

ON A GENERALIZATION OF HAAR SERIES

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Melvin A. Nyman

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Jan Mari'k

Major professor

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ABSTRACT

ON A GENERALIZATION OF HAAR SERIES

By

Melvin Andrew Nyman

For each natural number n let D_n be a set $\{t_{n,0}, t_{n,1}, \dots, t_{n,r_n}\}$, where

$$0 = t_{n,0} < t_{n,1} < \dots < t_{n,r_n} = 1.$$

Assume that $D_1 \subset D_2 \subset \dots$ and that $D = \bigcup_{n=1}^{\infty} D_n$ is dense in $[0,1]$. For each natural number n let $\mathcal{J}_n$ be the system of all intervals $[t_{n,j-1}, t_{n,j}]$ ($j = 1, \dots, r_n$). A function f is called regular if f is of bounded variation on $[0,1]$, $f(x) = \frac{f(x+) + f(x-)}{2}$ for $x \in (0,1)$, $f(0) = f(0+)$ and $f(1) = f(1-)$. For each $n \geq 1$ let $\mathfrak{g}_n$ be the vector space of all regular functions on $[0,1]$ which are constant on the interior of each interval from $\mathcal{J}_n$. Set $T_1 = \mathfrak{g}_1$ and

$$T_n = \{f \in \mathfrak{g}_n : \int_0^1 fg = 0 \text{ for all } g \in \mathfrak{g}_{n-1}\} \quad (n = 2, 3, \dots).$$

A K-series with respect to $\{D_n\}$ is any series of the form $\sum_{n=1}^{\infty} f_n$, where $f_n \in T_n$ ($n = 1, 2, \dots$). If

$\chi_0, \chi_1, \dots$ are the Haar functions and $a_0, a_1, a_2, \dots$ are numbers, then $\sum_{n=1}^{\infty} a_{n-1} \chi_{n-1}$ is a K-series (the corresponding

sequence of sets $\{D_n\}$ are easily constructed). An analogous assertion holds for the Rademacher functions.

The Fourier-K-series of a function which is Perron integrable on $[0,1]$ is defined. It is proved that if f is Perron integrable and $s_n(f)$ is the n^{th} partial sum of the Perron-Fourier-K-series for f , then $\int_J s_n(f) = \int_J f$ for every interval J of $\mathcal{D}_n$ ($n = 1, 2, \dots$). From this it may be shown that the Perron-Fourier-K-series of a Perron integrable function converges almost everywhere to the function. It is also proved that the Fourier-K-series of a Lebesgue integrable function converges to the function at every Lebesgue point of the function. It is shown that if $f \in L^p[0,1]$ ($1 \leq p < \infty$), then the Fourier-K-series of an arbitrary finite Borel measure on $[0,1]$ is defined and the behavior investigated.

A uniqueness theorem for K-series which generalizes those known for Haar series is proved.

Let f be a finite function on D . Derivates of f with respect to $\{D_n\}$ are defined. Using the properties of such derivates, a class of K-series is found such that no member of the class has an infinite sum on a set of positive Lebesgue measure.

Let f be a bounded function on $[0,1]$. Let $\sum_{n=1}^{\infty} f_n$ be a K-series with respect to $\{D_n\}$. Conditions on the sequence $\{D_n\}$ are found so that if

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

for every $x \in [0,1]$, then $\sum_{n=1}^{\infty} f_n$ is the Fourier-K-series for f . Examples are given to show that this need not always be the case.

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By

Melvin Andrew Nyman

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CHAPTER I

INTRODUCTION

Consider the Haar, Walsh and Rademacher systems of orthonormal step functions. The first two of these are complete in $L^2[0,1]$ and the last is not, Walsh's system being the completion of the Rademacher system. The similarities and dissimilarities of these systems with one another and with the trigonometric system have been investigated by several mathematicians.

The basic elements of the theory of Haar-Fourier series as well as a complete discussion of the relationship between the Haar, Rademacher and Walsh systems may be found in the book by Alexitis [1]. Fine [4],[5] has developed much of the theory of Walsh series.

In this paper the notion of K-series is defined and is shown to include series with respect to the systems of Haar and Rademacher. Since the partial sums of order 2^n ($n = 0,1,2,\dots$) of a Walsh series may be written as the 2^n th partial sum of a Haar series (see [1], pg.62) the convergence theory of K-series also includes these convergence questions.

Wade [14],[15] has given conditions under which a Haar series is the Haar-Fourier series of a Lebesgue integrable function. In Chapter III, we are able to prove more general theorems for K-series and deduce stronger versions of Wade's theorems as corollaries.

In Chapter IV, the question of whether a K-series may have an infinite sum on a set of positive measure is considered. Talalyan and Arutyunyan [13] have proved that this is impossible for Haar series. Skvorcov [12] deduces the same theorem by another method. Using the method of Skvorcov we can find a class of K-series for which the question has a negative answer. By applying directly a theorem of Gundy [7], we can find sufficient conditions for the answer to be affirmative.

Faber [3] has given an example of a Haar series which converges to zero at all but one point of $[0,1]$. Skvorcov [10] has shown that if a Haar series converges everywhere on $[0,1]$ to a bounded function then it is the Haar-Fourier series for the function. These two results suggest the analogous questions for K-series. Results in this direction are given in Chapter V.

Since the Perron integral is used extensively in this paper, we will include the definition and some of the most important properties. For a more detailed discussion of the Perron method of integration see [9].

Definition. Let f be a function (not necessarily finite) defined on an interval $[a,b]$. The function U is a majorant for f if

- a) $U(a) = 0$
- b) $\underline{DU}(x) = \liminf_{y \rightarrow x} \frac{U(y) - U(x)}{y - x} > -\infty$ for all $x \in [a,b]$
- c) $\underline{DU}(x) \geq f(x)$ for all $x \in [a,b]$.

The function V is a minorant for f if

- a) $V(a) = 0$
- b) $\bar{DV}(x) = \limsup_{y \rightarrow x} \frac{V(y) - V(x)}{y - x} < +\infty$ for all $x \in [a,b]$
- c) $\bar{DV}(x) \leq f(x)$ for all $x \in [a,b]$.

It can be shown that

$$(1) \quad \sup\{V(b) : V \text{ is a minorant for } f\} \leq \inf\{U(b) : U \text{ is a majorant for } f\}.$$

If f has at least one majorant, at least one minorant and equality holds in (1), then f is said to be Perron integrable on $[a,b]$ and the common value is denoted by $(P) \int_a^b f$. The following are some of the most useful properties of the Perron integral.

1. Let f be Perron integrable on $[a,b]$.
 Set $(P) \int_a^a f = 0$. Then for any $x \in [a,b]$
 f is Perron integrable on $[a,x]$ and on
 $[x,b]$. Furthermore $(P) \int_a^x f + (P) \int_x^b f = (P) \int_a^b f$
 for every $x \in [a,b]$ and the function
 $(P) \int_a^x f$ is continuous on $[a,b]$. Any
 function of the form $F(x) = C + (P) \int_a^x f$ is
 called an indefinite Perron integral of f .
2. Let f be Perron integrable on $[a,b]$ and
 k be a finite constant. Then the function
 kf is Perron integrable on $[a,b]$ and
 $(P) \int_a^b kf = k(P) \int_a^b f$.
3. Let f and g be Perron integrable functions
 on $[a,b]$ such that $f+g$ is defined on
 all of $[a,b]$. Then $f+g$ is Perron integrable
 on $[a,b]$ and $(P) \int_a^b (f+g) = (P) \int_a^b f + (P) \int_a^b g$.
4. Let f be Perron integrable on $[a,b]$, U
 a majorant for f , V a minorant for f
 and $F(x) = (P) \int_a^x f$. Then $U-F$ and $F-V$
 are non-decreasing functions on $[a,b]$.
5. If f is Lebesgue integrable on $[a,b]$,
 then f is Perron integrable on $[a,b]$
 and $(P) \int_a^b f = (L) \int_a^b f$.

6. If the function F possesses a finite derivative F' everywhere on $[a,b]$, then F' is Perron integrable and $F(b) - F(a) = (P) \int_a^b F'$.
7. Every Perron integrable function is measurable and is almost everywhere finite and equal to the derivative of its indefinite integral.
8. Let f and g be functions on $[a,b]$ such that $f=g$ a.e. Assume f is Perron integrable on $[a,b]$. Then g is Perron integrable and $(P) \int_a^b f = (P) \int_a^b g$.
9. Let f be Perron integrable on $[a,b]$ and g have finite variation on $[a,b]$. Then the product fg is Perron integrable.

When there is no danger of confusion we will write $\int_a^b f$ for $(P) \int_a^b f$.

Let g be a finite function on a set A . If $a, b \in A$, $a < b$ and $I = [a,b]$, denote $g(b) - g(a)$ by $g(I)$. If E is a set of real numbers, then $|E|$ will denote the outer Lebesgue measure of E ; in case E is measurable, then, of course, $|E|$ is the measure of E . Which of these is meant will be clear from the context.

CHAPTER II

DEFINITIONS AND SOME THEOREMS ON FOURIER-K-SERIES

We start by defining the notion of a K-series. This gives simultaneous generalization of series in the Haar and Rademacher orthonormal systems. Moreover, the concept of K-series includes as a special case the partial sums of order 2^n for series in the Walsh orthonormal system. We shall also define the Fourier-K-series of an integrable function and of a measure and investigate the convergence properties of such series.

1. Definition. Let D_n be a finite set $\{t_{n,0}, t_{n,1}, \dots, t_{n,r_n}\}$ where $0 = t_{n,0} < t_{n,1} < \dots < t_{n,r_n} = 1$ ($n = 1, 2, \dots$). Set $D = \bigcup_{n=1}^{\infty} D_n$. Assume $D_1 \subset D_2 \subset \dots$ and that D is dense in $[0, 1]$. For each $n > 0$ let $\mathcal{I}_n$ denote the system of all intervals of the form $[t_{n,j-1}, t_{n,j}]$ ($j = 1, 2, \dots, r_n$). Set $\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n$.

We say a function g is regular on $[0, 1]$ if g is of bounded variation and $g(0) = g(0+)$, $g(1) = g(1-)$, $g(x) = \frac{g(x+) + g(x-)}{2}$ for $x \in (0, 1)$. For each $n > 0$ denote by $\mathcal{R}_n$ the space of all regular functions on

$[0,1]$ which are constant on the interior of each interval of $\mathcal{J}_n$. Notice that $\mathcal{S}_n$ has dimension r_n .

Define $T_1 = \mathcal{S}_1$ and

$$T_n = \{f \in \mathcal{S}_n : \int_0^1 fg = 0 \text{ for all } g \in \mathcal{S}_{n-1}\} \quad (n = 2, 3, \dots).$$

A series of the form $\sum_{n=1}^{\infty} f_n$ where $f_n \in T_n$ ($n = 1, 2, \dots$)

will be called a K-series with respect to the sequence

$$\{D_n\}_{n=1}^{\infty}.$$

2. Lemma. Let f be a Perron integrable function on $[0,1]$. Let T be a finite dimensional vector space of regular functions on $[0,1]$. Then there is a unique function $g \in T$ such that $\int_0^1 (f-g)t = 0$ for every $t \in T$.

Proof. Since every non-zero regular function is non-zero on a set of positive measure, we may choose a basis $v_1, \dots, v_n$ for T such that

$$\int_0^1 v_i v_j = \delta_{ij} \quad (i, j = 1, \dots, n).$$

Set

$$\beta_i = \int_0^1 f v_i \quad (i = 1, \dots, n).$$

Then $g = \sum_{i=1}^n \beta_i v_i$ is the required function. Suppose $h \in T$ fulfills $\int_0^1 (f-h)t = 0$ for all $t \in T$. This is equivalent to

$$\int_0^1 f v_i = \int_0^1 h v_i \quad (i = 1, \dots, n).$$

Since $h \in T$ we have $h = \sum_{i=1}^n \gamma_i v_i$. Thus

$$\int_0^1 f v_i = \int_0^1 \left(\sum_{j=1}^n \gamma_j v_j \right) v_i = \sum_{j=1}^n \gamma_j \int_0^1 v_i v_j = \gamma_i \quad (i = 1, \dots, n).$$

Hence $h=g$. $\square$

3. Definition. Let T, f and g be as in lemma 2. Then g is called the orthogonal projection of f to T .

4. Definition. Let f be a Perron integrable function on $[0,1]$. For each $n \geq 1$ let f_n be the orthogonal projection of f to T_n . The K-series $\sum_{n=1}^{\infty} f_n$ is the Perron-Fourier-K-series for f , hereafter denoted by PFK-series for f . In case f is Lebesgue integrable we say that $\sum_{n=1}^{\infty} f_n$ is the LFK-series for f .

5. Lemma. Let S be a finite dimensional vector space of regular functions on $[0,1]$. Let T be a linear subspace of S . Let

$$V = \{f \in S : \int_0^1 fg = 0 \text{ for all } g \in T\}.$$

Let h be Perron integrable on $[0,1]$. Let ψ_1 and ψ_2 denote the orthogonal projections of h to T and V respectively. Then the orthogonal projection of h to S equals $\psi_1 + \psi_2$.

Proof. Choose an arbitrary $u \in S$. Let t be the orthogonal projection of u to T and let $v = u - t$. Obviously $v \in V$ so that

$$\begin{aligned} \int_0^1 (h - \psi_1 - \psi_2) u &= \int_0^1 (h - \psi_1 - \psi_2) (t + v) = \int_0^1 (h - \psi_1) t \\ &\quad - \int_0^1 \psi_2 t + \int_0^1 (h - \psi_2) v - \int_0^1 \psi_1 v = 0. \quad \square \end{aligned}$$

6. Lemma. Let f be Perron integrable on $[0, 1]$ and $\sum_{k=1}^{\infty} f_k$ be the PFK-series for f . Then $\sum_{k=1}^n f_k$ is the orthogonal projection of f to $\mathfrak{S}_n$ ($n = 1, 2, \dots$).

Proof. By induction. The assertion is obvious for $n=1$, since $\mathfrak{S}_1 = T_1$. Assume $\sum_{k=1}^n f_k$ is the orthogonal projection of f to $\mathfrak{S}_n$. Because $D_n \subset D_{n+1}$ we have $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$. Therefore we may apply lemma 5 with $S = \mathfrak{S}_{n+1}$, $T = \mathfrak{S}_n$, $V = T_n$ to see that the orthogonal projection of f to $\mathfrak{S}_{n+1}$ is $\sum_{k=1}^n f_k + f_{n+1}$. $\square$

7. Theorem. Let f be Perron integrable on $[0, 1]$. Let g be the orthogonal projection of f to $\mathfrak{S}_n$. Then

$$\int_J g = \int_J f \quad \text{for every } J \in \mathcal{J}_n \quad (n = 1, 2, \dots).$$

Proof. Fix an n . Let $J_1, \dots, J_{r_n}$ be an enumeration of the intervals of $\mathcal{J}_n$. Let v_i be a regular function such that $v_i = 1$ on $\text{int } J_i$ and $v_i = 0$ off J_i . Obviously $v_i \in \mathfrak{S}_n$ ($i = 1, \dots, r_n$).

Since g is the orthogonal projection of f to $\mathfrak{g}_n$, we have

$$\int_{J_i} (f-g) = \int_0^1 (f-g) v_i = 0 \quad (i = 1, \dots, r_n). \quad \square$$

Given any K-series $\sum_{k=1}^{\infty} f_k$, define $s_n = \sum_{k=1}^n f_k$ ($n = 1, 2, \dots$). In case $\sum_{k=1}^{\infty} f_k$ is the PFK-series for a Perron integrable function f write $s_n(f) = \sum_{k=1}^n f_k$. We will write $s_n(f, x)$ for $(s_n(f))(x)$.

We now present theorems on the behavior of the K-Fourier series of Perron integrable functions.

8. Theorem. Let f be Perron integrable on $[0, 1]$. Let $\sum_{n=1}^{\infty} f_n$ be the PFK-series for f . Then

$$s_n(f, x) = \frac{1}{|J|} \int_J f \text{ for } x \in \text{int } J, J \in \mathcal{J}_n \quad (n = 1, 2, \dots).$$

Proof. This is a direct consequence of lemma 6 and theorem 7. $\square$

9. Theorem. Let g be Perron integrable on $[0, 1]$. Let $\sum_{n=1}^{\infty} f_n$ be the PFK-series for g . Then

$$\sum_{n=1}^{\infty} f_n(x) = g(x)$$

for every x such that

$$\frac{d}{dx} \left((P) \int_0^x g \right) = g(x).$$

Proof. Let G be an indefinite Perron integral of g . By theorem 8 we have

$$s_n(g, x) = \frac{1}{|J|} \int_J g \text{ for } x \in \text{int } J, J \in \mathcal{D}_n$$

and

$$s_n(g, x) = \frac{1}{2|J|} \int_J g + \frac{1}{2|L|} \int_L g \text{ for } x \in D_n,$$

x a common endpoint of $J, L \in \mathcal{D}_n$. Therefore, if $G'(x) = g(x)$ we have $g(x) = \lim_{n \rightarrow \infty} s_n(g, x)$ since D is dense in $[0, 1]$. $\square$

10. Corollary. Let g and $\sum_{n=1}^{\infty} f_n$ be as in theorem 9. Then $\sum_{n=1}^{\infty} f_n(x)$ converges to $g(x)$ almost everywhere on $[0, 1]$.

Proof. Follows directly from theorem 9 and the fact that for any Perron integrable function ψ

$$\frac{d}{dx}((P) \int_0^x \psi) = \psi(x) \text{ a.e.}$$

11. Corollary. Let g and $\sum_{n=1}^{\infty} f_n$ be as in theorem 9. Then $\sum_{n=1}^{\infty} f_n(x)$ converges to $g(x)$ for every x which is a point of continuity of g . Furthermore, if g is Lebesgue integrable, then $\sum_{n=1}^{\infty} f_n(x) = g(x)$ at every Lebesgue point x of g .

12. Lemma. Let f be Lebesgue integrable on $[0, 1]$. Let $\sum_{n=1}^{\infty} f_n$ be the LFK-series for f . Assume f is continuous on an interval $I \subset [0, 1]$. Then for any

compact interval I_1 contained in the interior of I ,
 $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on I_1 . In case
 $I = [0,1]$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to f
on $[0,1]$.

