THE INTERACTION OF ELECTROMAGNETIC RADIATION WITH A BOUNDED PLASMA

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ABSTRACT

THE INTERACTION OF
ELECTROMAGNETIC RADIATION
WITH A
BOUNDED PLASMA

by Andrew Rostyslaw Melnyk

The interaction between electromagnetic waves and a bounded plasma is studied by extending the Fresnel equations of reflection and transmission to include plasma waves with irrotational electric fields. The properties of the plasma are incorporated in the dispersion relations for propagating waves. These dispersion relations are obtained from a frequency and complex wave vector dependent dielectric tensor $\underline{\underline{\varepsilon}}(\underline{k},\omega)$ calculated from a linearized Boltzmann transport equation. Results show structure in the absorptance, reflectance, and transmittance spectrum of thin plasma slabs ($d < 2\pi c/\omega_p$) due to resonance phenomena at frequencies where $kd = n\pi$; ($n = 1, 3, 5,\ldots$). These results suggest that the dispersion relations of metallic plasmas such as silver films may be optically measured.

THE INTERACTION OF ELECTROMAGNETIC RADIATION WITH A BOUNDED PLASMA

By

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I. INTRODUCTION

1. Fundamentals

Plasma physics came into existence in 1929 when Tonks and Langmuir[1] presented their now famous theory of plasma oscillations in an ionized gas to explain certain anomalies in arc discharges.[2] Idealizing the ionized gas as electrons imbedded in a uniform positive background, they found that a small displacement of a slab of these electrons from their equilibrium position produces, by Coulomb interactions, a restoring force which to first order is proportional to the displacement. Thus the electrons oscillate in simple harmonic motion with a characteristic frequency ω_p;

$$\omega_p^2 = \frac{4\pi Ne^2}{m} , \tag{I.1}$$

where N is the density, e the charge, and m the mass of the electrons.* Noting the similarity between these oscillations and the oscillations of a jelly plasma, Tonks

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* If the material in the positive background is polarizable, e.g. the lattice ions in a solid state plasma, the relation becomes $\omega_p = \frac{4\pi Ne^2}{\varepsilon_0 m}$, where ε_0 is the dielectric constant.

and Langmuir christened them "plasma oscillations" and named the nearly neutral part of the ionized gas a "plasma." Today a plasma is defined as: "the portion of a material body which is much larger than the shielding length which in turn is larger than the interparticle distance of the charged particles moving through the body." The shielding length λ_D is the distance in which the Coulomb field of a test charge will be screened out by the charged particles. For particles with a Maxwellian energy distribution,

$$\lambda_D^2 = \frac{\kappa T}{4\pi N e^2} = \frac{\kappa T/m}{\omega_p^2} \qquad \text{(I.2)}$$

and is called the Debye length;[3] and for particles with a Fermi-Dirac distribution,

$$\lambda_D^2 = \frac{V_F^2}{5\omega_p^2} \qquad \text{(I.3)}$$

and is called the Fermi-Thomas length. In Eqs. I.2 and I.3, κ is the Boltzmann constant, T is the temperature, and V_F is the Fermi velocity.

Solid State Plasmas. Until 1951, only gaseous plasmas, consisting of electrons and ions were known and studied. Besides laboratory plasmas, these included naturally occurring gaseous plasmas such as stellar atmospheres, interstellar gas clouds, the ionosphere, etc.. But the 1948 experiments of Ruthemann and Lang,[4] in which they observed that kev electrons upon passing through thin metallic foils lost energy in discrete steps characteristic of the metal, led to the discovery of the solid state plasma. By treating the metal as a plasma of electrons in a positive lattice, Bohm and Pines[5] explained the discrete energy losses as the result of exciting quantized plasma oscillations, called plasmons,[6] each quantum possessing energy equal to $\hbar\omega_p$.

While alike in their essential features, solid state and gaseous plasmas are dissimilar in one important respect: stability. Gas plasmas, because they are produced by violent means such as spark discharges, are far from thermal equilibrium, are not easily contained, and as a consequence of one or more of the many instabilities tend to break up. Thus the central problems in gaseous plasma research are containment and control of all the instabilities. The solid state plasma, however, is absolutely stable and contained by the neutralizing lattice. The problem is no longer how to produce and contain the plasma, but given the plasma, what to do with it; i.e., how to throw it out of equilibrium and produce instabilities.

