THE INFLUENCE ON A FINITE GROUP OF THE COFACTORS AND SUBCOFACTORS OF ITS SUBGROUPS

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ABSTRACT

THE INFLUENCE ON A FINITE GROUP OF THE
COFACTORS AND SUBCOFACTORS OF ITS SUBGROUPS

By

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There are a number of theorems of the form: If every proper subgroup of the finite group G has property X, then G has property Y. Examples are the classic results of Schmidt, Iwasawa, Ito, and Huppert. Such results have been extended by imposing condition X on only one maximal subgroup of G (Deskins, Huppert, Thompson) or on a certain class of subgroups (Rose). The major goal here is to extend results of this type by imposing condition X on only the "worst" parts of the "bad" subgroups of G (from the viewpoint of normality), namely, the cofactors or subcofactors of the self-normalizing or abnormal subgroups of G. In some cases, X is also imposed on the "good" subgroups, those which are normal or close to being normal in G. In the last chapter, the influence on G of X-outer cofactors of subgroups is examined.

Throughout, G denotes a finite group. For a subgroup H of G, the cofactor and subcofactor of H are $\mathrm{cof}_G H = H/\mathrm{cor}_G H$ and $\mathrm{scof}_G H = H/\mathrm{scor}_G H$ respectively, where $\mathrm{cor}_G H$ is the core of H in G, and $\mathrm{scor}_G H$ = the subnormal core of H is the largest G-subnormal subgroup of H.

For X = nilpotency, we have: G/F(G) is nilpotent if-f $\mathrm{cof}_G S$ is nilpotent for all maximal subgroups S of G, where F(G) is the Fitting subgroup of G. Also, if $\gamma_n(G)$ is the (n+1)st term of the descending

central series of G, the following are equivalent: (a) $G/F(G)$ has class $\leq n$. (b) $\mathrm{cof}_G S$ has class $\leq n$ for all maximal subgroups S of G. (c) $\gamma_n(G)$ is nilpotent. (d) $\gamma_n(H) \lhd\lhd G$ for all (abnormal) $H \lneqq G$.

Under the hypothesis that G is solvable, one can replace γ_n by the (n+1)st derived subgroup and "has class $\leq n$" by "has derived length $\leq n$" in (a)-(d) above. The resulting statements are equivalent.

In the preceding results, G need not have a normal Sylow subgroup. If, however, each $K \ntrianglelefteq G$ also is nilpotent, but G itself is not, then for some prime p_r where $|G| = \prod_{i=1}^{r} p_i^{a_i}$, we obtain, among other results: (1) $|G:F(G)| = p_r$ so that G has normal p_i-Sylow subgroups P_i for each $i \neq r$, and each $P_i \subseteq G'$. (2) $|H/\mathrm{scor}_G H| = 1$ or p_r for each $H \lneqq G$. (3) For each abelian P_i, $i \neq r$, G has $p_i^{a_i}$ p_i-complements, $C_G(P_i) = F(G)$, and $p_i^{a_i}$ divides the number of p_r-Sylow subgroups of G. (4) If all the P_i, $i \neq r$, are abelian, then G has $|G|/p_r^{a_r}$ p_r-Sylow subgroups.

Under the preceding hypotheses, $|\pi(G)|$ can be arbitrarily large. Defining $H < G$ to be nearly normal in G if $|H/\mathrm{cor}_G H| = 1$ or a prime, however, we have that if G is nonnilpotent but has all nearly normal maximal subgroups nilpotent as well as the cofactors of maximal subgroups, then $|\pi(G)| = 2$, all proper subgroups of G are nilpotent, and thus the Schmidt-Iwasawa conclusions hold.

For X = p-nilpotency, we have: If (a) $\mathrm{scof}_G H$ is p-nilpotent for each self-normalizing $H \lneqq G$, or if (b) $\mathrm{scof}_G H$ is p-nilpotent for each abnormal $H \lneqq G$ and either p is odd or the p-Sylows of G are abelian, then, in each case, G has a normal p-subgroup P with G/P p-nilpotent.

If in addition to (a) or (b), each $K \ntrianglelefteq G$ is p-nilpotent while G itself is not, then, among other things: (1) G has a normal p-Sylow subgroup P with $P \subseteq G'$. (2) If P is abelian, then $C_G(P) = F_p(G)$, the largest normal p-nilpotent subgroup of G. (3) For G solvable, $|G:F_p(G)|$ is equal to a prime $\neq$ p; and if also P is abelian, G has exactly p^a distinct p-complements, where $|G| = p^a m$ with $(p,m) = 1$.

If (a) or (b) as above holds and each proper somewhat normal subgroup of G is p-nilpotent, but G is not, where $H < G$ is somewhat normal in G if $H/\mathrm{cor}_G H$ is cyclic of prime-power order, then $|G| = p^a q^b$ for some prime $q \neq p$, and the Ito-Schmidt-Iwasawa conclusions hold.

Defining G to be (p:q)-nilpotent if (i) G is p-nilpotent, and (ii) G is q-nilpotent with $q \mid |G|$ in case $p \nmid |G|$ and $|G| > 1$, we obtain: If each $K \trianglelefteq G$ is $(p:q_K)$-nilpotent and $\mathrm{cof}_G H$ is $(p:q_H)$-nilpotent for each $H \lneqq G$, then G is solvable and has a normal Sylow subgroup (in addition to (1)-(3) above holding in case G is not p-nilpotent).

For X = supersolvable or Sylow-towered, $\mathrm{cof}_G S$ supersolvable for all maximal subgroups S of G does not imply that G is solvable. But for a fixed ordering σ of a set of primes containing $\pi(G)$, we have G solvable with $G/F(G)$ σ-Sylow-towered if either (a) $\mathrm{scof}_G H$ is σ-Sylow-towered for each self-normalizing $H \lneqq G$, or (b) $\mathrm{scof}_G H$ is σ-Sylow-towered for each abnormal $H \lneqq G$ and the 2-Sylows of G are abelian.

If (a) or (b) holds with "supersolvable" replacing "σ-Sylow-towered," or if (c) $\mathrm{scof}_G H$ is supersolvable for each abnormal $H \lneqq G$ and the abnormal maximal subgroups of G have prime-power index, then, in each case, G is solvable with $G/F(G)$ supersolvable and Fitting length of $G' = \mathfrak{f}(G') \leq 2$, $\mathfrak{f}(G) \leq 3$. These are the best possible bounds on $\mathfrak{f}(G')$ and $\mathfrak{f}(G)$.

In the last chapter, the outer cofactors of a subgroup H as a kind of dual to $\mathrm{cof}_G H$ are considered. These are of the form $C/\mathrm{cor}_G(C \cap H)$ where $C \not\subseteq H$ and $L \subseteq H$ for each proper G-normal subgroup L of C. We term this a normal, self-normalizing, or abnormal outer cofactor of H according as C is normal, self-normalizing, or abnormal in G. General results for nonnormal outer cofactors, from which corollaries parallel to the preceding results follow, are: Given a subgroup-inherited property θ which is invariant under homomorphisms. If (a) G has a θ-maximal subgroup whose self-normalizing (abnormal) outer cofactors are θ-groups, or if (b) the self-normalizing (abnormal) outer cofactors of each abnormal maximal subgroup of G are θ-groups, then, in either case, $cof_G H$ is a θ-group for all self-normalizing (abnormal) $H \lneqq G$.

The following are some of the properties of the normal outer cofactors. For a maximal subgroup S of G, the normal outer cofactors of S are isomorphic; the order of any one is called the normal index of S. The normal outer cofactors of S are p-solvable (solvable) if-f the normal index of S is a power of p or is prime to p (is a power of a prime); in the solvable case, the normal index and the index of S are equal.

The influence on G of normal outer cofactors is described by: (a) G is p-solvable if-f (b) G has a p-solvable maximal subgroup having p-solvable normal outer cofactors if-f (c) the normal outer cofactors of each (abnormal) maximal subgroup of G are p-solvable if-f (d) the normal index of each (abnormal) maximal subgroup of G is a power of p or is prime to p. If "p-solvable" is replaced by "solvable" and "is a power of a prime" replaces "is a power of p or is prime to p," the resulting statements are equivalent to (e) the normal index and the index of each (abnormal) maximal subgroup of G are equal.

Finally, the intersection of all the maximal subgroups of G with normal index divisible by both p and some prime $\neq$ p coincides with the intersection of all abnormal maximal subgroups having this property, and is equal to the largest normal p-solvable subgroup of G. Replacing "p-solvable" by "solvable" and "divisible by both p and a prime $\neq$ p" by "divisible by two distinct primes" yields an immediate corollary.

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TO MY WIFE
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CHAPTER ONE

INTRODUCTION; PRELIMINARY RESULTS

There are a number of known theorems of the type: For a finite group G, if every proper subgroup of G has property P, then G has property Q. For example, Schmidt [18] and Iwasawa [14] have shown that if every proper subgroup of a finite group G is nilpotent, then G is solvable. More precisely, if G itself is not nilpotent, then $|G| = p^a q^b$ for distinct primes p and q; G has a normal p-Sylow subgroup P with $\Phi(P) \subseteq Z(G)$, and thus P has class ≤ 2; $\exp(P) = p$ or $\exp(P) \leq 4$ according as p is odd or $p = 2$; each q-Sylow subgroup Q of G is cyclic with $\Phi(Q) \subseteq Z(G)$. Huppert [12] and Doerk [6] have obtained corresponding results for the case where the proper subgroups of G are supersolvable. Results are also known for the cases where the proper subgroups of G are p-nilpotent, or abelian, or σ-Sylow-towered for some fixed ordering σ of a set Σ of primes containing the prime divisors of $|G|$.

Extensions of such results have been obtained by imposing the conditions not on the totality of proper subgroups, but only on certain subgroups. Thus there are a number of theorems in which the conditions are imposed on only one maximal subgroup of G. For example, Deskins has shown in [4] that the finite group G is solvable if it possesses a nilpotent maximal subgroup having Sylow subgroups of class ≤ 2. There are also a number of results in which the conditions are imposed on the proper subgroups of a certain kind. Examples of this are the following two results established by Rose in [17]: (1) If all the proper abnormal subgroups of the finite group G are nilpotent, then G is solvable; in fact, G has a normal Sylow subgroup P such that G/P is nilpotent.

(2) If all the proper self-normalizing subgroups of the finite group G are supersolvable, then G is solvable.

In the following chapters, the major effort is directed at extending theorems of the type described. We shall, like Rose, consider the influence on a finite group G of conditions imposed on those proper subgroups of G which, from the viewpoint of normality, are the "bad" subgroups, namely, the self-normalizing subgroups of G, or the abnormal subgroups of G. However, we will not require that the conditions be satisfied by these subgroups themselves, but only by their "worst" parts (from the viewpoint of normality or subnormality), that is, their cofactors, or subcofactors (or, as in Chapter 3, their outer cofactors). In all the cases we consider, this will be enough to guarantee the solvability or p-solvability of G, and in several cases we can say more. To obtain still more information about the structure of G, we shall on occasion impose the conditions on the "good" subgroups of G also, that is, the normal subgroups, or the subgroups of G which are rather close to being normal in the sense that their cofactors are quite small. In the last chapter, we define the outer cofactors of subgroups of G, as a kind of dual to the cofactors, and investigate their influence on the group G.

To begin, therefore, we make the following definitions.

Definition 1.1: For a proper subgroup H of the finite group G, we define:

(i) the core of H in G, $cor_G H$, as

$$cor_G H = \bigcap_{x \in G} H^x$$

$$= \text{the largest } G\text{-normal subgroup of } H;$$

(ii) the subnormal core of H in G, $\mathrm{scor}_G H$, as

$$\mathrm{scor}_G H = \text{the largest } G\text{-subnormal subgroup of } H$$
$$= \langle L \mid L \subseteq H \text{ and } L \lhd \lhd G \rangle.$$

Note: It is a well-known property of subnormal subgroups (see, for example, Scott [19], 15.2.4) that if L_1 and L_2 are subnormal subgroups of a finite group G, then $\langle L_1 , L_2 \rangle$ is also subnormal in G. Thus the subnormal core of a subgroup H of G is well-defined.

Lemma 1.1: For a proper subgroup H of the finite group G, $\mathrm{scor}_G H$ is normal in H.

Proof: From the definition, $L = \mathrm{scor}_G H$ is subnormal in G, say $L \lhd N_1 \lhd \ldots \lhd N_r = G$. Now, let x be any element of H; then, clearly, $L^x \lhd N_1^x \lhd \ldots \lhd N_r^x = G$ so that $L^x \lhd \lhd G$. By the above note, $\langle L , L^x \rangle$ also is subnormal in G. From the definition of $L = \mathrm{scor}_G H$ it follows that $L^x = L$. Thus $(\mathrm{scor}_G H)^x = \mathrm{scor}_G H$ for each $x \in H$ so that $\mathrm{scor}_G H \lhd H$. ▯

The preceding lemma makes possible the second part of the following definition.

Definition 1.2: For a proper subgroup H of the finite group G, we define:

(i) the cofactor of H in G, $\mathrm{cof}_G H$, as

$$\mathrm{cof}_G H = H/\mathrm{cor}_G H;$$

(ii) the subcofactor of H in G, $\mathrm{scof}_G H$, as

$$\mathrm{scof}_G H = H/\mathrm{scor}_G H.$$

Since we shall be dealing only with finite groups, we assume at the outset that all groups considered here are finite. It might be mentioned, however, that the preceding definitions of the core and the cofactor of a subgroup of a given group G are still legitimate in the case that G is an infinite group. Also, one can show (see, for example, Scott [19], 3.3.5) that if G possesses a proper subgroup K of finite index, then $G/\text{cor}_G K$ is a finite group. Thus, the results obtained do give some information about such infinite groups. For suppose that some group-theoretic property θ, which is preserved under homomorphisms, is required of the cofactors (and/or cores) of subgroups of G, and that K is a proper subgroup of G of finite index. Then the cofactors (and/or cores) of proper subgroups of $G/\text{cof}_G K$ also are θ-groups (since (i) and (iii) of Lemma 1.4 hold for any group G). Thus, for example, supposing that the cofactors of all maximal subgroups of G are nilpotent as are the proper normal subgroups of G and that G has a proper subgroup K of finite index, we have that G is solvable and the conclusions of Theorem 2.11 hold for $G/\text{cor}_G K$.

Lemma 1.2: Let θ be a homomorphism-invariant property, that is, homomorphic images of θ-groups are θ-groups; and let H be a proper subgroup of the finite group G. If $\text{cof}_G H$ is a θ-group, then $\text{scof}_G H$ also is a θ-group.

Proof: This is immediate; for it follows from the definitions that $\text{cor}_G H \subseteq \text{scor}_G H$. Thus, $\text{scof}_G H = H/\text{scor}_G H$ is a homomorphic image of the θ-group $\text{cof}_G H = H/\text{cor}_G H$, and hence is also a θ-group. ▯

It follows from the preceding lemma that any theorem which gives information about G resulting from conditions imposed on the subcofactors of subgroups of G will automatically be true if these conditions are satisfied by the cofactors of these subgroups (provided, of course, that these conditions are homomorphism-invariant properties). Consequently, wherever possible, we will impose conditions on only the subcofactors of subgroups as opposed to their cofactors.

The subnormal core of a subgroup H of a finite group G is in general not equal to the core of H. One need only take H and G so that $H \lhd \lhd G$ but $H \ntrianglelefteq G$ to see this. For example, if $G = A_4$ = the alternating group of degree 4, and H is a subgroup of order 2, then $\mathrm{cor}_G H = \langle 1 \rangle$, but $\mathrm{scor}_G H = H$. For maximal subgroups, however, the core and subnormal core must always coincide.

Lemma 1.3: If S is a maximal subgroup of the finite group G, then

$\mathrm{cor}_G S = \mathrm{scor}_G S$.

Proof: From the definition, $\mathrm{scor}_G S$ is subnormal in G, say $\mathrm{scor}_G S \ntrianglelefteq N_1 \ntrianglelefteq N_2 \ntrianglelefteq \ldots \ntrianglelefteq N_r = G$; and no subgroup H of S properly containing $\mathrm{scor}_G S$ can be subnormal in G so that, in particular, $N_1 \not\subseteq S$ since $N_1 \lhd \lhd G$. From Lemma 1.1 we have $\mathrm{scor}_G S \lhd S$; also, $\mathrm{scor}_G S \lhd N_1$. Therefore, $\mathrm{scor}_G S \lhd \langle N_1, S \rangle = G$, which implies that $\mathrm{scor}_G S \subseteq \mathrm{cor}_G S$. Since the reverse inclusion is immediate from the definitions, this establishes the desired equality. ▯

Because many of our results will involve induction arguments, we must examine for a given group G the relationship between the core and the subnormal core of a subgroup of a homomorphic image of G and those of the corresponding subgroup of G. The following basic lemma does precisely this.

Lemma 1.4: Let H and K be proper subgroups of the finite group G with

$K \lhd G$ and $K \subseteq H$. Then:

(i) $\mathrm{cor}_{G/K}(H/K) = \mathrm{cor}_G H/K$;

(ii) $\mathrm{scor}_{G/K}(H/K) = \mathrm{scor}_G H/K$;

(iii) $\mathrm{cof}_{G/K}(H/K) \cong \mathrm{cof}_G H$;

(iv) $\mathrm{scof}_{G/K}(H/K) \cong \mathrm{scof}_G H$.

Proof: (i) Let $L/K = cor_{G/K}(H/K)$. Since $L \triangleleft G$ and $L \subseteq H$, we have $L \subseteq cor_G H$; thus $cor_{G/K}(H/K) \subseteq cor_G H/K$. Conversely, since $K \triangleleft G$ and $K \subseteq H$, we have $K \subseteq cor_G H$; and since $cor_G H \triangleleft G$ and $cor_G H \subseteq H$, we have $cor_G H/K \subseteq cor_{G/K}(H/K)$. Therefore, $cor_{G/K}(H/K) = cor_G H/K$.

(ii) is proved in a similar way.

(iii) From the definition of cofactor, part (i), and the Third Isomorphism Theorem, we have

$$cof_{G/K}(H/K) = \frac{H/K}{cor_{G/K}(H/K)} = \frac{H/K}{cor_G H/K} \cong H/cor_G H = cof_G H.$$

(iv) follows in a similar manner. ▯

The three results that follow illustrate the close connection between the cores and subnormal cores of subgroups of a finite group G and the normal structure of G.

Lemma 1.5: Let G be a finite group. Then:

(i) $cor_G H \neq \langle 1 \rangle$ for all proper subgroups $\langle 1 \rangle \neq H \lneqq G$ if and only if every minimal subgroup of G is normal in G.

(ii) $scor_G H \neq \langle 1 \rangle$ for all proper subgroups $\langle 1 \rangle \neq H \lneqq G$ if and only if every minimal subgroup of G is subnormal in G.

Proof: (i) Suppose first they every nontrivial proper subgroup of G has nontrivial core, and let $M \neq \langle 1 \rangle$ be any minimal subgroup of G. By hypothesis, $cor_G M \neq \langle 1 \rangle$; thus, by the minimality of M, we have $M = cor_G M$ is a normal subgroup of G.

Conversely, suppose that every minimal subgroup of G is normal in G, and let H be any nontrivial proper subgroup of G. Taking $x \in H$ of prime order, we have that since $\langle x \rangle$ is a minimal subgroup of G, it is, by hypothesis, normal in G. Thus, $\langle 1 \rangle \neq \langle x \rangle \subseteq cor_G H$ so that $cor_G H$ is nontrivial.

(ii) is proved in a similar manner. ▯

<u>Theorem</u> <u>1.6</u>: For a finite group G, $cor_G H$ is nontrivial and is Hall in H for all proper nontrivial subgroups H of G if and only if all subgroups of G are normal in G, that is, G is a Dedekind group.

<u>Proof</u>: Suppose first that every proper nontrivial subgroup H of G has a nontrivial core which is a Hall subgroup of H, and let K be any subgroup of G. Since the trivial subgroups $< 1 >$ and G are normal in G, we may assume that $< 1 > \neq K \lneq G$. Let $|K| = \prod_{i=1}^{r} p_i^{a_i}$ where the p_i are distinct primes dividing $|K|$; and for each $i = 1, \dots, r$, let $x_i \in K$ have order p_i. Then, from Lemma 1.5, $< x_i > \lhd G$ for each i, and thus $L = < x_1 >< x_2 > \dots < x_r > \lhd G$; since $L \subseteq K$, we have $L \subseteq cor_G K$. Consequently, $p_i \mid |cor_G K|$ for each $i = 1, \dots, r$. But since $cor_G K$ is Hall in K, no p_i can divide $|K:cor_G K|$; hence $|K:cor_G K| = 1$ so that $K = cor_G K$ is normal in G.

The converse is immediate, since if G is Dedekind, each subgroup of G is equal to its core. ▯

<u>Theorem</u> <u>1.7</u>: For a finite group G, $scor_G H$ is nontrivial and is Hall in H for all proper nontrivial subgroups H of G if and only if all subgroups of G are subnormal in G, that is, G is nilpotent.

The proof of this result is entirely similar to that of Theorem 1.6 and amounts to not much more than replacing "normal" by "subnormal" and cor_G by $scor_G$.

CHAPTER TWO

THE INFLUENCE ON A GROUP OF THE COFACTORS AND SUBCOFACTORS OF ITS SUBGROUPS

2.1 Basic Results

There are two quite general results due to Baer [1] and a well-known theorem of Ore [15] of which we shall make rather frequent use. These are given in Theorem 2.1. To state these, however, we need two preliminary definitions. In what follows, we shall assume that the trivial group is always a θ-group.

Definition 2.1: For θ a group-theoretic property, the θ-commutator subgroup, $[G,\theta]$, of a given group G is defined by

$$[G,\theta] = \bigcap\{K \lhd G \mid G/K \text{ is a } \theta\text{-group}\}.$$

From this definition it is easily seen that for θ a homomorphism-invariant property, the θ-commutator subgroup $[G,\theta]$ is a characteristic subgroup of G. In particular, one can form the factor group $G/[G,\theta]$ and make the following definition.

Definition 2.2: A group-theoretic property θ is said to be strictly homomorphism-invariant if:

(a) θ is homomorphism-invariant; that is, homomorphic images of θ-groups are θ-groups;

(b) θ is subgroup-inherited; that is, subgroups of θ-groups are θ-groups;

(c) $G/[G,\theta]$ is a θ-group.

It is quite easy to show (see, for example, Baer [1]) that in the presence of conditions (a) and (b), condition (c) is equivalent to:

(c') Direct products of θ-groups are θ-groups.

As an example, if we take θ = abelian, we then have $[G,\theta] = G'$, the ordinary commutator (or derived) subgroup of G. Clearly, θ = abelian is a strictly homomorphism-invariant property as are θ = nilpotent, p-nilpotent, supersolvable, or solvable.

<u>Theorem</u> <u>2.1</u>: Let θ be a strictly homomorphism-invariant property and G a finite group.

(i) (Baer) If θ-groups are nilpotent, then $[G,\theta]$ is nilpotent if and only if $cof_G S$ is a θ-group for all maximal subgroups S of G.

(ii) (Baer) $[G,\theta]$ is nilpotent and $G/[G,\theta]$ is solvable if and only if $cof_G S$ is a θ-group for all maximal subgroups S of G and equicore maximal subgroups of G are conjugate in G.

(iii) (Ore) If G is solvable, then equicore maximal subgroups of G are conjugate in G.

An immediate corollary to this theorem is obtained by considering the Fitting subgroup of a group. This characteristic subgroup is defined as follows.

<u>Definition</u> <u>2.3</u>: The <u>Fitting</u> <u>subgroup</u>, F(G), of a finite group G is the largest normal nilpotent subgroup of G; that is, F(G) = the product of all normal nilpotent subgroups of G.

<u>Corollary</u> <u>2.2</u>: Let θ be a strictly homomorphism-invariant property and G a finite group.

(i) If θ-groups are nilpotent, then G/F(G) is a θ-group if and only if $cof_G S$ is a θ-group for all maximal subgroups S of G.

(ii) If G is solvable, then G/F(G) is a θ-group if and only if $cof_G S$ is a θ-group for all maximal subgroups S of G.

