CONTRIBUTIONS TO CONSTRUCTION OF GENERALIZED YOUDEN DESIGN. ON CONSTRUCTION OF ORTHOGONAL LATIN SQUARES USING THE METHOD OF SUM COMPOSITION

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ABSTRACT

CONTRIBUTIONS TO CONSTRUCTION OF GENERALIZED YOUDEN DESIGN.
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The present thesis deals with two independent problems. In the first part (Chapters I and II) we investigate generalized Youden designs while in the second part (Chapter III) we further study the method of sum composition of Latin Squares introduced by Hedayat and Seiden (1969).

Generalized Youden designs were introduced by Kiefer (1958) who proved E-optimality and, in the presence of some divisibility conditions, D-optimality. In Chapter I we study optimality in detail and investigate relationships among the parameters; several necessary conditions for existence of GY-designs are found, and the chapter closes with the usual analysis of these designs.

Chapter II is devoted to the construction of GY-designs; using well-known combinatorial systems such as finite geometries, symmetric balanced incomplete block designs, Latin squares, etc. We construct several infinite families of GY-designs; the last construction of this chapter provides an infinite family of GY-designs whose parameters do not satisfy Kiefer's divisibility conditions and which are not D-optimum.

The method of sum composition of Latin Squares allows us in certain cases to construct $O(n,2)$ sets by composition of a $O(n_1,2)$ and a $O(n_2,2)$ set, $n = n_1 + n_2$. It is assumed that $O(n_1,2)$ is based on $GF(n_1)$ and formed by $A(x)$, $A(y)$, where for any $r \in GF(n_1)$, $r \neq 0$, $A(r)$ is the $n_1 \times n_1$ square with element $r\alpha_i + \alpha_j$ in its (i,j) cell, $\alpha_i, \alpha_j \in GF(n_1)$.

Hedayat and Seiden have further assumed that $xy = \alpha^2$ for some $\alpha \in GF(n_1)$; we free ourselves from that restriction and obtain further constructions. We also prove that the condition $xy = \alpha^2$ is a necessary one in 12 of the 24 possible patterns of composition of $O(p^\alpha,2)$ and $O(3,2)$.

Removal of the restriction $xy = \alpha^2$ produces compatibility equations which are non-linear in both x and y, therefore allowing the possibility of extending the method of sum composition to construction of $O(n,3)$ sets.

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CHAPTER I

ANALYSIS OF GENERALIZED YOUDEN SQUARES

1.1 Introduction

Frequently in scientific investigations the experimenter wishes to study the effect of several variables that he can control on a response or dependent variable which he can observe and measure. The variables under the control of the experimenter are called FACTORS and they would appear at various categories or LEVELS; a situation in which every factor appears at some level is a TREATMENT. Clearly if a design contains m factors $F_1, F_2, \ldots, F_m$, where F_i assumes s_i levels, $i = 1,2,\ldots,m$, there are $s_1 \cdot s_2 \ldots s_m$ possible treatments. A design which includes exactly one observation on each of the $s_1, \ldots, s_m$ possible treatments is called a COMPLETE FACTORIAL DESIGN; if several observations are made on each treatment it is called a FACTORIAL DESIGN WITH REPLICATES; if all factors assume the same number of levels (i.e. $s_i = s$, $i = 1,2,\ldots,m$) the design is SYMMETRIC; a COMPLETE SYMMETRIC DESIGN consists therefore of all s^m m-tuples of the s levels.

If the number of factors is large, the number of treatments necessary for a complete design becomes prohibitive; hence the need for fractional replication and confounding.

Fractional replication was studied, among others, by Finney (1945), Plackett and Burman (1946) and Plackett (1946). Essentially a $1/s^n$ replication of a complete s^m factorial design is a partition of the s^m treatments into blocks of s^{m-n} treatments each; the partitioning is said to be of STRENGTH t if no effect of interaction of t or fewer factors is confounded with the block effect. By using fractional replication the experimenter can discover cheaply at the early stages of his research which factors among many have an important effect on the product.

Balanced Incomplete Block designs are an example of fractional replication of complete two-factor designs, while Latin Squares, Youden Squares and Generalized Youden designs are fractional replications of three-factor designs.

Definition 1.1.1: A (v,b,k,r,λ) Balanced Incomplete Block (BIB) design is an arrangement of v elements (varieties) in b subsets (blocks) of k varieties each, such that any two distinct varieties occur together in λ blocks.

Then any variety occurs in r blocks and

$$vr = kb \ , \ \lambda(v-1) = k(r-1) \ .$$

If $v = b$ the BIB design is said to be Symmetric.

Definition 1.1.2: A Latin Square of order n is a square matrix of order n on a set of n varieties such that every row and every column is a permutation of the set of varieties.

Definition 1.1.3: A (v,k) Youden design is a $k \times v$ matrix on v varieties such that with the columns as blocks it is a (v,k,λ) symmetric BIB design, and each row is a permutation of the varieties.

Definition 1.1.4: A $(v,b,k,r,\lambda_1,\lambda_2)$ Generalized Youden (GY) design is a $k \times b$ matrix on a set of v varieties such that the following conditions are satisfied:

a) Every variety occurs r times.

b) Every variety occurs either m or $m+1$ times in each row, as well as either n or $n+1$ times in each column, where m is the integer part of $\frac{b}{v}$ and n is the integer part of $\frac{k}{v}$.

c) Every two distinct varieties occur together λ_1 times in the same row and λ_2 times in the same column.

Generalized Youden designs were first introduced by Kiefer (1958), who proved some optimality properties of those designs and gave two examples with two and four varities respectively; however he made no attempt to construct GY-designs.

In the next paragraph we examine closely the optimality properties of GY-designs.

1.2 Optimality of GY-designs

Let the linear hypothesis to be tested be $R\beta = 0$, where β is the p-rowed vector of parameters to be estimated and R is a $q \times p$ matrix of rank $q \leq p$; by means of an appropriate linear transformation this hypothesis can be reduced to the canonical form

$$\beta_1 = \beta_2 = \ldots = \beta_q = 0 \text{ .}$$

The covariance matrix of the best linear estimate of β is

$$\mathrm{cov}(\hat{\beta}) = (X^TX)^{-1}X^T \mathrm{cov}(Y)X(X^TX)^{-1} = (X^TX)^{-1}\sigma^2$$

where X is the matrix of the design and Y is the vector of observations with covariance matrix $\sigma^2 I$.

We restrict ourselves to the use of the F-test whose power function is a monotonically increasing function of the parameter

$$\lambda = \frac{1}{\sigma^2} \beta^T P^T [P(X^T X)^{-1} P^T] P\beta$$

where $P = (I_q, 0_{q,p-q})$, $0_{r,s}$ being a $r \times s$ matrix of zeros (see, for instance, Tang 1938).

It is known that the minimum value of $\sigma^2 \lambda$ on the unit sphere $(P\beta)^T(P\beta) = 1$ is equal to the smallest eigenvalue of $P(X^T X)^{-1} P^T$; similarly the greatest eigenvalue equals the maximum value of $\sigma^2 \lambda$ on the sphere. Therefore we maximize the minimum power of the F-test on the contour $(P\beta)^T(P\beta) = 1$ by maximizing the smallest eigenvalue of $P(X^T X)^{-1} P^T$.

For a given design d we will designate $A_d^* = P(X^T X)^{-1} P^T$ Remembering that the determinant of a square matrix equals the product of its eigenvalues, we are naturally led to the following criteria.

__Definition__ __1.2.1__: A design d is said to be E-optimum in a class Δ of available designs if

$$\min E(A_d^*) = \max_{d' \in \Delta} \min E(A_{d'}^*)$$

where for any square matrix A, $E(A)$ represents the set of eigenvalues of A.

__Definition__ __1.2.2__: A design is said to be D-optimum in a class Δ of available designs if

$$\det(A_d^*) = \max_{d'\in\Delta} \det(A_{d'}^*) \ .$$

Both criteria of optimality were introduced by Wald (1943), who also proved D-optimality of Latin Square designs. Keifer (1958) has proved that GY-designs are E-optimum, and also that they are D-optimum if either k or b is a multiple of v. We will show in the next chapter that if neither k nor b are mutliples of v the GY-design may fail to be D-optimum. The problem presents itself of determining, in the absence of the divisibility condition, which cases give D-optimum GY-designs and which cases do not; we were so far unsuccessful in solving this problem but hope that further research will overcome the difficulties.

1.3 Properties of GY-designs

The row-incidence matrix of a GY-design is a $v \times k$ matrix $A = (a_{ij})$, where a_{ij} is the number of times that the i-th variety appears in the j-th row; of course $a_{ij} \in \{m,m+1\}$.

Similarly, the column-incidence matrix of a GY-design is a $v \times b$ matrix $B = (b_{ij})$, where b_{ij} is the number of times that the i-th variety appears in the j-th column; evidently, $b_{ij} \in \{m,m+1\}$.

Notation: The quotient and remainder of the division of an integer a by another b will be written $[\frac{a}{b}]$ and $a_{(b)}$ respectively.

Theorem 1.3.1: In a GY-design

i) The number of rows containing a given variety $m+1$ times is the same for all the varieties, and equals $r_{(k)}$.

ii) The number of columns containing a given variety $m+1$ times is the same for all the varieties, and equals $r_{(b)}$.

iii) The number of varieties occurring $m+1$ times in a given row is the same for all rows, and equals $b_{(v)}$.

iv) The number of varieties occurring $n+1$ times in a given column is the same for all columns, and equals $k_{(v)}$.

Proof: Let $\alpha^{(i)}$ be the number of rows which contain the i-th variety $m+1$ times. We must have

$$\alpha^{(i)}(m+1) + (k - \alpha^{(i)})m = r \quad , \quad \text{or}$$

$$\alpha^{(i)} + km = r \tag{1}$$

therefore $\alpha^{(i)}$ is independent of i. Obviously $kb = rv$, therefore $[\frac{r}{k}] = [\frac{b}{v}] = m$, and thus $r = mk + r_{(k)}$. Substituting in (1) we obtain the desired result

$$\alpha^{(i)} = r_{(k)} \; .$$

The proofs of ii), iii) and iv) are entirely similar and therefore omitted.

Theorem 1.3.2: Let A be the row incidence matrix of a GY-design. Then

$$AA^T = (rb - \lambda_1 v)I_v + \lambda_1 J_v \tag{1}$$

$$Aj_k = rj_v \; , \; A^T j_v = bj_k \tag{2}$$

where A^T is the transpose of A, I_v is the unit matrix, $J_v = J_{v,v}$ is the $v \times v$ matrix of 1's, j_n is the $n \times 1$ vector of 1's.

Similarly for the column incidence matrix B,

$$BB^T = (rk - \lambda_2 v)I_v + \lambda_2 J_v$$

$$Bj_b = rj_v \quad , \quad B^T j_v = kj_v \; .$$

__Proof__: Let $a^{(i)}$ be the $1 \times k$ vector whose j-th component is a_{ij}; then the element in the i-th row and ℓ-th column of AA^T is the inner product $a^{(i)}a^{(\ell)}$; clearly $a^{(i)}a^{(\ell)} = \lambda_1$ if $i \neq \ell$. In order to obtain $a^{(i)}a^{(i)}$ let us count the number of occurrences in the same row of the design of pairs containing a particular variety v_i; on the one hand v_i is paired λ_1 times with each of the remainder v-1 varieties; on the other hand if v_i appears a_{ij} times in the j-th row of the design it will form pairs with each of the $b - a_{ij}$ varieties (not necessarily different) left in the row, each pair being counted a_{ij} times, for a total over the rows of $\sum_{j=1}^{k} (b - a_{ij})a_{ij} = (bj_k - a^{(i)})a^{(i)}$; the two counts provide us with

$$(bj_k - a^{(i)})a^{(i)} = \lambda_1(v-1)$$

which gives

$$a^{(i)}a^{(i)} = rb - \lambda_1(v-1)$$

and therefore the result

$$AA^T = (rb - \lambda_1 v)I_v + \lambda_1 J_v .$$

Part (2) is self evident.

__Theorem__ __1.3.3__: In any GY-design

(1) $$\lambda_1 k \leq r^2$$

(2) $$\lambda_2 b \leq r^2 .$$

__Proof__: For real numbers $a_1, a_2, \ldots, a_n$ it is always true that

$$(\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2 .$$

Applying this inequality to the components of the vector $a^{(i)}$ and using the previous theorem we obtain

$$r^2 \leq k(rb - \lambda_1(v-1))$$

which, remembering that $kb = rv$, gives

$$0 \leq (v-1)(r^2 - \lambda_1 k)$$

and therefore the theorem.

Part (2) is proven in the same fashion.

Theorem 1.3.4: In any GY-design

$$\lambda_1 v \leq rb$$

$$\lambda_2 v \leq rk .$$

Proof: Schwartz inequality gives

$$(a^{(i)}a^{(\ell)})^2 \leq (a^{(i)}a^{(i)})(a^{(\ell)}a^{(\ell)}) .$$

By Theorem 1.2.2 this gives

$$\lambda_1^2 \leq [rb - \lambda_1(v-1)]^2$$

or $$0 \leq [rb - \lambda_1(v-1)]^2 - \lambda_1^2$$

$$0 \leq [rb - \lambda_1(v-1) + \lambda_1][rb - \lambda_1 v] .$$

The first factor of the product is always positive, since $rb - \lambda_1(v-1) + \lambda_1 = a^{(i)}a^{(i)} + \lambda_1 > 0$, and hence so is the second, giving the theorem.

Part (2) is similarly proven.