<u>Acoustic plasma waves.</u> In addition to the high frequency electron-plasma oscillations, Tonks and Langmuir showed that a two component plasma, containing two types of mobile charge carriers or particles, has a low frequency

mode of oscillation. In such a two component plasma, the high frequency oscillation results from the two charge species oscillating out of phase with the frequency

$$\omega_p^2 = \omega_{p+}^2 + \omega_{p-}^2 \ , \tag{I.4}$$

where ω_{p-} and ω_{p+} are the plasma frequencies of each species, e.g., electrons and ions. If there are more than two species, e.g. different ions or electrons with different effective masses, Eq. I.4 is extended to include the plasma frequency of each species. Because in a gaseous plasma the ions are much heavier than the electrons; i.e. $\omega_{p-} >> \omega_{p+}$, their motion can be neglected and the plasma frequency is just the electron-plasma frequency, but in general the high frequency oscillation of a multi-component plasma is given by a generalization of I.4.

The lower frequency mode results from the two charge species oscillating in phase, and for the electron-ion plasma is

$$\omega^2 = \frac{\omega_{p+}^2 (k\lambda_{D-})^2}{1 + (k\lambda_{D-})^2} \ , \tag{I.5}$$

where k is the wave vector, $k = 2\pi/\lambda$, and λ_{D-} is the screening length for the electrons. Because the mode I.5 exists only for wavelengths larger than the screening length, i.e. for $(k\lambda_D)<1$, I.5 defines a band of frequencies up to the ion-plasma frequency ω_{p+}.

Plasma waves. The mathematically simple theory of Tonks and Langmuir led to a single frequency of oscillation ω_p, which will not propagate, but they pointed out this may not be true if the thermal motion of the electrons is included. The first attempt to calculate the dispersion spectrum for the high frequency mode was by J. J. and G. P. Thomson,[7] but Vlasov[8] is credited with obtaining the correct dispersion relation

$$\omega^2 = \omega_p^2\left[1 + 3\ (\lambda_D k)^2\right]. \qquad (I.6)$$

Equation I.6 shows that longitudinal plasma waves exist for a band of frequencies, the lower limit being ω_p and the upper limit is determined by the requirement that $k\lambda_D < 1$.

Because the theory of longitudinal plasma waves describes small disturbances from equilibrium whereas laboratory plasmas are unstable, direct experimental verification of I.6 for gaseous plasmas is very difficult and was accomplished only recently.[9] Indirect measurements of I.6 for solid state plasmas have been reported, for example Wantanabe's[10] studies of the variation of energy loss with angle of electrons scattered by a foil, but no known direct measurements of wavelengths in solid state plasmas have yet been made. Based on the theory developed here, such an experiment will be proposed later.

2. Interactions of Plasmas with Electromagnetic Waves

There are two basic methods for studying plasmas: by their interaction with charged particles and by their interaction with electromagnetic (em) fields, especially with em waves. Although the original discovery of plasma oscillations was aided by the discovery that arc discharges emitted radio waves at a characteristic frequency,[11] the problem of radiation by a plasma was neglected because the original theory showed the oscillations to be longitudinal and hence non-radiating. Interest in the radiation problem was revived by Shklovsky,[12] Martyn,[13] and Haeff[14] who suggested that some radio bursts from the sun originated in coronal plasma oscillations. Field[15] and others[16] developed the theory of plasma radiation, showed that longitudinal plasma waves may couple with transverse em waves at density or temperature gradients or in the presence of magnetic fields, and pointed out that at finite temperatures longitudinal plasma oscillations may propagate as dispersive waves. The excitation of em waves by solid state plasma waves was first considered by Ferrell.[17] He noted that the energy loss method of measuring plasma frequencies[18] was limited in accuracy and resolution* and suggested a more direct method. Under

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* Because the mean free path of the incident charged particles must exceed the thickness of the metal foil to avoid complicated multiple scattering processes, particle

suitable conditions the plasmons generated by the beam of charged particles would decay by emitting em waves, and the plasma frequency can be measured directly by detecting this radiation.[19] The predicted radiation has been observed by several investigators in silver foils[20] and also in aluminum and magnesium foils.[21]