Proof. Let I_1 be a compact subinterval of
int I . Let I_2 be a compact interval such that
 $I_1 \subset \text{int } I_2$ and $I_2 \subset \text{int } I$. Let $\epsilon > 0$. By the
uniform continuity of f on I_2 there is a $\delta > 0$
such that $|f(x) - f(y)| < \epsilon$ for every pair $x, y \in I_2$
with $|x - y| < \delta$. Because of the density of D and
because $D_n \subset D_{n+1}$ ($n = 1, 2, \dots$) there is an integer
 N_0 so large that if $n > N_0$ and $J \in \mathcal{D}_n$, then
 $|J| < \delta$.

Now there is an N_1 such that if $n > N_1$,
 $J \in \mathcal{D}_n$ and $J \cap I_1 \neq \emptyset$, then $J \subset I_2$. Let
 $n > \max(N_0, N_1)$ and let $t \in I_1$. If $t \notin D_n$, then
there is a $J \in \mathcal{D}_n$ such that $t \in \text{int } J$ and by theorem 8

$$\begin{aligned} (1) \quad |s_n(f, t) - f(t)| &= \frac{1}{|J|} \left| \int_J [f(u) - f(t)] du \right| \\ &\leq \frac{1}{|J|} \int_J |f(u) - f(t)| du < \epsilon. \end{aligned}$$

If $t \in D_n \cap (0,1)$, then there are two intervals
 $J_1, J_2 \in \mathcal{D}_n$ for which t is a common endpoint and by
theorem 8

$$\begin{aligned}
 (2) \quad |s_n(f, t) - f(t)| &\leq \frac{1}{2|J_1|} \int_{J_1} |f(u) - f(t)| du \\
 &+ \frac{1}{2|J_2|} \int_{J_2} |f(u) - f(t)| du < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
 \end{aligned}$$

Assume that $I = [0, 1]$. Take $I_1 = I_2 = [0, 1]$ in the first part of the proof. Then there is an N such that if $n > N$ and $J \in \mathcal{J}_n$, then $|J| < \delta$. Let $n > N$ and $t \notin D_n$. Apply theorem 8 as above to conclude that for $J \in \mathcal{J}_n$ and $t \in \text{int } J$

$$(1') \quad |s_n(f, t) - f(t)| \leq \frac{1}{|J|} \int_J |f(u) - f(t)| du < \epsilon.$$

If $t \in (0, 1) \cap D_n$, then applying theorem 8 as we did to obtain (2) we get

$$\begin{aligned}
 (2') \quad |s_n(f, t) - f(t)| &\leq \frac{1}{2|J_1|} \int_{J_1} |f(u) - f(t)| du \\
 &+ \frac{1}{2|J_2|} \int_{J_2} |f(u) - f(t)| du < \epsilon.
 \end{aligned}$$

In case $t=0$ or $t=1$, theorem 8 gives (1') where t is an endpoint of J . $\square$

The inequality which is derived in theorem 16 is the direct analogue of one which is known for the Haar-Fourier series of an L^p function ($1 < p < \infty$) [6, pg. 72].

We begin with some lemmas.

13. Lemma. Let $p \geq 1$. Let f be a Lebesgue integrable function on $[a, b]$. Let γ be a number such that $\int_a^b \gamma = \int_a^b f$. Then

$$\int_a^b |\gamma|^p \leq \int_a^b |f|^p.$$

Proof. We have

$$\begin{aligned} |\gamma|(b-a) &= \int_a^b |\gamma| = \left| \int_a^b \gamma \right| = \left| \int_a^b f \right| \leq \left(\int_a^b 1 \right)^{1-1/p} \left(\int_a^b |f|^p \right)^{1/p} \\ &= (b-a)^{p-1/p} \left(\int_a^b |f|^p \right)^{1/p}. \end{aligned}$$

Taking p^{th} power of both sides of this inequality we see $|\gamma|^p (b-a)^p \leq (b-a)^{p-1} \int_a^b |f|^p$. Thus

$$\int_a^b |\gamma|^p = |\gamma|^p (b-a) \leq \int_a^b |f|^p. \quad \square$$

14. Proposition. Let $\sum_{n=1}^{\infty} f_n$ be a K-series.

For each $x \in [0, 1]$ and each $n \geq 1$ set

$$s_n^*(x) = \max_{i \leq j \leq n} |s_j(x)|.$$

Then for all $\lambda > 0$

$$\begin{aligned} |\{x: s_n^*(x) > \lambda\}| &\leq \frac{1}{\lambda} \int_{\{x: s_n^*(x) > \lambda\}} |s_n(t)| dt \\ &\leq \frac{1}{\lambda} \int_0^1 |s_n(t)| dt. \end{aligned}$$

Proof. Take $\lambda > 0$ and a natural number n .

Let $A = \{x: s_n^*(x) > \lambda\}$ and $A_j = \{x: |s_1(x)| \leq \lambda, \dots, |s_{j-1}(x)| \leq \lambda, |s_j(x)| > \lambda\}$ ($j = 1, \dots, n$). Obviously $A_j \cap A_i = \emptyset$ for $i \neq j$ and $A = \bigcup_{j=1}^n A_j$.

Let $\mathcal{J}_j$ be the algebra generated by the elements of $\mathcal{D}_j$ ($j = 1, 2, \dots$). Now $s_j \in \mathcal{G}_j$ and hence is constant on interiors of intervals of $\mathcal{D}_j$. It follows that A_j is composed of interiors of intervals of $\mathcal{D}_j$ together with some points of $\mathcal{D}_j$. Thus $A_j \in \mathcal{J}_j$ ($j = 1, 2, \dots, n$). If $j \leq n$ and $J \in \mathcal{D}_j$, then $\int_J s_j = \int_J s_n$, so that

$$\int_J |s_j| = \left| \int_J s_j \right| \leq \int_J |s_n|.$$

Since $A_j \in \mathcal{J}_j$ we have

$$\int_{A_j} |s_j| \leq \int_{A_j} |s_n| \quad (j = 1, \dots, n).$$

Hence

$$\begin{aligned} \int_A |s_n| &= \sum_{j=1}^n \int_{A_j} |s_n| \geq \sum_{j=1}^n \int_{A_j} |s_j| \geq \sum_{j=1}^n \lambda |A_j| \\ &= \lambda |A|. \end{aligned}$$

This completes the proof. $\square$

15. Lemma. Let $1 < p < \infty$. Let $f \in L^p[0, 1]$ and let $\sum_{j=1}^{\infty} f_j$ be the LFK-series for f . Let s_n^* be

as in proposition 14. Then

$$\int_0^1 (s_n^*)^p \leq \left(\frac{p}{p-1}\right)^p \int_0^1 |s_n(f)|^p \leq \left(\frac{p}{p-1}\right)^p \int_0^1 |f|^p \quad (n = 1, 2, \dots).$$

Proof. Let $\lambda > 0$ and $n \geq 1$. Define

$$\psi(\xi, \lambda) = \begin{cases} 1 & \text{if } \xi > \lambda \\ 0 & \text{if } 0 \leq \xi \leq \lambda \end{cases}.$$

Then the inequality

$$\lambda |\{x: s_n^*(x) > \lambda\}| \leq \int_{\{x | s_n^*(x) > \lambda\}} |s_n(x)| dx$$

of proposition 14 becomes

$$\lambda \int_0^1 \psi(s_n^*(x), \lambda) dx \leq \int_0^1 \psi(s_n^*(x), \lambda) |s_n(x)| dx.$$

Thus

$$\int_0^\infty \lambda^{p-1} \left(\int_0^1 \psi(s_n^*(x), \lambda) dx \right) d\lambda \leq \int_0^\infty \lambda^{p-2} \left(\int_0^1 \psi(s_n^*(x), \lambda) |s_n(x)| dx \right) d\lambda.$$

Since we may interchange the order of integration, we have

$$\begin{aligned} (1) \quad \int_0^1 \frac{1}{p} (s_n^*(x))^p dx &= \int_0^1 \left(\int_0^{s_n^*(x)} \lambda^{p-1} d\lambda \right) dx \\ &= \int_0^1 \left(\int_0^\infty \lambda^{p-1} \psi(s_n^*(x), \lambda) d\lambda \right) dx \\ &\leq \int_0^1 \left(\int_0^\infty \lambda^{p-2} \psi(s_n^*(x), \lambda) |s_n(x)| d\lambda \right) dx \\ &= \int_0^1 \left(\int_0^{s_n^*(x)} \lambda^{p-2} |s_n(x)| d\lambda \right) dx \\ &= \int_0^1 \frac{1}{p-1} (s_n^*(x))^{p-1} |s_n(x)| dx. \end{aligned}$$



By Hölder's inequality we have

$$(2) \quad \int_0^1 (s_n^*(x))^{p-1} |s_n(x)| dx \\ \leq \left[\int_0^1 ([s_n^*(x)]^{p-1})^{p/(p-1)} dx \right]^{1-1/p} \left[\int_0^1 |s_n(x)|^p dx \right]^{1/p}.$$

From (1) and (2) we get

$$\frac{1}{p} \int_0^1 (s_n^*(x))^p dx \leq \frac{1}{p-1} \left(\int_0^1 [s_n^*(x)]^p dx \right)^{1-1/p} \left(\int_0^1 |s_n(x)|^p dx \right)^{1/p}.$$

Hence

$$\left(\int_0^1 (s_n^*(x))^p dx \right)^{1/p} \leq \frac{p}{p-1} \left(\int_0^1 |s_n(x)|^p dx \right)^{1/p}.$$

Taking p^{th} powers we have the first inequality of our assertion.

The second inequality follows from theorem 8 and lemma 13. $\square$

16. Theorem. Let $1 < p < \infty$; $f \in L^p[0,1]$.

Let $\sum_{j=1}^{\infty} f_j$ be the LFK-series for f . Define

$$s^*(x) = \sup_n |s_n(f, x)| \text{ for all } x \in [0,1].$$

Then

$$\int_0^1 (s^*(x))^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^1 |f(x)|^p dx.$$

Proof. Let s_n^* be as in 14 and 15 ($n = 1, 2, \dots$). Then it is easily seen that $s_n^* \nearrow s^*$. Hence we have

$$\int_0^1 (s^*)^p = \lim_{n \rightarrow \infty} \int_0^1 (s_n^*)^p$$

by the Lebesgue Monotone Convergence Theorem. This combined with lemma 15 proves our assertion. $\square$

Next we show that the LFK-series for a function in $L^p[0,1]$ ($1 \leq p < \infty$) converges to f in L^p norm.

17. Theorem. Let $1 \leq p < \infty$. Assume $f \in L^p[0,1]$. Then the LFK-series for f converges to f in L^p norm.

Proof. Let $\epsilon > 0$. There is a continuous function g such that $\|f-g\|_p < \frac{\epsilon}{3}$. By lemma 12, $s_n(g) \rightarrow g$ uniformly on $[0,1]$. Thus there is an integer N such that $|s_n(g, x) - g(x)| < \frac{\epsilon}{3}$ for $n \geq N$. Therefore $\|s_n(g) - g\|_p = (\int_0^1 |s_n(g) - g|^p)^{1/p} < \frac{\epsilon}{3}$ for $n \geq N$.

Combining theorem 8 and lemma 13 we have

$$\begin{aligned} \|s_n(f-g)\|_p^p &= \int_0^1 |s_n(f-g)|^p = \sum_{J \in \mathcal{J}_n} \int_J |s_n(f-g)|^p \\ &\leq \sum_{J \in \mathcal{J}_n} \int_J |f-g|^p = \int_0^1 |f-g|^p \\ &= \|f-g\|_p^p \quad (n = 1, 2, \dots). \end{aligned}$$

From this and the triangle inequality we get

$$\begin{aligned}\|s_n(f) - f\|_p &\leq \|s_n(f-g)\|_p + \|s_n(g) - g\|_p + \|g-f\|_p \\ &\leq 2\|f-g\|_p + \|s_n(g) - g\|_p < \epsilon \quad \text{for } n \geq N.\end{aligned}$$

The following two sections contain the definition of the formal integral of a K-series and one of the elementary properties.

18. Definition. Let $\sum_{n=1}^{\infty} f_n$ be a K-series. The sum of the series $\sum_{n=1}^{\infty} \int_0^x f_n$ will be denoted by $F(x)$ at points of convergence.

19. Lemma. Let $\sum_{j=1}^{\infty} f_j$ be a K-series. Let n be a natural number; let $x \in D_n$. Then $F(x) = \int_0^x s_n$.

Proof. By orthogonality we have $\int_0^x s_i = \int_0^x s_n$ for $i \geq n$. Thus

$$\sum_{j=1}^{\infty} \int_0^x f_j = \sum_{j=1}^n \int_0^x f_j = \int_0^x s_n = F(x). \quad \square$$

Next a short investigation of Fourier-Stieltjes-K-series for measures is undertaken. By a measure we will mean a finite, signed measure defined on the Borel sets in $[0,1]$.

20. Definition. For any $J = [\alpha, \beta] \in \mathcal{D}$ define J^* by $J^* = [\alpha, \beta)$ if $\beta < 1$ and $J^* = J$ if $\beta = 1$. Let u be a measure. For each $n \geq 1$ define $s_n \in \mathcal{S}_n$ by $\int_J s_n = u(J^*)$ for each $J \in \mathcal{D}_n$. Set $f_1 = s_1$ and

$f_n = s_n - s_{n-1}$ for $n \geq 2$. Obviously $f_1 \in T_1$. Take $n \geq 2$ and let $I_1, \dots, I_{r_{n-1}}$ be an enumeration of $\mathcal{D}_{n-1}$. Define v_j to be regular and

$$v_j = \begin{cases} 1 & \text{on int } I_j \\ 0 & \text{off } I_j \end{cases} \quad (j = 1, 2, \dots, r_{n-1}).$$

Then $\{v_1, \dots, v_{r_{n-1}}\}$ is a basis for $\mathcal{S}_{n-1}$. Fix a j .

Then $I_j = J_1 \cup \dots \cup J_p$ where $J_i \in \mathcal{D}_n$ ($i = 1, \dots, p$) and $J_i^* \cap J_\ell^* = \emptyset$ for $i \neq \ell$. (Note: it may happen that $p=1$). Now applying the definitions of v_j , f_n , s_n and s_{n-1} it follows that

$$\begin{aligned} \int_0^1 f_n v_j &= \int_{I_j} f_n = \int_{I_j} (s_n - s_{n-1}) = \sum_{i=1}^p \int_{J_i} s_n - \int_{I_j} s_{n-1} \\ &= \sum_{i=1}^p \mu(J_i^*) - \mu(I_j^*) = \mu(I_j^*) - \mu(I_j^*) = 0. \end{aligned}$$

Therefore $f_n \in T_n$. Hence $\sum_{n=1}^{\infty} f_n$ is a K-series with respect to $\{D_n\}$.

The K-series $\sum_{n=1}^{\infty} f_n$ is the Fourier-Stieltjes-K-series for μ , hereafter denoted by FSK-series for μ .

21. Definition. Let $\{D_n\}$ and $\{\mathcal{D}_n\}$ be as in definition 1. Let $x \in [0, 1)$ and $n \geq 1$. Define $\alpha_n(x)$ and $\beta_n(x)$ by $\alpha_n(x) \leq x < \beta_n(x)$ and $[\alpha_n(x), \beta_n(x)] \in \mathcal{D}_n$. For $x \in (0, 1]$, define $\alpha'_n(x)$ and $\beta'_n(x)$ by $\alpha'_n(x) < x \leq \beta'_n(x)$ and

$$[\alpha'_n(x), \beta'_n(x)] \in \mathcal{D}_n \quad (n = 1, 2, \dots).$$

Set $\alpha_n(1) = \beta_n(1) = \beta'_n(1) = 1$ and

$$\alpha_n'(0) = \beta_n'(0) = \alpha_n(0) = 0 \quad (n = 1, 2, \dots).$$

For any $x \in [0, 1]$ set $J_n(x) = [\alpha_n(x), \beta_n(x)]$ and $J_n'(x) = [\alpha_n'(x), \beta_n'(x)]$, where $[0, 0] = \{0\}$ and $[1, 1] = \{1\}$.

Note that if $x \notin D_n$, then $J_n(x) = J_n'(x)$ ($n = 1, 2, \dots$). Furthermore, if $x \in [0, 1] \cap D_n$ then $\alpha_n(x) = \beta_n'(x) = x$ ($n = 1, 2, \dots$).

22. Proposition. Let μ be a measure. Let $\sum_{n=1}^{\infty} f_n$ be the FSK-series for μ . Then

$$\lim_{n \rightarrow \infty} \int_{J_n(y)} s_n = \mu(\{y\}) \quad \text{for each } y \in [0, 1]$$

and

$$\lim_{n \rightarrow \infty} \int_{J_n'(1)} s_n = \mu(\{1\})$$

Proof. Let $y \in [0, 1]$. Then it follows directly from the definition that for n so large that $\beta_n(y) < 1$

$$\int_{J_n(y)} s_n = \mu[\alpha_n(y), \beta_n(y)).$$

The first assertion follows from this since

$$\bigcap_{n=1}^{\infty} [\alpha_n(y), \beta_n(y)) = \{y\} \quad \text{and} \quad |\mu[0, 1]| < \infty. \quad \text{Similarly}$$

$$\lim_{n \rightarrow \infty} \int_{J_n'(1)} s_n = \mu(\{1\}). \quad \square$$

23. Proposition. Let n be a natural number and $x \in D_n$. Let μ be a measure and $\sum_{j=1}^{\infty} f_j$ its FSK-series. Let F be as in definition 18. Then $\mu[0, x) = F(x)$, if $x < 1$ and $\mu[0, 1] = F(1)$.