<u>Theorem 1.3.5</u>: Let (v_i, v_ℓ) be a pair of distinct varieties; let α_0, α_1, 2α be the number of rows containing the pair (v_i, v_ℓ) m^2, $(m+1)^2$, $m(m+1)$ times respectively, and similarly let β_0, β_1, 2β be the number of columns containing the pair (v_i, v_ℓ) n^2, $(n+1)^2$, $n(n+1)$ times respectively. Then α_0, α_1, α, β_0, β_1, β are independent of the pair (v_i, v_ℓ) and

$$\alpha = rb - \lambda_1 v \qquad \beta = rk - \lambda_2 v$$

$$\alpha_0 = k - r_{(k)} - rb + \lambda_1 v \qquad \beta_0 = b - r_{(b)} - rk + \lambda_2 v$$

$$\alpha_1 = r_{(k)} - rb + \lambda_1 v \qquad \beta_1 = r_{(b)} - rk + \lambda_2 v \ .$$

<u>Proof</u>: Looking at the row-incidence matrix we easily establish the following relations among α, α_0, α_1:

$$(1) \qquad \begin{aligned} &2\alpha + \alpha_0 + \alpha_1 = k \\ &\alpha + \alpha_1 = r_{(k)} \\ &\alpha_0 m^2 + \alpha_1 (m+1)^2 + 2\alpha\, m(m+1) = \lambda_1 \\ &(\alpha + \alpha_0) m^2 + (\alpha + \alpha_1)(m+1)^2 = rb - \lambda_1 (v-1) \ . \end{aligned}$$

The first equation expresses the fact that the row-incidence matrix has k columns, the second gives the number of rows which contain the i-th variety $m+1$ times, which we know by Theorem 1.2.2 to be $r_{(k)}$, while the last two are immediate consequences of

$$a^{(i)} a^{(\ell)} = \lambda_1 \qquad a^{(i)} a^{(i)} = rb - \lambda_1 (v-1) \ .$$

Subtracting the 3rd equation from the last one we obtain directly

$$\alpha = rb - \lambda_1 v$$

independent of i,ℓ; substituting in the first two equations we obtain

$$\alpha_1 = r_{(k)} - rb + \lambda_1 v \qquad \alpha_0 = k - r_{(k)} - rb + \lambda_1 v \; .$$

In exactly the same way we will obtain the corresponding expressions for β, β_0, β_1.

<u>Theorem</u> <u>1.3.6</u>: In any GY-design

(I)
a) $vr_{(k)} = kb_{(v)} \qquad br_{(k)} = rb_{(v)}$

b) $(v-1)(rb - \lambda_1 v) = b_{(v)}(k - r_{(k)})$

c) $\lambda_1 = m(r + r_{(k)}) + \dfrac{r_{(k)}(b_{(v)} - 1)}{v-1}$

(II)
a) $vr_{(b)} = bk_{(v)} \qquad kr_{(b)} = rk_{(v)}$

b) $(v-1)(rk - \lambda_2 v) = k_{(v)}(b - r_{(b)})$

c) $\lambda_2 = m(r + r_{(b)}) + \dfrac{r_{(b)}(k_{(v)} - 1)}{v-1}$.

<u>Proof</u>: If A is the row-incidence matrix of the GY-design, then $A - m\,J_{k,v}$ is the incidence matrix of a BIB design with parameters (r',b',k',r',λ') where

$$v' = v \quad b' = k \quad k' = b_{(v)} \quad r' = r_{(k)} \quad \lambda' = \alpha_1 \; .$$

Therefore we must have

(1) $$vr_{(k)} = kb_{(v)}$$

(2) $$\alpha_1(v-1) = r_{(k)}(b_{(v)} - 1) \; .$$

Equation (1) together with $rv = kb$ gives

$$br_{(k)} = rb_{(v)}$$

therefore proving part a) of I.

By previous Theorem 1.3.5 we know that

$$\alpha_1 = r_{(k)} - rb + \lambda_1 v \ . \tag{3}$$

Substituting in (2) we obtain

$$(v-1)(rb - \lambda_1 v) = r_{(k)}(v - b_{(v)}) \ .$$

Using the first result $r_{(k)}v = kb_{(v)}$ we obtain part b) of (I).

From (2) and (3) we obtain

$$\lambda_1 v = \frac{r_{(k)}[b_{(v)} - 1]}{v - 1} + rb - r_{(k)} \ . \tag{4}$$

Noting that $b_{(v)} = b - mv$ and $r = mK + r_{(k)}$ and substituting in (4) we obtain after simplifying

$$\lambda_1 = \frac{r_{(k)}(b - m - 1)}{v - 1} + mr \ .$$

Obsering that $b - m = m(v-1) + b_{(v)}$ we obtain the desired result

$$\lambda_1 = m(r + r_{(k)}) + \frac{r_{(k)}(b_{(v)} - 1)}{v - 1} \ .$$

Part (II) is similarly proven.

<u>Corollary 1.3.1</u>: In any GY-design $rb = \lambda_1 v$, or equivalently $rk = \lambda_2 v$, if and only if $b, (k)$, is a multiple of v; otherwise $rb > \lambda_1 v$, $(rk > \lambda_2 v)$.

<u>Proof</u>: It is an immediate consequence of part b) and Theorem 1.2.4.

Corollary 1.3.2: The matrix $AA^T, (BB^T)$, is singular if and only if $b, (k)$, is a multiple of v.

Proof: By Theorem 1.2.2 we know that

$$AA^T = (rb - \lambda_1 v)I_v + \lambda_1 J_v .$$

Since the eigenvalues of $\lambda_1{}_v$ are 0 and $\lambda_1 v$ with multiplicities $v-1$ and 1 respectively, the eigenvalues of AA^T are $rb - \lambda_1 v$ and $rb - \lambda_1 v + \lambda_1 v$, with multiplicities $v-1$ and 1, therefore its product, which coincides with $\det(AA^T)$, is $rb(rb - \lambda_1 v)^{v-1}$, which by previous corollary is zero if and only if b is a multiple of v.

Corollary 1.3.3: A necessary condition for the existence of a $(v,b,k,r,\lambda_1,\lambda_2)$ GY-design is that

$$\frac{r_{(k)}(b_{(v)} - 1)}{v - 1} \quad \text{and} \quad \frac{r_{(b)}(k_{(v)} - 1)}{v - 1}$$

both be integers.

Proof: It is an immediate consequence of part c) of the theorem.

Corollary 1.3.4: In any GY-design $k \geq v$.

Proof: It is Fisher inequality for the BIB design with incidence matrix $A - mJ_v$.

Assumption: Since the transpose of a GY-design is also a GY-design we will assume from now on that $b \geq k$; we will also assume $b > v$, since $b = v$ reduces the GY-design to an ordinary Youden design.

1.4 Analysis of GY-designs

$(v,b,k,r,\lambda_1,\lambda_2)$ - GY-designs serve as designs for factorial experiments with three factors (row, column and variety) at k,b,v

levels respectively. The row factor at the i-th level will be spoken of as the i-th row, and similarly for the other factors. Every entry of the GY-design represents a treatment, so we have only $N = kb$ treatments instead of the kbv which will appear in a complete design.

Let $y_{ij\ell}$ be the observation on the ℓ-th variety in the i-th row and j-th column; of course ℓ is uniquely determined by i,j, that is $\ell = \ell(i,j)$ where the function ℓ is given by the design. The model is

$$y_{ij\ell} = \mu + \alpha_i + \beta_j + \gamma_\ell + e_{ij\ell}$$

where μ is the overall mean, α_i, β_j, γ_ℓ are main effects and $e_{ij\ell}$ is the random error, that is we assume that no interactions of two or more factors are present. We further assume that the random errors are normally distributed around zero with covariance matrix $\sigma^2 I$.

The model can also be described in matrix notation:

$$Y = X\beta + e$$

where Y is the N-rowed vector of observations, β is the p-rowed vector of parameters to be estimated (overall mean plus $k + b + v$ main effects), e is the N-rowed vector of errors and X is the $N \times p$ matrix of the design; the normality assumption can be expressed as $Y \sim N(X\beta,\sigma I)$ that is, Y has a multivariate normal distribution with mean $X\beta$ and covariance matrix $\sigma^2 I$.

We will associate the positive integer $\alpha(i,j) = (i-1)b + j$ with the (i,j) cell, or <u>plot</u>, of the GY-design. Clearly for every positive integer $\alpha \leq kb$ there is exactly one pair (i,j), or plot, such that $\alpha = \alpha(i,j)$; accordingly, the plot in the i-th row and the j-th column will be called the α-th plot, and the corresponding observation the α-th observation, with $\alpha = \alpha(i,j)$.

The matrix X is a matrix of zeros and ones whose rows correspond to the plots and whose columns correspond to the parameters to be estimated. Clearly the column corresponding to μ is a column of 1's only, which we will write as last. Since each of the k levels of the row factor appears exactly once with each of the b levels of the column factor, each of the k columns corresponding to the parameters α_i, $i = 1,\ldots,k$ contains exactly b ones, each of the b columns corresponding to the parameters β_j, $j = 1,2,\ldots,b$ contains exactly k ones and the product of the columns corresponding to α_i and β_j is always 1 for any $i = 1,\ldots,k$, any $j = 1,\ldots,b$. Finally the column corresponding to the parameter γ_h, $h = 1,2,\ldots,v$, has a one in the $\alpha(i,j)$ row if and only if $\ell(i,j) = h$ and contains exactly r ones. The matrix X therefore looks like:

$$X = \begin{bmatrix}
1 & 0 \ldots 0 & 1 & 0 \ldots 0 & 0 \ldots 1 \ldots 0 & 1 \\
1 & 0 \ldots 0 & 0 & 1 \ldots 0 & 0 \ldots 1 \ldots 0 & 1 \\
\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \\
1 & 0 \ldots 0 & 0 & 0 \ldots 1 & & 1 \\
0 & 1 \ldots 0 & 1 & 0 \ldots 0 & & 1 \\
0 & 1 \ldots 0 & 0 & 1 \ldots 0 & & 1 \\
\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & & \\
0 & 1 \ldots 0 & 0 & 0 \ldots 1 & & 1 \\
 & \vdots & & & & \\
0 & 0 \ldots 1 & 1 & 0 \ldots 1 & & 1 \\
0 & 0 \ldots 1 & 0 & 1 \ldots 0 & & 1 \\
\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & & \\
0 & 0 \ldots 1 & 1 & 0 \ldots 1 & & 1
\end{bmatrix}$$

Let us introduce the matrices A^i, B^i, $i = 1,\dots,k$ where A^i is a $b \times k$ matrix with 1's in the i-th column and zeros elsewhere, and B^i is a $b \times v$ matrix with a one in the cells $(j,\ell(i,j))$, $j = 1,2,\dots,b$ and zeros elsewhere. J will be a matrix of ones, j a column vector of ones; we will indicate the dimension of a matrix with subindices when necessary. Using these matrices X can be expressed as

$$X = \begin{bmatrix} A^1 & I & B^1 & j \\ A^2 & I & B^2 & j \\ \vdots & & & \\ A^k & I & B^k & j \end{bmatrix}$$

The normal equations are

$$X^TY = X^TX\,\hat{\beta}$$

so that if X^TX is non-singular the least square estimates $\hat{\beta}$ are given by

$$\hat{\beta} = (X^TX)^{-1}\,X^TY\ .$$

It is a straightforward calculation to arrive at

$$X^TX = \begin{bmatrix} bI_k & J_{k,b} & A^T_{k,v} & bj_k \\ J_{bK} & kI_b & B^T_{b,v} & kj_b \\ A_{vk} & B_{v,b} & rI_v & rj_v \\ bj^T & kj_b^T & rj_v^T & kv \end{bmatrix}$$

where A, B are, of course, the row-incidence and column-incidence matrices of the GY-design.

The normal equations can also be obtained directly from the model. With the usual notation and side conditions

$$\sum_i \alpha_i = 0 \qquad \sum_j \beta_j = 0 \qquad \sum_\ell \gamma_\ell = 0$$

we obtain

$$\begin{aligned}
T_{\cdots} &= kb\,\hat{\mu} \\
T_{i\cdot\cdot} &= b\,\hat{\mu} + b\,\hat{\alpha}_i + \sum_\ell a_{\ell i}\,\hat{\gamma}_\ell \\
T_{\cdot j\cdot} &= k\,\hat{\mu} + k\,\hat{\beta}_j + \sum_\ell b_{\ell j}\,\hat{\gamma}_\ell \\
T_{\cdot\cdot\ell} &= r\,\hat{\mu} + \sum_i a_{\ell i}\,\hat{\alpha}_i + \sum_j b_{\ell j}\,\hat{\beta}_j + r\,\hat{\gamma}_\ell
\end{aligned}$$

where, of course, $a_{\ell i}$, $b_{\ell j}$ are the general entries in the row-incidence and column-incidence matrices of the GY-design.

In order to eliminate the row and column effects let us compute

$$kb\,T_{\cdot\cdot\ell} - b\sum_j b_{\ell j}T_{\cdot j\cdot} - k\sum_i a_{\ell i}T_{i\cdot\cdot} =$$

$$= kbr\,\hat{\gamma}_\ell - b\sum_j b_{\ell j}\sum_\ell b_{\ell j}\,\hat{\gamma}_\ell - k\sum_i a_{\ell i}\sum_\ell a_{\ell i}\,\hat{\gamma}_\ell - br\,\hat{\mu} =$$

$$= kbr\,\hat{\gamma}_\ell - kbr\,\hat{\mu} - b\,\hat{\gamma}_\ell(rk - \lambda_2 v) - k\,\hat{\gamma}_\ell(rb - \lambda_1 v) =$$

$$= \hat{\gamma}_\ell v[k\,\lambda_1 + b\,\lambda_2 - r^2] - kbr\,\hat{\mu}\ .$$

Dividing by $r^2 v$,

$$\frac{k\,\lambda_1 + b\,\lambda_2 - r^2}{r^2}\,\hat{\gamma}_\ell = y_{\cdots} + y_{\cdot\cdot\ell} - \frac{1}{r}\sum_j b_{\ell j}y_{\cdot j\cdot} - \frac{1}{r}\sum_i a_{\ell i}y_{i\cdot\cdot}$$

which gives the variety effects $\hat{\gamma}_\ell$; the row effects $\hat{\alpha}_i$ and column effects $\hat{\beta}_j$ can now be easily obtained; from the first normal equation we obviously obtain $\hat{\mu} = y_{\cdots}$.

CHAPTER II

CONSTRUCTION OF GENERALIZED YOUDEN DESIGNS

2.1 GY-designs with $b_{(v)} = 0$ or $k_{(v)} = 0$

Theorem 2.1.1: There exist GY-designs with $b = mv$, $k = nv$ for any positive integers m, n and v.

Proof: Let $\{L_{i,j} | i = 1,\ldots,n;\ j = 1,\ldots,m\}$ be a collection of mn Latin Squares of order v, not necessarily different. Then the $nv \times mv$ matrix

$$D = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1m} \\ \\ L_{n1} & L_{n2} & \cdots & L_{nm} \end{bmatrix}$$

is a GY-design, since clearly every variety occurs m times in each row and n times in each column, and every pair of distinct varieties occur together in the same row m^2k times and n^2b times in the same column.

Theorem 2.1.2: If there exists a symmetric BIB design with parameters (v,k',λ), $k' \le v$, then there exists GY-designs with parameters $v = v$, $b = mv$, $k = nv + k'$ for arbitrary positive integers m, n.