Proof: Statement (i) follows immediately from the definition of $[G,\theta]$, the normality of $[G,\theta]$ and $F(G)$, and Theorem 2.1. For $G/F(G)$ is a θ-group if-f $[G,\theta] \subseteq F(G)$ which is true if-f $[G,\theta]$ is nilpotent, and by the theorem, this in turn is true if-f $\mathrm{cof}_G S$ is a θ-group for all maximal subgroups S of G. Statement (ii) follows in the same manner. ‖

2.2 Nilpotent Cofactors or Subcofactors

In this section we seek to determine the influence on a group G of nilpotent subcofactors of subgroups of G, and to investigate what additional structure is forced upon G if one then also requires that the proper normal subgroups of G be nilpotent, and finally, that the nearly normal maximal subgroups be nilpotent. The classic results in this direction are due to Schmidt [18] and Iwasawa [14] (also proved by Huppert in [11], 5.1, 5.2) and are given in the following theorem.

Theorem 2.3: If all proper subgroups of a finite group G are nilpotent, but G itself is not nilpotent, then:

(i) $|G| = p^a q^b$ for distinct primes p and q.

(ii) G has a normal p-Sylow subgroup P.

(iii) P has class ≤ 2; in fact, $\Phi(P) \subseteq Z(G)$.

(iv) If p is odd, $\exp(P) = p$; if $p = 2$, $\exp(P) \leq 4$.

(v) Each q-Sylow subgroup Q of G is cyclic; also, $\Phi(Q) \subseteq Z(G)$.

In this same direction, Rose [17] has established the following result.

Theorem 2.4: If all the proper abnormal subgroups of the finite group G are nilpotent, then G has a normal Sylow subgroup P such that G/P is nilpotent.

Since we will be concerned chiefly with the subcofactors and cofactors of abnormal or self-normalizing subgroups, let us make the following definitions.

Definition 2.4: A subgroup H of a given group G is said to be self-normalizing in G if $N_G(H) = H$, that is, H is its own normalizer in G. H is said to be abnormal in G, denoted $H \bowtie G$, if $x \in \langle H , H^x \rangle$ for every $x \in G$, or equivalently, if every subgroup of G containing H is self-normalizing in G and H is not contained in two distinct conjugate subgroups of G.

The following remarks are immediate consequences of this definition and familiar results from Sylow theory:

(1) If $H \bowtie G$ and $H \subseteq K \leqq G$, then $K \bowtie G$.

(2) A maximal subgroup S of G is abnormal in G if-f S is self-normalizing in G if-f S is nonnormal in G.

(3) For every Sylow subgroup P of G, $N_G(P) \bowtie G$.

As a first result describing the structure forced upon a finite group by nilpotent cofactors or subcofactors, we take θ = nilpotent in Corollary 2.2-(i). Since nilpotence is clearly a strictly homomorphism-invariant property, we obtain the following result.

Theorem 2.5: For G a finite group, G/F(G) is nilpotent if and only if $cof_G S = S/cor_G S$ is nilpotent for all maximal subgroups S of G.

In slightly different terminology, this theorem states that:

$\mathfrak{f}(G) \leqq 2$ if-f $cof_G S$ is nilpotent for all maximal subgroups S of G,

where $\mathfrak{f}(G)$ denotes the Fitting length of G and is defined as follows.

Definition 2.5: The ascending Fitting series of the group G,

$$\langle 1 \rangle = F_0(G) \subseteq F_1(G) \subseteq \dots \subseteq F_i(G) \subseteq F_{i+1}(G) \subseteq \dots$$

is defined by $F_0(G) = \langle 1 \rangle$, and $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$, the Fitting subgroup of $G/F_i(G)$. The Fitting length of a finite solvable group G, $\mathcal{f}(G)$, is the least integer n for which $F_n(G) = G$.

One would hope to be able to say more if the cofactors of the maximal subgroups are all required to be nilpotent of the same class. Before stating the result, however, we first recall the definition of the class of a nilpotent group.

Definition 2.6: The descending central series of a group G,

$G = \gamma_0(G) \supseteq \gamma_1(G) \supseteq \dots \supseteq \gamma_i(G) \supseteq \gamma_{i+1}(G) \supseteq \dots$

is defined by $\gamma_0(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ = the subgroup generated by all commutators $[x,y] = x^{-1}y^{-1}xy$, where $x \in \gamma_i(G)$ and $y \in G$. The class of a finite nilpotent group G, cl(G), is the least integer n for which $\gamma_n(G) = \langle 1 \rangle$.

The following remarks are immediate consequences of this definition:

(1) If ϕ is a homomorphism of G onto $\bar{G}$, then $\gamma_i(\bar{G}) = \phi(\gamma_i(G))$ for each i.

(2) $\gamma_i(G/\gamma_i(G)) = \langle 1 \rangle$ for each i.

Lemma 2.6: Let Γ_n denote the property "nilpotent of class $\leqq$ n."

Then:

(i) For any group G, $[G,\Gamma_n] = \gamma_n(G)$, where $[G,\Gamma_n]$ is the Γ_n-commutator subgroup of G.

(ii) Γ_n is a strictly homomorphism-invariant property.

Proof: (i)

$$\begin{aligned}[G,\Gamma_n] &= \cap\{K \lhd G \mid G/K \text{ is a } \Gamma_n\text{-group}\}\\ &= \cap\{K \lhd G \mid \gamma_n(G/K) = \langle 1 \rangle\}\\ &= \cap\{K \lhd G \mid \gamma_n(G)K/K = \langle 1 \rangle\} \quad \text{(by remark (1) above)}\\ &= \cap\{K \lhd G \mid \gamma_n(G) \subseteq K\}\\ &= \gamma_n(G)\end{aligned}$$

(ii) Let G be any Γ_n-group, that is, $\gamma_n(G) = \langle 1 \rangle$, and let H be a subgroup of G. It is clear from the definition that $\gamma_i(H) \subseteq \gamma_i(G)$ for each i; thus, $\gamma_n(H) = \langle 1 \rangle$ also so that H is a Γ_n-group. Therefore, Γ_n is a subgroup-inherited property. From remark (1) above, it follows that Γ_n is homomorphism-invariant. And from remark (2) above and part (i), we have $\gamma_n(G/[G,\Gamma_n]) = \gamma_n(G/\gamma_n(G)) = \langle 1 \rangle$ so that $G/[G,\Gamma_n]$ is a Γ_n-group. Consequently, Γ_n is a strictly homomorphism-invariant property. ▯

<u>Theorem 2.7</u>: For a finite group G, the following are equivalent:

(i) $G/F(G)$ is nilpotent of class $\leq$ n.

(ii) $\mathrm{cof}_G S = S/\mathrm{cor}_G S$ is nilpotent of class $\leq$ n for all maximal subgroups S of G.

(iii) $\gamma_n(G)$ is nilpotent.

(iv) $\gamma_n(H) \lhd\lhd G$ for all subgroups H of G.

(v) $\gamma_n(H) \lhd\lhd G$ for all proper abnormal subgroups H of G.

<u>Proof</u>: The equivalence of (i) and (ii) is an immediate consequence of Lemma 2.6 and Corollary 2.2-(i); and (ii) → (iii) follows from Theorem 2.1-(i) and Lemma 2.6.

(iii) → (iv). If H is any subgroup of G, then $\gamma_n(H) \subseteq \gamma_n(G)$. Since $\gamma_n(G)$ is nilpotent, $\gamma_n(H)$ is subnormal in $\gamma_n(G)$ which is normal in G; hence, $\gamma_n(H) \lhd\lhd G$.

(iv) → (v) is trivially true.

(v) → (ii): Let S be any maximal subgroup of G. If $S \lhd G$, then $\mathrm{cof}_G S = \langle 1 \rangle$ obviously has class $\leq$ n. Thus suppose $S \ntriangleleft G$, and hence $S \rtimes\!\!\lhd G$. Then, since $\gamma_n(S) \lhd\lhd G$, we have, using Lemma 1.4, that $\gamma_n(S) \subseteq \mathrm{scor}_G S = \mathrm{cor}_G S$. Now, by the remark (2) above, $\gamma_n(S/\gamma_n(S)) = \langle 1 \rangle$ so that $S/\gamma_n(S)$ is a Γ_n-group, and hence, by Lemma 2.6-(ii), so also is its homomorphic image $S/cor_G S$. Therefore, $cof_G S = S/cor_G S$ is nilpotent of class $\leq n$. ▯

We obtain a similar result using the property Δ_n: solvable of derived length $\leq n$. Here we mean, as usual, by the derived length of a finite solvable group G, the least integer n for which the term $G^{(n)}$ of the derived series of G is trivial; the derived series of G,

$$G = G^{(0)} \supseteq G^{(1)} \supseteq \dots \supseteq G^{(i)} \supseteq G^{(i+1)} \supseteq \dots$$

is defined by $G^{(0)} = G$ and $G^{(i+1)} = (G^{(i)})'$, the derived subgroup of $G^{(i)}$.

As before, the following remarks are immediate consequences:

(1) If ϕ is a homomorphism of G onto $\bar{G}$, then $(\bar{G})^{(i)} = \phi(G^{(i)})$ for each i.

(2) $(G/G^{(i)})^{(i)} = \langle 1 \rangle$ for each i.

Lemma 2.8: Let Δ_n denote the property "solvable of derived length $\leq n$." Then:

(i) For any group G, $[G,\Delta_n] = G^{(n)}$, where $[G,\Delta_n]$ is the Δ_n-commutator subgroup of G.

(ii) Δ_n is a strictly homomorphism-invariant property.

Proof: (i)

$$\begin{aligned}[G,\Delta_n] &= \cap\{K \lhd G \mid G/K \text{ is a } \Delta_n\text{-group}\}\\ &= \cap\{K \lhd G \mid (G/K)^{(n)} = \langle 1 \rangle\}\\ &= \cap\{K \lhd G \mid G^{(n)}K/K = \langle 1 \rangle\} \quad \text{(by remark (1) above)}\\ &= \cap\{K \lhd G \mid G^{(n)} \subseteq K\}\\ &= G^{(n)}\end{aligned}$$

(ii) Clearly Δ_n is subgroup-inherited; and by remark (1) above, it is homomorphism-invariant. From part (i) and remark (2) above, we have $(G/[G,\Delta_n])^{(n)} = (G/G^{(n)})^{(n)} = \langle 1 \rangle$, so that $G/[G,\Delta_n]$ is a Δ_n-group. Thus, Δ_n is a strictly homomorphism-invariant property. ▯

<u>Theorem 2.9</u>: Let G be a finite solvable group. Then the following are equivalent:

(i) G/F(G) has derived length $\leq$ n.

(ii) $\text{cof}_G S = S/\text{cor}_G S$ has derived length $\leq$ n for all maximal subgroups S of G.

(iii) $G^{(n)}$ is nilpotent.

(iv) $H^{(n)} \lhd\lhd G$ for all subgroups H of G.

(v) $H^{(n)} \lhd\lhd G$ for all proper abnormal subgroups H of G.

<u>Proof</u>: (i) $\leftrightarrow$ (ii) $\rightarrow$ (iii) is immediate from Lemma 2.8 and Corollary 2.2. (iii) $\rightarrow$ (iv) $\rightarrow$ (v) $\rightarrow$ (ii) follows as in the proof of Theorem 2.7, replacing $\gamma_n(\ )$ by $(\ )^{(n)}$ and "nilpotent of class $\leq$ n" by "solvable of derived length $\leq$ n." ‖

We have seen that if the cofactors of all the maximal subgroups of a finite group G are nilpotent, then G is solvable and of Fitting length at most 2; and in the preceding results, we have seen the effect on G of requiring these cofactors to all be nilpotent of class at most n. These conditions are not sufficient, however, to guarantee that G has a normal Sylow subgroup; in particular, the conclusions of Theorem 2.4 need not hold. In fact, it is not even sufficient to require that the cofactors of all proper subgroups of G be abelian, as the following example shows.

<u>Example 2.10</u>: Let $G = S_3 \times A_4$, where S_3 is the symmetric group on 3 letters and A_4 is the alternating group of degree 4. Then:

(i) G has no normal Sylow subgroups.

(ii) $H/\text{cor}_G H$ is abelian for all proper subgroups H of G.

<u>Proof</u>: (i) is immediate since S_3 has no normal 2-Sylow subgroup and A_4 has no normal 3-Sylow subgroup.

(ii) Let $\bar{S}_3 = S_3 \times \langle 1 \rangle$, $\bar{A}_4 = \langle 1 \rangle \times A_4$, $\bar{U} = U \times \langle 1 \rangle$ = the normal 3-Sylow subgroup of $\bar{S}_3$, and $\bar{V} = \langle 1 \rangle \times V$ = the normal 2-Sylow subgroup of $\bar{A}_4$. We note first that $G/F(G) = G/\bar{U}\bar{V}$ is isomorphic to the direct product of a cyclic group of order 2 with a cyclic group of order 3, hence is abelian, so that from Corollary 2.2-(i), $T/\mathrm{cor}_G T$ is abelian for all maximal subgroups T of G.

Now let H be a proper subgroup of G; we wish to show that $H/\mathrm{cor}_G H$ is abelian. Since this is trivially true if $H \triangleleft G$, we may assume that $H \ntriangleleft G$. Also, in view of the comment above and the fact that groups of order p or p^2 (p a prime) are abelian, the only cases that need be checked are for $|H| = 6, 8, 12$, or 18.

Case 1: $|H| = 6$, $H \ntriangleleft G$.—Let $x = (a,b) \in H$ have order 2, and $z = (p,q) \in H$ have order 3. Then at least one, but not both, of b and q is 1. For if $b \neq 1$ and $q \neq 1$, then $\langle b, q \rangle = A_4$, which implies that $|H| \geq 12$, a contradiction. And if $b = 1 = q$, then $\langle x, z \rangle = \bar{S}_3 \subseteq H$ so that $H = \bar{S}_3 \triangleleft G$, also a contradiction.

Suppose $q = 1$. Then $\bar{U} = \langle z \rangle \subseteq H$, thus $\bar{U} \subseteq \mathrm{cor}_G H$, and hence $H/\mathrm{cor}_G H$ is cyclic of order 2.

Suppose $q \neq 1$, thus $b = 1$ and $a \neq 1$. Then ap has order 2; consequently, (ap,q) has order 6. And since $(ap,q) \in H$, this means that H is cyclic of order 6 so that $H/\mathrm{cor}_G H$ also is cyclic.

Case 2: $|H| = 8$, $H \ntriangleleft G$.—In this case, the normal subgroup $\bar{V}$ of G is contained in H, hence is contained in $\mathrm{cor}_G H$. Therefore, $H/\mathrm{cor}_G H$ is cyclic of order 2.

Case 3: $|H| = 12$, $H \ntriangleleft G$.—Since A_4 has no subgroups of order 6 and $|H \cap \bar{A}_4| = |H||\bar{A}_4|/|H\bar{A}_4| = 144/|H\bar{A}_4|$, it follows that $|H \cap \bar{A}_4| = 2$ or 4. In the latter case where $|H \cap \bar{A}_4| = 4$, the normal subgroup $\bar{V}$

of G is contained in H, hence in $cor_G H$, so that $H/cor_G H$ is cyclic of order 3.

Thus, suppose that $|H \cap \bar{A}_4| = 2$, and let $x = (1,b) \in H \cap \bar{A}_4$ have order 2. Then $H \cap \bar{S}_3 \neq <1>$; for otherwise, $H \cong H\bar{S}_3/\bar{S}_3 = G/\bar{S}_3 \cong A_4$, from which it follows that H has no normal subgroup of order 2; however, $H \cap \bar{A}_4$ is normal in H and of order 2.

Now, suppose first that $2 \mid |H \cap \bar{S}_3|$; let $y = (a,1) \in H \cap \bar{S}_3$ have order 2, and $z = (p,q) \in H$ have order 3. Then ap has order 2, and $(1,q^2) = (ap,q)^2 \in H$. It follows then that $q = 1$, since otherwise we would have $< (1,q^2) , (1,b) > = \bar{A}_4 \subseteq H$, which implies that $H = \bar{A}_4$ is normal in G. Thus, $< (a,1) , (p,1) > = \bar{S}_3$ is contained in H, hence in $cor_G H$, so that $H/cor_G H$ is cyclic of order 2.

Suppose now that $2 \nmid |H \cap \bar{S}_3|$, and thus $3 \mid |H \cap \bar{S}_3|$. Then the normal subgroup $\bar{U}$ of G is contained in H, hence in $cor_G H$, so that $H/cor_G H$ has order 4 and is therefore abelian.

Case 4: $|H| = 18$, $H \ntrianglelefteq G$.—In this case, the normal subgroup $\bar{U}$ of G must be contained in H, hence in $cor_G H$, so that $|H/cor_G H| \leq 6$. We may assume that $|H/cor_G H| = 6$ (since in the other cases $H/cor_G H$ is clearly cyclic), and thus that $\bar{U} = cor_G H$. Then $G/cor_G H = G/\bar{U}$ is isomorphic to $C_2 \times A_4$ where C_2 is a cyclic group of order two. It is easily checked that this group has no proper subgroup isomorphic to S_3; consequently, $H/cor_G H$ cannot be isomorphic to S_3 and is, therefore, a cyclic group of order 6. ▯

If we now require that in addition to the cofactors of all maximal subgroups of G being nilpotent, the proper normal subgroups of G also be nilpotent, we would certainly hope to be able to say more about the structure of G. Because of Example 2.13, we cannot hope to recover all the results of the theorems of Schmidt, Iwasawa, and Rose (Theorems 2.3 and 2.4). Nevertheless, we do find that the structure of G is quite severely restricted, and that if G is not itself nilpotent, then, in comparison with Theorem 2.3, it is within a prime of being nilpotent; more precisely, G/F(G) is of prime order. To state the complete result, the following definitions are needed.

Definition 2.7: A finite group G is said to be p-nilpotent if it has a normal p-complement; that is, there exists $K \lhd G$ such that $p \nmid |K|$ and $|G:K|$ is a power of p.

It is quite easy to show as a consequence of this definition that G is nilpotent if and only if G is p-nilpotent for all primes p which divide $|G|$.

Definition 2.8: A subgroup H of a given finite group G will be said to be nearly normal in G if $\mathrm{cof}_G H = H/\mathrm{cor}_G H$ is trivial or of prime order. H will be said to be nearly subnormal in G if $\mathrm{scof}_G H = H/\mathrm{scor}_G H$ is trivial or of prime order.

Theorem 2.11: Let $|G| = \prod_{i=1}^{r} p_i^{a_i}$ where the p_i are distinct primes dividing $|G|$. Suppose that $\mathrm{cof}_G S = S/\mathrm{cor}_G S$ is nilpotent for all maximal subgroups S of G and that all proper normal subgroups of G are nilpotent, but that G itself is not nilpotent. Then the following hold.

(a) G is solvable.

(b) F(G) is the unique maximal normal subgroup of G.

(c) There exists a prime, say p_r, dividing $|G|$ for which the following hold.

(1) G is p_r-nilpotent.

(2) For all $i \neq r$, G is non-p_i-nilpotent.

(3) $|G:F(G)| = p_r$.

(4) For any proper subgroup H of G, $scof_G H = H/scor_G H$ has order 1 or p_r; in particular, all proper subgroups of G are nearly subnormal in G, and all maximal subgroups are nearly normal in G.

(5) For each $i \neq r$, G has a normal p_i-Sylow subgroup P_i.

(6) For each $i \neq r$, $P_i \subseteq G'$; thus, G/G' is a p_r-group.

(7) p_r divides $d_{p_i} = \prod_{j=1}^{a_i} (p_i^{\,j} - 1)$ for all $i \neq r$.

(8) For each p_r-Sylow subgroup Q of G, $Q^G = G$, that is, Q has G as its normal closure.

(9) For each $i \neq r$ such that P_i is abelian,

(i) G has exactly $p_i^{a_i}$ distinct p_i-complements,

(ii) $C_G(P_i) = F(G)$, and thus G induces in P_i a cyclic group of automorphisms of order p_r,

(iii) the number of p_r-Sylow subgroups of G is a multiple of $p_i^{a_i}$.

(10) If P_i is abelian for all $i \neq r$, then:

(i) G has exactly $|G|/p_r^{a_r}$ distinct p_r-Sylow subgroups, each of which is abnormal in G.

(ii) The set of p_r-Sylow subgroups of G = the set of system normalizers of G = the set of Carter subgroups of G.

(iii) If, in addition, $a_r = 1$, then $G = X \cup X'$ where X is the set of p_r-elements of G and X' is the set of p_r'-elements; thus, G is a Frobenius group with kernel = $F(G) = G'$ = the normal p_r-complement of G; also, $Z(G) = \langle 1 \rangle$.

Proof: (a) follows from Theorem 2.5; and (b) is immediate since every proper normal subgroup of G is, by hypothesis, nilpotent and hence is contained in F(G).

(c) (3) Since G is solvable and F(G) is a maximal normal subgroup of G, G/F(G) is of prime order. Relabelling if necessary, we may assume that $|G:F(G)| = p_r$.

(1) We have $|G:F(G)| = p_r$, and F(G) is nilpotent, hence p_r-nilpotent. Let T be the normal p_r-complement of F(G). Then T is characteristic in the normal subgroup F(G) of G so that $T \lhd G$; also, $|G:T| = |G:F(G)||F(G):T| = p_r^{a_r}$. Thus T is a normal p_r-complement of G.

(4) Let H be any proper subgroup of G. If $H \subseteq F(G)$, then by the nilpotence of F(G), $H \lhd\lhd F(G) \lhd G$, so that $H \lhd\lhd G$. Thus $\mathrm{scor}_G H = H$ and $|H/\mathrm{scor}_G H| = 1$. Now suppose $H \not\subseteq F(G) = F$. Then by the maximality of F, we have HF = G so that $|H \cap F| = \frac{|H||F|}{|HF|} = \frac{|H|}{|G|} \cdot \frac{|G|}{p_r} = \frac{|H|}{p_r}$. From the nilpotence of F, we have $H \cap F \lhd\lhd F \lhd G$, that is, $H \cap F \lhd\lhd G$; it follows that $H \cap F \subseteq \mathrm{scor}_G H$. Hence, since $|H:H \cap F| = p_r$, $H/\mathrm{scor}_G H$ has order 1 or p_r. In either case, therefore, $|\mathrm{scof}_G H| = 1$ or p_r as we wished to show. If S is a maximal subgroup of G, then by Lemma 1.3, $\mathrm{scor}_G S = \mathrm{cor}_G S$. By what we have just shown, $S/\mathrm{cor}_G S = S/\mathrm{scor}_G S$ has order 1 or p_r so that S is nearly normal in G.

(5) If $i \neq r$ and P_i is a p_i-Sylow subgroup of F(G), then by the nilpotence of F(G), we have P_i char $\leqq F(G) \lhd G$, hence $P_i \lhd G$. And since $|G:F(G)| = p_r$ is prime to p_i, P_i is a normal p_i-Sylow subgroup of G.

(2) Suppose that for some $i \neq r$, G is p_i-nilpotent. Then there exists a normal subgroup T_i of G with $|G:T_i| = p_i^{a_i}$. By hypothesis, T_i is nilpotent, and thus has a characteristic p_r-Sylow

subgroup Q. Then $Q \lhd G$; and since $|G:T| = p_i^{a_i}$ is prime to p_r, Q is a normal p_r-Sylow subgroup of G. However, this together with (5) implies that all the Sylow subgroups of G are normal in G, contradicting the nonnilpotence of G. Therefore, G is non-p_i-nilpotent for each $i \neq r$.

(6) Let $G'_{(p_i)}$ denote the smallest normal subgroup of G for which the factor group is an abelian p_i-group. We show first that for $i \neq r$, $G'_{(p_i)} = G$. For suppose that $G'_{(p_i)} \neq G$ for some $i \neq r$. Then $G'_{(p_i)}$ is a proper normal subgroup of G, hence is nilpotent by hypothesis, and in particular, is p_i-nilpotent. This means that there exists a characteristic subgroup T_i of $G'_{(p_i)}$ with T_i a p_i'-group and $|G'_{(p_i)}:T_i|$ = a power of p_i. Then $T_i \lhd G$ and $|G:T_i| = |G:G'_{(p_i)}||G'_{(p_i)}:T_i|$ is a power of p_i so that G is p_i-nilpotent, in contradiction to (2).

Thus, $G'_{(p_i)} = G$ for each $i \neq r$. From one of the basic transfer theorems (see, for example, Scott [19], 13.5.2), it follows that $\langle 1 \rangle = G/G'_{(p_i)} \cong P_i/P_i \cap G'$. For each $i \neq r$, therefore, $P_i \cap G' = P_i$, that is, $P_i \subseteq G'$.

