raised by J. P. Serre [AMM 73 P. 89, 1966] concerning the existence of non-planar Sylvester-Gallai configurations in complex 3-space. We succeed only in establishing a rather high lower bound on the cardinality of such sets if, in fact, they do exist.

In Chapter 3 we attack the problem of characterizing generalized Sylvester Gallai configurations in ordered projective three space. It has been conjectured that the classical desmic configuration is the lone element of this class but we were unable to prove or disprove this.

However, we state and prove a number of unpublished theorems of Edelstein throwing considerable light on the structure of GSG 's in ordered projective 3-space and go on from there to show that there are no analogues of the classical configuration in which each of the three sets has 5, 6, 7 or 8 points. The 8 point analysis involved a lengthy case-analysis which seems to preclude a generalization by these methods to higher orders.

Chapter 4 is devoted to a study of the existence and characterization of generalized desmic configurations (GDC) in a variety of projective spaces. There is considerable emphasis on the problem of obtaining configurations which span spaces of various dimensions. For example we show that a desmic triad in which each set consists of precisely 7 points does not exist in any projective space, while one in which each set consists of exactly 6 points has a very special structure [see theorem 4.3] and exists only in a projective 3-space over a field of characteristic 3. Detailed analysis of the desmic triads resulted in the concept of multiply perspective and self perspective sets.

As a final example of the variety of results in this chapter we cite the following theorems:

Theorem 1.4:

There exists desmic triads of order $q^{d-1} - q^{d-2}$ for any prime p , where $q = p^k$, d is dimension of the space of order q spanned by the configuration.

Theorem 1.5:

There is no desmic triad of order 10 spanning a projective space of 4 dimensions.

CHAPTER 2
Sylvester - Gallai Configurations

1. Configurations:

In our introductory remarks it did not seem necessary to abstract the notion of configurations. However, we cannot conveniently proceed much further without a more precise and a more general definition.

Definition 2.1:

A structure consisting of two sets, P (points) and L (lines) together with a symmetric binary relation (incidence) between points and lines such that two points are incident with at most one line is a linear configuration.

It is usual in such studies to identify a line in L with the set of points of P incident with that line.

Definition 2.2:

Two linear configurations are abstractly (combinatorially) isomorphic if there is a 1-1 correspondence between their points and lines respectively which preserves the relation of incidence.

Definition 2.3:

A configuration is embeddable (or realizable) in a projective space if there is a 1-1 mapping of the points of

the configuration onto a set of points of the space and a 1-1 mapping of the set of lines of the configuration onto a set of lines of the space such that incidence is preserved.

Definition 2.4:

A configuration in which each two points are incident with a line is line complete. A point complete configuration is defined dually.

In this new terminology we might recast the Sylvester-Gallai theorem thus:

Theorem 2.1:

A necessary condition that a finite line complete configuration be embeddable in an ordered projective space is that some line of the configuration be incident with two and only two points.

Since this theorem is so basic to this study we reproduce a proof here adapted from that of Kelly and Moser [6] which employs the useful concept of residence of a point. This concept will recur in subsequent proofs.

Definition 2.5:

A line complete configuration of n points, $n-1$ of which are on a line is a near pencil.

Definition 2.6:

The set of lines incident with a point of a configuration is called a pencil of the configuration. The point is

the vertex of the pencil and each line a ray. If the number of rays is m , the pencil is an m -pencil. If the numbers of points on the rays, arranged in descending order of magnitude, are $n_1, n_2, \dots, n_m$, the pencil is said to be of type $(n_1, n_2, \dots, n_m)$.

Definition 2.7:

A finite line complete configuration in which no line is incident with exactly two points is a Sylvester-Gallai configuration or an S.G. configuration.

Definition 2.8:

If C is a finite line complete configuration in an ordered projective plane, with point set T , then the set of lines of C not through $P \in T$ either consists of a single element or partition the plane into regions. In the latter case P is in one of these regions which is called the residence of P relative to C . A sideline of a residence is called a neighbor of P .

We now proceed to prove theorem 2.1 stated earlier.

There is no SG configuration in an ordered projective space.

Case (i):

The lines of the configuration form a partition of the plane which is a near pencil. Let P be the point not on i -th line of the pencil. Then the join of P to any point on l is a 2-secant.

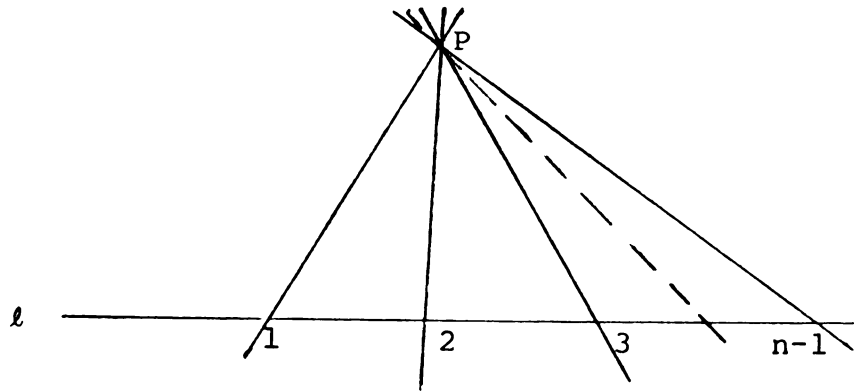


Figure 2.1

Case (ii):

The lines of the configuration, not through P , form a proper partition of the plane such that P has a residence with neighbor ℓ .

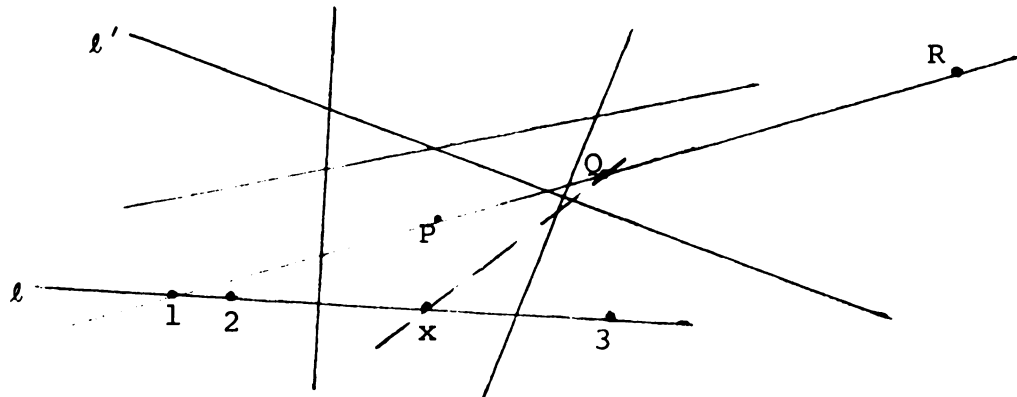


Figure 2.2

Suppose ℓ has three points on it $1, 2, 3$. Let x be a point of the boundary of the residence of P such that $[1x, 23]$. Claim that $x \neq i$, $i = 1, 2, 3$. Suppose not; i.e. $x = 3$ (say). Let ℓ' be another neighbor of P . ℓ' has at least two points $z, 4$ on it. Now either $\overline{z3}$ intersects the residence of P or $\overline{43}$ intersects the residence of P and $z, 3, P$ are collinear or $z, 3, P$ are

collinear and $1,2,3,4$ are collinear. In the last case ℓ' contains a third point w such that either $\overline{3w}$ intersects the residence or ℓ' is a 2-secant as required.

We now suppose $x \neq i$, $i = 1,2,3$. The line $\overline{1P}$ has a third point Q on it otherwise we have a 2-secant as required. Thus $[\overline{1Q} \overline{xQ}, \overline{3Q} \overline{wQ}]$, i.e. $\overline{1Q} = \overline{1P}$ and $\overline{xQ}$ intersect the residence $\therefore \overline{3Q}$ intersects the residence of P or $\overline{2Q}$ intersects the residence of P according as $[1Q, PR]$ or $[1P, QR]$ where $1, P, Q, R$ are collinear as indicated. Therefore in every case we have a contradiction of the definition of residence of P , which implies ℓ is a 2-secant as required. ■

Remark 2.2 (a)

There is no S.G. configuration in E_n for all n .

2.2 (b):

There is no S.G. configuration in any real Banach space.

2.2 (c):

Every finite projective space is an S.G. configuration.

Definition of a Scheme 2.9:

Following the usual practice of using rectangular schemes to represent configurations of type (a_α) in projective geometry, we describe the following variant of it to represent any S.G. configuration of order n .

In constructing an S.G. configuration of order n the following method will be found convenient: We label the n points with the numbers 1 through n and label the n pencils at each point, similarly with the numbers (1) through (n) . Then we set up a scheme of lines in which the lines incident with any given pencil are arranged in a column and the points incident with any given line are arranged in a row of a column. There will be n columns corresponding to the n pencils, but the number of rows per column will vary according to the number of lines in a pencil. The scheme must satisfy the following conditions

- (1) The numbers written in the rows of any one column must contain the number representing the pencil corresponding to the column to ensure that all lines of a pencil contain its vertex.
- (2) The numbers of a row of a column must be more than three in cardinality to avoid 2-secants.
- (3) Two different rows of a column cannot have two numbers in common, as this would make the straight lines corresponding to the rows coincide.

Remark 2.3:

We observe that a row of length k is repeated in the columns k times.

An Illustration:

An S.G. configuration of order 7 corresponds to the following scheme:

(1)	(2)	(3)	(4)	(5)	(6)	(7)
124	214	317	412	516	615	713
137	235	325	436	523	627	726
156	267	346	457	547	634	745

where each row in the columns after the vertical bar are repeats. The resulting configuration is seen easily to be isomorphic to the Fano plane.

Definition 2.10:

Two S.G. configurations are equivalent iff there is a 1-1 correspondence between the points and lines of the configurations which preserves incidence.

Definition 2.11:

Two schemes are equivalent if one can be obtained from the other by means of one or all of the following operations:

- (a) Interchange of any two numbers in each row of any column, corresponds to relabelling the points of a line.
- (b) Interchange of any two columns. This corresponds to the relabelling of the pencils hence of the points.
- (c) Interchange of any two rows of a column. This corresponds to relabelling the lines of a pencil.

Definition 2.12:

Two S.G. configurations are said to be schematically equivalent if they have equivalent schemes.

Theorem 2.5:

Two S.G. configurations are equivalent iff they are schematically equivalent.

Proof:

Suppose the two configurations are denoted by C_1 and C_2 respectively. Furthermore suppose C_1 and C_2 are equivalent, i.e., there is a 1-1 correspondence between their points which preserves incidence. The 1-1 correspondence between points is an operation of type (b) i.e relabeling the points. The preservation of incidences are operations of types (a) and (c); i.e. C_1 and C_2 are schematically equivalent. Conversely if C_1 and C_2 are schematically equivalent all the operations involved are 1-1 correspondences and they preserve incidence; i.e. C_1 and C_2 are equivalent. ■

We now derive necessary numerical conditions for an S.G. to exist.

Lemma 2.6:

If a k -pencil F has the sequence $(n_1, n_2, \dots, n_k)$ associated with it. Then

$$\sum_i^k n_i = n+k-1, \text{ where } n \text{ is order of S.G.}$$

Proof:

We have, counting all points but the vertex of the pencil

$$(n_1-1) + (n_2-1) + \dots + (n_k-1) = n-1$$

$$\Rightarrow \sum_{i=1}^k n_i - k = n-1$$

$$\text{i.e. } \boxed{\sum_{i=1}^k n_i = n+k-1} \quad \blacksquare$$

2. Complete characterizations of S. G. configurations for low orders not exceeding 12.

Our procedure is as follows: For any given n as the order of an S.G. configuration we first of all seek the solution of the following system of equations and inequalities.

$$\sum_{i=1}^k n_i = n+k-1$$

$$n_i \geq 3 \quad i = 1, 2, \dots, k$$

then for each solution investigate the construction of an S. G. configuration.

$$\underline{n = 7}$$

Lemma 2.7:

An S. G. configuration of order 7 is not a subset of a 5-pencil.

Proof:

This follows from the fact that there are no integer solutions to the following system of equations and inequalities

$$\sum_{i=1}^5 n_i = 11$$

$$n_i \geq 3 \quad i = 1, 2, 3, 4, 5.$$

Remark 2.8:

Similarly, an S.G. configuration of order 7 is not a subset of a 4-pencil.

Remark 2.9:

For a 3-pencil the corresponding system of equations and inequalities has the solution (3,3,3). An S.G. configuration realizing this solution has already been constructed in the illustration above. Any two line complete configuration, each vertex of which has order 3, we easily see to be equivalent.

$$\underline{n = 9}$$

Lemma 2.10:

An S.G. configuration of order 9 is not a subset of a k -pencil for $k \geq 5$.

Proof:

This follows from the fact that there are, for $k = 5$ (say), no integer solutions to the system

$$\sum_{i=1}^5 n_i = 13$$

$$n_i \geq 3, \quad i = 1, 2, 3, 4, 5.$$

$$n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5 \quad \blacksquare$$

Remark 2.11:

There are solutions $(3, 3, 3, 3)$ and $(5, 3, 3)$ $(4, 4, 3)$ corresponding to a 4-pencil and two 3-pencils respectively.

Remark 2.2:

- (a) We observe that if an S.G. configuration is a subset of a 3-pencil then we have

$$n_1 = n_2 = n_3$$

- (b) It is also clear that an S.G. configuration is not a subset of a 2-pencil. Later we shall prove a theorem which determines completely all S.G. configurations which are subsets of a 3-pencil.
- (c) Solutions $(5, 3, 3)$ and $(4, 4, 3)$ are impossible by (a).

Theorem 2.13:

There exists an S. G. configuration of order 9.

Proof:

This contains the 4-pencil corresponding to the solution $(3, 3, 3, 3)$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
123	213	312	415	514	617	716	819	918
145	249	346	429	527	628	725	826	924
167	257	358	436	538	634	739	835	937
189	268	379	478	569	659	748	847	956

The above table satisfies all three conditions

- (a) Each column contains the integer representing the pencil in each of its rows.
- (b) Each row has length at least 3.
- (c) No two rows of a column have two integers in common.

$\therefore$ The scheme represents an S.G. configuration of order 9. ■

Remark 2.14:

We observe that the above S.G. configuration is equivalent to the configuration of type $(9_4, 12_3)$.

It is always possible to represent a configuration by points in a projective plane in which the lines of two pencils of the configuration are faithfully represented by subsets of points on projective lines. If the projective line joining these two points is taken as an ideal line the pencils in the resulting affine plane are two independent sets of parallel lines and the resulting planar set of points and lines is called a grid diagram of the configuration.

Further linearities are sometimes conveniently suggested by means of simple arcs passing through a set of points of the grid diagram which correspond to sets of points which are linear in the configuration.

Thus we might represent the seven point projective plane by a grid diagram in the real affine plane as follows:

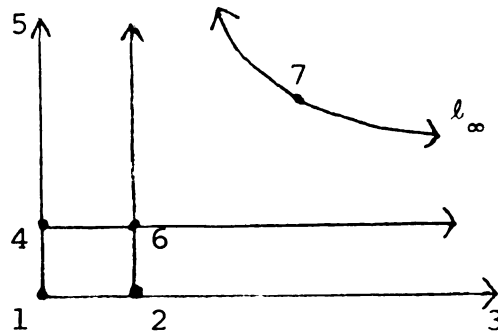


Fig. 2.4

As a second example consider a grid diagram representing the configuration $(9_4, 12_3)$

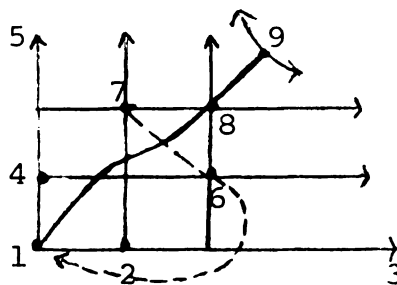


Fig. 2.5

In this second example the grid diagram is only a partial representation of the configuration while in the first example we have a complete representation of the seven point plane.

Proceeding from the linearities of the grid diagram in example 2 we can construct a tabular scheme as follows:

312	593	123	634	716	856	213	415	918
346	514	145	658	725	819	257	428	926
378	527	167	617	738	837	269	436	935
359	568	189	629	749	824	284	479	947

It should be clear that up to the equivalence previously described the tabular scheme is unique and hence we claim that the configuration $(9_4, 12_3)$ is combinatorially unique.

Definition 2.13:

We now ask whether it is possible in some affine plane to complete the grid diagram so that we will have a complete representation of the $(9_4, 12_3)$ configuration with all lines of the configuration represented by linear subsets of the affine plane. Such a representation we refer to as an embedding of the configuration in the particular affine plane.

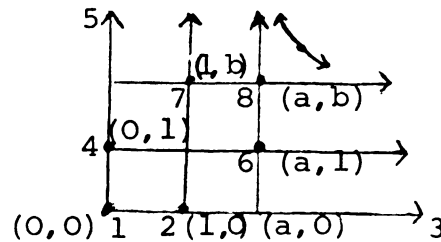


Fig. 2.6

We may coordinatise the grid diagram over a division ring as shown above.

The equation of the line 42 is $x+y = 1$ and the linearity 428 implies that $a+b = 1$

The equation of line 16 is $x = ya$

and linearity 167 gives $1 = ba$

Thus $b(1-b) = 1 \Rightarrow 0 = b^2 - b + 1$ and

if the ring does not have characteristic 2 then

$$b = \frac{1 \pm \sqrt{-3}}{2}, \quad a = \frac{1 \mp \sqrt{-3}}{2}$$

It follows that, since an easy checking shows the remaining linearities are consistent with the above values of a and b ,

Theorem 2.16:

There exists a unique embedding of $(9_4, 12_3)$ in a Desarguesian plane iff the coordinate ring contains $\sqrt{-3}$.

Unique means here that any two embeddings in a plane are projectively equivalent. It is possible to embed a $(9_4, 12_3)$ configuration in planes over division rings of characteristic 2 but we do not analyze this further.

It follows from the above analysis of $(9_4, 12_3)$ configuration that

Corollary 2.15:

Any two embeddings of an S.G. configuration of order 9 are projectively equivalent.

Corollary 2.16:

An S.G. configuration of order 9 is uniquely realizable in the complex projective plane.

$$\underline{n = 8}$$

We now show that there exists no S.G. configuration S with $|S| = 8$

Theorem 2.17:

There does not exist an S.G. configuration S with $|S| = 8$.

Proof:

We have

$$n_1 + n_2 + \cdots + n_k = 8 + k - 1$$

$$n_1 \geq n_2 \geq \cdots \geq n_k \geq 3 \quad k \geq 3$$

For $k \geq 4$ these inequalities have no solution, suppose $k = 3$ then $n_1 = n_2 = n_3$ which is impossible. ■

The analysis of S.G. configurations of orders 7, 8, and 9 is thus complete.

$$\underline{n = 10}$$

Now let S be an S.G. configuration with $|S| = 10$.

Theorem 2.18:

A 10 point S.G. configuration with a 3-pencil exists and is combinatorially unique.

Proof:

$$\text{Now } \sum_{i=1}^k n_i = 9 + k \quad n_1 \geq n_2 \geq \cdots \geq n_k \geq 3, \quad k \geq 3$$

have no solutions for $k \geq 5$

Let $k = 4$. $n_1 + n_2 + n_3 + n_4 = 13$ has solution $(4, 3, 3, 3)$, thus 4-pencils in S must be of type $(4, 3, 3, 3)$

Let $k = 3$, $n_1 + n_2 + n_3 = 12$

$$n_1 = n_2 = n_3 = 4$$

i.e. 3-pencils must be of type $(4, 4, 4)$.

Suppose S has a 3-pencil

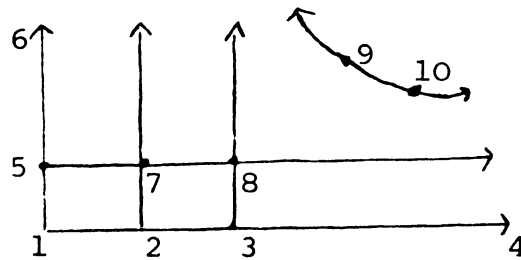


Fig. 2.7

The scheme below easily follows and is unique

4123	615	156	7458	836	9,10,64
4578	627	1234	726	8457	917
49106	638	179	719	829	928
	69104	18,10	73,10	8110	935
2134	3124	5478	10964		
267	359	516	1025		
289	368	539	1037		
25,10	37,10	52,10	1018	■	

We claim that

Theorem 2.19:

There is no 10 point S.G. configuration containing only 4-pencils

Proof:

Suppose S were such a pencil. Its grid diagram is shown below

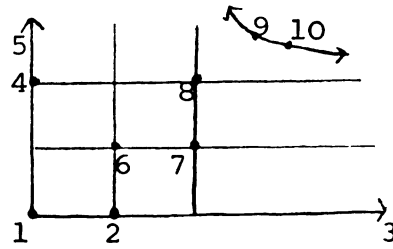


Fig. 2.8

The pencil with vertex 1 must contain a 4-line and this 4-line will certainly contain either point 9 or 10. Thus the pencil with either vertex 9 or vertex 10 is not a 4-pencil. The unique S.G. configuration of order 10 has been combinatorially characterized.

We now consider its possible embeddings.

Theorem 2.23:

A 10 point S.G. is embeddable in a Desarguesian plane, iff the coordinate ring has characteristic 3.

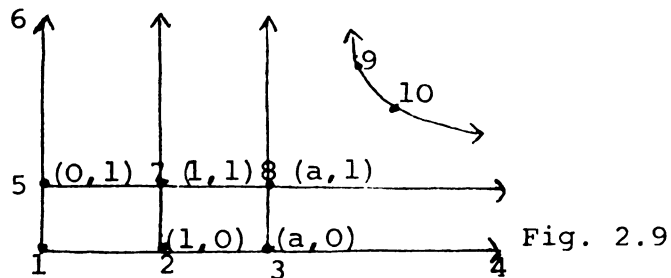


Fig. 2.9

Line 17: $x = y$

Line 28: $x - y(a-1) - 1 = 0$

Since these lines are parallel, $a = 2$

Line 25: $x + y = 1$

Line 18: $x = ya$

Since these lines are parallel, $a = -1$. Thus $2 = -1 \Rightarrow 3 = 0$

It is easy to check that this necessary condition is also sufficient. ■

$$\underline{n = 11}$$

Suppose now that S is an S.G. configuration of order 11.

Theorem 2.21:

An 11 point S.G. configuration with a 4-pencil exists and it is combinatorially unique.

Proof:

We have

$$\sum_{i=1}^k n_i = 10 + k \quad n_1 \geq n_2 \geq \dots \geq n_k \geq 3, \quad k \geq 3.$$

For $k \geq 6$ these relations have no solution. If $k = 5$

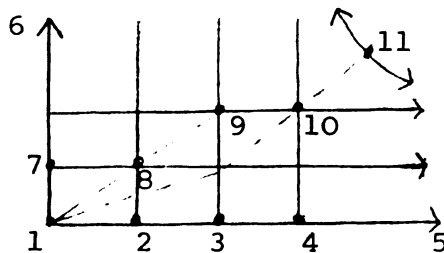
$n_1 + n_2 + n_3 + n_4 + n_5 = 15$ has the unique solution $(3, 3, 3, 3, 3)$.

If $k = 4$ $n_1 + n_2 + n_3 + n_4 = 14$ has solutions $(5, 3, 3, 3)$

and $(4, 4, 3, 3)$. If $k = 3$, $n_1 + n_2 + n_3 = 13$ which is

clearly impossible. Suppose S contains a pencil of

type $(5, 3, 3, 3)$



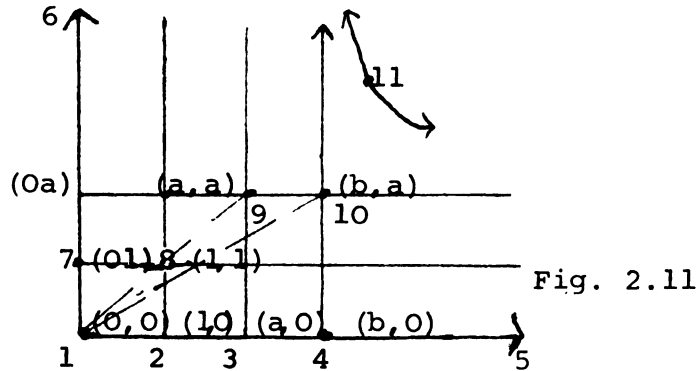
The scheme below follows and it is unique.

Fig. 2.10

51234	617	12345	21345	1059	
578	628	167	268	1046	
5910	639	189	2710	1027	
56,11	64,10	11011	2911	1038	
	65,11				
11,1,10	31245	41235	758	826	9510
11,56	369	4610	716	8310	918
11,29	38,10	4811	7210	819	9211
11,37	37,11	497	7311	857	936
11,48			749	8411	947

Embedding:Theorem 2.22:

An 11-point S.G. is embeddable in a Desarguesian plane iff the coordinate ring has characteristic 2.



$$\text{Line } 1, 10: \quad x b^{-1} a - y = 0 \quad \Rightarrow \quad (x b^{-1} a) (a^{-1} (a-1)) = x - 1$$

$$\text{Line } 29: \quad y a^{-1} (a-1) = x - 1$$

$$\text{i.e. } x b^{-1} (a-1) = x - 1$$

$$\text{i.e. } x [b^{-1} (a-1) - 1] = -1$$

But lines 1, 10 and 29 are parallel.

$$\therefore b^{-1} (a-1) = 1$$

$$\text{i.e. } \boxed{a-1 = b}$$

Now the linearity 72, 10 gives

$$-1 = \frac{2}{a-2} \quad \Rightarrow \quad -a+2 = a$$

$$\Rightarrow 2 = 2a$$

$$\Rightarrow 1 = a \quad \text{impossible}$$

$$\text{or } 2 = 0 \quad \text{as required}$$

It is easy to check that this condition is also sufficient. ■

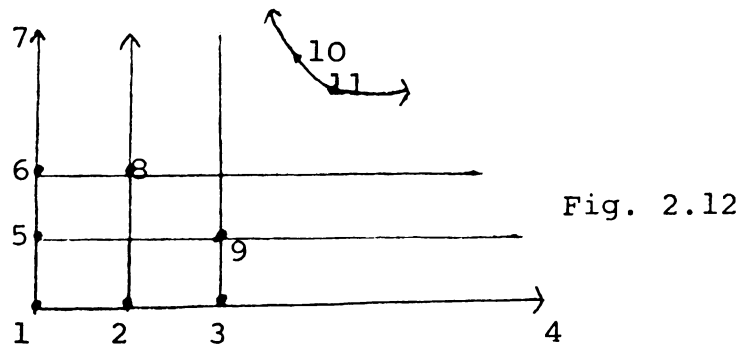
We claim that

Theorem 2.23:

There is no 11 point S.G. configuration containing a 4-pencil of type $(4,4,3,3)$

Proof:

Suppose S were such a pencil. Its grid diagram is shown below



If we consider the vertex 2 then since it already has one 4-line, it must have a second 4-line. i.e. this second 4-line must hit the lines $\overline{59}$, $\overline{68}$ and $\overline{10,11}$. Similarly for the vertex 3. But this is impossible. ■

Lemma 2.24:

There is no 11 point S.G. configuration containing pencils of type $(3,3,3,3,3)$ at each vertex.