Proof. Let k be such that $x = t_{n,k}$. Then

$$\begin{aligned} F(x) &= \int_0^x s_n = \sum_{j=1}^k \int_{t_{n,j-1}}^{t_{n,j}} s_n = \sum_{j=1}^k \mu[t_{n,j-1}, t_{n,j}) \\ &= \mu[0, x) \quad \text{if } x < 1, \end{aligned}$$

and

$$\begin{aligned} F(1) &= \sum_{j=1}^{r_n} \int_{t_{n,j-1}}^{t_{n,j}} s_n = \sum_{j=1}^{r_n-1} \mu[t_{n,j-1}, t_{n,j}) + \mu[t_{n,r_n-1}, 1] \\ &= \mu[0, 1]. \end{aligned}$$

24. Proposition. Let $x \notin D$. Assume μ is a non-atomic measure. Let $\sum_{n=1}^{\infty} f_n$ be the FSK-series for μ . Then $F(x) = \mu[0, x)$.

Proof. Choose a natural number n . Then

$$\begin{aligned} \int_0^x s_n &= \int_0^{\alpha_n(x)} s_n + \int_{\alpha_n(x)}^x s_n = F(\alpha_n(x)) + \int_{\alpha_n(x)}^x s_n \\ &= \mu[0, \alpha_n(x)) + \int_{\alpha_n(x)}^x s_n, \end{aligned}$$

by proposition 23. Now

$$\lim_{n \rightarrow \infty} \mu[0, \alpha_n(x)) = \mu\left(\bigcup_{n=1}^{\infty} [0, \alpha_n(x))\right) = \mu[0, x).$$

Moreover,

$$\left| \int_{\alpha_n(x)}^x s_n \right| = \left| \frac{x - \alpha_n(x)}{\beta_n(x) - \alpha_n(x)} \mu[\alpha_n(x), \beta_n(x)) \right| \leq |\mu[\alpha_n(x), \beta_n(x))|$$

since $\left| \frac{x - \alpha_n(x)}{\beta_n(x) - \alpha_n(x)} \right| < 1$. But

$$\mu[\alpha_n(x), \beta_n(x)) \rightarrow \mu(\{x\}) = 0.$$

This proves the assertion. $\square$

We will next show how the singular portion of the measure may be recovered.

The following theorem is well known (see for example [8], pg.154).

25. Theorem. Let μ be a Borel measure on $[0,1]$. Set $g(x) = \mu[0,x]$ for $x \in (0,1]$ and $g(0) = \mu\{0\}$. Then

- a) g is differentiable a.e. with respect to Lebesgue measure,
- b) g' is Lebesgue integrable on $[0,1]$,
- c) there is a measure μ_s such that

$$\mu(E) = \mu_s(E) + \int_E g'(x) dx \quad \text{for every}$$
 Borel set E and μ_s is singular with respect to Lebesgue measure.

26. Theorem. Let μ be a measure and $\sum_{n=1}^{\infty} f_n$ its FSK-series. Let g be as in theorem 25. Then

- (1) $g'(x) = \sum_{n=1}^{\infty} f_n(x)$ except for a set of Lebesgue measure zero, and if μ is non-atomic,

$$(2) \quad \mu_s(0, x) = \sum_{n=1}^{\infty} \int_0^x f_n - \int_0^x \sum_{n=1}^{\infty} f_n \quad \text{for all } x \in [0, 1].$$

Proof. Let $x \in (0, 1)$ be such that $g'(x)$ exists. Then we have

$$\begin{aligned} g'(x) &= \lim_{n \rightarrow \infty} \frac{g(\beta_n(x)) - g(\alpha_n(x))}{\beta_n(x) - \alpha_n(x)} = \lim_{n \rightarrow \infty} \frac{\mu[\alpha_n(x), \beta_n(x)]}{\beta_n(x) - \alpha_n(x)} \\ &= \lim_{n \rightarrow \infty} s_n(x+). \end{aligned}$$

But $s_n(x+) = s_n(x)$ for $x \in D_n$. Thus $g'(x) = \lim_{n \rightarrow \infty} s_n(x)$

for almost all x with respect to Lebesgue measure.

This proves (1).

If u is non-atomic, we have $\mu[0, x] = \sum_{n=1}^{\infty} \int_0^x f_n(t)$ for all $x \in [0, 1]$ by proposition 23 and 24. Combining this with (1) and part (c) of theorem 25 we have (2). $\square$

CHAPTER III

A UNIQUENESS THEOREM FOR K-SERIES

In this chapter we will show that a Haar series is a K-series and prove a uniqueness theorem for K-series which generalizes those known for Haar series. We will start by defining the Haar functions.

Set $\chi_0(x) = 1$ for all $x \in [0, 1]$. Set

$$\chi_1(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ 0 & \text{if } x = \frac{1}{2} \\ -1 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Given any integer $n \geq 1$, write it uniquely as $n = 2^m + k$ where $0 \leq k < 2^m$. Define χ_n to be regular and

$$\chi_n(x) = \begin{cases} \sqrt{2^m} & \text{for } x \in (\frac{k}{2^m}, \frac{2k+1}{2^{m+1}}) \\ -\sqrt{2^m} & \text{for } x \in (\frac{2k+1}{2^{m+1}}, \frac{k+1}{2^m}) \\ 0 & \text{for } x \notin [\frac{k}{2^m}, \frac{k+1}{2^m}], \end{cases}$$

Note that this gives χ_1 as defined above if we take $n=1$. Any series of the form $\sum_{n=0}^{\infty} a_n \chi_n$, where a_n are real numbers, is called a Haar series.

In papers on Haar series the following condition is often imposed [11], [14], [15].

1. Definition. A Haar series $\sum_{n=0}^{\infty} a_n \chi_n$ satisfies condition GH if for every $x \in [0,1]$

$$\lim_{k \rightarrow \infty} \frac{a_{n_k}}{\chi_{n_k}(x)} = 0$$

where $\{n_k\}$ is the sequence of integers m for which $\chi_m(x) \neq 0$.

The corresponding conditions for K-series are the following:

2. Definition. A K-series $\sum_{n=1}^{\infty} f_n$ satisfies condition G if

$$\lim_{n \rightarrow \infty} \int_{J_n(x)} f_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{J'_n(x)} f_n = 0$$

for every $x \in [0,1]$, where $J_n(x)$ and $J'_n(x)$ are as in section 21 of Chapter II.

The K-series $\sum_{n=1}^{\infty} f_n$ satisfies condition H if

$$\lim_{n \rightarrow \infty} \int_{J_n(x)} s_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{J'_n(x)} s_n = 0$$

for every $x \in [0,1]$. (Note that $J_n(x) = J'_n(x)$ if $x \notin D_n$.)

The next objective is to show that a Haar series is a K-series and that for Haar series conditions GH, G and H are equivalent. Define a sequence of finite sets $\{\Delta_n\}$ as follows. Set $\Delta_1 = \{0, 1\}$, $\Delta_2 = \{0, \frac{1}{2}, 1\}$. For $n \geq 2$ write $n-1 = 2^m + k$ where $0 \leq k < 2^m$ as in the definition of the Haar functions. Set

$$\Delta_n = \{0, \frac{1}{2^{m+1}}, \frac{2}{2^{m+1}}, \dots, \frac{2k+1}{2^{m+1}}, \frac{k+1}{2^m}, \frac{k+2}{2^m}, \dots, 1\}.$$

For instance $\Delta_3 = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$, $\Delta_4 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$,

$\Delta_5 = \{0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, $\Delta_6 = \{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1\}$, etc.

3. Proposition. Let $\sum_{n=0}^{\infty} a_n \chi_n$ be a Haar series. Then $\sum_{n=0}^{\infty} a_n \chi_n$ is a K-series with respect to $\{\Delta_n\}$. Moreover, for Haar series conditions GH and G are equivalent.

Proof. Define $f_{n+1} = a_n \chi_n$ ($n = 0, 1, 2, \dots$). Then $f_n \in T_n$ for each n . Hence $\sum_{n=1}^{\infty} f_n$ is a K-series with respect to $\{\Delta_n\}$.

Let $\sum_{n=0}^{\infty} a_n \chi_n$ be a Haar series. Define f_j ($j = 1, 2, \dots$) as above. Let $y \in [0, 1]$ and let $\{n_k\}_{k=1}^{\infty}$ be the sequence of integers i such that $\chi_i(y) \neq 0$. Fix a $k > 1$ and write $n_k = 2^j + m$ where $0 \leq m < 2^j$. If $y \notin \Delta_{n_k}$, then

$$\begin{aligned}
 (1) \quad \left| \int_{J_{n_k+1}(y)} f_{n_k+1} \right| &= \left| \int_{J'_{n_k+1}(y)} f_{n_k+1} \right| = \frac{1}{2^{j+1}} |a_{n_k}| \sqrt{2^j} \\
 &= \frac{|a_{n_k}|}{2 |\chi_{n_k}(y)|} .
 \end{aligned}$$

If $y \in \Delta_{n_k} \setminus \{0,1\}$, then y is an endpoint of the support of χ_{n_k} since $\chi_{n_k}(y) \neq 0$. Therefore, of the two integrals

$$\int_{J'_{n_k+1}(y)} f_{n_k+1} \quad \text{and} \quad \int_{J_{n_k+1}(y)} f_{n_k+1}$$

one is zero and the other has absolute value

$$(2) \quad \frac{1}{2^{j+1}} |a_{n_k}| \sqrt{2^j} = \frac{|a_{n_k}|}{4 |\chi_{n_k}(y)|} .$$

If $y=0$ or 1 , then one of the two integrals

$$\int_{J_{n_k+1}(y)} f_{n_k+1} \quad \text{and} \quad \int_{J'_{n_k+1}(y)} f_{n_k+1}$$

is zero and the other has absolute value

$$(3) \quad \frac{1}{2^{j+1}} |a_{n_k}| \sqrt{2^j} = \frac{|a_{n_k}|}{2 |\chi_{n_k}(y)|} .$$

We will show that there is an N such that

$$\begin{aligned}
 (4) \quad \int_{J_{n+1}(y)} f_{n+1} &= \int_{J'_{n+1}(y)} f_{n+1} = 0 \quad \text{for } n > N \quad \text{and} \\
 n &\neq n_k \quad (k = 1, 2, \dots) .
 \end{aligned}$$

There are two cases.

Case 1. Suppose $y \notin \Delta = \bigcup_{j=1}^{\infty} \Delta_j$. Let $n \neq n_k$ ($k = 1, 2, \dots$). Then $y \notin \Delta_{n+1}$. Consequently $J_{n+1}(y) = J'_{n+1}(y)$. Furthermore χ_n is zero on $\text{int } J_{n+1}(y)$ since $\chi_n(y) = 0$ and χ_n is constant on the interior of $J_{n+1}(y)$. Thus

$$\int_{J_{n+1}(y)} f_{n+1} = \int_{J'_{n+1}(y)} f_{n+1} = 0.$$

So in this case (4) holds for any $n \neq n_k$ ($k = 1, 2, 3, \dots$).

Case 2. Assume $y \in \Delta$. Then $y \in \Delta_p$ for some p . Let n be an integer such that $\chi_n(y) = 0$. Then either y is the midpoint of the support of χ_n , or y is outside the support of χ_n . Suppose y is the midpoint of the support of χ_n . Then $y \in \Delta_{n+1}$ and $y \notin \Delta_i$ for $i \leq n$. Therefore, since $\Delta_i \subset \Delta_{i+1}$ ($i = 1, 2, \dots$), we have $n+1 \leq p$. From this consideration we see that if $n \geq p$ and $\chi_n(y) = 0$, then y is outside the support of χ_n . Let $n \geq p$ and $n \neq n_k$ ($k = 1, 2, \dots$). Since the support of χ_n is the union of two adjacent closed intervals with endpoints in Δ_{n+1} and since y is outside the support of χ_n , the support of χ_n does not overlap with $J_{n+1}(y)$ and $J'_{n+1}(y)$. Hence

$$\int_{J_{n+1}(y)} f_{n+1} = \int_{J'_{n+1}(y)} f_{n+1} = 0$$

for $n \neq n_k$ ($k = 1, 2, \dots$). This establishes (4).

Now assume that $\sum_{n=0}^{\infty} a_n \chi_n$ satisfies condition GH. Let $\epsilon > 0$. There is a K such that

$$\left| \frac{a_{n_k}}{\chi_{n_k}(y)} \right| < \epsilon \quad \text{for } k \geq K.$$

Combining this with (1), (2) and (3) we have

$$\left| \int_{J_{n_k+1}(y)} f_{n_k+1} \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| \int_{J'_{n_k+1}(y)} f_{n_k+1} \right| < \frac{\epsilon}{2}$$

for $k \geq K$.

Therefore by (4) we see that

$$\left| \int_{J_n(y)} f_n \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| \int_{J'_n(y)} f_n \right| < \frac{\epsilon}{2}$$

for all $n > \max(n_K+1, N+1)$. Thus $\sum_{n=1}^{\infty} f_n$ satisfies condition G.

Conversely, suppose $\sum_{n=1}^{\infty} f_n$ satisfies condition G. Let $\epsilon > 0$. Then there is an N such that

$$\left| \int_{J_n(y)} f_n \right| < \frac{\epsilon}{4} \quad \text{and} \quad \left| \int_{J'_n(y)} f_n \right| < \frac{\epsilon}{4} \quad \text{for } n > N.$$

Choose K such that $n_K > N$. Then from (1), (2) and (3) it follows that

$$\left| \frac{a_{n_k}}{\chi_{n_k}(y)} \right| < \epsilon \quad \text{for } k \geq K.$$

Therefore $\sum_{n=0}^{\infty} a_n \chi_n$ satisfies condition GH. $\square$

The equivalence of conditions G and H for Haar series follows from a more general assertion.

4. Definition. We say $\{D_n\}$ satisfies condition Q_0 if there exists $q \in (0,1)$ with the following property: if $J \in \mathcal{D}_n$, $L \in \mathcal{D}_{n+1}$, $L \subset J$, then either $L=J$ or $|L| < q|J|$ ($n = 1, 2, \dots$).

5. Theorem. Let $\{D_n\}$ satisfy condition Q_0 . Assume $\sum_{n=1}^{\infty} f_n$ is a K-series with respect to $\{D_n\}$. Then $\sum_{n=1}^{\infty} f_n$ satisfies condition G if and only if it satisfies condition H.

Proof. Assume $\sum_{n=1}^{\infty} f_n$ satisfies condition H. Let $y \in [0,1]$. Then

$$\lim_{n \rightarrow \infty} \int_{J_n(y)} s_n = \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{J_n(y)} f_j = 0 \text{ implies}$$

$$\lim_{n \rightarrow \infty} \int_{J_n(y)} f_n = 0.$$

Similarly

$$\lim_{n \rightarrow \infty} \int_{J'_n(y)} s_n = 0 \text{ implies } \lim_{n \rightarrow \infty} \int_{J'_n(y)} f_n = 0.$$

Therefore $\sum_{n=1}^{\infty} f_n$ satisfies condition G.

Conversely, suppose $\sum_{n=1}^{\infty} f_n$ satisfies condition G. Let $x \in [0,1]$. Set

$$\beta_j = \sup_{j \leq k} \int_{J_k(x)} |f_k| \quad (j = 1, 2, \dots).$$

Let q be the number in definition 4 and let

$$b_m = \beta_1 q^{m-1} + \dots + \beta_{m-1} q + \beta_m \quad (m = 1, 2, \dots).$$

Since $\beta_{j+1} \leq \beta_j$ for $j \geq 1$, we have

$$b_m \leq \beta_1 (q^{m-1} + \dots + q + 1) \leq \frac{\beta_1}{1-q} \quad \text{for all } m \geq 1.$$

An easy induction shows that $b_{m+1} = qb_m + \beta_{m+1}$
($m = 1, 2, \dots$). Therefore

$$\limsup_{m \rightarrow \infty} b_m = q \limsup_{m \rightarrow \infty} b_m + \lim_{m \rightarrow \infty} \beta_m = q \limsup_{m \rightarrow \infty} b_m.$$

Since $\limsup_{m \rightarrow \infty} b_m < \infty$ it follows that $\limsup_{m \rightarrow \infty} b_m = \lim_{m \rightarrow \infty} b_m = 0$.

For each $j \geq 1$ let c_j be the number of distinct intervals occurring in $J_1(x), \dots, J_j(x)$. For each $n \geq 1$ we have

$$(1) \quad \int_{J_n(x)} |f_j| = \frac{|J_n(x)|}{|J_j(x)|} \int_{J_j(x)} |f_j| \leq q^{c_n - c_j} \int_{J_j(x)} |f_j|$$

($j = 1, \dots, n$)

by condition Q_0 . If $J_j(x) = J_{j+1}(x)$ for any j , then $f_{j+1} = 0$ on $\text{int } J_{j+1}(x)$, because of the orthogonality. Combining this fact with (1) we have

$$(2) \quad \begin{aligned} \int_{J_n(x)} |f_1| + \dots + \int_{J_n(x)} |f_n| &\leq q^{c_n - 1} \int_{J_1(x)} |f_1| \\ &+ q^{c_n - c_2} \int_{J_2(x)} |f_2| + \dots + q^{c_n - c_{n-1}} \int_{J_{n-1}(x)} |f_{n-1}| \\ &+ q^{c_n - c_n} \int_{J_n(x)} |f_n| \leq q^{c_n - 1} \beta_1 + q^{c_n - 2} \beta_2 + \dots \\ &+ q \beta_{c_n - 1} + \beta_{c_n} = b_{c_n} \end{aligned}$$

since $J_{j+1}(x) \neq J_j(x)$ implies $c_{j+1} = c_j + 1$.

Now

$$\begin{aligned} \left| \int_{J_n(x)} s_n \right| &= \int_{J_n(x)} |s_n| \leq \int_{J_n(x)} |f_1| + \dots \\ &+ \int_{J_n(x)} |f_n| \quad (n = 1, 2, \dots). \end{aligned}$$

Thus it follows from (2) that

$$(3) \quad \left| \int_{J_n(x)} s_n \right| \leq b_{c_n} \quad (n = 1, 2, \dots).$$

From the density of D it follows that $\lim_{n \rightarrow \infty} c_n = \infty$.

Hence, given $\epsilon > 0$, there is an N such that $n \geq N$ implies $b_{c_n} < \epsilon$. From this and (3) it follows that $\left| \int_{J_n(x)} s_n \right| < \epsilon$ for $n \geq N$.