The inverse of the radiation problem is the excitation of plasma waves by incident em waves and the associated reflection and transmission of em waves by a plasma. Compared to the incoherent scattering of radio waves by a plasma,[22] and more recently parametric excitation of plasma waves,[23] the linear coherent coupling of em waves to plasma waves has received little attention. On the basis of Ferrell's[17] physical picture for the radiation peak, Ferrell and Stern,[24] predicted that p-polarized (with the electric field in the plane of incidence) em radiation obliquely incident on a thin metallic film would be anomalously transmitted and reflected at the plasma frequency. McAlister and Stern[25] found that the transmission spectrum from a silver foil had a dip at the plasma frequency for p-polarized radiation but showed no such structure for s-polarized radiation. But they explained their results as due to a surface plasma wave

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energies of 10 kev or more are necessary. But the energy loss is only a few electron volts, thus the quantity of interest is the difference of two large and practically equal energies.

(surface plasmon) and not bulk plasma waves. Fedorchenko[26] studied coupling between em and bulk plasma waves by essentially inverting Field's calculation.[15] But his results, which were based on the hydrodynamic approximation of a plasma, were incorrect because of an inappropriate boundary condition.[27]

In what follows we present a general theory of linear, coherent coupling between em waves and waves in a plasma. The calculations are based on classical electromagnetic theory and quantum mechanical effects are essentially neglected. In the second chapter the usual Fresnel equations of reflection and transmission are shown to be inadequate for media with longitudinal polarization waves and more general expressions are derived for slabs of infinite and finite thickness. Because waves in conducting media such as plasmas are inhomogeneous, corresponding to complex wave vectors, we develop appropriate dispersion relations for inhomogeneous plasma waves in the third chapter. The fourth chapter presents numerical results for some typical solid state plasmas and the last chapter discusses the theory and results.

II. THEORY OF REFLECTION AND TRANSMISSION

The problem of reflection and transmission of em waves by a plasma is a special case of the propagation of em waves across discontinuities in the electric properties of matter. To be more specific, it is a special case of reflection and transmission by a conducting medium, a problem treated by most textbooks on electromagnetic theory.[28] But these treatments are generally incomplete since they usually neglect waves with irrotational electric fields and thereby exclude the effects of such waves as plasma waves or longitudinal optical phonons. It is for this reason that the theory used by McAlister and Stern[25] to explain their measurements considers only surface plasmons. Their expressions for transmission and reflection are the usual Fresnel equations which are derived for divergence free fields only, thus excluding plasma waves or bulk plasmons. But in their theory, the non-propagating plasma oscillations produce surface charge densities oscillating at the plasma frequency. Because normal electric fields of em waves couple to such surface charge densities their theory predicts anomalous behavior in the p-polarized transmission spectrum due to these surface plasmons.

The exclusion of irrotational fields in the usual derivation of Fresnel equations for conducting media has been obscured by an ambiguous description of the electric field in the medium. Thus for obliquely incident, p-polarized waves the electric field in the conducting medium is often described as no longer purely transverse, but possessing longitudinal components. At first glance a longitudinal electric field suggests an irrotational field due to net charges. The difficulty is that for inhomogeneous waves the terms "transverse" and "longitudinal" are not equivalent to "divergence free" and "irrotational" respectively, and it is well known that for oblique incidence the waves in a lossy medium are in fact inhomogeneous.

To avoid ambiguities, the following definitions will be used: An electric field (or the wave associated with it) will be called <u>divergence free</u> or <u>em</u> if its divergence vanishes; i.e. if $\nabla \cdot \underline{E} = 0$, and it will be called <u>irrotational</u> if its curl vanishes; i.e. if $\nabla \times \underline{E} = 0$. The terms <u>longitudinal</u> and <u>transverse</u> will be applied respectively to the components of the fields parallel and normal to the direction of phase propagation. Finally, a wave will be called <u>inhomogeneous</u> if its surfaces of constant phase and constant amplitude do not coincide. If a plane wave is represented by*

$$\underline{E}(\underline{r},t) = \mathrm{Re}[\underline{\tilde{E}}(\underline{r},t)]$$
$$= \mathrm{Re}[\underline{\tilde{E}}_0 \exp(i\underline{\tilde{k}}\cdot\underline{r} - i\omega t)] \qquad (II.1)$$
$$= \mathrm{Re}[\underline{\tilde{E}}_0 \exp(i\underline{k}_1\cdot\underline{r} - i\omega t)]\exp(-\underline{k}_2\cdot\underline{r}),$$

the surface of constant amplitude, a plane in this case, is $\underline{k}_2\cdot\underline{r}$ = constant; while the plane of constant phase at a given time is $\underline{k}_1\cdot\underline{r}$ = constant. Since only harmonic plane waves will be considered here, a wave is inhomogeneous whenever the real and imaginary parts of the wave vector $\underline{\tilde{k}}$ have different directions. Thus inhomogeneous waves can exist in any medium.