(7) Case 1: For some $i \neq r$, the normal p_i-Sylow subgroup P_i of G is not minimal normal in G.—Let $M \lhd G$ with $\langle 1 \rangle \neq M \lneqq P_i$; we show that G/M is not p_i-nilpotent. For suppose that it is; then there exists $T \lhd G$ with $|G:T| = |G/M:T/M| = p_i^{a_i - m_i}$ where $|M| = p_i^{m_i}$. Since M is properly contained in P_i, p_i divides $|G/M|$ so that T is a proper normal subgroup of G, hence is nilpotent by hypothesis; in particular, T is p_i-nilpotent. There exists, therefore, a characteristic subgroup U of T with $|T:U| = p_i^{m_i}$. Then $U \lhd G$ and $|G:U| = |G:T||T:U| = p_i^{a_i}$, which means that U is a normal p_i-complement of G. This, however, contradicts (2).

Thus, G/M is not nilpotent. And since all proper normal subgroups of G are nilpotent, the proper normal subgroups of G/M are all nilpotent. Also, the cofactors of the maximal subgroups of G/M are nilpotent. For if S/M is a maximal subgroup of G/M, then S is a maximal subgroup of G. By hypothesis, $\mathrm{cof}_G S$ is nilpotent so that, by Lemma 1.4, $\mathrm{cof}_{G/M}(S/M) \cong \mathrm{cof}_G S$ also is nilpotent.

The hypotheses, therefore, hold for G/M; consequently, since each p_i divides $|G/M|$, we have by induction that for each $i \neq r$, p_r divides $d^*_{p_i} = \prod_{j=1}^{b_i} (p_i^{\,j} - 1)$ where $|G/M| = p_1^{b_1} \cdots p_r^{b_r}$. Hence, p_r also divides $d_{p_i} = \prod_{j=1}^{a_i} (p_i^{\,j} - 1)$ for each $i \neq r$.

Case 2. $a_r > 1$.—Let Q be a p_r-Sylow subgroup of G. From (4) we have $|Q/\mathrm{cor}_G Q| = p_r$, and thus, since $a_r > 1$, $\mathrm{cor}_G Q \neq \langle 1 \rangle$. Now suppose that $G/\mathrm{cor}_G Q$ is p_i-nilpotent for some $i \neq r$. Then there exists $T_i \lhd G$ with $|G/\mathrm{cor}_G Q : T_i/\mathrm{cor}_G Q| = p_i^{a_i}$ so that $|G{:}T| = p_i^{a_i}$, which implies that G is p_i-nilpotent. This, however, contradicts (2).

Therefore, $G/\mathrm{cor}_G Q$ is not nilpotent; and as in Case 1, all proper normal subgroups of $G/\mathrm{cor}_G Q$ and all cofactors of maximal subgroups of $G/\mathrm{cor}_G Q$ are nilpotent. Hence, by induction, since all the p_i divide $|G/\mathrm{cor}_G Q|$, we have the result as in Case 1.

Case 3: $a_r = 1$ and for all $i \neq r$ the p_i-Sylow subgroup P_i of G is minimal normal in G.—In this case, each P_i for $i \neq r$ is elementary abelian; thus $|\mathrm{Aut}(P_i)| = p_i^{e_i} . d_{p_i}$ where $e_i = a_i(a_i - 1)/2$ and $d_{p_i} = \prod_{j=1}^{a_i} (p_i^{\,j} - 1)$. Now from (9), which will be proved independently of (7), we have $C_G(P_i) = F(G)$ for all $i \neq r$, so that $|G{:}C_G(P_i)| = p_r$ for $i \neq r$. Since $G/C_G(P_i)$ is isomorphic to a subgroup of $\mathrm{Aut}(P_i)$, this means that p_r divides $|\mathrm{Aut}(P_i)|$, and hence divides d_{p_i} for all

$i \neq r$.

(8) Suppose that $Q^G \lneq G$ for some p_r-Sylow subgroup Q of G. Then Q^G is a proper normal subgroup of G, hence is nilpotent so that Q is characteristic in Q^G. But this implies, since $Q^G \lhd G$, that Q is normal in G, which together with (5) means that all the Sylow subgroups of G are normal in G, contradicting the nonnilpotence of G. Therefore, $Q^G = G$ for each p_r-Sylow subgroup Q of G.

(9) (i) This is an immediate consequence of Theorem 2.25: Given a solvable non-p-nilpotent group G having all proper normal subgroups p-nilpotent, and with $|G| = p^a m$ where $(p,m) = 1$. If G has an abelian normal p-Sylow subgroup, then G has exactly p^a distinct p-complements.

(ii) Let P_i be abelian for some $i \neq r$. Since $P_i \subseteq F(G)$ and F(G) is nilpotent, it follows that F(G) centralizes P_i. Thus, $F(G) \subseteq C_G(P_i)$; from the maximality of F(G), we have $C_G(P_i) = F(G)$ or G. Now, $P_i \not\subseteq Z(G)$; otherwise, P_i would centralize and hence normalize a p_i-complement T_i of G, which would imply that T_i is normal in G, contradicting (2). Therefore, $C_G(P_i) \neq G$ so that $C_G(P_i) = F(G)$.

(iii) Let P_i be abelian for some $i \neq r$; then, from (i), G has exactly $p_i^{a_i}$ distinct p_i-complements. We show first that each p_r-Sylow subgroup Q of G is contained in some p_i-complement. For this, we have F(G) nilpotent, hence p-nilpotent, and thus has a unique p_i-complement W which is normal in G. Since $|G:F(G)| = p_r$ and $|F(G):W| = p_i^{a_i}$, it follows that $|W| = p_r^{a_r - 1} \prod_{j \neq i,r} p_j^{a_j}$.

Now, since $W \lhd G$, $Q \cap W$ is a p_r-Sylow subgroup of W so that $Q \cap W$ has order $p_r^{a_r - 1}$. Therefore

$$|QW| = \frac{|Q|\,|W|}{|Q \cap W|} = \frac{p_r^{a_r}\,|W|}{p_r^{a_r - 1}} = p_r|W| = \prod_{j \neq i} p_j^{a_j}$$

which shows that QW is a p_i-complement of G.

We note next that $p_r^{a_r}$ divides the order of each p_i-complement of G for $i \neq r$. Consequently, each p_i-complement of G, for $i \neq r$, contains a p_r-Sylow subgroup of G.

Also, no two distinct p_i-complements $X \neq Y$ of G can contain the same p_r-Sylow subgroup of G. For since F(G) is normal in G, $X \cap F(G)$ and $Y \cap F(G)$ are p_i-complements of F(G); thus, $X \cap F(G) = Y \cap F(G) = W$, the unique p_i-complement of F(G). Now,

$$|X:W| = \frac{|G|/p_i^{a_i}}{|F(G)|/p_i^{a_i}} = |G:F(G)| = p_r$$

and similarly, $|Y:W| = p_r$. Therefore, since $W \subseteq X \cap Y$ and $X \cap Y \neq X, Y$, we must have $X \cap Y = W$. Since $p_r^{a_r}$ does not divide $|W| = |X \cap Y|$, it follows that $X \cap Y$ contains no p_r-Sylow subgroup of G so that X and Y can have no p_r-Sylow subgroup of G in common.

Finally, since G is solvable, any two distinct p_i-complements X and Y of G are conjugate. Consequently, X and Y must contain the same number, say λ, of p_r-Sylow subgroups of G.

We therefore have the following: each p_r-Sylow subgroup of G is contained in some p_i-complement of G; no two distinct p_i-complements have any p_r-Sylow subgroup of G in common; each p_i-complement of G contains λ p_r-Sylow subgroups of G; G has exactly $p_i^{a_i}$ distinct p_i-complements. From these it follows that G has exactly $\lambda p_i^{a_i}$ distinct p_r-Sylow subgroups.

(10) (i) Let n = the number of distinct p_r-Sylow subgroups of G. Since P_i is abelian for all $i \neq r$, from (9)-(iii) we have that $p_i^{a_i} \mid n$ for all $i \neq r$, so that $n = p_r^{b_r} \prod_{i \neq r} p_i^{a_i}$ for some $b_r \geq 0$. However, from the Sylow theorems, $n \equiv 1 \pmod{p_r}$. This implies that

$b_r = 0$, and hence $n = \prod_{i \neq r} p_i^{a_i} = |G|/p_r^{a_r}$.

Now, if Q is any p_r-Sylow subgroup of G, by what we have just shown, $|G:Q|$ = the number of distinct p_r-Sylow subgroups of G, which in turn is equal to $|G:N_G(Q)|$. Hence, $Q = N_G(Q)$ so that Q is abnormal in G, since the normalizers of Sylow subgroups are abnormal subgroups.

(ii) Since F(G) is nilpotent as in G/F(G), it follows by a result of Carter [3] that the Carter subgroups of G are identical with the system normalizers in G.

Now, G has a unique p_r-complement since it is p_r-nilpotent; and for each $i \neq r$, G has $p_i^{a_i}$ distinct p_i-complements. Thus G has $|G|/p_r^{a_r}$ distinct Sylow (complement) systems, and hence has at most $|G|/p_r^{a_r}$ distinct system normalizers. But each p_r-Sylow subgroup is a nilpotent self-normalizing subgroup of G, that is, a Carter subgroup of G, and by the preceding comments, is, therefore, a system normalizer of G. It now follows that since G has exactly $|G|/p_r^{a_r}$ distinct p_r-Sylow subgroups, these must be all the system normalizers of G.

(iii) If $a_r = 1$, then by (i), G has exactly $|G|/p_r$ distinct p_r-Sylow subgroups. Since each of these has order p_r and every p_r-element of G belongs to some p_r-Sylow subgroup, the number n_r of non-identity p_r-elements of G is given by $n_r = \frac{|G|}{p_r}(p_r - 1) = |G| - \frac{|G|}{p_r}$. Since G is p_r-nilpotent, the number n_r' of p_r'-elements different from 1 is $n_r' = \frac{|G|}{p_r} - 1$. Since $n_r + n_r' = |G| - 1$, the first conclusion of (iii) now follows.

From (i) we have $Q = N_G(Q)$ for each p_r-Sylow subgroup Q of G. Since $|Q| = p_r$, this implies that $Q \cap Q^x = \langle 1 \rangle$ for all $x \in G - Q$, and hence that G is a Frobenius group with kernel = the set of p_r'-elements of G, which by (3) is equal to F(G). Also, from (6), G/G' is a p_r-group so that $|G/G'| = 1$ or p_r. Since G is solvable, $G' \neq G$, hence

$|G/G'| = p_r$. Therefore, since $G' \subseteq F(G)$ and $|G:F(G)| = p_r$, we have $G' = F(G)$.

Finally, it follows from a well-known property of solvable groups having all Sylow subgroups abelian (Taunt [20], also proved by Huppert in [11], 14.3) that $G' \cap Z(G) = \langle 1 \rangle$; thus, $|Z(G)| = 1$ or p_r since $|G/G'| = p_r$. But since the p_r-Sylow subgroups of G are non-normal, we have $|Z(G)| \neq p_r$ so that $Z(G) = \langle 1 \rangle$. ▯

Statement (c)-(4) of the preceding theorem shows that if we extend the requirement of nilpotence of normal subgroups to the nearly normal maximal subgroups of G, we then have all the maximal subgroups of G being nilpotent so that Theorem 2.3 holds. This gives the following corollary to Theorem 2.11.

Corollary 2.12: Suppose that the cofactors of all maximal subgroups of G are nilpotent as are the nearly normal maximal subgroups of G, but that G itself is not nilpotent, say G not p-nilpotent. Then the following hold.

(i) All proper subgroups of G are nilpotent.

(ii) $|G| = p^a q^b$ for some prime $q \neq p$, and the conclusions of Theorem 2.3 hold.

(iii) The conclusions of Theorem 2.11 hold with $r = 2$, $p_1 = p$, $p_2 = q$, $a_1 = a$, and $a_2 = b$.

The condition imposed in Theorem 2.11 that a nonnilpotent group G have the cofactors of all its maximal subgroups and all its normal subgroups nilpotent does not impose any bounds on $|\pi(G)|$ = the number of distinct prime factors of $|G|$. Neither does it guarantee that G has a normal Sylow subgroup for which the factor group is nilpotent. In fact, it is not even sufficient to require that the cofactors of all proper subgroups and all normal subgroups of G be cyclic. More precisely, we have the following example.

<u>Example 2.13</u>: For all $n \geq 3$, there exists a finite nonnilpotent group G such that:

(i) the cofactors of all proper subgroups of G and all proper normal subgroups of G are cyclic;

(ii) $|G|$ is divisible by n distinct primes;

(iii) G has no normal Sylow subgroup Q for which G/Q is nilpotent.

<u>Proof</u>: Let $\Sigma = \{p_1, \ldots, p_{n-1}\}$ be any collection of n-1 distinct odd primes and $P_i = \langle x_i \rangle$ be a cyclic group of order p_i for each i. Then each P_i has an automorphism α_i of order 2. Let $K = P_1 \times \ldots \times P_{n-1}$; $\alpha = \alpha_1 \times \ldots \times \alpha_{n-1}$ is an automorphism of K of order 2. Now let G be the extension of K by α. G is not nilpotent; for if it were, then $\langle\alpha\rangle$ would be a normal subgroup of G, hence would centralize K, which contradicts the fact that $|\alpha| = 2$. We show now that G satisfies the three conditions. Since (ii) is clear, only (i) and (iii) require proof.

(i) Let H be any proper subgroup of G. Suppose first that $2 \nmid |H|$, say $|H| = p_{i_1} \cdots p_{i_k}$ where each $p_{i_j} \in \Sigma$. In this case, we must have $H \cong P_{i_1} \times \ldots \times P_{i_k}$ which is cyclic.

Suppose now that $2 \mid |H|$. Then since $H \neq G$, some $p_t \nmid |H|$ so that $P_t \cap H = \langle 1 \rangle$. Now, H is not normal in G; for suppose that it is. Then since $2 \mid |H|$, the 2-Sylow subgroups of G are contained in H; in particular, $\alpha \in H$. Since H is normal in G, we have $x_t^{-1}\alpha x_t \in H$, and hence, $[x_t, \alpha] = x_t^{-1}\alpha x_t \alpha \in H$. But since $P_t = \langle x_t \rangle$ is a characteristic subgroup of G, we have $[x_t, \alpha] \in P_t$. Thus, $[x_t, \alpha] \in P_t \cap H = \langle 1 \rangle$, which implies that $(x_t)^{\alpha} = x_t$. This, however, contradicts the fact that $\alpha_t = \alpha|_{P_t}$ has order 2. Therefore, $H \ntriangleleft G$; and it is clear that

$cor_G H$ is the product of the P_i for p_i dividing $|H|$. Consequently, $H/cor_G H$ is cyclic of order 2.

(iii) The P_i are the normal Sylow subgroups of G. If for some i, G/P_i is nilpotent, then TP_i is a proper normal subgroup of G for T a 2-Sylow subgroup of G. However, this contradicts the proof in (i) that if H is a proper subgroup of G and $2 \mid |H|$, then H is not normal in G. Therefore, G has no normal Sylow subgroup for which the factor group is nilpotent. ▯

2.3 p-nilpotent Cofactors or Subcofactors

We now turn to a consideration of those finite groups G for which the cofactors or subcofactors of certain proper subgroups of G are p-nilpotent, that is, they possess a normal p-complement. As in the preceding section, we will later require that the proper normal subgroups of G also be p-nilpotent in order to further delimit the structure of G, and finally, that the somewhat normal subgroups of G also be p-nilpotent. One of the major results that we seek to extend is the classic theorem due to Ito [13] (also proved by Huppert in [11], 5.4).

<u>Theorem</u> <u>2.14</u>: If all the proper subgroups of a finite group G are p-nilpotent, but G itself is not, then

(i) all proper subgroups of G are nilpotent.

Thus, the conclusions of Theorem 2.3 hold; that is,

(ii) $|G| = p^a q^b$ for some prime $q \neq p$;

(iii) G has a normal p-Sylow subgroup P; P has class ≤ 2, and in fact, $\Phi(P) \subseteq Z(G)$; if p is odd, $\exp(P) = p$, and if $p = 2$, $\exp(P) \leq 4$;

(iv) the q-Sylow subgroups of G are cyclic; and if Q is any such, $\Phi(Q) \subseteq Z(G)$.

Rose [17] has also established some results in this direction. The following are two such theorems.

Theorem 2.15: If every proper self-normalizing subgroup of G is p-nilpotent, then G has a normal p-subgroup P_0 (which may be trivial) such that G/P_0 is p-nilpotent.

Theorem 2.16: If every proper abnormal subgroup of G is p-nilpotent and either p is odd or the p-Sylow subgroups of G are abelian, then the conclusion of Theorem 2.15 holds.

That the added conditions on p in the preceding theorem cannot be omitted is shown by the following example of Rose, the details of which appear in [17].

Example 2.17: Let H be the simple group of order 168 ($H = PGL(3,2) = GL(3,2) = PSL(3,2)$), and G the split extension of H by the automorphism α of H defined by $\alpha: x \to (x^{-1})^t$, where y^t denotes the transpose of the matrix y. Then every proper abnormal subgroup of G is supersolvable, hence 2-nilpotent, but G is not solvable, hence not 2-solvable.

As we now show, the two results of Rose (Theorems 2.15 and 2.16) can be extended by requiring not that the self-normalizing or abnormal subgroups themselves be p-nilpotent, but only their subcofactors. To establish this, we will use the following well-known results due to Burnside [2] and the following lemma.

Theorem 2.18: (1) If the finite group G is not p-nilpotent, then G has a nontrivial p-subgroup P_0 and a p'-element x such that $x \in N_G(P_0)$ but $x \notin C_G(P_0)$.

(2) If G is a finite group with P a p-Sylow subgroup, and if $P \subseteq Z(N_G(P))$, then G is p-nilpotent.

Lemma 2.19: Let θ be a group-theoretic property such that products of normal θ-subgroups of a group are again θ-groups. If the finite group G has a nontrivial subnormal θ-subgroup, then G has a nontrivial normal θ-subgroup.

Proof: Let $K \neq \langle 1 \rangle$ be a θ-subgroup of G, and

$$K = K_0 \ntrianglelefteq K_1 \ntrianglelefteq K_2 \ntrianglelefteq \dots \ntrianglelefteq K_{r-1} \ntrianglelefteq K_r = G$$

where r is the minimal length of subnormal chains from K to G. We use induction on r. The result is trivially true if $r = 1$; thus, suppose $r > 1$. By induction, K_{r-1} has a nontrivial normal θ-subgroup. Let K^* be the product of all such. Then K^* is a nontrivial θ-group, is clearly characteristic in K_{r-1}, and hence is normal in G. ▯

Definition 2.9: A finite group G is said to be p-solvable if it has a normal series $\langle 1 \rangle = K_0 \subseteq K_1 \subseteq \dots \subseteq K_n = G$ in which each factor K_i/K_{i-1} is either a p-group or a p'-group. For a p-solvable group G, the ascending p-series

$$\langle 1 \rangle = P_0 \subseteq N_0 \subset P_1 \subset N_1 \subset P_2 \subset \dots \subset P_t \subseteq N_t = G$$

is defined by taking N_i/P_i to be the largest normal p'-subgroup of G/P_i, and P_{i+1}/N_i the largest normal p-subgroup of G/N_i. The p-length of G, $\ell_p(G)$, is the least integer t such that $N_t = G$.

The following are well-known consequences of this definition:

(1) G p-nilpotent → G is p-solvable.

(2) G is solvable if-f G is p-solvable for all primes p which divide $|G|$.

(3) $\ell_p(G)$ is the smallest number of p-factors that can occur in a normal series of G for which the factor groups are either p-groups or p'-groups.

Definition 2.10: For a subgroup H of a finite group G, the hypernormalizer of H in G, denoted $N_G^{\infty}(H)$, is defined to be the subgroup in which the ascending chain $H = H_0 \lhd H_1 \lhd H_2 \lhd \dots$, defined by $H_i = N_G(H_{i-1})$, terminates.

Theorem 2.20: If the subcofactor $H/\mathrm{scor}_G H$ of each proper self-normalizing subgroup H of G is p-nilpotent, then there exists a normal p-subgroup P_0 of G (P_0 may be trivial) such that G/P_0 is p-nilpotent. In particular, G is p-solvable of p-length ≤ 2.

Proof: The proof is by induction on $|G|$. We may assume that G is not p-nilpotent, since the result is trivially true otherwise. It suffices to show that G has a nontrivial normal p-subgroup P_0. For if $\overline{H} = H/P_0$ is any proper self-normalizing subgroup of $\overline{G} = G/P_0$, then clearly H is a proper self-normalizing subgroup of G; thus, by hypothesis, $H/\mathrm{scor}_G H$ is p-nilpotent; hence, by Lemma 1.4, so also is $\overline{H}/\mathrm{scor}_{\overline{G}}(\overline{H}) \cong H/\mathrm{scor}_G H$. The hypotheses therefore hold for $\overline{G} = G/P_0$ so that, by induction, there exists a normal p-subgroup $\overline{P}_1 = P_1/P_0$ of $\overline{G}$ such that $\overline{G}/\overline{P}_1$ is p-nilpotent. Then P_1 is a normal p-subgroup of G and $G/P_1 \cong \overline{G}/\overline{P}_1$ is p-nilpotent.

Since G is not p-nilpotent, it follows by Theorem 2.18 that G has a nontrivial p-subgroup P and a p'-element x such that $x \in N_G(P) - C_G(P)$. Let $N = N_G^{\infty}(P)$ be the hypernormalizer of P in G. If $N = G$, we then have $\langle 1 \rangle \neq P \lhd \lhd G$ so that from Lemma 2.19, G has a nontrivial normal p-subgroup. The result then follows from our comments above.

Thus, suppose that $N \neq G$. Since N is clearly a self-normalizing subgroup of G, we then have, by hypothesis, that $\overline{N} = N/\mathrm{scor}_G N$ is p-nilpotent. Now let $\overline{P} = P\,\mathrm{scor}_G N/\mathrm{scor}_G N$ and $\overline{x} = x\,\mathrm{scor}_G N$. Since x normalizes P, we have $\overline{P} \lhd \langle \overline{P}, \overline{x} \rangle$; and since $\langle \overline{P}, \overline{x} \rangle$ is p-nilpotent as a subgroup of $\overline{N}$, we also have $\langle \overline{x} \rangle \lhd \langle \overline{P}, \overline{x} \rangle$. Since x is a p'-element of G, it follows that $\langle \overline{P}, \overline{x} \rangle = \overline{P} \times \langle \overline{x} \rangle$, hence that $\overline{x}$ centralizes $\overline{P}$.

Since $x \notin C_G(P)$, there exists $u \in P$ such that $[u,x] = u^{-1}x^{-1}ux \neq 1$. And since $\overline{x}$ centralizes $\overline{P}$, we have $[u,x] \in \mathrm{scor}_G N$; also, $[u,x] \in P$ since $x \in N_G(P)$. Thus $1 \neq [u,x] \in P \cap \mathrm{scor}_G N$ so that $P \cap \mathrm{scor}_G N$ is a nontrivial p-subgroup of G; and since $P \lhd\lhd N$, we have $P \cap \mathrm{scor}_G N \lhd\lhd \mathrm{scor}_G N \lhd\lhd G$, hence, $P \cap \mathrm{scor}_G N \lhd\lhd G$. From Lemma 2.19, we conclude that G has a nontrivial normal p-subgroup; and the result now follows as above.

That G has p-length ≤ 2 is now an immediate consequence of the remark (3) following Definition 2.9. For we have shown that G has a normal p-subgroup P_0 (perhaps trivial) such that G/P_0 is p-nilpotent. Letting T/P_0 be the normal p-complement of G/P_0, we have the normal series $G \supseteq T \supseteq P_0 \supseteq \langle 1 \rangle$ with factors that are either p-groups or p'-groups, at most two of which are nontrivial p-groups. ▯

<u>Corollary 2.21</u>: If the subcofactor $H/\mathrm{scor}_G H$ of each proper self-normalizing subgroup H of G is 2-nilpotent, then G is solvable, and there exist normal subgroups H, K of G such that H/K is isomorphic to a 2-complement of G.