Proof:

If all pencils of S were of type $(3,3,3,3,3)$ the configuration would be regular and the equation $11(5) = 3x$ would have an integral solution. Since it does not the lemma follows. ■

This completes the analysis of S.G. configuration of order 11.

$$\underline{n = 12}$$

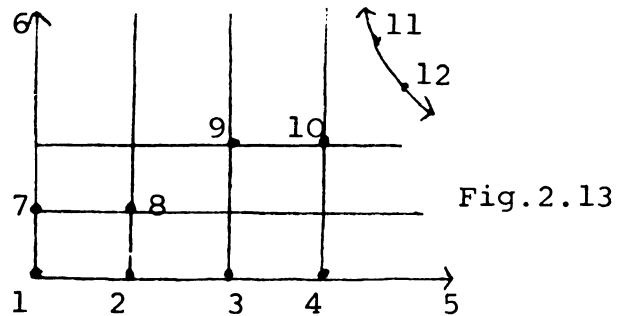
Suppose finally that S is an S.G. configuration of order 12.

Theorem 2.25:

There is no 12 point S.G. configuration containing a 4-pencil of type $(5,4,3,3)$

Proof:

Suppose S were such a pencil. Its grid diagram is shown below



There is a 4-line through 1 and it has to contain either 11 or 12. But each of them is a vertex of a 5-pencil. i.e. there is no second 4-line through them.

Hence theorem ■

We obtain the other solutions as follows

Remark 2.26:

$$\sum_{i=1}^k n_i = 11 + k \quad n_1 \geq n_2 \geq \dots \geq n_k \geq 3, \quad k \geq 3$$

For $k \geq 6$ these relations have no solution. If $k = 5$

$n_1 + n_2 + n_3 + n_4 + n_5 = 16$ has the unique solution $(4,3,3,3,3)$.

If $k = 4$, $n_1 + n_2 + n_3 + n_4 = 15$ has solutions $(5, 4, 3, 3)$, $(4, 4, 4, 3)$, $(6, 3, 3, 3)$ where the last one is clearly impossible. If $k = 3$, $n_1 + n_2 + n_3 = 14$ with $n_1 = n_2 = n_3$ is impossible; since 3 cannot divide 14.

Theorem 2.27:

A 12-point S.G. configuration containing two pencils of type $(4, 4, 4, 3)$ exists and is combinatorially unique, and infact all pencils are of same type.

Proof:

The scheme below follows and it is unique.

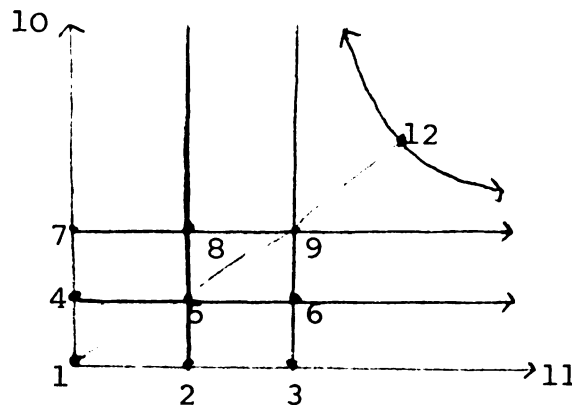


Fig. 2.14

11123	10147	12311	54611	97811	65411
11456	10258	14710	52810	93610	63910
11784	10369	15912	51912	91512	618
11, 10, 12	101112	186	573	924	62712

87911	25810	31211	41710	71410
82510	23111	34812	429	72612
816	249	357	43812	735
83412	26712	36910	45611	78911

12159
12267
12348
12, 10, 11. ■

EmbeddingTheorem 2.28:

If the above 12 point S.G. is embeddable in a Desarguesian plane the coordinate ring must have characteristic 3.

Proof:

$$186: y = xa$$

$$1 = a^2 \Rightarrow a = -1 \text{ since } a \neq 1.$$

$$429: \left. \begin{array}{l} x + y = 1 \\ a + a = 1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} 2a = 1 \\ 4a^2 = 1 \end{array} \right\} \Rightarrow 4 = 1 \Rightarrow 3 = 0$$

$$48312: (a-1)^2 = -a$$

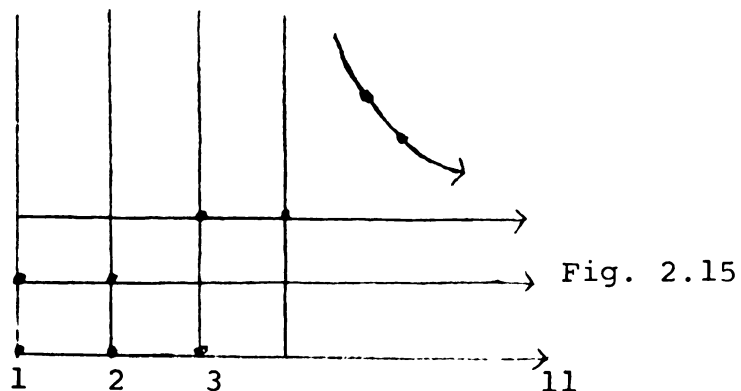
$$\Rightarrow a^2 - a + 1 = 0$$

$$\Rightarrow a = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2}$$

which is consistent with $a = 2 = -1$. ■

Theorem 2.29:

There is no 12 point S.G. containing pencils of type $(4,4,4,3)$ and $(4,3,3,3,3)$.

Proof:

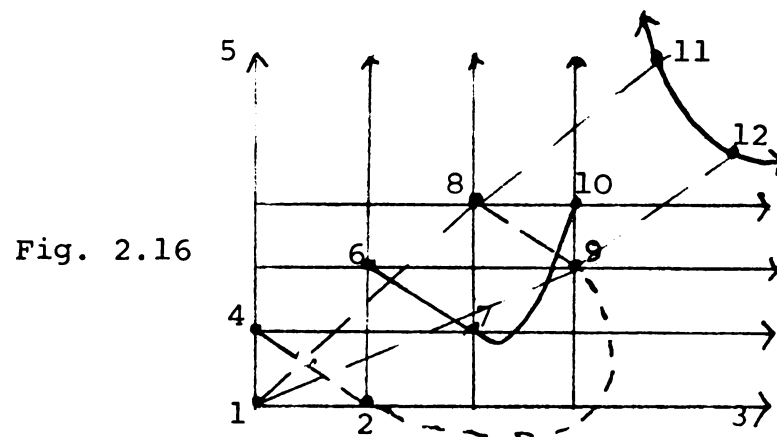
From the above diagram we observe that it is impossible to form the grid to begin with. For the 4-pencil with vertex 11 has not got the 3rd 4-line. ■

Lemma 2.30:

An S. G. configuration of order 12, with each pencil of type $(4,3,3,3,3)$ contains a complete quadrilateral.

Proof:

With suitable labelling we have the grid diagram shown:



The following table lists these linearities

123	312	514
145	347	526
16710	369	578
1811	3810	5910
1912	311125	511123

Additional linearities implied by these are:

67110	415	213
639	437	256
625	4289	2489
6114	4116	21110
6128	41210	2127

But now points 2, 6, 7, 8, 5, 12 are vertices of a complete quadrilateral.



We observe that each diagonal line of the complete quadrilateral is a 4-line. It will now be shown that there exist two distinct S.G. configurations of order 12 all of whose pencils are of type $(4,3,3,3,3)$.

Definition 2.14:

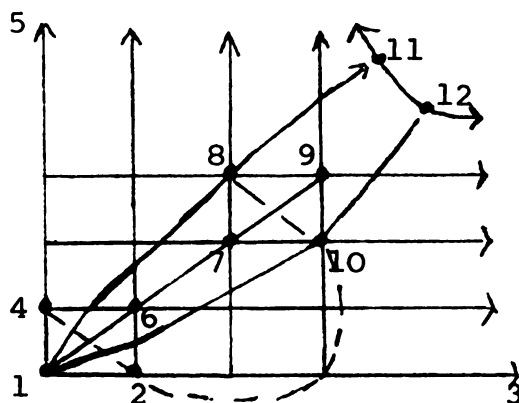
Let λ_s = number of complete quadrilaterals contained in an S.G.

Theorem 2.31:

If one pencil is of type $(4,3,3,3,3)$ hence all pencils are of this type by theorem 2.32, then there exists exactly two distinct non-equivalent 12-point S.G.'s.

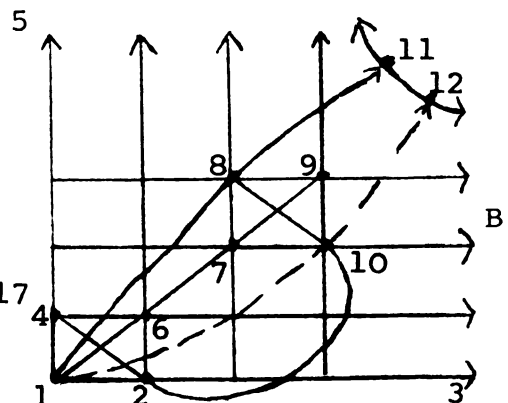
Proof:

Let 3 and 5 be two opposite vertices of a complete quadrilateral of each of the S's which is assured by lemma 2.33. Subject to proper labelling the two S.G.'s have the grid diagrams shown.



A

Fig. 2.17



B

Table for A

312	514	123	213	415	857	1095	634
346	526	145	256	436	839	1073	625
3710	578	1679	24810	48102	82410	10248	6179
389	5910	1811	2911	4711	8111	10611	6812
351112	531112	11012	2712	4912	8126	10112	61011
7310	9167	1118	12110				
758	9211	1129	1227				
7212	938	113512	123511				
7169	9412	1147	1249				
7411	9510	11610	1268				

Table B

312	514	123	213	415	857
346	526	145	256	436	839
3710	578	1679	24810	48102	82410
389	5910	1811	2117	4119	8111
31112	511123	11012	2129	4127	8126
1095	643	7310	9167	1118	12110
1073	625	758	9212	1127	1229
10248	6179	7211	938	113512	123511
10116	6812	7169	9411	1149	1247
10121	61110	7412	9510	11610	1268

It is easy to check that $\lambda_A = 6$ and $\lambda_B = 2$. (see following diagram)

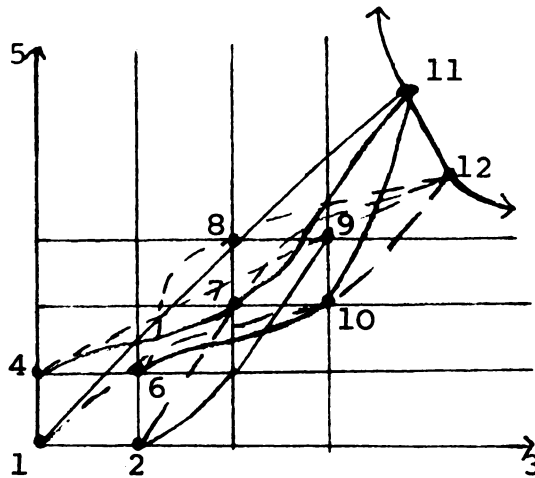
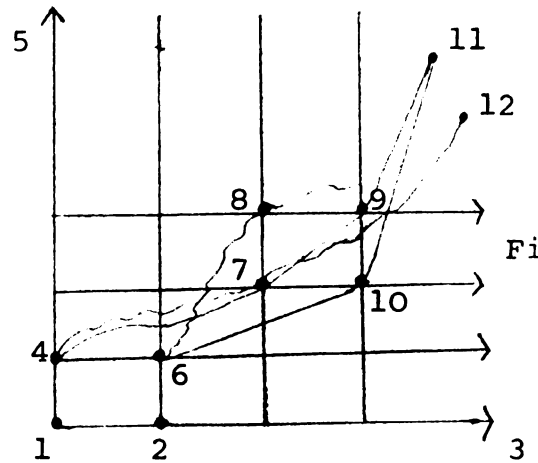
A: $\lambda_A = 6$ B: $\lambda_B = 2$

Fig. 2.

i.e.; the S.G. configurations of order 12 represented by tables A and B are combinatorially distinct.

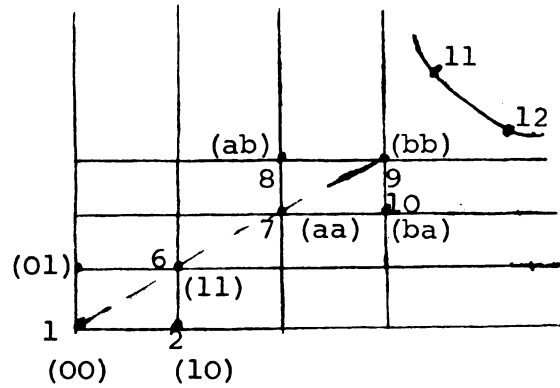
Embedding of A

Fig. 2.19

$$\text{Line 42: } x + y = 1 \Rightarrow a + b = 1 \quad \text{-----} \quad (1)$$

$$\text{For linearity } 1, 8, 11 \quad x = yb^{-1}a$$

$$4, 7, 11 \quad x = (y-1)(a-1)^{-1}a$$

$$\Rightarrow yb^{-1}a = (y-1)(a-1)^{-1}a$$

$$yb^{-1}a = y(a-1)^{-1}a - (a-1)^{-1}a$$

$$y[b^{-1}a - (a-1)^{-1}a] = -(a-1)^{-1}a$$

Since 1, 8, 11 is parallel to 4, 7, 11

we have

$$b^{-1}a = (a-1)^{-1}a$$

$$\Rightarrow b = a - 1 \quad \text{-----} \quad (2)$$

From (1) and (2) we have

$$\left. \begin{array}{l} a + b = 1 \\ a - b = 1 \end{array} \right\} \Rightarrow (2a-2) = 0$$

$$\Rightarrow a = 1 \text{ impossible}$$

$$\text{or } \boxed{2 = 0}$$

Embedding of B

$$\text{Linearity } 42810: \quad a + b = 1$$

$$\begin{aligned}
\text{Line } 1 \ 8 \ 11 \quad x &= yb^{-1}a \\
2 \ 9 \ 11 \quad x-1 &= y[a^{-1}(a-1)] \\
\text{i.e. } yb^{-1}a &= y[a^{-1}(a-1)] + 1 \\
\text{i.e. } y\{b^{-1}a - 1 + a^{-1}\} &= 1
\end{aligned}$$

Since 1,8,11 and 2,9,11 are parallel

$$\begin{aligned}
b^{-1}a + a^{-1} &= 1 \\
\text{i.e. } (1-a)^{-1}a &= 1 - a^{-1} \\
&= -a^{-1}(1-a) \\
\text{i.e. } [a^{-1}(1-a)]^{-1} &= -a^{-1}(1-a) \\
\text{i.e. } -1 &= [a^{-1}(1-a)]^2 \\
-a^2 &= 1 - 2a + a^2 \\
\text{i.e. } \boxed{a^2 - 2a + 1 = 0} &\Rightarrow a = \frac{1+i}{2} \Rightarrow b = \frac{-1+i}{2}
\end{aligned}$$

Thus we observe that S.G. configuration represented by table A embeds in a Desarguesian plane iff the ring is of characteristic 2, while that represented by table B embeds in a Desarguesian plane iff the ring contains 4th roots of unity. ■

Corollary 2.32:

There is one and only one S.G. configuration of order 12 in the complex plane whose structure is determined as a subset of a 5-pencil coordinatized by 4th roots of unity.

This completes the analysis of S.G. configuration of order 12.

3. S.G. configurations as subsets of 3 lines

Theorem 2.33:

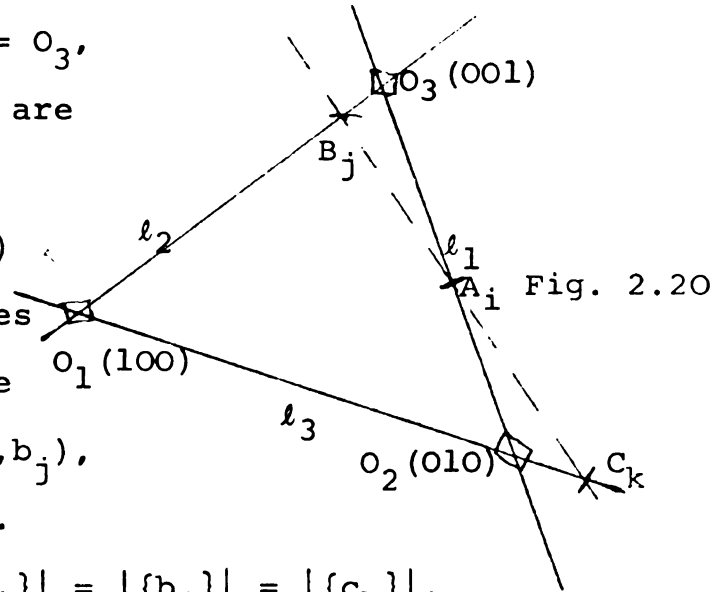
A subset, S, of 3 non-concurrent lines in a projective

plane coordinatized over a field is a finite S.G. configuration if and only if S is projectively equivalent to the set of points $\{(1, a_i, 0), (0, 1, -a_j), (1, 0, a_k)\}$ $i, j, k = 1, 2, \dots, n$ where $\{a_\alpha\}$, $\alpha = 1, 2, 3, \dots, n$ are n th roots of unity.

Proof:

Suppose S , a subset of 3 non-concurrent lines l_1, l_2, l_3 in a Pappian plane, is a finite S.G.

configuration. Now if $l_1 \cap l_2 = O_3$, $l_2 \cap l_1 = O_1$ and $l_3 \cap l_1 = O_2$ are the vertices of the triangle of reference with $O_1(1, 0, 0)$, $O_2(0, 1, 0)$ and $O_3(0, 0, 1)$ then the coordinates of A_i, B_j, C_k may respectively be taken to be $A_i(0, 1, -a_i)$, $B_j(1, 0, b_j)$, $C_k(1, c_k, 0)$ with $a_1 = 1, c_1 = 1$.



It is easily seen that $|\{a_i\}| = |\{b_j\}| = |\{c_k\}|$.

Now suppose A_i, B_j, C_k are collinear

$$\text{Then } \begin{vmatrix} 0 & 1 & -a_i \\ 1 & 0 & b_j \\ 1 & c_k & 0 \end{vmatrix} = 0 \Rightarrow (-1)(-b_j) - a_i c_k = 0$$

$$\Rightarrow \boxed{b_j = a_i c_k} \quad *$$

With the assumption that $C_1 = 1$ we see that the following points are collinear: $(1, 1, 0)$, $(1, 0, b)$ and $(0, 1, -a)$

i.e. $\exists a, b \in \{b_j\} \rightarrow$ for each $a \in \{a_i\}$

$b = a$ from the above equation (*)

$$\Rightarrow \{b_j\} = \{a_i\}$$

Similarly $\{b_j\} = \{c_k\}$

i.e. $\{a_i\} = \{b_j\} = \{c_k\}$.

Since $\{a_i\} = \{b_j\} = \{c_k\} \exists a_3 \in \{a_i\}$: for $a_1, a_2 \in \{a_i\}$

$$a_3 = a_1 a_2$$

i.e. $\{a_i\}$ is a subset of non-zero element of a field F which is closed under multiplication, therefore it is a multiplicative subgroup of F^* . Therefore $\{a_i\}$ is a cyclic group (see Artin [12] page 49).

i.e. $\exists$ a positive integer n such that $a^n = 1$ for some $a \in \{a_i\}$

i.e. the set $\{a_i\} = \{b_j\} = \{c_k\}$ are n th roots of unity.

For the 2nd part of the proof, suppose $\{a_i\}$ is a cyclic subgroup of F^* of some field F . Consider the following set of points $\{(1, a_i, 0), (0, 1, -a_j), (1, 0, a_k)\}$. We observe that the indicated coordinates satisfy

$$a_j = a_i a_k$$

$\therefore$ whenever any point $A_i(1, a_i, 0)$ is joined by a line to a point $A_k(1, 0, a_k)$ the line passes through a third point $A_j(0, 1, -a_j)$ i.e. the indicated set form a finite S.G. configuration on three non-concurrent lines. ■

Theorem 2.34:

A subset, S , of 3 concurrent lines in a projective plane coordinatized over a field F is a finite S.G. configuration iff F has finite characteristic and S is projectively equivalent to the set of points $\{(1, 0, a_i), (1, 1, a_j), (0, 1, a_k)\}$

$i, j, k = 1, 2, \dots, n$ where $\{a_\alpha\}$ $\alpha = 1, 2, \dots, n$ is an additive subgroup of the coordinate field.

Proof:

Suppose S a subset of 3 concurrent lines ℓ_1, ℓ_2, ℓ_3 in a Pappian plane is a finite S.G. configuration.

If $O_3 = \ell_1 \cap \ell_2 \cap \ell_3$ and $O_1 \in \ell_1, O_2 \in \ell_2$ are the vertices of the triangle of reference with $O_1(100)$

$O_2(010)$ and $O_3(001)$ then

the coordinates of A_i, B_j, C_k may respectively be taken to

be $A_i(01a_i), B_j(10b_j)$ and $C_k(11c_k)$, with $a_1 = 0, b_1 = 0$ it is easily seen that

$$|\{a_i\}| = |\{b_j\}| = |\{c_k\}|$$

Now suppose A_i, B_j, C_k are collinear

$$\begin{array}{l} \text{Then } \begin{vmatrix} 0 & 1 & a_i \\ 1 & 0 & b_j \\ 1 & 1 & c_k \end{vmatrix} = 0 \Rightarrow -[c_k - b_j] + a_i = 0 \\ \Rightarrow \boxed{c_k = a_i + b_j} \end{array}$$

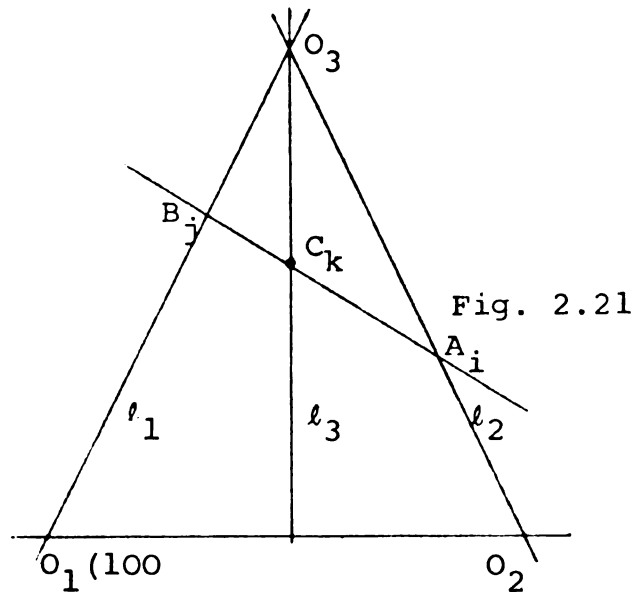
With the assumption that $a_1 = 0$ we see that the following points are collinear: $(010), (10b),$ and $(11b)$

$\exists c \in \{c_k\}$ such that for each $b \in \{b_j\}$

$$c = b \Rightarrow \{c_k\} = \{b_j\}$$

Similarly

$$\{c_k\} = \{a_i\}.$$



$$\therefore \{a_i\} = \{b_j\} = \{c_k\}.$$

Since $\{a_i\} = \{b_j\} = \{c_k\} \ni a_3 \in \{a_i\}$: for $a_1, a_2 \in \{a_i\}$

$$a_3 = a_1 + a_2$$

i.e. $\{a_i\}$ is a subset of the coordinate field F closed under addition, therefore it is an additive subgroup of the coordinate field. Conversely, suppose $\{a_i\}$ is an additive subgroup of F , then whenever we join the point $A_i(01a_i)$ on ℓ_2 to the point $B_j(10b_j)$ on ℓ_1 we get a point $C_k(11c_k)$ on ℓ_3 . Also if we join $B_j(10b_j)$ on ℓ_1 to a point $C_k(11c_k)$ on ℓ_3 we get a point $A_i(01a_i)$ on ℓ_2 since $a_i = c_k - b_j$ is an element of the additive group $\{a_i\}$. i.e. A_i, B_j, C_k are point of an S.G. and a subset of 3 concurrent lines. For the remaining part of the proof we have to show that the field has finite characteristic.

We have that $C(11a), B(100)$ and $A(0,1,a)$ are collinear. Also $(11a), (010)$ and $(10a)$ are collinear and $(10a), (112a)$ and $(01a)$ are collinear. Similarly $(100), (112a), (012a)$ are collinear; $(010), (112a), (102a)$ are collinear and $(102a), (114a), (012a)$ are collinear and so on, for some $a \in F$.