1. Equations of Transmission and Reflection for a Medium Bounded by a Plane

Before the equations of reflection and transmission are derived it will be instructive to rederive Snell's law, consider its consequences for inhomogeneous waves, and discuss the boundary conditions.

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* Where it is desirable to distinguish between real and complex quantities, a complex quantity will be indicated by a tilde and its real and imaginary parts by the subscripts 1 and 2; e.g. $\tilde{a} = a_1 + ia_2$

Snell's Law. If $\underline{n}$ is a unit normal to the plane interface separating the two media, let $\underline{n}\cdot\underline{r} = 0$ define the surface of the interface. The existence of boundary conditions on the fields at any point on $\underline{n}\cdot\underline{r} = 0$ at any time, requires that the space and time variation of all fields be the same on $\underline{n}\cdot\underline{r} = 0$. Consequently, the phase factors in II.1 for all the waves must be equal at $\underline{n}\cdot\underline{r} = 0$ and independent of the nature of the boundary conditions. Since the time factors are trivially equal we require,

$$(\underline{\tilde{k}}_0\cdot r)_{\underline{n}\cdot\underline{r}=0} = (\underline{\tilde{k}}_j\cdot\underline{r})_{\underline{n}\cdot\underline{r}=0} , \tag{II.2}$$

where $\underline{k}_0$ is the wave vector of the incoming wave and $\underline{k}_j$ is the wave vector of any of the possible reflected or refracted waves. But

$$\underline{r} = (\underline{n}\cdot\underline{r})\underline{n} - \underline{n} \times (\underline{n} \times \underline{r}), \tag{II.3}$$

so II.2 becomes

$$\underline{\tilde{k}}_0\cdot\underline{n} \times (\underline{n} \times \underline{r}) = \underline{\tilde{k}}_j\cdot\underline{n} \times (\underline{n} \times \underline{r}) \tag{II.4}$$

which by means of a vector identity can be written as

$$(\underline{\tilde{k}}_0 \times \underline{n} - \underline{\tilde{k}}_j \times \underline{n})\cdot(\underline{n} \times \underline{r}) = 0. \tag{II.5}$$

Thus Snell's law, as expressed by II.5, states that $\underline{n}$ and separately the real and imaginary parts of the wave vectors $\underline{\tilde{k}}$ are coplanar; and the components of the wave vectors parallel to the interface are equal.

For example, consider a wave in vacuum incident with an angle θ_0 on a semi-infinite medium in which the wave-vector is complex, as in Fig. 1. Snell's law requires

$$\begin{aligned} k_0 \sin \theta_0 &= k_1 \sin \theta \\ 0 &= k_2 \sin \phi \end{aligned} \tag{II.6}$$

II.6 demonstrate how an obliquely incident homogeneous wave can produce inhomogeneous waves, and furthermore II.6, quite graphically, shows that the planes of constant amplitude are parallel to the interface in a lossy medium such as a metal.[28]

Boundary Conditions. On the microscopic scale real systems do not have abrupt boundaries; their electrical properties change smoothly over some short but finite distance. With all charges, both free and bound, taken explicitly into account we need only introduce the total electric vector $\underline{E}$; the concept of a displacement vector then has no role. For example, at a plasma-dielectric interface the plasma extends partially into the dielectric and vice versa, creating a transition region in which the plasma changes gradually into the dielectric. If we construct