<u>Proof</u>: From the theorem, there exists a normal 2-subgroup K of G such that G/K is 2-nilpotent. Thus there exists $H \lhd G$ such that H/K is a 2-complement of G/K (and hence is isomorphic to a 2-complement

of G since K is a 2-group). Now, H/K has odd order. By the Feit-Thompson Theorem, therefore, H/K is solvable. Since the 2-groups K and G/H are solvable, it follows that G is solvable. ▯

To extend Theorem 2.17, we make use of the Glaubermann-Thompson Theorem concerning the Thompson subgroup J(P) of a p-Sylow subgroup P. Various definitions of J(P) have been given; we shall use the following (given by Gorenstein in [8], in which a proof of the Glaubermann-Thompson Theorem also appears).

<u>Definition 2.11</u>: For a given p-group P, the <u>Thompson subgroup</u> J(P) of P is defined by $J(P) = \langle A \mid A \in \mathcal{A}(P) \rangle$, where $\mathcal{A}(P)$ is the collection of all abelian subgroups of P of maximal order.

<u>Note</u>: For P and J(P) as in the definition, we have $Z(P) \subseteq Z(J(P))$. For if $A \in \mathcal{A}(P)$ and $x \in Z(P)$, then $\langle A, x \rangle$ is abelian; thus, by the maximality of $|A|$, we have $x \in A$, from which this inclusion follows.

<u>Theorem 2.22</u>: (Glaubermann-Thompson Theorem) Let P be a p-Sylow subgroup of the finite group G with p odd. Then, if $N_G(Z(J(P)))$ is p-nilpotent, so also is G.

<u>Theorem 2.23</u>: If the subcofactor $H/\mathrm{scor}_G H$ of each proper abnormal subgroup H of G is p-nilpotent and either p is odd or the p-Sylow subgroups of G are abelian, then there exists a normal p-subgroup P_0 of G (P_0 may be trivial) such that G/P_0 is p-nilpotent. In particular, G is p-solvable of p-length ≤ 2.

<u>Proof</u>: The last statement follows as in the proof of Theorem 2.20; thus only the existence of P_0 requires proof. For this, we proceed by induction on $|G|$. Suppose that G has a nontrivial normal p-subgroup P_0^*. If $\bar{H} = H/P_0^*$ is any proper abnormal subgroup of $\bar{G} = G/P_0^*$, clearly H is a proper abnormal subgroup of G so that, by hypothesis,

$H/\mathrm{scor}_G H$ is p-nilpotent, hence so also is $\bar{H}/\mathrm{scor}_{\bar{G}}(\bar{H}) \cong H/\mathrm{scor}_G H$ (by Lemma 1.4). Since the other hypotheses obviously hold for $\bar{G}$, we have by induction that $\bar{G}$ possesses a normal p-subgroup $\bar{P}_0 = P_0/P_0^*$ such that $\bar{G}/\bar{P}_0$ is p-nilpotent. Then P_0 is a normal p-subgroup of G and $G/P_0 \cong \bar{G}/\bar{P}_0$ is p-nilpotent. Thus, we may assume that G has no nontrivial normal p-subgroup, and hence, by Lemma 2.19, no nontrivial subnormal p-subgroup; and we must show that G is p-nilpotent.

Let P be a p-Sylow subgroup of G. We consider first the case where P is abelian. Let $N = N_G(P)$. Then N is an abnormal subgroup of G, and since $P \ntriangleleft G$, $N \neq G$; by hypothesis, therefore, $N/\mathrm{scor}_G N$ is p-nilpotent. Let $T/\mathrm{scor}_G N$ be the normal p-complement of $N/\mathrm{scor}_G N$. Now, since $P \triangleleft N$, we have $P \cap \mathrm{scor}_G N \triangleleft \mathrm{scor}_G N \triangleleft \triangleleft G$ so that $P \cap \mathrm{scor}_G N \triangleleft \triangleleft G$. Since G has no nontrivial subnormal p-subgroups, we have $P \cap \mathrm{scor}_G N = \langle 1 \rangle$ and hence that $\mathrm{scor}_G N$ is a p'-group. This implies then that T is a normal p-complement of N, from which it follows that $N = P \times T$. Therefore, $T \subseteq C_G(P)$; and since P is abelian, $P \subseteq C_G(P)$. From this it follows that $P \subseteq Z(N_G(P))$ so that, by Theorem 2.18, G is p-nilpotent.

Now consider the case where p is odd, and let $\tilde{\tilde{N}} = N_G(Z(J(P)))$, where J(P) is the Thompson subgroup of P. Since Z(J(P)) is characteristic in P, we have $Z(J(P)) \triangleleft N_G(P)$, and thus $N_G(P) \subseteq \tilde{\tilde{N}}$. Since $N_G(P)$ is abnormal in G, it follows that $\tilde{\tilde{N}}$ also is abnormal in G. And since G has no nontrivial normal p-subgroups, $\tilde{\tilde{N}} \neq G$ so that $\tilde{\tilde{N}}$ is a proper abnormal subgroup of G. By hypothesis, therefore, $\tilde{\tilde{N}}/\mathrm{scor}_G \tilde{\tilde{N}}$ is p-nilpotent.

Suppose now that $P_1 = P \cap \mathrm{scor}_G N \neq \langle 1 \rangle$. Then, since $\mathrm{scor}_G \tilde{\tilde{N}} \triangleleft \tilde{\tilde{N}}$, we have $P_1 = P \cap \mathrm{scor}_G \tilde{\tilde{N}} \triangleleft P$, and hence P_1 must intersect Z(P) nontrivially. Therefore, since $Z(P) \subseteq Z(J(P))$ by the above note, this means that $P_2 = P_1 \cap Z(J(P)) \neq <1>$ also.

Now, $P_2 = P_1 \cap Z(J(P)) = P \cap \text{scor}_G\tilde{\tilde{N}} \cap Z(J(P)) = Z(J(P))) \cap \text{scor}_G\tilde{N}$. And since $Z(J(P)) \lhd \tilde{\tilde{N}}$, we have $P_2 = Z(J(P)) \cap \text{scor}_G\tilde{N} \lhd \text{scor}_G\tilde{\tilde{N}} \lhd \lhd G$, so that $P_2 \lhd \lhd G$. Thus P_2 is a nontrivial subnormal p-subgroup of G, which is a contradiction to the fact that G has no such subgroup.

Consequently, $P \cap \text{scor}_G\tilde{\tilde{N}} = <1>$ so that $\text{scor}_G\tilde{N}$ is a p'-group. Since $\tilde{N}/\text{scor}_G\tilde{N}$ is p-nilpotent, it now follows, as in the previous case, that $\tilde{N} = N_G(Z(J(P)))$ is p-nilpotent. By the Glaubermann-Thompson Theorem, therefore, G is p-nilpotent. ▯

<u>Corollary 2.24</u>: If the subcofactor $H/\text{scor}_G H$ of each proper abnormal subgroup H of G is 2-nilpotent and the 2-Sylow subgroups of G are abelian, then G is solvable, and there exist normal subgroups H, K of G such that H/K is isomorphic to a 2-complement of G.

<u>Proof</u>: This follows from the preceding theorem and the Feit-Thompson Theorem in the same manner as Corollary 2.21 was proved. ▯

Example 2.10 of the preceding section shows that a non-p-nilpotent group having the cofactors of all its proper subgroups p-nilpotent need not have a normal p-Sylow subgroup. This is no longer the case if we require that, in addition, all the proper normal subgroups of G be p-nilpotent. Before establishing this and other properties of G, however, we first prove the following result (which we have already used in Theorem 2.11).

<u>Theorem 2.25</u>: Let G be a solvable non-p-nilpotent group having all proper normal subgroups p-nilpotent, and let $|G| = p^a m$ where $(p,m) = 1$. If G has an abelian normal p-Sylow subgroup P, then G has exactly p^a distinct p-complements.

Proof: Extend $G \supset P \supset \langle 1 \rangle$ to a chief series

$$G = G_0 \supset \ldots \supset G_m = P \supset G_{m+1} \supset \ldots \supset G_{m+n} = \langle 1 \rangle$$

and set $P_i = G_{m+i}$ for $i = 0, 1, \ldots, n$. We assert first that for each $i \geqq 1$, G/P_i is not p-nilpotent. For suppose that for some $i \geqq 1$, G/P_i is p-nilpotent, say with normal p-complement K/P_i. Then $K \triangleleft G$; and since $P_i \subsetneqq P$ for $i \geqq 1$, $p \mid |G/P_i|$ so that K is a proper normal subgroup of G. By hypothesis, therefore, K is p-nilpotent. If T is the normal p-complement of K, then T char $\leqq K \triangleleft G$, which implies that $T \triangleleft G$; also, $|G:T| = |G:K||K:T| = |G/P_i:K/P_i||K:T| = p^a$. Thus T is a normal p-complement of G; but this contradicts the non-p-nilpotence of G.

Let now $(G/P_i)'_{(p)}$ denote the least normal subgroup of G/P_i for which the factor group is an abelian p-group. Then $(G/P_i)'_{(p)} = G/P_i$ for all $i \geqq 1$. For suppose not and that $(G/P_i)'_{(p)} = L/P_i$ where $L \ntriangleleft G$ for some $i \geqq 1$. Then, by hypothesis, L is p-nilpotent, and hence so also is L/P_i. Let U/P_i be the normal p-complement of L/P_i. Since U/P_i is a characteristic subgroup of L/P_i, we have $U/P_i \triangleleft G/P_i$; also, $|G/P_i:U/P_i| = |G/P_i:L/P_i||L/P_i:U/P_i|$ is a power of p. This means that G/P_i is p-nilpotent, a contradiction to what we have shown above.

Now, for each $i \geqq 1$, let τ_i be the transfer of G/P_i into its abelian normal p-Sylow subgroup P/P_i. From the basic properties of the transfer (see, for example, Scott [19], 13.5.2, 13.5.5), we have that $\ker \tau_i = (G/P_i)'_{(p)}$, and since P/P_i is an abelian normal p-Sylow subgroup of G/P_i, $\tau_i(G/P_i) = (P/P_i) \cap Z(G/P_i)$. Since we have just shown that $(G/P_i)'_{(p)} = G/P_i$ for each $i \geqq 1$, we have $\tau_i(G/P_i) = \langle 1 \rangle$, and hence that $(P/P_i) \cap Z(G/P_i) = \langle 1 \rangle$ for all $i \geqq 1$.

For each $i = 1, 2, \ldots, n$, therefore, since $P_{i-1} \subseteq P$ and $(P/P_i) \cap Z(G/P_i) = \langle 1 \rangle$, we have $(P_{i-1}/P_i) \cap Z(G/P_i) = \langle 1 \rangle$. In alternate terminology, this means that each of P/P_1, P_1/P_2, ... , P_{n-1}/P_n is an eccentric chief factor of G. A result of P. Hall [10] states that for a solvable group G, the number of p-complements of G is equal to the product of the orders of the eccentric p-chief factors of G. Thus we have that the number of distinct p-complements of G is equal to $\prod_{i=1}^{n} |P_{i-1}/P_i| = |P| = p^a$. ▯

We now examine the structure of a non-p-nilpotent finite group G having all its proper normal subgroups p-nilpotent as well as the subcofactors of its self-normalizing or abnormal subgroups. Although Example 2.13 shows that we cannot hope to recover all of Theorem 2.14, we do discover a considerable amount of structure in G.

<u>Theorem</u> <u>2.26</u>: Let G be a finite non-p-nilpotent group having all of its proper normal subgroups p-nilpotent, and let $|G| = p^a m$ with $(p,m) = 1$. Suppose also that one of the following two conditions holds:

(a) The subcofactor of each proper self-normalizing subgroup of G is p-nilpotent.

(b) The subcofactor of each proper abnormal subgroup of G is p-nilpotent and either p is odd or the p-Sylow subgroups of G are abelian.

Then, the following hold:

(i) G has a normal p-Sylow subgroup P.

(ii) $P \subseteq G'$, and thus G/G' is an abelian p'-group.

(iii) $F_p(G)$, the largest normal p-nilpotent subgroup of G, is the unique maximal normal subgroup of G, and $G/F_p(G)$ is a p'-group.

(iv) For all $K \ntrianglelefteq G$, $(|G/K|, d_p) \neq 1$ where $d_p = \prod_{i=1}^{a} (p^i - 1)$.

(v) If P is abelian, then $C_G(P) = F_p(G)$; thus G induces a p'-group of automorphisms in P.

(vi) If G is solvable, then $|G:F_p(G)| = q$ for some prime q dividing $|G|$ and d_p. If also P is abelian, then G has exactly p^a distinct p-complements.

<u>Proof</u>: (i) Suppose that G has no normal p-Sylow subgroup. From Theorem 2.20 or 2.23, there exists a normal p-subgroup P_0 of G such that G/P_0 is p-nilpotent, say with normal p-complement K/P_0. Since P_0 is not a p-Sylow subgroup of G, we have $K \ntrianglelefteq G$. By hypothesis, therefore, K is p-nilpotent, say with normal p-complement T. Since T is characteristic in the normal subgroup K of G, we have $T \triangleleft G$; also, $|G:T| = |G:K||K:T| = |G/P_0:K/P_0||K:T|$ is a power of p. However, this means that T is a normal p-complement of G, contradicting the non-p-nilpotence of G. Therefore, G does have a normal p-Sylow subgroup.

For (ii), the proof of part (c)-(6) of Theorem 2.11 carries over with $p_i = p$.

(iii) is immediate. For if K is any proper normal subgroup of G, then, by hypothesis, K is p-nilpotent and thus is contained in $F_p(G)$, from which it follows that $F_p(G)$ is the unique maximal normal subgroup of G. In particular, the normal p-Sylow subgroup P of G must be contained in $F_p(G)$ so that $G/F_p(G)$ is a p'-group.

(iv) We suppose this result to be false, and let G be a minimal counterexample. Then there exists $L \ntrianglelefteq G$ with $(|G/L|, d_p) = 1$. Let K be a maximal normal subgroup of G containing L and let $|G:K| = n$; then n also is relatively prime to d_p. Now, from (iii), $F_p(G)$ is the unique maximal normal subgroup of G. Consequently, we have $K = F_p(G)$. And from (iii) again, $F_p(G)$ has index prime to p. Therefore, G/K is a p'-group so that $|G:K| = n$ is prime to p also, and the normal p-Sylow subgroup P of G is contained in K.

Suppose now that P is not minimal normal in G. Then there exists $M \lhd G$ with $\langle 1 \rangle \neq M \lneqq P$. If G/M were p-nilpotent, say with normal p-complement U/M, then $U \lhd G$, and $U \neq G$ since $p \mid |G/M|$; by hypothesis, therefore, U is p-nilpotent. Then the normal p-complement V of U is normal in G, and $|G:V| = |G:U||U:V| = |G/M:U/M||U:V|$ is a power of p so that V is a normal p-complement of G. But this contradicts the non-p-nilpotence of G. Thus, G/M is not p-nilpotent. Also, the proper normal subgroups of G/M are clearly p-nilpotent. Since, from Lemma 1.4, $\mathrm{scof}_{G/M}(H/M) = \mathrm{scof}_G H$ for each proper subgroup H/M of G/M, and since it is immediate from the definitions that H is self-normalizing (abnormal) in G if H/M is self-normalizing (abnormal) in G/M, it follows that (a) or (b) holds in G/M according as (a) or (b) holds in G. In addition, $K/M \ntrianglelefteq G/M$ since $K \ntrianglelefteq G$, and $|G/M:K/M| = |G:K| = n$ is prime to d_p, hence is prime to $d_p^* = \prod_{i=1}^{s} (p^i - 1)$ where p^s is the highest power of p which divides $|G/M|$. But this means that G/M is a counterexample to this result with $|G/M| < |G|$, which contradicts the minimality of G. Therefore, P is a minimal normal subgroup of G, hence is elementary abelian. The order of the automorphism group of P, Aut(P), is thus equal to $p^e.d_p$ where $e = a(a - 1)/2$.

Now, since T is a normal p-complement of K and $P \subseteq K$, we have $K = P \times T$ so that T centralizes P; and since P is abelian, $P \subseteq C_G(P)$ also. Thus, $K = P \times T \subseteq C_G(P)$ from which it follows that the order of $G/C_G(P)$ divides $|G:K| = n$ and hence is prime to both p and d_p. However, $G/C_G(P)$ is isomorphic to a subgroup of Aut(P) so that $|G/C_G(P)|$ must divide $|\mathrm{Aut}(P)| = p^e.d_p$. Thus we have a contradiction so that no such minimal counterexample to (iv) can exist.

(v) $F_p(G)$ is, by definition, the largest normal p-nilpotent subgroup of G; let W be its normal p-complement. Since P is normal in G and is trivially p-nilpotent, we have $P \subseteq F_p(G)$. It follows that $F_p(G) = P \times W$ so that W centralizes P. Since P is abelian, $P \subseteq C_G(P)$ also, and hence $F_p(G) \subseteq C_G(P)$. From Theorem 2.18, since G is not p-nilpotent, $P \not\subseteq Z(G) = Z(N_G(P))$. Thus, by the maximality of $F_p(G)$ established in (iii), we have $F_p(G) = C_G(P)$.

(vi) For G solvable, we have $|G/F_p(G)| = q$ for some prime q, since $F_p(G)$ is a maximal normal subgroup of G; and from (iv), we have that $q | d_p$. The last statement of (vi) follows from Theorem 2.25. ▯

<u>Corollary 2.27</u>: Let G be a finite group having all of its proper normal subgroups p-nilpotent and $|G| = p^a m$ where $(p,m) = 1$ and $a \geq 1$. Suppose also that either condition (a) or (b) of Theorem 2.26 holds. Then G is p-nilpotent if and only if there exists a proper normal subgroup K of G with $(|G:K|, d_p) = 1$, where $d_p = \prod_{i=1}^{a} (p^i - 1)$.

<u>Proof</u>: If G is p-nilpotent, then there exists $K \ntrianglelefteq G$ with $|G:K| = p^a$ which is prime to d_p. On the other hand, if G is not p-nilpotent, then by (iv) of the preceding theorem, every proper normal subgroup of G has index prime to d_p. ▯

In Theorem 2.26 and its corollary, we have required that all the proper normal subgroups of G be p-nilpotent. If we now extend this requirement of p-nilpotence to the larger class of somewhat normal subgroups of G, as defined below, we recover all of Theorem 2.14.

<u>Definition 2.12</u>: Let H be a proper subgroup of a given finite group G. H will be said to be <u>somewhat normal</u> in G if $\mathrm{cof}_G H = H/\mathrm{cor}_G H$ is cyclic of prime-power order.

Theorem 2.28: Let G be a finite non-p-nilpotent group having all of its proper somewhat normal subgroups p-nilpotent. Suppose also that one of the following two conditions is satisfied:

(a) The subcofactor of each proper self-normalizing subgroup of G is p-nilpotent.

(b) The subcofactor of each proper abnormal subgroup of G is p-nilpotent and either p is odd or the p-Sylow subgroups of G are abelian.

Then:

(i) $|G| = p^a q^b$ for some prime $q \neq p$; in particular, G is solvable.

(ii) All proper subgroups of G are nilpotent.

(iii) The conclusions of Theorems 2.14 and 2.26 hold.

Proof: (i) It follows from Theorem 2.26 that G has a normal p-Sylow subgroup P. We consider first the case where P is not a minimal normal subgroup of G. Then there exists $M \triangleleft G$ with $\langle 1 \rangle \neq M \lneq P$. Since $\mathrm{scof}_{G/M}(H/M) \cong \mathrm{scof}_G H$ for each proper subgroup H/M of G/M (by Lemma 1.4), and since H is self-normalizing (abnormal) in G if H/M is self-normalizing (abnormal) in G/M, it follows that hypothesis (a) or (b) holds in G/M according as (a) or (b) holds in G. Also, the proper normal subgroups of G/M are clearly p-nilpotent. Finally, G/M is not p-nilpotent. For if T/M were a normal p-complement of G/M, then T would be a proper normal subgroup of G since $p \mid |G/M|$, and hence p-nilpotent by hypothesis, say with normal p-complement U. As in the proof of the preceding theorems, it then follows that U would be a normal p-complement of G; but this contradicts the non-p-nilpotence of G.

The hypotheses are thus satisfied by G/M. Since $p \mid |G/M|$, we have by induction, that $|G/M| = p^k q^b$ for some prime $q \neq p$. Therefore, $|G| = p^a q^b$ where $|M| = p^{a-k}$.

Now consider the case where P is minimal normal in G and thus elementary abelian. Let $|G| = p^a \prod_{i=1}^{r} q_i^{b_i}$ where p and the q_i are distinct primes dividing $|G|$. Suppose now that $r > 1$. Then for each $i = 1, \ldots, r$, if Q_i is any q_i-Sylow subgroup of G, PQ_i is a proper subgroup of G, and thus so also is $P\langle x \rangle$ for each $x \in Q_i$. Now, since $P \lhd G$, we have $P \subseteq \mathrm{cor}_G(P\langle x \rangle)$ so that $P\langle x \rangle/\mathrm{cor}_G(P\langle x \rangle)$ is a homomorphic image of $P\langle x \rangle/P \cong \langle x \rangle$ and is therefore a cyclic q_i-group. Hence, for each $i = 1, \ldots, r$ and for each $x \in Q_i$, $P\langle x \rangle$ is a somewhat normal subgroup of G and, consequently, is p-nilpotent by hypothesis. This implies that $\langle x \rangle$ is normal in $P\langle x \rangle$ so that x centralizes P; and since this is true for each $x \in Q_i$, we have $Q_i \subseteq C_G(P)$ for each $i = 1, \ldots, r$. Since P is abelian, we also have $P \subseteq C_G(P)$. It follows that $G = \langle P, Q_1, \ldots, Q_r \rangle \subseteq C_G(P)$, that is, $P \subseteq Z(G)$. But by (2) of Theorem 2.18, this implies that G is p-nilpotent, a contradiction. Therefore, $r = 1$, and $|G| = p^a q_1^{b_1}$.

The solvability of G now follows from the well-known theorem of Burnside that groups of order $p^a q^b$, where p and q are primes, are solvable.

(ii) Since $P \lhd G$ and $|G| = p^a q^b$, G is q-nilpotent so that all subgroups of G also are q-nilpotent. Let K be a maximal normal subgroup of G containing P. Then K is q-nilpotent, and by hypothesis, K is p-nilpotent; consequently, K is nilpotent.

Since G is solvable, G/K is of prime order; and since $P \subseteq K$, we have $|G/K| = q$. Now let S be any maximal subgroup of G; by what we have shown, S is q-nilpotent. Either S = K, in which case S is nilpotent, or SK = G. In this latter case we have $|S \cap K| = \frac{|S||K|}{|G|} = \frac{|S|}{q}$. By the nilpotence of K, $S \cap K \lhd \lhd K \lhd G$, that is, $S \cap K \lhd \lhd G$, and thus $S \cap K \subseteq \mathrm{scor}_G S = \mathrm{cor}_G S$. Hence, $S/\mathrm{cor}_G S$ has order 1 or q so that S is somewhat normal (in fact, nearly normal) in G. By hypothesis, therefore, S is p-nilpotent; and since S is also q-nilpotent, this means that S is nilpotent. Thus all the maximal subgroups (and hence all proper subgroups) of G are nilpotent.

(iii) now follows immediately from (ii). ▯

We conclude this section with the following result in which, as in the preceding theorem, we again strengthen the conditions imposed in Theorem 2.26. Since the p-nilpotence of a subgroup or of the subcofactor of a subgroup provides no useful information in the case that this subgroup or subcofactor has order prime to p, we would hope to obtain more of the structure of G if we impose some additional condition on these. Although, by Example 2.13, we cannot hope to recover all of Theorem 2.14 under the conditions imposed in the following theorem, we do, nevertheless, obtain some additional information about G. To state the result we need the following definition.

Definition 2.13: A finite group G will be said to be (p:q)-nilpotent if:

(i) G is p-nilpotent;

(ii) $q \mid |G|$ and G is q-nilpotent in case $p \nmid |G|$ and $|G| > 1$.

Theorem 2.29: Let G be a finite group with p a prime factor of $|G|$ for which every proper normal subgroup K is $(p:q_K)$-nilpotent for some prime q_K depending on K. Suppose also that the cofactor $H/\mathrm{cor}_G H$ of each proper subgroup H of G is $(p:q_H)$-nilpotent for some prime q_H depending on H. Then the following hold:

(i) G is solvable.

(ii) G has a normal Sylow subgroup.