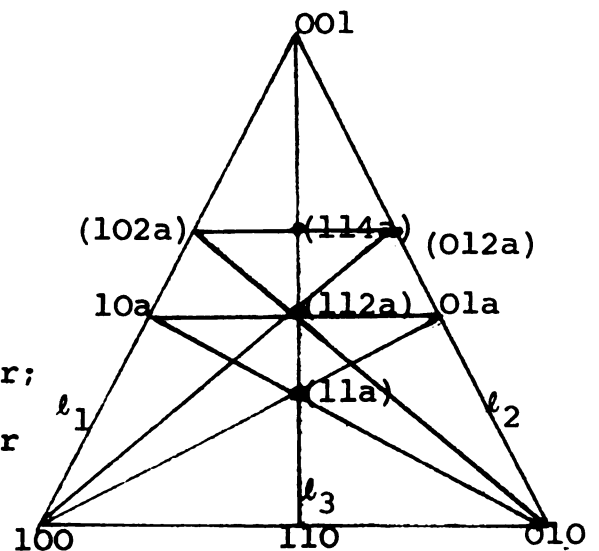


Fig. 2.2

For the configuration to close we need the last point generated on ℓ_3 to coincide with some earlier point on ℓ_3 . Thus we need $2^k a = 2^l a$ for some positive integers k and l where $l < k$ (say).

$$\Rightarrow 2^{\ell}(2^{k-\ell}-1)a = 0$$

$$\Rightarrow 2^{\ell}(2^m-1)a = 0 \quad \text{for some positive integer } m.$$

Since $a \neq 0$ we need either $2 = 0$ i.e. coordinate field has characteristic 2 or $p = 0$ where p is a prime divisor of 2^m-1 (since 2^m-1 contains a prime factor $\neq 2$) i.e. characteristic p . Thus in all cases the coordinate field has finite characteristic. This completes the proof of the theorem. ■

Remark 2.35:

Theorem 2.36 corrects the mistake in the theorem on page 460 of Motzkin [8].

4. On the existence of S.G. configurations
in complex projective 3-space

Motivated by the question raised by J. P. Serre (AMM 73, 1966) we, in this section, show that there is no non-planar Sylvester-Gallai configuration in complex projective 3-space with fewer than 40 points. It is convenient to confine the argument to the C_3 setting but the reasoning for the most part is applicable to more general spaces and permits us eventually to characterize all non-planar Sylvester-Gallai configurations of fewer than 40 points in any projective 3-space.

Lemma 2.36:

If a Sylvester-Gallai configuration, S , is a subset of a 4-pencil having lines $\ell_1, \ell_2, \ell_3, \ell_4$ with vertex P

and if the associated sequence (n_1, n_2, n_3, n_4) has the properties $n_1 - 1 = x \geq n_2 - 1 = y \geq n_3 - 1 = z \geq n_4 - 1 = w$ then $|S| \geq x + 9$.

Proof:

Observe that $x = |\ell_1 \cap (S-P)|$; $y = |\ell_2 \cap (S-P)|$; $z = |\ell_3 \cap (S-P)|$; $w = |\ell_4 \cap (S-P)|$. If Q is a point of $\ell_2 \cap (S-P)$, then the x lines joining Q to points of $\ell_1 \cap (S-P)$ intersect $(\ell_3 \cup \ell_4) \cap (S-P)$ and hence $z+w \geq x$. Similarly $y+z \geq x$ and $y+w \geq x$. Thus $y+z+w \geq \frac{3}{2}x$ and $|S| \geq x + \frac{3}{2}x$.

Now if $x \geq 6$, $|S| \geq x + 9$. If $x = 5$ and $w \geq 3$ then $y + z \geq 6$, $\Rightarrow |S| \geq x + 9$. If $x = 5$ and $w = 2$, again $y + z + 2w \geq 2x$, $y + z \geq 6$. Keeping in mind that P must be an element of S we again have $|S| \geq x + 9$. If $x = 4$, it follows from theorem 2.28 that $|S| \geq 13 = 4 + 9$. Finally if $x = 3$ then y, z, w are all 3 and $|S| > x + 9$. ■

Lemma 2.37:

If in complex projective 3-space a Sylvester-Gallai configuration, S , is a subset of a pencil of four planes having axis ℓ , and if ℓ' is a line in one of these planes α with $|\alpha \cap S| \geq 13$, $|\ell' \cap S| \geq k \geq 3$, and $|\ell \cap \ell' \cap S| = 0$ then $|S| \geq 40$.

Proof:

Let β be any plane containing ℓ' and a point of S not on α . $S \cap \beta$ is a planar Sylvester-Gallai configuration and is a subset of 4-pencil with vertex $P = \ell \cap \ell'$. It follows from lemma 2.48 that $|\beta \cap S| \geq k + 9$.

Now there are at least three distinct planes through ℓ' intersecting S in points not on ℓ . Hence

$$|S| \geq 3(9) + |\alpha \cap S| \geq 27 + 13 = 40. \quad \blacksquare$$

Theorem 2.38:

If S is a Sylvester-Gallai configuration spanning complex projective 3-space, then $|S| \geq 40$.

Proof:

Let ℓ be a line of C_3 .

Case 1: $|\ell \cap S| = 7$.

There is clearly a line ℓ' skew to ℓ such that $|\ell' \cap S| = k \geq 3$. There are at least 7 distinct planes through ℓ' intersecting points of S not on ℓ . If α is any one of these planes then $|\alpha \cap S| \geq k + 6$. Thus $|S| \geq 7(6) + k \geq 45$.

Case 2: $|\ell \cap S| = 4$.

If there are 5 planes containing ℓ and intersecting $S - \ell$ then $|S| \geq 5(8) + 4 = 44$. Since no Sylvester-Gallai configuration spanning C_3 is a subset of a pencil of

3 planes (vide lemma) we may assume that S is a subset of precisely four planes containing ℓ . Suppose α and β are two of these planes with $|\alpha \cap S| = |\beta \cap S| = 12$.

From lemma 232 detailing the structure of Sylvester-Gallai configuration of order 12 it is clear that there are then lines ℓ_1 in α , and ℓ_2 in β such that $|S \cap \ell_1| = |S \cap \ell_2| = 4$ and $(\ell_1 \cap \beta) \cap S = \emptyset$. There are thus five distinct planes through ℓ_2 intersecting $S - \ell_2$ and hence $|S| \geq 5(8) + 4 = 44$.

Suppose then, that one of the four planes through ℓ contains 12 points of S . The only possible distribution of points of S on these four planes that does not immediately lead to the conclusions that $|S| \geq 40$ is that in which three of the planes contains precisely 13 points and the remaining plane contains precisely 12 points. Suppose α is one of the planes with 13 points of S and β the one with 12 points. If P is a point of $S \cap (\alpha - \ell)$ and Q is a point on $S \cap (\alpha - \ell)$ not on any of the four lines joining P to points of $S \cap \ell$, then it follows from lemma 2.49 that $|S| \geq 40$. If no such Q exists, then for at least one of the lines, say ℓ' , joining P to a point of $S \cap \ell$ $|S \cap \ell'| \geq 4$. There are now at least 5 planes containing ℓ' and points of $\beta \cap S$ and $|S| \geq 40$.

Case 3: $|\ell \cap S| = 5$

As in the previous cases we can proceed at once to assume that there are precisely four planes containing ℓ and points of $S - \ell$. If each of these contain more than

13 points of S then $|S| \geq 4(9) + 5 = 41$. We may thus assume that for one of the planes α , $|\alpha \cap S| = 12$. It now easily follows that $S \cap \alpha$ is a subset of a 4-pencil two of which contain 5 points of S and the other two of which contain 3 points each. In such a configuration there is a line ℓ' with either $|\ell' \cap S| = 4$ or $|\ell' \cap S| = 3$ and $\ell' \cap \ell \cap S = \emptyset$. In the first instance the theorem follows from case 2 and in the latter from lemma 2.40.

Case 4: $|\ell \cap S| = 6$.

First observe that there is a line ℓ' skew to ℓ such that $|\ell' \cap S| \geq 3$. Now through ℓ' there are at least six planes intersecting $S - \ell'$. Hence $|S| \geq 6(6) + 3 = 39$.

Now suppose as usual that S is a subset of a pencil of four planes through ℓ . For at least one of these planes, α , $|\alpha \cap S| > 13$. Let $P \in S \cap (\alpha - \ell)$. If Q is a point of $S \cap (\alpha - \ell)$ not on any of the six lines joining P to the points of $S \cap \ell$ then $|S| \geq 40$ by lemma 2.49.

If no such Q exists one of the six lines joining P to the points of $S \cap \ell$ may be assumed to contain precisely 6 points of S . Thus $|\alpha \cap S| \geq 16$. Since $3(7) + 16 = 37 < 39$ it follows that either $|\alpha \cap S| \geq 19$ or for some second plane β , $|\beta \cap S| \geq 16$. In either case $|S| \geq 40$.

Case 5: For all ℓ , $|\ell \cap S| \leq 3$.

If for any plane, α , $|\alpha \cap S| > 13$, then for any point $P \in S \cap \alpha$ there are at least 7 lines containing P and

points of $(S \cap \alpha) - P$. If ℓ' is a line joining P to a point of $S - \alpha$ then ℓ' is contained in at least 7 distinct planes intersecting $S - \ell'$. Hence $|S| \geq 7(6) + 3 = 45$. We are thus led to assume that for all planes, α , containing at least 3 non-linear points of S , $|\alpha \cap S| = 13$ or $|\alpha \cap S| = 9$.

If $|\alpha \cap S| = 13$, then any point P of $S \cap \alpha$ is on 6 distinct lines containing points of $(S \cap \alpha) - P$. If ℓ' , as above, is a line joining P to a point of $S - \alpha$, then ℓ' is on six planes containing points of $S - \ell'$. Thus $|S| \geq 6(6) + 3 = 39$.

Suppose, in fact, that $|S| = 39$, $\ell \cap S \subset \alpha$, $|S \cap \alpha| = 13$. There must be $m+n$ planes each containing and intersecting $S - \ell$, with $39 = 10m + 6n + 3$, $m \geq 1$. This means $m = 3$, $n = 1$. That is S is a subset of a pencil of four planes through ℓ three of which contain 13 points of S and one of which contains 6 points.

If Q is a point of $\ell \cap S$ and ℓ^* a line joining Q to a point of $S \cap (\beta - \ell)$, where β is one of the planes containing 13 points of S , then since $|\alpha \cap S| = 13$, there are 6 lines through Q in α each intersecting S . Hence, there are 6 planes containing ℓ^* and points of $S - \ell^*$. One of these planes is β . Thus $|S| \geq 5(6) + 10 + 3 = 43$. It remains to consider the case in which for all planes α intersecting S in three non collinear points $|\alpha \cap S| = 9$. We will show that such configurations do not exist.

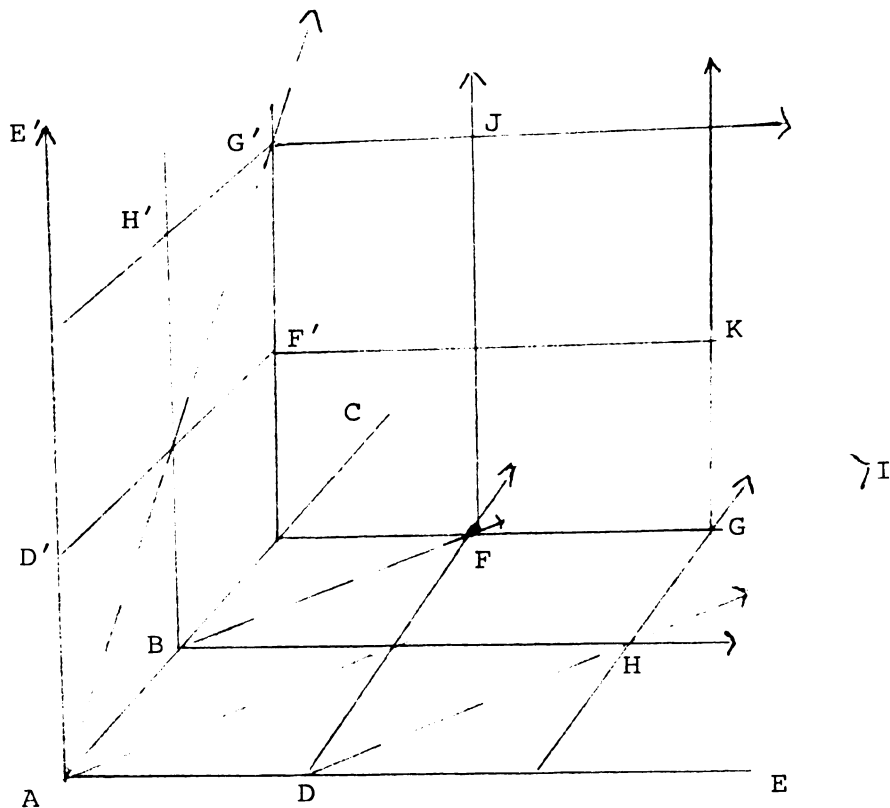


Fig. 2.3

Suppose, in fact, that A, B, C are three collinear points of such a configuration S , and α and β two planes containing A, B, C and intersecting $S - \{A, B, C\}$. Let A, D, E be collinear points of $S - \{A, B\}$ in α and A, D', E' be collinear points in $S - \{A, B\}$ in β .

We now introduce affine coordinates, taking the plane E, E', C as ideal plane, in such a way that $A(0, 0, 0)$, $D(1, 0, 0)$, $B(0, 1, 0)$, $D'(0, 0, 1)$. From the known structure of a 9 point planar S.G. configuration (see page 20) it follows that the remaining four points of $\alpha \cap S$ are $H(a, 1, 0)$, $F(1, \frac{1}{a}, 0)$, $G(a, \frac{1}{a}, 0)$ and the ideal point $I = AG \cap BF \cap DH \cap CE$. Similarly in plane β point $H'(0, 1, a)$, $F'(0, \frac{1}{a}, 1)$, $G'(0, \frac{1}{a}, a)$, and the ideal point $I' = AG' \cap D'H' \cap BF' \cap CE'$ are all points of $\beta \cap S$.

Now F, G, E, F', G', E' are in a plane γ . An easy computation shows that $J(1, \frac{1}{a}, a)$ and $K(a, \frac{1}{a}, 1)$ are also in $\gamma \cap S$.

Finally we observe that B, H', E', F, J, L are in a plane S , and that they are the six vertices of a complete quadrilateral. But a planar S.G. configuration of 9 points does not contain a complete quadrilateral as a subconfiguration so it follows that $|\delta \cap S| \neq 9$.

This contradiction shows that no S.G. configuration exists in C_3 all of whose plane sections are 9 point S.G. configurations and completes the proof of the theorem. ■

Chapter 3

The structure of a GSG spanning ordered 3-space.

Introduction:

Definition 1.2 of the introduction described generalized Sylvester-Gallai configurations in the context of projective or linear spaces. We proceed now to define an abstract GSG.

Definition 3.0:

If the point set, S , of a finite linear configuration, C , is partitioned into sets $\{S_i\}$, $i \geq 2$ such that each pair of points from two different sets S_i and S_j is on a line of C and if no line of C intersects exactly two of the sets $\{S_i\}$, then C is a generalized Sylvester-Gallai configuration or simply a GSG. The lines of the configuration are its secants and the sets S_i are called the (constituent)sets of C .

We first reproduce the Kelly-Edelstein proof that a GSG cannot span ordered projective k -space for $k > 3$. The remainder of the chapter is designed to throw light on their conjecture that the only GSG spanning ordered projective k -space, $k > 2$, is the 12 point desmic configuration (see introduction, page 4) discovered by Stephanos in 1890.

Theorem 3.1:

There exists no GSG spanning ordered projective 4-space.

Proof:

Suppose, in fact, that S is a GSG spanning an ordered space, S^4 , of 4-dimensions. Let $\{A_i\}$ be the constituent point sets of the configuration. Consider the line L through $p_1 \in A_1$ and $p_2 \in A_2$ and let S_1^3 be a 3-space not containing L . Then S_1^3 together with p_1 spans S^4 .

The line L and each point of $S-L$ determines a plane. All such planes may be described in a picturesque way as forming a book of planes with back L . Now $L \cap S_1^3 = P$ and the planes of the book intersect S_1^3 in a bundle of lines with vertex P .

Motzkin observed in his 1951 paper [8] that for such a bundle in 3-space there exists a plane containing precisely two lines of the bundle. This follows, at once, if we consider a section of the bundle by a plane and apply theorem 2.1.

The 3-space, S_2^3 , spanned by these two lines and L contains precisely two pages of our book. It is now an easy matter to check that if a collection of two or more finite non-empty and disjoint sets in S_2^3 lie on two planes and not on one, then there is a line intersecting precisely two of the sets. This contradiction establishes the theorem. ■

PROOF:

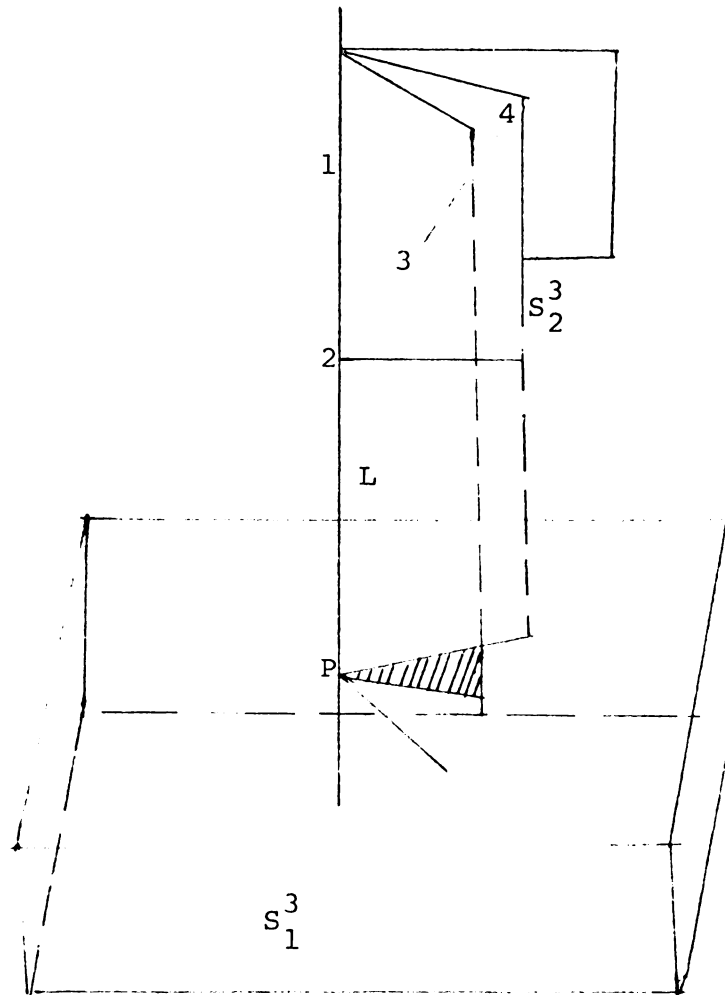


Fig. 3.1

Theorem 3.1 implies that GSG's in ordered projective spaces are confined to two and three dimensions. The next two theorems, due to M. Edelstein are the only ones known to us which throw much light on the structure of such sets.

Definition 3.1:

A k -secant of a collection of sets is simple if it intersects no set of the collection in more than one point.

Theorem 3.2:

If a finite family, $\{S_i\}$, of disjoint finite sets spans an ordered projective space of dimension 3 and has no 2-secants, then all k -secants, $k > 2$, are simple.

That is to say all k -secants of the sets of a GSG $k > 1$ are simple.

Proof:

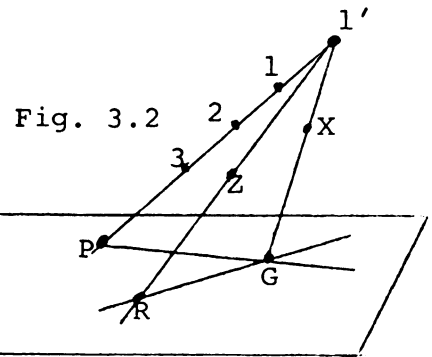
Suppose, to the contrary, that ℓ_1 is a k -secant with $k > 2$ which is not simple. We may assume the labelling so that points 1 and 1' are in $S_1 \cap \ell_1$, $2 \in S_2 \cap \ell_1$, $3 \in S_3 \cap \ell_1$. Let π be a plane such that $\pi \cap S_i = \emptyset$ for $i = 1, 2, \dots, n$. Now centrally project all points of $\bigcup_{i=2}^n S_i$ from 1' onto π and let T be the set of these projections in π , and P the projection of 2. Let $Q \in T$ and $X \in \bigcup_{i=2}^n S_i$ which projects into Q . Since $X \notin S_1$ the line $1X$ contains a point $Y \in \bigcup_{i=2}^n S_i$ other than X . The projection of Y onto π is a point of T on PQ . Hence PQ is not a 2-secant of T .

It is now clear that T is not the point set of a near pencils and hence that the set of lines defined by the pairs of points of T which do not pass through P partition π into polygonal regions. It is shown in [6] that if there are no 2-secants in T through P then the side lines of the residence of P are all 2-secants of T .

Now if QR is a side line of the residence of P , with $Q, R \in T$, there is a point $Z \in \bigcup_{i=2}^n S_i$ whose projection is

R, where X and Z are in different sets.

Hence the line joining X and Y must contain a point of S_1 since if it contains a point from $\bigcup_{i=2}^n S_i$, RQ would not be a 2-secant of T. Thus we can assume a point $1'' \in S_1$ whose projection S^* is on RQ.



Now consider $T \cup S^*$. It is easy to verify that the lines defined by pairs of points of this set are precisely those defined by pairs of points of T. Hence the residence of P relative to $T \cup S^*$ is the same as the residence of P relative to T. Every line of $T \cup S^*$ through P contains at least three points of $T \cup S^*$, but one of the side lines of the residence of P, namely QR, is not a 2-secant of $T \cup S^*$. This is a contradiction and proves the theorem. ■

Theorem 3.3:

If G is a GSG whose point sets $\{S_i\}$ span an ordered projective 3-space then no line intersects more than three of the sets.

Proof:

Suppose G has a k-secant ℓ with $k > 3$. From theorem 3.2 we know that ℓ must be simple and the notation may be chosen so that $\ell \cap S_i = i$, $i = 1, 2, 3, 4, \dots, k$, with points $1, 2, \dots, k$ in cyclic order.

As in the proof of theorem 3.2 we centrally project $\bigcup_{i=1}^n S_i$ from l onto a plane π with P the projection of l and T the set of projection of point of $\bigcup_{i=1}^n S_i$. If $Q \neq P$ is any point of T we again wish to show that the line PQ intersects T in yet another point. The argument then will proceed precisely as in theorem 3.2.

To show that PQ contains a third point of T we consider a point $X \in l' \cap \bigcup_{i=1}^n S_i$, where $l' = lQ$, such that l and X are not separated by any points of $l' \cap \bigcup_{i=1}^n S_i$. Suppose $X \notin S_3$. The line $3X$ must then contain a point of $\bigcup_{i=1}^n S_i$. If not there is a point $X' = S_1 \cap 3X$. If $X' \cap l' = \alpha \in \bigcup_{i=1}^n S_i$, $X' \cap l' = \beta \in \bigcup_{i=1}^n S_i$ then $l \alpha X \beta$ and l and X are separated by two points of $\bigcup_{i=1}^n S_i$ contrary to the definition of X . If $X \in S_3$ but $k > 4$ the same argument shows that the line $4X$ has a third point of $\bigcup_{i=1}^n S_i$ on it. Thus in these cases there is a third point of T on line PQ .

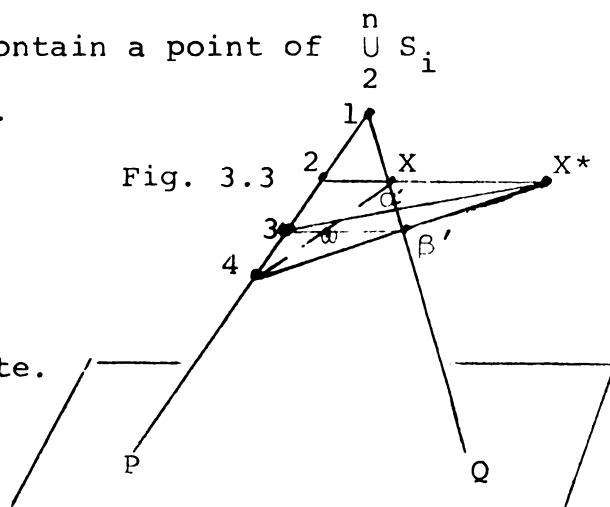
It remains to consider the case in which $X \in S_3$ and $k = 4$.

If the line $2X$ does not contain a point of $\bigcup_{i=1}^n S_i$ it must contain a point $X^* \in S_1$.

Let $X^* \cap l' = \alpha' \in \bigcup_{i=1}^n S_i$
 $X^* \cap l' = \beta' \in \bigcup_{i=1}^n S_i$. Again

line $3\beta'$ contains a point w of S_1 or the proof is complete.

Since $w \notin S_3$, XW contains a third point of $\bigcup_{i=1}^n S_i \cap l$. This



point is clearly not 1, 2, or 3 and hence must be 4. Let $\alpha' \cap \ell = u \in US_i$. Now $1\ 2\ 3\ 4 \ \wedge\ 1\ \alpha'\beta' \wedge 1\ 4\ u\ 3$. Thus $u \neq 2$ and we have a contradiction to the assumption that $k = 4$.

Thus in all cases the lines PQ intersect T in at least three points.

The proof now proceeds in a completely analogous fashion to that of theorem 3.2 producing a contradiction to the assumption that $|\ell \cap US_i| \geq 3$. ■

We have previously noted that the only known example of a three dimensional GSG in an ordered projective space is the 12 point desmic configuration consisting of three sets of four points each i.e. the configuration of type $(12_4, 16_3)$.

Since our efforts to characterize generalized Sylvester-Gallai configurations were not too fruitful we turned to the special case in which the number of sets in the configuration is three. It is easy to see that in this case $|S_1| = |S_2| = |S_3|$. We refer to such configurations as desmic triads or generalized desmic configurations (GDC).

Definition 3.2:

A GSG having exactly three sets all of whose 3-secants are simple is a desmic triad.

In a desmic triad each point of set S_i is a center of perspectivity for S_j and S_k (i, j, k) a permutation of the integers 1, 2, 3. Thus sets S_j and S_k are

multiply perspective under $|S_i|$ different perspectivities. Thus the problem of constructing desmic triads in various spaces may be looked at as that of constructing two sets of n points which are perspective from n different centers. We will now show that a necessary condition that this be possible in a given space is that it be possible to find an n -point set in a plane of that space which is completely self perspective.

The theorem showing the techniques used in the subsequent analysis will now be proved.

Theorem 3.4:

Suppose A, B, C are three sets of k points each, in a projective 3-space S which have no 2-secants. Let π be a plane of S with $A \cap \pi = B \cap \pi = C \cap \pi = \emptyset$. Let all secants be simple. Let φ be a central projection of S onto π with center $a_1 \in A$. Define $\varphi(b_i) = \varphi(c_i) = p_i \in \pi$, $\varphi(B) = \varphi(C) = P$ and $\varphi(a_i) = q_i$, $i = 2, 3, \dots, n$. Then P is self-perspective from each of the points q_i .