Similarly, we may show $\lim_{n \rightarrow \infty} \int_{J'_n(x)} s_n = 0$.

Therefore $\sum_{n=1}^{\infty} f_n$ satisfies condition H. $\square$

It should be remarked that assumption of condition Q_0 is not used in proving the "if" part of the assertion.

Combining theorem 5 and proposition 3 we see that for Haar series conditions GH, G and H are equivalent. In view of the following proposition, condition H seems to be the more natural condition to impose for the study of K-series.

6. Proposition. Let g be a Perron integrable function on $[0,1]$. Then the PFK-series for g satisfies condition H.

Proof. Let G be an indefinite Perron integral of g . Then

$$\int_{J_n(x)} s_n(g) = \int_{J_n(x)} g = G(\beta_n(x)) - G(\alpha_n(x))$$

and

$$\int_{J_n'(x)} s_n(g) = \int_{J_n'(x)} g = G(\beta_n'(x)) - G(\alpha_n'(x))$$

$$(n = 1, 2, \dots).$$

Since D is dense, $\lim_{n \rightarrow \infty} \beta_n(x) = \lim_{n \rightarrow \infty} \alpha_n(x) = x$ and

$$\lim_{n \rightarrow \infty} \beta_n'(x) = \lim_{n \rightarrow 0} \alpha_n'(x) = x. \text{ Therefore } \lim_{n \rightarrow \infty} \int_{J_n(x)} s_n(g) = 0$$

and $\lim_{n \rightarrow \infty} \int_{J_n'(x)} s_n(g) = 0$ by the continuity of G . $\square$

It should be remarked that proposition 6 remains valid if the Perron integral is replaced by any method of integration for which the indefinite integral is continuous.

7. Lemma. Let $\sum_{n=1}^{\infty} f_n$ be a K-series, F as in definition 13 of Chapter II. Then the following properties hold:

(i) for $x \notin D_n$

$$s_n(x) = \frac{F(\beta_n(x)) - F(\alpha_n(x))}{\beta_n(x) - \alpha_n(x)} \quad (n = 1, 2, \dots).$$

(ii) for $x \in D_n \cap (0, 1)$

$$s_n(x) = \frac{1}{2} \left[\frac{F(x) - F(\alpha_n'(x))}{x - \alpha_n'(x)} + \frac{F(\beta_n(x)) - F(x)}{\beta_n(x) - x} \right]$$

(iii) if $\sum f_n$ fulfills condition H, then

$$\lim_{n \rightarrow \infty} [F(\beta_n(x)) - F(\alpha_n(x))] = 0 \quad \text{for all } x \in (0, 1)$$

and

$$\lim_{n \rightarrow \infty} F(\alpha_n'(x)) = \lim_{n \rightarrow \infty} F(\beta_n(x)) = F(x) \quad \text{for } x \in D.$$

Proof. For any $x \in [0, 1]$

$$(1) \quad \int_{\alpha_n(x)}^{\beta_n(x)} s_n = \int_0^{\beta_n(x)} s_n - \int_0^{\alpha_n(x)} s_n = F(\beta_n(x)) - F(\alpha_n(x))$$

by lemma 19 of Chapter II. For $x \notin D_n$ we have

$$\int_{\alpha_n(x)}^{\beta_n(x)} s_n = s_n(x) [\beta_n(x) - \alpha_n(x)]$$

since s_n is constant on $(\alpha_n(x), \beta_n(x))$. This proves (i).

If $x \in D_n \cap (0, 1)$, then $2s_n(x) = s_n(x+) + s_n(x-)$ by definition. Now $s_n(x-) = s_n(t)$ for any $t \in (\alpha_n'(x), x)$ and $s_n(x+) = s_n(\tau)$ for any $\tau \in (x, \beta_n(x))$. Now we apply part (i) in order to compute $s_n(x+)$ and $s_n(x-)$. This proves (ii).

Assume $\sum_{n=1}^{\infty} f_n$ satisfies condition H. From
(1) we observe that

$$(2) \quad |F(\beta_n(x)) - F(\alpha_n(x))| = \left| \int_{\alpha_n(x)}^{\beta_n(x)} s_n \right|.$$

Thus $F(\beta_n(x)) - F(\alpha_n(x)) \rightarrow 0$ as $n \rightarrow \infty$.

If $x \in D_m$ for some m , then $\alpha_n(x) = x$ for $n \geq m$. Therefore for $n \geq m$ (2) becomes

$$|F(\beta_n(x)) - F(x)| = \left| \int_x^{\beta_n(x)} s_n \right|.$$

Similarly,

$$|F(x) - F(\alpha_n(x))| = \left| \int_{\alpha_n(x)}^x s_n \right| \quad \text{for } n \geq m.$$

This completes the proof of (iii). $\square$

8. Proposition. Let $\sum_{n=1}^{\infty} f_n$ be a K-series. Suppose $F(x) = 0$ for every $x \in D$. Then $s_n = 0$ ($n = 1, 2, 3, \dots$).

Proof. This is an immediate consequence of lemma 7, (ii) and the regularity, for we have $s_n = 0$ on the interior of any interval of I_n ($n = 1, 2, \dots$). $\square$

9. Proposition. Let f be a Perron integrable function on $[0, 1]$. Let $\sum_{n=1}^{\infty} f_n$ be the PFK-series for f . Then for each $x \in [0, 1]$, $F(x) = \sum_{n=1}^{\infty} \int_0^x f_n$ exists as a finite number and

$$F(x) = (P) \int_0^x f(t) dt.$$

Proof. Let $x \in D_m$. Then $x = t_{m,k}$ for some k . Hence by lemma 19 of Chapter II

$$\begin{aligned} F(x) &= \sum_{j=1}^k \int_{t_{m,j-1}}^{t_{m,j}} s_m = \sum_{j=1}^k (P) \int_{t_{m,j-1}}^{t_{m,j}} f \\ &= (P) \int_0^x f(t) dt. \end{aligned}$$

Let $x \in [0,1] \setminus D$. Fix an n . Then by the first paragraph

$$F(\alpha_n(x)) = (P) \int_0^{\alpha_n(x)} f(t) dt.$$

Thus

$$\begin{aligned} \int_0^x s_n(t) dt &= \int_0^{\alpha_n(x)} s_n + \int_{\alpha_n(x)}^x s_n = F(\alpha_n(x)) \\ &+ \int_{\alpha_n(x)}^x s_n = (P) \int_0^{\alpha_n(x)} f + \int_{\alpha_n(x)}^x s_n. \end{aligned}$$

Now

$$\left| \int_{\alpha_n(x)}^x s_n \right| \leq \left| \int_{\alpha_n(x)}^{\beta_n(x)} s_n \right| = \left| (P) \int_{\alpha_n(x)}^{\beta_n(x)} f \right|$$

since s_n is constant on $(\alpha_n(x), \beta_n(x))$. By the continuity of the integral we have $(P) \int_{\alpha_n(x)}^{\beta_n(x)} f \rightarrow 0$ and $(P) \int_0^{\alpha_n(x)} f \rightarrow (P) \int_0^x f$ (as $n \rightarrow \infty$). Therefore

$$\lim_{n \rightarrow \infty} \int_0^x s_n(t) dt = (P) \int_0^x f. \quad \square$$

10. Definition. Let $a, b \in D$, $a < b$. Let $\mathfrak{F}_{a,b}'$ be the class of all finite real-valued functions on $D \cap [a,b]$ which fulfill the following properties:

P_1 : If $x \in [a, b) \cap D$ and $\lambda = \lim_{n \rightarrow \infty} f(\beta_n(x))$,
finite or infinite, exists, then $f(x) \geq \lambda$.

P_2 : If $x \in (a, b] \cap D$ and $\mu = \lim_{n \rightarrow \infty} f(\alpha_n(x))$,
finite or infinite, exists, then $f(x) \leq \mu$.

11. Remark. Let $a, b \in D$, $f \in \mathfrak{F}_{a,b}$, and let g be continuous on $D \cap [a, b]$. Then $f+g \in \mathfrak{F}_{a,b}$.

Proof. Let $x \in [a, b) \cap D$ and let the limit $\gamma = \lim_{n \rightarrow \infty} [f(\beta_n(x)) + g(\beta_n(x))]$ exist. Then also the limit $\delta = \lim_{n \rightarrow \infty} f(\beta_n(x))$ exists and $\gamma = \delta + g(x) \leq f(x) + g(x)$. So $f+g$ fulfills P_1 .

Similarly we have property P_2 for $f+g$. $\square$

12. Lemma. Let $a, b \in D$, $a < b$. Assume $f \in \mathfrak{F}_{a,b}$, $J \in \mathcal{A}_n$, $J \subset [a, b]$ and that $f(J) \geq 0$. Then there exists an $m > n$ and intervals $J_k \in \mathcal{A}_k$ such that $J = J_n \supset J_{n+1} \supset \dots \supset J_m$, $f(J_k) \geq 0$ ($k = n, \dots, m$) and $J_m \subset \text{int } J_n$.

Proof. Denote $J = [x, y]$. Among all the intervals of $\mathcal{A}_{n+1}$ contained in J there must be at least one, say L , such that $f(L) \geq 0$. In this way we may construct intervals $L_i \in \mathcal{A}_i$ such that $J = L_n \supset L_{n+1} \supset \dots$, $f(L_i) \geq 0$ ($i \geq n$). Let p be the smallest integer greater than n for which $L_p \neq J$. The existence of such a p follows from the density of D .

Suppose the assertion is false. Then we have either $x \in L_i$ or $y \in L_i$ for any $i \geq n$.

1) Assume that $x \in L_p$. Then $x \in L_i$ for all $i \geq n$, and, therefore, $L_i = [x, \beta_i(x)]$ for all i . Choose an $i > p$. Denote the set $D_i \cap L_{i-1}$ by $\{\tau_0, \dots, \tau_s\}$, where $x = \tau_0 < \dots < \tau_s = \beta_{i-1}(x)$. Clearly $\tau_1 = \beta_i(x)$. If $s=1$, then $f(\beta_i(x)) = f(\beta_{i-1}(x))$. If $s > 1$, then we must have $f(\tau_{j-1}) > f(\tau_j)$ for $j = 2, \dots, s$, for otherwise the assertion would be true with $m=i$, $J_k = L_k$ ($k = n, \dots, i-1$), $J_i = [\tau_{j-1}, \tau_j]$; thus $f(\beta_i(x)) = f(\tau_1) > f(\tau_s) = f(\beta_{i-1}(x))$. We have therefore $f(\beta_i(x)) \geq f(\beta_{i-1}(x)) \geq f(x)$ for any $i > p$. Because of the density of D there are infinitely many numbers $i > p$ for which $s > 1$ and, therefore, $f(\beta_i(x)) > f(\beta_{i-1}(x))$. It follows that $\lim_{i \rightarrow \infty} f(\beta_i(x)) > f(x)$ which contradicts P_1 .

2) If $y \in L_p$ then we prove analogously that $\lim_{i \rightarrow \infty} f(\alpha_i'(y)) < f(y)$ which contradicts P_2 . This proves our assertion. $\square$

13. Lemma. Let $a, b \in D$, $a < b$. Assume that $[a, b] \in \mathcal{B}_j$ for some j . Let $f \in \mathfrak{F}_{a,b}$ and $f(a) \leq f(b)$. Then there exists an $x \in (a, b) \setminus D$ such that $f(\alpha_n(x)) \leq f(\beta_n(x))$ ($n = j, j+1, \dots$).

Proof. Take $n=j$, $J = [a,b]$ in l2 and construct the intervals $J_j, \dots, J_m$. Set $p_0 = j$, $p_1 = m$. Now apply l2 with $n=m$ and $J=J_m$, getting intervals $J_{p_1} \supset J_{p_1+1} \supset \dots \supset J_{p_2}$ such that $J_k \in \mathcal{A}_k$, $f(J_k) \geq 0$ ($k = p_1, \dots, p_2$) and $J_{p_2} \subset \text{int } J_{p_1}$. Continuing in this fashion we construct sequences of intervals $J_j \supset J_{j+1} \supset \dots$ and integers $p_0 < p_1 < \dots$ such that

$$J_k \in \mathcal{A}_k, f(J_k) \geq 0 \quad (k = j, j+1, \dots)$$

and $J_{p_i} \subset \text{int } J_{p_{i-1}}$ ($i = 1, 2, \dots$).

Since each J_k is closed and D is dense in $[0,1]$ there is a point x_0 such that $\bigcap_{k=j}^{\infty} J_k = \{x_0\}$. It is obvious that $J_k = [\alpha_k(x_0), \beta_k(x_0)]$ for each k and that $x_0 \in \text{int } J_k$ for each k , so $x_0 \notin D$. Having remarked that $f(J_k) \geq 0$ ($k = j, j+1, \dots$), the proof is complete. $\square$

The following lemma is a generalization of a lemma in [2], due in its present form, to J. Marik.

14. Lemma. Let $a, b \in D$, $a < b$. Assume $f \in \mathfrak{F}_{a,b}$. Let C be a countable set in (a,b) . Let $\liminf_{n \rightarrow \infty} f(J_n(x)) \leq 0$ for $x \in (a,b) \setminus D$. Let $\liminf_{n \rightarrow \infty} \frac{f(J_n(x))}{|J_n(x)|} \leq 0$ for $x \in (a,b) \setminus (C \cup D)$. Then f is non-increasing on $D \cap [a,b]$.

Proof. Suppose not. Then there exists $y, z \in D \cap [a, b]$ such that $y < z$ and $f(y) < f(z)$. There is a j such that $y, z \in D_j$. Then we can find $J = [v, w] \in \mathcal{D}_j$ for which $f(J) > 0$. Choose an $\epsilon > 0$ such that $f(v) + 3\epsilon < f(w)$. We may assume C is an infinite subset of $(v, w) \setminus D$. Let $\{c_1, c_2, \dots\}$ be an enumeration of C . Define functions

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \operatorname{sgn}(x - c_n),$$

$$R(x) = \varphi(x) + \epsilon x, \quad \text{and}$$

$$\psi(x) = f(x) - R(x).$$

Now $R(w) - R(v) \leq R(1) - R(0) = \varphi(1) + \epsilon - \varphi(0) = 3\epsilon$, so $\psi(v) < \psi(w)$. Since φ is continuous on $[0, 1] \setminus C$, R is continuous on D . Thus $\psi \in \mathfrak{U}_{v, w}$.

For each $x \in (0, 1) \setminus (C \cup D)$ we have

$$\liminf_{n \rightarrow \infty} \frac{\psi(\beta_n(x)) - \psi(\alpha_n(x))}{\beta_n(x) - \alpha_n(x)} \leq \liminf_{n \rightarrow \infty} \left[\frac{f(\beta_n(x)) - f(\alpha_n(x))}{\beta_n(x) - \alpha_n(x)} - \epsilon \right] \leq -\epsilon$$

since φ is non-decreasing.

If $x \in C$, then

$$\begin{aligned} \liminf_{n \rightarrow \infty} [\psi(\beta_n(x)) - \psi(\alpha_n(x))] &= \liminf_{n \rightarrow \infty} [f(\beta_n(x)) - f(\alpha_n(x))] \\ &\quad - \lim_{n \rightarrow \infty} [\varphi(\beta_n(x)) - \varphi(\alpha_n(x))] \\ &\leq \lim_{n \rightarrow \infty} [\varphi(\alpha_n(x)) - \varphi(\beta_n(x))] < 0. \end{aligned}$$

Therefore, for each $x \in (0,1) \setminus D$ there is an n such that $\psi(\beta_n(x)) < \psi(\alpha_n(x))$. This is a contradiction with lemma 13. $\square$

15. Theorem. Let $\sum_{n=1}^{\infty} f_n$ be a K-series fulfilling condition H. Let $a, b \in D$, $a < b$. Let C be a countable set in (a,b) . Let f be a Perron integrable function on (a,b) . Assume

- (1) $\limsup_{n \rightarrow \infty} s_n(x) \geq f(x)$ a.e. on (a,b) ,
- (2) $\limsup_{n \rightarrow \infty} s_n(x) > -\infty$ and $\liminf_{n \rightarrow \infty} s_n(x) < +\infty$ for $x \in (a,b) \setminus (C \cup D)$.

Then

- (3) $\sum_{n=1}^{\infty} f_n(x)$ exists and is finite a.e. on (a,b) ,
- (4) $\sum_{n=1}^{\infty} f_n(x)$ is Perron integrable on (a,b) , and
- (5) $F(x) - (P) \int_0^x \sum_{n=1}^{\infty} f_n$ is constant on $D \cap [a,b]$.

Proof. We may assume $\limsup_{n \rightarrow \infty} s_n(x) \geq f(x)$ everywhere on (a,b) . Let G be a minorant of f . Define H on $D \cap [a,b]$ by $H(x) = G(x) - F(x)$.

Set $\psi(x) = (P) \int_a^x f$ for $x \in [a,b]$. Then ψ is continuous and $G - \psi$ is non-increasing. Thus $G(x+) \leq G(x)$ for any $x \in [a,b]$. If $x \in [a,b] \cap D$,

then $\lim_{n \rightarrow \infty} F(\beta_n(x)) = F(x)$ by lemma 7, part (iii), so that $\lim_{n \rightarrow \infty} H(\beta_n(x)) \leq H(x)$. Similarly $\lim_{n \rightarrow \infty} H(\alpha_n'(x)) \geq H(x)$ for any $x \in (a,b] \cap D$. Hence $H \in \mathfrak{U}_{a,b}$.

Furthermore, for each $x \in (a,b)$

$$\begin{aligned} \liminf_{n \rightarrow \infty} [H(\beta_n(x)) - H(\alpha_n(x))] &= \liminf_{n \rightarrow \infty} [G(\beta_n(x)) - G(\alpha_n(x)) \\ &\quad - F(\beta_n(x)) + F(\alpha_n(x))] \leq 0 \end{aligned}$$

since $\bar{D}G(x) \neq +\infty$ and $\lim_{n \rightarrow \infty} [F(\beta_n(x)) - F(\alpha_n(x))] = 0$ by lemma 7.