(iii) If G is not p-nilpotent, then the conclusions of Theorem 2.26 hold; in particular, G has a normal p-Sylow subgroup $P \subseteq G'$; $F_p(G)$ is the unique maximal normal subgroup of G and $|G/F_p(G)| = q$ for some prime $q \neq p$; if P is abelian, then G has exactly p^a distinct p-complements where $|G| = p^a m$ with $(p,m) = 1$.

<u>Proof</u>: (iii) follows immediately from Lemma 1.2, part (i), and Theorem 2.26; only (i) and (ii) require proof. For these, we consider two cases.

Case 1: G is p-nilpotent.—Let $|G| = \prod_{i=1}^{r} p_i^{a_i}$ where $p_1 = p$ and the p_i are distinct primes dividing $|G|$. We may assume that $r > 1$, since the result is trivially true otherwise. Then there exists $T_1 \triangleleft G$ with $|G:T_1| = p_1^{a_1}$ since G is p_1-nilpotent. Since $T_1 \neq \langle 1 \rangle$ is a proper normal p_1'-subgroup of G, it is, by hypothesis, p_i-nilpotent for some $i \geq 2$, say for $i = 2$. Thus there exists T_2 characteristic in T_1, hence normal in G, with $|T_1:T_2| = p_2^{a_2}$. Continuing gives a normal series of G,

$$G = T_0 \supset T_1 \supset T_2 \supset \dots \supset T_{r-1} \supset T_r = \langle 1 \rangle,$$

where for each $i = 1, 2, \dots, r$, $|T_{i-1}/T_i| = p_i^{a_i}$. It follows that G is solvable with a normal p_r-Sylow subgroup T_{r-1}.

Case 2: G is not p-nilpotent.—Then by Theorem 2.26, G has a normal p-Sylow subgroup P so that (ii) holds. We consider separately the two possibilities that P is or is not a minimal normal subgroup of G.

(a) Suppose P is not minimal normal in G. Then there exists $M \triangleleft G$ with $\langle 1 \rangle \neq M \subsetneq P$. We show that the hypotheses are satisfied by G/M.

If H/M is any proper subgroup of G/M, then H is a proper subgroup of G. By hypothesis, $\mathrm{cof}_G H = H/\mathrm{cor}_G H$ is $(p{:}q_H)$-nilpotent for some prime q_H depending on H. From Lemma 1.4, we have that $\mathrm{cof}_{G/M}(H/M)$ is isomorphic to $\mathrm{cof}_G H$ and is therefore $(p{:}q_H)$-nilpotent relative to this same prime q_H.

Clearly, all proper normal subgroups of G/M are p-nilpotent. Now suppose $K/M \neq \langle 1 \rangle$ is a proper normal p'-subgroup of G/M. Then $K \trianglelefteq\!\!\!\!\neq G$, hence is p-nilpotent by hypothesis, say with normal p-complement T. Since T is characteristic in K, T is normal in G; and since M is the p-Sylow subgroup of K, $K = M \times T$. Now since T is a nontrivial proper normal p'-subgroup of G, it is q-nilpotent for some prime q dividing $|T|$. Hence, since $K/M \cong T$, q divides $|K/M|$ and K/M is q-nilpotent.

The hypotheses thus hold for G/M so that, by induction, G/M is solvable. And since the p-group M is solvable, it follows that G also is solvable.

(b) Suppose now that P is minimal normal in G. Then either there exists a minimal normal subgroup L of G which is distinct from P, or else P is the unique minimal normal subgroup of G.

In the first case, we have $L \cap P = \langle 1 \rangle$ so that L is a proper normal p'-subgroup of G, hence is q-nilpotent for some prime q dividing $|L|$. Since the normal q-complement of L is characteristic in L, it follows from the minimality of L that L is a q-group.

As in (a), the conditions on the cofactors of subgroups of G/L are satisfied, and all proper normal subgroups of G/L are p-nilpotent. Now let $K/L \neq \langle 1 \rangle$ be a proper normal p'-subgroup of G/L. Then since L

is a p'-group, K is a proper normal p'-subgroup of G, and hence is, by hypothesis, q_1-nilpotent for some prime q_1 dividing $|K|$. Let U be the normal q_1-complement of K. Now, if $q_1 \mid |K/L|$, we have exhibited a prime q_1 dividing $|K/L|$ for which K/L is q_1-nilpotent. Suppose, therefore, that q_1 does not divide $|K/L|$. This means that $q_1 = q$, from which it follows that $K = U \times L$. Now since U is a nontrivial proper normal p'-subgroup of G, it is q_2-nilpotent for some prime q_2 dividing $|U|$. Since $K/L \cong U$, we have that $q_2 \mid |K/L|$ and that K/L is q_2-nilpotent.

The hypotheses thus hold for G/L so that, by induction, G/L is solvable. Since the q-group L is solvable, this means that G also is solvable.

There remains to consider only the possibility that P is the unique minimal normal subgroup of G. By the Schur-Zassenhaus Theorem, P has a complement T in G. Now if H is any subgroup of T (not necessarily proper), H is a proper p'-subgroup of G; and since it does not contain the unique minimal normal subgroup P of G, we have $\text{cor}_G H = \langle 1 \rangle$ so that $H = \text{cof}_G H$. By hypothesis, therefore, each nontrivial subgroup H of T (including T itself) is q_H-nilpotent for some prime q_H dividing $|H|$ (where q_H depends on H).

Let $|G| = p^a \prod_{i=1}^{t} q_i^{b_i}$, where p and the q_i are distinct primes dividing $|G|$. Then T, being a p-complement of G, has order $\prod_{i=1}^{t} q_i^{b_i}$. Since T is q_i-nilpotent for some i, say for i = 1, there exists $T_1 \lhd T$ with $|T:T_1| = q_1^{b_1}$. T_1 is q_i-nilpotent for some $i \geqq 2$, say for i = 2, and thus there exists T_2 characteristic in T_1, hence normal in T, such that $|T_1:T_2| = q_2^{b_2}$. Continuing gives a normal series of T,

$$T = T_0 \supset T_1 \supset T_2 \supset \dots \supset T_{t-1} \supset T_t = \langle 1 \rangle,$$

with $|T_{i-1}/T_i| = q_i^{b_i}$ for each i = 1, 2, ..., t. Therefore, T is solvable. Since $G/P \cong T$, we have G/P solvable; and since the p-group P is solvable, it follows that G also is solvable. ▯

2.4 Sylow-towered and Supersolvable Cofactors or Subcofactors

In this section we examine the influence on a finite group G of supersolvable subcofactors of certain subgroups of G and, more generally, of σ-Sylow-towered subcofactors where σ is some fixed ordering of a set Σ of primes containing $\pi(G)$ = the set of prime factors of $|G|$. Our goal is to extend the well-known theorem of Huppert__ If all the proper subgroups of a finite group G are supersolvable, then G is solvable__ and some extensions of this result by Rose [16, 17], who required that only the self-normalizing or abnormal subgroups of G be supersolvable, or, more generally, σ-Sylow-towered.

The concept of a group being σ-Sylow-towered is defined as follows.

<u>Definition 2.14</u>: Let G be a given finite group and $\pi(G)$ the set of prime factors of $|G|$. Let $\sigma = (p_1, p_2, \ldots, p_t)$ be a fixed ordering of a set Σ of primes containing $\pi(G)$. Then G is said to have a <u>σ-Sylow-tower</u> if there exists a normal series $\langle 1 \rangle = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_t = G$ such that for each $i = 1, 2, \ldots, t$, G_i/G_{i-1} is isomorphic to a p_i-Sylow subgroup of G (which we allow to be trivial in case $p_i \nmid |G|$).

For example, let $\Sigma = \{p_1, p_2, \ldots, p_n\} \supseteq \pi(G)$ and σ the natural descending order of Σ, say $\sigma = (p_1, p_2, \ldots, p_n)$ where $p_1 > p_2 > \ldots > p_n$. It is well-known (see, for example, M. Hall [9], 10.5.3) that if G is supersolvable, then G has a σ-Sylow tower for this σ.

We might mention that Doerk in [6] has extended the theorem of Huppert, which was stated above, by describing much of the structure of G. Several of his results parallel those of the Schmidt-Iwasawa Theorem (Theorem 2.3); some of these are given in the following theorem.

Theorem 2.30: Let G be a finite group all of whose proper subgroups are supersolvable. Then:

(i) G is solvable;

(ii) G has a σ-Sylow tower where σ is the natural descending order of $\pi(G)$, or G is a nonnilpotent group having all its proper subgroups nilpotent.

If G itself is not supersolvable, then the following also are true:

(iii) G has exactly one normal Sylow subgroup P.

(iv) $\Phi(P) \subseteq Z(G)$ so that $cl(P) \leqq 2$; $exp(P) = p$ for p odd and $exp(P) \leq 4$ for $p = 2$, where P is a p-group; $\Phi(P)$ is supersolvably embedded in G, that is, there exist normal subgroups N_i of G such that $<1> = N_0 \subset N_1 \subset \ldots \subset N_m = \Phi(P)$ and $|N_i/N_{i-1}| = p$ for each $i = 1, \ldots, m$.

(v) $|G|$ is divisible by at most three distinct primes.

Our first result in this direction follows from Corollary 2.2 of Section 1. Before stating it, however, we first establish the following lemma.

Lemma 2.31: For a given group G, $F_2(G') = G'$, that is, G' has Fitting length $\leqq 2$, if and only if $G/F_2(G)$ is abelian.

Proof: We show first that $F(G') = F(G) \cap G'$. For this, we have that since $F(G')$ is characteristic in G' which is normal in G, $F(G')$ is normal in G and is nilpotent, hence is contained in $F(G)$; consequently, $F(G') \subseteq F(G) \cap G'$. But $F(G) \cap G'$ is normal in G' and is nilpotent, hence is contained in $F(G')$. Thus, $F(G') = F(G) \cap G'$.

Using this equality and the normality of G' and $F(G)$, we now have the following chain of equivalent statements:

$F_2(G') = G'$ if-f $G'/F(G')$ is nilpotent

if-f $G'/F(G) \cap G'$ is nilpotent

if-f $G'F(G)/F(G)$ is nilpotent

if-f $G'F(G)/F(G) \subseteq F(G/F(G)) = F_2(G)/F(G)$

if-f $G'F(G) \subseteq F_2(G)$

if-f $G' \subseteq F_2(G)$

if-f $G/F_2(G)$ is abelian. ▯

Theorem 2.32: Let G be a finite solvable group for which the cofactors of all maximal subgroups are supersolvable. Then:

(i) $G/F(G)$ is supersolvable.

(ii) $\int(G') \leq 2$; that is, $F_2(G') = G'$, or equivalently, $G/F_2(G)$ is abelian.

(iii) $\int(G) \leq 3$; that is, $F_3(G) = G$.

Proof: (i) Supersolvability is clearly a strictly homomorphism-invariant property in the sense of Definition 2.2. Thus, since G is solvable and cofactors of maximal subgroups are supersolvable, it follows from Corollary 2.2-(ii) that $G/F(G)$ is supersolvable.

(ii) Since $G/F(G)$ is supersolvable, its derived subgroup $(G/F(G))'$ is nilpotent. By the remark (1) preceding Lemma 2.8, $(G/F(G))' = G'F(G)/F(G)$, which is isomorphic to $G'/G' \cap F(G)$, and which is in turn equal to $G'/F(G')$ by the proof of Lemma 2.31. Therefore, $G'/F(G')$ is nilpotent so that $F_2(G') = G'$.

(iii) From Lemma 2.31, it now follows that $G/F_2(G)$ is abelian, hence nilpotent, and thus G has Fitting length $\leqq 3$. ▯

That $m = 2$ and $n = 3$ are the best possible integers for which $G' = F_m(G')$ and $G = F_n(G)$ in the preceding theorem is shown by the following example.

Example 2.33: Let S_4 be the symmetric group on 4 letters. Then S_4 is solvable, the cofactors of all proper subgroups of S_4 are supersolvable, the Fitting length of $S_4' = A_4$ (the alternating group of degree 4) is 2, and the Fitting length of S_4 is 3.

Proof: The solvability of S_4 is well-known. The Fitting subgroup of $S_4' = A_4$ is the four-group V, from which it is clear that $\mathcal{f}(A_4) = 2$. Also, it is immediate that $F(S_4) = F_1(S_4) = V$ and $F_2(S_4) = A_4$ so that $\mathcal{f}(S_4) = 3$.

The subgroups of S_4 having order 4 or 12 are normal in S_4 and hence have trivial cofactor. The subgroups of order 1, 2, or 3 are obviously supersolvable, and thus so also are their cofactors. The only other subgroups of S_4 that need be checked are those of order 6. These have trivial core and are isomorphic to the group S_3 which is supersolvable. Consequently, the cofactors of all proper subgroups of S_4 are supersolvable. ▯

In Example 2.17 of the preceding section, a group G was constructed which was not solvable, but in which all the proper abnormal subgroups were supersolvable. This shows that the hypothesis of G being solvable in Theorem 2.32 cannot be omitted; that is, the supersolvability of the cofactors of all maximal subgroups of G is not sufficient to guarantee that G is solvable. As we now show, however, if we enlarge the class of subgroups which are to have supersolvable cofactors or subcofactors (more generally, σ-Sylow-towered subcofactors) from the nonnormal maximal subgroups to the collection of all self-normalizing subgroups of G, then G is solvable. This extends the following result due to Rose [16].

<u>Theorem</u> <u>2.34</u>: Let σ be a fixed ordering of a set Σ of primes containing $\pi(G)$. If every proper self-normalizing subgroup H of G has a σ-Sylow tower, then G is solvable.

<u>Theorem</u> <u>2.35</u>: Let σ be a fixed ordering of a set Σ of primes containing $\pi(G)$. If the subcofactor $H/\mathrm{scor}_G H$ of each self-normalizing subgroup H of G has a σ-Sylow tower, then G is solvable. Moreover, $G/F(G)$ has a σ-Sylow tower.

<u>Proof</u>: We first establish the solvability of G, using induction on $|G|$. If G is simple, then every proper subgroup has subnormal core $= \langle 1 \rangle$, and hence has a σ-Sylow tower. The solvability of G then follows by Theorem 2.34.

Thus, suppose that G is not simple, and let M be a minimal normal subgroup of G. If $\bar{H} = H/M$ is any proper self-normalizing subgroup of $\bar{G} = G/M$, then clearly H is a proper self-normalizing subgroup of G. By hypothesis, $H/\mathrm{scor}_G H$ is σ-Sylow-towered; hence, from Lemma 1.4, so also is $\bar{H}/\mathrm{scor}_{\bar{G}}(\bar{H}) \cong H/\mathrm{scor}_G H$. Thus, since $\pi(G/M) \subseteq \pi(G) \subseteq \Sigma$, the hypotheses hold for $\bar{G} = G/M$ so that, by induction, G/M is solvable.

We show now that the hypotheses hold for M. For this, let H be any self-normalizing (in M) proper subgroup of M. Then $N = N_G^{\infty}(H) =$ the hypernormalizer of H in G is a self-normalizing subgroup of G. Now, $M \not\subseteq N$; for otherwise, since $H \lhd \lhd N$, we would have $H \lhd \lhd M$, in contradiction to the fact that H is self-normalizing in M. Thus, $N \neq G$; by hypothesis, therefore, $N/\mathrm{scor}_G N$ has a σ-Sylow tower and hence, so also does its subgroup $H\,\mathrm{scor}_G N/\mathrm{scor}_G N$.

Now, since $H \lhd\lhd N$, we have $H \cap \mathrm{scor}_G N \lhd\lhd \mathrm{scor}_G N \lhd\lhd G$ so that $H \cap \mathrm{scor}_G N$ is subnormal in G and thus must be contained in $\mathrm{scor}_G H$. Hence, $H \cap \mathrm{scor}_G N \subseteq \mathrm{scor}_G H \subseteq \mathrm{scor}_M H$ (the last inclusion being true since $\mathrm{scor}_G H$ subnormal in G implies that it is subnormal in M). Since $H/H \cap \mathrm{scor}_G N$, being isomorphic to $H\,\mathrm{scor}_G N/\mathrm{scor}_G N$, has a σ-Sylow tower, so also does its homomorphic image $H/\mathrm{scor}_M H$.

Thus, since $\pi(M) \subseteq \pi(G) \subseteq \Sigma$, the hypotheses hold for M so that, by induction, M is solvable. And since G/M is solvable, it follows that G also is solvable.

To show that G/F(G) has a σ-Sylow tower, we need only show that the property T_σ of having a σ-Sylow tower is a strictly homomorphism-invariant property in the sense of Definition 2.2. For the hypotheses of the theorem imply that $S/\mathrm{scor}_G S = S/\mathrm{cor}_G S = \mathrm{cof}_G S$ is a T_σ-group for all maximal subgroups S of G, since each maximal subgroup of G is either normal, and thus has trivial cofactor, or is self-normalizing in G. Also, we have proved that G is solvable. Therefore, if T_σ is a strictly homomorphism-invariant property, then by Corollary 2.2-(ii), G/F(G) is a T_σ-group.

The fact that T_σ is strictly homomorphism-invariant is almost immediate. For let $\langle 1\rangle = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_t = G$ be a σ-Sylow tower of G. If ϕ is a homomorphism of G onto $\overline{G}$, then clearly

$$\langle 1\rangle = \phi(G_0) \subseteq \phi(G_1) \subseteq \ldots \subseteq \phi(G_t) = \overline{G}$$

is a σ-Sylow tower for $\overline{G}$; consequently, T_σ is homomorphism-invariant. Also, if H is a subgroup of G, then

$$\langle 1\rangle = H \cap G_0 \subseteq H \cap G_1 \subseteq \ldots \subseteq H \cap G_t = H$$

is a σ-Sylow tower for H; thus, T_σ is subgroup-inherited. Finally, if $<1> = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_t = K$ is a σ-Sylow tower of the group K, then

$$<1> = G_0 \times K_0 \subseteq G_1 \times K_1 \subseteq \ldots \subseteq G_t \times K_t = G \times K$$

is a σ-Sylow tower for $G \times K$. Therefore, T_σ is a strictly homomorphism-invariant property, and the result now follows. ▯

We have already commented that a supersolvable group G has a σ-Sylow tower for σ the natural descending order of a set Σ of primes containing $\pi(G)$. The following corollary now is an immediate consequence of the preceding theorem and Theorem 2.32.

<u>Corollary 2.36</u>: If the subcofactor $H/\mathrm{scor}_G H$ of each proper self-normalizing subgroup H of G is supersolvable, then G is solvable. Moreover, $G/F(G)$ is supersolvable and $G/F_2(G)$ is abelian; thus, $\int(G') \leqq 2$, $\int(G) \leqq 3$.

It might be mentioned that Rose in [17] has shown that, in comparison with Doerk's result (Theorem 2.30), for all $n > 1$, there exists a group G such that $|\pi(G)| = n$, G is not supersolvable, but every self-normalizing proper subgroup of G is cyclic. Thus, assuming that G is not supersolvable, or more generally, not σ-Sylow-towered, in these results imposes no bounds on the number of prime factors of $|G|$.

As we have already seen, the group G in Example 2.17, constructed by extending the simple group of order 168 by an automorphism of order 2, is not solvable but has every proper abnormal subgroup supersolvable. One cannot, therefore, replace "self-normalizing" by "abnormal" in the three preceding results without imposing some additional condition. Rose in [16] has shown, however, that the following is true.

<u>Theorem 2.37</u>: Let σ be a fixed ordering of a set Σ of primes containing $\pi(G)$. If every proper abnormal subgroup H of G has a σ-Sylow

tower and the 2-Sylow subgroups of G are abelian, then G is solvable.

<u>Theorem</u> <u>2.38</u>: Let $\sigma = (p_1, p_2, \ldots, p_n)$ be a fixed ordering of the set $\{p_1, p_2, \ldots, p_n\}$ of primes containing $\pi(G)$. If the subcofactor $H/\mathrm{scor}_G H$ of each proper abnormal subgroup H of G has a σ-Sylow tower and the 2-Sylow subgroups of G are abelian, then G is solvable. Moreover, $G/F(G)$ has a σ-Sylow tower.

<u>Proof</u>: The last statement follows as in the proof of Theorem 2.35 so that only the solvability of G requires proof. For this, we suppose that it is not solvable, and let G be a minimal counterexample. Then G has no proper nontrivial solvable normal subgroup. For if K were such a subgroup, then by the minimality of G, G/K would be solvable, since it is easily checked that the hypotheses hold for G/K. Since K is solvable, this would mean that G is solvable. Thus, G has no nontrivial normal solvable subgroup and hence, from Lemma 2.19, no nontrivial subnormal solvable subgroup.

Now define the integer $r \leqq n$ by the following conditions: There exists a normal chain $G_r \subseteq G_{r+1} \subseteq \cdots \subseteq G_n = G$ such that G_r is not p_r-nilpotent; and in case $r < n$, G_i/G_{i-1} is isomorphic to a p_i-Sylow subgroup of G for each $i = r + 1, \ldots, n$. Since G is not solvable and groups having a σ-Sylow tower are solvable, we have $r > 0$, so that $H = G_r \neq \langle 1 \rangle$. There are two possibilities: (1) $p_r = 2$, and (2) p_r is odd.

Case 1: $p_r = 2$.—Let P be a 2-Sylow subgroup of $H = G_r$. Then, since 2 does not divide $|G:H|$, P is a 2-Sylow subgroup of G. Thus, $N = N_G(P)$ is abnormal in G, and $N \neq G$ since G has no nontrivial normal solvable subgroups; by hypothesis, therefore, $N/\mathrm{scor}_G N$ has a σ-Sylow tower. Now, since $P \lhd N$, we have $P \cap \mathrm{scor}_G N \lhd \mathrm{scor}_G N \lhd \lhd G$ so that $P \cap \mathrm{scor}_G N \lhd \lhd G$. Since G has no nontrivial subnormal solvable subgroups, $P \cap \mathrm{scor}_G N = \langle 1 \rangle$, that is, $\mathrm{scor}_G N$ is of odd order, hence is solvable by the Feit-Thompson Theorem. Again, since $\mathrm{scor}_G N \lhd \lhd G$ and G has no nontrivial subnormal solvable subgroups, we have $\mathrm{scor}_G N = \langle 1 \rangle$. Therefore, $N = N/\mathrm{scor}_G N$ has a σ-Sylow tower, and thus so also does its subgroup $N_H(P) = N \cap H$.

Now, since $p_{r+1}, \ldots, p_n$ do not divide $|H|$, it follows that $N_H(P)$ has a $(p_1, \ldots, p_r)$-Sylow tower; in particular, $N_H(P)$ is p_r-nilpotent, say with normal p_r-complement (= 2-complement) T. Since both P and T are normal in $N_H(P)$, we have $N_H(P) = P \times T$; consequently, T centralizes P. But P is abelian by hypothesis so that $P \subseteq C_G(P)$. It now follows that $P \subseteq Z(N_H(P))$, and hence, by Theorem 2.18, $H = G_r$ is p_r-nilpotent. This, however, contradicts the choice of r.

Case 2: p_r is odd.—Let P be a p_r-Sylow subgroup of $H = G_r$ and thus, as in Case 1, a p_r-Sylow subgroup of G also. Let $J(P)$ be the Thompson subgroup of P, as defined in Def. 2.11, and let $\tilde{\tilde{N}} = N_G(Z(J(P)))$. Now $N_G(P)$ is abnormal in G; and $Z(J(P)) \text{ char } \leqq P \lhd N_G(P)$ implies that $Z(J(P))$ is normal in $N_G(P)$ so that $N_G(P) \subseteq \tilde{\tilde{N}}$. Consequently, $\tilde{\tilde{N}}$ also is abnormal in G. And since G has no nontrivial normal solvable subgroups, we have $\tilde{\tilde{N}} \neq G$. By hypothesis, therefore, $\tilde{\tilde{N}}/\mathrm{scor}_G \tilde{\tilde{N}}$ has a σ-Sylow tower, and hence so also does its subgroup $(\tilde{\tilde{N}}_H)(\mathrm{scor}_G \tilde{\tilde{N}})/\mathrm{scor}_G \tilde{\tilde{N}}$ where $\tilde{\tilde{N}}_H = N_H(Z(J(P))) = \tilde{\tilde{N}} \cap H$. It follows that $\tilde{\tilde{N}}_H / \tilde{\tilde{N}}_H \cap \mathrm{scor}_G \tilde{\tilde{N}}$, being isomorphic to $(\tilde{\tilde{N}}_H)(\mathrm{scor}_G \tilde{\tilde{N}})/\mathrm{scor}_G \tilde{\tilde{N}}$, also has a σ-Sylow tower.