Proof: The other parameters of the GY-design are easily established to be

$$r = mk\ ,\ \lambda_1 = m^2k\ ,\ \lambda_2 = m[\lambda + n(k + k')]\ .$$

Let $\{B_i \mid i = 1,\dots,m\}$ be a collection of symmetric BIB designs, not necessarily different, with common parameters (v,k',λ); each B_i can be converted to a Youden Square Y_i by reordering the varieties within each block (Smith and Hartley, 1958). Let $\{L_{ij} \mid i = 1,\dots,n;\ j = 1,\dots,m\}$ be a collection of Latin Squares of order v, not necessarily different. We claim that the matrix

$$D = \begin{bmatrix} Y_1 & \cdots & Y_m \\ L_{11} & \cdots & L_{1m} \\ \vdots & & \\ L_{n1} & \cdots & L_{n,m} \end{bmatrix}$$

is a GY-design.

a) Every variety appears v times in each Latin Square and k' times in each Youden Square, for a total of $mnv + mk' = mk = r$ times.

b) Every variety appears once in each row of each Youden Square and of each Latin Square, therefore any pair of distinct varieties occurs together in the same row of D, $m^2k = \lambda_1$ times.

c) Let x,y be two distinct varieties; each Youden Square has λ columns containing both x and y, $(k' - \lambda)$ columns containing x but no y, another $(k' - \lambda)$ columns containing y but no x, and the remainder $v - \lambda - 2(k' - \lambda)$ columns will contain neither x nor y. Therefore the two varieties x,y will appear together in the same column in D, $[\lambda(n+1)^2 + 2(k'-\lambda)n(n+1) + [v - \lambda - 2(k'-\lambda)]n^2]m = m[\lambda + n(k + k'+] = \lambda_2$ times, which concludes the proof.

Note that the existence of a symmetric BIB design is needed to carry out the construction but is by no means necessary for the existence of the GY-design, as the following theorem shows.

Theorem 2.1.3: Let $s = p^n$ be a power of a prime. Then there exists GY-designs with parameters

$$v = s^2 + 1 \ , \ b = s(s^2 + 1) \ , \ k = s + 1 \ .$$

Proof: The other parameters are easily computed

$$r = s(s + 1) \ , \ \lambda_1 = s^2(s + 1) \ , \ \lambda_2 = (s+1)(2s^2 + 2s + 1) \ .$$

Now let Q be a non-degenerate elliptic quadric in $PG(3,s)$; it contains $s^2 + 1$ points and each plane of the geometry intersects the quadric Q in either one single point (tangent plane) or in exactly $s + 1$ points forming a non-degenerate quadric Since Q contains $s^2 + 1$ points, there are $s^2 + 1$ tangent planes and $s^3 + s^2 + s+1 - (s^2 + 1) = s(s^2 + 1)$ non-tangent planes. Taking the points of Q as varieties and the non-tangent planes as blocks we obtain a BIB design with parameters

$$v = s^2 + 1, \ b = s(s^2 + 1), \ k = s + 1, \ \lambda = s + 1 \ .$$

This design has the property that every triple of varieties occurs in exactly one block, which is a translation of the fact that any three points of the quadric determine a unique non-tangent plane.

Agrawal (1966) has proved that in any BIB design with $b = mv$, the varieties can be rearranged within each block (column) so that every variety appears in a row m times; the rearrangement is achieved using systems of distinct representatives, which in turn can be constructed with Hall's algorithm (Hall, 1956). After the rearrangement of varieties, the BIB design becomes the desired GY-design.

Note that no symmetric BIB design exists with $v = s^2 + 1$, $k = s + 1$.

2.2 Geometric construction of GY-designs

In this section we will consistently make use of the following conventions and notation.

s will designate a power of a prime number, $s = p^n$; GF(s) will stand for the Galois field with s elements; EG(2,s) will designate the Euclidean plane based on GF(s).

Let $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2, \ldots, \alpha_{s-1}$ be the s elements of GF(s) in some order; let ℓ_i be the line with equation $x = \alpha_i$, $i = 0,1,\ldots,s-1$ and similarly let $\ell_{j,i}$ be the line with equation $\alpha_j x + y = \alpha_i$, $i,j = 0,1,\ldots,s-1$; the s parallel lines ℓ_i, $i = 0,1,\ldots,s-1$ form a pencil X, and for each $\alpha_j \in GF(s)$ the s parallel lines $\ell_{j,i}$, $i = 0,1,\ldots,s-1$, form also a pencil Y_j; the order in GF(s) induces an order of the lines within each pencil as follows: for any $\alpha_i, \alpha_j, \alpha_u \in GF(s)$,

$$\ell_i < \ell_u \quad \text{it and only if} \quad \alpha_i < \alpha_u$$

$$\ell_{j,i} < \ell_{j,u} \quad \text{if and only if} \quad \alpha_i < \alpha_u .$$

The lines ℓ_i and $\ell_{j,i}$ will be referred to as the i-th lines of pencils X and Y_j respectively.

Any point P of EG(2,s) is uniquely determined as the intersection of a line of the pencil X and a line of the pencil

Y_0. We can therefore order the points of $EG(2,s)$ as follows:

Let P, P' be two distinct points of $EG(2,s)$ given by

$$P = \ell_i \cap \ell_{0,j} , \quad P' = \ell_{i'} \cap \ell_{0,j'}$$

then $P < P'$ if and only if $\ell_i < \ell_{i'}$ or $i = i'$ and $\ell_{0,j} < \ell_{0,j'}$.

We will assign the numbers $0,1,\ldots,s^2-1$ to the s^2 points of $EG(2,s)$ in that order. In all the algebraic manipulations applied subsequently these serial numbers of the points will be treated as actual numbers.

Lines will be viewed as s-tuples of their points enumerated in increasing order, and pencils as square matrices of points whose i-th row is the i-th line of the pencil, $i = 0,1,\ldots,s-1$.

We will use the $n \times n$ permutation matrices τ_n and ζ_n defined as follows:

$$\tau_n = \begin{bmatrix} 0_{n-1,1} & I_{n-1} \\ 1 & 0_{1,n-1} \end{bmatrix} \qquad \zeta_n = \tau_n^T .$$

By premultiplying a $m \times n$ matrix A by τ_m we achieve a cyclic permutation of its rows; by postmultiplying A by ζ_n we achieve a cyclic permutation of its columns. The subindices will be dropped whenever the dimensions of the matrices involved are clear.

We will also introduce the transformation σ defined on the points of $EG(2,s)$ as follows:

$$\sigma(x,y) = (y,x) \quad \forall \ (x,y) \in EG(2,s) .$$

Y will denote the $s^2 \times s$ matrix

$$Y = \begin{bmatrix} Y_0 \\ Y_1\zeta \\ \vdots \\ Y_{s-1}\zeta^{s-1} \end{bmatrix}$$

and G will be the $s^2 + s \times s$ matrix

$$G = \begin{bmatrix} X \\ Y \end{bmatrix} .$$

<u>Theorem 2.2.1</u>: There exist GY-designs with parameters $v = s^2$, $b = k = s(s + 1)$.

<u>Proof</u>: The other parameters are

$$r = (s + 1)^2 \text{ , } m = n = 1 \text{ , } \lambda_1 = \lambda_2 = s^2 + 3s + 3$$

$$\alpha = s \qquad , \alpha_0 = s^2 - s - 1 \text{ , } \alpha_1 = 1 \text{ .}$$

We will take the varieties of the design to be the points of $EG(2,s)$.

We claim that each column of the matrix Y is a permutation of the set of the s^2 points.

Suppose that the point a appears twice in the j-th column of Y for some j; then we must have

$$\{a\} = \ell_{\alpha,i} \cap \ell_{j+\alpha} = \ell_{\beta,k} \cap \ell_{j+\beta}$$

for some α,β,i,k, $\alpha \neq \beta$, which is impossible since the lines $\ell_{j+\alpha}$ and $\ell_{j+\beta}$ are different and parallel.

Similarly each row of σY^T is also a permutation of the points of $EG(2,s)$.

We now claim that the matrix

$$D = \begin{bmatrix} X & \sigma Y^T \\ Y & L \end{bmatrix}$$

where L is any Latin Square of order s^2, is the desired GY-design.

First note that $\sigma X^T = X$, therefore the first s rows of D are the lines of EG(2,s) written vertically, and we have natural one-to-one correspondence between the lines of EG(2,s) and the rows and the columns of D.

Note that a point occurs twice in a row or column of D if and only if it belongs to the corresponding line; consequently since no two lines have more than one point in common any two rows or columns will have at most one point occurring twice in common. Therefore $\alpha_1 = \beta_1 = 1$ and we conclude that D is a GY-design.

<u>Example 2.2.1</u>: For $s = 4$ we have

$$v = 16 \text{ , } b = k = 20 \text{ , } r = 25 \text{ , } \lambda_1 = \lambda_2 = 31$$

$$X = \begin{matrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{matrix} \qquad Y_0 = \begin{matrix} 0 & 4 & 8 & 12 \\ 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \end{matrix}$$

$$Y_1 = \begin{matrix} 0 & 5 & 10 & 15 \\ 1 & 4 & 11 & 14 \\ 2 & 7 & 8 & 13 \\ 3 & 6 & 9 & 12 \end{matrix} \qquad Y_2 = \begin{matrix} 0 & 6 & 11 & 13 \\ 1 & 7 & 10 & 12 \\ 2 & 4 & 9 & 15 \\ 3 & 5 & 8 & 14 \end{matrix}$$

$$Y_3 = \begin{matrix} 0 & 7 & 9 & 14 \\ 1 & 6 & 8 & 15 \\ 2 & 5 & 11 & 12 \\ 3 & 4 & 10 & 13 \end{matrix}$$

$$D = \begin{bmatrix}
0 & 1 & 2 & 3 & 0 & 4 & 8 & 12 & 5 & 1 & 13 & 9 & 14 & 10 & 6 & 2 & 11 & 15 & 3 & 7 \\
4 & 5 & 6 & 7 & 1 & 5 & 9 & 13 & 10 & 14 & 2 & 6 & 7 & 3 & 15 & 11 & 0 & 4 & 8 & 12 \\
8 & 9 & 10 & 11 & 2 & 6 & 10 & 14 & 15 & 11 & 7 & 3 & 0 & 4 & 8 & 12 & 13 & 9 & 5 & 1 \\
12 & 13 & 14 & 15 & 3 & 7 & 11 & 15 & 0 & 4 & 8 & 12 & 9 & 13 & 1 & 5 & 6 & 2 & 14 & 10 \\
0 & 4 & 8 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 5 & 9 & 13 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 \\
2 & 6 & 10 & 14 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 \\
3 & 7 & 11 & 15 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 \\
5 & 10 & 15 & 0 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 \\
4 & 11 & 14 & 1 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 \\
7 & 8 & 13 & 2 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 \\
6 & 9 & 12 & 3 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
11 & 13 & 0 & 6 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
10 & 12 & 1 & 7 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 15 & 2 & 4 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
8 & 14 & 3 & 5 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
14 & 0 & 7 & 9 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
15 & 1 & 6 & 8 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
12 & 2 & 5 & 11 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
13 & 3 & 4 & 10 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14
\end{bmatrix}$$

Remark: It seems worthwhile to explain the main idea behind the construction of the above GY-designs. The points, the lines and the points within the lines were ordered in such a way that the corresponding columns of each of the matrices representing the parallel pencils Y_j, $j = 0,1,\ldots,s-1$, consist of all elements of the same row of the matrix X. Moreover since for $x = 0$ the equation $y = \alpha_i$ is the same as $\alpha_j x + y = \alpha_i$ the columns of each of these matrices consisting of the elements of the row of the X matrix for which $x = 0$ are also identical with respect to the order of their elements within the columns. Consequently, since no two lines of distinct parallel pencils can have more than one point in common, the remaining $s - 1$ sets consisting of s columns whose elements belong to the same row of the X matrix $x = \alpha_i$, $\alpha_i \neq 0$, form distinct permutations of these elements of a specific structure. Namely each element will belong to one and only one set of s columns and will occupy within the set all the distinct s positions of a column. Hence the s distinct powers of the ζ operation, which permutes cyclically the columns of each of the Y_j parallel pencils, will place each element in each of the distinct columns of the matrix Y.

Analogous reasoning applies to the σY^T matrix with y and τ replacing the roles of x and ζ respectively.

For the next construction we need the following lemma.

Lemma 2.2.1: There exist Latin Squares of order s^2 which can be split into s groups of s columns in such a way that every row in each group is a line of $EG(2,s)$.

Proof: We claim that

$$L = \begin{bmatrix} Y_0 & \tau Y_0 & \cdots & \tau^{s-1} Y_0 \\ Y_1 \zeta & \tau Y_1 \zeta & \cdots & \tau^{s-1} Y_1 \zeta \\ \vdots & \vdots & & \vdots \\ Y_{s-1}\zeta^{s-1} & \tau Y_{s-1}\zeta^{s-1} & \cdots & \tau^{s-1} Y_{s-1}\zeta^{s-1} \end{bmatrix}$$

is the desired Latin Square.

We have already shown that each column of Y is a permutation of the s^2 points, therefore so is every column of L.

We must show now that each row of L is also a permutation of the s^2 points; but since τ^i is not the identity if $0 < i < s-1$ each row of L is made out of s different lines belonging to the same parallel pencil and therefore no point can occur twice in the same row.