For each $x \in (a,b) \setminus (C \cup D)$ we have by lemma 7, part (i),

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{H(\beta_n(x)) - H(\alpha_n(x))}{\beta_n(x) - \alpha_n(x)} &= \liminf_{n \rightarrow \infty} \left[\frac{G(\beta_n(x)) - G(\alpha_n(x))}{\beta_n(x) - \alpha_n(x)} - s_n(x) \right] \\ &\leq \bar{D}G(x) - \limsup_{n \rightarrow \infty} s_n(x) \leq 0 \end{aligned}$$

since $\bar{D}G(x) < +\infty$, $\limsup_{n \rightarrow \infty} s_n(x) > -\infty$, and

$\bar{D}G(x) \leq f(x) \leq \limsup_{n \rightarrow \infty} s_n(x)$. Thus we may apply lemma 14

to conclude that H is non-increasing on $D \cap [a,b]$.

Since D is dense in $[0,1]$, H may be extended to a non-increasing function $\tilde{H}$ on all of $[a,b]$. Since $\tilde{H}$ is monotone, $\tilde{H}'(x)$ exists and is finite almost everywhere on (a,b) . Furthermore

$$(L) \int_a^b \tilde{H}' \geq H(b) - H(a)$$

which implies $\tilde{H}'$ is Lebesgue integrable on (a,b) .

Recall $\psi(x) = (P) \int_a^x f$. Since $\psi - G$ is non-decreasing there is a set A of measure zero such that for any $x \in (a,b) \setminus A$, $\psi - G$ has a finite derivative. Furthermore, the function $\bar{D}(\psi - G)$ is Lebesgue integrable on (a,b) . Since $\psi'(x)$ exists and is almost everywhere $f(x)$, there is a set B of measure zero such that $G'(x) = \psi'(x) - (\psi - G)'(x)$ exists as a finite number for every $x \in (a,b) \setminus B$. Thus $\bar{D}G$ is Perron integrable on (a,b) since $\bar{D}\psi$ is Perron integrable and $\bar{D}(\psi - G)$ is Lebesgue integrable on (a,b) . For any $x \in (a,b) \setminus D_n$ we have

$$(6) \quad s_n(x) = \frac{G(\beta_n(x)) - G(\alpha_n(x))}{\beta_n(x) - \alpha_n(x)} - \frac{H(\beta_n(x)) - H(\alpha_n(x))}{\beta_n(x) - \alpha_n(x)}$$

$$(n = 1, 2, 3, \dots).$$

Since $\tilde{H}'(x)$ and $G'(x)$ exist and are finite for almost every $x \in (a,b)$, it follows from (6) that $\lim_{n \rightarrow \infty} s_n(x)$ exists and is finite for almost every $x \in (a,b)$.

Furthermore the function $\varphi(x) = \limsup_{n \rightarrow \infty} s_n(x)$ being equal almost everywhere to the difference of two Perron integrable functions, is a Perron integrable function on (a,b) . This proves (3) and (4).

To prove (5) construct sequences of minorants $\{U_n\}_{n=1}^{\infty}$ and majorants $\{L_n\}_{n=1}^{\infty}$ for φ such that

$$(7) \quad \lim_{n \rightarrow \infty} U_n(x) = \lim_{n \rightarrow \infty} L_n(x) = (P) \int_0^x \varphi$$

for every $x \in (a, b)$.

Since we may take $f = \varphi$ in the first part of the proof, we see that the function $U_n - F$ fulfills the conditions of lemma 14 ($n = 1, 2, \dots$).

Part (iii) of lemma 7 and the fact that L_n is a majorant ($\underline{DL}_n(x) \neq -\infty$ for all x) imply that the function $K = F - L_n$ belongs to $\mathfrak{F}_{a,b}$ and that

$$\liminf_{j \rightarrow \infty} [K(\beta_j(x)) - K(\alpha_j(x))] = 0 \quad (n = 1, 2, \dots).$$

Moreover, for each $x \in (a, b) \setminus (C \cup D)$

$$\begin{aligned} \liminf_{j \rightarrow \infty} \frac{K(\beta_j(x)) - K(\alpha_j(x))}{\beta_j(x) - \alpha_j(x)} \\ = \liminf_{j \rightarrow \infty} \left[s_j(x) - \frac{L_n(\beta_j(x)) - L_n(\alpha_j(x))}{\beta_j(x) - \alpha_j(x)} \right] \\ \leq \liminf_{j \rightarrow \infty} s_j(x) - \underline{DL}_n(x) \leq 0 \quad (n = 1, 2, \dots) \end{aligned}$$

because L_n is a majorant for φ , $\underline{DL}_n(x) > -\infty$,

$$\liminf_{j \rightarrow \infty} s_j(x) < +\infty, \quad \text{and} \quad \liminf_{j \rightarrow \infty} s_j(x) \leq \varphi(x) \leq \underline{DL}_n(x).$$

Therefore for each $n > 0$, $F - L_n$ and $U_n - F$ fulfill the conditions of lemma 14. Hence $F - L_n$

and $U_n - F$ are non-increasing on $D \cap [a, b]$ for each n . From this and (7) it follows that $F(x) - \int_a^x \varphi$ and $\int_a^x \varphi - F(x)$ are non-increasing on $D \cap [a, b]$.

Whence, $F(x) - \int_a^x \lim_{j \rightarrow \infty} s_j$ is constant on $[a, b] \cap D$. $\square$

16. Theorem. Let $\sum_{n=1}^{\infty} f_n$ be a K-series satisfying condition H. Let C be a countable set in $(0, 1)$, f be a Perron integrable function on $(0, 1)$. Assume

$$\limsup_{n \rightarrow \infty} s_n(x) \geq f(x) \quad \text{a.e. on } (0, 1),$$

$$\limsup_{n \rightarrow \infty} s_n(x) > -\infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} s_n(x) < +\infty$$

for all $x \in (0, 1) \setminus (C \cup D)$. Then $\sum_{n=1}^{\infty} f_n$ converges a.e. to a Perron integrable function φ and $\sum_{n=1}^{\infty} f_n$ is the PFK-series for φ . Moreover

$$\sum_{n=1}^{\infty} \int_0^x f_n = (P) \int_0^x \varphi \quad \text{for all } x \in [0, 1].$$

Proof. Take $a=0, b=1$ in theorem 15. Then by 15, $\sum_{n=1}^{\infty} f_n$ converges a.e. on $[0, 1]$ to a Perron integrable function φ . Furthermore $F(x) - (P) \int_0^x \varphi$ is constant on D . But $F(0) = 0$, so $F(x) = (P) \int_0^x \varphi$ for $x \in D$.

Let $\sum_{n=1}^{\infty} g_n$ be the PFK-series for φ . By proposition 9, $\sum_{n=1}^{\infty} \int_0^x g_n = (P) \int_0^x \varphi$ for all $x \in [0,1]$.

Therefore, proposition 8 applied to the K-series

$\sum_{n=1}^{\infty} (f_n - g_n)$ gives $f_n = g_n$ ($n = 1, 2, \dots$). This proves the assertion. $\square$

17. Theorem. Let $\sum_{n=1}^{\infty} f_n$ be a K-series satisfying condition H. Let g be a Perron integrable function on $[0,1]$. Let C be a countable subset of $[0,1]$. Let $\{a_{n,k}\}$ and $\{b_{n,k}\}$ be two non-negative, limit preserving matrices such that $\{k : a_{n,k} \neq 0\}$ and $\{k : b_{n,k} \neq 0\}$ are finite ($n = 1, 2, \dots$). Define $\sigma_n = \sum_{k=1}^{\infty} a_{nk} s_k$ and $\sigma'_n = \sum_{k=1}^{\infty} b_{nk} s_k$. Let $\{n_j\}$ and $\{n'_j\}$ be two increasing sequences of natural numbers.

Assume

$$\lim_{j \rightarrow \infty} \sigma_{n_j} = g \text{ in measure,}$$

$$\limsup_{j \rightarrow \infty} \sigma'_{n'_j}(x) > -\infty \text{ for } x \in (0,1) \setminus C, \text{ and}$$

$$\liminf_{j \rightarrow \infty} \sigma'_{n'_j}(x) < +\infty \text{ for } x \in (0,1) \setminus C.$$

Then $\sum_{n=1}^{\infty} f_n(x) = g(x)$ for almost every x in $[0,1]$ and $\sum_{n=1}^{\infty} f_n$ is the PFK-series for g .

Proof. There are integers $j_1 < j_2 < \dots$ such that putting $m_l = n_{j_l}$, we have $\lim_{l \rightarrow \infty} \sigma_{m_l}(x) = g(x)$

for almost all x . Since the matrices are non-negative it follows that $\limsup_{n \rightarrow \infty} s_n \geq \limsup_{n \rightarrow \infty} \sigma_n$, $\limsup_{n \rightarrow \infty} s_n \geq \limsup_{n \rightarrow \infty} \sigma'_n$, and $\liminf_{n \rightarrow \infty} s_n \leq \liminf_{n \rightarrow \infty} \sigma'_n$. Because of this $\limsup_{n \rightarrow \infty} s_n(x) \geq \limsup_{n \rightarrow \infty} \sigma_n(x) \geq \lim_{l \rightarrow \infty} \sigma_{m_l}(x) = g(x)$ almost everywhere. For $x \in (0,1) \setminus C$

$$\limsup_{n \rightarrow \infty} s_n(x) \geq \limsup_{n \rightarrow \infty} \sigma'_n(x) \geq \limsup_{j \rightarrow \infty} \sigma'_{n_j}(x) > -\infty$$

and

$$\liminf_{n \rightarrow \infty} s_n(x) \leq \liminf_{n \rightarrow \infty} \sigma'_n(x) \leq \liminf_{j \rightarrow \infty} \sigma'_{n_j}(x) < +\infty.$$

Therefore theorem 16 may be applied to see that $\sum_{n=1}^{\infty} f_n$ converges a.e. to a Perron integrable function φ for which it is the PFK-series. But $\lim_{n \rightarrow \infty} s_n(x) = \varphi(x)$ a.e. implies $\lim_{n \rightarrow \infty} \sigma_n(x) = \varphi(x)$ a.e. since $\{a_{n,k}\}$ is limit preserving. Therefore $g = \varphi$ a.e. by the uniqueness of the limits. This completes the proof. $\square$

Generalizations of the theorems of Wade [14], [15] for Haar and Walsh series may be obtained as corollaries to proposition 3 and theorems 5, 16 and 17. In each of these corollaries C will denote an arbitrary countable set in $[0,1]$ and Δ the set of dyadic rationals in $[0,1]$. If $\sum_{k=0}^{\infty} a_k \chi_k$ is a Haar series, then we will denote $\sum_{k=0}^{n-1} a_k \chi_k$ by s_n ($n = 1, 2, \dots$).

If we put $a_{n,k} = b_{n,k} = 0$ for $n \neq k$ and $a_{n,n} = b_{n,n} = 1$ ($n, k = 1, 2, \dots$) in theorem 17, then we have the following assertion.

18. Corollary. Let $\sum_{n=0}^{\infty} a_n \chi_n$ be a Haar series satisfying condition GH. Let g , $\{n_j\}$ and $\{n'_j\}$ be as in theorem 17. Set $2^{n_j} = m_j$ and $2^{n'_j} = m'_j$. Assume

$$\lim_{j \rightarrow \infty} s_{m_j} = g \text{ in measure,}$$

$$\limsup_{j \rightarrow \infty} s_{m'_j} > -\infty \text{ on } (0,1) \setminus C \text{ and}$$

$$\liminf_{j \rightarrow \infty} s_{m'_j} < +\infty \text{ on } (0,1) \setminus C.$$

Then $\sum_{n=0}^{\infty} a_n \chi_n(x) = g(x)$ for almost all x in $[0,1]$ and $\sum_{n=0}^{\infty} a_n \chi_n$ is the Perron-Haar-Fourier series for g .

Let $\psi_0, \psi_1, \dots$ be the Walsh functions and $a_0, a_1, \dots$ be real numbers. Denote $\sum_{j=0}^{n-1} a_j \psi_j$ by s_n ($n = 1, 2, \dots$), as we did above for Haar series. It is well known (see for example [1] or [14]) that there are real numbers $\gamma_0, \gamma_1, \dots$ such that

$$\sum_{j=0}^{2^n-1} a_j \psi_j(x) = \sum_{i=0}^{2^n-1} \gamma_i \chi_i(x) \text{ for } x \notin \Delta$$

and $n \geq 0$. Wade [14] has shown that if $\lim_{k \rightarrow \infty} a_k = 0$, then $\sum_{i=0}^{\infty} \gamma_i \chi_i$ satisfies condition GH. Wade [14] also has shown that if $\sum_{i=0}^{\infty} \gamma_i \chi_i$ is the Haar-Fourier series of a function g , then $\sum_{j=0}^{\infty} a_j \psi_j$ is the Walsh-Fourier series for g . Putting $C \cup \Delta$ for C in corollary 18 we get the following corollary to theorem 17.

19. Corollary. Let $\sum_{n=0}^{\infty} a_n \psi_n$ be a Walsh series. Let $g, \{n_j\}$ and $\{n'_j\}$ be as in theorem 17. Set $2^{n_j} = m_j$ and $2^{n'_j} = m'_j$. Assume

$$\lim_{j \rightarrow \infty} s_{m_j} = g \text{ in measure,}$$

$$\limsup_{j \rightarrow \infty} s_{m'_j} > -\infty \text{ on } (0,1) \setminus (C \cup \Delta),$$

$$\liminf_{j \rightarrow \infty} s_{m'_j} < +\infty \text{ on } (0,1) \setminus (C \cup \Delta), \text{ and}$$

$$\lim_{k \rightarrow \infty} a_k = 0.$$

Then $\sum_{n=0}^{\infty} a_n \psi_n$ is the Perron-Walsh-Fourier series for g and $\lim_{n \rightarrow \infty} s_{2^n}(x) = g(x)$ for almost all $x \in [0,1]$.

CHAPTER IV

EXISTENCE OF INFINITE SUMS FOR K-SERIES ON SETS OF POSITIVE MEASURE

In a 1965 paper, A.A. Talalan and F.G. Arutyunyan [13] have shown that a Haar or Walsh series cannot have an infinite sum on a set of positive measure. This problem has been considered by R.F. Gundy [7] and V.A. Skvorcov [12]. In this chapter we will show that under suitable restrictions on the sequence $\{D_n\}$, a K-series with respect to $\{D_n\}$ can not have an infinite sum on a set of positive measure. The method is similar to that used by Skvorcov.

1. Definition. Let g be a finite function on D . Recall that if $[x, y] = J \in \mathcal{D}$, we write $g(J) = g(y) - g(x)$. For $x \notin D$, define the symmetric D-derivates of g at x to be the limit points of the sequence

$$\frac{g(J_n(x))}{|J_n(x)|}.$$

In particular, define

$$\underline{S}_D g(x) = \liminf_{n \rightarrow \infty} \frac{g(J_n(x))}{|J_n(x)|}$$

and

$$\bar{S}_D g(x) = \limsup_{n \rightarrow \infty} \frac{g(J_n(x))}{|J_n(x)|}.$$

2. Lemma. Let $\sum_{n=1}^{\infty} f_n$ be a K-series. Let $x \in [0,1] \setminus D$. Then the set of limit points of the sequence $\{s_n(x)\}_{n=1}^{\infty}$ is the same as the set of symmetric D-derivates of F . In particular

$$\liminf_{n \rightarrow \infty} s_n(x) = \underline{S}_D F(x), \quad \limsup_{n \rightarrow \infty} s_n(x) = \bar{S}_D F(x) \quad \text{and}$$

at points of convergence $\sum_{n=1}^{\infty} f_n(x) = S_D F(x)$.

Proof. This is an immediate consequence of lemma 7, part (ii) of Chapter III.

3. Lemma. Let G be a finite function on D . Let $a > 0$. Let $E \subset [0,1] \setminus D$ be a set of positive outer measure. Assume $\underline{S}_D G(x) < a$ for every $x \in E$. Then for each $\epsilon > 0$ there is an integer $n \geq 1$ and an interval $Q \in \mathcal{J}_n$ such that

$$|Q| < \epsilon; \quad |E \cap Q| > (1-\epsilon)|Q| \quad \text{and} \\ G(Q) < a|Q|.$$

Proof. Let x be a point of outer density for E . There is a $\delta > 0$ such that $|E \cap J| > (1-\epsilon)|J|$ for all intervals J such that $x \in J$ and $|J| < \delta$.

By the hypothesis we have

$$\liminf_{n \rightarrow \infty} \frac{G(J_n(x))}{|J_n(x)|} < a.$$

Choose an i such that

$$\frac{G(J_i(x))}{|J_i(x)|} < a \quad \text{and} \quad |J_i(x)| < \min(\delta, \epsilon).$$

Take $Q = J_i(x)$. $\square$

4. Definition. We say the sequence $\{D_n\}$ satisfies condition Q_1 if there is a $q \in (0,1)$ with the property: if $J \in \mathcal{D}_n$, $L \in \mathcal{D}_{n+1}$ and $L \subset J$, then $\frac{|L|}{|J|} > q$ ($n = 1, 2, \dots$).

5. Lemma. Let $\{D_n\}$ satisfy condition Q_1 ; let q be the corresponding number from definition 4. Let G be a finite function on D . Let $E \subset [0,1]$, $Q \in \mathcal{D}$. Let $\epsilon > 0$ and b be arbitrary numbers. Assume that

$$(1) \quad |E \cap Q| > (1-\epsilon) |Q|$$

$$(2) \quad G(J) > 0 \quad \text{for every } J \in \mathcal{D} \text{ such that} \\ J \subset Q \text{ and } J \cap E \neq \emptyset$$

$$(3) \quad \bar{S}_D G(x) > b \quad \text{for all } x \in E.$$

Then $G(Q) > b(1 - \frac{2\epsilon}{q}) |Q|$.

Proof. We may assume $E \subset \text{Int } Q \setminus D$ and that all points of E are points of outer density. Since

$$(1) \quad \limsup_{n \rightarrow \infty} \frac{G(J_n(x))}{|J_n(x)|} = \bar{S}_D G(x) > b$$

for each $x \in E$, we may associate with each $x \in E$ and $\eta > 0$ a $J \in \mathcal{J}$ such that

$$(2) \quad G(J) > b|J|, \quad x \in J \quad \text{and} \quad |J| < \eta.$$

The collection of all intervals $J \in \mathcal{J}$ such that

$$(3) \quad J \cap E \neq \emptyset, \quad J \subset Q \quad \text{and} \quad G(J) > b|J|$$

is therefore a Vitali covering for E . Set $\epsilon_1 = \epsilon|Q|$.