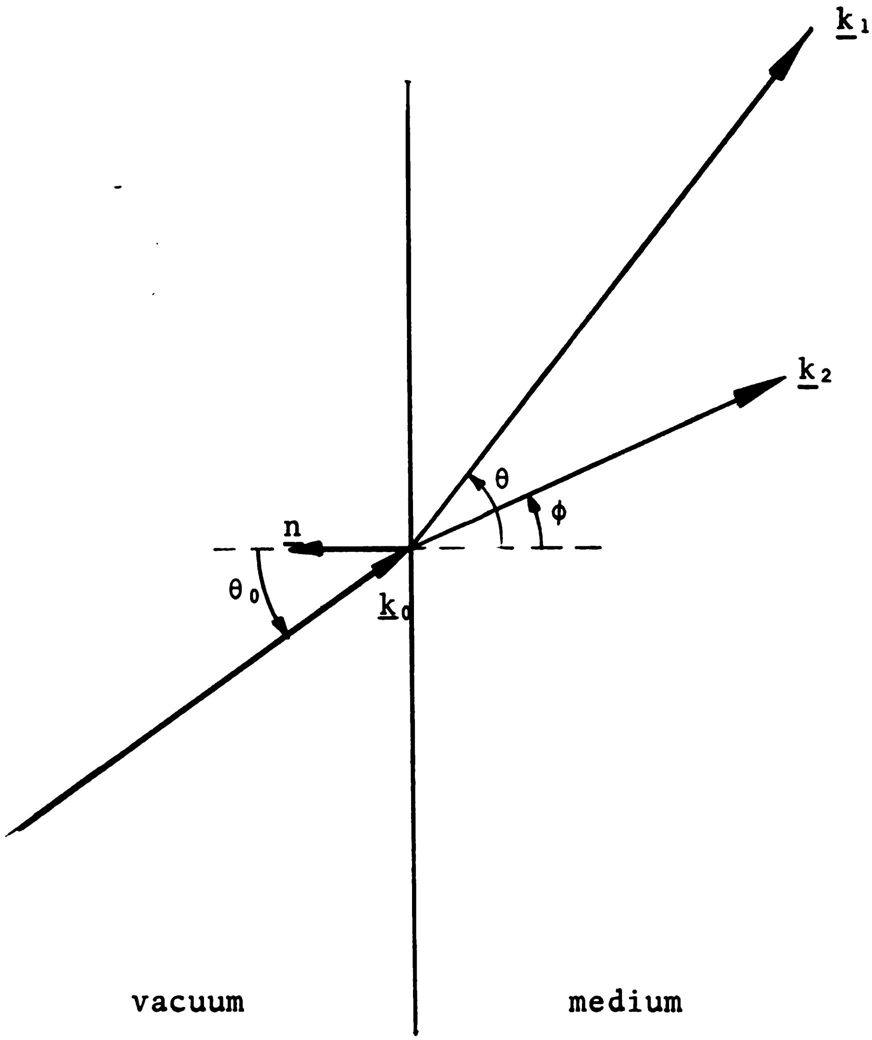

Fig. 1. Wave $\underline{k}_0$ incident on some lossy medium excites a wave $\underline{\tilde{k}} = \underline{k}_1 + i\underline{k}_2$.

an imaginary surface lying in this narrow transition region then all components of the total E and H are continuous across this surface. This follows from the standard arguments which apply Maxwell's equations to pillboxes and infinitesimal circuits passing through the imaginary surface, provided only volume distributions of charge and current density are present. For the case of a transition region over which the properties of the system change smoothly but rapidly from pure plasma to pure dielectric only volume distributions of charge and current density can exist. Thus we shall adopt the boundary condition that all components of E and H are continuous, keeping in mind that we *explicitly* regard whatever charges are bound to the dielectric as able to carry alternating currents in the same way as the free electrons of the plasma with which they communicate through the electromagnetic field. The continuity of the field components and Snell's law are thus fundamental to the solution of the problem. We complete our idealization of the system by postulating that the bulk properties of the plasma and dielectric as they express themselves in the respective dispersion relations may be extended up to the imaginary boundary surface in the transition region. Expressed in brief terms, our boundary condition essentially postulates that there is no surface charge density and that all charge distributions are volume distributions which can alter the normal component of the electric field only over finite distances.