<u>Example 2.2.3</u>: We have already constructed $EG(2,4)$. The Latin Square can now be exhibited as follows:

$$L = \begin{array}{cccc|cccc|cccc|cccc} 0 & 4 & 8 & 12 & 1 & 5 & 9 & 13 & 2 & 6 & 10 & 14 & 3 & 7 & 11 & 15 \\ 1 & 5 & 9 & 13 & 2 & 6 & 10 & 14 & 3 & 7 & 11 & 15 & 0 & 4 & 8 & 12 \\ 2 & 6 & 10 & 14 & 3 & 7 & 11 & 15 & 0 & 4 & 8 & 12 & 1 & 5 & 9 & 13 \\ 3 & 7 & 11 & 15 & 0 & 4 & 8 & 12 & 1 & 5 & 9 & 13 & 2 & 6 & 10 & 14 \\ \hline 5 & 10 & 15 & 0 & 4 & 11 & 14 & 1 & 7 & 8 & 13 & 2 & 6 & 9 & 12 & 3 \\ 4 & 11 & 14 & 1 & 7 & 8 & 13 & 2 & 6 & 9 & 12 & 3 & 5 & 10 & 15 & 0 \\ 7 & 8 & 13 & 2 & 6 & 9 & 12 & 3 & 5 & 10 & 15 & 0 & 4 & 11 & 14 & 1 \\ 6 & 9 & 12 & 3 & 5 & 10 & 15 & 0 & 4 & 11 & 14 & 1 & 7 & 8 & 13 & 2 \\ \hline 11 & 13 & 0 & 6 & 10 & 12 & 1 & 7 & 9 & 15 & 2 & 4 & 8 & 14 & 3 & 5 \\ 10 & 12 & 1 & 7 & 9 & 15 & 2 & 4 & 8 & 14 & 3 & 5 & 11 & 13 & 0 & 6 \\ 9 & 15 & 2 & 4 & 8 & 14 & 3 & 5 & 11 & 13 & 0 & 6 & 10 & 12 & 1 & 7 \\ 8 & 14 & 3 & 5 & 11 & 13 & 0 & 6 & 10 & 12 & 1 & 7 & 9 & 15 & 2 & 4 \\ \hline 14 & 0 & 7 & 9 & 15 & 1 & 6 & 8 & 12 & 2 & 5 & 11 & 13 & 3 & 4 & 10 \\ 15 & 1 & 6 & 8 & 12 & 2 & 5 & 11 & 13 & 3 & 4 & 10 & 14 & 0 & 7 & 9 \\ 12 & 2 & 5 & 11 & 13 & 3 & 4 & 10 & 14 & 0 & 7 & 9 & 15 & 1 & 6 & 8 \\ 13 & 3 & 4 & 10 & 14 & 0 & 7 & 9 & 15 & 1 & 6 & 8 & 12 & 2 & 5 & 11 \end{array}$$

An attractive feature of this family of Latin Squares is that they are split into s^2 subsquares each of which contains each of the varieties once. We conjecture that they are orthogonally mateless, but were so far unsuccessful in proving it. Orthogonally mateless Latin Squares of order k are known to exist for arbitrarily large and even k, but their existence is unknown for arbitrarily large k when k is odd. Our conjecture, if true, will give a construction of an orthogonally mateless Latin Square for all k of the form $k = p^{2n}$, p a prime number.

Theorem 2.2.2: There exist GY-designs with parameters $v = s^2$, $b = s(s^2 - 1)$, $k = s(s+1)$.

Proof: The other parameters are

$$r = (s + 1)^2(s - 1) \qquad m = s - 1 \qquad n = 1$$

$$r_{(b)} = s^2 - 1 \qquad r_{(k)} = s^2 - 1$$

$$\lambda_1 = (s-1)(s^2 - 1)(s + 2) + (s^2 - s - 1)$$

$$\lambda_2 = (s^2 - 1)(s + 2) + (s-1)$$

$$\alpha = s \qquad \alpha_0 = 1 \qquad \alpha_1 = s^2 - s - 1$$

$$\beta = s^2 - s \qquad \beta_0 = s^3 - 2s^2 + 1 \qquad \beta_1 = s - 1 .$$

Let L be the Latin Square of order s^2 constructed as in the previous lemma. For every point a, let $p_L(a)$ be the transpose of the column vector of L whose first component is a with that first component missing, this notation is consistent since each row of L is a permutation of the points of $EG(2,s)$. Thus $p_L(a)$ is a (s^2-1)-tuple of distinct points and it does not contain the point a; p_L is a mapping defined through the Latin Square L; in matrix notation

$$p_L(a) = c_L(a)^T \begin{bmatrix} 0_{1,s^2-1} \\ \\ I_{s^2-1} \end{bmatrix}$$

where $c_L(a)$ is the column of L whose first element is a.

For any $m \times n$ matrix $A = (a_{ij})$, $p_L(A)$ will be naturally understood as the $m \times n(s^2-1)$ matrix $p_L(A) = (p_L(a_{ij}))$.

Now let $G = \begin{bmatrix} X \\ Y \end{bmatrix}$ and consider the $s(s+1) \times s(s^2-1)$ matrix $D = p_L(G)$.

We will prove first that the rows of D satisfy the requirements for a GY-design.

Any row of D contains every point of the geometry s times, except for the s points in the corresponding row of G, which will occur $s-1$ times. Furthermore, since the rows of G are the lines of $EG(2,s)$ the two elements of every pair of distinct points occur $s-1$ times in the same row of D exactly once. Therefore $\alpha_0 = 1$ and the row conditions are satisfied.

Let $x_{i,j}$, $y_{i,j}$ be the (i,j) entries in the matrices X and Y respectively; let G_j, $j = 0,1,\ldots,s-1$, be the $s \times s^2-1$ matrix whose i-th row is $p_L(x_{ij})$, $i = 0,1,\ldots,s-1$, and similarly let L_j, $j = 0,1,\ldots,s-1$, be the $s^2 \times s^2-1$ matrix whose i-th row is $p_L(y_{ij})$, $i = 0,1,\ldots,s^2-1$. Note that there are no repeated points in any row or column of L_j, $j = 0,\ldots,s-1$, but it is not a Latin Square since each row has only s^2-1 points.

The matrix D can be written

$$D = p_L(G) = \begin{bmatrix} G_0 & G_1 & \cdots & G_{s-1} \\ \\ L_0 & L_1 & \cdots & L_{s-1} \end{bmatrix} .$$

Observe that since $X^T = Y_0$,

$$G_j = \begin{bmatrix} \tau^j Y_0 \\ \tau^j Y_1 \zeta \\ \vdots \\ \tau^j Y_{s-1} \zeta^{s-1} \end{bmatrix}^T \begin{bmatrix} 0_{1,s^2-1} \\ \\ I_{s^2-1} \end{bmatrix} \qquad j = 0,1,\ldots,s-1$$

that is, the matrix G_j is the transpose of the j-th block of s columns of L with the first row missing, and that missing first row is $\ell_{0,j}$, the j-th line of the pencil Y_0. Therefore the columns of G_j are the lines of EG(2,s) written vertically except for the line $\ell_{0,j}$ and the s lines ℓ_i, $i = 0,1,\ldots,s-1$, of the pencil X. Hence in each G_j there are $s + 1$ missing lines.

The idea of the construction is to use one of the matrices G_j consisting of $s^2 - 1 = (s+1)(s-1)$ s-tuple columns to complete each of the remaining s-1 G_j's to a full geometry. We shall show that this can be achieved by permuting the elements within each row of the chosen G_j and keeping the rows constant which will preserve the already established GY-design property for the rows.

The lines to be recovered by the chosen G_j are the s lines of the pencil X each replicated s-1 times plus the lines of the pencil Y_0 except $\ell_{0,j}$, a total of

$$s(s-1) + s-1 = s^2 - 1 \text{ lines} .$$

Let the lines of the pencil X be written vertically. Since $X^T = Y_0$, if we apply the cyclic permutation τ^i to the i-th line of the pencil X, each row of the resulting matrix X^* will contain one point from each line of Y_0; indeed

$$X^* = [\ell_0, \tau\ell_1, \ldots, \tau^{s-1}\ell_{s-1}]$$

where ℓ_i, $i = 0,1,\ldots,s-1$, is the i-th line $x = \alpha_i$ of X written vertically. Consequently each row of the $s \times s(s-1)$ matrix

$$[X^*, \tau X^*, \ldots, \tau^{s-2}X^*]$$

will contain $s-1$ points from each line of Y_0.

We shall add to each row of the above matrix $s-1$ points chosen in such a way that all the lines except $\ell_{0,j}$ will be completed. Notice that this must be done in a unique way since each of the lines had exactly one point missing. We obtain this way the $s \times s^2 - 1$ matrix G_j^* which is characterized by the fact that only the line $\ell_{0,j}$ of Y_0 is not complete.

It is clear from the way G_j^* was constructed that the i-th point of $\ell_{0,j}$ will appear in the $j+i$-th ($j+i$ taken mod s) row of G_j^* as well as in the $s-2$ preceding rows $j+i-1 \pmod s, \ldots, j+i-(s-2) \pmod s$, but not in the following row $j+i+1 \pmod s$, $i = 0,1,\ldots,s-1$. Therefore the matrix $\tau^{j+1}G_j^*$ is such that its i-th row does not contain the i-th point of $\ell_{0,j}$, $i = 0,1,\ldots,s-1$, which is also the case with G_j. Thus the i-th rows of $\tau^{j+1}G_j^*$ and of G_j contain the same points, but in a different order.

Substituting $\tau^{j+1}G_j^*$ for G_j in D we obtain

$$D_j^* = \begin{bmatrix} G_0 & \cdots & \tau^{j+1}G_j^* & \cdots & G_{s-1} \\ & & & & \\ L_0 & \cdots & L_j & \cdots & L_{s-1} \end{bmatrix}$$

which we claim is a GY-design.

We need only to verify the conditions regarding the columns.

Since every column of L_i, $i = 0,1,\ldots,s-1$ is a row of a Latin Square, and since each column of G_i, $i = 0,1,\ldots,s-1$, and G_j^* is a line of EG(2,s) we see that a point occurs twice in a column as many times as it appears in a line; since each point belongs to $s+1$ lines in the geometry and we have $s-1$ replicated geometries, we conclude that any given point occurs twice in $(s+1)(s-1) = s^2 - 1 = r_{(b)}$ columns.

Two distinct points will appear each twice in the same column if they belong to the same line; since a pair of distinct points determine a unique line and there are $s-1$ replicated geometries, $\beta_1 = s-1$ and we can conclude that D_j^* is a GY-design.

Example 2.2.3: For $s = 4$ we have

$$\sigma = 16 \quad b = 60 \quad k = 20 \quad r = 75 \quad \lambda_1 = 281 \quad \lambda_2 = 93 \; .$$

From Example 2.3.2 we directly write

$$G_3 = \begin{matrix} 0 & 1 & 2 & & 6 & 5 & 4 & 7 & & 8 & 11 & 10 & 9 & & 13 & 14 & 15 & 12 \\ 4 & 5 & 6 & & 9 & 10 & 11 & 8 & & 14 & 13 & 12 & 15 & & 3 & 0 & 1 & 2 \\ 8 & 9 & 10 & & 12 & 15 & 14 & 13 & & 3 & 0 & 1 & 2 & & 4 & 7 & 6 & 5 \\ 12 & 13 & 14 & & 3 & 0 & 1 & 2 & & 5 & 6 & 7 & 4 & & 10 & 9 & 8 & 11 \end{matrix}$$

We have already obtained

$$X = \begin{matrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{matrix} \qquad Y_0 = \begin{matrix} 0 & 4 & 8 & 12 \\ 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \end{matrix}$$

We directly obtain

$$X^* = \begin{matrix} 0 & 7 & 10 & 13 \\ 1 & 4 & 11 & 14 \\ 2 & 5 & 8 & 15 \\ 3 & 6 & 9 & 12 \end{matrix}$$

$$G_3^* = \begin{matrix} 0 & 7 & 10 & 13 & 1 & 4 & 11 & 14 & 2 & 5 & 8 & 15 & 12 & 9 & 6 \\ 1 & 4 & 11 & 14 & 2 & 5 & 8 & 15 & 3 & 6 & 9 & 12 & 0 & 13 & 10 \\ 2 & 5 & 8 & 15 & 3 & 6 & 9 & 12 & 0 & 7 & 10 & 13 & 4 & 1 & 14 \\ 3 & 6 & 9 & 12 & 0 & 7 & 10 & 13 & 1 & 4 & 11 & 14 & 8 & 5 & 2 \end{matrix}$$

Since τ^4 is the identity, the rows of G_3^* correspond to the rows of G_3, so there is no need to reorder these rows.

The GY-design D_3^* would be

$$D_3^* = \begin{bmatrix} G_0 & G_1 & G_2 & G_3^* \\ & & & \\ L_0 & L_1 & L_2 & L_3 \end{bmatrix} .$$

<u>Theorem 2.2.3</u>: There exist GY-designs with parameters $v = s^2$, $b = k = s(s^2 - 1)$.

<u>Proof</u>: The other parameters are

$$r = (s^2-1)^2 \qquad\qquad m = n = s-1$$

$$r_{(b)} = r_{(k)} = (s^2-1)(s-1) \qquad\qquad b_{(v)} = s(s-1)$$

$$\lambda_1 = \lambda_2 = s^5 - 3s^3 + 3s - 1$$

$$\alpha = \beta = s(s-1) \qquad \alpha_1 = \beta_1 = s^3 - 2s^2 + 1 \qquad \alpha_0 = \beta_0 = s-1 .$$

Let us permute cyclically the lines within the same parallel pencil in $\tau^{j+1}G_j^*$; this can be accomplished by matrix multiplication as follows:

$$\tau^{j+1}G_j^* \begin{bmatrix} \zeta_s & & & & \\ & \zeta_s & & & \\ & & \ddots & & \\ & & & \zeta_s & \\ & & & & \zeta_{s-1} \end{bmatrix}^j = G_j^{**}$$

where there are $s-1$ matrices ζ_s and all the off diagonal matrices are zero.

We claim that the $s(s^2-1) \times s(s^2-1)$ square matrix

$$D^{**} = \begin{bmatrix} G_0 & G_1 & \cdots & G_{s-1}^{**} \\ L_0 & L_1 & \cdots & L_{s-1} \\ G_1 & G_2 & \cdots & G_0^{**} \\ L_1 & L_2 & \cdots & L_0 \\ \cdots & \cdots & \cdots & \cdots \\ G_{s-2} & G_{s-1} & \cdots & G_{s-3}^{**} \\ L_{s-2} & L_{s-1} & \cdots & L_{s-3} \end{bmatrix}$$

is a GY-design.

Using the same argument as in the previous theorem we will prove that the row conditions for GY-designs are satisfied.

Any given column of D^{**} is made out of $s-1$ rows of the Latin Square L, corresponding to the matrices L_i, plus $s-1$ different parallel lines, cooresponding to either the matrices G_i or to the matrices G_i^{**} as the case may be. Therefore a point occurs in each column either $s+1$ or s times; it will occur s times if and only if it belongs to one of the $s-1$

parallel lines. Since these parallel lines contain $s(s-1)$ points, the number of points repeated in the column $s = n+1$ times is $s(s-1) = s^2 - s = k_{(v)}$. Furthermore, the missing lines from each column of D^{**} are the columns of the missing G_j, (G_j^{**}), matrix in each block of s^2-1 columns; these matrices are

$$G_{s-1}, G_0, \ldots, G_{s-2}^{**}$$

and they constitute, as we have seen in the previous theorem, the full geometry $EG(2,s)$ replicated $s-1$ times. Therefore each member of a pair of points will appear $s-1$ times in the same column if and only if both points belong to the line missing from that column, and $\beta_0 = s-1$. This concludes the proof that D^{**} is a GY-design.