Proof:

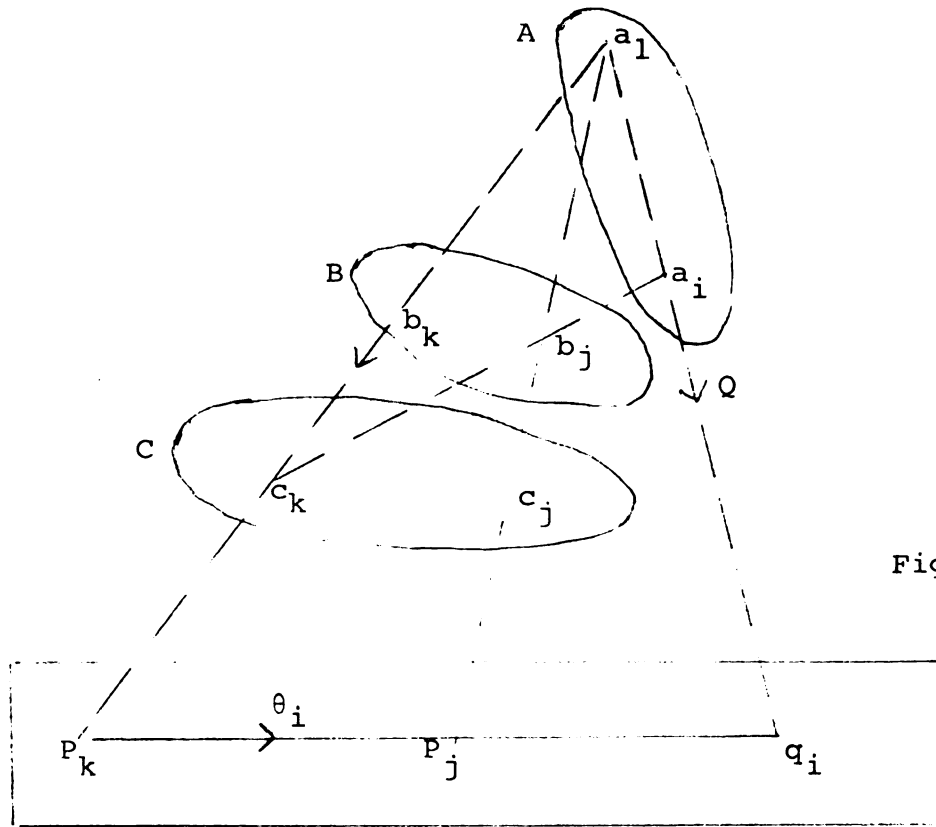


Fig. 3.4

Define $\theta_i: P \rightarrow P$ as follows:

Let $\overline{c_k a_i} \cap B = b_j$, then $\theta_i(p_k) = \varphi(b_j)$ (where $\overline{c_k a_i}$ is the line joining c_k to a_i). Indeed θ_i is uniquely defined, otherwise $\overline{c_k a_i}$ will be non-simple. Also θ_i is 1-1 since $\theta_i(p_k) = \theta_i(p_\ell) \Rightarrow \overline{c_k a_i} \cap B = \overline{c_\ell a_i} \cap B \Rightarrow \overline{c_k a_i} = \overline{c_\ell a_i} \Rightarrow c_k = c_\ell \Rightarrow p_k = p_\ell$. Finally θ_i is a self-perspectivity of P with center q_i . Clearly p_k, p_j, q_i are collinear since they are images of the collinear points c_k, b_j, a_i under φ . ■

Remark 3.8:

If no three points of P are collinear then the θ_i 's are involutory.

Indeed; let $\theta_i(p_k) = p_j$. Now for $\theta_i(p_j)$ we consider $\overline{c_j a_i} \cap B$, since $\overline{c_j a_i}$ is coplanar with $\overline{p_k p_j q_i}$ and no other points of P are in this plane we must have $\overline{c_j a_i} \cap B = b_k$. i.e. $\theta_i(p_j) = p_k$ i.e. $\theta_i^2 = 1, \forall i$ ■

Lemma 3.6:

For each i , $\theta_i(p) \neq p \forall p \in P$

Proof:

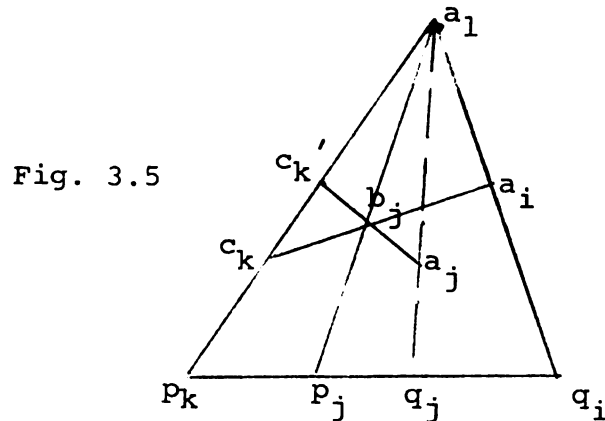
Suppose the statement is false, then the points a_1, a_i, b_j , and c_k are collinear with p_k . i.e. $\overline{a_1 a_i}$ is a non-simple secant in $A \leftrightarrow$ ■

Lemma 3.7:

$\theta_i(p) \neq \theta_j(p)$ if $i \neq j$

Proof:

Suppose the statement is false; consider the accompanying figure. We observe that $a_1 p_k$ is a non-simple secant cutting C at c_k and c_k' . $\leftrightarrow$ ■



Theorem 3.8:

A necessary condition for a GDC to exist in a projective 3-space is that in π the set P be self perspective from $n-1$ centers.

Proof:

In defining the self-perspectivities θ_i of P we obtain a 1-1 correspondence between the centers q_i and the points a_i , $i = 2, \dots, n$ of A . ■

Remark 3.9:

The q_i 's must be distinct from the p_k 's otherwise we have a non-simple secant in A , namely since $\overline{c_k b_j a_i}$ is mapped by θ onto $\overline{p_k p_j q_i}$ if $p_j = q_i$ (say) then a_i lies on the line $\overline{a_i b_j c_j}$. The q_i 's need not be distinct points.

The above theorem 3.12 holds true in any space, not only in ordered projective spaces since no order is involved in theorem 3.8 - 3.11 leading to it. Thus we shall make use of it in the more general analysis to be undertaken in chapter 4.

3.2 Completely self perspective sets of low orders.

Suppose S a completely self perspective set in an ordered projective space and P, Q two distinct centers of perspectivity.

S is then a subset of two pencils $[P]$ and $[Q]$ formed by joining P and Q respectively to points of S . Let p and q be the number of rays in these pencils. If $S^* = [P] \cap [Q]$ then $S \subset S^*$ and $|S^*| = pq$. It is convenient to consider the line PQ as the ideal line and to introduce affine coordinates. S may be then viewed as a subset of the "grid" $S^* = \{(a_i, b_j)\}$, $a_1 < a_2 < \dots < a_p$, $b_1 < b_2 < \dots < b_q$.

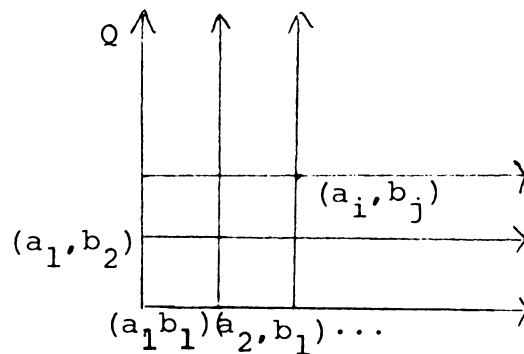


Fig. 3.6

It is, of course, clear that any quadruple of points no three collinear is a completely self perspective set, while no set of five points is of this type.

We now proceed to show that there are no completely self perspective sets, S , in an ordered projective plane with $|S| = 6, 7, 8$. More general conclusions can be drawn about completely self perspective sets of orders 6 and 7 in more general spaces. This will be done in chapter 4. However the analysis of the case $|S| = 8$ by strictly algebraic arguments seems to be very involved.

Theorem 3.10:

There are no completely self-perspective sets, S , in an ordered projective plane with $|S| = 6, 7, 8$.

3.10.1 Completely self perspective sets of order 6. $|S| = 6$.

In the associated grid diagram either $|S^*| = 6$ or $|S^*| = 9$. In the first instance we may assume $p = 2, q = 3$ and $S^* = S = \{(a_1, b_1), (a_2, b_1), (a_3, b_1), (a_1, b_2), (a_2, b_2), (a_3, b_2)\}$.

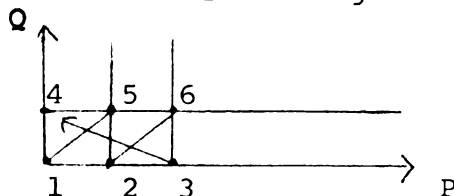


Fig. 3.7

Let $(a_1, b_1) = 1, (a_2, b_1) = 2, (a_3, b_1) = 3, (a_1, b_2) = 4, (a_2, b_2) = 5, (a_3, b_2) = 6$.

If θ_{15} denotes a perspectivity such that $\theta_{15}(1) = 5$ then $\theta_{15}(2) = 6$ or $\theta_{15}(2) = 4$. The latter is immediately ruled out since $\theta_{15}(2) = 4 \Rightarrow \theta_{15}(3) = 6$ which violates the assumption that the perspectivities are disjoint.

If $\theta_{15}(2) = 6$ then $\theta_{15}(3) = 4$. But obvious order relations show that lines 15, 26, 34 are not concurrent and θ_{15} could not have a center.

Hence $|S| = |S^*| = 6$ is impossible

Now suppose $|S^*| = 9, p = 3, q = 3$.

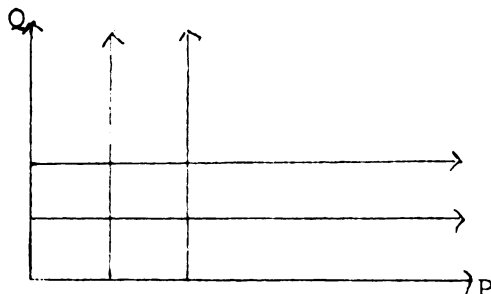


Fig. 3.8

Let $T = \{(a_1, b_1), (a_1, b_3), (a_3, b_1), (a_3, b_3)\}$, i.e. T is the set of "corners" of the grid.

It is immediately clear that S must contain at least two points of T and at least two opposite corners.

Case 1: $S \cap T = \{(a_1, b_1), (a_3, b_3)\}$.

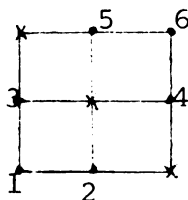


Fig. 3.9

$$\Rightarrow S = \{(a_1, b_1), (a_2, b_1), (a_3, b_2), (a_2, b_3), (a_3, b_3)\}$$

Heavy dots indicate points of S , crosses points of $S^* \setminus S$.

$\theta_{23}(1) = 4$, or $\theta_{23}(1) = 5$, or $\theta_{23}(1) = 6$. The following three grid diagrams show that none of these is possible.

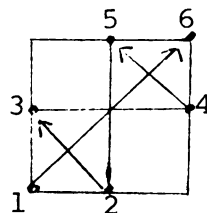
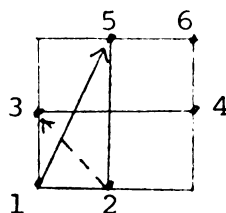
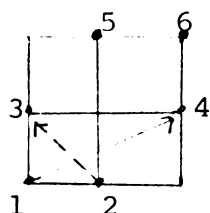


Fig. 3.10

In the first two respectively θ_{23} and θ_{46} , θ_{23} and θ_{56} are not disjoint while in the third diagram obvious betweenness relations make it impossible for θ_{23} to have a center.

Case 2: $S \cap T = \{(a_1, b_1), (a_3, b_3), (a_3, b_1)\}$

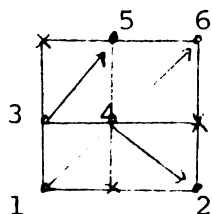


Fig. 3.11

$$\Rightarrow S = \{(a_1, b_1), (a_3, b_3), (a_3, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3)\}$$

$$\theta_{16}(4) = 2 \quad \theta_{16}(3) = 5$$

But lines 16, 35, 24 do not concur and θ_{16} has no center. All other cases are isomorphic to either case 1 or case 2.

Thus there are no completely self perspective sets of order 6 in an ordered projective space.

Again we remark that for $|S| = 6$ a more algebraic analysis can and will be made. But the presentation given here paves the way for the much more complicated treatment of $|S| = 8$.

3.10.2. Completely self perspective sets of order 7. $|S| = 7$.

The situations in which $|S^*| = 8$ or $|S^*| = 12$ are easily seen to be combinatorially impossible. For $|S^*| = 9$, $p = q = 3$ there are 3 non-isomorphic case depicted by the diagrams below

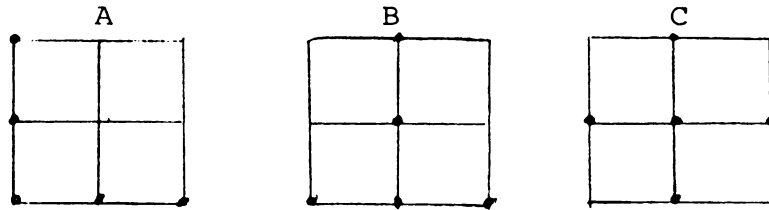


Fig. 3.12

Associated with the A diagram there are two possibilities for S

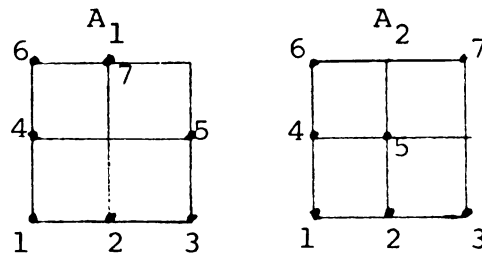


Fig. 3.13

Since in A_1 no triples other than $\{1,2,3\}$ and $\{1,4,6\}$ are linear, S has at most two self perspectivities.

For similar reasons there can be at most two self perspectivities of S in the A_2 case.

Diagrams B and C give rise to essentially one possibility for S in each case

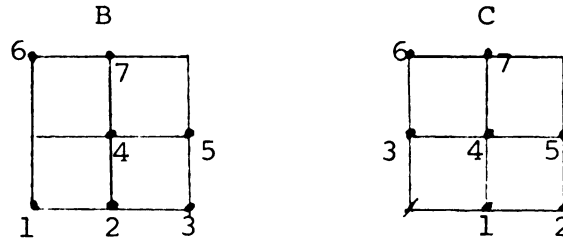


Fig. 3.14

The analysis of linear triples shows that S in B has at most four possible self perspectivities while S in C can have at most two.

Thus if S is a subset of an ordered projective space and $|S| = 7$ then S is not a completely self perspective set.

3.1Q3. Completely self perspective sets of order 8. $|S| = 8$.

If $|S^*| = 8$ we have the essentially unique diagram shown

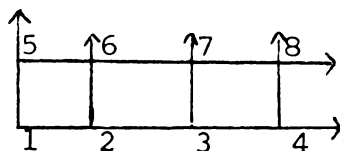


Fig. 3.15

The following diagrams depict the possibilities for θ_{16}

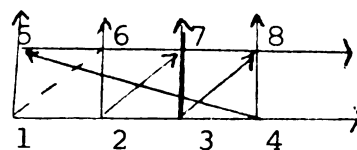
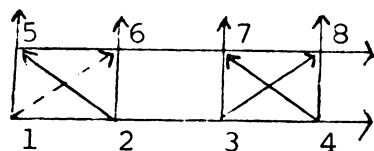


Fig. 3.16

In neither case can θ_{16} have a center, so $|S^*| = 8$ is impossible.

The case $|S^*| = 12$ is very easily handled by methods discussed in chapter four while its analysis by ordered grid diagrams is a little awkward. Accordingly we refer the reader to chapter four for a discussion of this case. The discussion there shows that in an ordered projective space $|S^*| = 12$, $|S| = 8$ is impossible.

We turn to the final and most difficult case in which $|S^*| = 16$.

Let $S^* = \{(a_i, b_j)\} \quad i, j = 1, 2, 3, 4.$

$T = \{(a_1, b_i)\} \cup \{(a_i, b_1)\} \quad i = 1, 2, 3, 4.$

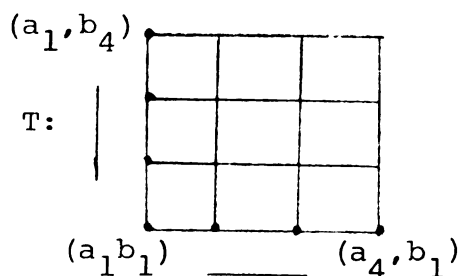


Fig. 3.17

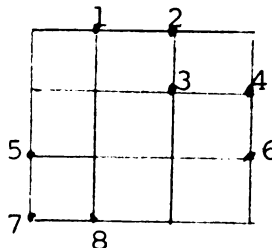
Our analysis is organized into a number of cases which can be described informally as follows:

1. 1 corner, 2 adjacent points
2. 1 corner, 1 adjacent, 1 non adjacent point
3. 1 corner, no adjacent points.
4. 2 corners adjacent, 2 non adjacent points.
- 5a,b,c. 2 corners adjacent, 1 adjacent point.
- 6a,b,c. 2 corners opposite, 2 adjacent points.
- 7a,b. 2 corners opposite, non adjacent points.
- 8a,b,c. 2 corners opposite, 1 adjacent point.
- 9a,b. 3 corners, 1 adjacent point.
10. 3 corners, non adjacent point.
11. 4 corners.
12. No corners.

Case 1:

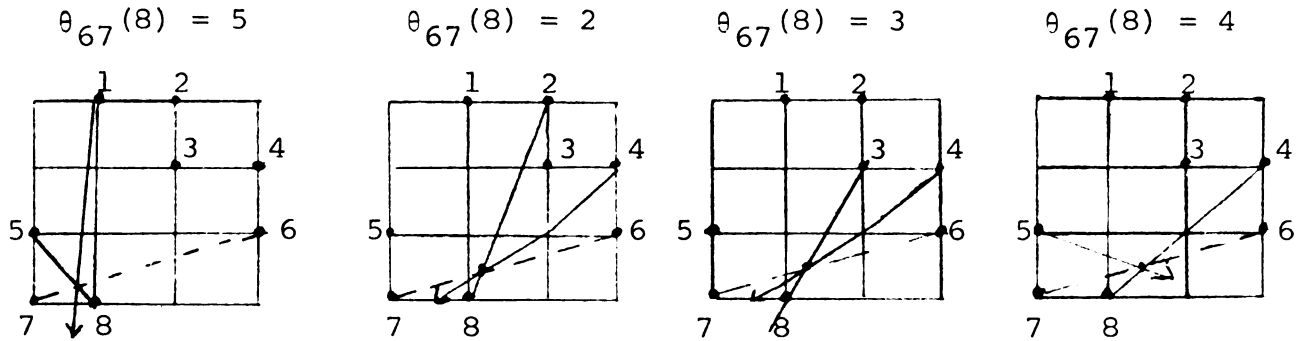
$$S \cap T = \{(a_1, b_1), (a_2, b_1), (a_1, b_2)\}.$$

It follows that $S = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_2, b_4), (a_3, b_3), (a_3, b_4), (a_4, b_2), (a_4, b_3)\}.$



Let θ_{ij} represent the perspectivity in a complete set of perspectivities taking i into j . This notation is, of course, far from unique.

We propose to show, in this case, that there are no possibilities for θ_{67} .

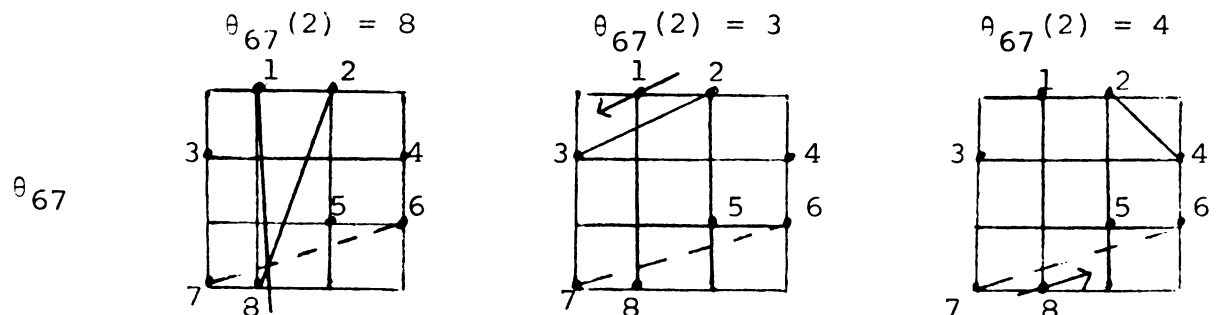
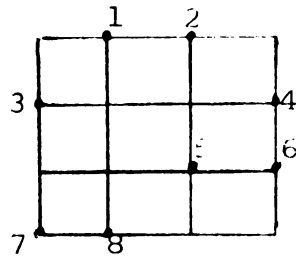


In each case the only possible center of the perspectivity θ_{67} is seen to be inconsistent with the ordered arrangements of the remaining points. The impossibility is strongly suggested by the line with arrows.

Case 2:

$$S \cap T = \{(a_1, b_1), (a_2, b_1), (a_1, b_3)\}.$$

$$S = \{(a_1, b_1), (a_2, b_1), (a_4, b_2), (a_3, b_2), (a_4, b_3), (a_1, b_3), (a_2, b_4), (a_3, b_4)\}$$

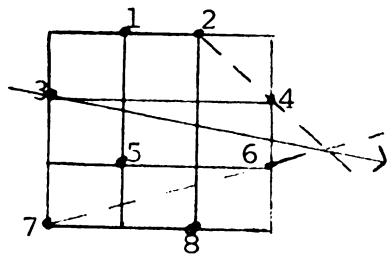


Hence θ_{67} is impossible.

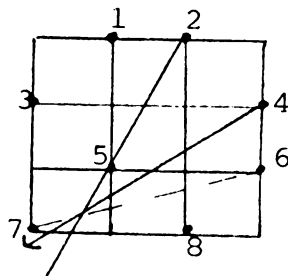
Case 3:

$$S \cap T = \{(a_1, b_1), (a_3, b_1), (a_1, b_3)\}.$$

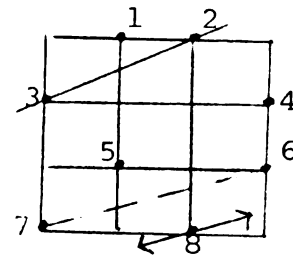
$$\Rightarrow S = \{(a_1, b_1), (a_3, b_1), (a_1, b_3), (a_2, b_2), (a_4, b_2), (a_4, b_3), (a_3, b_4), (a_2, b_4)\}.$$



$$\theta_{67}(2) = 4$$



$$\theta_{67}(2) = 5$$



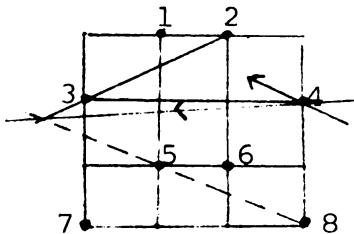
$$\theta_{67}(2) = 3$$

Hence θ_{67} is impossible. ■

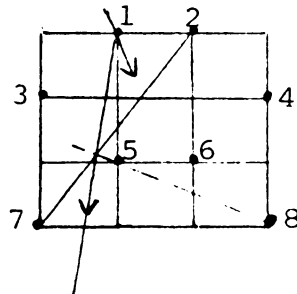
Case 4:

$$S \cap T = \{(a_1, b_1), (a_4, b_1), (a_1, b_3)\}$$

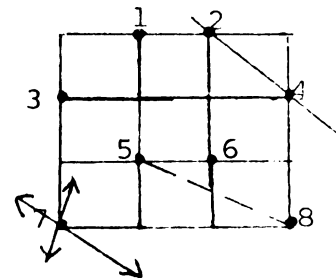
$$\Rightarrow S = \{(a_1, b_1), (a_4, b_1), (a_1, b_3), (a_2, b_2), (a_2, b_4), (a_3, b_2), (a_3, b_4), (a_4, b_1), (a_4, b_3)\}$$



$$\theta_{58}(2) = 3$$



$$\theta_{58}(2) = 7$$



$$\theta_{58}(2) = 4$$

θ_{58} is impossible. ■

Case 5:

$$S \cap T = \{(a_1, b_1), (a_4, b_1), (a_1, b_2)\}$$

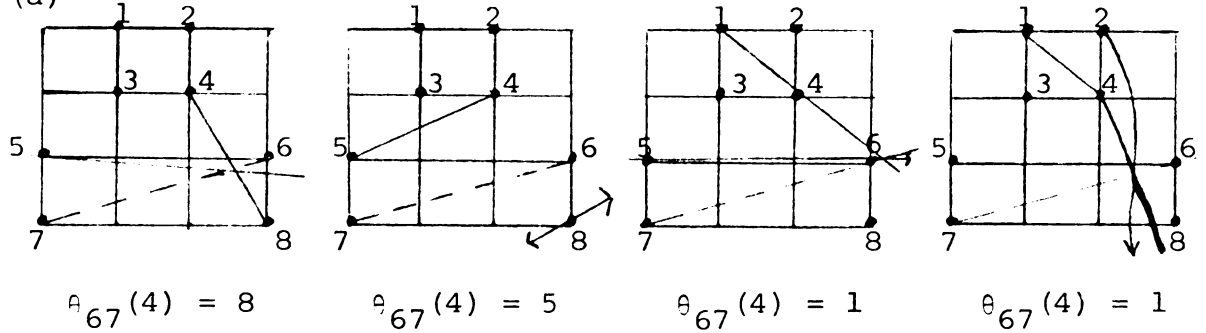
There are three possibilities for S :

$$\text{a: } S = \{(a_1, b_1), (a_4, b_1), (a_1, b_2), (a_4, b_2), (a_2, b_3), (a_3, b_3), (a_2, b_4), (a_3, b_4)\}$$

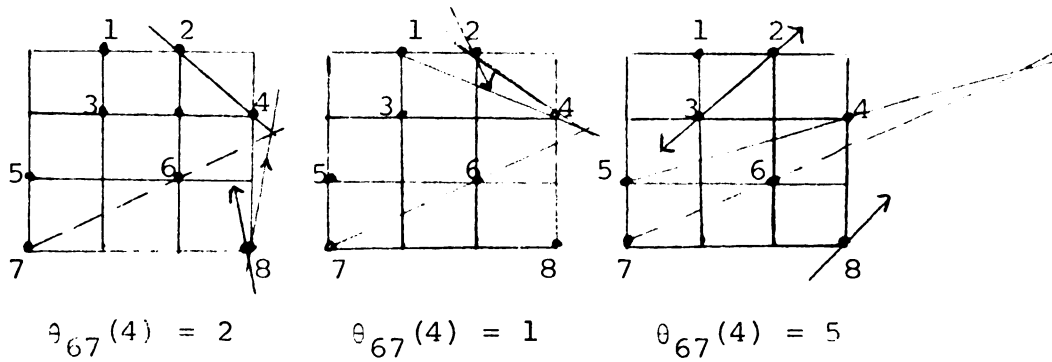
$$\text{b: } S = \{(a_1, b_1), (a_4, b_1), (a_1, b_2), (a_3, b_2), (a_2, b_3), (a_4, b_3), (a_2, b_4), (a_3, b_4)\}$$

$$\text{c: } S = \{(a_1, b_1), (a_4, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_4), (a_3, b_3), (a_3, b_4), (a_4, b_3)\}.$$

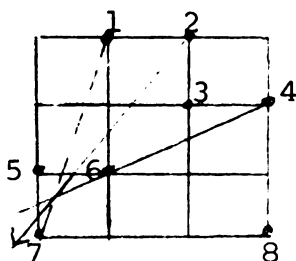
(a)



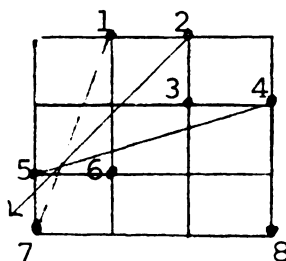
(b)



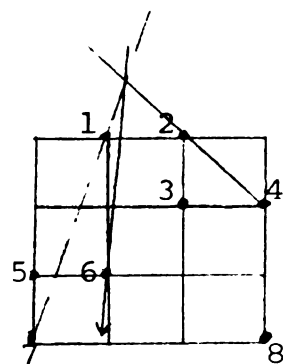
(c)



$$\theta_{17}(4) = 6$$



$$\theta_{17}(4) = 5$$



$$\theta_{17}(4) = 2$$

Case 6:

$$S \cap T = \{(a_1, b_1), (a_2, b_1), (a_1, b_2)\}$$

$$6(a) \ S = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_2, b_3), (a_3, b_2), (a_3, b_4), (a_1, b_3), (a_4, b_4)\}$$

$$6(b) \ S = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_3), (a_2, b_2), (a_3, b_4), (a_4, b_3), (a_4, b_4)\}.$$

$$6(c) \ S = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_2, b_4), (a_3, b_2), (a_3, b_3), (a_4, b_3), (a_4, b_4)\}.$$

Case 6a is the first of several which cannot be dismissed by virtue of the fact that for some i, j there is no corresponding θ_{ij} . In fact for each i, j there often exist several possible perspectivities and it is only after comparing the geometry of the configurations that it becomes clear that a complete set of perspectivities cannot exist.