Fresnel Equations. To facilitate comparison between the Fresnel equations and the new equations incorporating irrotational waves, we will derive the former first. Consider a plane, linearly polarized em wave incident on a semi-infinite medium at an angle θ_0, as in Fig. 2. Let $\underline{s}$, $\underline{n}$, and $\underline{p}$ be a triplet of orthogonal unit vectors: $\underline{n}$ being normal to the surface separating the two media, and $\underline{p}$ in the plane of incidence. We assume medium 0 to be vacuum or at worst a dispersionless dielectric while medium 1 is quite general. If $\underline{k}_0$, $\underline{k}_r$, and $\underline{k}_t$ are the wave vectors of the incident, reflected, and refracted em waves, Snell's Law requires

$$\begin{aligned} \underline{k}_0 \cdot \underline{p} &= \underline{k}_r \cdot \underline{p} = \underline{k}_t \cdot \underline{p} \\ \underline{k}_0 \cdot \underline{n} &= -\underline{k}_r \cdot \underline{n} \end{aligned} \qquad \text{(II.7)}$$

By the superposition principle, an arbitrary direction of polarization can be resolved into two cases: one with the E-field in the plane of incidence (p-polarized) and the other with the E-field normal to the plane of incidence (s-polarized). For the p-polarized case let the incident, reflected, and transmitted waves be

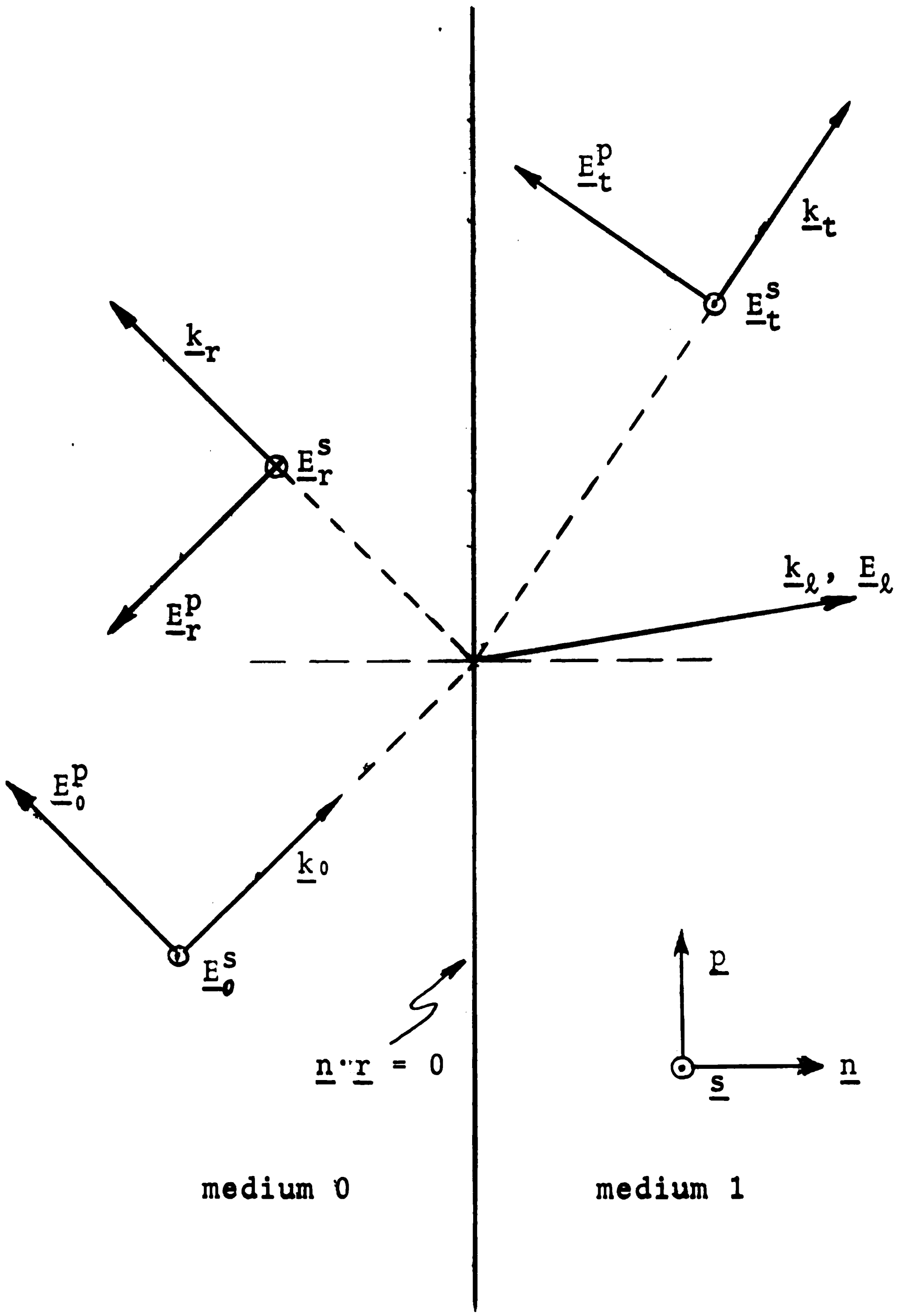

Fig. 2. Reflection and refraction of s and p polarized waves at a boundary.