Now, $p_{r+1}, \ldots, p_n$ do not divide $|H|$ and thus do not divide $|\tilde{\tilde{N}}_H|$. Consequently, $\tilde{\tilde{N}}_H / \tilde{\tilde{N}}_H \cap \mathrm{scor}_G \tilde{\tilde{N}}$ has a $(p_1, \ldots, p_r)$-Sylow tower. In particular, $\tilde{\tilde{N}}_H / \tilde{\tilde{N}}_H \cap \mathrm{scor}_G \tilde{\tilde{N}}$ is p_r-nilpotent.

Suppose now that $P_1 = P \cap \mathrm{scor}_G \tilde{\tilde{N}} \neq \langle 1 \rangle$. Then since $\mathrm{scor}_G \tilde{\tilde{N}} \lhd \tilde{\tilde{N}}$, we have $\langle 1 \rangle \neq P_1 \lhd P$ so that P_1 must intersect $Z(P)$ nontrivially. Since $Z(P) \subseteq Z(J(P))$ by the note following Definition 2.11, this implies that $P_2 = P_1 \cap Z(J(P))$ is also nontrivial. Therefore,

$$\langle 1 \rangle \neq P_2 = P_1 \cap Z(J(P)) = P \cap \mathrm{scor}_G \tilde{\tilde{N}} \cap Z(J(P)) = Z(J(P)) \cap \mathrm{scor}_G \tilde{\tilde{N}}.$$

But since $Z(J(P)) \lhd \tilde{\tilde{N}}$, we have $\langle 1 \rangle \neq P_2 = Z(J(P)) \cap \mathrm{scor}_G \tilde{\tilde{N}} \lhd \mathrm{scor}_G \tilde{\tilde{N}} \lhd \lhd G$ so that $\langle 1 \rangle \neq P_2 \lhd \lhd G$; and this contradicts G having no nontrivial subnormal solvable subgroups.

Thus $P \cap \mathrm{scor}_G \tilde{\tilde{N}} = \langle 1 \rangle$ so that $\mathrm{scor}_G \tilde{\tilde{N}}$ is a p_r'-group, and hence so also is $\tilde{\tilde{N}}_H \cap \mathrm{scor}_G \tilde{\tilde{N}}$. But since $\tilde{\tilde{N}}_H / \tilde{\tilde{N}}_H \cap \mathrm{scor}_G \tilde{\tilde{N}}$ is p_r-nilpotent, this implies that $\tilde{\tilde{N}}_H = N_H(Z(J(P)))$ is also p_r-nilpotent. By the Glaubermann-Thompson Theorem (Thm. 2.22), it follows that $H = G_r$ is p_r-nilpotent, which again contradicts the choice of r.

Each case, therefore, leads to a contradiction; and we conclude that no such minimal counterexample can exist. $\square$

From this theorem and Theorem 2.32, we have the following corollary.

<u>Corollary 2.39</u>: If the subcofactor $H/\mathrm{scor}_G H$ of each proper abnormal subgroup H of G is supersolvable and the 2-Sylow subgroups of G are abelian, then G is solvable. Moreover, $G/F(G)$ is supersolvable and $G/F_2(G)$ is abelian; thus, $\ell(G') \leq 2$ and $\ell(G) \leq 3$.

Rose has also established in [17] the following result:

<u>Thoerem 2.40</u>: If every proper abnormal subgroup H of G is supersolvable and the abnormal maximal subgroups have prime-power index, then G is solvable.

Here again it is sufficient to require only that the subcofactors of the abnormal subgroups of G be supersolvable. To establish this, we need the following lemmas, the first of which is due to Gaschütz [7].

Lemma 2.41: For a given group G, let $\Gamma(G)$ be the intersection of all the abnormal maximal subgroups of G, and $\Phi(G)$ the Frattini subgroup of G. Then $\Gamma(G)/\Phi(G) = Z(G/\Phi(G))$; in particular, $\Gamma(G)$ is a normal nilpotent subgroup of G.

Lemma 2.42: Suppose that G is a simple group and that G = HK where H and K are proper subgroups of G. Then $\mathrm{cor}_H(H \cap K) = \mathrm{cor}_K(H \cap K) = \langle 1 \rangle$.

Proof: Let $C = \mathrm{cor}_H(H \cap K)$. Then,

$$C^G = \langle g^{-1}Cg \mid g \in G \rangle$$
$$= \langle (hk)^{-1}C(hk) \mid h \in H,\ k \in K \rangle$$
$$= \langle k^{-1}Ck \mid k \in K \rangle \qquad (\text{since } C \lhd H)$$

so that $C^G \subseteq K \subsetneqq G$. Thus, C^G is a proper normal subgroup of G. Since G is simple, we have $C^G = \langle 1 \rangle$. Therefore, $C = \langle 1 \rangle$ also. Similarly, $\mathrm{cor}_K(H \cap K) = \langle 1 \rangle$. ▯

Theorem 2.43: If the subcofactor $H/\mathrm{scor}_G H$ of each proper abnormal subgroup H of G is supersolvable and the abnormal maximal subgroups of G have prime-power index, then G is solvable. Moreover, $G/F(G)$ is supersolvable and $G/F_2(G)$ is abelian; thus, $\int(G') \leqq 2$ and $\int(G) \leqq 3$.

Proof: The last part follows as before, and only the solvability of G requires proof. For this, we proceed by induction on $|G|$. If G is simple, then all maximal subgroups of G are abnormal in G and have subnormal core 1, hence are supersolvable by hypothesis, so that G is solvable by Theorem 2.30 or the preceding theorem.

Thus, suppose that G is not simple, and let M be a minimal normal subgroup of G. Now, if $\overline{H} = H/M$ is any proper abnormal subgroup of $\overline{G} = G/M$, then clearly H is a proper abnormal subgroup of G; by hypothesis, $H/\mathrm{scor}_G H$ is supersolvable, and hence, by Lemma 1.4, so also is $\overline{H}/\mathrm{scor}_{\overline{G}}(\overline{H}) \cong H/\mathrm{scor}_G H$. And if $\overline{S} = S/M$ is any abnormal maximal subgroup of $\overline{G}$, then S is an abnormal maximal subgroup of G so that $|\overline{G}:\overline{S}| = |G:S|$ = a power of a prime. The hypotheses therefore hold for G/M; consequently, G/M is solvable by induction. If G has another minimal normal subgroup $M^* \neq M$, then G/M^* is likewise solvable by induction, hence so also is $G/M \times G/M^*$. And since $G = G/M \cap M^*$ is isomorphically embedded in $G/M \times G/M^*$, it follows that G also is solvable.

We may assume, therefore, that M is the unique minimal normal subgroup of G, and we need only show that M is solvable. We assume that it is not, and will show that this leads to a contradiction.

So suppose that M is not solvable. Then $M = M_1 \times \ldots \times M_k$, where the M_i are isomorphic simple nonabelian groups. From Lemma 2.41, $\Gamma(G)$ = the intersection of all abnormal maximal subgroups of G is a normal nilpotent subgroup of G. Since M is not solvable, we have $M \not\subseteq \Gamma(G)$, from which it follows that there exists an abnormal maximal subgroup S of G not containing M. By the uniqueness of M, we have $\mathrm{scor}_G S = \mathrm{cor}_G S = \langle 1 \rangle$, and thus, G possesses maximal subgroups of core 1.

Suppose now that there exists only one conjugacy class $\mathcal{C}$ of maximal subgroups of core 1. Then, by hypothesis, for some power p^a of a prime p, $|G:S| = p^a$ for all $S \in \mathcal{C}$. Since $M \not\subseteq S$ for $S \in \mathcal{C}$, we have $MS = G$ so that $p^a = |G:S| = |MS:S| = |M:M \cap S|$, and hence $p \mid |M|$. Let P be a p-Sylow subgroup of M; then, $< 1 > \neq P \lneq M$ since M is not solvable and $p \mid |M|$. Now, from the Frattini argument, $G = MN_G(P)$; and by the minimality of M, $N_G(P) \neq G$. Thus $N_G(P) \subseteq T$ for some maximal subgroup T of G. Since $G = MN_G(P) = MT$, we have $M \not\subseteq T$, hence $cor_G T = < 1 >$ by the uniqueness of M. It follows that $T \in \mathcal{C}$, so that $p^a = |G:T| = |MT:T| = |M:M \cap T|$. This, however, contradicts the fact that since $P \subseteq M \cap N_G(P) \subseteq M \cap T$, $p \nmid |M:M \cap T|$. Therefore, G has at least two distinct conjugacy classes of maximal subgroups of core 1.

Now let S be a maximal subgroup of G with $cor_G S = scor_G S = < 1 >$, and let P be a p-Sylow subgroup of S, where $p = \max(\pi(S))$. Since $S \ntrianglelefteq G$, S is abnormal in G, and hence, $S = S/scor_G S$ is supersolvable by hypothesis. This means, in particular, that since p is the greatest prime factor of $|S|$, P is normal in S so that $S \subseteq N_G(P)$. And since G has no nontrivial solvable normal subgroups, it follows from the maximality of S that $S = N_G(P)$; also, it is now clear that P is a p-Sylow subgroup of G.

Thus, if S and T are maximal subgroups of G with core 1 and $p = \max(\pi(S)) = \max(\pi(T))$, then $S = N_G(P)$ and $T = N_G(P^*)$ for some p-Sylow subgroups P and P^* of G. Since P and P^* are conjugate in G, so also are S and T. Therefore, if S is any maximal subgroup of G with core 1 and $p = \max(\pi(S))$, then the conjugacy class of $S = \mathcal{C}(p) = \{N_G(P) \mid P \text{ a p-Sylow subgroup of } G\}$.

Let now $\mathcal{C}(p_1)$ and $\mathcal{C}(p_2)$ be two such conjugacy classes of maximal subgroups of core 1, with $p_1 > p_2$. Then $p_1 = p$ = the greatest prime factor of $|G|$. For if $T \in \mathcal{C}(p_2)$, then $p_2 = \max(\pi(T))$; and since $p \geqq p_1 > p_2$, this implies that both p and p_1 divide $|G:T|$. By hypothesis, $|G:T|$ is a power of a prime, and thus $p = p_1$.

Let $S \in \mathcal{C}(p_1) = \mathcal{C}(p)$ and $T \in \mathcal{C}(p_2)$. Then $|G:T|$ = power of p and $p \nmid |T|$ so that T is a p-complement of G. Let $q \mid |G:S|$ and U be a q-complement of S, hence of G. (U exists since $S = S/\text{scor}_G S$ is supersolvable, hence solvable.) Then, since $M \lhd G$, $T \cap M$ and $U \cap M$ are p- and q-complements respectively of M; and since M is a direct product of the M_i, $T \cap M_i$ and $U \cap M_i$ are p- and q-complements respectively of M_i, for each $i = 1, \ldots, k$. From this it follows that $M_i = (T \cap M_i)(U \cap M_i)$; and since $T \cap M_i$ and $U \cap M_i$ are supersolvable while M is not, these must be proper subgroups of M_i for each i. By Lemma 2.42, therefore, $A_i = T \cap U \cap M_i$ contains no nontrivial normal subgroup of either $T \cap M_i$ or $U \cap M_i$.

Now, A_i is a Hall $\{p, q\}'$-subgroup of M_i for each i; for p and q do not divide $|A_i|$, and $|A_i| = |T \cap M_i||U \cap M_i|/|M_i|$ so that $|M_i:A_i| = |M_i:T \cap M_i||M_i:U \cap M_i| = p^a q^b$ for some a and b. It follows that A_i is a q-complement of $T \cap M_i$ and a p-complement of $U \cap M_i$.

We assert now that A_i is abelian. For this, let P be a p-Sylow subgroup of $U \cap M_i$. Since $p = \max(\pi(U \cap M_i))$ and $U \cap M_i$ is supersolvable, we have $P \lhd U \cap M_i$ so that $P \subseteq F(U \cap M_i)$, the Fitting subgroup of $U \cap M_i$. If P were properly contained in $F(U \cap M_i)$, then $F(U \cap M_i)$ would have a nontrivial normal p-complement K which, being characteristic in $F(U \cap M_i)$, is then normal in $U \cap M_i$. But then K must be contained in the p-complement A_i of $U \cap M_i$, which contradicts the fact that A_i contains no nontrivial normal subgroup of $U \cap M_i$. Thus, $P = F(U \cap M_i)$; and since $U \cap M_i$ is supersolvable, $(U \cap M_i)' \subseteq F(U \cap M_i) = P$ so that $(U \cap M_i)/P \cong A_i$ is abelian.

Also, we have $q = \max(\pi(T \cap M_i))$. For let q' be the greatest prime factor of $|T \cap M_i|$, and let Q be a q'-Sylow subgroup of $T \cap M_i$; since $T \cap M_i$ is supersolvable, Q is normal in $T \cap M_i$. And since A_i contains no nontrivial normal subgroup of $T \cap M_i$, $Q \not\subseteq A_i$; hence $q' \mid |T \cap M_i : A_i|$, which means that $q' = q$. From this fact that $q = \max(\pi(T \cap M_i))$, it follows, in particular, that $q \neq \min(\pi(M_i))$. For if it were, then $T \cap M_i$ would be a q-group, making M_i a $\{p, q\}$-group and hence solvable by Burnside's Theorem.

Now let $A = T \cap U \cap M$; then A is a Hall $\{p, q\}'$-subgroup of M, and A is abelian as the direct product of the abelian groups A_i. Let $r = \min(\pi(M))$; by what we have just shown, $p > q > r$. Let R be an r-Sylow subgroup of A, hence of M; R is abelian. We claim that $N = N_G(R)$ is a proper abnormal subgroup of G. That $N \neq G$ is clear, since G contains no nontrivial solvable normal subgroups. Now, since $M \lhd G$, $R = R^* \cap M$ for some r-Sylow subgroup R^* of G. If $x \in N_G(R^*)$, then $R^x = (R^*)^x \cap M^x = R^* \cap M = R$, thus $x \in N_G(R)$. This shows then that $N_G(R^*) \subseteq N_G(R)$; and since $N_G(R^*) \rtimes\!\!\lhd\, G$, $N_G(R)$ also is abnormal in G.

By hypothesis, therefore, $N/\mathrm{scor}_G N$ is supersolvable, and thus so also is its subgroup $(N_M)(\mathrm{scor}_G N)/\mathrm{scor}_G N$ where $N_M = N_M(R) = M \cap N$. Hence, $N_M/N_M \cap \mathrm{scor}_G N \cong (N_M)(\mathrm{scor}_G N)/\mathrm{scor}_G N$ is supersolvable; and since $r = \min(\pi(M))$, $N_M/N_M \cap \mathrm{scor}_G N$ is r-nilpotent, say with normal r-complement $L/N_M \cap \mathrm{scor}_G N$.

Now, since R is normal in $N = N_G(R)$, we have $R \cap \mathrm{scor}_G N \lhd \mathrm{scor}_G N$; and since $\mathrm{scor}_G N \lhd \lhd G$, it follows that $R \cap \mathrm{scor}_G N \lhd \lhd G$. Because G has no nontrivial solvable normal subgroups and hence, by Lemma 2.19, no nontrivial solvable subnormal subgroups, we have $R \cap \mathrm{scor}_G N = \langle 1 \rangle$.

This means then that $N_M \cap \mathrm{scor}_G N$ is an r'-group, and thus that L is a normal r-complement of $N_M = N_M(R)$. It follows that $N_M(R) = R \times L$ so that L centralizes R. And since R is abelian, $R \subseteq C_G(R)$; hence $R \subseteq Z(N_M(R))$. But, by Theorem 2.18, this implies that M is r-nilpotent, and thus has a proper characteristic subgroup, which contradicts the minimality of M.

Therefore, M is solvable, and the result now follows. ▯

As an addition to Corollaries 2.36, 2.39, and the preceding theorem, we have the following result.

Theorem 2.44: Let G be a finite nonsupersolvable group having all proper normal subgroups supersolvable. Suppose also that one of the following two conditions holds:

(a) The subcofactors of the proper self-normalizing subgroups of G are supersolvable.

(b) The subcofactors of the proper abnormal subgroups of G are supersolvable and either the 2-Sylow subgroups of G are abelian or the abnormal maximal subgroups of G have prime-power index.

Then the conclusions of Theorem 2.43 hold, and G has a normal p-Sylow subgroup for p the least or largest prime factor of $|G|$.

Proof: Let $p = \min(\pi(G))$. Then the proper normal subgroups and the subcofactors of the self-normalizing (abnormal) subgroups of G are p-nilpotent. If G is not p-nilpotent, then by Theorem 2.20 or 2.23, G has a normal p-Sylow subgroup. On the other hand, if G is p-nilpotent, say with normal p-complement T, then T is, by hypothesis, supersolvable. Thus if q is the largest prime dividing $|G|$, hence the greatest prime factor of $|T|$, T has a normal q-Sylow subgroup Q. Then Q, being characteristic in T, is normal in G; and since $|G:T|$ is prime to q, Q is a normal q-Sylow subgroup of G. ▯

CHAPTER THREE

THE INFLUENCE ON A GROUP OF THE OUTER COFACTORS OF ITS SUBGROUPS

3.1 Introduction; Definitions and Basic Properties

For H a proper subgroup of a given finite group G, $cor_G H$ was defined as the maximal G-normal subgroup contained in H. In the preceding chapter we have considered the effect on G of conditions imposed on $cor_G H$ and $cof_G H = H/cor_G H$ (or $scof_G H$), where H ranges over a certain class of proper subgroups of G. Now, one might hope to be able to "dualize" some of the results obtained. Thus, we consider for a given proper subgroup H of G, those subgroups which are outside H, or at least not contained in H, and which are in some sense minimal with respect to the normal structure of G. Following basically the ideas suggested by Deskins in [5], we make the following definitions.

Definition 3.1: For H a proper subgroup of a finite group G we let

> $\mathcal{O}_G(H)$ denote the collection of subgroups C of G which satisfy the following conditions:
>
> (i) $C \not\subseteq H$;
>
> (ii) each proper G-normal subgroup of C is contained in H.

Notice that if $C \in \mathcal{O}_G(H)$ and $C \triangleleft G$, then C is minimal with respect to being normal in G and not contained in H; thus, in a sense, we have dualized the notion of the core of H.

In this "outer family" $\mathcal{O}_G(H)$ of H, we single out certain subcollections as given in the following definition.

Definition 3.2: For H a proper subgroup of a finite group G, we define:

(1) $\mathcal{O}_{\lhd G}(H) = \{C \in \mathcal{O}_G(H) \mid C \lhd G\}$;

(2) $\mathcal{O}_{\ntriangleleft G}(H) = \{C \in \mathcal{O}_G(H) \mid C \ntriangleleft G\}$;

(3) $\mathcal{O}_{(sn)G}(H) = \{C \in \mathcal{O}_{\ntriangleleft G}(H) \mid C$ is self-normalizing in $G\}$;

(4) $\mathcal{O}_{\rtimes G}(H) = \{C \in \mathcal{O}_{\ntriangleleft G}(H) \mid C$ is abnormal in $G\}$.

Lemma 3.1: Let H be a proper subgroup of a given finite group G, $C \in \mathcal{O}_G(H)$, and D the maximal G-normal proper subgroup of C. Then:

(i) $D = \mathrm{cor}_G(C \cap H)$.

(ii) If $C \lhd G$, then $D = \mathrm{cor}_G(C \cap H) = C \cap \mathrm{cor}_G H$.

(iii) If $C \ntriangleleft G$, then $D = \mathrm{cor}_G(C \cap H) = \mathrm{cor}_G C$.

Proof: (i) Since $D \lhd G$ and D is properly contained in C, we have, by definition of $\mathcal{O}_G(H)$, $D \subseteq H$. Thus, $D \subseteq C \cap H$, and hence $D \subseteq \mathrm{cor}_G(C \cap H)$. On the other hand, since $\mathrm{cor}_G(C \cap H) \lhd G$ and $\mathrm{cor}_G(C \cap H)$ is properly contained in C (since $C \cap H \neq C$), we have from the maximality of D that $\mathrm{cor}_G(C \cap H) \subseteq D$. Therefore, the equality in (i) holds.

(ii) For $C \lhd G$, we have that $C \cap \mathrm{cor}_G H$ is normal in G and is contained in $C \cap H$; thus $C \cap \mathrm{cor}_G H \subseteq \mathrm{cor}_G(C \cap H)$. But clearly, $\mathrm{cor}_G(C \cap H) \subseteq C$ since $C \cap H \subseteq C$; and since $\mathrm{cor}_G(C \cap H)$ is normal in G and is contained in $C \cap H \subseteq H$, we have $\mathrm{cor}_G(C \cap H) \subseteq \mathrm{cor}_G H$ so that $\mathrm{cor}_G(C \cap H) \subseteq C \cap \mathrm{cor}_G H$. Therefore, $\mathrm{cor}_G(C \cap H) = C \cap \mathrm{cor}_G H$ for $C \lhd G$.

(iii) Since $C \cap H \subseteq C$, the inclusion $\mathrm{cor}_G(C \cap H) \subseteq \mathrm{cor}_G C$ is immediate. Now, since $C \ntriangleleft G$, $\mathrm{cor}_G C$ is properly contained in C; thus by the maximality of D, $\mathrm{cor}_G C \subseteq D = \mathrm{cor}_G(C \cap H)$. Therefore, $\mathrm{cor}_G(C \cap H) = \mathrm{cor}_G C$ for $C \ntriangleleft G$. ▯

Using the properties established in the preceding lemma, we can now make the following definitions of the various outer cofactors of a subgroup H.

Definition 3.3: Let H be a proper subgroup of a finite group G. For $C \in \mathcal{O}_G(H)$, we call $C/\mathrm{cor}_G(C \cap H)$ an *outer cofactor of H in G*. More precisely, for $C \in \mathcal{O}_{\lhd G}(H)$, we call $C/\mathrm{cor}_G(C \cap H) = C/C \cap \mathrm{cor}_G H$ a *normal outer cofactor of H*. If $C \in \mathcal{O}_{\not\lhd G}(H)$, we will say that $C/\mathrm{cor}_G(C \cap H) = C/\mathrm{cor}_G C$ is a *nonnormal outer cofactor of H*; in particular, if $C \in \mathcal{O}_{(sn)G}(H)$, we will say that $C/\mathrm{cor}_G C$ is a *self-normalizing outer cofactor of H*, and if $C \in \mathcal{O}_{\bowtie G}(H)$, $C/\mathrm{cor}_G C$ is an *abnormal outer cofactor of H*.

3.2 Influence on a Group of the Nonnormal Outer Cofactors of Subgroups

In this section, our goal is to parallel the results of the preceding chapter by investigating the properties of a finite group G which arise from conditions imposed on the self-normalizing (or abnormal) outer cofactors of maximal subgroups of G. One such result is given by Deskins in [5]; with the terminology and notation that we have adopted, it can be stated as follows.

Theorem 3.2: If the finite group G contains a maximal subgroup S which is supersolvable and if for each $C \in \mathcal{O}_G(S)$ with $C \not\lhd G$ or $C \cap S \neq \langle 1 \rangle$, $C/\mathrm{cor}_G(C \cap S)$ is supersolvable, then G is solvable.

We will now establish two general theorems from which results parallel to those of the preceding chapter are immediate corollaries. In these two theorems, we shall assume that the trivial group is always a θ-group, and hence, in particular, that θ-groups do exist.

Theorem 3.3: Let θ be a subgroup-inherited homomorphism-invariant property. If the finite group G has a maximal subgroup S such that S and all its $\left\{\begin{matrix}(1)\ \text{nonnormal}\\(2)\ \text{self-normalizing}\\(3)\ \text{abnormal}\end{matrix}\right\}$ outer cofactors are θ-groups, then $\text{cof}_G H = H/\text{cor}_G H$ is a θ-group for all proper $\left\{\begin{matrix}(1)\ \text{nonnormal}\\(2)\ \text{self-normalizing}\\(3)\ \text{abnormal}\end{matrix}\right\}$ subgroups H of G.