Applying the Vitali covering theorem, we can find

a finite number of non-overlapping intervals

$J_1, \dots, J_N \in \mathcal{J}$, $J_i \subset Q$ so that

$$(4) \quad G(J_i) > b|J_i|$$

$$(5) \quad J_i \cap E \neq \emptyset \quad (i = 1, 2, \dots, N), \quad \text{and}$$

$$(6) \quad |E \setminus \bigcup_{i=1}^N J_i| < \epsilon_1.$$

But $|E| > (1-\epsilon)|Q|$ by assumption and

$$|E \setminus \bigcup_{i=1}^N J_i| \geq |E| - \sum_{i=1}^N |J_i|.$$

Combining this with (6) we see that

$$\begin{aligned} (7) \quad \left| \bigcup_{i=1}^N J_i \right| &= \sum_{i=1}^N |J_i| > |E| - \epsilon_1 > (1-\epsilon)|Q| - \epsilon|Q| \\ &= (1-2\epsilon)|Q|. \end{aligned}$$

Choose K so large that none of the intervals $J_1, \dots, J_N$ belongs to $\mathcal{A}_n$ for every $n \geq K$. Let $\mathcal{U}$ be the system of all intervals L such that (i) $L \subset Q$, (ii) $L \in \mathcal{A}_k$ for some $k \leq K$, (iii) either L is one of $J_1, \dots, J_N$ or $L \cap E \neq \emptyset$ and L has the property that of the intervals belonging to $\mathcal{A}_{k+1}$ whose union is L , at least one doesn't overlap with any of the intervals $J_1, \dots, J_N$.

We first show that $Q = \bigcup_{L \in \mathcal{U}} L$. For if not, then we could find an interval $L_0 \in \mathcal{A}_K$ which is contained in Q but not contained in any interval of $\mathcal{U}$. Now $J_j \in \mathcal{U}$ for $j = 1, \dots, N$ by the definition of $\mathcal{U}$. Suppose L_0 overlaps with J_j for some j . Then since $\mathcal{D}_k \subset \mathcal{D}_{k+1}$ for each k , either $L_0 \subset J_j$ or $J_j \subset L_0$. From the choice of K it follows that $J_j \subset L_0$ is impossible. So $L_0 \subset J_j$. But L_0 is not contained in any interval from $\mathcal{U}$. Hence L_0 overlaps with none of the intervals $J_1, \dots, J_N$. Therefore, if $L_0 \cap E \neq \emptyset$, then L_0 fulfills (i), (ii) and (iii) in the definition of $\mathcal{U}$. But $L_0 \notin \mathcal{U}$. Hence $L_0 \cap E = \emptyset$.

Consider the interval $L_1 \in \mathcal{A}_{K-1}$ which contains L_0 . We have $L_1 \subset Q$ because $Q \in \mathcal{A}$, so if $L_1 \cap E \neq \emptyset$, then L_1 satisfies conditions (i), (ii) and (iii) of the definition of $\mathcal{U}$. But this implies

$L_0 \subset \bigcup_{L \in \mathcal{U}} L$, a contradiction. So $L_1 \cap E = \emptyset$. If $L_1 = Q$, the construction terminates. In case $L_1 \neq Q$ proceed as follows. We claim L_1 overlaps with no interval belonging to $\mathcal{U}$. To see this recall that $D_k \subset D_{k+1}$ for every k . Therefore, if L_1 overlaps with any interval I of $\mathcal{U}$, either $I \subset L_1$ or $L_1 \subset I$. If $L_1 \subset I$, then $L_1 \subset \bigcup_{L \in \mathcal{U}} L$, which implies $L_0 \subset \bigcup_{L \in \mathcal{U}} L$, a contradiction. On the other hand, if $I \subset L_1$, then $L_1 \cap E \neq \emptyset$ since $I \cap E \neq \emptyset$ by the definition of $\mathcal{U}$, again a contradiction. Hence L_1 overlaps with none of the intervals from $\mathcal{U}$. In particular, L_1 overlaps with none of $J_1, \dots, J_N$. Now consider $L_2 \in \mathcal{D}_{K-2}$ such that $L_1 \subset L_2$. Clearly L_2 fulfills (i) and (ii) in the definition of $\mathcal{U}$. If $L_2 \cap E \neq \emptyset$, then L_2 fulfills (iii) in the definition of $\mathcal{U}$. But this implies $L_0 \subset L_1 \subset L_2 \subset \bigcup_{L \in \mathcal{U}} L$, a contradiction. So $L_2 \cap E = \emptyset$. In this way we could construct a finite ascending sequence of intervals disjoint from E and belonging respectively to $\mathcal{D}_K, \mathcal{D}_{K-1}, \dots, \mathcal{D}_n$ where $Q \in \mathcal{D}_n$. The last term in this sequence is Q . Therefore $Q \cap E = \emptyset$, contradiction. Hence $Q = \bigcup_{L \in \mathcal{U}} L$.

For any pair of intervals in $\mathcal{U}$ which overlap, one is contained in the other. Thus we may replace the system $\mathcal{U}$ by a system $\mathcal{U}_1 \subset \mathcal{U}$ such that $Q = \bigcup_{L \in \mathcal{U}_1} L$ and no two intervals of $\mathcal{U}_1$ overlap.

Set $A = \bigcup_{J_i \in \mathcal{U}_1} J_i$. Then by (4) we have

$$(8) \quad \sum_{J_i \in \mathcal{U}_1} G(J_i) > \sum_{J_i \in \mathcal{U}_1} b |J_i| = b |A|.$$

Set $\mathcal{U}_2 = \{L \in \mathcal{U}_1 : L \neq J_i, 1 \leq i \leq N\}$. Then

$$(9) \quad \overline{Q \setminus A} = \bigcup_{L \in \mathcal{U}_2} L$$

and $L \cap E \neq \emptyset$ for each $L \in \mathcal{U}_2$, so that by (2)

$$(10) \quad 0 < \sum_{L \in \mathcal{U}_2} G(L).$$

If $L \in \mathcal{U}_2$, then $L \neq J_i$ ($i = 1, \dots, N$) by the construction of $\mathcal{U}_2$. Since $\mathcal{U}_2 \subset \mathcal{U}$ any $L \in \mathcal{U}_2$ satisfies (iii) in the definition of $\mathcal{U}$. Therefore, if $L \in \mathcal{U}_2$, say $L \in \mathcal{A}_j$, there is an $L^{(1)} \in \mathcal{A}_{j+1}$ such that $L^{(1)}$ overlaps with none of $J_1, \dots, J_N$ because of (iii). By condition Q_1 , $|L| < \frac{1}{q} |L^{(1)}|$ for each such pair L and $L^{(1)}$. Therefore

$$|Q \setminus A| = \sum_{L \in \mathcal{U}_2} |L| < \frac{1}{q} \sum_{L \in \mathcal{U}_2} |L^{(1)}|.$$

However, the union of all such $L^{(1)}$ is contained in

$\overline{(Q \setminus \bigcup_{i=1}^N J_i)}$. Thus

$$(11) \quad |Q \setminus A| < \frac{1}{q} \left| \bigcup_{L \in \mathcal{U}_2} L^{(1)} \right| \leq \frac{1}{q} |Q \setminus \bigcup_{i=1}^N J_i| < \frac{1}{q} 2\epsilon |Q|$$

in view of (7). That is

$$(12) \quad |A| > (1 - \frac{2\epsilon}{q}) |Q|.$$

Now

$$G(Q) = \sum_{J_i \in \mathcal{U}_1} G(J_i) + \sum_{L \in \mathcal{U}_2} G(L) > \sum_{J_i \in \mathcal{U}_1} G(J_i)$$

by (10). This combined with (8) and (12) gives

$$G(Q) > b|A| > b(1 - \frac{2\epsilon}{q})|Q|. \quad \square$$

6. Theorem. Let $\{D_n\}$ satisfy condition Q_1 .

Let G be a finite function on D . Let

$$A = \{x \in [0,1] : \underline{S}_D G(x) > -\infty \text{ or } \bar{S}_D G(x) < +\infty\}.$$

Then $\underline{S}_D G(x) = \bar{S}_D G(x)$ and is finite a.e. on A .

Proof. Let $A_1 = \{x : \bar{S}_D G(x) > \underline{S}_D G(x) > -\infty\}$.

Suppose $|A_1| > 0$. There is a number $a > 0$ and a set $B \subset A_1$ such that $|B| > 0$ and $\bar{S}_D G(x) - \underline{S}_D G(x) > a$ for all $x \in B$. Let q be the number of definition 4. Choose $\epsilon > 0$ such that

$$(1) \quad 2\epsilon < a(1 - \frac{2\epsilon}{q}).$$

For each integer n set

$$B_n = \{x \in B : n\epsilon < \underline{S}_D G(x) \leq (n+1)\epsilon\}.$$

Since $|B| > 0$, there exists a p such that $|B_p| > 0$.

For each $x \in B_p$ there is an integer $m(x)$ such that for $m \geq m(x)$

$$(2) \quad p\epsilon < \frac{G(J_m(x))}{|J_m(x)|}.$$

Now define

$$E_m = \{x \in B_p : m(x) \leq m\} \quad (m = 1, 2, \dots).$$

Then $B_p = \bigcup_{m=1}^{\infty} E_m$, so there is an m_0 such that $|E_{m_0}| > 0$. Denote E_{m_0} by E and set $\delta = \min_{J \in \mathcal{A}_{m_0}} |J|$.

We have for all $x \in E$

$$(3) \quad 0 < \underline{S}_D[G(x) - p\epsilon x] = \underline{S}_D G(x) - p\epsilon \leq \epsilon < 2\epsilon$$

since $E \subset B_p$ and $p\epsilon < \underline{S}_D G(x) \leq (p+1)\epsilon$ on B_p . Further, we have for all $x \in E$

$$(4) \quad \begin{aligned} \bar{S}_D[G(x) - p\epsilon x] &> \bar{S}_D[G(x) - p\epsilon x] - \underline{S}_D[G(x) - p\epsilon x] \\ &= \bar{S}_D G(x) - \underline{S}_D G(x) > a. \end{aligned}$$

Also

$$(5) \quad G(J_m(x)) - p\epsilon |J_m(x)| > 0 \text{ for all } x \in E$$

and $m \geq m_0$.

Applying lemma 3 and (3) we can find an interval $Q \in \mathcal{A}$ such that

$$(6) \quad |Q| < \min(\epsilon, \delta)$$

$$(7) \quad |E \cap Q| > (1 - \min(\epsilon, \delta)) |Q| \quad \text{and}$$

$$(8) \quad G(Q) - p\epsilon |Q| < 2\epsilon |Q|.$$

If $J \in \mathcal{A}_m$, $J \cap E \neq \emptyset$ and $J \subset Q$, then (6) implies that $m \geq m_0$. Thus from (5), (4) and (7) we see that lemma 5 may be applied to the function $G(x) - p\epsilon x$.

Hence

$$(9) \quad G(Q) - p\epsilon|Q| > a(1 - \frac{2\epsilon}{q})|Q|.$$

From (8) and (9) we see that

$$2\epsilon|Q| > a(1 - \frac{2\epsilon}{q})|Q|,$$

which implies $2\epsilon > a(1 - \frac{2\epsilon}{q})$. This is a contradiction

with (1). Therefore $|A_1| = 0$. Similarly we show

$\{x : \underline{S}_D G(x) < \bar{S}_D G(x) < +\infty\}$ has measure zero.

It remains to show that $M = \{x : S_D G(x) = +\infty\}$ has measure zero. For each $x \in M$ there is a natural number $m(x)$ such that $m \geq m(x)$ implies

$$\frac{G(J_m(x))}{|J_m(x)|} > 0.$$

Set

$$M_j = \{x \in M : m(x) \leq j\} \quad (j = 1, 2, \dots).$$

Assume that $|M| > 0$. Then since $M = \bigcup_{j=1}^{\infty} M_j$ there is an n such that $|M_n| > 0$. Let $y \in M_n$ be a point of outer density and choose $m > n$ such that

$$|J_m(y) \cap M_n| > (1 - \frac{1}{2})|J_m(y)|.$$

Set $P = J_m(y)$. From the definition of M_n it follows that if $x \in M_n$ and $j \geq n$, then $G(J_j(x)) > 0$.

Therefore, if $J \in \mathcal{A}$, $J \subset P$ and $J \cap M_n \neq \emptyset$, then $G(J) > 0$.

Let $x \in M_n$. Since $x \in M$ we have $S_D G(x) = \bar{S}_D G(x) > b$ for every real number b . Therefore

conditions (1), (2) and (3) of lemma 5 are fulfilled with $\epsilon = \frac{1}{2}$, from which it follows that

$$G(P) > b(1 - \frac{1}{q}) |P|$$

for every real number b . But this is impossible.

Hence M has measure zero. Similarly, we show

$\{x : S_D G(x) = -\infty\}$ has measure zero. $\square$

7. Theorem. Let $\{D_n\}$ satisfy condition Q_1 . Let $\sum_{n=1}^{\infty} f_n$ be a K-series with respect to $\{D_n\}$. Let E be a set of positive measure. Assume that for each $x \in E$, either

$$\limsup_{n \rightarrow \infty} s_n(x) < +\infty$$

or

$$\liminf_{n \rightarrow \infty} s_n(x) > -\infty.$$

Then $\sum_{n=1}^{\infty} f_n$ converges to a finite number almost everywhere on E .

Proof. This follows immediately from lemma 2 and theorem 6. $\square$

8. Corollary. Let $\{D_n\}$ satisfy condition Q_1 . Then a K-series with respect to $\{D_n\}$ cannot diverge to either $+\infty$ or $-\infty$ on a set of positive measure.

9. Definition. We say $\{D_n\}$ satisfies condition Q_2 if for each natural number n there is an interval $J \in \mathcal{D}_n$ such that $D_{n+1} \setminus D_n \subset J$. Note that if D_n has $n+1$ points ($n = 1, 2, \dots$), then $\{D_n\}_{n=1}^{\infty}$ satisfies condition Q_2 .

R.L. Gundy [7] has considered the problem of representing arbitrary measurable functions on a probability space by means of series with respect to complete orthonormal sequences of step functions. If we take $[0, 1]$ with Lebesgue measure as the probability space and construct an H^* -system $\{u_n\}_{n=1}^{\infty}$ according to Gundy's definition, then there is a sequence $\{D_n\}_{n=1}^{\infty}$ fulfilling condition Q_2 such that any series $\sum_{n=1}^{\infty} a_n u_n$ is a K-series with respect to $\{D_n\}$. Let $\{D_n\}$ be such a sequence. Then applying theorem 2.2 of [7] we see that there exists a K-series with respect to $\{D_n\}$ with an infinite limit on a set of positive measure if and only if $\{D_n\}$ does not satisfy condition Q_1 .

This leads us to conjecture that the assumption of condition Q_1 is essential in theorem 7.

CHAPTER V

REMARKS ON EVERYWHERE CONVERGENCE

V.A. Skvorcov [10] has shown that if a Haar series converges pointwise everywhere on $[0,1]$ to a bounded function g , then the given Haar series is the Haar-Fourier series for g . On the other hand, L. Faber [3] has constructed a nontrivial Haar series which converges to zero at all but one point in $[0,1]$. In this chapter the corresponding questions for K-series are considered.

First an example of a nontrivial K-series which converges to zero everywhere on $[0,1]$ is constructed. The partial sums of this example are a subsequence of the partial sums in Faber's Haar series example.

1. Example. The example is constructed by defining the partial sums. Let $\{D_n\}_{n=1}^{\infty}$ be given by $D_n = \{\frac{j}{2^n} : j = 0, 1, \dots, 2^n\}$ ($n = 1, 2, \dots$). Define s_n to be regular and

$$s_n = \begin{cases} 2^{n-1} & \text{on } (\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2}) \\ -2^{n-1} & \text{on } (\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}) \\ 0 & \text{off } [\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} + \frac{1}{2^n}] \end{cases}$$

($n = 1, 2, \dots$). Set $f_1 = s_1$, $f_n = s_n - s_{n-1}$ for $n \geq 2$.

Take an $n \geq 2$. The elements of $\mathcal{D}_{n-1}$ are of the form

$$J_j = [\frac{(j-1)}{2^{n-1}}, \frac{j}{2^{n-1}}] \quad (j = 1, \dots, 2^{n-1}).$$

Let v_j be regular and satisfy

$$v_j = \begin{cases} 1 & \text{on } \text{int } J_j \\ 0 & \text{off } J_j \end{cases} \quad (j = 1, 2, \dots, 2^{n-1}).$$

Then $\{v_1, \dots, v_{2^{n-1}}\}$ is a basis for $\mathcal{S}_{n-1}$. In order to show $f_n \in T_n$ we need only show $\int_0^1 f_n v_j = \int_{J_j} f_n = 0$

for $j = 1, \dots, 2^{n-1}$. But

$$f_n = \begin{cases} 2^{n-2} & \text{on } (\frac{1}{2} - \frac{1}{2^{n-1}}, \frac{1}{2} - \frac{1}{2^n}) \cup (\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}) \\ -2^{n-2} & \text{on } (\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2}) \cup (\frac{1}{2} + \frac{1}{2^n}, \frac{1}{2} + \frac{1}{2^{n-1}}) \\ 0 & \text{off } [\frac{1}{2} - \frac{1}{2^{n-1}}, \frac{1}{2} + \frac{1}{2^{n-1}}]. \end{cases}$$

Therefore, $\int_{J_j} f_n = 0 \quad (j = 1, \dots, 2^{n-1}).$

Since $f_n \in T_n$ ($n = 1, 2, \dots$) the series $\sum_{n=1}^{\infty} f_n$ is a K-series with respect to $\{D_n\}$.

By the regularity $s_n(\frac{1}{2}) = 0$ for all n . Fix $x \neq \frac{1}{2}$; there is a j such that $s_n(x) = 0$ for $n > j$.