For example a glance at the A_1 and B_1 diagrams of case 6a shows that the betweenness relations implied by lines 46 are incompatible. Hence we conclude that the perspectivities $(74)(25)(13)(68)$ and $(73)(26)(48)(15)$ cannot exist. We indicate this $A_1 \Psi B_1(46)$.

On the other hand A_1 and B_3 are incompatible because their perspectivities are not disjoint having transformations (25) and (68) in common. This is symbolized: $A_1 \cap B_3 = (25)(68)$.

The analysis of 6a then proceeds as follows:

$$A_1 \Psi B_1 (46), A_1 \Psi B_2 (73), A_1 \cap B_3 = (25)(68).$$

$$A_2 \Psi B_1 (46), A_2 \Psi B_2 (73), A_2 \cap B_3 = (68)$$

$$A_3 \cap B_1 = (26) \quad A_3 \Psi B_2 (46)$$

Thus a complete set of perspectivities must include A_3 and B_3 . We proceed now to see if any of the possibilities for θ_{72} are compatible with these two.

$$A_3 \cap C_1 = (58)(13), \quad B_3 \cap C_3 = (14)(86)$$

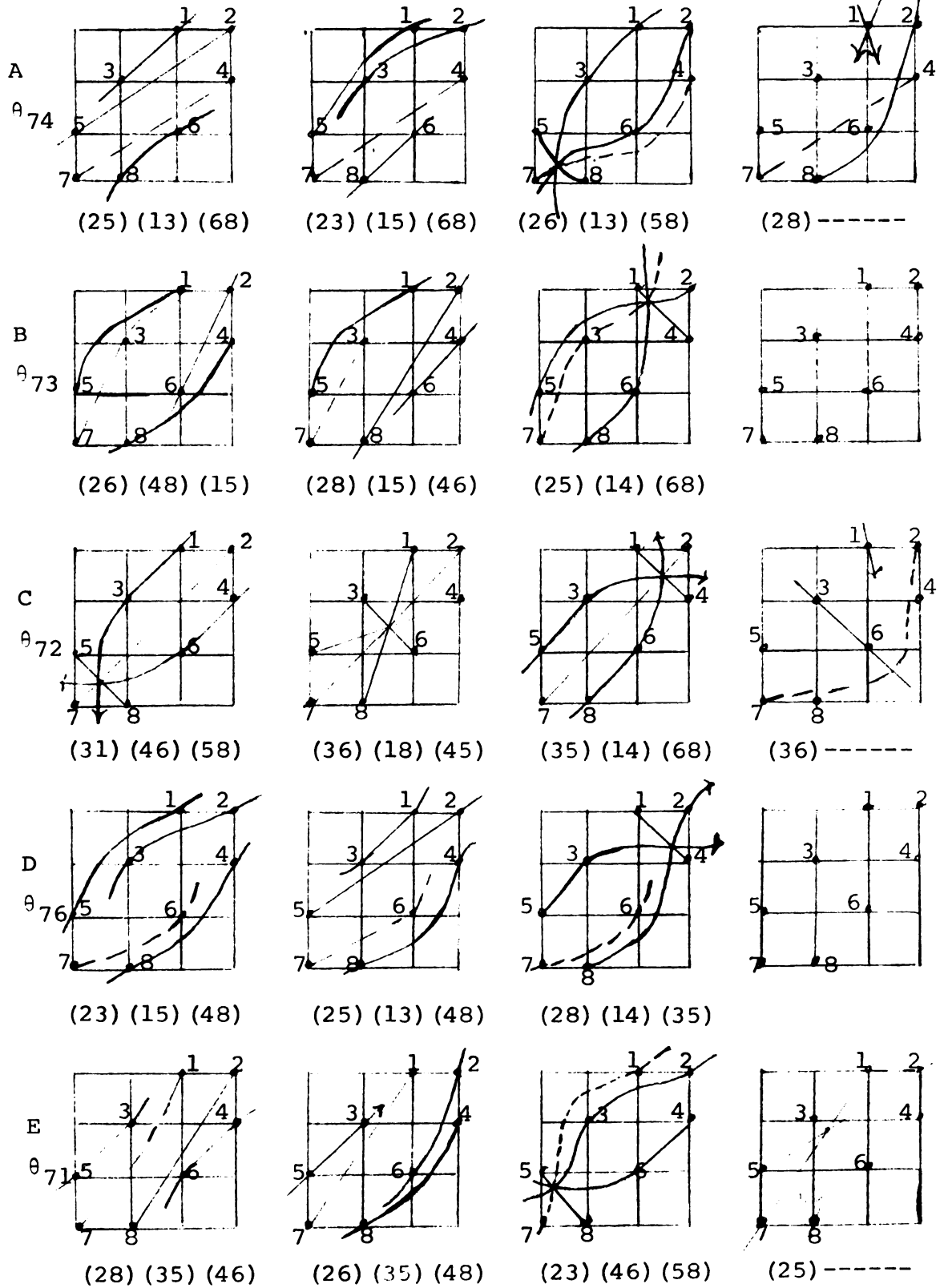
At this point we know that the only possibilities for a complete set of perspectivities must include the perspectivities represented by diagrams A_3, B_3, C_2 . We now examine θ_{76} .

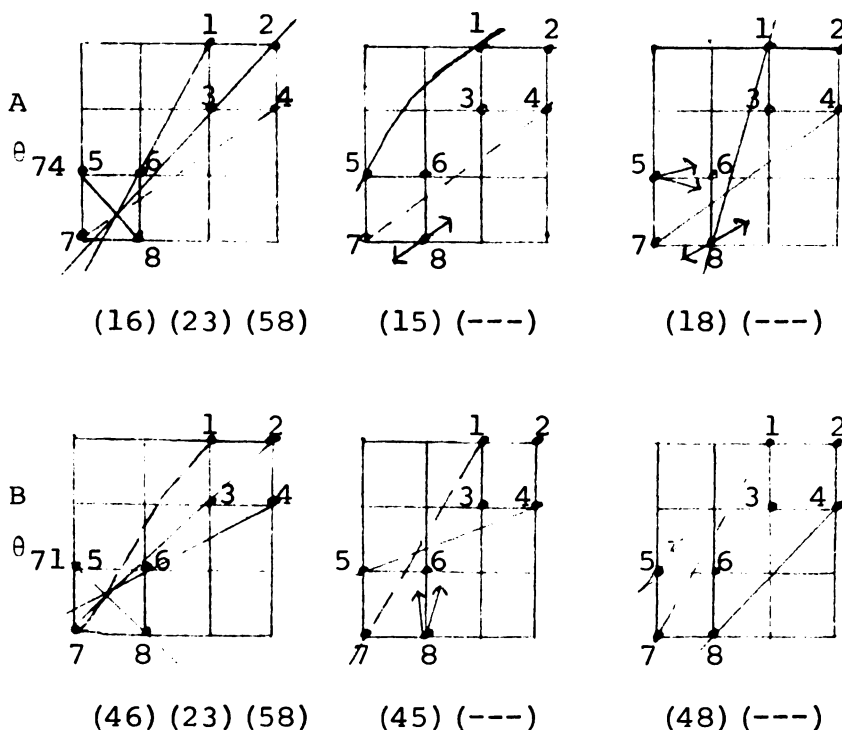
$$D_2 \cap A_3 = (13), \quad D_3 \cap B_3 = (14)$$

Now a complete set of perspectivities must include A_3, B_3, C_2, D_1 .

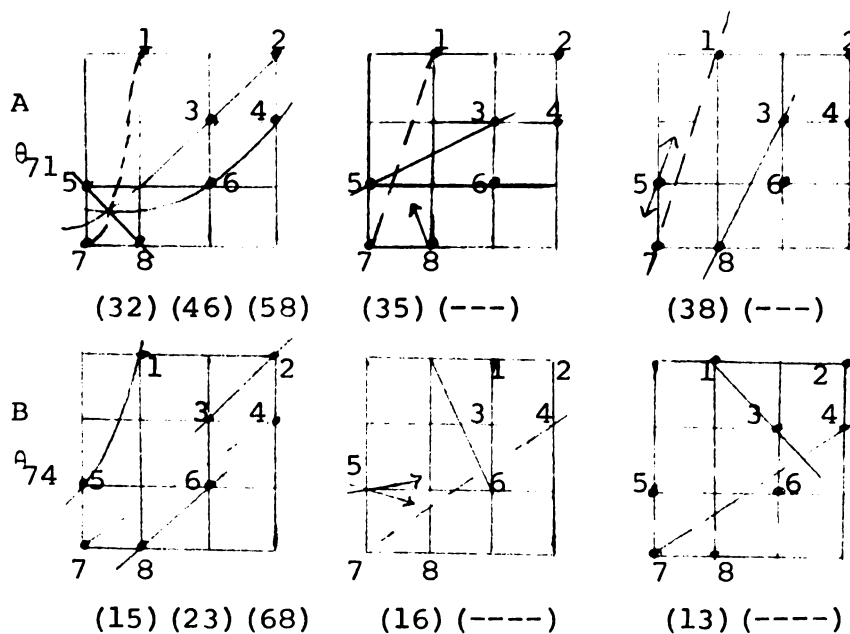
$$E_1 \Psi D_1 (46), \quad E_2 \cap A_3 = (26), \quad E_3 \cap A_3 = (58).$$

Thus the diagram of case 6a is not a completely self perspective set. ■

Case 6a:

Case 6b:

Since $A_1 \cap B_1 = (58)(23)$, θ_{71} and θ_{74} cannot coexist and the set of perspectivities in this case is incomplete. ■

Case 6c:

$A_1 \cap B_1 = (23) \Rightarrow$ Diagram not completely self perspective. ■

Case 7:

$$S \cap T = \{(a_1, b_1), (a_1, b_3), (a_3, b_1)\}$$

$$(a): S = \{(a_1, b_1), (a_1, b_3), (a_3, b_1), (a_2, b_4), (a_4, b_2), (a_4, b_4), (a_3, b_3), (a_2, b_2)\}$$

$$(b): S = \{(a_1, b_1), (a_1, b_3), (a_3, b_1), (a_2, b_4), (a_4, b_2), (a_4, b_4), (a_2, b_3), (a_3, b_2)\}.$$

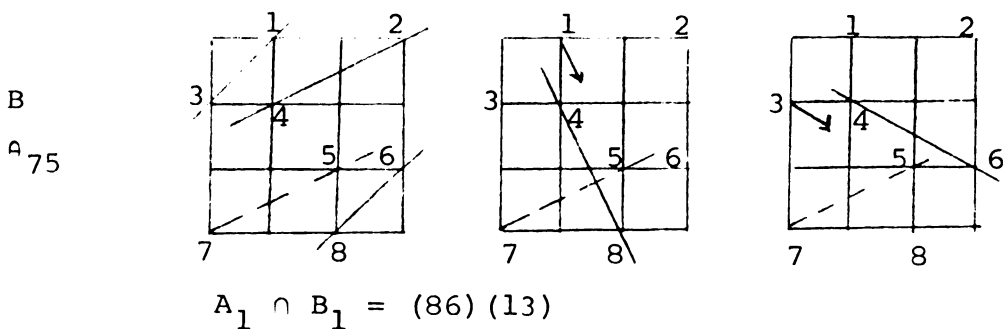
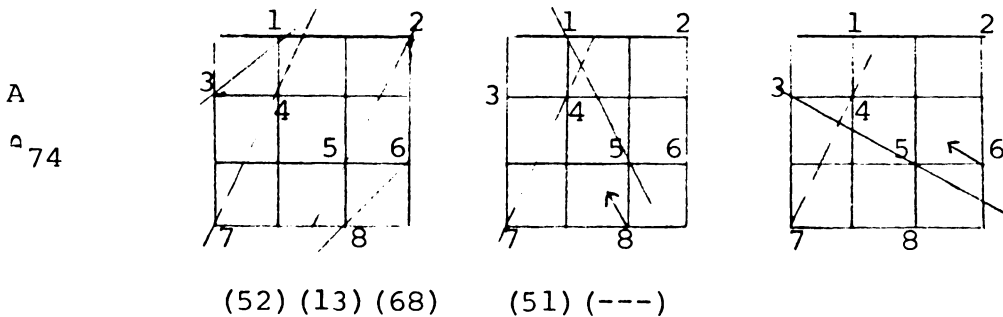
Case 7a (see grid diagrams)

$$A_1 \Rightarrow B_1, \quad A_1 \cup B_1 \Rightarrow C_1, \quad (A_1 \cup B_1 \cup C_1) \cap D_i \neq \emptyset \quad i = 1, 2$$

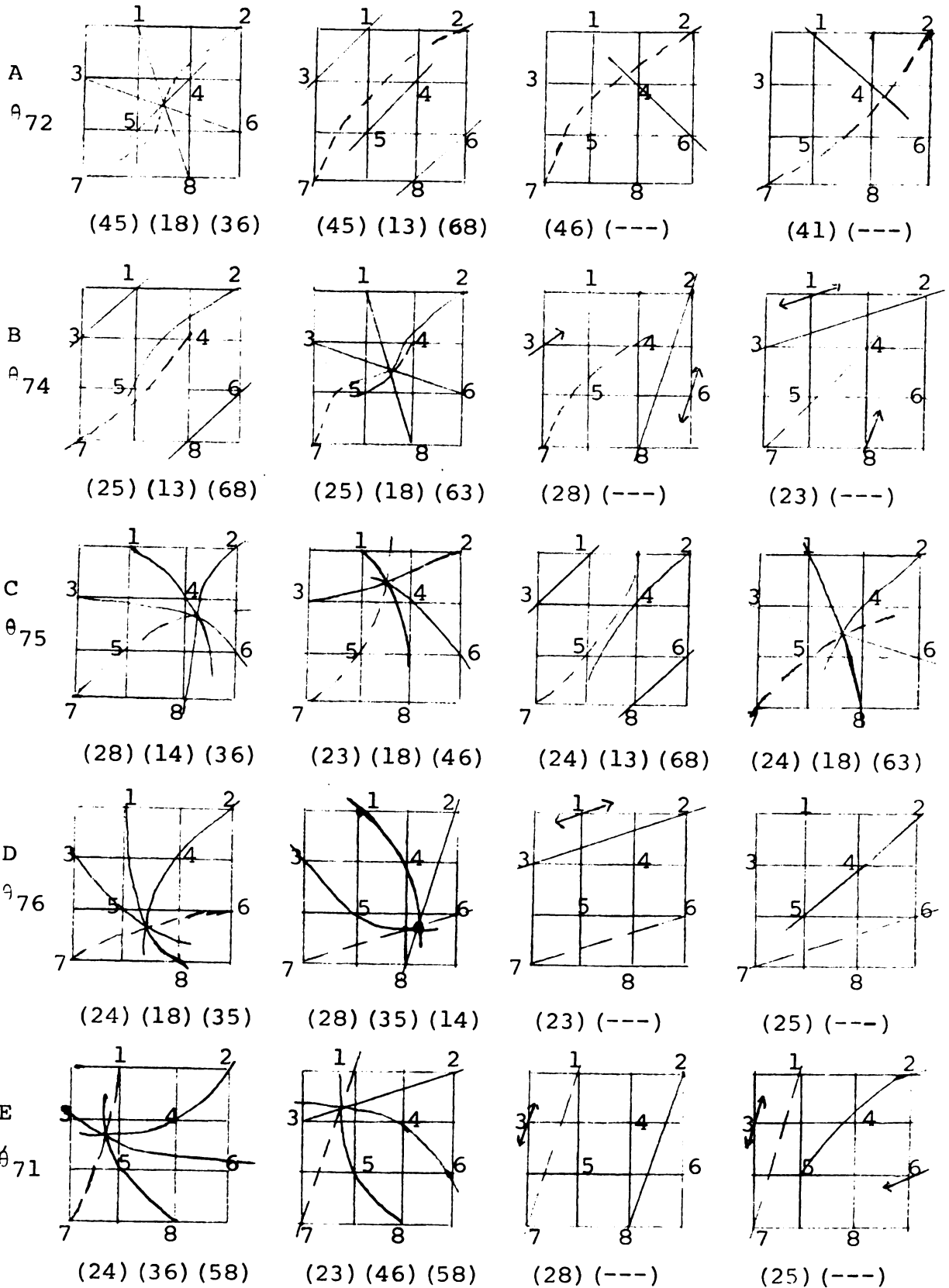
$$A_2 \Rightarrow B_2, \quad A_2 \cup B_2 \Rightarrow C_1, \quad (A_2 \cup B_2 \cup C_1) \cap E_i \neq \emptyset \quad i = 1, 2$$

■

Case 7b



■

Case 7a: Grid diagrams

Case 8a:

$$S \cap T = \{(a_1, b_1), (a_2, b_1), (a_1, b_3)\}.$$

$$(a): S = \{(a_1, b_1), (a_2, b_1), (a_1, b_3), (a_3, b_4), (a_4, b_4), (a_4, b_2), (a_2, b_2), (a_3, b_3)\}$$

$$(b): S = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_2, b_4), (a_3, b_2), (a_3, b_3), (a_4, b_2), (a_4, b_4)\}$$

$$(c): S = \{(a_1, b_1), (a_2, b_1), (a_1, b_3), (a_2, b_3), (a_3, b_2), (a_3, b_4), (a_4, b_2), (a_4, b_4)\}.$$

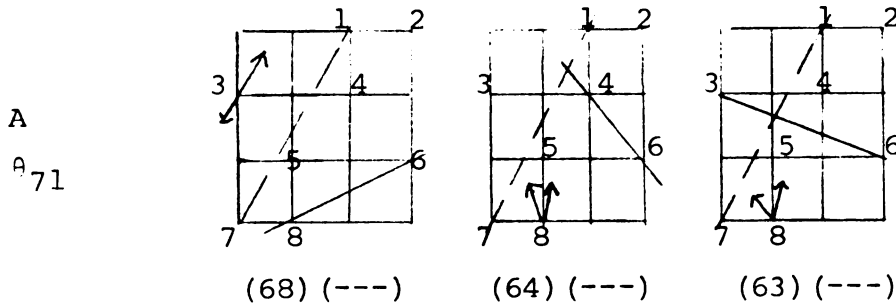
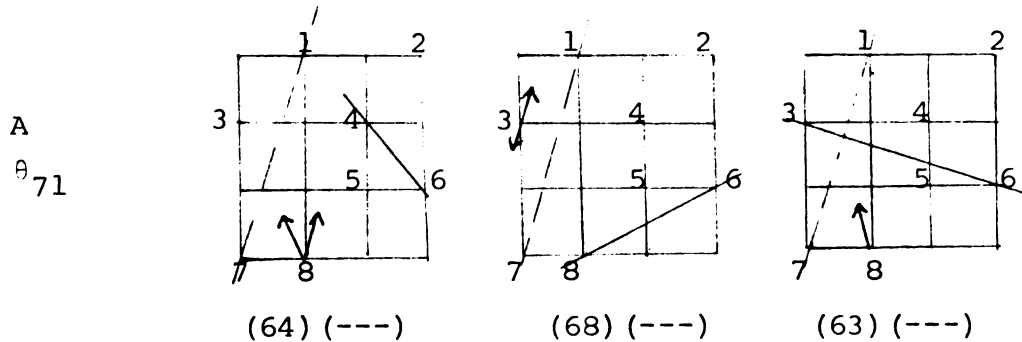
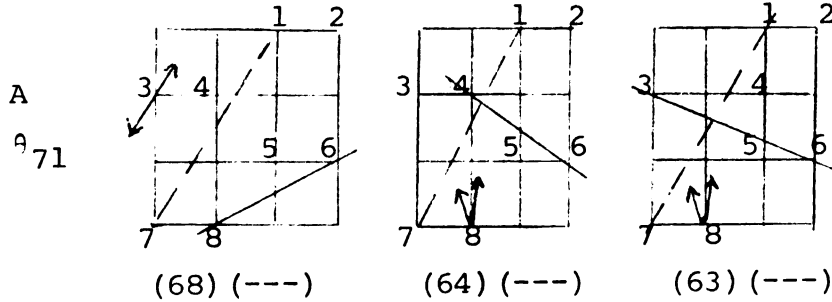


Fig. 3.31

Thus θ_{71} does not exist and configuration 8a is not completely self perspective.