Proof: Let H be any proper (k)-subgroup of G, where k = 1, 2, or 3 (that is, (k) denotes one of the three properties (1) nonnormal, (2) self-normalizing, or (3) abnormal). Now, if $H \subseteq S$, then trivially $H/\text{cor}_G H$ is a θ-group since S is a θ-group and θ is subgroup-inherited and homomorphism-invariant. Thus, suppose $H \not\subseteq S$. If $\text{cor}_G H \subseteq S$, then H is an element of $\mathcal{O}_{\ntriangleleft G}(S)$, $\mathcal{O}_{(sn)G}(S)$, or $\mathcal{O}_{\rtimes G}(S)$ according as k = 1, 2, or 3 so that, by hypothesis, $H/\text{cor}_G H$ is a θ-group. If $\text{cor}_G H \not\subseteq S$, then $S\,\text{cor}_G H = G$ by the maximality of S. In this case, since S is a θ-group and θ is homomorphism-invariant, $G/\text{cor}_G H = S\,\text{cor}_G H/\text{cor}_G H$ which is isomorphic to $S/S \cap \text{cor}_G H$ also is a θ-group; hence, since θ is subgroup-inherited, $H/\text{cor}_G H$ is a θ-group. $\square$

Note: The proof of the preceding theorem also shows that if S and all its (k)-outer cofactors $C/\text{cor}_G C$, with C a maximal subgroup of G, are θ-groups, then $H/\text{cor}_G H$ is a θ-group for all (k)-maximal subgroups of G.

The following results are now immediate consequences of the preceding theorem or the note, Lemma 1.2, and the corresponding results of Chapter 2. Part (b)-(vi) strengthens Theorem 3.2 by removing the condition "or $C \cap S \neq \langle 1 \rangle$" and by giving information about the Fitting lengths of G' and G. These results also show that there is nothing especially significant about the condition of supersolvability imposed in Theorem 3.2, but that it can be replaced by a variety of other conditions.

Corollary 3.4: Suppose that the finite group G has a maximal subgroup S for which one of the following nine conditions holds:

(a) S and all its abnormal outer cofactors $T/\mathrm{cor}_G T$ with T a maximal subgroup of G are

(i) nilpotent.

(ii) nilpotent of class $\leqq$ n.

(iii) solvable of derived length $\leqq$ n.

(b) S and all its self-normalizing outer cofactors are

(iv) p-nilpotent.

(v) σ-Sylow-towered for σ some fixed ordering of a set Σ of primes containing $\pi(G)$.

(vi) supersolvable.

(c) S and all its abnormal outer cofactors are

(vii) p-nilpotent, and either p is odd or the p-Sylow subgroups of G are abelian.

(viii) σ-Sylow-towered for σ some fixed ordering of a set Σ of primes containing $\pi(G)$, and the 2-Sylow subgroups of G are abelian.

(ix) supersolvable, and either the 2-Sylow subgroups of G are abelian or the abnormal maximal subgroups of G all have prime-power index.

Then, in the respective cases, the following hold:

(a) (i) G is solvable with $G/F(G)$ nilpotent.

(ii) G is solvable with $\gamma_n(G)$ nilpotent (and the other conclusions of Theorem 2.7 hold).

(iii) If G is solvable, then $G^{(n)}$ is nilpotent (and the other conclusions of Theorem 2.9 hold).

(b) (iv) G has a normal p-subgroup P_0 (which may be trivial) such that G/P_0 is p-nilpotent; in particular, G is p-solvable of p-length $\leqq 2$.

(v) G is solvable with $G/F(G)$ σ-Sylow-towered.

(vi) G is solvable with $G/F(G)$ supersolvable and $G/F_2(G)$ abelian; thus $\mathfrak{f}(G') \leqq 2$, $\mathfrak{f}(G) \leqq 3$.

(c) (vii) Same as (iv).

(viii) Same as (v).

(ix) Same as (vi).

Theorem 3.5: Let θ be a subgroup-inherited homomorphism-invariant property. Then the following are equivalent:

(a) For all abnormal maximal subgroups S of G, the $\left\{\begin{array}{l}(1) \text{ self-normalizing} \\ (2) \text{ abnormal}\end{array}\right\}$ outer cofactors of S are θ-groups.

(b) $H/\mathrm{cor}_G H$ is a θ-group for all $\left\{\begin{array}{l}(1) \text{ self-normalizing} \\ (2) \text{ abnormal}\end{array}\right\}$ subgroups H of G.

Proof: (b) $\rightarrow$ (a) is immediate.

(a) $\rightarrow$ (b): Suppose this to be false, and let G be a minimal counterexample. Then there exists some proper (k)-subgroup $H \neq \langle 1 \rangle$ of G such that $H/\mathrm{cor}_G H$ is not a θ-group, where k = 1 or 2.

Suppose first that $\mathrm{cor}_G H = \langle 1 \rangle$. Then $H \not\subseteq \Gamma(G)$ = the intersection of all abnormal maximal subgroups of G. For, by Lemma 2.41, $\Gamma(G)$ is a normal nilpotent subgroup of G, and if $H \subseteq \Gamma(G)$, then $H \lhd \lhd \Gamma(G) \lhd G$, hence $H \lhd \lhd G$; however, if H is a (k)-group, it cannot be subnormal

in G. Thus, $H \nsubseteq \Gamma(G)$, and there exists an abnormal maximal subgroup S of G not containing H. Since $cor_G H = \langle 1 \rangle \subseteq S$, H belongs to $\mathcal{O}_{(sn)G}(S)$ or $\mathcal{O}_{\rtimes G}(S)$ according as k = 1 or 2, so that $H/cor_G H$ is a θ-group by hypothesis. This, however, contradicts the choice of H.

Suppose now that $cor_G H \neq \langle 1 \rangle$ and consider $\bar{G} = G/cor_G H$. We assert that (a) holds for $\bar{G}$. For this, let $\bar{S} = S/cor_G H$ be any abnormal maximal subgroup of $\bar{G}$, and let $\bar{C} = C/cor_G H$ be any element of $\mathcal{O}_{(sn)\bar{G}}(\bar{S})$ or $\mathcal{O}_{\rtimes \bar{G}}(\bar{S})$ according as k = 1 or 2. Then S is an abnormal maximal subgroup of G, and C is a (k)-subgroup of G; also, $C \nsubseteq S$ since $\bar{C} \nsubseteq \bar{S}$. Now, from Lemma 1.4, we have $cor_{\bar{G}}(\bar{C}) = cor_G C/cor_G H$. Since $cor_{\bar{G}}(\bar{C}) \subseteq \bar{S}$ (from the definition of an outer cofactor of $\bar{S}$), it follows that $cor_G C \subseteq S$, and thus that C belongs to $\mathcal{O}_{(sn)G}(S)$ or $\mathcal{O}_{\rtimes G}(S)$ according as k = 1 or 2. By hypothesis, therefore, $C/cor_G C$ is a θ-group, and hence so also is $\bar{C}/cor_{\bar{G}}(\bar{C}) \cong C/cor_G C$ (by Lemma 1.4).

Since (a) thus holds for $\bar{G}$ and $|\bar{G}| < |G|$, it follows from the minimality of G that (b) also must hold for $\bar{G}$, that is, the cofactors of all proper (k)-subgroups of $\bar{G}$ are θ-groups. In particular, $\bar{H} = H/cor_G H$ is a proper (k)-subgroup of $\bar{G}$ and $cor_{\bar{G}}(\bar{H}) = \langle \bar{1} \rangle$ so that $\bar{H} = \bar{H}/cor_{\bar{G}}(\bar{H})$ is a θ-group. However, this again contradicts the choice of H.

We conclude, therefore, that no such minimal counterexample can exist; and the result now follows. ▯

<u>Note</u>: An obvious modification of the preceding proof shows that the following are equivalent:

(a) For all abnormal maximal subgroups S of G, the (k)-outer cofactors $T/cor_G T$ of S with T a maximal subgroup of G are θ-groups.

(b) $H/\text{cor}_G H$ is a θ-group for all (k)-maximal subgroups H of G.

Corresponding to Corollary 3.4, we have the following results which are direct consequences of the preceding theorem or the note, Lemma 1.2, and the corresponding results of Chapter 2.

Corollary 3.6: Suppose that for each abnormal maximal subgroup S of the finite group G, one of the following nine conditions holds:

(a) The abnormal outer cofactors $T/\text{cor}_G T$ of S with T a maximal subgroup of G are:

(i) nilpotent.

(ii) nilpotent of class $\leq n$.

(iii) solvable of derived length $\leq n$.

(b) The self-normalizing outer cofactors of S are:

(iv) p-nilpotent.

(v) σ-Sylow-towered for σ some fixed ordering of a set Σ of primes containing $\pi(G)$.

(vi) supersolvable.

(c) The abnormal outer cofactors of S are:

(vii) p-nilpotent, and either p is odd or the p-Sylow subgroups of G are abelian.

(viii) σ-Sylow-towered for σ some fixed ordering of a set Σ of primes containing $\pi(G)$, and the 2-Sylow subgroups of G are abelian.

(ix) supersolvable, and either the 2-Sylow subgroups of G are abelian or the abnormal maximal subgroups of G all have prime-power index.

Then, in the respective cases, the conclusions (a)-(i) through (c)-(ix) of Corollary 3.4 hold.

3.3 Influence on a Group of the Normal Outer Cofactors of Subgroups

We turn now to a consideration of the normal outer cofactors of the maximal subgroups of a finite group G and investigate what effect properties imposed on these will have on G. Such an approach is suggested by Deskins in [5]. As mentioned there, while every maximal subgroup of a finite solvable group has prime-power index, the converse is not true, as the simple group of order 168 shows. Deskins then defines the normal index of a maximal subgroup; it is precisely this that must be of prime-power for the group to be solvable.

Let us recall that for a maximal subgroup S of a finite group G, $\mathcal{O}_{\lhd G}(S)$ consists of all those subgroups $H \subseteq G$ which satisfy

(1) $H \not\subseteq S$, that is, $HS = G$,

(2) $H \lhd G$ (where we allow $H = G$), and

(3) $L \subseteq S$, that is, $LS = S$, for all proper G-normal subgroups L of H.

Also, the normal outer cofactors of S are the groups $H/\mathrm{cor}_G(H \cap S) = H/H \cap \mathrm{cor}_G S$ with $H \in \mathcal{O}_{\lhd G}(S)$. Using this terminology and notation, we can state the theorem of Deskins, which makes possible the definition of normal index, as follows.

<u>Theorem 3.7</u>: Let S be a maximal subgroup of the finite group G. Then:

(i) All normal outer cofactors of S have the same order.

(ii) If $|G:S|$ = a power of a prime, then there exists a unique $H \in \mathcal{O}_{\lhd G}(S)$.

<u>Definition 3.4</u>: The <u>normal index</u> of a maximal subgroup S of a finite group G is the order of any normal outer cofactor of S.

The following theorem extends statement (i) of the preceding theorem and also shows that if we impose a condition on one of the

normal outer cofactors of a maximal subgroup S, then, in fact, we are imposing it on all of them.

<u>Theorem 3.8</u>: Let S be a maximal subgroup of a finite group G. Then all the normal outer cofactors of S are isomorphic.

<u>Proof</u>: Let H and K be distinct elements of $\mathcal{O}_{\lhd G}(S)$. We are to show that $H/H \cap \mathrm{cor}_G S$ and $K/K \cap \mathrm{cor}_G S$ are isomorphic.

Case 1: $\mathrm{cor}_G S = \langle 1 \rangle$.—In this case, H and K are minimal normal subgroups of G. For suppose $L \lhd G$ with $L \subsetneq H$. Then, since $H \in \mathcal{O}_{\lhd G}(S)$, we have $L \subseteq S$; and since $L \lhd G$, this means that $L \subseteq \mathrm{cor}_G S = \langle 1 \rangle$, so that $L = \langle 1 \rangle$. Consequently, H contains properly no nontrivial normal subgroup of G, and is therefore minimal normal in G. Similarly, K is shown to be minimal normal in G. It follows then that H and K centralize each other.

Also, we have $H \cap S = K \cap S = \langle 1 \rangle$. For since H is normal in G, $H \cap S$ is normal in S, and thus $S \subseteq N_G(H \cap S)$. Now, K centralizes H, hence centralizes $H \cap S$ so that $K \subseteq N_G(H \cap S)$ also. Consequently, $G = KS \subseteq N_G(H \cap S)$, that is $H \cap S \lhd G$. But since $\mathrm{cor}_G S = \langle 1 \rangle$, this means that $H \cap S = \langle 1 \rangle$. In the same way we obtain $K \cap S = \langle 1 \rangle$.

Now, from Dedekind's Law, $H(HK \cap S) = HK \cap HS = HK \cap G = HK$, and $K(HK \cap S) = HK \cap KS = HK \cap G = HK$ also. It follows that

$$H = H/H \cap K \cong HK/K = K(HK \cap S)/K \cong (HK \cap S)/(HK \cap S \cap K) = HK \cap S,$$

and similarly, $K \cong HK \cap S$. Thus $H/H \cap \mathrm{cor}_G S = H \cong K = K/K \cap \mathrm{cor}_G S$.

Case 2: $\mathrm{cor}_G S \neq \langle 1 \rangle$.—In this case we consider $\overline{G} = G/\mathrm{cor}_G S$, and we let $\overline{S} = S/\mathrm{cor}_G S$, $\overline{H} = H\,\mathrm{cor}_G S/\mathrm{cor}_G S$, and $\overline{K} = K\,\mathrm{cor}_G S/\mathrm{cor}_G S$. We show first that $\overline{H}$ and $\overline{K}$ belong to $\mathcal{O}_{\lhd \overline{G}}(\overline{S})$.

For this, suppose that $\overline{L} = L/\mathrm{cor}_G S$ is a normal subgroup of $\overline{G}$ which is properly contained in $\overline{H}$. Then, since $H \cap L \lhd G$, $H \cap L \subsetneq H$, and

$H \in \mathcal{O}_{\lhd G}(S)$, we have that $H \cap L \subseteq S$, and hence $H \cap L \subseteq \mathrm{cor}_G S$. But since $\mathrm{cor}_G S \subseteq L$, we have $H \cap \mathrm{cor}_G S \subseteq H \cap L$; thus, $H \cap L = H \cap \mathrm{cor}_G S$ and

$$\frac{|H||L|}{|HL|} = |H \cap L| = |H \cap \mathrm{cor}_G S| = \frac{|H||\mathrm{cor}_G S|}{|H\,\mathrm{cor}_G S|}.$$

And since $HL = H(L\,\mathrm{cor}_G S) = (H\,\mathrm{cor}_G S)L = H\,\mathrm{cor}_G S$, it follows that $|L| = |\mathrm{cor}_G S|$ so that $\overline{L} = <\overline{1}>$. This shows that $\overline{H} \in \mathcal{O}_{\lhd \overline{G}}(\overline{S})$; and in a similar manner, one shows that $\overline{K} \in \mathcal{O}_{\lhd \overline{G}}(\overline{S})$.

Now, since $\mathrm{cor}_{\overline{G}}(\overline{S}) = <\overline{1}>$, it follows from Case 1 that $\overline{H} \cong \overline{K}$. Therefore, $H/H \cap \mathrm{cor}_G S \cong \overline{H} \cong \overline{K} \cong K/K \cap \mathrm{cor}_G S$. ▯

Statement (iii) of the following result coincides with statement (ii) of Theorem 3.7. We include a proof of it here since none is given in [5].

Theorem 3.9: Let S be a maximal subgroup of a finite group G. Then:

(i) The normal outer cofactors of S are p-solvable if and only if the normal index of S is either a power of p or is prime to p.

(ii) The normal outer cofactors of S are solvable if and only if the normal index of S is a power of a prime. Then, the normal index of S = the index of S.

(iii) If S has prime-power index, or prime-power normal index, then there exists a unique H in $\mathcal{O}_{\lhd G}(S)$.

Proof: (i) Suppose first that the normal outer cofactors of S are p-solvable, and let $H \in \mathcal{O}_{\lhd G}(S)$. As in the proof of Theorem 3.8, $\overline{H} = H\,\mathrm{cor}_G S/\mathrm{cor}_G S \in \mathcal{O}_{\lhd \overline{G}}(\overline{S})$ where $\overline{S} = S/\mathrm{cor}_G S$ and $\overline{G} = G/\mathrm{cor}_G S$; and since $\mathrm{cor}_{\overline{G}}(\overline{S}) = <\overline{1}>$, $\overline{H}$ is minimal normal in $\overline{G}$. Now, by hypothesis, $H/H \cap \mathrm{cor}_G S$ is p-solvable, hence so also is $\overline{H} \cong H/H \cap \mathrm{cor}_G S$. Thus, $\overline{H}$ is either a p-group or a p'-group so that the normal index of S = $|H/H \cap \mathrm{cor}_G S| = |\overline{H}|$ is either a power of p or is prime to p.

The converse implication is an immediate consequence of the definition of normal index.

(ii) The equivalence of the two statements given here follows from (i) together with the fact that a finite group G is solvable if-f G is p-solvable for all primes p dividing $|G|$. For the second part, let $H \in \mathcal{O}_{\triangleleft G}(S)$ with $H/H \cap \mathrm{cor}_G S$ solvable. For $\bar{G}$, $\bar{S}$, and $\bar{H}$ as above, we have $\bar{H} \in \mathcal{O}_{\triangleleft \bar{G}}(\bar{S})$ and $\bar{H}$ is minimal normal in $\bar{G}$. Since $\bar{H} \cong H/H \cap \mathrm{cor}_G S$, $\bar{H}$ is solvable, hence is elementary abelian. Now $\bar{H} \cap \bar{S} \triangleleft \bar{S}$ since $\bar{H} \triangleleft \bar{G}$; and $\bar{H} \cap \bar{S} \triangleleft \bar{H}$ since $\bar{H}$ is abelian; thus, we have $\bar{H} \cap \bar{S} \triangleleft \bar{H}\bar{S} = \bar{G}$. But $\mathrm{cor}_{\bar{G}}(\bar{S}) = \langle \bar{1} \rangle$ so that $\bar{H} \cap \bar{S} = \langle \bar{1} \rangle$. Therefore, $|G:S| = |\bar{G}:\bar{S}| = |\bar{H}\bar{S}:\bar{S}| = |\bar{H}:\bar{H} \cap \bar{S}| = |\bar{H}| = |H/H \cap \mathrm{cor}_G S|$ = the normal index of S.

(iii) We note first that if $H \in \mathcal{O}_{\triangleleft G}(S)$, then $H/H \cap \mathrm{cor}_G S$ is divisible by $|H:H \cap S| = |HS:S| = |G:S|$, so that the index of S divides the normal index of S. If the normal index of S is a power of a prime, therefore, so also is the index of S. Thus, suppose that S has prime-power index, say $|G:S| = p^a$; and let $H \in \mathcal{O}_{\triangleleft G}(S)$.

Case 1: $\mathrm{cor}_G S = \langle 1 \rangle$.—In this case, H is minimal normal in G, as shown in the proof of Theorem 3.8. If $C_G(H) = \langle 1 \rangle$, then H is the unique element of $\mathcal{O}_{\triangleleft G}(S)$. For if $K \in \mathcal{O}_{\triangleleft G}(S)$ and $H \neq K$, then K also is minimal normal in G so that $H \cap K = \langle 1 \rangle$, and H and K centralize each other; thus $K \subseteq C_G(H) = \langle 1 \rangle$, which is impossible since $K \in \mathcal{O}_{\triangleleft G}(S)$ implies that $K \not\subseteq S$.

So suppose that $C = C_G(H) \neq \langle 1 \rangle$. Since $C \triangleleft G$, we have $C \not\subseteq S$ since $\mathrm{cor}_G S = \langle 1 \rangle$, so that $CS = G$. Then $H \cap S = \langle 1 \rangle$. For since $H \triangleleft G$, we have $H \cap S \triangleleft S$; also, $C \subseteq N_G(H \cap S)$ since C centralizes H, hence centralizes $H \cap S$. Thus $G = CS \subseteq N_G(H \cap S)$, that is $H \cap S \lhd G$. But since $\text{cor}_G S = \langle 1 \rangle$, this means that $H \cap S = \langle 1 \rangle$.

Therefore, $|H| = |H:H \cap S| = |HS:S| = |G:S| = p^a$ so that H is solvable, hence elementary abelian, and $H \subseteq C = C_G(H)$. It follows from this that $C \cap S = \langle 1 \rangle$. For $C \cap S$ is normal in S, and H centralizes C, hence centralizes $C \cap S$; thus $G = HS \subseteq N_G(C \cap S)$, that is, $C \cap S \lhd G$. However, $\text{cor}_G S = \langle 1 \rangle$, so that $C \cap S = \langle 1 \rangle$.

It now follows by Dedekind's Law that

$$H = H(C \cap S) = C \cap HS = C \cap G = C.$$

This then implies that H is the unique element of $\mathcal{O}_{\lhd G}(S)$. For any other $K \in \mathcal{O}_{\lhd G}(S)$ would centralize H as above, which would contradict $H = C_G(H)$.

Case 2: $\text{cor}_G S \neq \langle 1 \rangle$.—In this case, we let $\overline{G} = G/\text{cor}_G S$ and $\overline{S} = S/\text{cor}_G S$. Now let $H \in \mathcal{O}_{\lhd G}(S)$. As before, $\overline{H} = H \,\text{cor}_G S/\text{cor}_G S \in \mathcal{O}_{\lhd \overline{G}}(\overline{S})$. Thus, by Case 1, since $|\overline{G}:\overline{S}| = |G:S| = p^a$ and $\text{cor}_{\overline{G}}(\overline{S}) = \langle \overline{1} \rangle$, $\overline{H}$ is the unique element of $\mathcal{O}_{\lhd \overline{G}}(\overline{S})$.

Now suppose there exists $K \in \mathcal{O}_{\lhd G}(S)$ with $K \neq H$. Then, as before, $\overline{K} = K \,\text{cor}_G S/\text{cor}_G S \in \mathcal{O}_{\lhd \overline{G}}(\overline{S})$. We will show that $\overline{K} \neq \overline{H}$, which contradicts the uniqueness of $\overline{H}$, and thus establishes the result. For this, suppose that $\overline{K} = \overline{H}$ so that $H \,\text{cor}_G S = K \,\text{cor}_G S$. Let $x = hk$, where $h \in H$, $k \in K$, be an arbitrary element of HK. Then $h^{-1}s_1 = ks_2$ for some s_1, s_2 belonging to $\text{cor}_G S$. Hence, $x = hk = s_1 s_2^{-1} \in \text{cor}_G S$. This shows then that $HK \subseteq \text{cor}_G S$. But this is impossible, since $H \in \mathcal{O}_{\lhd G}(S)$ implies that $H \not\subseteq S$. Therefore, $\overline{K} \neq \overline{H}$ as we wished to show. []

Deskins has shown in [5] the following equivalences.

<u>Theorem 3.10</u>: For G a given finite group, the following are equivalent:

(i) G is solvable.

(ii) Each maximal subgroup of G has prime-power normal index.

(iii) The index and normal index are equal for each maximal subgroup of G.

Our next two theorems are extensions of this result. The following lemma will prove useful in establishing not only these two theorems but later results as well.

Lemma 3.11: Let M be a normal subgroup of the given group G and S/M a maximal subgroup of G/M. Then each normal outer cofactor of S/M is isomorphic to every normal outer cofactor of S.

Proof: Let $\bar{G} = G/M$, $\bar{S} = S/M$, and let $\bar{K}/\bar{K} \cap \mathrm{cor}_{\bar{G}}(\bar{S})$ be a normal outer cofactor of $\bar{S}$, where $\bar{K} = K/M$. We have immediately that S is a maximal subgroup of G, $K \lhd G$, and $K \not\subseteq S$. It follows that $L \subseteq K$ for some $L \in \mathcal{O}_{\lhd G}(S)$.

Now, $ML = K$. For $ML \subseteq K$ so that $\overline{ML} = ML/M \subseteq K/M = \bar{K}$; also, $\overline{ML} \lhd \bar{G}$, and $\overline{ML} \not\subseteq \bar{S}$ (since $L \not\subseteq S$). Thus, since $\bar{K} \in \mathcal{O}_{\lhd \bar{G}}(\bar{S})$, we have $\overline{ML} = \bar{K}$, and hence, $ML = K$.