Therefore $\sum_{n=1}^{\infty} f_n(x) = 0$ for every x . Notice that

$$\int_{J_n} \left(\frac{1}{2}\right)^{s_n} = \frac{1}{2}$$

for all n , so $\sum_{n=1}^{\infty} f_n$ does not satisfy condition H.

Example 1 shows that for general K-series there can be no analogue to Skvorcov's Haar series theorem. In order to obtain a class of K-series for which there is an analogue to Skvorcov's theorem we will impose the following conditions.

Recall from Chapter IV that the sequence $\{D_n\}$ satisfies condition Q_2 if for any n , there is an interval $J \in \mathcal{D}_n$ such that $D_{n+1} \setminus D_n \subset J$.

2. Definition. We say $\{D_n\}$ satisfies condition Q_3 if there is a number K with the property that for each pair of intervals $J_1, J_2 \in \mathcal{D}_{n+1}$ with $J_1 \cup J_2 \subset L \in \mathcal{D}_n$ we have

$$\frac{|J_1|}{|J_2|} < K \quad (n = 1, 2, \dots).$$

3. Lemma. Let $\{D_n\}$ satisfy condition Q_2 . Let $\sum_{n=1}^{\infty} f_n$ be a K-series with respect to $\{D_n\}$. Let $P = [a, b] \in \mathcal{D}_p$, $a_1 \in \text{int } P$, $P_1 = [a_1, b] \in \mathcal{D}_q$ and $s_q \geq s_p$ on $\text{int } P_1$. Assume that for each $n > p$ and each $J \in \mathcal{D}_n$ with the property $J \subset \text{int } P$ we have

$s_n \leq s_p$ on $\text{int } J$. Then there are numbers $q = m_1 < m_2 < \dots$ and $a_1 < a_2 < \dots < b$ such that $P_j = [a_j, b] \in \mathcal{A}_{m_j}$, $f_{m_j} \leq 0$ off P_j , and $f_{m_j} \geq 0$ on $\text{int } P_j$. Moreover $\{m_j\}$ may be chosen so that $f_{m_j+1} = f_{m_j+2} = \dots = f_{m_{j+1}-1} = 0$ on $\text{int } P_j$ ($j = 1, 2, \dots$).

Proof. Because of condition Q_2 and the orthogonality there exists $L \in \mathcal{A}_n$ such that the support of f_{n+1} is contained in L ($n = 1, 2, \dots$).

If $f_n = 0$ on $\text{int } P_1$ for all $n > q$, the assertion is trivial. Otherwise proceed as follows. Let n be the smallest integer $i > q$ such that f_i is not identically zero on $\text{int } P_1$. By orthogonality and condition Q_2 there is at least one interval $I \in \mathcal{A}_n$ such that $I \subset P_1$, $I \neq P_1$ and $f_n > 0$ on $\text{int } I$. From our assumptions it follows that we cannot have $I \subset \text{int } P$, because $s_n > s_{n-1} = \dots = s_q \geq s_p$ on $\text{int } I$. Let $I = [\alpha, \beta]$. Then $a < a_1 \leq \alpha < \beta \leq b$. If $\beta < b$, then $I \subset \text{int } P$. Therefore $\beta = b$ and $a_1 < \alpha$. Set $m_2 = n$, $a_2 = \alpha$, and $P_2 = I$.

Now assume that $m_1, \dots, m_j$ and $P_1, \dots, P_j$ have been constructed. Repeating the argument above with m_j in place of q , P_j in place of P_1 and P_{j-1} in place of P , we construct m_{j+1} and P_{j+1} . $\square$

4. Theorem. Let f be a bounded function on $[0,1]$. Let $\{D_n\}$ satisfy conditions Q_2 and Q_3 . Let $\sum_{n=1}^{\infty} f_n$ be a K-series with respect to $\{D_n\}$. Assume $\sum_{n=1}^{\infty} f_n(x) = f(x)$ for all $x \in [0,1]$. Then $\sum_{n=1}^{\infty} f_n$ is the LFK-series for f .

Proof. Let $|f(x)| \leq C$ for all $x \in [0,1]$. We will show that $|s_n(x)| \leq C$ for all x and n .

Suppose not. Then there is an n_0 and $J_0 \in \mathcal{D}_{n_0}$ such that $|s_{n_0}| > C$ on $\text{int } J_0$. For the sake of definiteness, assume $s_{n_0} > C$ on $\text{int } J_0$.

It will be shown that there exists a sequence of intervals $\{J_j\}$ and an increasing sequence of integers $\{n_j\}$ with the properties

- (1) $J_{j+1} \subseteq \text{int } J_j$,
- (2) $J_j \in \mathcal{D}_{n_j}$,
- (3) $s_{n_{j+1}} \geq s_{n_j} > C$ on $\text{int } J_{j+1}$ ($j = 0, 1, 2, \dots$).

The sequences are constructed by induction. We have n_0 and J_0 . Assume that $n_0, n_1, \dots, n_k$ and $J_0, J_1, \dots, J_k$ have been constructed. Suppose that for every $n > n_k$ and every $I \in \mathcal{D}_n$ with $I \subset \text{int } J_k$ we have $s_n < s_{n_k}$ on $\text{int } I$. (We wish to deduce a contradiction.)

Obviously there exists an index $i > n_k$ such that f_i is not identically zero on $\text{int } J_k$. Let m be the smallest such i . By property Q_2 and orthogonality there is an $I \in \mathcal{D}_m$ such that $I \subset J_k$, $I \neq J_k$ and $f_m > 0$ on $\text{int } I$. Since $s_m > s_{m-1} = \dots = s_{n_k}$ on $\text{int } I$, the interval I must, by the assumption, have a common endpoint with J_k . For definiteness, assume the common endpoint of J_k and I is the right, denoted by y .

Now apply lemma 3 with $p = n_k$, $P = J_k$, $q = m$ and $P_1 = I$ to find integers $m = m_1 < m_2 < \dots$ and intervals $P_j \in \mathcal{D}_{m_j}$ ($j = 1, 2, \dots$), all having y as a common right endpoint, such that $f_{m_j} \leq 0$ off P_j , $f_{m_j} \geq 0$ on $\text{int } P_j$, $f_{m_j+1} = f_{m_j+2} = \dots = f_{m_{j+1}-1} = 0$ on $\text{int } P_j$ ($j = 1, 2, \dots$).

Define $M_j = -\min_t f_{m_j}(t)$ ($j = 1, 2, \dots$).

Suppose for $j \geq 2$

$$(4) \quad M_j \geq f_{m_1}(y-).$$

Let P_j' denote a member of $\mathcal{D}_{m_j}$ upon which $-M_j$ is attained by f_{m_j} . Then we have

$$(5) \quad M_j |P_j'| \leq f_{m_j}(y-) |P_j| \quad (j = 2, 3, \dots).$$

Because of condition Q_2 we have $f_{m_j}(y-) = 2f_{m_j}(y)$

($j = 1, 2, \dots$). This fact combined with (4), (5) and condition Q_3 gives

$$(6) \quad 0 < 2f_{m_1}(y) \leq M_j \leq 2f_{m_j}(y) \frac{|P_j|}{|P'_j|} < 2f_{m_j}(y)K$$

for $j \geq 2$, where K is a constant independent of j . But the convergence of $\sum_{n=1}^{\infty} f_n(y)$ implies $\lim_{j \rightarrow \infty} f_{m_j}(y) = 0$, a contradiction with (6). Therefore, there exists $j \geq 2$ for which $M_j < f_{m_1}(y-)$.

By condition Q_2 and orthogonality the support of f_{m_j} is contained in P_{j-1} ($j = 2, 3, \dots$). Therefore $P'_j \subset P_{j-1}$ ($j = 2, 3, \dots$). Now P'_j is disjoint from and to the left of y so $P'_j \subset \text{int } J_k$ ($j = 2, 3, \dots$). Let $j_1 > 1$ be such that $|f_{m_{j_1}}(t)| < f_{m_1}(y-)$ for every $t \in \text{int } P'_{j_1}$. Then $f_{m_1} + f_{m_{j_1}} > 0$ on $\text{int } P'_{j_1}$. Since $P'_j \subset P_{j-1}$ for $j \geq 2$, $f_{m_j} \geq 0$ on $\text{int } P'_{j_1}$ ($j = 1, 2, \dots, j_1-1$). Combining this fact with the choice of $\{m_j\}$ we have

$$\begin{aligned} s_{m_{j_1}} &= s_{m_1-1} + f_{m_1} + \dots + f_{m_{j_1-1}} + f_{m_{j_1}} > s_{m_1-1} \\ &= s_{n_k} \quad \text{on } \text{int } P'_{j_1}, \end{aligned}$$

a contradiction.

Therefore there is an $n > n_k$ and $J \in \mathcal{D}_n$ such that $J \subset \text{int } J_k$ and $s_n \geq s_{n_k}$ on $\text{int } J$. Set $n_{k+1} = n$ and $J_{k+1} = J$. This completes the construction.

From the density of D and from (1), (2), and (3) we have $\bigcap_{k=1}^{\infty} J_k = \{z\}$ and $\lim_{k \rightarrow \infty} s_{n_k}(z) > C \geq f(z)$, a contradiction. Therefore $|s_n(x)| \leq C$ for all $x \in [0,1]$ and all n .

Fix an integer $m \geq 1$ and $J \in \mathcal{D}_m$. Set $g = \text{characteristic function of } J$. Then

$$\int_0^1 g s_n = \int_J s_n \quad \text{for } n \geq m.$$

By the Lebesgue Dominated Convergence Theorem we have

$$\lim_{n \rightarrow \infty} \int_J s_n = \lim_{n \rightarrow \infty} \int_0^1 g s_n = \int_0^1 g f = \int_J f.$$

Because of the orthogonality

$$\int_J s_n = \int_J s_m \quad \text{for } n \geq m.$$

Combining these last two facts with theorem 8 of Chapter II proves that $\sum_{n=1}^{\infty} f_n$ is the LFK series for f . $\square$

In the remainder of this chapter it will be assumed that $\mathcal{D}_n$ has $n+1$ points. Note that this is the case for Haar series, that condition Q_2 is automatically

fulfilled and that conditions Q_0 , Q_1 and Q_3 are equivalent under this assumption. Furthermore if D_n has $n+1$ points, then $\mathcal{D}_n$ has n elements and $\mathcal{D}_n \setminus \mathcal{D}_{n-1}$ has two elements ($n = 2, 3, \dots$).

5. Definition. Let $\mathcal{D}_n \setminus \mathcal{D}_{n-1}$ have two elements, say J and K . Set

$$q_n = \max\left(\frac{|J|}{|K|}, \frac{|K|}{|J|}\right) \quad (n = 2, 3, \dots).$$

Under the assumption that D_n has $n+1$ points, condition Q_3 is equivalent to boundedness of the sequence $\{q_n\}$. In the next example a sequence $\{D_n\}$ and a non-trivial K -series $\sum_{n=1}^{\infty} f_n$ with respect to $\{D_n\}$ are constructed such that $\sum_{n=1}^{\infty} f_n(x) = 0$ for every $x \in [0, 1]$. This example satisfies all the hypotheses of theorem 4 except for the boundedness of $\{q_n\}$.

6. Example. Let $2 = A_1 < A_2 < \dots$, $\lim_{n \rightarrow \infty} A_n = \infty$, $\lim_{n \rightarrow \infty} (A_{n+1} - A_n) = 0$. Define $J_n = [\frac{1}{2} - \frac{1}{A_n}, \frac{1}{2}]$, $L_n = [\frac{1}{2}, \frac{1}{2} + \frac{1}{A_n}]$ ($n = 1, 2, \dots$). It is easy to see that there is a sequence $\{D_n\}$ fulfilling the requirements of definition 1 of Chapter II and a sequence of natural numbers $2 = k_1 < k_2 < \dots$ with the properties

$$(1) \quad D_n \text{ has } n+1 \text{ points}$$

$$(2) \quad J_n, L_n \in \mathcal{A}_{k_n}$$

$$(3) \quad J_{n+1} \in \mathcal{A}_{k_n+1}$$

$$(4) \quad L_{n+1} \in \mathcal{A}_{k_n+2}.$$

Set $s_1 = 0$. For each $n \geq 1$ define s_{k_n} to be regular and

$$s_{k_n} = \begin{cases} A_n & \text{on } \text{int } J_n \\ -A_n & \text{on } \text{int } L_n \\ 0 & \text{on } [0,1] \setminus (J_n \cup L_n). \end{cases}$$

Let s_{k_n+1} be regular and

$$s_{k_n+1} = \begin{cases} A_{n+1} & \text{on } \text{int } J_{n+1} \\ -A_n & \text{on } \text{int } L_n \\ 0 & \text{on } [0,1] \setminus (J_{n+1} \cup L_n). \end{cases}$$

Let s_{k_n+2} be regular and

$$s_{k_n+2} = \begin{cases} A_{n+1} & \text{on } \text{int } J_{n+1} \\ -A_{n+1} & \text{on } \text{int } L_{n+1} \\ 0 & \text{on } [0,1] \setminus (J_{n+1} \cup L_{n+1}). \end{cases}$$

Set $s_{k_{n+1}} = s_{k_{n+1}-1} = \dots = s_{k_n+3} = s_{k_n+2}$. Define

$f_1 = s$, and $f_m = s_m - s_{m-1}$ ($m = 2, 3, \dots$). Then f_m

is identically zero for $m \neq k_n+1$ and $m \neq k_n+2$

($n = 1, 2, \dots$). Therefore $f_m \in T_m$ for $m \neq k_n+1$ and

$m \neq k_n+2$ ($n = 1, 2, \dots$). Choose an $n \geq 1$, let

$I_1, \dots, I_{k_n}$ be an enumeration of the intervals in $\mathcal{A}_{k_n}$.

Define v_j to be regular and

$$v_j = \begin{cases} 1 & \text{on } \text{int } I_j \\ 0 & \text{on } [0,1] \setminus I_j \end{cases}$$

($j = 1, \dots, k_n$). Then $\{v_1, \dots, v_{k_n}\}$ is a basis for $\mathfrak{g}_{k_n}$. In order to show $f_{k_n+1} \in T_{k_n+1}$ we need only show

$$\int_0^1 f_{k_n+1} v_j = \int_{I_j} f_{k_n+1} = 0$$

for $j = 1, \dots, k_n$. Obviously $f_{k_n+1} = 0$ on $\text{int } I_j$, if $I_j \neq J_n$. Since $\int_{J_n} s_{k_n} = 1 = \int_{J_n} s_{k_n+1}$, we have $\int_{J_n} f_{k_n+1} = 0$. Hence $f_{k_n+1} \in T_{k_n+1}$.

Similarly let $P_1, \dots, P_{k_n+1}$ be an enumeration of the intervals of $\mathcal{A}_{k_n+1}$. Define u_j to be regular and

$$u_j = \begin{cases} 1 & \text{on } \text{int } P_j \\ 0 & \text{off } P_j \end{cases} \quad (j = 1, \dots, k_n+1).$$

Then $\{u_1, \dots, u_{k_n+1}\}$ is a basis for $\mathfrak{g}_{k_n+1}$. To show $f_{k_n+2} \in T_{k_n+2}$ we need only show

$$\int_0^1 f_{k_n+2} u_j = \int_{P_j} f_{k_n+2} = 0$$

for $j = 1, \dots, k_n+2$. Obviously $f_{k_n+2} = 0$ on $\text{int } P_j$ if $P_j \neq L_n$. Since $\int_{L_n} s_{k_n+1} = 1 = \int_{L_n} s_{k_n+2}$, we have $\int_{L_n} f_{k_n+2} = 0$.

Consequently, $f_m \in T_m$ for every m . So $\sum_{n=1}^{\infty} f_m$ is a K-series with respect to $\{D_m\}$. Let $x = \frac{1}{2}$. Then there is an integer m such that $s_n(x) = 0$ for $n \geq m$. For $x = \frac{1}{2}$ we have $s_{k_n+1}(\frac{1}{2}) = A_{n+1} - A_n$ and

$$s_{k_n+2}(\frac{1}{2}) = s_{k_n+3}(\frac{1}{2}) = \dots = s_{k_{n+1}}(\frac{1}{2}) = 0$$

($n = 1, 2, \dots$). Therefore $\lim_{m \rightarrow \infty} s_m(\frac{1}{2}) = 0$.

Hence $\sum_{m=1}^{\infty} f_m(x) = 0$ for every x , but f_{k_n+1} and f_{k_n+2} are non-zero functions ($n = 1, 2, \dots$).

The example constructed in section 6 has several interesting properties which are the subject of the next remarks.

7. Remark. Let $\{A_n\}$, $\{k_n\}$, $\{L_n\}$, $\{J_n\}$, be as in example 6. Set

$$K_n = [\frac{1}{2} - \frac{1}{A_{n-1}}, \frac{1}{2} - \frac{1}{A_n}] \quad (n = 2, 3, \dots).$$

Then $q_{k_n+1} = q_{k_n+2} = \frac{|J_{n+1}|}{|K_{n+1}|} = \frac{A_n}{A_{n+1} - A_n} \quad (n = 1, 2, \dots).$

Moreover, there does not exist an integer N such that $q_{k_n+1} \leq n$ for $n > N$. To see that this is the case, suppose that there were such an $N > 1$. Then for any $n > N$ we have $\frac{1}{n} \leq \frac{A_{n+1}}{A_n} - 1$. That is

$$(1) \quad A_n \left(1 + \frac{1}{n}\right) \leq A_{n+1}.$$

There is an $\alpha > 0$ for which

$$(2) \quad A_n \geq \alpha n \quad (n = 1, \dots, N+1).$$

Let n be an integer such that $A_n \geq \alpha n$. If $n \leq N+1$ then (2) holds. If $n > N$, then by (1) we have

$A_{n+1} \geq \alpha(n+1)$. So that (2) holds in any case. From

(1) and the fact that $\lim_{n \rightarrow \infty} (A_{n+1} - A_n) = 0$ it follows

that $\lim_{n \rightarrow \infty} \frac{A_n}{n} = 0$, a contradiction.