$$\underline{E}_0(\underline{r},t) = (\underline{s} \times \frac{c\underline{k}_0}{\omega})E_0\exp(i\underline{k}_0\cdot\underline{r} - i\omega t)$$

$$\underline{E}_r(\underline{r},t) = (\underline{s} \times \frac{c\underline{k}_r}{\omega})R_pE_0\exp(i\underline{k}_r\cdot r - i\omega t) \qquad \text{(II.8)}$$

$$\underline{E}_t(\underline{r},t) = (\underline{s} \times \frac{c\underline{k}_t}{\omega})T_pE_0\exp(i\underline{k}_t\cdot\underline{r} - i\omega t)$$

Note that the fields in II.8 satisfy $\underline{k}\cdot\underline{E} = 0$ and therefore are divergence free. For our harmonic waves the magnetic fields are related to the electric by

$$\underline{H} = (c/\omega)\underline{k} \times \underline{E} \qquad \text{(II.9)}$$

so

$$\underline{H}_0(\underline{r},t) = \underline{s}\varepsilon_0E\ \exp(i\underline{k}_0\cdot\underline{r} - i\omega t)$$

$$\underline{H}_r(\underline{r},t) = \underline{s}\varepsilon_0R_pE_0\exp(i\underline{k}_r\cdot\underline{r} - i\omega t) \qquad \text{(II.10)}$$

$$\underline{H}_t(\underline{r},t) = \underline{s}\varepsilon_tT_pE_0\exp(i\underline{k}_t\cdot\underline{r} - i\omega t)$$

where

$$\varepsilon_\alpha \equiv \left(\frac{c}{\omega}\right)^2\underline{k}_\alpha\cdot\underline{k}_\alpha \qquad \text{(II.11)}$$

Applying the condition of continuity of tangential E,

$$\underline{n} \times (\underline{E}_0 + \underline{E}_r)_{\underline{n}\cdot\underline{r}=0} = \underline{n} \times (\underline{E}_t)_{\underline{n}\cdot\underline{r}=0} \qquad \text{(II.12)}$$

to II.8, we get after some algebra,

$$\left(\frac{ck_0}{\omega}\right)_n \left(1-R_p\right) = \left(\frac{ck_t}{\omega}\right)_n T_p \tag{II.13}$$

where $\left(\frac{ck}{\omega}\right)_n$ is the normal component of the wave vector k in units of the vacuum wave vector ω/c.* Similarly the continuity of the magnetic fields II.10 yields

$$\varepsilon_0 (1+R_p) = \varepsilon_t T_p. \tag{II.14}$$

Combining II.13 and II.14 we get the Fresnel equations of reflection and transmission for p-polarized waves

$$R_p = \frac{\alpha-\beta}{\alpha+\beta}$$
$$T_p = \frac{2\varepsilon_0\left(\frac{ck_0}{\omega}\right)_n}{\alpha+\beta}, \tag{II.15}$$

where

$$\alpha \equiv \varepsilon_t\left(\frac{ck_0}{\omega}\right)_n$$
$$\beta \equiv \varepsilon_0\left(\frac{ck_t}{\omega}\right)_n \tag{II.16}$$

---

* In some derivations $(ck_\alpha/\omega)_n$ is written as $\sqrt{\varepsilon_\alpha}\cos\theta_\alpha$, with $(ck_\alpha/\omega)_p = \sqrt{\varepsilon_\alpha}\sin\theta_\alpha$, but for complex wave vectors, i.e. for inhomogeneous waves, the angle of refraction becomes complex and loses its meaning, so we avoid this notation.