Case 8b:

θ_{71} is impossible. ■

Case 8c:

θ_{71} is impossible. ■

Case 9:

$$S \cap T = \{(a_1, b_1), (a_4, b_1), (a_1, b_4)\}$$

$$(a): S = \{(a_1, b_1), (a_4, b_1), (a_1, b_4), (a_2, b_3), (a_2, b_4), (a_3, b_2), (a_3, b_3), (a_4, b_2)\}.$$

$$(b): S = \{(a_1, b_1), (a_4, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_3), (a_3, b_2), (a_4, b_2)\}.$$

Case 9a:

$A_1 \Psi B_1 (64)$	$A_1 \cap B_2 = 16$	$A_1 \Psi B_3 (64)$	$A_1 \Psi B_4 (64)$	$A_1 \cap B_5 = (16) (24)$
$A_2 \cap B_1 = (28)$	$A_2 \sim B_2$	$A_2 \sim B_3$	$A_2 \cap B_4 = (14)$	$A_2 \Psi B_5 (85)$
$A_3 \Psi B_1 (48)$	$A_3 \sim B_2$	$A_2 \cap B_3 = (18)$	$A_3 \Psi B_4 (46)$	$A_3 \cap B_5 = (24)$
$A_4 \Psi B_1 (64)$	$A_4 \cap B_2 = (48)$	$A_4 \Psi B_3 (64)$	$A_4 \cap B_4 = (26)$	$A_4 \Psi B_5 (64)$
$A_5 \cap B_1 = 28$	$A_5 \Psi B_2 (85)$	$A_5 \cap B_3 = (46)$	$A_5 \Psi B_4 (64)$	$A_5 \Psi B_5 (85)$

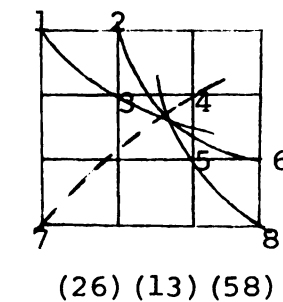
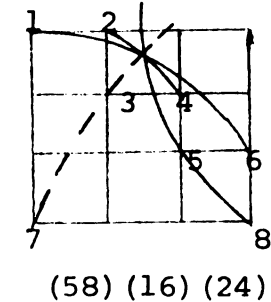
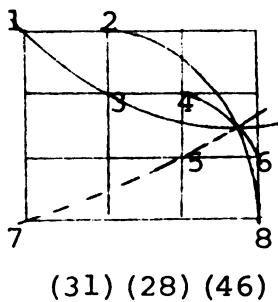
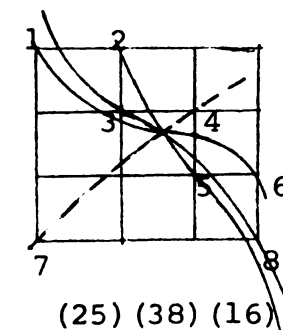
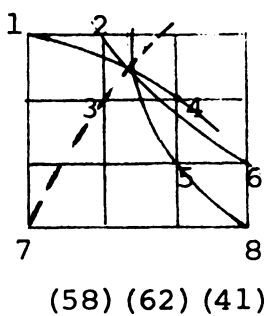
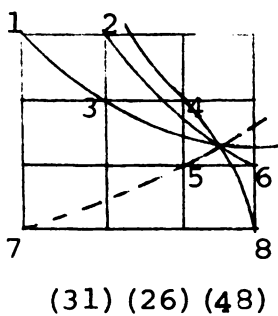
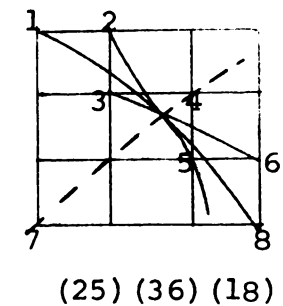
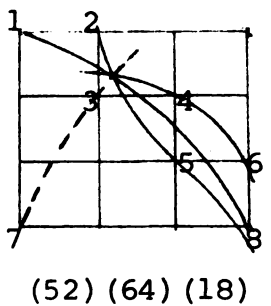
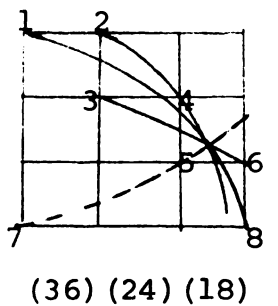
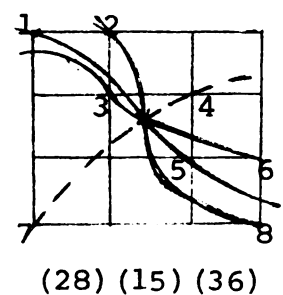
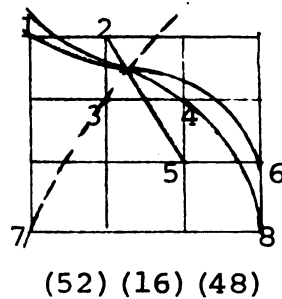
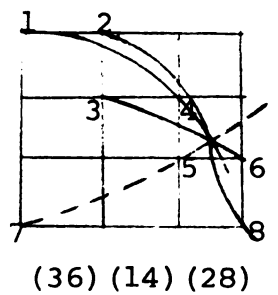
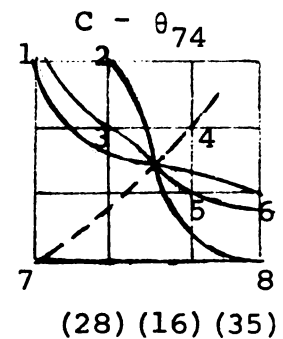
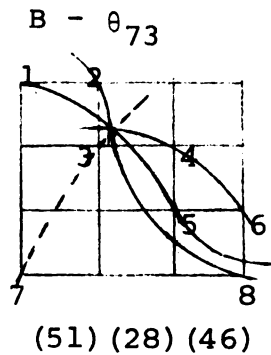
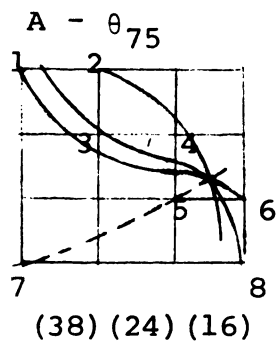
Thus the only AB combinations are: A_2, B_2 ; A_2, B_3 ; A_3, B_2 .

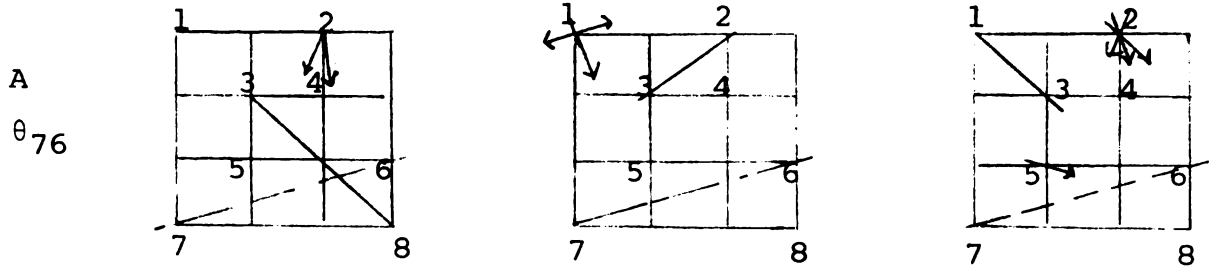
But $A_2 \cap C_1 = (28)$, $A_2 \cap C_2 = (28)$, $A_2 \cap C_3 = (36)$, $A_2 \Psi C_4 (64)$, $A_2 \Psi C_5 (85)$

$B_2 \cap C_1 = 16$, $B_2 \Psi C_2 (85)$, $B_2 \cap C_3 = (25)$, $B_2 \cap C_4 = (16)(25)$, $B_2 \Psi C_5 85$

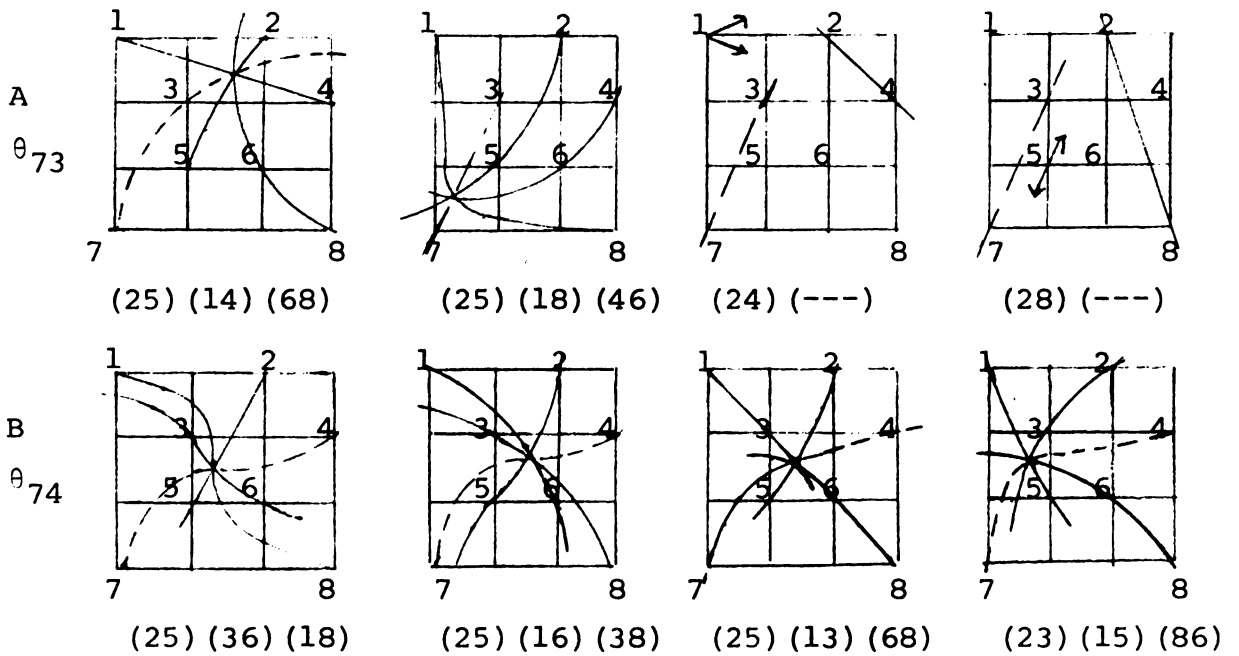
Hence θ_{75} , θ_{73} and θ_{74} cannot coexist.

Grid diagrams Case 9a:



Case 9b:

θ_{76} is impossible. ■

Case 10:

$A_i \cap B_j = \emptyset \Rightarrow i = 2, j = 4$. But $A_2 \Psi B_4 (86)$

Hence θ_{73} and θ_{74} cannot coexist.

Case 11:

$$S = \{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_2), (a_3, b_4), (a_4, b_1), (a_4, b_4)\}$$

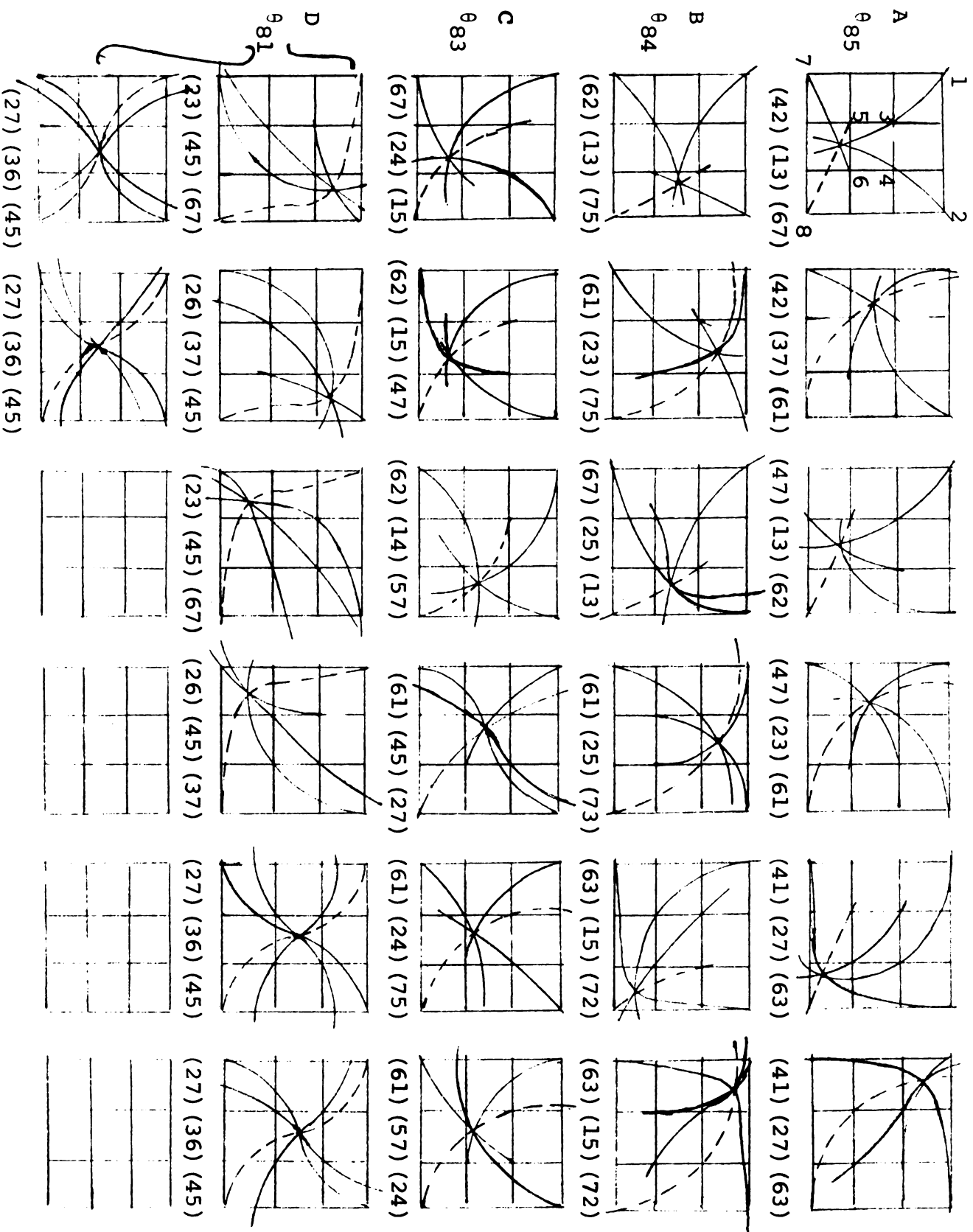
	A_1	A_2	A_3	A_4	A_5	A_6
B_1	(13)	D_i	(62)	D_i	27	$D_3 \cup C_i$
B_2	D_i	18	D_i	(61)	D_i	18
B_3	(67)	D_i	(13)	27	18	27
B_4	D_i	(61)	27	(16)	27	18
B_5	27	D_i	18	27	(27)	27
B_6	$D_2 \cup C_i$	18	27	18	(27)	18

Explantion of the table.

If the B_i, A_j entry is (ℓ, m) then B_k, A_j have the transposition (ℓ, m) in common. If the entry is ℓm then B_k and A_j are incompatible by virtue of the betweenness relations implied by the line joining points ℓ and m .

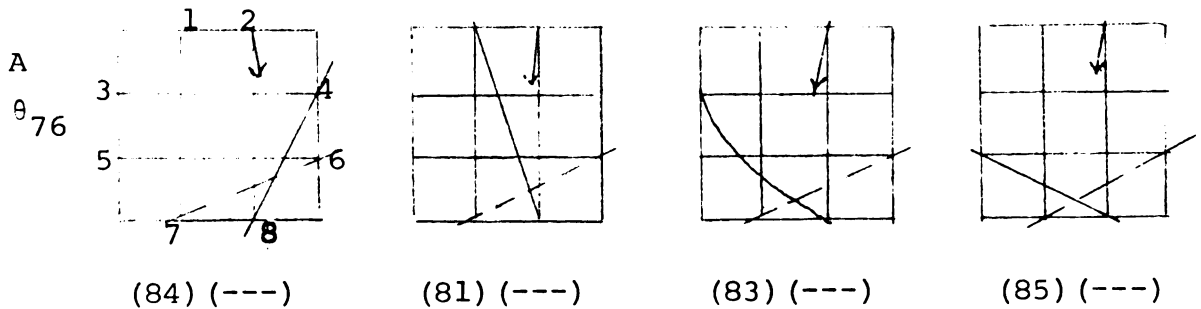
If the entry is D_i then A_k and B_j are compatible but no D_i is compatible with both.

Finally, the entry $D_2 \cup C_i$ corresponding to B_6, A_1 means that A_1, B_6, D_2 are compatible but together they are compatible with no C_i . Similarly for the B_1, A_6 entry



Case 12:

$$S = \{(a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_4), (a_3, b_1), (a_4, b_4), (a_4, b_2), (a_4, b_3)\}.$$



θ_{76} is impossible. ■

This completes the analysis of the situation in which $|S^*| = 16$. All other cases are isomorphic to one of those analyzed above.

Subject to completing the consideration of case in which $|S^*| = 12$ in chapter 4 this theorem shows that there is no desmic triad of 24 points spanning an ordered projective k -space, $k > 2$.

CHAPTER 4

Desmic triads in general spaces

4.1 Introduction

In this chapter we continue the study of desmic triads but now in general spaces and not necessarily spanning three dimensions. If Σ_1^{n-1} , Σ_2^{n-1} , Σ_3^{n-1} are three hyperplanes in a finite projective space of dimension n then quite clearly the three sets $\Sigma_1^{n-1} - (\Sigma_2^{n-1} \cup \Sigma_3^{n-1})$, $\Sigma_2^{n-1} - (\Sigma_1^{n-1} \cup \Sigma_3^{n-1})$, $\Sigma_3^{n-1} - (\Sigma_1^{n-1} \cup \Sigma_2^{n-1})$ constitute a desmic triad. How much more general such triads can be is still not clear and a large segment of this chapter is concerned with this question.

Many of the techniques of Chapter 3 are applicable to this more general setting with obvious modifications. Thus theorem 3.8 which asserts that a necessary condition for the existence of a desmic triad spanning a 3-space is the existence of a completely self-perspective set in the 2-spaces of that 3-space clearly extends to n -spaces.

In Chapter 3 we found it occasionally convenient to express self-perspectivities as permutations in "factored" form. In this chapter much more extensive use is made of this technique in completely characterizing desmic triads of order 7 or less in all dimensions. The use of fixed sets in analyzing the equivalence of self-perspective arrangements

seems to be very closely related to some work of L. D. Cummings [11] concerning steiner triple systems. We do not yet completely understand these connections but propose to look into the matter further.

We have already observed in Chapter 3 that if A, B, C are three sets of a desmic triad then sets A and B are perspectively related by $|C|$ disjoint perspectivities. Thus for the rest of this chapter we shall concentrate on the language of multiply perspective arrangements (to be defined shortly). The concept of multiply perspective arrangements is the same as that of multiply perspective sets but the new terminology is more convenient for the general treatment in this chapter.

4.2 Multiply perspective arrangements

Definition 4.1:

Let $\{A, B, \Gamma\}$ denotes two disjoint sets A and B in a projective space Σ and a set Γ of disjoint perspective mappings of A onto B . Let c_i be the center of $\gamma_i \in \Gamma$ and $C = \{c_1, c_2, \dots, c_{p-1}\}$. If $|\Gamma| = p$, $|A| = |B| = k$ we say that $\{A, B, \Gamma\}$ is a multiply perspective arrangement of class $[k, p, d]$, $d > 1$ where $A \cup B$ spans a subspace of Σ of dimension d and $A \cap B = B \cap C = A \cap C = \emptyset$.

Notation:

$[k, p, d]$ MPA.

Definition 4.2:

Suppose $\{A, \Gamma\}$ denotes a set A in a projective space Σ and a set Γ of disjoint self perspective mappings of A onto A including the identity. If $|\Gamma| = p$ $|A| = k$ we say that $\{A, \Gamma\}$ is a self perspective arrangement of class $[k, p, d]$ where A spans a subspace of Σ of dimension $d \geq 2$, $A \cap C = \emptyset$ and $C = \{c_1, c_2, \dots, c_{p-1}\}$ is the set of centers.

It should be clear that if γ be expressed as a factored permutation then the points of A corresponding to each cycle must be linear.

Keeping this in mind the condition $A \cap C = \emptyset$ can be described more explicitly in terms of permutation cycles of Γ as follows:

(1) If a point in one cycle is on the line of a second cycle of the same $\gamma_i \in \Gamma$ then the lines associated with

the two cycles are identical.

(2) If two cycles have a pair of points in common then their associated lines are identical.

Notation:

$[k, p, d]$ SPA

Definition 4.3:

Two multiply perspective arrangements $\{S_1, S_2, \Gamma\}$ and $\{\bar{S}_1, \bar{S}_2, \bar{\Gamma}\}$ are said to be equivalent if there exist 1-1 mappings $\sigma_1: S_1 \rightarrow \bar{S}_1$, $\sigma_2: S_2 \rightarrow \bar{S}_2$, $g: \Gamma \rightarrow \bar{\Gamma}$, with $g(\gamma_i) = \bar{\gamma}_i$, $\gamma_i \in \Gamma$ such that the following diagram commutes

$$\begin{array}{ccc} S_1 & \xrightarrow{\gamma_1} & S_2 \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ \bar{S}_1 & \xrightarrow{\bar{\gamma}_1} & \bar{S}_2 \end{array}$$

Definition 4.4:

Two self perspective arrangements $\{S, \Gamma\}$ and $\{\bar{S}, \bar{\Gamma}\}$ are equivalent if there exist 1-1 maps $\sigma: S \rightarrow \bar{S}$ and $g: \Gamma \rightarrow \bar{\Gamma}$ with $g(\gamma_i) = \bar{\gamma}_i \in \bar{\Gamma}$, such that $\bar{\gamma}_i = \sigma^{-1} \gamma_i \sigma$. Since $\sigma, \gamma_i, \bar{\gamma}_i$ may be regarded as permutations, on $|S| = k$ symbols this means that Γ and $\bar{\Gamma}$ are conjugate sets of permutations in the symmetric group on k symbols.

$$\boxed{\bar{\Gamma} = \sigma \Gamma \sigma^{-1}}$$

Definition 4.4:

If in the self-perspective arrangement $\{S, \Gamma\}$, $|S| = |\Gamma|$ then the arrangement is called completely self perspective.

Remark 4.2:

If $\{S, \Gamma\}$ is a completely self perspective arrangement then the set of permutations is simply transitive on S . Γ , in general, is not a group.

In chapter 3 we showed that it was impossible to realize completely self perspective arrangements of class $[k, k, 2]$ $k = 5, 6, 7, 8$ in ordered projective spaces. We now propose to conduct a similar analysis for $k \leq 7$ in general Desarguesian spaces. In Chapter 3 our technique involved making a few combinatorial observations and then showing that these were incompatible with the order requirements of the embedding space. Now we are going to make similar combinatorial observations and then complete the analysis by making use of the conditions contained in the following theorem.

Theorem 4.3:

Let $\bar{S} = \{1, 2, \dots, n\}$ denote the points of a completely self perspective arrangement $\{S, \Gamma\}$. Let $\bar{\Gamma} = \{\text{the permutations of } \bar{S} \text{ representing the self-perspectivities } \Gamma\}$. Let $C = \{c_1, c_2, \dots, c_n\}$ denote the centers of the perspectivities Γ . Then $\{\bar{S}, \bar{\Gamma}\}$ satisfies the following conditions:

- (1) $\bar{\Gamma}$ is simply transitive on $\bar{S}$.
- (2) Elements of $\bar{\Gamma}$ are pairwise disjoint.
- (3) $1 \in \bar{\Gamma}$ where 1 is the identity permutation of $\bar{S}$.

Proof:

Condition (1) follows from lemma 3.6 of Chapter 3.
 Condition (2) follows from the definition of completely self-perspective arrangement. Condition (3) is a consequence of $|\Gamma| = |S|$ and the fact that the elements of Γ are disjoint.

Remark 4.4:

From now on we shall use Γ to represent both the self-perspectivities and their representing permutations.

An illustration:

If (S, Γ) is a self perspective arrangement of class $[4, 4, d]$ in a projective space Σ , and $S = \{1, 2, 3, 4\}$ then $d = 2$ and

$$\Gamma = \{I, (12)(34), (14)(32), (13)(42)\}.$$

It is clear that no cycle of any γ can be of length three since γ would then have a fixed element and γ and I would fail to be disjoint.

Theorem 4.6:

If (S, Γ) is a completely self perspective arrangement then no cycle C , of $\gamma \in \Gamma$ has length exceeding $\frac{n}{2}$ where $|S| = n$.

Proof:

Suppose C has length $k > \frac{n}{2}$. The points of C are on a line ℓ . Let $S \cap \ell = P$. Since Γ is simply transitive there are at least $k-1$ distinct elements $\gamma_1, \gamma_2, \dots, \gamma_{k-1}$ of Γ taking a point $t \in P$ into points of P .

The center of each γ_i is on ℓ . If $\gamma_i(p) = q$, $p \in P$, $q \in \Sigma - \ell$ then the center of γ_i is on line pq . Hence the center of γ_i is $\ell \cap pq = p \in S$. But this contradicts the definition of self-perspective arrangements.

Thus $\gamma_i(P) = P$ and $\gamma_i(S-P) = S-P$. But $0 < |S-P| < \frac{n}{2}$. Hence the $k > \frac{n}{2}$ mappings γ_i cannot be disjoint. ■

The next theorem establishes the result stated at the beginning of the introduction to this chapter.

Theorem 4.7:

If $\Sigma_1, \Sigma_2, \Sigma_3$ are three distinct hyperplanes in a finite projective space Σ^d , $d \geq 3$, then $\Sigma_1 - (\Sigma_2 \cup \Sigma_3)$, $\Sigma_2 - (\Sigma_3 \cup \Sigma_1)$, $\Sigma_3 - (\Sigma_2 \cup \Sigma_1)$ form a desmic triad.

Proof:

$\Sigma_i - (\Sigma_j \cup \Sigma_k)$ is not empty since each line in Σ_i contains at least 3 points. Let $p \in \Sigma_1 - (\Sigma_2 \cup \Sigma_3)$, $q \in \Sigma_2 - (\Sigma_3 \cup \Sigma_1)$. $pq \cap \Sigma_3 = X$ and $k \nmid \Sigma_2 \cup \Sigma_1$.

Q.E.D.

Corollary 4.7.1:

There exists multiply perspective arrangement $\{A, B, \Gamma\}$ of order $|A| = |B| = |\Gamma| = q^{d-1} - q^{d-2}$ where $q = p^k$, p any prime and k a positive integer.

Theorem 4.8:

If (S_1, S_2, Γ) is a multiply perspective arrangement of class $[k, k, d]$ then $k \geq 2^{d-1}$.

Proof:(Induction on d)

For $d = 2$ the theorem is trivially true. Let c be a center of one of the perspectivities of Γ . There exists a set P of $d-1$ points in S_1 such that $c \cup P$ spans a space Σ^{d-1} . Let $S_1 \cap \Sigma^{d-1} = S_1^*$, $S_2 \cap \Sigma^{d-1} = S_2^*$, if Γ^* is the set of restrictions

to S_1 of the perspectivities of Γ which have centers in Σ^{d-1} then it is easily seen that (S_1^*, S_2^*, Γ^*) is a multiply perspective arrangement of class $(l, l, d-1)$. Hence by the induction hypothesis $l \geq 2^{d-2}$.

Since the γ_i are disjoint and

$$|S_1| = |S_2| = |\Gamma| \text{ if } x \in S_1^*$$

and $y \in S_2 - S_2^*$ then there is a $\gamma_i \in \Gamma$ such that $\gamma_i(x) = y$.

We claim that $\gamma_i(S_1^*) \subset S_2 - S_2^*$. For suppose $z \in S_1^*$ and $\gamma_i(z) = w \in S_2^*$. Then the center of γ_i is on the lines xy

and zw . Thus the only possibility for the center is the point $x \in S_1$. But this is also impossible. Hence

$$|S_2 - S_2^*| = |S_1^*| = 2^{d-2} \Rightarrow |S_1| = 2^{d-1} \text{ which was to be proved. } \blacksquare$$

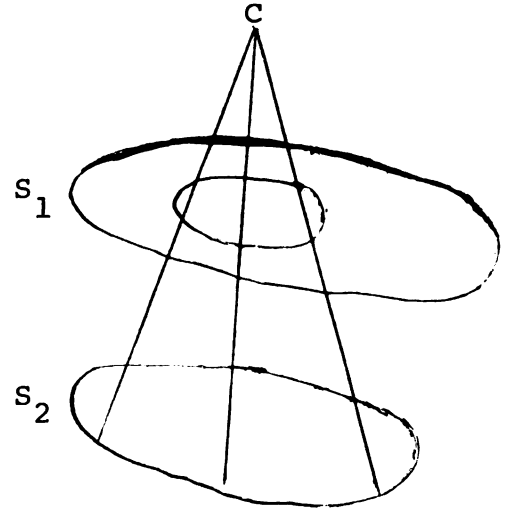


Fig. 4.1

One of the difficulties in analyzing self-perspective arrangements (S, Γ) for $|S| = 8$ comes in trying to list all conjugate classes of admissible sets of permutations. For example after some effort we discovered the following three sets of admissible perspectivities for $n = 8$.