Example 2.2.4: For $s = 3$ we have

$v = 9$, $b = k = 24$, $r = 64$, $\lambda_1 = \lambda_2 = 170$, $\alpha = \beta = 6$, $\alpha_0 = \beta_0 = 2$, $\alpha_1 = \beta_1 = 10$

$$X = \begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{matrix} \quad Y_0 = \begin{matrix} 0 & 3 & 6 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{matrix} \quad Y_1 = \begin{matrix} 0 & 5 & 7 \\ 1 & 3 & 8 \\ 2 & 4 & 6 \end{matrix} \quad Y_2 = \begin{matrix} 0 & 4 & 8 \\ 1 & 5 & 6 \\ 2 & 3 & 7 \end{matrix}$$

$$L = \begin{matrix} 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 8 \\ 1 & 4 & 7 & 2 & 5 & 8 & 0 & 3 & 6 \\ 2 & 5 & 8 & 0 & 3 & 6 & 1 & 4 & 7 \\ 5 & 7 & 0 & 3 & 8 & 1 & 4 & 6 & 2 \\ 3 & 8 & 1 & 4 & 6 & 2 & 5 & 7 & 0 \\ 4 & 6 & 2 & 5 & 7 & 0 & 3 & 8 & 1 \\ 8 & 0 & 4 & 6 & 1 & 5 & 7 & 2 & 3 \\ 6 & 1 & 5 & 7 & 2 & 3 & 8 & 0 & 4 \\ 7 & 2 & 3 & 8 & 0 & 4 & 6 & 1 & 5 \end{matrix}$$

$$G_0 = \begin{matrix} 1 & 2 & 5 & 3 & 4 & 8 & 6 & 7 \\ 4 & 5 & 7 & 8 & 6 & 0 & 1 & 2 \\ 7 & 8 & 0 & 1 & 2 & 4 & 5 & 3 \end{matrix} \qquad G_1 = \begin{matrix} 2 & 0 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 3 & 8 & 6 & 7 & 1 & 2 & 0 \\ 8 & 6 & 1 & 2 & 0 & 5 & 3 & 4 \end{matrix}$$

$$G_2 = \begin{matrix} 0 & 1 & 4 & 5 & 3 & 7 & 8 & 6 \\ 3 & 4 & 6 & 7 & 8 & 2 & 0 & 1 \\ 6 & 7 & 2 & 0 & 1 & 3 & 4 & 5 \end{matrix}$$

$$\tau G_0^* = \begin{matrix} 1 & 5 & 6 & 2 & 3 & 7 & 4 & 8 \\ 0 & 4 & 8 & 1 & 5 & 6 & 7 & 2 \\ 2 & 3 & 7 & 0 & 4 & 8 & 1 & 5 \end{matrix} \qquad \tau^2 G_1^* = \begin{matrix} 2 & 3 & 7 & 0 & 4 & 8 & 6 & 5 \\ 1 & 5 & 6 & 2 & 3 & 7 & 0 & 8 \\ 0 & 4 & 8 & 1 & 5 & 6 & 3 & 2 \end{matrix}$$

$$G_2^* = \begin{matrix} 0 & 4 & 8 & 1 & 5 & 6 & 3 & 7 \\ 2 & 3 & 7 & 0 & 4 & 8 & 6 & 1 \\ 1 & 5 & 6 & 2 & 3 & 7 & 0 & 4 \end{matrix}$$

$$G_0^{**} = \tau G_0^* \qquad G_1^{**} = \begin{matrix} 3 & 7 & 2 & 4 & 8 & 0 & 5 & 6 \\ 5 & 6 & 1 & 3 & 7 & 2 & 8 & 0 \\ 4 & 8 & 0 & 5 & 6 & 1 & 2 & 3 \end{matrix}$$

$$G_2^{**} = \begin{matrix} 8 & 0 & 4 & 6 & 1 & 5 & 3 & 7 \\ 7 & 2 & 3 & 8 & 0 & 4 & 6 & 1 \\ 6 & 1 & 5 & 7 & 2 & 3 & 0 & 4 \end{matrix}$$

1 2 5 3 4 8 6 7   2 0 3 4 5 6 7 8   8 0 4 6 1 5 3 7
4 5 7 8 6 0 1 2   5 3 8 6 7 1 2 0   7 2 3 8 0 4 6 1
7 8 0 1 2 4 5 3   8 6 1 2 0 5 3 4   6 1 5 7 2 3 0 4

1 2 5 3 4 8 6 7   4 5 7 8 6 0 1 2   7 8 0 1 2 4 5 3
2 0 3 4 5 6 7 8   5 3 8 6 7 1 2 0   8 6 1 2 0 5 3 4
0 1 4 5 3 7 8 6   3 4 6 7 8 2 0 1   6 7 2 0 1 3 4 5
3 4 6 7 8 2 0 1   8 6 1 2 0 5 3 4   1 2 5 3 4 8 6 7
4 5 7 8 6 0 1 2   6 7 2 0 1 3 4 5   2 0 3 4 5 6 7 8
5 3 8 6 7 1 2 0   7 8 0 1 2 4 5 3   0 1 4 5 3 7 8 6
6 7 2 0 1 3 4 5   1 2 5 3 4 8 6 7   5 3 8 6 7 1 2 0
7 8 0 1 2 4 5 3   2 0 3 4 5 6 7 8   3 4 6 7 8 2 0 1
8 6 1 2 0 5 3 4   0 1 4 5 3 7 8 6   4 5 7 8 6 0 1 2

D^{**} = 2 0 3 4 5 6 7 8   0 1 4 5 3 7 8 6   1 5 6 2 3 7 4 8
5 3 8 6 7 1 2 0   3 4 6 7 8 2 0 1   0 4 8 1 5 6 7 2
8 6 1 2 0 5 3 4   6 7 2 0 1 3 4 5   2 3 7 0 4 8 1 5

4 5 7 8 6 0 1 2   7 8 0 1 2 4 5 3   1 2 5 3 4 8 6 7
5 3 8 6 7 1 2 0   8 6 1 2 0 5 3 4   2 0 3 4 5 6 7 8
3 4 6 7 8 2 0 1   6 7 2 0 1 3 4 5   0 1 4 5 3 7 8 6
8 6 1 2 0 5 3 4   1 2 5 3 4 8 6 7   3 4 6 7 8 2 0 1
6 7 2 0 1 3 4 5   2 0 3 4 5 6 7 8   4 5 7 8 6 0 1 2
7 8 0 1 2 4 5 3   0 1 4 5 3 7 8 6   5 3 8 6 7 1 2 0
1 2 5 3 4 8 6 7   5 3 8 6 7 1 2 0   6 7 2 0 1 3 4 5
2 0 3 4 5 6 7 8   3 4 6 7 8 2 0 1   7 8 0 1 2 4 5 3
0 1 4 5 3 7 8 6   4 5 7 8 6 0 1 2   8 6 1 2 0 5 3 4

We finish this section exhibiting another GY-design D_* with parameters $v = 9$, $b = k = 24$ which is non-isomorphic to D^{**} with the same parameters.

<u>Definition 2.1</u>: Two GY-designs with the same parameters are said to be isomorphic if one can be obtained from the other by renaming the varieties, reordering the rows or reordering the columns.

The GY-design D_*, which follows, was constructed using the unique geometry $EG(2,3)$ and trial and error.

$$D_* = \begin{array}{ccccccc}
1~1~2~2~3~3 & 4~4~4 & 5~5~7 & 6~6~6 & 7~7~9 & 8~8~8 & 9~9~5 \\
4~4~5~5~6~6 & 8~8~1 & 2~2~2 & 3~3~3 & 7~7~7 & 1~8~1 & 9~9~9 \\
7~7~8~8~9~9 & 1~1~1 & 6~2~2 & 3~3~3 & 4~4~4 & 5~5~5 & 6~6~2 \\
1~1~5~5~9~9 & 2~2~2 & 3~3~3 & 4~4~4 & 6~6~6 & 7~7~7 & 8~8~8 \\
2~2~6~6~7~7 & 1~1~1 & 8~3~3 & 9~9~9 & 5~5~5 & 8~3~8 & 4~4~4 \\
3~3~4~4~8~8 & 2~2~2 & 1~1~6 & 5~5~5 & 6~6~1 & 7~7~9 & 9~9~7 \\
1~1~4~4~7~7 & 2~2~8 & 3~3~3 & 5~5~5 & 9~6~6 & 9~9~6 & 2~8~8 \\
2~2~5~5~8~8 & 3~3~3 & 1~1~1 & 4~4~4 & 9~9~9 & 6~6~7 & 7~7~6 \\
3~3~6~6~9~9 & 8~8~8 & 2~2~2 & 4~4~1 & 1~1~4 & 7~7~7 & 5~5~5 \\
8~8~1~1~6~6 & 9~9~9 & 7~7~7 & 5~5~4 & 3~4~3 & 3~5~4 & 2~2~2 \\
9~9~2~2~4~4 & 8~8~8 & 5~5~5 & 6~6~6 & 7~7~7 & 3~3~3 & 1~1~1 \\
3~3~7~7~5~5 & 6~6~6 & 9~9~9 & 2~2~2 & 8~8~8 & 1~1~1 & 4~4~4 \\
4~4~3~3~2~2 & 5~5~5 & 7~7~8 & 9~9~7 & 1~1~6 & 6~1~6 & 8~8~9 \\
6~6~7~7~5~5 & 9~9~9 & 4~4~4 & 8~8~8 & 3~3~3 & 2~2~2 & 1~1~1 \\
8~8~9~9~1~1 & 7~7~7 & 6~6~6 & 2~2~2 & 5~5~3 & 4~4~3 & 5~3~4 \\
2~2~6~6~1~1 & 3~3~3 & 5~5~5 & 7~7~7 & 8~8~8 & 9~4~9 & 4~4~9 \\
8~8~3~3~7~7 & 4~4~4 & 9~9~9 & 2~2~2 & 5~5~5 & 1~1~1 & 6~6~6 \\
5~5~9~9~4~4 & 7~7~7 & 6~6~6 & 1~1~1 & 8~8~8 & 3~3~3 & 2~2~2 \\
5~5~8~8~2~2 & 7~7~7 & 4~4~4 & 9~9~9 & 1~1~1 & 6~6~6 & 3~3~3 \\
6~6~9~9~3~3 & 5~5~5 & 7~7~7 & 8~8~8 & 4~4~4 & 2~2~2 & 1~1~1 \\
7~7~1~1~4~4 & 6~6~3 & 8~8~8 & 3~3~6 & 2~2~2 & 9~9~9 & 5~5~5 \\
9~9~7~7~2~2 & 5~5~5 & 8~8~8 & 1~1~3 & 3~3~1 & 4~4~4 & 6~6~6 \\
5~5~3~3~1~1 & 6~6~6 & 4~4~9 & 8~8~8 & 9~9~9 & 2~2~2 & 7~7~7 \\
6~6~4~4~8~8 & 9~9~9 & 1~1~1 & 7~7~7 & 2~2~2 & 5~5~5 & 3~3~3
\end{array}$$

That D^{**} and D_* are not isomorphic is evident since D has several columns identical, while D^{**} has not two identical columns.

2.3 A class of non D-optimum GY-designs

As stated in Chapter I, J. Kiefer proved in his 1958 paper that GY-designs are D-optimum if either $b_{(v)} = 0$ or $k_{(v)} = 0$. We will show now that if the divisibility condition is not satisfied the GY-design may not be D-optimum.

Theorem 2.3.1: There exists GY-designs with $v = 4$, $b = k = 6t$ for any odd integer t.

Proof: The other parameters are

$$r = 9t^2 \qquad b_{(v)} = k_{(v)} = 2 \qquad r_{(b)} = r_{(k)} = 3t$$

$$m = n = \frac{3t-1}{2} \qquad \lambda_1 = \lambda_2 = \frac{27t^3 - t}{2}$$

$$\alpha = \beta = 2t \qquad \alpha_0 = \beta_0 = t \qquad \alpha_1 = \beta_1 = t \ .$$

Let the set of varieties be $V = \{A,1,2,3\}$ and let ζ be a permutation on V^b defined as follows:

$$\zeta(a_1,\ldots,a_b) = (a_b,a_1,\ldots,a_{b-1}), \ \forall(a_1,\ldots,a_b) \in V^b.$$

Let τ be a transformation on V which leaves exactly one variety fixed; by renaming the varieties if necessary we may assume without loss of generality that

$$\tau(A) = A \ , \ \tau(1) = 2 \ , \ \tau(2) = 3 \ , \ \tau(3) = 1 \ .$$

Finally let $p \in V^b$ be

$$p = (A \overset{m}{\ldots} A,\ 1 \overset{m+1}{\ldots} 1,\ 2 \overset{m+1}{\ldots} 2,\ 3 \overset{m}{\ldots} 3)$$

and let D be a $k \times b$ matrix whose first row is p and such that every row and column is the transformed of the preceding one by $\zeta \circ \tau$.

Since τ leaves A fixed, A will occur m times in each row and column of D; since τ^3 is the identity every variety other than A will appear $m+1$ times in two out of every three consecutive rows or columns.

Let d_{ij}, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,b$ be the (i,j) entry of the matrix D. We claim that if we make $d_{i,3t+i} = A$, $i = 1,2,\ldots,3t$, the resulting matrix D^* is a GY-design.

Variety A appears $m+1$ times in each of the first $3t = r_{(k)}$ rows; any other variety $x \neq A$ appears $m+1$ times in one out of every three consecutive rows for the first $3t$ rows, and in two out of every three consecutive rows for the last $3t$ rows, that is in a total of $\frac{3t}{3} + \frac{3t}{3}2 = 3t = r_{(k)}$ rows. Moreover, the pair of distinct varieties A,x $(x \neq A)$ appear $m+1$ times each in the same row $t = \alpha_1$ times.

A pair of distinct varieties other than A can occur $m+1$ times each in the same row only in the last $3t$ rows and in exactly one out of every three consecutive rows, that is in $t = \alpha_1$ rows.

The same arguments applied to the columns would allow us to conclude that D^* is a GY-design.