Since $M \subseteq \mathrm{cor}_G S$, it follows that

$$L/L \cap \mathrm{cor}_G S \cong L\,\mathrm{cor}_G S/\mathrm{cor}_G S = LM\,\mathrm{cor}_G S/\mathrm{cor}_G S$$
$$= K\,\mathrm{cor}_G S/\mathrm{cor}_G S \cong K/K \cap \mathrm{cor}_G S;$$

and from Lemmas 3.1 and 1.4, we have

$$K/K \cap \mathrm{cor}_G S = K/\mathrm{cor}_G(K \cap S) \cong \frac{K/M}{\mathrm{cor}_G(K \cap S)/M}$$
$$= \bar{K}/\mathrm{cor}_{\bar{G}}(\bar{K} \cap \bar{S}) = \bar{K}/\bar{K} \cap \mathrm{cor}_{\bar{G}}(\bar{S}).$$

Therefore, the normal outer cofactor $\bar{K}/\bar{K} \cap \mathrm{cor}_{\bar{G}}(\bar{S})$ of $\bar{S} = S/M$ is isomorphic to the normal outer cofactor $L/L \cap \mathrm{cor}_G S$ of S, and hence, by Theorem 3.8, to every normal outer cofactor of S. ▯

Theorem 3.12: For a finite group G, the following are equivalent:

(i) G is p-solvable.

(ii) G has a maximal subgroup S such that S and its normal outer cofactors are p-solvable.

(iii) For each abnormal maximal subgroup S of G, the normal outer cofactors of S are p-solvable.

(iv) For each abnormal maximal subgroup S of G, the normal index of S is either a power of p or is prime to p.

Note: As the proof shows, the word "abnormal" can be omitted in (iii) and (iv).

Proof of Theorem 3.12: (i) → (ii) is trivially true.

(ii) → (i): We use induction on $|G|$. Let S be a maximal subgroup of G which is p-solvable, and let $K \in \mathcal{O}_{\lhd G}(S)$ with $K/K \cap \mathrm{cor}_G S$ p-solvable.

Case 1: $K \cap \mathrm{cor}_G S = \langle 1 \rangle$.—Then K is a minimal normal subgroup of G. For if L is a normal subgroup of G with $L \subsetneqq K$, then since $K \in \mathcal{O}_{\lhd G}(S)$, we have $L \subseteq S$. Thus $L \subseteq K \cap S$; and since $L \lhd G$, this means that $L \subseteq \mathrm{cor}_G(K \cap S) = K \cap \mathrm{cor}_G S = \langle 1 \rangle$, hence $L = \langle 1 \rangle$.

Now, by hypothesis, $K = K/K \cap \mathrm{cor}_G S$ is p-solvable. This implies that since K is minimal normal in G, K is either a p-group or a p'-group. Since $KS = G$, we have $G/K = KS/K \cong S/S \cap K$ is p-solvable, since S is p-solvable. Therefore, since K is either a p-group or a p'-group, G is p-solvable.

Case 2: $K \cap \mathrm{cor}_G S \neq \langle 1 \rangle$.—Let M be a minimal normal subgroup of G contained in $K \cap \mathrm{cor}_G S$. Since $M \subseteq S$, M is p-solvable, and hence is either a p-group or a p'-group.

Now, (ii) holds for G/M. For S/M is a p-solvable maximal subgroup of G/M; and by Lemma 3.11, each normal outer cofactor of S/M is isomorphic to the normal outer cofactors of S, hence is p-solvable. By induction, therefore, G/M is p-solvable. And since M is either a p-group or a p'-group, it follows that G is p-solvable.

(i) → (iii) is clear.

(iii) → (i): If G is simple, then every maximal subgroup of G is abnormal in G. Let S be any such, and let $H \in \mathcal{O}_{\lhd G}(S)$. Then $H \not\subseteq S$, $H \lhd G$, and G simple imply that $H = G$ so that $\mathcal{O}_{\lhd G}(S) = \{G\}$. Therefore, since $\mathrm{cor}_G S = \langle 1 \rangle$, $G = G/G \cap \mathrm{cor}_G S$ is p-solvable.

So suppose that G is not simple and let M be a minimal normal subgroup of G. Then M is p-solvable, hence is either a p-group or a p'-group. For if $M \subseteq \Gamma(G)$ = the intersection of all abnormal maximal subgroups of G, then since $\Gamma(G)$ is nilpotent by Lemma 2.41, M is nilpotent, thus is p-solvable. On the other hand, if $M \not\subseteq \Gamma(G)$, then there exists an abnormal maximal subgroup S of G which does not contain M. Then $M \in \mathcal{O}_{\lhd G}(S)$ so that $M = M/M \cap \mathrm{cor}_G S$ is p-solvable by hypothesis.

Now consider $\bar{G} = G/M$. Then (iii) holds for $\bar{G}$. For let $\bar{S} = S/M$ be any abnormal maximal subgroup of $\bar{G}$, and let $\bar{K} = K/M \in \mathcal{O}_{\lhd \bar{G}}(\bar{S})$; we must show that $\bar{K}/\bar{K} \cap \mathrm{cor}_{\bar{G}}(\bar{S})$ is p-solvable. S is clearly an abnormal maximal subgroup of G; by hypothesis, therefore, all normal outer cofactors of S are p-solvable. By Lemma 3.11, $\bar{K}/\bar{K} \cap \mathrm{cor}_{\bar{G}}(\bar{S})$ is isomorphic to the normal outer cofactors of S, and hence is p-solvable.

The hypotheses thus hold for $\bar{G} = G/M$ so that, by induction, G/M is p-solvable; and since M is either a p-group or a p'-group, it follows that G is p-solvable.

(iv) → (i) is immediate since (iv) clearly implies (iii).

(i) → (iv): Let G be p-solvable, S a maximal subgroup of G, and $K \in \mathcal{O}_{\lhd G}(S)$. To show that $K/K \cap \mathrm{cor}_G S$ is either a p-group or a p'-group, it suffices to show that it is a chief factor of G. And this is immediate. For if $L \lhd G$ with $K \cap \mathrm{cor}_G S \subseteq L \subsetneq K$, then $L \subseteq S$ since $K \in \mathcal{O}_{\lhd G}(S)$, and thus $L \subseteq \mathrm{cor}_G S$ since $L \lhd G$. Therefore $L \subseteq K \cap \mathrm{cor}_G S$ so that $L = K \cap \mathrm{cor}_G S$. ▯

<u>Theorem 3.13</u>: For a finite group G, the following are equivalent:

(i) G is solvable.

(ii) G has a maximal subgroup S such that S and its normal outer cofactors are solvable.

(iii) For each abnormal maximal subgroup S of G, the normal outer cofactors of S are solvable.

(iv) For each abnormal maximal subgroup S of G, the normal index of S is a power of a prime.

(v) For each abnormal maximal subgroup S of G, the normal index of S = the index of S.

(vi) For each abnormal maximal subgroup S of G, $K \cap S$ is normal in G for all $K \in \mathcal{O}_{\lhd G}(S)$.

(vii) For each abnormal maximal subgroup S of G, $K \cap S$ is subnormal in G for all $K \in \mathcal{O}_{\lhd G}(S)$.

<u>Note</u>: As in the preceding theorem, the proof shows that the word "abnormal" can be omitted in (iii)-(vii).

<u>Proof of Theorem 3.13</u>: The equivalence of (i)-(iv) is immediate from Theorem 3.12 and the fact that a finite group G is solvable if-f G is p-solvable for all primes p dividing $|G|$. (i) → (v) follows from Theorem 3.9.

(v) → (i): If G is simple, then all the maximal subgroups of G are abnormal in G. Let S be any such and $K \in \mathcal{O}_{\lhd G}(S)$. Then $K \lhd G$, $K \not\subseteq S$, and G simple imply that K = G, and thus that $\mathcal{O}_{\lhd G}(S) = \{G\}$. Also, $\mathrm{cor}_G S = \langle 1 \rangle$ since G is simple. By hypothesis, $|G:S|$ = the normal index of S = $|G/G \cap \mathrm{cor}_G S| = |G|$ so that $|S| = 1$. Therefore, G has no nontrivial maximal subgroups, hence is cyclic of prime order, and thus is solvable.

So suppose G is not simple, and let M be a minimal normal subgroup of G. Then (v) holds for G/M. For if S/M is an abnormal maximal subgroup of G/M, then S is an abnormal maximal subgroup of G; and by Lemma 3.11, each normal outer cofactor of S/M is isomorphic to the normal outer cofactors of S so that the normal indices of S/M and S are equal. Thus, $|G/M:S/M| = |G:M|$ = the normal index of S = the normal index of S/M.

By induction, therefore, G/M is solvable. If G has a minimal normal subgroup $M^* \neq M$, then G/M^* is likewise solvable by induction, hence so also is $G/M \times G/M^*$. Since $G = G/M \cap M^*$ is isomorphically embedded in $G/M \times G/M^*$, it follows that G also is solvable.

We may assume, therefore, that M is the unique minimal normal subgroup of G; and we need only show that M is solvable. This is clear if $M \subseteq \Phi(G)$; so suppose that $M \not\subseteq \Phi(G)$, and hence that there exists a maximal subgroup of G not containing M. Let the prime p divide $|M|$. Now, if S is maximal in G and $M \not\subseteq S$, then $S \ntriangleleft G$ by the uniqueness of M so that S is abnormal in G; by hypothesis, therefore, since $M \in \mathcal{O}_{\lhd G}(S)$, $|G:S|$ = the normal index of S = $|M/M \cap \mathrm{cor}_G S| = |M|$, so that $p \mid |G:S|$. Thus $M \subseteq \cap \{S \mid S$ is a maximal subgroup of G with $p \nmid |G:S|\}$; and this latter group is a normal solvable subgroup of G (see, for example, Deskins [5]). Consequently, M is solvable, and the result follows.

(v) ↔ (vi): Let S be a maximal subgroup of G and $K \in \mathcal{O}_{\triangleleft G}(S)$. Then,

$$
\begin{aligned}
K \cap S \triangleleft G \quad &\text{if-f} \quad K \cap cor_G S = cor_G(K \cap S) = K \cap S \\
&\text{if-f} \quad |K/K \cap cor_G S| = |K:K \cap S| \ (\text{since } K \cap cor_G S \subseteq K \cap S) \\
&\text{if-f} \quad |K/K \cap cor_G S| = |KS:S| = |G:S| \\
&\text{if-f} \quad \text{normal index of } S = \text{index of } S.
\end{aligned}
$$

(vi) ↔ (vii): We show first that if T is any proper subgroup of G and $K \in \mathcal{O}_{\triangleleft G}(T)$, then $scor_G(K \cap T) = K \cap scor_G T$. For since $K \triangleleft G$ and $scor_G T \triangleleft\triangleleft G$, we have $K \cap scor_G T \triangleleft\triangleleft G$; hence, $K \cap scor_G T$ is contained in $scor_G(K \cap T)$. On the other hand, since $scor_G(K \cap T)$ is contained in T and is subnormal in G, it must be contained in $scor_G T$; consequently, $scor_G(K \cap T) \subseteq K \cap scor_G T$. The equality now follows.

Now let S be a maximal subgroup of G and $K \in \mathcal{O}_{\triangleleft G}(S)$. By what we have just shown, we have, as a consequence of Lemmas 1.3 and 3.1, that $scor_G(K \cap S) = K \cap scor_G S = K \cap cor_G S = cor_G(K \cap S)$. Therefore, since $K \cap S \triangleleft\triangleleft G$ if-f $scor_G(K \cap S) = K \cap S$, it follows that $K \cap S \triangleleft\triangleleft G$ if-f $K \cap S = cor_G(K \cap S)$, which is true if-f $K \cap S$ is normal in G. ▯

Our last two results are mild extensions of the following theorem of Deskins [5].

<u>Theorem 3.14</u>: The intersection of those maximal subgroups of G whose normal indices are divisible by two distinct primes is the largest normal solvable subgroup of G.

(Here, and in the following, the intersection of an empty collection of subgroups is, as usual, understood to be the entire group G.)

Theorem 3.15: Let p be a prime dividing $|G|$, $R_p(G)$ = the largest normal p-solvable subgroup of G, and $\mathcal{M}_p(G)$ = the collection of maximal subgroups of G with normal index divisible by both p and some prime distinct from p. Then:

$$R_p(G) = \cap \{S \mid S \in \mathcal{M}_p(G)\}$$
$$= \cap \{S \mid S \rtimes\!\!\!\lhd G \text{ and } S \in \mathcal{M}_p(G)\}.$$

Note: In view of Theorem 3.9-(i), the collection $\mathcal{M}_p(G)$ could as well be defined as the family of maximal subgroups whose normal outer cofactors are not p-solvable.

Proof of Theorem 3.15: We show (i) $R_p(G) = \cap \{S \mid S \in \mathcal{M}_p(G)\}$, and then (ii) $R_p(G) = \cap \{S \mid S \rtimes\!\!\!\lhd G \text{ and } S \in \mathcal{M}_p(G)\}$.

(i) Let $T_p(G) = \cap \{S \mid S \in \mathcal{M}_p(G)\}$; we wish to show that $R_p(G) = T_p(G)$. We note first that every p-solvable minimal normal subgroup M of G (which is thus either a p-group or a p'-group) is contained in $T_p(G)$. For suppose $M \not\subset T_p(G)$ for such an M; then there exists $S \in \mathcal{M}_p(G)$ with $M \not\subset S$, and thus, $M \not\subset \mathrm{cor}_G S$. Now, $M \in \mathcal{O}_{\lhd G}(S)$ so that the normal index of $S = |M/M \cap \mathrm{cor}_G S| = |M|$, and hence $|M|$ is divisible by both p and some prime $\neq$ p. Thus, M is not p-solvable.

It follows then that if $T_p(G) = <1>$, then $R_p(G) = <1>$ also, so that $T_p(G) = R_p(G)$. We may assume, therefore, that $T_p(G) \neq <1>$. Now, $T_p(G)$ is a normal subgroup of G. For, from the definition of $\mathcal{O}_{\lhd G}(S)$ for S a maximal subgroup of G, if $x \in G$, then $H \in \mathcal{O}_{\lhd G}(S)$ if-f $H \in \mathcal{O}_{\lhd G}(S^x)$; thus, the normal index of S = the normal index of S^x. Therefore, $S \in \mathcal{M}_p(G)$ if-f $S^x \in \mathcal{M}_p(G)$ for each $x \in G$.

Now, let M be a minimal normal subgroup of G contained in $T_p(G)$. We show first that M is p-solvable, hence is either a p-group or a p'-group and is contained in $R_p(G)$. For suppose that M is not p-solvable. Then M is not contained in the nilpotent subgroup $\Gamma(G)$ = the intersection of all abnormal maximal subgroups of G. Thus there exists an abnormal maximal subgroup S of G not containing M. As above, $M \in \mathcal{O}_{\triangleleft G}(S)$ and the normal index of $S = |M|$. Since M is not p-solvable and hence is neither a p-group nor a p'-group, there exists a prime $q \neq p$ such that both p and q divide $|M|$. Then both p and q divide the normal index of S so that $S \in \mathfrak{M}_p(G)$. And since $M \not\subseteq S$, this means that $M \not\subseteq T_p(G)$.

Now, $R_p(G/M) = R_p(G)/M$. For since $R_p(G)$ is a normal p-solvable subgroup of G containing M, we have $R_p(G)/M$ is p-solvable and normal in G/M; thus $R_p(G)/M \subseteq R_p(G/M)$. On the other hand, if $R_p(G/M) = K/M$, then K is normal in G, and K is p-solvable since K/M is p-solvable and M is either a p-group or a p'-group. Thus $K \subseteq R_p(G)$ so that $R_p(G/M) = K/M \subseteq R_p(G)/M$, which establishes the equality.

Also, we have $T_p(G/M) = T_p(G)/M$. For as was shown in the proof of Theorem 3.11, if S/M is a maximal subgroup of G/M, then the normal outer cofactors of S/M are isomorphic to the normal outer cofactors of S and the normal index of S/M = the normal index of S. And since $M \subseteq T_p(G)$, we have $M \subseteq S$ for all $S \in \mathfrak{M}_p(G)$. It follows, therefore, that $S \in \mathfrak{M}_p(G)$ if-f $S/M \in \mathfrak{M}_p(G/M)$. Thus,

$$\mathfrak{M}_p(G/M) = \{S/M \mid S \in \mathfrak{M}_p(G)\},$$

so that

$$T_p(G/M) = \cap\,\{S/M \mid S \in \mathfrak{M}_p(G)\} = [\cap\,\{S \mid S \in \mathfrak{M}_p(G)\}]/M = T_p(G)/M.$$

Now, by induction, $T_p(G/M) = R_p(G/M)$. By what we have just shown, therefore, $T_p(G)/M = R_p(G)/M$, and hence, $T_p(G) = R_p(G)$.

(ii) Let $\tilde{\tilde{T}}_p(G) = \cap\,\{S \mid S \bowtie G \text{ and } S \in \mathfrak{M}_p(G)\}$; we wish to show now that $\tilde{\tilde{T}}_p(G) = R_p(G)$. First, it is immediate from the definitions that $T_p(G) \subseteq \tilde{\tilde{T}}_p(G)$; thus, if $\tilde{\tilde{T}}_p(G) = \langle 1 \rangle$, then $T_p(G) = \langle 1 \rangle$, and from (i), $R_p(G) = \langle 1 \rangle$ so that $\tilde{\tilde{T}}_p(G) = R_p(G)$.

So suppose that $\tilde{\tilde{T}}_p(G) \neq \langle 1 \rangle$. $\tilde{\tilde{T}}_p(G)$ is a normal subgroup of G. For from (i), we have $S \in \mathcal{M}_p(G)$ if-f $S^x \in \mathcal{M}_p(G)$ for each $x \in G$; and from the definition of abnormality, it is clear that $S \bowtie G$ if-f $S^x \bowtie G$ for each $x \in G$.

Now let M be a minimal normal subgroup of G contained in $\tilde{\tilde{T}}_p(G)$. From (i), $R_p(G/M) = R_p(G)/M$. Also, we have $\tilde{\tilde{T}}_p(G/M) = \tilde{\tilde{T}}_p(G)/M$. For letting $\tilde{\mathcal{M}}_p(G) = \{S \mid S \bowtie G \text{ and } S \in \mathcal{M}_p(G)\}$, we have from the proof of (i) that

$$\tilde{\mathcal{M}}_p(G/M) = \{S/M \mid S/M \bowtie G/M \text{ and } S \in \mathcal{M}_p(G)\};$$

and since it is immediate from the definition of abnormality that $S/M \bowtie G/M$ if-f $S \bowtie G$ and $M \subseteq S$, we have

$$\begin{aligned}\tilde{\mathcal{M}}_p(G/M) &= \{S/M \mid S \bowtie G,\ S \in \mathcal{M}_p(G), \text{ and } M \subseteq S\}\\ &= \{S/M \mid S \in \tilde{\mathcal{M}}_p(G) \text{ and } M \subseteq S\}.\end{aligned}$$

But since $M \subseteq \tilde{\tilde{T}}_p(G)$, we have that $M \subseteq S$ for all $S \in \tilde{\mathcal{M}}_p(G)$; thus

$$\tilde{\mathcal{M}}_p(G/M) = \{S/M \mid S \in \tilde{\mathcal{M}}_p(G)\}.$$

Hence,

$$\tilde{\tilde{T}}_p(G/M) = \cap \{S/M \mid S \in \tilde{\mathcal{M}}_p(G)\} = [\cap \{S \mid S \in \tilde{\mathcal{M}}_p(G)\}]/M = \tilde{\tilde{T}}_p(G)/M.$$

Now, by induction, $R_p(G/M) = \tilde{\tilde{T}}_p(G/M)$; therefore, $R_p(G)/M = \tilde{\tilde{T}}_p(G)/M$, that is, $R_p(G) = \tilde{\tilde{T}}_p(G)$. ▯

<u>Corollary 3.16</u>: Let R(G) = the largest normal solvable subgroup of G and $\mathcal{M}(G)$ = the collection of all maximal subgroups of G with normal index divisible by two distinct primes. Then,

$$\begin{aligned}R(G) &= \cap \{S \mid S \in \mathcal{M}(G)\}\\ &= \cap \{S \mid S \bowtie G \text{ and } S \in \mathcal{M}(G)\}.\end{aligned}$$

Note: As in the theorem, we can, by Theorem 3.9-(ii), let $\mathfrak{M}(G)$ be the family of all maximal subgroups whose normal outer cofactors are nonsolvable.

Proof of Corollary: Let $\pi(G) = \{p_1, p_2, \ldots, p_m\}$. From the fact that a group H is solvable if-f it is p-solvable for all primes p dividing $|H|$, it follows that $R(G) = \bigcap_{i=1}^{m} R_{p_i}(G)$. Now, for each $i = 1, 2, \ldots, m$, let $\mathfrak{M}_{p_i}(G) = \{S_{ij} | j = 1, \ldots, n_i\}$. Then, from the preceding theorem, we obtain

$$R(G) = \bigcap_{i=1}^{m} R_{p_i}(G) = \bigcap_{i=1}^{m} \bigcap_{j=1}^{n_{p_i}} S_{ij} = \bigcap \{S | S \in \mathfrak{M}(G)\}.$$

In a similar manner, $R(G) = \bigcap \{S | S \rtimes G \text{ and } S \in \mathfrak{M}(G)\}$. ▯

3.4 Addendum

Since the preparation of this thesis, two papers relevant to the results given here have come to my attention. The first of these, "The $\mathfrak{F}$-normalizers of a finite soluble group" by R. Carter and T. Hawkes, appears in the Journal of Algebra, 5 (1967), 175-202, and is a study of finite solvable groups from the viewpoint of the theory of formations. Although there is little actual overlap with this thesis, some of the results are related to those of this thesis. For example, the equivalence of the first two statements of Theorem 2.9 is a consequence of one of the results of this paper. And since the normal outer cofactors of a subgroup of a finite group G are chief factors of G, the theorems of Carter and Hawkes dealing with chief factors of a finite solvable group have some bearing on the results of Chapter 3 and, more particularly, on those of Section 3.3.

The second paper is that of J. Beidleman and A. Spencer, "The normal index of maximal subgroups in finite groups", which, as of this date, has not yet been published but has been submitted to the *Illinois Journal of Mathematics*. The results are closely related to those of Section 3.3. There are, for example, statements which can be added to the lists of equivalent statements in Theorems 3.12 and 3.13.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. R. Baer, "Classes of finite groups and their properties," Illinois Jour. Math., 1 (1957), 115-187.

2. W. Burnside, Theory of Groups of Finite Order (2nd ed.), 1911, New York: Dover, 1955.

3. R. W. Carter, "On a class of finite soluble groups," Proc. London Math. Soc., (3) 9 (1959), 623-640.

4. W.E. Deskins, "A condition for the solvability of a finite group," Illinois Jour. Math., 5 (1961), 306-313.

5. ___________, "On maximal subgroups," Proc. Symp. Pure. Math., Amer. Math. Soc., Pasadena, 1959.

6. K. Doerk, "Minimal nicht überauflösbare, endliche Gruppen," Math. Zeit., 91 (1966), 198-205.

7. W. Gaschütz, "Über die Φ-Gruppen der endlicher Gruppen," Math. Zeit., 58 (1953), 160-170.

8. D. Gorenstein, Finite Groups, New York: Harper & Row, 1968.

9. M. Hall, The Theory of Groups, New York: Macmillan, 1959.

10. P. Hall, "On the system normalizers of a soluble group," Proc. London Math. Soc., 43 (1937), 507-528.

11. B. Huppert, Endliche Gruppen I, Berlin: Springer-Verlag, 1967.

12. __________, "Normalteiler und maximale Untergruppen endlicher Gruppen," Math. Zeit., 60 (1954), 409-434.

13. N. Ito, "Note on (LM)-groups of finite orders," Kodai Math. Seminar Report, 1951, 1-6.

14. K. Iwasawa, "Über die Struktur der endlichen Gruppen, deren echte Untergruppen sämtlich nilpotent sind," Proc. Phys. Math. Soc. Japan, (3) 23 (1941), 1-4.

15. O. Ore, "Contributions to the theory of groups of finite order," Duke. Math. Jour., 5 (1938), 431-460.

16. J. S. Rose, "Finite groups with prescribed Sylow towers," Proc. London Math. Soc., (3) 16 (1966), 577-589.

17. __________, "The influence on a group of its proper abnormal structure," Jour. London Math. Soc., 40 (1965), 348-361.

18. O. Schmidt, "Über Gruppen, deren sämtliche Teiler spezielle Gruppen sind," Rec. Math. Moscou, 31 (1924), 366-372.

19. W. R. Scott, Group Theory, Englewood Cliffs, N. J.: Prentice-Hall, 1964.

20. D. Taunt, "On A-groups," Proc. Cambridge Phil. Soc., 12 (1937), 82-85.