	I		I
	(12) (34) (56) (78)		(12) (34) (56) (78)
	(13) (24) (57) (68)		(13) (25) (47) (78)
	(14) (23) (58) (67)		(14) (26) (38) (57)
Γ^1	(15) (26) (37) (48)	Γ^2	(15) (28) (37) (46)
	(16) (25) (38) (47)		(16) (27) (35) (48)
	(17) (28) (35) (46)		(17) (24) (36) (58)
	(18) (27) (36) (45)		(18) (23) (45) (67)
	I		
	(12) (34) (56) (78)		
	(13) (24) (57) (68)		
	(14) (25) (38) (67)		
Γ^3	(15) (26) (37) (48)		
	(16) (23) (47) (58)		
	(17) (28) (35) (46)		
	(18) (27) (36) (45).		

To decide whether these are conjugate sets we could seek a permutation exhibiting the conjugacy. Another common scheme is to study the action of the sets Γ^i on the set $S = (1, 2, \dots, n)$. If for example Γ^1 is simply transitive on the elements of S while Γ^2 is not then Γ^1 and Γ^2 are not conjugate sets. However, in the case at hand all our admissible sets are simply transitive by construction.

Another technique is to study fixed sets of Γ or subsets of Γ . Note for example the set $\{1, 2, 3, 4\}$ is fixed under the first three mappings of Γ^1 while Γ^2 contains no three mappings which fix a four element subset of S . Thus Γ^1 and Γ^2 are not conjugate sets. Similarly, the first two maps in Γ^3 fix the set $\{1, 2, 3, 4\}$ while no two transformations of Γ^2 fix any four element set. In this way

we see that $\Gamma^1, \Gamma^2, \Gamma^3$ are in different conjugate classes. Thus if completely self perspective arrangements (S_i, Γ_i) exists $i = 1, 2, 3$ they will be pairwise inequivalent.

Our investigations of completely self perspective arrangements of class $(8, 8, 2)$ are still incomplete and we proceed now to an analysis of those of class $(k, k, 2)$ for $k \leq 7$.

3. Multiply perspective arrangements of class $[k,k,d]$, $k \leq 7$

Our general procedure is as follows: If (S,Γ) is an SPA of class $(k,p,2)$ then the sum of the lengths of the cycles in a particular $\gamma \in \Gamma$ is k . Thus we first seek possible integral solutions of

$$\sum k_i = k, \quad k_i \geq 2.$$

For small k this gives relatively few possible cycle lengths. The construction of the arrangement proceeds initially as in Chapters 2 and 3, making use of a grid diagram and then making extensive use of the conditions contained in theorem 4.3.

If such a combinatorial representation exists then we try to find the nature of the embedding space by coordination.

Finally we investigate the possibility of "lifting" the various completely self perspective arrangements of class $[k,k,d]$ to multiply perspective arrangements of class $[k,k,d+1]$. That is to say, we attempt to create desmic triads of order k .

$$k = 4$$

Theorem 4.9:

In any projective space Σ , $\{S,\Gamma\}$ is an SPA of class $[4,4,d]$ iff $d = 2, S = \{1,2,3,4\}$ is a quadruple of points no three of which are linear and $\Gamma = \{I, (12)(34), (13)(24), (14)(23)\}$.

Proof:

Suppose $\{S, \Gamma\}$ is an SPA of class $[4, 4, d]$ in a projective space Σ . Since each of the 4 points has to be joined to the remaining 3, we have that $d = 2$. No three of the points are collinear since $d > 1$ by definition, and using disjointness and transitivity we have $\Gamma = \{I, (12)(34), (13)(24), (14)(23)\}$.

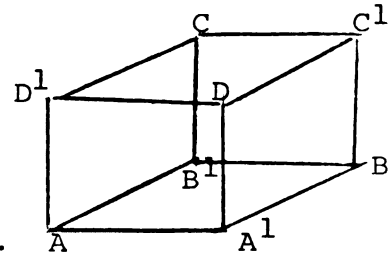
The sufficiency is clear.

In an affine 3-space over a field let

$$S_1 = \{A(0,0,0), B(1,1,0), C(0,1,1), D(1,0,1)\}$$

$$S_2 = \{A^1(1,0,0), B^1(0,1,0), C^1(1,1,1), D^1(0,0,1)\}.$$

$$\Gamma: \begin{cases} \gamma_1(A) = A^1, \gamma_1(B) = B^1, \gamma_1(C) = C^1, \gamma_1(D) = D^1. \\ \gamma_2(A) = B^1, \gamma_2(B) = A^1, \gamma_2(C) = D^1, \gamma_2(D) = C^1 \\ \gamma_3(A) = C^1, \gamma_3(B) = D^1, \gamma_3(C) = A^1, \gamma_3(D) = B^1 \\ \gamma_4(A) = D^1, \gamma_4(B) = C^1, \gamma_4(C) = B^1, \gamma_4(D) = A^1 \end{cases}$$



$\{S_1, S_2, \Gamma\}$ is an MPA of type $[4, 4, 3]$.

It should be clear that a configuration isomorphic to $\{S_1, S_2, \Gamma\}$ is realizable in any affine space.

Now let $\{\bar{S}_1, \bar{S}_2, \bar{\Gamma}\}$ be an MPA of type $(4, 4, 3)$ in a Desarguesian affine space.

$$\bar{S}_1 = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}\} \quad \bar{S}_2 = \{\bar{A}^1, \bar{B}^1, \bar{C}^1, \bar{D}^1\}, \quad \bar{\Gamma} = \{\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4\}.$$

We may assume that $\bar{\gamma}_1(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = (\bar{A}^1, \bar{B}^1, \bar{C}^1, \bar{D}^1)$. As a consequence of theorem 3.8 and 4.10 we can conclude that no three of the lines $\bar{A}\bar{A}^1, \bar{B}\bar{B}^1, \bar{C}\bar{C}^1, \bar{D}\bar{D}^1$ are coplanar. Thus if $\gamma_2(\bar{A}) = \bar{B}^1$ then $\gamma_2(\bar{B}) = \bar{A}^1, \gamma_2(\bar{D}) = \bar{C}^1, \gamma_2(\bar{C}) = \bar{D}^1$ and if $\gamma_3(\bar{A}) = \bar{D}^1$ then $\gamma_3(\bar{D}) = \bar{A}^1, \gamma_3(\bar{C}) = \bar{B}^1, \gamma_3(\bar{B}) = \bar{C}^1$. Finally, $\gamma_4(\bar{A}) = \bar{C}^1, \gamma_4(\bar{B}) = \bar{D}^1, \gamma_4(\bar{C}) = \bar{A}^1, \gamma_4(\bar{D}) = \bar{B}^1$.

Now there is an affine transformation T such that $T(\bar{A}) = A, T(\bar{A}^1) = A^1, T(\bar{D}^1) = D^1, T(\bar{B}^1) = B^1, T(\bar{A}^1) = A^1$. Thus $T\{\bar{S}_1, \bar{S}_2, \bar{\Gamma}\} = \{S_1, S_2, \Gamma\}$. This means that an MPA in Desarguesian affine space is affinely unique.

Theorem 4.10:

Any two multiply perspective arrangements of class $(4,4,3)$ are combinatorially isomorphic.

Theorem 4.11:

Any two multiply perspective arrangements in a Desarguesian projective (affine) space are projectively (affinely) equivalent.

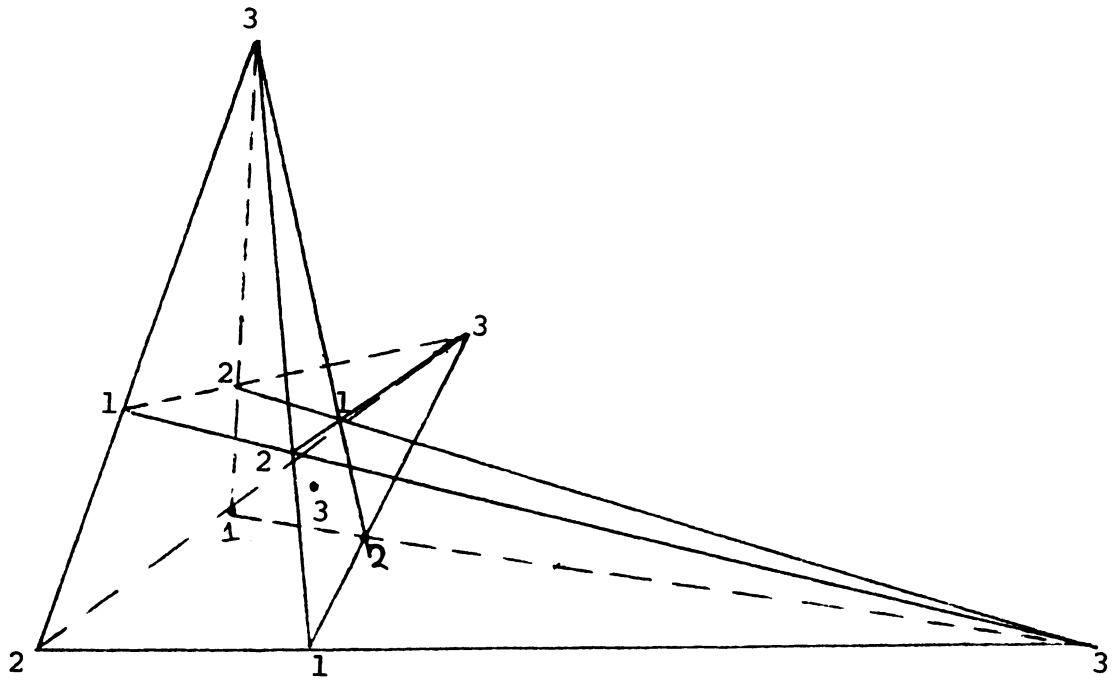


Fig. 4.2:

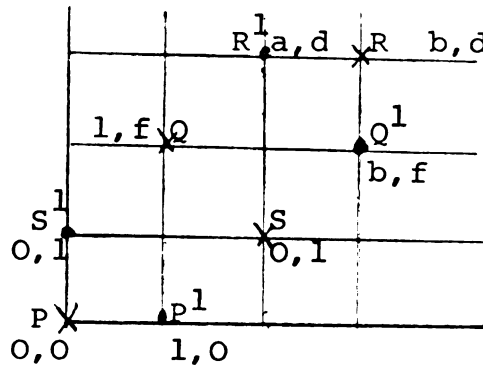
A geometric representation of the multiply perspective arrangement of class $[4,4,3]$.

In the case of $\{S_1, S_2, \Gamma\}$ of class $[k, k, 2]$ it has been conjectured by M. Edelstein that in the real projective plane the figure must be rigid in the sense that the set of centers must lie on a line. We now show for $k = 4$ that this is not so by the following theorem.

Theorem 4.12:

An MPA of type $[4, 4, 2]$ exists and is combinatorially unique.

Case 1:



$$S_1 = \{P, Q, R, S\}$$

$$S_2 = \{P^1, Q^1, R^1, S\}.$$

$$\gamma_1: P \rightarrow P^1, Q \rightarrow Q^1, R \rightarrow R^1, S \rightarrow S^1.$$

Center: ideal pt on x axis

$$\gamma_2: P \rightarrow S^1, Q \rightarrow P^1, R \rightarrow Q^1, S \rightarrow R^1$$

Center: ideal pt on y axis

$$\gamma_3: P \rightarrow R^1, Q \rightarrow S^1, R \rightarrow P^1, S \rightarrow Q^1$$

$$\text{Center: } \frac{a}{a-b+1} \frac{d}{a-b+1}$$

$$\gamma_4: P \rightarrow Q^1, Q \rightarrow R^1, R \rightarrow S^1, S \rightarrow P^1$$

$$\text{Center: } \frac{b}{f-d+1} \frac{f}{f-d+1}$$

$$af = b+d-1$$

Select a, b, d, f such that $af = b+d-1, a-b+1 \neq 0$

$$f-d+1 \neq 0, d \neq f, a \neq b$$

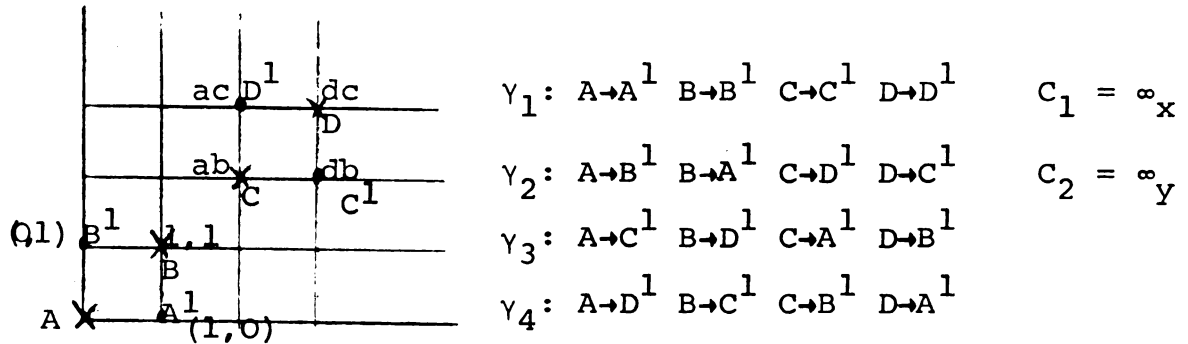
$$a, b, f, d \neq 0, 1$$

For example $a = 2, b = 4, d = 7, f = 5$ produces a multiply perspective arrangement of type $(4, 4, 2)$ in the real affine plane such that no three of the four centers of perspectivity are on a line. For other properly chosen values of a, b, d

and if the four centers will be linear. Thus there exists two combinatorially equivalent MPA's of type $[4,4,2]$ in real affine space which are not affinely equivalent.

Case 2:

To complete the picture of the multiply perspective arrangements of type $[4,4,3]$ we examine the remaining combinatorially distinct arrangement and show that is not realizable in any Pappian plane.



$$C_3: y = \frac{b}{d}x, \quad y = \frac{c-1}{a}x + 1 \Rightarrow x = \frac{ad}{ab-cd+d} \quad y = \frac{ab}{ab-cd+d}$$

If $ab-cd+d \neq 0$

$$\begin{vmatrix} ad & ab & ab-cd+d \\ 1 & 1 & 1 \\ a & a & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} ad-ab & ab & d-cd \\ 0 & 1 & 0 \\ a-c & c & 1-c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} ad-ab & ab & d \\ 0 & 1 & 0 \\ a-c & c & 1 \end{vmatrix} = 0$$

$$\Rightarrow ad-ab = d(a-c) \Rightarrow ab=cd. \Rightarrow x = a \quad y = \frac{ab}{d}$$

$$\begin{vmatrix} ad & ab & ab-cd+d \\ 1 & 0 & 1 \\ a & b & 1 \end{vmatrix} = 0 \quad \begin{vmatrix} ad & ab & d \\ 1 & 0 & 1 \\ a & b & 1 \end{vmatrix} = 0 \quad \begin{vmatrix} ad & a & d \\ 1 & 0 & 1 \\ a & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} d(a-1) & a & d \\ 0 & 0 & 1 \\ a-1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow d(a-1) = a(a-1) \Rightarrow a = d.$$

But this is impossible

Suppose then that $ab - cd + d = 0$.

Then

$$\frac{b}{d} = \frac{c-1}{a-1} = \frac{b}{a-1} \Rightarrow b = c-1 \text{ and } d = a-1.$$

$$\Rightarrow ab - d(c-1) = 0 \Rightarrow ab - db = 0 \Rightarrow a = d$$

and this is impossible.

$$k = 5$$

In this case if an SPA of class $[5,5,2]$ exists the possible cycle length combination is $(3,2)$. But this is impossible by Theorem 4.6.

Thus there is no (S, Γ) of class $[5,5,d]$ for all d , hence there is no MPA $\{S_1, S_2, \Gamma\}$ of class $[5,5,d]$ $d \geq 3$.

However MPA's of class $[5,5,2]$ do exist. For example, the following figure exhibits an $\{S_1, S_2, \Gamma\}$ of class $[5,5,2]$

Fig. 4.3

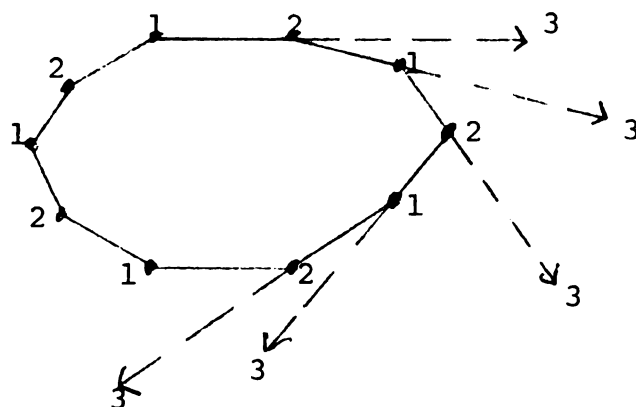


Fig. 4.3

The complete analysis of this class is not attempted in this thesis.

$$k = 6$$

Theorem 4.17:

There are two distinct $\{S, \Gamma\}$'s as SPA's of class $[6,6,2]$.

Proof:

If $\{S, \Gamma\}$ is an SPA of class $[6,6,2]$ then the numerical conditions gives the possible cycle lengths combinations $(3,3)$ and $(2,2,2)$.

Case 1:

Suppose $\{S, \Gamma\}$ contains permutations having cycles of lengths $(3,3)$. Its grid diagram is shown below

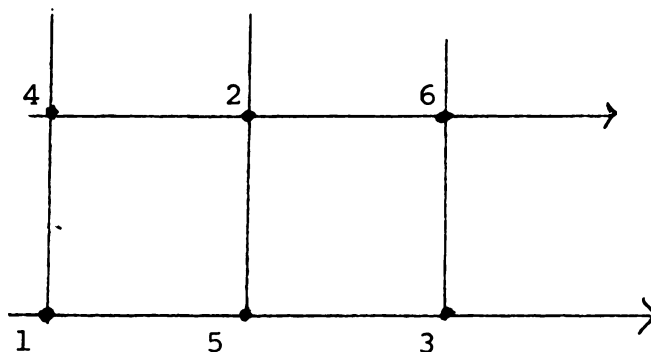


Fig. 4.4

A representation for Γ follow easily from the grid diagram as

$$\Gamma = \{I, (135)(246), (153)(264), (14)(25)(36), (16)(45)(23), (12)(34)(56)\}.$$

and it is clear that for this diagram this $\{S, \Gamma\}$ of class $[6,6,2]$ is unique up to an equivalence.

Case 2:

Suppose $\{S, \Gamma\}$ contains permutations having all cycles of lengths $(2,2,2)$. The grid diagram in this case is given by

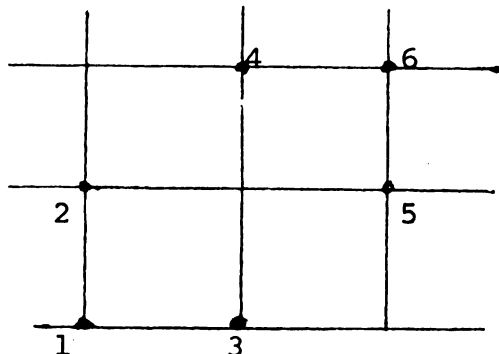


Fig. 4.5

We thus derive a representation for Γ as

$$\Gamma = \{I, (12)(34)(56), (13)(25)(46), (14)(26)(35), (15)(24)(36), (16)(23)(45)\}$$

which again is unique up to equivalence. Furthermore, the two SPA's of cases 1 and 2 are inequivalent by virtue of their cycle structure. ■

Remark 4.18:

From the grid diagram of case 1, it is clear that we cannot have a Γ in which all cycles of the permutations have lengths (3,3).

Theorem 4.19:

A Desarguesian affine plane Σ contains a unique self perspective arrangement (S, Γ) of class $[6,6,2]$ with 3 points of S linear iff the associated coordinate ring contains $\sqrt{-3}$. Two such arrangements in Σ are affinely equivalent.

Proof:

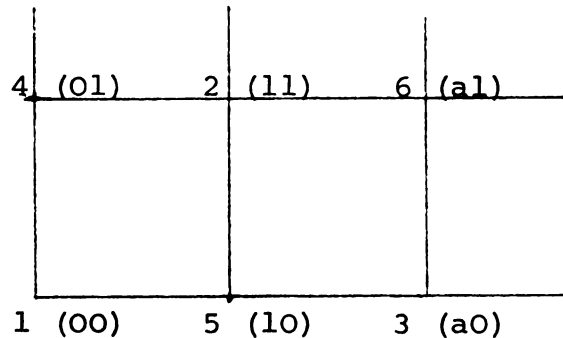


Fig. 4.6

$$\begin{aligned} \text{Line 16: } ay - x &= 0 & ay &= 1-y \\ \text{Line 45: } y + x - 1 &= 0 & y &= (a+1)^{-1} \\ \text{Line 23: } (1-a)y - (x-a) &= 0 & x &= a(a+1)^{-1} \\ \Rightarrow \boxed{a^2 - a + 1} &= 0 \end{aligned}$$

Using lines $\overline{12}$, $\overline{34}$, $\overline{56}$ we observe that the above equations are consistent. Thus we can choose $a = \frac{-1 \pm \sqrt{-3}}{2}$ and this is possible in any division ring containing $\sqrt{-3}$. That this condition is also sufficient is easily checked. ■

Theorem 4.20:

A Desarguesian affine plane Σ contains a self perspective arrangement (S, Γ) of class $[6, 6, 2]$ with no three points of S linear iff the associated coordinate ring has characteristic 2. Two such arrangements in Σ are affinely equivalent.

Proof:

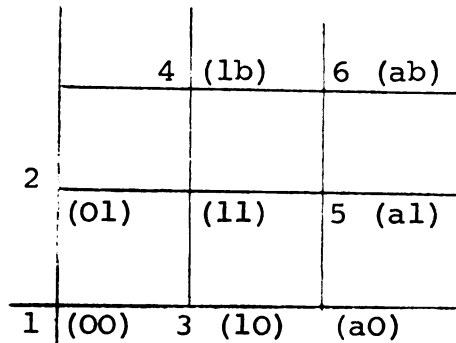


Fig. 4.7

From lines $\overline{15}$, $\overline{24}$, $\overline{36}$ we have

$$\boxed{ab^2 + 1 - a - b = 0} \quad (1)$$

From lines $\overline{16}$, $\overline{23}$, $\overline{45}$ we have

$$\boxed{ab[-b+1 - a+1] = 0} \quad (2)$$

From lines $\overline{14}$, $\overline{26}$, $\overline{35}$ we have

$$\boxed{a^2b + 1 - b - a = 0} \quad (3)$$

(1) and (3) give $a^2b = ab^2 \Rightarrow a = b$

Since $ab \neq 0$, (2) gives $-a^2[2-2a] = 0 \Rightarrow a = 1$ or $2 = 0$

But $a \neq 1$, hence $2 = 0$. That this condition is also sufficient is easily checked. ■

Lifting:

Theorem 4.21:

An MPA (S_1, S_2, Γ) of class $[6, 6, 3]$ exists in any projective 3-space whose coordinate ring is of characteristic 3.

Proof:

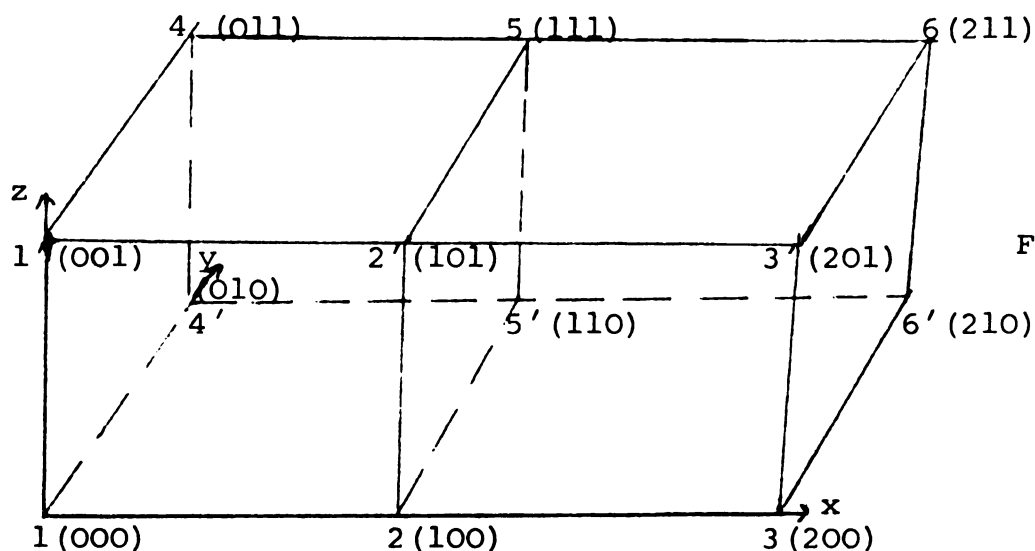


Fig. 4.8

Consider the above prism coordinatised from a division ring of characteristic 3 and labelled as shown. Let $S_1 = \{1, 2, 3, 4, 5, 6\}$ and $S_2 = \{1', 2', 3', 4', 5', 6'\}$. The centers of the perspectivities Γ are now obtained from the following calculations.

On the front plane we have

Line $\overline{12'}$; $x : y : z = 1 : 0 : 1$

Line $\overline{23'}$; $x-1 : y : z = 1 : 0 : 1$

Line $\overline{1'3}$; $x-2 : y : z = -2 : 0 : 1 = 1 : 0 : 1$

i.e. these lines are parallel with direction numbers $(1,0,1)$. Similar calculations on the rear plane give a set of three parallel lines with direction numbers $(1,0,1)$. Thus we have a center on the ideal plane in the direction with direction numbers $(1,0,1)$. Call this center C_{12} . Now the center C_{13} is obtained similarly:

$$\overline{13'} ; \text{ dir. nos. } = (2,0,1) = (2,0,1)$$

$$\overline{32'} ; \text{ dir. nos. } = (-1,0,1) = (2,0,1)$$

$$\overline{21'} ; \text{ dir. nos. } = (-1,0,1) = (2,0,1)$$

$$\overline{4'6} ; \text{ dir. nos. } = (2,0,1) = (2,0,1)$$

$$\overline{6'5} ; \text{ dir. nos. } = (-1,0,1) = (2,0,1)$$

$$\overline{45'} ; \text{ dir. nos. } = (-1,0,1) = (2,0,1)$$

i.e. C_{13} is a point on the ideal plane in the direction with dir. nos. $(2,0,1)$. Similarly C_{15} has dir. nos. $(1,1,0)$ and C_{16} has dir. nos. $(2,1,0)$. C_{11} and C_{14} are known to be on the ideal plane in the direction of the z -axis and the y -axis respectively.