<u>Example 2.3.1</u>: For $t = 3$, we have

$v = 4$, $b = k = 18$, $r = 81$, $\lambda_1 = \lambda_2 = 363$, $m = n = 4$, $r_{(b)} = r_{(k)} = 9$, $b_{(v)} = k_{(v)} = 2$, $\alpha = \beta = 6$, $\alpha_0 = \beta_0 = 3$, $\alpha_1 = \beta_1 = 3$.

$$D = \begin{array}{cccccccccccccccccc}
A & A & A & A & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
1 & A & A & A & A & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 \\
2 & 2 & A & A & A & A & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
3 & 3 & 3 & A & A & A & A & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 & A & A & A & A & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\
1 & 2 & 2 & 2 & 2 & A & A & A & A & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & 3 & 3 & A & A & A & A & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
3 & 3 & 3 & 1 & 1 & 1 & 1 & A & A & A & A & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & A & A & A & A & 3 & 3 & 3 & 3 & 3 & 1 \\
2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & A & A & A & A & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & A & A & A & A & 2 & 2 & 2 & 2 \\
3 & 3 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & A & A & A & A & 3 & 3 & 3 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & A & A & A & A & 1 & 1 \\
2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & A & A & A & A & 2 \\
3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & A & A & A & A \\
A & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & A & A & A \\
A & A & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & A & A \\
A & A & A & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & A
\end{array}$$

$$
D^{*} = \begin{array}{cccccccccccccccccc}
A & A & A & A & 1 & 1 & 1 & 1 & 1 & A & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
1 & A & A & A & A & 2 & 2 & 2 & 2 & 2 & A & 3 & 3 & 3 & 3 & 1 & 1 & 1 \\
2 & 2 & A & A & A & A & 3 & 3 & 3 & 3 & 3 & A & 1 & 1 & 1 & 1 & 2 & 2 \\
3 & 3 & 3 & A & A & A & A & 1 & 1 & 1 & 1 & 1 & A & 2 & 2 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 & A & A & A & A & 2 & 2 & 2 & 2 & 2 & A & 3 & 3 & 3 & 3 \\
1 & 2 & 2 & 2 & 2 & A & A & A & A & 3 & 3 & 3 & 3 & 3 & 3 & A & 1 & 1 \\
2 & 2 & 3 & 3 & 3 & 3 & A & A & A & A & 1 & 1 & 1 & 1 & 1 & A & 2 & 2 \\
3 & 3 & 3 & 1 & 1 & 1 & 1 & A & A & A & A & 2 & 2 & 2 & 2 & 2 & A & 3 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & A & A & A & A & 3 & 3 & 3 & 3 & 3 & A \\
2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & A & A & A & A & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & A & A & A & A & 2 & 2 & 2 & 2 \\
3 & 3 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & A & A & A & A & 3 & 3 & 3 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & A & A & A & A & 1 & 1 \\
2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & A & A & A & A & 2 \\
3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & A & A & A & A \\
A & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & A & A & A \\
A & A & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & A & A \\
A & A & A & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & A
\end{array}
$$

We will show now that the GY-design D^{*} is not D-optimum, by comparing it with the non-symmetrical design D.

The hypothesis to be tested is that variety has no effect on yield, that is

$$\gamma_A = \gamma_1 = \gamma_2 = \gamma_3 \ .$$

In the two-way heterogeneity setting where we have v varieties and a $k \times b$ array of plots, the covariance matrix is given by (see for instance, Kiefer, 1958)

$$c_{ij} = \delta_{ij} r_i - \frac{\lambda_{ij}^{(1)}}{b} - \frac{\lambda_{ij}^{(2)}}{k} + \frac{r_i r_j}{kb}$$

where δ_{ij} is the Kronecker delta, r_i is the number of replications of the i-th variety and

$$\lambda_{ij}^{(1)} = \sum_{\ell=1}^{k} n_{i\ell}^{(1)} \; n_{j\ell}^{(1)}$$

$$\lambda_{ij}^{(2)} = \sum_{\ell=1}^{b} n_{i\ell}^{(2)} \; n_{j\ell}^{(2)}$$

with $n_{i\ell}^{(q)}$ equal to the number of occurrences of the i-th variety in the ℓ-th row $(q = 1)$ or the ℓ-th column $(q = 2)$.

It is a straightforward but long computation to obtain in the case of D^*

$$c_{ii}^* = \frac{27t^2 - 2}{4} \qquad c_{ij}^* = \frac{2 - 27t^2}{12}$$

for $i \neq j$, $i,j = A,1,2,3$.

For the design D one would obtain

$$c_{AA} = \frac{27t^2 - 6t - 1}{4} \qquad c_{Ai} = -\frac{27t^2 - 6t - 1}{12}$$

$$c_{ii} = \frac{243t^2 + 18t - 17}{36} \qquad c_{ij} = -\frac{81t^2 - 18t + 7}{36}$$

$i \neq j$, $i,j = 1,2,3$

and for the corresponding determinants Δ^* and Δ,

$$\Delta^* = \left[\frac{27t^2 - 2}{3}\right]^3$$

$$\Delta = \frac{[27t^2 + 3t - 2]^2 [27t^2 - 6t - 1]}{3^3} .$$

The difference $\Delta - \Delta^* = \dfrac{108t^3 - 45t^2 - 12t + 4}{3^3}$ is positive for any positive t, therefore D^* is not D-optimum.

Note however that for the eigenvalues we still have $\dfrac{27t^2 - 2}{3} > \dfrac{27t^2 - 6t - 1}{3}$, that is the smallest eigenvalue of D^* is larger than the smallest eigenvalue of D, as it should be.

<u>Example 2.3.2</u>: For $t = 1$

$$D = \begin{array}{cccccc} A & 1 & 1 & 2 & 2 & 3 \\ 1 & A & 2 & 2 & 3 & 3 \\ 1 & 2 & A & 3 & 3 & 1 \\ 2 & 2 & 3 & A & 1 & 1 \\ 2 & 3 & 3 & 1 & A & 2 \\ 3 & 3 & 1 & 1 & 2 & A \end{array} \qquad D^* = \begin{array}{cccccc} A & 1 & 1 & A & 2 & 3 \\ 1 & A & 2 & 2 & A & 3 \\ 1 & 2 & A & 3 & 3 & A \\ 2 & 2 & 3 & A & 1 & 1 \\ 2 & 3 & 3 & 1 & A & 2 \\ 3 & 3 & 1 & 1 & 2 & A \end{array}$$

$$\Delta^* = \left[\frac{25}{3}\right]^3 = \frac{15625}{25}$$

$$\Delta = \left[\frac{28}{3}\right]^2 \frac{20}{3} = \frac{15680}{27}$$

$$\Delta - \Delta^* = \frac{55}{27} > 0 .$$

<u>Final Remark</u>: Other sets of parameters satisfying the necessary conditions for GY-designs were obtained but they did not lead to suggestive combinatorial configurations. Further research is now in progress to construct other classes of GY-designs using different combinatorial structures.

CHAPTER III

SUM COMPOSITION OF LATIN SQUARES

3.1 Introduction and Definitions

The different methods of composition are among the most powerful techniques of construction of combinatorial systems. Those methods permit the construction of a new combinatorial system out of known ones.

However the methods known so far are of the product type, in the sense that the parameters of the new system are some sort of product of the parameters of the initial systems; for instance the existence of orthogonal arrays $(\lambda_i v_i^t, q_i, v_i, t)$, $i = 1,2,\ldots,r$, implies the existence of the orthogonal array $(\lambda v^t, q, v_i, t)$, where $\lambda = \lambda_1 \cdot \lambda_2 \cdots \lambda_r$, $v = v_1 \cdot v_2 \cdots v_r$ and $q = \min(q_1, q_2, \ldots, q_r)$.

In this chapter we will be dealing with a new sum type method of composition of Latin Squares due to Hedayat and Seiden (1969).

Definition 3.1.1: Two Latin Squares of order n are orthogonal if upon superimposition each of the n^2 pairs of distinct varieties occur exactly once.

A system of two orthogonal Latin Squares or order n will be referred to as a $O(n,2)$ set. If A and B are orthogonal Latin Squares we will write $A \perp B$.

Definition 3.1.2: t Latin Squares of order n are mutually orthogonal if any two of them are orthogonal.

A system of t mutually orthogonal Latin Squares of order n will be referred to as a $O(n,t)$ set.

Definition 3.1.3: A Latin Square L of order n is orthogonally mateless if for any other Latin Square L_1 of order n the pair (L,L_1) is not a $O(n,2)$ set.

Definition 3.1.4: A transversal of a Latin Square of order n is a collection of n cells whose entries exhaust the set of varieties and such that no two cells belong to the same row or to the same column.

Two transversals are parallel if they have no cell in common.

Definition 3.1.5: A common transversal for a $O(n,t)$ set is a collection of n cells which is a transversal for each of the t Latin Squares in the set.

Example 3.1.1:

$$L_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} (1) & 2 & 3 & \underline{4} \\ \underline{2} & 1 & 4 & (3) \\ 3 & (4) & \underline{1} & 2 \\ 4 & \underline{3} & (2) & 1 \end{bmatrix} \qquad L_3 = \begin{bmatrix} (1) & 2 & 3 & \underline{4} \\ \underline{3} & 4 & 1 & (2) \\ 4 & (3) & \underline{2} & 1 \\ 2 & \underline{1} & (4) & 3 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \qquad L_5 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

L_1 is the only Latin Square of order 2; it has no transversals at all.

L_2, L_3, L_4 form a $O(4,3)$ set; L_5 is orthogonally mateless, the underlined and paranthesized cells in L_2, L_3 form two common parallel transversals of the $O(4,2)$ set formed by L_1, L_2; the $O(4,3)$ set formed by L_1, L_2, L_3 has no common transversals.

3.2 The Method of Sum Composition

This method was first introduced by Hedayat and Seiden (1969).

Let L_1, L_2 be two Latin Squares of orders n_1 and n_2 on disjoint sets of varieties $\{a_1, a_2, \ldots, a_{n_1}\}$ and $\{b_1, b_2, \ldots, b_{n_2}\}$, $n_1 \geq n_2$, and let L_1 have at least n_2 parallel transversals.

Select arbitrarily n_2 parallel transversals from L_1 and name them $1, 2, \ldots, n_2$; in a $n_1 + n_2$ square fill the $n_1 \times n_1$ upper left corner with L_1 and the $n_2 \times n_2$ lower right corner with L_2. Fill the cells $(i, n_1 + k)$, $k = 1, 2, \ldots, n_2$, with that element of transversal k which appears in row i, $i = 1, 2, \ldots, n_1$; similarly fill the cells $(n_1 + k, j)$, $k = 1, 2, \ldots, n_2$, with that element of transversal k which appears in column j, $j = 1, 2, \ldots, n_1$. Finally substitute b_k for the n_1 elements of transversal k, $k = 1, 2, \ldots, n_2$.

The resulting $n_1 + n_2$ square matrix L is easily seen to be a Latin Square.

The procedure just described of filling the first n_1 entries of column (row) $n_1 + k$ is called horizontal (vertical) projection of transversal k on column (row) $n_1 + k$.

Remark: It is by no means required that the ordering of transversals be the same for both horizontal and vertical projections. Therefore, if N is the total number of parallel transversals of L_1 we can construct by this method

$$\binom{N}{n_2}(n_2!)^2$$

different Latin Squares of order $n_1 + n_2$.

Example 3.2.1:

$$L_1 = \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{matrix} \qquad L_2 = \begin{matrix} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{matrix}$$

In L_1 the cells (i,j) such that $i + j \equiv k(\text{mod } 7)$ form a transversal for each value of k, $k = 0,1,\ldots,6$. Let us use those corresponding to $k = 0,2,4,6$, in that order, for horizontal projection, in reverse order, $(6,4,2,0)$, for vertical projection and in alternate order $(0,4,2,6)$ for substitution. The result is the Latin Square L of order 11.

$$L = \begin{array}{ccccccccccc}
A & 1 & C & 3 & B & 5 & D & 0 & 2 & 4 & 6 \\
2 & C & 4 & B & 6 & D & A & 1 & 3 & 5 & 0 \\
C & 5 & B & 0 & D & A & 3 & 2 & 4 & 6 & 1 \\
6 & B & 1 & D & A & 4 & C & 3 & 5 & 0 & 2 \\
B & 2 & D & A & 5 & C & 0 & 4 & 6 & 1 & 3 \\
3 & D & A & 6 & C & 1 & B & 5 & 0 & 2 & 4 \\
D & A & 0 & C & 2 & B & 4 & 6 & 1 & 3 & 5 \\
5 & 4 & 3 & 2 & 1 & 0 & 6 & A & B & C & D \\
1 & 0 & 6 & 5 & 4 & 3 & 2 & B & C & D & A \\
4 & 3 & 2 & 1 & 0 & 6 & 5 & C & D & A & B \\
0 & 6 & 5 & 4 & 3 & 2 & 1 & D & A & B & C
\end{array}$$

3.3 Sum Composition of $O(n,2)$ Sets

Under certain conditions it is possible to use the method of sum composition to obtain $O(n,2)$ sets from known $O(n_1,2)$ and $O(n_2,2)$ sets, $n = n_1 + n_2$.

Let $\{A_1,A_2\}$ be a $O(n_1,2)$ set on the set of varieties $A = \{a_1,a_2,\ldots,a_{n_1}\}$ with at least $2n_2$ common parallel transversals, and $\{B_1,B_2\}$ a $O(n_2,2)$ set on the set of varieties $B = \{b_1,b_2,\ldots,b_{n_2}\}$, $A \cap B = \phi$.

Select $2n_2$ common parallel transversals from the first set and use half of them to compose A_1 and B_1 to obtain a Latin Square L_1 of order $n_1 + n_2 = n$; use the remainder n_2 transversals to compose A_2 and B_2 to obtain a Latin Square L_2 of order n.

It is obvious from the construction that upon superimposition of L_1 on L_2 the elements of $A \times B$ and $B \times A$ will appear along the $2n_2$ transversals in the $n_1 \times n_1$ upper left corner; the elements of $B \times B$ will appear in the $n_2 \times n_2$ lower right corner, since B_1 and B_2 are orthogonal. However some of the elements of $A \times A$ will be missing, but by properly choosing the $2n_2$ transversals and the order of projection we may achieve that the pairs (a_i, a_k) lost by substituting elements of B in transversals of A_1 and A_2 be recovered on projection.

Although we do not have a unified rule to achieve this we do have procedures which are applicable in several cases.

Example 3.3.1:

Let $n_1 = p^\alpha$ be a power of a prime number p, and number the rows and columns of a $n_1 \times n_1$ square matrix $0,1,2,\ldots,n_1-1$; for a fixed $x \in GF(n_1)$, $x \neq 0$, fill cell (i,j) of the matrix with $ix + j \in GF(n_1)$; the resulting square is a Latin Square $A(x)$. Furthermore the $n_1 - 1$ Latin Squares $A(x)$, $x \in GF(n_1)$, $x \neq 0$, constitute a $O(n_1, n_1-1)$ set; the cells (i,j) such that $i + j = k$, $k \in GF(n_1)$ constitute a set of n_1 common parallel transversals of the $O(n, n_1-1)$ set.