8. Remark. We have seen in section 7 that no matter how the sequence $\{A_n\}$ is chosen it is impossible for q_{k_n+1} to be eventually bounded above by n . However, choose $\beta > 1$ and $\alpha \in (\beta^{-1}, 1)$. Define $A_n = 2n^\alpha$ ($n = 1, 2, \dots$). Then

$$q_{k_n+1}^{-1} = \left(\frac{n+1}{n}\right)^\alpha - 1.$$

Now

$$\left(\frac{n+1}{n}\right)^\alpha - 1 = \left(1 + \frac{1}{n}\right)^\alpha - 1 = \frac{\alpha}{n} b_n,$$

where $b_n = \left(1 + \frac{a_n}{n}\right)^{\alpha-1}$ and $0 < a_n < 1$. Therefore

$\lim_{n \rightarrow \infty} b_n = 1$, so there exists an N such that $b_n > \frac{1}{\beta\alpha}$

for $n \geq N$. For $n > N$ we have then $q_{k_n+1}^{-1} > \frac{\alpha}{n} \frac{1}{\beta\alpha}$.

Thus $q_{k_n+1} < n\beta$ for $n \geq N$.

9. Remark. Let $1 < b_1 < b_2 < \dots$, $\lim_{n \rightarrow \infty} b_n = \infty$. It is clear that the sequence $\{k_n\}$ in example 6 may

be chosen so that $q_{k_n+1} < b_{k_n+1}$ and $q_{k_n+2} < b_{k_n+2}$; at the same time we may choose the sets D_n so that $q_m = 1$ for $m \neq k_n+1, k_n+2$ ($n = 1, 2, \dots$). Then $q_m < b_m$ for any m .

In view of example 6 and these remarks, it appears that the assumption of condition Q_3 in theorem 4 cannot be substantially weakened. A natural question is whether the imposition of conditions on the rate at which the lengths of the smallest and largest intervals in the n^{th} partition tend to zero might imply the conclusion of theorem 4.

10. Definition. Let $\{D_n\}$ be given. Denote

$$v_n = \max_{J \in D_n} |J| \quad \text{and} \quad \mu_n = \min_{J \in D_n} |J| \quad (n = 1, 2, \dots).$$

11. Lemma. Assume that there exist numbers α, β such that $n\mu_n > \alpha$ and $nv_n < \beta$ ($n = 1, 2, \dots$). Then $\{D_n\}$ satisfies condition Q_3 .

Proof. Take $n > 1$ and $J, L \in D_n$. Then

$$\frac{|J|}{|L|} \leq \frac{v_n}{\mu_n} < \frac{\beta}{\alpha}. \quad \square$$

12. Corollary (to theorem 4). Let $\{D_n\}$ satisfy the assumptions of lemma 11. Assume D_n has $n+1$ points for each $n \geq 1$. Let $\sum_{n=1}^{\infty} f_n$ be a K-series

with respect to $\{D_n\}$ which converges everywhere on $[0,1]$ to a bounded function f . Then $\sum_{n=1}^{\infty} f_n$ is the LFK-series for f .

The next example shows that if we only assume $\{nu_n\}$ is bounded away from zero the conclusion of theorem 4 need not hold.

13. Example. Let $0 < \alpha < 1$. Set $A_n = 2n^\alpha$ ($n = 1, 2, \dots$). Define sets $E_0 \subset E_1 \subset E_2 \subset \dots$ as follows. Set $E_0 = [0, 1]$, $E_1 = [0, \frac{1}{2}, 1]$, $E_2 = [0, \frac{1}{2} - \frac{1}{A_2}, \frac{1}{2}, \frac{1}{2} + \frac{1}{A_2}, 1]$. Let $m \geq 2$ and let finite sets $E_0 \subset E_1 \subset \dots \subset E_m$ be defined. Define $E_m^{(1)} = \{t \in E_m : t < \frac{1}{2}\}$, $E_m^{(2)} = \{t \in E_m : t > \frac{1}{2}\}$. Let $\{t_1 < \dots < t_r\}$ be an enumeration of $E_m^{(1)}$ and $\{t_{r+1} < t_{r+2} < \dots < t_s\}$ be an enumeration of $E_m^{(2)}$. Let $\delta_m^{(1)} = \{[t_{j-1}, t_j] : 2 \leq j \leq r\}$ and $\delta_m^{(2)} = \{[t_{j-1}, t_j] : r+2 \leq j \leq s\}$. Denote $\delta_m = \delta_m^{(1)} \cup \delta_m^{(2)}$. Let F_m be the set of midpoints of intervals from δ_m . Now define

$$E_{m+1} = E_m \cup F_m \cup \left\{ \frac{1}{2} - \frac{1}{A_{m+1}}, \frac{1}{2} + \frac{1}{A_{m+1}} \right\}.$$

Now we prove $E_m \subset [0, 1]$ ($m = 0, 1, 2, \dots$). The assertion is clear for $m = 0, 1, 2$. Assume $n \geq 2$ and $E_n \subset [0, 1]$. Then certainly $F_n \subset [0, 1]$. Since

$A_{n+1} > 2$ it follows that $0 < \frac{1}{2} - \frac{1}{A_{n+1}} < \frac{1}{2} + \frac{1}{A_{n+1}} < 1$.

Therefore $E_{n+1} \subset [0, 1]$. This establishes the assertion.

Next we prove that $\max\{t \in E_n : t < \frac{1}{2}\} = \frac{1}{2} - \frac{1}{A_n}$

($n = 2, 3, \dots$). This is obvious for $n=2$. Assume $n \geq 2$

and $\max\{t \in E_n : t < \frac{1}{2}\} = \frac{1}{2} - \frac{1}{A_n}$. Thus

$\max\{t \in E_n \cup F_n : t < \frac{1}{2}\} = \frac{1}{2} - \frac{1}{A_n}$. Thus $\max\{t \in E_{n+1} :$

$t < \frac{1}{2}\} = \frac{1}{2} - \frac{1}{A_{n+1}}$ since $\frac{1}{2} - \frac{1}{A_{n+1}} > \frac{1}{2} - \frac{1}{A_n}$ and

$E_{n+1} = E_n \cup F_n \cup \{\frac{1}{2} - \frac{1}{A_{n+1}}, \frac{1}{2} + \frac{1}{A_{n+1}}\}$. This proves the

assertion. Similarly it may be shown that

$$\min\{t \in E_n : t > \frac{1}{2}\} = \frac{1}{2} + \frac{1}{A_n} \quad (n = 2, 3, \dots).$$

From these two facts it follows that

$$\{\frac{1}{2} - \frac{1}{A_{n+1}}, \frac{1}{2} + \frac{1}{A_{n+1}}\} \cap E_n = \emptyset \quad (n = 2, 3, \dots)$$

because $A_{n+1} > A_n$ ($n = 1, 2, \dots$).

Now we may show that E_n has 2^{n+1} points

($n = 0, 1, 2, \dots$). This is clear for $m = 0, 1, 2$. Suppose

$n \geq 2$ and E_n has 2^{n+1} points. It follows from

the definition of F_n that $F_n \cap E_n = \emptyset$. Moreover

F_n has 2^{n-2} points. We have seen above that

$$\{\frac{1}{2} - \frac{1}{A_{n+1}}, \frac{1}{2} + \frac{1}{A_{n+1}}\} \cap E_n = \emptyset.$$

Therefore

$$E_{n+1} = E_n \cup F_n \cup \left\{ \frac{1}{2} - \frac{1}{A_{n+1}}, \frac{1}{2} + \frac{1}{A_{n+1}} \right\}$$

has $(2^{n+1}) + (2^n - 2) + 2$ points. This establishes the assertion.

Set $D_{2^n} = E_n$ ($n = 0, 1, 2, \dots$). In order to define D_k for any natural number k , choose an n . Let $J_1, \dots, J_{2^n}$ be an enumeration of the elements of $\mathcal{S}_{2^n}$. Further assume that the indices are so chosen that the right endpoint of J_i is the left endpoint of J_{i+1} ($i = 1, \dots, 2^n - 1$). Set $D_{2^{n+1}} = E_n \cup \{\text{midpoint of } J_1\}$, and $D_{2^{n+j}} = D_{2^{n+j-1}} \cup \{\text{midpoint of } J_j\}$ for $1 \leq j < 2^{n-1}$.

Define

$$D_{2^{n+2^{n-1}}} = D_{2^{n+2^{n-1}-1}} \cup \left\{ \frac{1}{2} + \frac{1}{A_{n+1}} \right\}$$

and

$$D_{2^{n+2^{n-1}+1}} = D_{2^{n+2^{n-1}}} \cup \left\{ \frac{1}{2} + \frac{1}{A_{n+1}} \right\}.$$

Define $D_{2^{n+j}} = D_{2^{n+j-1}} \cup \{\text{midpoint of } J_j\}$ for $2^{n-1}+1 < j \leq 2^n$.

Since $\left(\frac{n}{n-1}\right)^\alpha < 2$ ($n = 2, 3, \dots$) and $A_n = 2n^\alpha$ ($n = 1, 2, \dots$), we have

$$\frac{2}{A_n} = \frac{2}{2n^\alpha} > \frac{1}{2(n-1)^\alpha} = \frac{1}{A_{n-1}} \quad (n = 2, 3, \dots).$$

Therefore

$$(1) \quad \frac{1}{A_n} > \frac{1}{A_{n-1}} - \frac{1}{A_n} \quad (n = 2, 3, \dots).$$

Now we prove that

$$(2) \quad \frac{1}{2} \left(\frac{1}{A_{n-1}} - \frac{1}{A_n} \right) < \frac{1}{A_n} - \frac{1}{A_{n+1}} \quad \text{for } n > 4.$$

To see this rewrite the inequality as

$$3 - 2 \left(1 + \frac{1}{n} \right)^{-\alpha} > \left(1 - \frac{1}{n} \right)^{-\alpha}.$$

Now using the Taylor expansion for $(1+x)^{-\alpha}$ we see that there are sequences $\{c_n\}$ and $\{d_n\}$ satisfying

$$0 < c_n < \frac{1}{n}, \quad -\frac{1}{n} < d_n < 0,$$

$$\left(1 + \frac{1}{n} \right)^{-\alpha} = 1 - \frac{\alpha}{n} + \frac{\alpha(\alpha+1)}{2n^2(1+c_n)^{\alpha+2}}, \quad \text{and}$$

$$\left(1 - \frac{1}{n} \right)^{-\alpha} = 1 + \frac{\alpha}{n} + \frac{\alpha(\alpha+1)}{2n^2(1+d_n)^{\alpha+1}} \quad (n = 2, 3, \dots).$$

Hence

$$(3) \quad \left(1 + \frac{1}{n} \right)^{-\alpha} < 1 - \frac{\alpha}{n} + \frac{\alpha(\alpha+1)}{2n^2}.$$

Now let $n > 4$. Since $n \geq 5$ and $0 < \alpha < 1$, we have

$$d_n > -\frac{1}{5}, \quad (1+d_n)^{\alpha+2} > \left(\frac{4}{5}\right)^{\alpha+2} > \left(\frac{4}{5}\right)^3,$$

that is

$$\frac{1}{(1+d_n)^{\alpha+2}} < \frac{125}{64} < 2$$

so that

$$(4) \quad \left(1 - \frac{1}{n} \right)^{-\alpha} < 1 + \frac{\alpha}{n} + \frac{\alpha(\alpha+1)}{n^2}.$$

From (3) and (4) it follows that

$$\begin{aligned}
3-2\left(1 + \frac{1}{n}\right)^{-\alpha} - \left(1 - \frac{1}{n}\right)^{-\alpha} &> 3-2 + \frac{2\alpha}{n} - \frac{\alpha(\alpha+1)}{n^2} - 1 \\
&- \frac{\alpha}{n} - \frac{\alpha(\alpha+1)}{n^2} = \frac{\alpha}{n} - \frac{2\alpha(\alpha+1)}{n^2} \\
&= \frac{\alpha}{n^2} [n - 2\alpha(\alpha+1)] > 0.
\end{aligned}$$

This establishes (2).

$$\begin{aligned}
\text{Set } J_n &= \left[\frac{1}{2} - \frac{1}{A_n}, \frac{1}{2}\right], \quad L_n = \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{A_n}\right], \\
K_{n+1} &= \left[\frac{1}{2} - \frac{1}{A_n}, \frac{1}{2} - \frac{1}{A_{n+1}}\right], \quad I_{n+1} = \left[\frac{1}{2} + \frac{1}{A_{n+1}}, \frac{1}{2} + \frac{1}{A_n}\right] \\
(n = 1, 2, \dots). \quad &\text{Now we prove}
\end{aligned}$$

$$(5) \quad \mu_{2^n} = \min\left\{\frac{1}{2} \mu_{2^{n-1}}, \frac{1}{A_{n-1}} - \frac{1}{A_n}\right\} \quad (n = 2, 3, \dots).$$

Obviously $\mu_2 = \frac{1}{2}$. By (1) we have $\frac{1}{2} - \frac{1}{A_2} < \frac{1}{A_2}$;

therefore

$$\mu_4 = \min\left\{\frac{1}{2} - \frac{1}{A_2}, \frac{1}{A_2}\right\} = \frac{1}{2} - \frac{1}{A_2}.$$

Since $\mu_4 + \frac{1}{A_2} = \mu_2$ and $\mu_4 < \frac{1}{A_2}$, we have $\mu_4 < \frac{1}{2} \mu_2$

which is (5) for $n=2$. Now suppose that (5) holds for

some $n \geq 2$. Let $J \in \mathcal{D}_{2^n}$ be such that $|J| = \mu_{2^n}$.

Because of (1)

$$\mu_{2^n} = \min\left\{\frac{1}{2} \mu_{2^{n-1}}, \frac{1}{A_{n-1}} - \frac{1}{A_n}\right\} < \frac{1}{A_n}.$$

Therefore J is not J_n or L_n . Now in constructing

$\mathcal{D}_{2^{n+1}}$ from $\mathcal{D}_{2^n}$, each interval of $\mathcal{D}_{2^n} \setminus \{J_n, L_n\}$ is divided in half. Thus for each $I \in \mathcal{D}_{2^{n+1}} \setminus \{J_{n+1}, K_{n+1}, L_{n+1}, I_{n+1}\}$ we have $|I| \geq \frac{1}{2} |J|$. By (1) we have

$$|L_{n+1}| = |J_{n+1}| > |K_{n+1}| = |I_{n+1}|.$$

Thus

$$u_{2^{n+1}} = \min\{\frac{1}{2} |J|, |K_{n+1}|\} = \min\{\frac{1}{2} u_{2^n}, \frac{1}{A_n} - \frac{1}{A_{n+1}}\}.$$

This proves (5).

Let $u_{32} = u$. Now we prove that $u_{2^n} = 2^{5-n}u$ for $n \geq 5$. The relation is clear for $n=5$. Suppose that for some $n \geq 5$ $u_{2^n} = 2^{5-n}u$. From (5) it follows that $u_{2^n} \leq \frac{1}{A_{n-1}} - \frac{1}{A_n}$. Thus

$$\frac{1}{2} u_{2^n} \leq \frac{1}{2} \left(\frac{1}{A_{n-1}} - \frac{1}{A_n} \right).$$

Combining this with (2) gives

$$(6) \quad \frac{1}{2} u_{2^n} \leq \frac{1}{A_n} - \frac{1}{A_{n+1}}.$$

From (5) and (6) it follows that

$$u_{2^{n+1}} = \min\{\frac{1}{2} u_{2^n}, \frac{1}{A_n} - \frac{1}{A_{n+1}}\} = \frac{1}{2} u_{2^n}.$$

Therefore $u_{2^{n+1}} = \frac{1}{2} (2^{5-n}u) = 2^{5-(n+1)}u$. This proves the assertion.

Let $n > 4$ and $2^n < j < 2^{n+1}$. Then

$$u_{2^{n+1}} \leq u_j \leq u_{2^n}, \text{ so that } ju_j \geq ju_{2^{n+1}} > 2^n u_{2^{n+1}}.$$

Thus $ju_j > 16u$. There is a number $c > 0$ such

that $mu_m > 2c$ for $m = 1, \dots, 32$. Thus

$ju_j > 16u = \frac{1}{2} 32u_{32} > c$. Therefore $ku_k > c$ for every $k \geq 1$.

Set $s_1 = 0$. Define s_2 to be regular and

$$s_2 = \begin{cases} A_1 & \text{on int } J_1 \\ -A_1 & \text{on int } L_1. \end{cases}$$

Let s_3 be regular and

$$s_3 = \begin{cases} A_2 & \text{on int } J_2 \\ -A_1 & \text{on int } L_1 \\ 0 & \text{off } J_2 \cup L_1. \end{cases}$$

For $n \geq 2$ set $p_n = 3 \cdot 2^{n-2} + 1$ and define s_{p_n} to be regular and

$$s_{p_n} = \begin{cases} A_n & \text{on int } J_n \\ -A_n & \text{on int } L_n \\ 0 & \text{off } J_n \cup L_n. \end{cases}$$

Set $s_{p_n} = s_{p_{n+1}} = \dots = s_{p_{n+1}-2}$. Let $s_{p_{n+1}-1}$ be regular and

$$s_{p_{n+1}-1} = \begin{cases} A_{n+1} & \text{on int } J_{n+1} \\ -A_n & \text{on int } L_n \\ 0 & \text{off } J_{n+1} \cup L_n. \end{cases}$$

Then $s_{p_n} \in \mathfrak{g}_{p_n}$ and $s_{p_{n+1}-1} \in \mathfrak{g}_{p_{n+1}-1}$ since $J_n, L_n \in \mathcal{J}_{p_n}$ and $J_{n+1}, L_n \in \mathcal{J}_{p_{n+1}-1}$.

Take $f_1 = s_1$ and $f_n = s_n - s_{n-1}$ for $n \geq 2$.

Then it follows in exactly the same way as in example 6

that $f_n \in T_n$ for each $n \geq 1$ and that $\sum_{n=1}^{\infty} f_n$ is a non-trivial K-series with respect to $\{D_n\}$ which converges to zero at every point in $[0,1]$. However, as was observed above, there is $c > 0$ such that $nu_n > c$ for every n .

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