Thus the above figure is an $\{S_1, S_2, \Gamma\}$ of class $[6,6,3]$ and clearly it is a lifting of an $\{S, \Gamma\}$ of class $[6,6,2]$ with three points on a line. This completes the proof of the theorem. ■

Remark 4.22:

This theorem can be generalized to an arbitrary prime number p , where in the above $p = 3$, is the characteristic of the ring.

Before we prove the uniqueness of the above $\{S_1, S_2, \Gamma\}$ of class $[6, 6, 3]$ we derive the following

Lemma 4.21:

In an MPA $\{S_1, S_2, \Gamma\}$ of class $[6, 6, 3]$ having permutations of Γ in a corresponding $\{S, \Gamma\}$ in Σ^2 with cycle lengths $(3, 3)$, if $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ denotes the set of centers, then each of the sets S_1, S_2 and C lie on two lines.

Proof:

As in chapter 3, we project the $\{S_1, S_2, \Gamma\}$ from one of its centers to obtain an $\{S, \Gamma\}$ in Σ^2 .

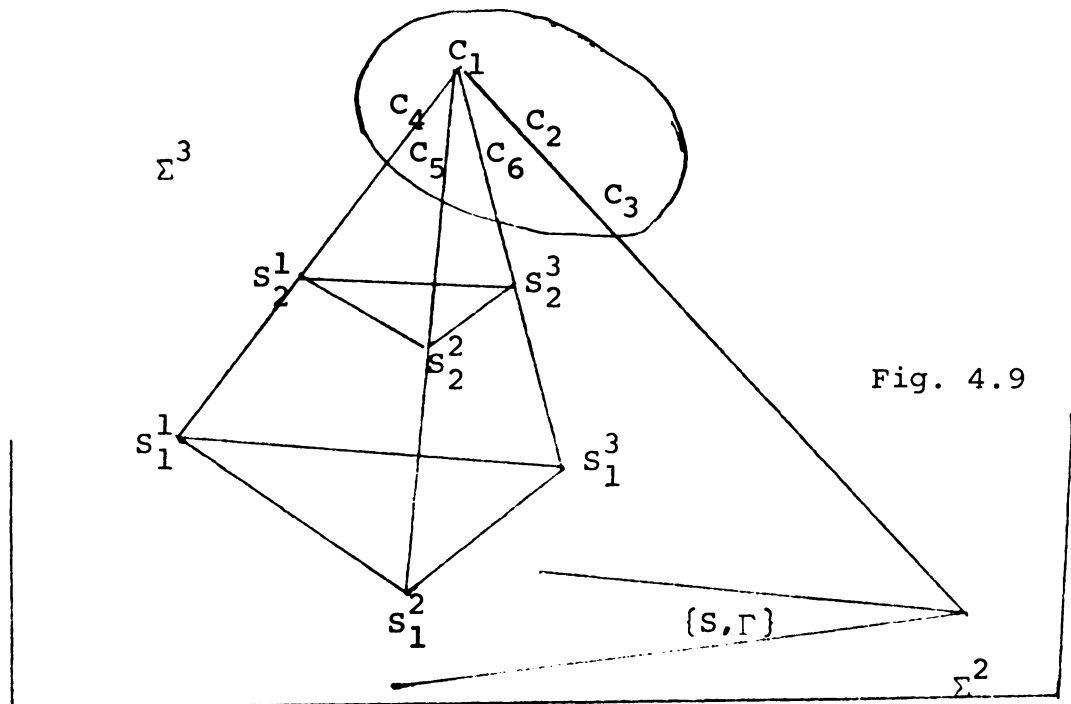


Fig. 4.9

Suppose the theorem is false. Then there exist three points in set C not on a line since we already have three points on a line by the structure of $\{S, \Gamma\}$ in Σ^2 . Hence, since

$|C| = |S_1| = |S_2|$ sets S_1 and C are multiply perspective from the points of set S_2 , sets S_1 and S_2 have each three points not on a line. Let these points be labelled s_1^1, s_1^2, s_1^3 and s_2^1, s_2^2, s_2^3 respectively as shown in the figure. Then C_1 with $\{s_1^1, s_1^2, s_1^3\}$ and $\{s_2^1, s_2^2, s_2^3\}$ span a plane. By our permutational representation of the perspectivities if γ_{12} corresponds to the case when $s_1^1 \rightarrow s_2^2$ with center C_2 (say) then $\{s_1^2, s_1^3\}$ and $\{s_2^1, s_2^3\}$ have to be mapped into each other for a new center other than C_1 and C_2 . Similarly for the sets of points in the plane containing $\{s_1^4, s_1^5, s_1^6\}$. But any such further perspectivity γ_{23} (say) is not disjoint from γ_{12} .

This contradiction proves the theorem. ■

Theorem 4.22:

(Uniqueness)

The multiply perspective arrangement $\{S_1, S_2, \Gamma\}$ of class $[6, 6, 3]$ constructed in theorem 4.21 is unique.

Proof:

By the above lemma all 6 points of each set S_1, S_2, C lie on two lines. Thus such an $\{S_1, S_2, \Gamma\}$ of class $[6, 6, 3]$ lies on two planes each containing nine points. Let us take one of these planes as the xy -plane and the other as the xz -plane. Furthermore, if we choose a point of the set of centers C at infinity then we have the following coordinate system.

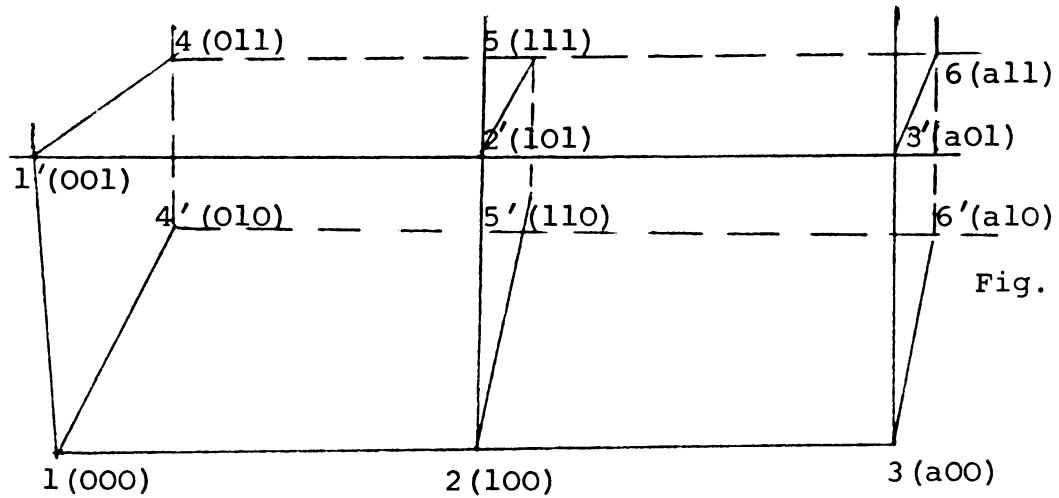


Fig. 4.10

Comparing direction numbers we observe that plane $\overline{36'5}$ is parallel to the plane $\overline{3'65}$. Hence we can take plane $\overline{3'65}$ at a unit distance above $\overline{36'5'}$. Furthermore, $\overline{12'}$ has dir. nos. (1,0,1) and $\overline{23'}$ has dir. nos. (a-1,0,1)

$\overline{31'}$ has dir. nos. (-a, 0, 1)

$\overline{4'5}$ has dir. nos. (1, 0, 1)

$\overline{5'6}$ has dir. nos. (a-1, 0, 1)

$\overline{6'4}$ has dir. nos. (-a, 0, 1)

$$\Rightarrow a-1 = 1 \text{ and } -a = 1 \text{ ' } a = 2 \text{ and } 3 = 0$$

This is exactly the configuration of Theorem 4. Hence the theorem. ■

Lemma 4.23:

If in an $\{S_1, S_2, \Gamma\}$ of class [6,6,3] one of its associated $\{S, \Gamma\}$ of class [6,6,2] has permutations containing cycles all of lengths (2,2,2) then S_1, S_2 , and the set of centers C are all planar.

Proof:

An $\{S, \Gamma\}$ of class $[k, k, d-1]$ is said to be associated with an $\{S_1, S_2, \Gamma\}$ of class $[k, k, d]$ if a projection of the $\{S_1, S_2, \Gamma\}$ of class $[k, k, d]$ from one of its centers onto a $d-1$ -projective subspace Σ^{d-1} is an $\{S, \Gamma\}$ of class $[k, k, d-1]$.

It is easy to show by some calculations that all the centers of an $\{S, \Gamma\}$ of class $[6, 6, 2]$ and of the prescribed cycle structure lie on the ideal line, hence the set C of centers of $\{S_1, S_2, \Gamma\}$ is planar. Claim S_1 and hence S_2 is also planar. Suppose the claim is false. Let the C and S_2 be perspective from the points of S_1 . These are two cases to be considered.

(i) A projection from a point of S_1 gives an $\{S, \Gamma\}$ with permutations having all cycle length $(2, 2, 2)$, then this implies S_1 is planar.

(ii) If the cycle lengths are $(3, 3)$ then three points of S_1 are linear, now if we revert back and project from a point of set C we obtain an $\{S, \Gamma\}$ with three points on a line, this is a contradiction. Therefore S_1 and hence S_2 is planar. ■

Theorem 4.24:

An MPA $\{S_1, S_2, \Gamma\}$ of class $[6, 6, 3]$ one of whose associated $\{S, \Gamma\}$ of class $[6, 6, 2]$ has permutations with all cycles of lengths 2 does not exist.

Proof:

From theorem 4.20 an associated $\{S, \Gamma\}$ can only exist in a Pappian plane over a field of characteristic 2. The coordinates of the centers are found to be

$$\left. \begin{array}{l} y = ax \\ x = \frac{a}{a^2 - a + 1} \end{array} \right\} \Rightarrow q_3 = (\infty, \infty)$$

for some a in the field and choice of coordinates as in theorem 4.20. Similarly $q_4 = (\infty, \infty)$ and $q_5 = (\infty, \infty)$. This means all five non identity centers are on the ideal line. This implies that the set C and each of the sets S_1 and S_2 is planar (see lemma 4.23). Choosing now the plane of the set C as the base plane π we proceed to show that the plane of set S_2 is parallel to π .

$$\text{Let } S_2 = \{1', 2', 3', 4', 5', 6'\}, C = \{1, 2, 3, 4, 5, 6\}$$

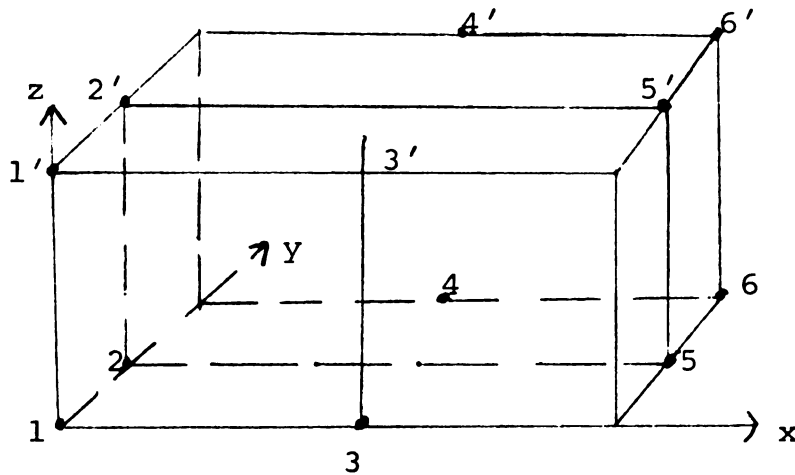
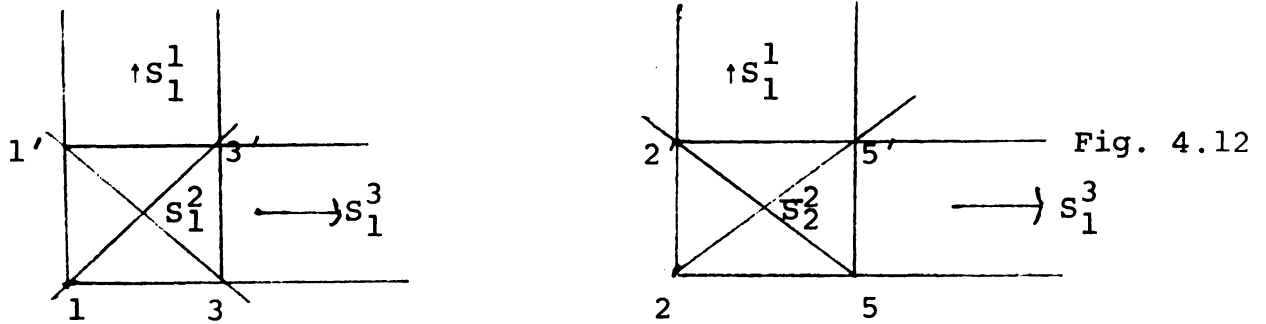


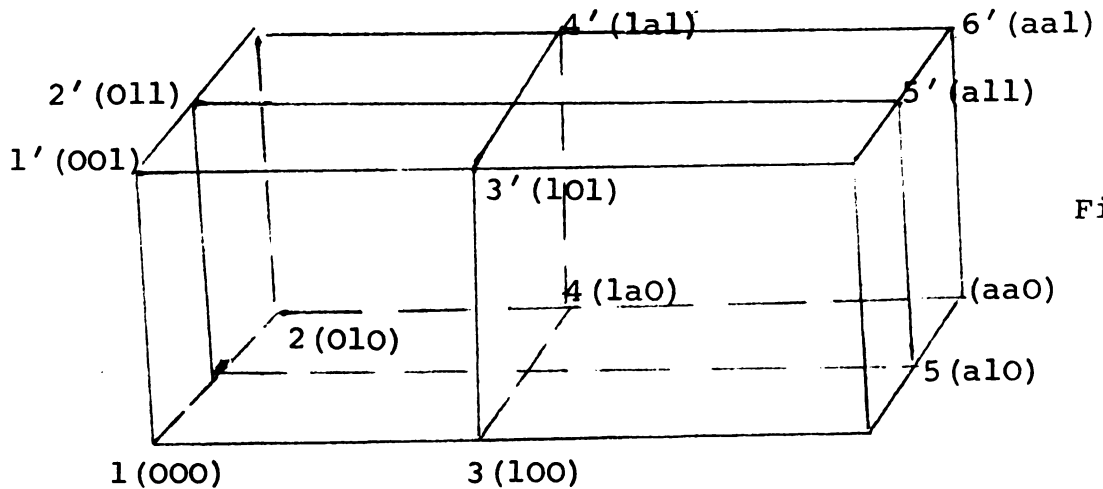
Fig. 4.11

Now we show that $\overline{1'3'} \parallel \overline{13}$

Let the point $S_1^1 \in S_1$ be chosen at infinity on the z-axis.
Consider the following two complete quadrangles



S_1^1 is at infinity by the choice of coordinate axis.
 $S_1^2 = \bar{S}_2^2 \Rightarrow S_1^2$ is at infinity. But S_1^1, S_1^2, S_1^3 are collinear since diagonal points of a complete quadrangle. Therefore S_1^3 is at infinity. Therefore $\overline{1'3'} \parallel \overline{13}$. Similarly $\overline{4'6'} \parallel \overline{46}$. This implies the plane of S_2 is parallel to the plane of C . Thus we can choose S_2 at height 1 above C , giving the following coordinates.



Considering vectors $\overline{13'}: (1,0,1)$, $\overline{1'3} = (-1,0,1) = (1,0,1)$
and $\overline{25'} = (a,0,1) = (101) \Rightarrow a = 1 \quad \leftrightarrow \blacksquare$

From the above results 4.17 - 4.24 we have the following result.

Theorem 4.25:

There is one and only one MPA, $\{S_1, S_2, \Gamma\}$ of class $[6, 6, 3]$ up to equivalence.

Theorem 4.26:

A Desarguesian affine space Σ contains a unique self-perspective arrangement $\{S, \Gamma\}$ of class $[7, 7, 2]$ iff the characteristic of the ring is 2.

Proof: It is clear that the only solution to our numerical equations is $(3, 2, 2)$. Thus all permutations of Γ have cycle lengths $(3, 2, 2)$. Let $C = \{c_2, c_3, c_4, c_5, c_6, c_7\}$ be the set of non-identity centers. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$.

Consider the following grid diagram

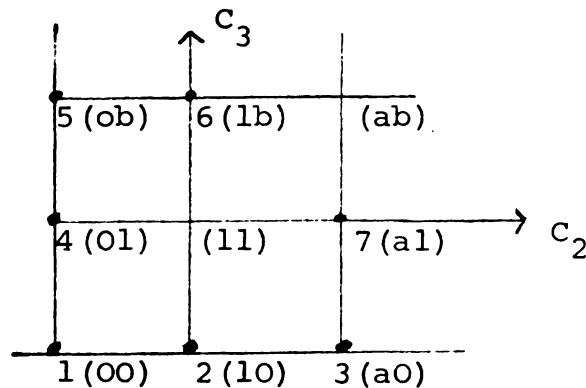


Fig. 4.14

From the linearities in the grid diagram we can derive the following set for Γ

$$\Gamma = \{I, (123)(47)(65), (132)(46)(57), (145)(26)(37), \\ (154)(27)(36), (167)(24)(35), (176)(25)(34)\}.$$

The set $\{S, \Gamma\}$ is clearly an SPA of class $[7, 7, 2]$.

After choosing coordinates as shown and performing a few simple calculations we obtain the following relations

$$b = a^{-1}$$

$$\text{and } (2a)^{-1} = (-b + a^{-1})^{-1}(1-b).$$

These imply that

$$2a = 0$$

and since $a \neq 0 \Rightarrow 2 = 0$. The sufficiency of this condition follow from the proof which is easily reversible. The uniqueness follows from the uniqueness of the Γ . ■

Lifting:

Theorem 4.27:

An MPA $\{S_1, S_2, \Gamma\}$ of class $[7, 7, 3]$ does not exist.

Proof:

Suppose the theorem is false. Then the projection of $\{S_1, S_2, \Gamma\}$ to a projective 2-subspace Σ^2 of its embedding space Σ^3 is an SPA $\{S, \Gamma\}$ of class $[7, 7, 2]$. From the preceding theorem, a necessary condition for the coordinatising ring is that

$$a = b^{-1}$$

Now if we compute the centers of the $\{S, \Gamma\}$ we have (with notation as in Theorem 4.26) $q_4 = (\frac{1}{1+b}, 0)$, $q_5 = (0, \frac{1}{a+1})$, $q_6 = (\frac{1}{1+b}, \frac{b}{1+b})$, $q_7 = (\frac{a}{1+ab}, \frac{1}{1+ab}) = (\infty, \infty)$. Thus we have the following configuration for the set of centers $\{q_i\}$,

$i = 2, 3, 4, 5, 6, 7$.

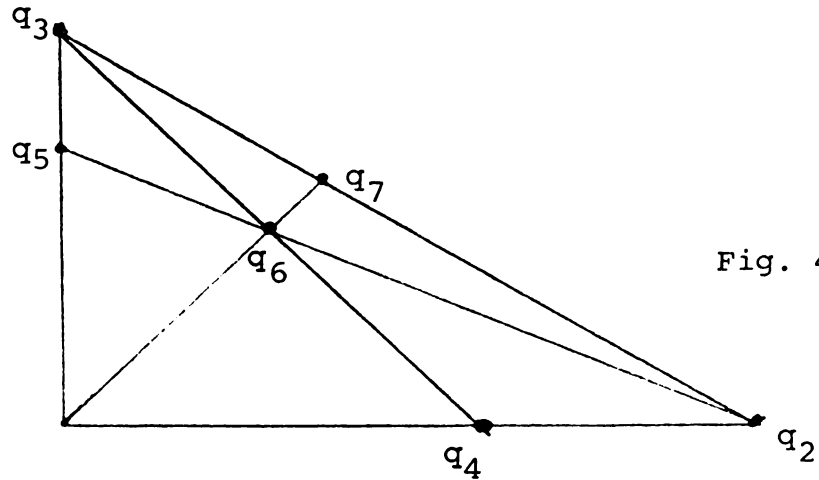


Fig. 4.15

This implies that in the set C and hence (in set S_1 and S_2) the points lie in 4's in three planes incident at C_1 (say). Now representing these three planes through C_1 each with 4 points we have the following tetrahedron.

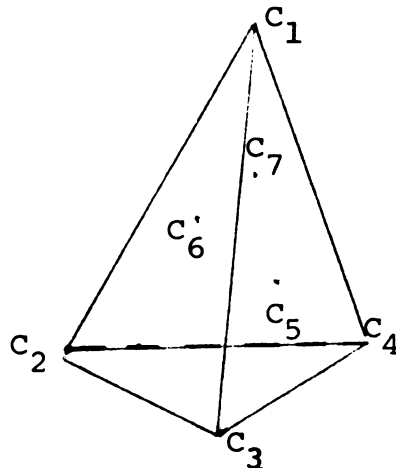


Fig. 4.16

But the projection used is independent of the choice of center in Set C . However, the above configuration of Set C is not true of the points C_2, C_3 , and C_4 as centers of projection. Since the plane $\overline{C_2, C_3, C_4}$ has no point on it. Therefore, an MPA of class $[7, 7, 3]$ cannot exist. ■

Remark 4.28:

There is no MPA $\{S_1, S_2, \Gamma\}$ of class $[7, 7, 2]$, $d > 3$, from theorem 4.9.

§ 4. Some general results

In this last short section we give some general results, with particular reference to $k = 8$.

Theorem 4.29:

In a finite projective 4-space Σ^4 , there exists an MPA of class $[p^3, p^3, 4]$ where p is a prime and is also the order of the underlying field.

Proof:

This follows from theorem 4.9 since $p^3 \geq 8 \quad \forall p$ ■

Corollary 4.30:

There exists an MPA of class $[8, 8, 4]$.

Corollary 4.31:

There exists an SPA of class $[8, 8, 3]$. This follows by projecting the MPA of class $[8, 8, 4]$ from one of its centers onto 3-subspace of its embedding space.

Remark 4.32:

The possible solutions to our numerical relations in case of $k = 8$ are $(2, 2, 2, 2)$, $(4, 2, 2)$, $(3, 2, 2)$ and $(4, 4)$. We have already made some remarks about the solution $(2, 2, 2, 2)$. For solution $(4, 2, 2)$ we have

Theorem 4.33:

If an SPA $\{S, \Gamma\}$ of class $[8, 8, 2]$ containing permutations with cycle lengths $(4, 2, 2)$ exists then the coordinating field must have characteristic 2.

Proof:

Consider the following representatives for the γ 's.

$$\gamma_{56} = (1234)(56)(78)$$

$$\gamma_{57} = (1432)(57)(68)$$

$$\gamma_{58} = (13)(24)(58)(67)$$

Let their corresponding centers be denoted by q_{56}, q_{57} , and q_{58} . These centers are the diagonal points of the complete quadrangle with vertices 5,6,7,8 and furthermore they lie on the line $\overline{1234}$. Hence the result. ■

Theorem 4.34:

There is no SPA of class $[8,8,2]$ containing permutations with cycle lengths $(3,3,2)$.

Proof:

This follows from theorem 4.6 since when we compare one cycle of length 3 against the rest of symbols we have an unequal splitting which is impossible by Theorem 4.6 ■

CONCLUDING REMARKS

Abstract Sylvester-Gallai configurations are very weak and inhomeogeneous structures. The class includes finite projective and affine spaces among others. Hence any effective classification or characterization of S.G. configuration is too much to expect. However, their complete absence in ordered projective spaces and their seeming scarcity in complex projective space suggests further efforts to understand their relation to the concept of order and to whatever it is that inhibits their occurence in complex spaces.

Certainly the curious behavior of generalized Sylvester-Gallai configurations in ordered projective spaces warrants more investigation. The Edelstein theorems cannot be the final general conclusions in this direction.

The related conjecture that completely self perspective arrangements of more than four points fail to exist in an ordered projective plane does not seem beyond hope of settling though our painful case examinations have failed to reveal the proper general methods. We propose to pursue this matter further.

There seems to be a resurgence of interest by algebraists and others in the study of general configurations and we can only hope that this investigation may contribute in some way to these developments.

BIBLIOGRAPHY

1. J. J. Sylvester - London Times Educational Supplement, 1893.
2. M. Edelstein, F. Herzog, L. M. Kelly, A further theorem of the Sylvester type, Proc. Amer. Math. Soc. 14 (1963) 359-363.
3. M. Edelstein and L. M. Kelly, Bisecants of finite collections of sets in linear spaces, Can. J. Math. 18 (1966) 375-380.
4. B. Grunbaum, A generalization of the problem of Sylvester, Riveon Lematematiks 10 (1956) 46-48.
5. D. Hilbert and S. Cohn-Vossen, Geometry and the imagination, Chelsea Publishing Co., N. Y., 1952.
6. L. M. Kelly and W. O. J. Moser, On the number of ordinary lines determined by n points, Can. J. Math. 10 (1958) 210-219.
7. F. Levi, Geometrische Konfigurationen, Verlag von S. Hirzel in Leipzig - 1929.
8. T. Motzkin, The lines and planes connecting the points of a finite set, Trans. Am. Math. Soc. 70 (1951) 451-464.
9. M. C. Stephanos, Comptes Rendus XCIII 1881 p. 578 and p. 634.
10. O. Veblen and J. W. Young, Projective Geometry (1916), 2 vols, Cunn and Co., Boston.
11. L. D. Cummings "On a method of comparison for Triple Systems", TAMS 15 (1914) 311-327.
12. E. Artin, Galois Theory (1959) Notre Dame, Indiana.
13. J. P. Serre, Problem AMM 73, Pg. 89 (1966).
14. P. Dembowski, Finite Geometries, Springer-Verlag (1968) New York.
15. F. G. Stockton, A Set of Triply Perspective Triangles Associated with Projective Triads, Slaught-Memorial Papers 4 (1955), 41.
16. N. A. Court, Modern Pure Solid Geometry, MacMillan 1935.

MICHIGAN STATE UNIVERSITY LIBRARIES



3 1293 03169 0724