Now, let $GF(7)$ be represented as the residue classes modulo 7, and let $A_1 = A(3)$, $A_2 = A(4)$ and similarly, for $GF(3)$ let $B_1 = B(1)$, $B_2 = B(2)$. To compose A_1 and B_1 use the transversals given by $k = 0,5,4$ in that order for both projections and substitution and obtain L_1; to compose A_2 and B_2 use the transversals given by $k = 1,2,6$ and obtain L_2; (L_1, L_2) is a $O(10,2)$.

$$A_1 = \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{array} \qquad A_2 = \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \end{array}$$

$$B_1 = \begin{array}{ccc} A & B & C \\ B & C & A \\ C & A & B \end{array} \qquad B_2 = \begin{array}{ccc} A & B & C \\ C & A & B \\ B & C & A \end{array}$$

$$L_1 = \begin{array}{cccccccccc} A & 1 & 2 & 3 & C & B & 6 & 0 & 5 & 4 \\ 3 & 4 & 5 & C & B & A & 1 & 2 & 0 & 6 \\ 6 & 0 & C & B & 3 & A & 5 & 4 & 2 & 1 \\ 2 & C & B & 5 & A & 0 & 1 & 6 & 4 & 3 \\ C & B & 0 & A & 2 & 3 & 4 & 1 & 6 & 5 \\ B & 2 & A & 4 & 5 & 6 & C & 3 & 1 & 0 \\ 4 & A & 6 & 0 & 1 & C & B & 5 & 3 & 2 \\ 0 & 5 & 3 & 1 & 6 & 4 & 2 & A & B & C \\ 1 & 6 & 4 & 2 & 0 & 5 & 3 & B & C & A \\ 5 & 3 & 1 & 6 & 4 & 2 & 0 & C & A & B \end{array}$$

$$L_2 = \begin{array}{cccccccccc} 0 & A & B & 3 & 4 & 5 & C & 1 & 2 & 6 \\ A & B & 6 & 0 & 1 & C & 3 & 4 & 5 & 2 \\ B & 2 & 3 & 4 & C & 6 & A & 0 & 1 & 5 \\ 5 & 6 & 0 & C & 2 & A & B & 3 & 4 & 1 \\ 2 & 3 & C & 5 & A & B & 1 & 6 & 0 & 4 \\ 6 & C & 1 & A & B & 4 & 5 & 2 & 3 & 0 \\ C & 4 & A & B & 0 & 1 & 2 & 5 & 6 & 3 \\ 4 & 1 & 5 & 2 & 6 & 3 & 0 & A & B & C \\ 1 & 5 & 2 & 6 & 3 & 0 & 4 & C & A & B \\ 3 & 0 & 4 & 1 & 5 & 2 & 6 & B & C & A \end{array}$$

Hedayat and Seiden (1969) have proved the following results.

<u>Theorem 3.3.1</u>: Let $n_1 = p^\alpha \geq 7$, where p is any odd prime number, α a positive integer, $n_1 \neq 13$. Then there exists an $O(n,2)$ set which can be constructed by composition of two $O(n_1,2)$ and $O(n_2,2)$ sets for $n_2 = \frac{n_1 - 1}{2}$ and $n = n_1 + n_2$.

<u>Theorem 3.3.2</u>: Let $n_1 = 2^\alpha \geq 8$ for any positive integer α. Then there exists an $O(n,2)$ set which can be constructed by composition of two $O(n_1,2)$ and $O(n_2,2)$ sets for $n_2 = \frac{n_1}{2}$ and $n = n_1 + n_2$.

The same authors have also proved in 1970

<u>Theorem 3.3.3</u>: If a prime number p has one of the following forms:

I	$3m + 1$
II	$8m + 1$
III	$8m + 3$
IV	$24m + 11$
V	$60m + 23$
VI	$60m + 47$

then using the method of sum composition it is possible to construct a pair of orthogonal Latin Squares of order $p^\alpha + 3$. The method of construction depends on the form of p, but does not depend on its specific value.

<u>Theorem 3.3.4</u>: If p is a prime of the form $8m + 1$ or $8m + 3$, $m \neq 0$, then one can compose an $O(4,2)$ with an $O(p^\alpha,2)$ based on Galois field, to obtain an $O(p^\alpha + 4,2)$.

3.4 Construction of $O(n,2)$ Sets by the Method of Sum Composition

In what follows we assume the following: $n_1 = p^\alpha$, a power of a prime p; the $O(n_1,2)$ set is based in $GF(p^\alpha)$ and formed, with the notation introduced in Example 3.3.1, by $A_1 = A(x)$, $A_2 = A(y)$, $x,y \in GF(n_1)$, $x \neq y$, $\{x,y\} \cap \{0,1\} = \phi$. We will use common parallel transversals given by cells (i,j) such that $i + j = k$, $k \in GF(n_1)$ and named by k. We further call $S = \{s_1,s_2,\ldots,s_{n_2}\}$ and $T = \{t_1,t_2,\ldots,t_{n_2}\}$ the disjoint sets of n_2 transversals each used to obtain L_1 and L_2.

We have seen that the only difficulty of the method of sum composition is to make it sure that every element of $A \times A$ appears on superimposition of L_1 on L_2; the missing pairs are the $2n_2n_1$ pairs of the form

$$(ix + j,\ iy + j)\ ,\ i + j \in S \cup T$$

which correspond to the entries in the $2n_2$ transversals used in the composition.

If transversal s of $A(x)$ is projected horizontally on the same column as transversal t of $A(y)$, on superimposition we will obtain along that column the n_1 pairs

$$(ax + b,\ ay + c)\ ,\ a + b = s\ ,\ a + c = t\ .$$

If those pairs are to be some of the lost ones we must have:

$$\begin{array}{lll} ix + j = ax + b & a + b = s \in S & a + c = t \in T \\ iy + j = ay + c & i + j = k \in S \cup T & \end{array}$$

or
$$i(x-1) + k = a(x-1) + s$$
$$i(y-1) + k = a(y-1) + t \ .$$

Eliminating i we obtain

$$k(y-x) = s(y-1) - t(x-1)$$

or
$$k(y-x) = s(y-x) + (s-t)(x-1) \ .$$

Making $\frac{x-1}{y-x} = \mu$ we finally get

$$k = (1 + \mu)s - \mu t$$

that is, by projecting horizontally transversal s of $A(x)$ on the same column as transversal t of $A(y)$ we obtain on superimposition the n_1 pairs

$$(ix + j, iy + j) \qquad i + j = (1 + \mu)s - \mu t \ .$$

Similarly, if transversals s and t of $A(x), A(y)$ are projected veritcally on the same row, we will obtain along that row the n_1 pairs

$$(ax + b, cy + b) \qquad a + b = s \qquad c + b = t \ .$$

If those pairs are to be some of the lost ones we must have

$$ix + j = ax + b \qquad a + b = s \in S \qquad c + b = t \in T$$
$$iy + j = cx + b \qquad i + j = k \in S \cup T$$

or
$$i(x-1) + k = a(x-1) + s$$
$$i(y-1) + k = c(y-1) + t \ .$$

Eliminating i we obtain

$$k(y-x) = (x-1)(y-1)(a-c) + s(y-1) - t(x-1) .$$

Since $a-c = s-t$, we get

$$k(y-x) = s(y-x) + (s-t)(x-1)y$$

and finally

$$k = (1 + y\mu)s - y\mu t$$

that is, by projecting vertically transversal s of $A(x)$ on the same row as transversal t of $A(y)$ we obtain on superimposition the n_1 pairs

$$(ix + j,\ iy + j) \qquad i + j = (1 + y\mu)s - y\mu t .$$

From now on we will use the following functions on $S \times T$

$$K_h(s,t) = (1 + \mu)s - \mu t$$
$$K_v(s,t) = (1 + y\mu)s - y\mu t .$$

By properly choosing x,y and the pattern of pairing transversals from S and T we may be able to recover all the lost pairs and thus obtain a $O(n,2)$ set with $n = p^{\alpha} + n_2$.

Hedayat and Seiden assume in all their work, $xy = 1$.

<u>Theorem 3.4.1</u>: If p is a prime of the form $p = 4m + 1$, $m > 1$, then it is possible to compose $O(p^{\alpha},2)$ based on $GF(p^{\alpha})$ with $O(4,2)$ to obtain a $O(p^{\alpha} + 4,2)$.

<u>Proof</u>: Consider the pattern

$$s_{i+1} = K_h(s_i,t_i) \qquad i = 1,2,3 \qquad s_1 = K_h(s_4,t_4)$$
$$t_{i-1} = K_v(s_i,t_i) \qquad i = 2,3,4 \qquad t_4 = K_v(s_1,t_1)$$

that is

$$
\begin{aligned}
s_2 &= (1+\mu)s_1 - \mu t_1 & \qquad t_4 &= (1+y\mu)s_1 - y\mu t_1 \\
s_3 &= (1+\mu)s_2 - \mu t_2 & t_1 &= (1+y\mu)s_2 - y\mu t_2 \\
s_4 &= (1+\mu)s_3 - \mu t_3 & t_2 &= (1+y\mu)s_3 - y\mu t_3 \\
s_1 &= (1+\mu)s_4 - \mu t_4 & t_3 &= (1+y\mu)s_4 - y\mu t_4 .
\end{aligned}
$$

Solving this linear system in terms of s_1 and t_1, we obtain as a solution

$$
\begin{aligned}
s_2 &= (1+\mu)s_1 - \mu t_1 \\
s_3 &= (1+\mu)[1+\mu - \tfrac{1}{y}(1+y\mu)]s_1 - [\mu(1+\mu) - \tfrac{1}{y}[\mu(1+y\mu)+1]]t_1 \\
s_4 &= [\mu(1+y\mu)+1]\frac{1}{1+\mu}s_1 - \frac{y\mu^2}{1+\mu}t_1 \\
t_2 &= [(1+y\mu)(1+\mu)\frac{1}{y\mu}]s_1 - [\mu(1+y\mu)+1]\frac{1}{y\mu}t_1 \\
t_3 &= [(1+y\mu)\frac{1}{1+\mu}[\mu(1+y\mu)+1] - y\mu(1+y\mu)]s_1 - \\
&\qquad - [(1+y\mu)y\mu^2\frac{1}{1+\mu} - y^2\mu^2]t_1 \\
t_4 &= (1+y\mu)s_1 - y\mu t_1 .
\end{aligned}
$$

The compatibility conditions are

$$\frac{1}{1+\mu}[\mu(1+y\mu)+1] = (1+\mu)^2[1+\mu - \frac{1}{y}(1+y\mu)] - \mu(1+y\mu)[\frac{1}{1+\mu}(1+\mu+y\mu^2) - y\mu]$$

$$-\frac{y\mu^2}{1+\mu} = -(1+\mu)[\mu(1+\mu) - \frac{1}{y}[1+\mu+y\mu^2]] + y\mu^3[(1+y\mu)\frac{1}{1+\mu} - y]$$

$$\frac{(1+\mu)(1+y\mu)}{y\mu} = (1+y\mu)(1+\mu)[1+\mu - \frac{1}{y}(1+y\mu)] - y\mu(1+y\mu)[\frac{1}{1+\mu}(1+\mu+y\mu^2) - y\mu]$$

$$-\frac{(1+\mu+y\mu^2)}{y\mu} = -(1+y\mu)[\mu(1+\mu) - \frac{1}{y}(1+\mu+y\mu^2)] + y^2\mu^3[\frac{1+y\mu}{1+\mu} - y] .$$

These compatibility conditions reduce to

$$(1+\mu)^3 - (1+\mu)^2 y\mu + (1+\mu)y^2\mu^2 - y^3\mu^3 = 0 \ .$$

Dividing by $y^3\mu^3$ and making $\frac{1+\mu}{y\mu} = \lambda$ we obtain as compatibility condition

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0 \quad \text{or} \quad (\lambda-1)(\lambda^2+1) = 0 \ .$$

$\lambda = 1$ would give $s_3 = s_1$, therefore we must have $\lambda^2 + 1 = 0$, that is -1 has to be a quadratic residue in $GF(p^\alpha)$ which is possible only if p is of the form $p = 4m + 1$.

Calling $i^2 = -1$, the compatibility condition becomes

$$y(1 \pm i(1-x)) = 1 \ .$$

which is satisfied by the pair $x = 2$, $y = \frac{1 \pm i}{2}$. Using $s_1 = 0$, $t_1 = 1$ we obtain as solution of the system

$$s_2 = \frac{3 \pm i}{5} \qquad t_2 = \frac{-3 \pm 4i}{5}$$

$$s_3 = \frac{4 \mp 2i}{5} \qquad t_3 = \frac{-1 \mp 2i}{5}$$

$$s_4 = \frac{-1 \pm 3i}{5} \qquad t_4 = \frac{1 \pm 2i}{5}$$

We must investigate now under what condition those solutions are all different. One can easily see that

$s_1 = s_2$ if $10 \equiv 0 \pmod p$, that is $p = 2,5$

$s_1 = s_3$ if $20 \equiv 0 \pmod p$, that is $p = 2,5$

$s_1 = s_4$ if $10 \equiv 0 \pmod p$, that is $p = 2,5$

$s_1 = t_2$ if $25 \equiv 0 \pmod p$, that is $p = 5$

$s_1 = t_3$ if $5 \equiv 0 \pmod p$, that is $p = 5$

$s_1 = t_4$ if $5 \equiv 0 \pmod p$, that is $p = 5$

$s_2 = s_3$ if $10 \equiv 0 \pmod p$, that is $p = 2,5$

$s_2 = s_4$ if $20 \equiv 0 \pmod p$, that is $p = 2,5$

$s_2 = t_1$ if $5 \equiv 0 \pmod p$, that is $p = 5$

$s_2 = t_2$ if $45 \equiv 0 \pmod p$, that is $p = 3,5$

$s_2 = t_3$ if $25 \equiv 0 \pmod p$, that is $p = 5$

$s_2 = t_4$ if $5 \equiv 0 \pmod p$, that is $p = 5$

$s_3 = s_4$ if $50 \equiv 0 \pmod p$, that is $p = 2,5$

$s_3 = t_1$ if $5 \equiv 0 \pmod p$, that is $p = 5$

$s_3 = t_2$ if $85 \equiv 0 \pmod p$, that is $p = 5,17$

$s_3 = t_3$ if $5 \equiv 0 \pmod p$, that is $p = 5$

$s_3 = t_4$ if $25 \equiv 0 \pmod p$, that is $p = 5$

$s_4 = t_1$ if $45 \equiv 0 \pmod p$, that is $p = 3,5$

$s_4 = t_2$ if $5 \equiv 0 \pmod p$, that is $p = 5$

$s_4 = t_3$ if $25 \equiv 0 \pmod p$, that is $p = 5$

$s_4 = t_4$ if $5 \equiv 0 \pmod p$, that is $p = 5$

$t_1 = t_2$ if $80 \equiv 0 \pmod p$, that is $p = 2,5$

$t_1 = t_3$ if $40 \equiv 0 \pmod p$, that is $p = 2,5$

$t_1 = t_4$ if $20 \equiv 0 \pmod p$, that is $p = 2,5$

$t_2 = t_3$ if $40 \equiv 0 \pmod p$, that is $p = 2,5$

$t_2 = t_4$ if $20 \equiv 0 \pmod p$, that is $p = 2,5$

$t_3 = t_4$ if $20 \equiv 0 \pmod p$, that is $p = 2,5$.

Therefore the solutions are all different when $p = 4m+1$, $m > 1$, provided $p \neq 17$.

If $p = 17$, the pair $x = 5$, $y = 9$ satisfies the compatibility equation; using again $s_1 = 0$, $t_1 = 1$ we obtain the solutions

$s_2 = 16 \qquad t_2 = 12$

$s_3 = 3 \qquad t_3 = 2$

$s_4 = 4 \qquad t_4 = 8$

which are all different in GF(17).

The limitation $m > 1$ is due, of course, to the fact that the method requires at least 8 parallel transversals in order to compose a $O(4,2)$ set.

Note that $xy = 1$ is incompatible with $y(1 \pm i(1-x)) = 1$; indeed, the only common solution is $x = y = 1$.

Theorem 3.4.2: If $p \equiv 1,2,4 \pmod 7$, $p \geq 11$ it is possible to compose $O(p^\alpha,2)$ based on $GF(p^\alpha)$ with $O(4,2)$ to obtain a $O(p^\alpha + 4,2)$.

Proof: Consider the pattern

$$\begin{aligned} s_1 &= K_h(s_2,t_2) & t_1 &= K_v(s_2,t_2) \\ s_2 &= K_h(s_3,t_3) & t_2 &= K_v(s_3,t_3) \\ s_3 &= K_h(s_4,t_4) & t_3 &= K_v(s_1,t_4) \\ s_4 &= K_h(s_1,t_1) & t_4 &= K_v(s_4,t_1) . \end{aligned}$$

Solving this linear system in terms of s_2 and t_2, we obtain as solutions:

$$s_1 = (1 + \mu)s_2 - \mu t_2$$

$$\begin{aligned} s_3 = {} & [(1+\mu)[(1+\mu)^2-\mu(1+y\mu)]-\mu[(1+y\mu)[(1+\mu)^2-\mu(1+y\mu)] - y\mu(1+y\mu)]]s_2 \\ & - [(1+\mu)[\mu(1+\mu)-y\mu^2] - \mu[(1+y\mu)[\mu(1+\mu)-y\mu^2] - y^2\mu^2]]t_2 \end{aligned}$$

$$s_4 = [(1+\mu)^2 - \mu(1+y\mu)]s_2 - [\mu(1+\mu) - y\mu^2]t_2$$

$$t_1 = (1+y\mu)s_2 - y\mu t_2$$

$$t_3 = [(1+y\mu)(1+\mu) - y\mu[(1+y\mu)[(1+\mu)^2 - \mu(1+y\mu)] - y\mu(1+y\mu)]]s_2$$

$$- [(1+y\mu)\mu - y\mu[(1+y\mu)(\mu(1+\mu) - y\mu^2) - y^2\mu^2]]t_2$$

$$t_4 = [(1+y\mu)[(1+\mu)^2 - \mu(1+y\mu)] - y\mu(1+y\mu)]s_2$$

$$- [(1+y\mu)[\mu(1+\mu) - y\mu^2] - y^2\mu^2]t_2 \ .$$

The compatibility conditions are

$$1 = (1+\mu)[(1+\mu)[(1+\mu)^2 - \mu(1+y\mu)] - \mu[(1+y\mu)[(1+\mu)^2 - \mu(1+y\mu)] - y\mu(1+y\mu)]]$$

$$- \mu[(1+y\mu)(1+\mu) - y\mu[(1+y\mu)[(1+\mu)^2 - \mu(1+y\mu)] - y\mu(1+y\mu)]]$$

$$0 = -(1+\mu)[(1+\mu)[\mu(1+\mu) - y\mu^2] - \mu[(1+y\mu)[\mu(1+\mu) - y\mu^2] - y^2\mu^2]]$$

$$+ \mu[\mu(1+y\mu) - y\mu[(1+y\mu)(\mu(1+\mu) - y\mu^2) - y^2\mu^2]]$$

$$0 = (1+y\mu)[(1+\mu)[(1+\mu)^2 - \mu(1+y\mu)] - \mu[(1+y\mu)[(1+\mu)^2 - \mu(1+y\mu)] - y\mu(1+y\mu)]]$$

$$- y\mu[(1+y\mu)(1+\mu) - y\mu[(1+y\mu)[(1+\mu)^2 - \mu(1+y\mu)] - y\mu(1+y\mu)]]$$

$$1 = -(1+y\mu)[(1+\mu)[\mu(1+\mu) - y\mu^2] - \mu[(1+y\mu)[\mu(1+\mu) - y\mu^2] - y^2\mu^2]]$$

$$+ y\mu[\mu(1+y\mu) - y\mu[(1+y\mu)(\mu(1+\mu) - y\mu^2) - y^2\mu^2]]$$

which reduce to

$$1 - \mu(y-1) - \mu^2(y-1)^2(\mu^2 y + \mu y - 1) = 0 \ .$$

Making $x-1 = u$, $y-1 = v$ we get

$$v^4(u-1)(u^2+1) + v^3u(3u^2 - 3u + 4) - v^2u^2(u^2 - 3u + 6) -$$

$$- v\,u^3(u-4) - u^4 = 0 \ .$$

For $u = 1$ the equation becomes

$$4v^3 - 4v^2 + 3v - 1 = 0$$

which can be factorized

$$(v - \tfrac{1}{2})(2v^2 - v + 1) = 0 .$$

However $u = 1$, $v = \tfrac{1}{2}$ gives $t_2 = t_4$, so we have to look for the roots of $2v^2 - v + 1 = 0$.

To solve that equation it is necessary that -7 be a quadratic residue, and this is so if $p \equiv 1,2,4 \pmod 7$.

Calling $i^2 = -7$, $u = 1$ gives $x = 2$, $y = \frac{5 \pm i}{4}$ and using $s_2 = 1$, $t_2 = 0$ we obtain as solution of the system

$$s_1 = \frac{1 \mp i}{4} \qquad t_1 = \frac{1 \mp i}{2}$$

$$s_3 = \frac{3 \mp i}{2} \qquad t_3 = 2$$

$$s_4 = \frac{7 \mp 3i}{8} \qquad t_4 = \frac{9 \mp 5i}{8} .$$

It is easily seen that

$s_1 = s_2$ if $121 \equiv 0 \pmod p$, that is $p = 11$

$s_1 = s_3$ if $32 \equiv 0 \pmod p$, that is $p = 2$

$s_1 = s_4$ if $32 \equiv 0 \pmod p$, that is $p = 2$

$s_1 = t_1$ if $8 \equiv 0 \pmod p$, that is $p = 2$

$s_1 = t_2$ if $8 \equiv 0 \pmod p$, that is $p = 2$

$s_1 = t_3$ if $56 \equiv 0 \pmod p$, that is $p = 2,7$

$s_1 = t_4$ if $112 \equiv 0 \pmod p$, that is $p = 2,7$

$$
\begin{array}{llll}
s_2 = s_3 & \text{if} & 8 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_2 = s_4 & \text{if} & 64 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_2 = t_1 & \text{if} & 8 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_2 = t_4 & \text{if} & 176 \equiv 0 \pmod{p}, \text{ that is} & p = 2,11 \\
s_3 = s_4 & \text{if} & 32 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_3 = t_1 & \text{if} & 2 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_3 = t_2 & \text{if} & 16 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_3 = t_3 & \text{if} & 8 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_3 = t_4 & \text{if} & 16 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_4 = t_1 & \text{if} & 16 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
s_4 = t_2 & \text{if} & 112 \equiv 0 \pmod{p}, \text{ that is} & p = 2,7 \\
s_4 = t_3 & \text{if} & 144 \equiv 0 \pmod{p}, \text{ that is} & p = 2,3 \\
s_4 = t_4 & \text{if} & 32 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
t_1 = t_2 & \text{if} & 8 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
t_1 = t_3 & \text{if} & 16 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
t_1 = t_4 & \text{if} & 32 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
t_2 = t_3 & \text{if} & 2 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
t_2 = t_4 & \text{if} & 256 \equiv 0 \pmod{p}, \text{ that is} & p = 2 \\
t_3 = t_4 & \text{if} & 224 \equiv 0 \pmod{p}, \text{ that is} & p = 2,7 .
\end{array}
$$

Therefore the solutions are all different when $p \equiv 1,2,4 \pmod{7}$, provided $p \neq 11$.

For $p = 11$ we obtain, using $y = \frac{5 + i}{4}$

$$
\begin{array}{llll}
s_1 = 8 & s_2 = 1 & s_3 = 6 & s_4 = 7 \\
t_1 = 5 & t_2 = 0 & t_3 = 2 & t_4 = 4
\end{array}
$$

which are all different in $GF(11)$.

Theorem 3.4.3: If $n_2 \neq 6$ is even, then for any prime number $p \geq \frac{n_2}{2}$ it is always possible to compare $O(p^\alpha,2)$ based on $GF(p^\alpha)$ with $O(n_2,2)$ to obtain a $O(p^\alpha + n_2,2)$ set.

Proof: Consider the pattern

$$s_1 = K_h(s_2,t_2) \qquad t_1 = K_v(s_2,t_2)$$
$$s_2 = K_h(s_1,t_1) \qquad t_2 = K_v(s_1,t_1) \; .$$

Solving this system in terms of s_1, t_1, the compatibility conditions are

$$1 = (1+\mu)^2 - \mu(1+y\mu)$$
$$0 = -\mu(1+\mu) + y\mu^2$$
$$0 = (1+\mu)(1+y\mu) - y\mu(1+y\mu)$$
$$1 = -\mu(1+y\mu) + y^2\mu^2$$

which reduce to

$$y\mu = 1 + \mu \; .$$

Taking $t_1 = s_1 + 1$ we obtain

$$s_2 = s_1 - \mu \qquad t_2 = s_1 - y\mu = s_2 - 1$$

that is, t_2, s_2 are also consecutive numbers. By properly choosing y, which uniquely determines x, since the equation of compatability is of first degree in x, we may achieve that $t_2 = t_1 + 1$; the choice is $\mu = -3$ which provides $y = \frac{2}{3}$ and $x = \frac{1}{2}$. The sets S and T are therefore

$$S = \{s_1, s_1 + 3\}$$

$$T = \{s_1 + 1, s_1 + 2\} .$$

By starting with $s_1 = 0$ and repeating the above process $\frac{n_2}{2}$ times, we obtain the sets of transversals

$$S = \{0,3; 4,7;\ldots; 2n_2 - 4, 2n_2 - 1\}$$

$$T = \{1,2; 5,6;\ldots; 2n_2 - 3, 2n_2 - 2\} .$$

We could also have considered the pattern

$$s_1 = K_h(s_2,t_2) \qquad t_1 = K_v(s_1,t_2)$$

$$s_2 = K_h(s_1,t_1) \qquad t_2 = K_v(s_2,t_1) .$$

Taking s_1,t_1 as independent unknowns, the compatibility condition reduces to

$$y\mu(1 + \mu) = 1 .$$

Using again $t_1 = s_1 + 1$ we obtain

$$s_2 = s_1 - \mu \qquad t_2 = s_1 - (1 + \mu) = s_2 - 1$$

that is, t_2,s_2 are also consecutive numbers; $t_2 = t_1 + 1$ would imply as before $\mu = -3$, $y = \frac{1}{6}$, $x = -\frac{1}{4}$ and we will get

$$S = \{s_1, s_1 + 3\}$$

$$T = \{s_1 + 1, s_1 + 2\} .$$

Again by starting with $s_1 = 0$ and repeating the process $\frac{n_2}{2}$ times we obtain

$$S = \{0,3;\ 4,7;\ldots;\ 2n_2 - 4,\ 2n_2 - 1\}$$

$$T = \{1,2;\ 5,6;\ldots;\ 2n_2 - 3,\ 2n_2 - 2\}$$

however this time we have to reverse the order of the set T before projecting vertically.

Note that although all the computations have been carried out in $GF(p)$, that is mod p, the theorem can be extended to p^α since any $GF(p^\alpha)$ has a subfield isomorphic to $GF(p)$; this is also the reason to impose the limitation $p \geq \frac{n_2}{2}$ on p rather than on p^α.

Note that if $xy = 1$ the compatibility conditions are not satisfied.

Unlike in previous theorems, where for each value of x we could obtain at least two values of y satisfying the compatibility conditions, this method cannot be extended to the construction of $O(n,3)$ sets because the value of y uniquely determines x.

3.5 Composition of $O(p^\alpha,2)$ and $O(3,2)$ Sets

The smallest non-trivial n for which a $O(n,2)$ set exists is $n = 3$; there are 24 possible patterns to compose a $O(n_1,2)$ and a $O(3,2)$ set. We assume, without loss of generality, that the pairs (s,t) of transversals horizontally projected on the same column are (s_i,t_i), $i = 1,2,3$. The sets S and T are now $S = \{s_1,s_2,s_3\}$, $T = \{t_1,t_2,t_3\}$.

<u>Theorem 3.5.1</u>: If a pattern for composition of a $O(p^{\alpha},2)$ and a $O(3,2)$ set is such that horizontal projection recovers transversals from both sets S and T, then $xy = 1$.

<u>Proof</u>: For any pattern, of the six equations which determine the pattern, three will involve the function K_h and the other three equations will involve the function K_v. Adding the six equations we will always obtain, no matter what the pattern is,

$$\Sigma s_i + \Sigma t_i = (1 + \mu + 1 + y\mu)\Sigma s_i - (\mu + y\mu)\Sigma t_i$$

or $$(\Sigma s_i - \Sigma t_i)(1 + \mu + y\mu) = 0 \ .$$

If horizontal projection recovers transversals from both S and T adding the three equations involving K_h we will obtain in the l.h.s. the sum of either two s's and one t, or one s and two t's; in the r.h.s. we will obtain $\Sigma s_i - \mu(\Sigma t_i - \Sigma s_i)$. Therefore if $\Sigma t_i - \Sigma s_i = 0$ we will have $s_i = t_j$ for some i,j. We must then have $1 + \mu + y\mu = 0$; but $1 + \mu + y\mu = xy - 1$, thus the result.

This theorem applies to 12 of the 24 possible patterns to compose $O(p^{\alpha},2)$ and $O(3,2)$ sets; they have been fully investigated by Hedayat and Seiden (1970).

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