The expression for T_p in II.15 differs slightly from the usual Fresnel equation because, while the usual definition for T is $T = E_t/E_0$, we have defined T as $[(c/\omega)\underline{k}_t x \underline{E}_t]/[(c/\omega)\underline{k}_0 x \underline{E}_0]$. That particular form was chosen because it explicitly shows the divergence free property of the waves in the derivation. Our expression for T_p may be changed to the usual Fresnel expression by multiplying with the factor $\sqrt{\varepsilon_t/\varepsilon_0}$.

For the s-polarized case let

$$
\begin{aligned}
\underline{E}_0 &= \underline{s}E_0 \exp(i\underline{k}_0 \cdot r - i\omega t) \\
\underline{E}_r &= \underline{s}R_s E_0 \exp(i\underline{k}_r \cdot \underline{r} - i\omega t) \\
\underline{E}_t &= \underline{s}T_s E_0 \exp(i\underline{k}_t \cdot \underline{r} - i\omega t)
\end{aligned}
\qquad \text{(II.17)}
$$

Upon applying the condition of continuity of tangential E and H to II.17 and the corresponding magnetic fields, we find the Fresnel equations for the s-polarized case,

$$
\begin{aligned}
R_s &= \frac{\left(\frac{c\underline{k}_0}{\omega}\right)_n - \left(\frac{c\underline{k}_t}{\omega}\right)_n}{\left(\frac{c\underline{k}_0}{\omega}\right)_n + \left(\frac{c\underline{k}_t}{\omega}\right)_n} \\
T_s &= \frac{2\left(\frac{c\underline{k}_0}{\omega}\right)_n}{\left(\frac{c\underline{k}_0}{\omega}\right)_n + \left(\frac{c\underline{k}_t}{\omega}\right)_n}
\end{aligned}
\qquad \text{(II.18)}
$$

The above derivations explicitly show that the Fresnel equations consider only em waves with divergence free fields and cannot be applied to systems with so-called "longitudinal waves" or waves with irrotational electric fields.

General Equations. To eliminate the restriction placed on the Fresnel equations let us rederive the coefficients assuming the most general field in the medium has irrotational as well as divergence free components. Let us further assume that the irrotational and divergence free (em) waves are non-interacting in the bulk; hence, labeling them with subscripts ℓ and t, they satisfy

$$\underline{k}_\ell \times \underline{E}_\ell = 0$$
$$\underline{k}_t \cdot \underline{E}_t = 0 \qquad \text{(II.19)}$$

This assumption is satisfied by any isotropic homogeneous medium and in the next chapter we will show that a collisional but non-drifted plasma satisfies the non-interacting condition even for inhomogeneous waves.

For the p-polarized case, in addition to the em fields II.8 and II.10 let the irrotational electric field be

$$\underline{E}_\ell(\underline{r},t) = (\frac{c\underline{k}}{\omega}\ell)L_p E_0 \exp(i\underline{k}_\ell \cdot \underline{r} - i\omega t), \qquad \text{(II.20)}$$

which has no associated magnetic field. With the additional field II.20 the continuity condition on tangential E now yields

$$\left(\frac{ck_0}{\omega}\right)_n\left[1 - R_p\right] = \left(\frac{ck_t}{\omega}\right)_n T_p + \left(\frac{ck_\ell}{\omega}\right) L_p \qquad \text{(II.21)}$$

where $(ck_l/\omega)_p$ is the p component (see Fig. 2) of $(c/\omega)\underline{k}_\ell$, and by Snell's Law is equal to $(ck_0/\omega)_p$, hence the subscript will be dropped from now on.

The condition that normal E is continuous results in

$$\left(\frac{ck}{\omega}\right)_p\left[1 + R_p\right] = \left(\frac{ck}{\omega}\right)_p T_p - \left(\frac{ck_\ell}{\omega}\right)_n L_p \qquad \text{(II.22)}$$

Solving II.21, II.22, and II.14 we obtain the general equations of reflection and transmission

$$\begin{aligned} R_p &= \frac{\alpha - \beta - \gamma}{\alpha + \beta + \gamma} \\ T_p &= \frac{2\varepsilon_0\left(\frac{ck_0}{\omega}\right)_n}{\alpha + \beta + \gamma} \\ L_p &= \frac{2\left[\left(\frac{ck_0}{\omega}\right)_n \Big/ \left(\frac{ck}{\omega}\right)_p\right]\gamma}{\alpha + \beta + \gamma} \end{aligned} \qquad \text{(II.23)}$$

where

$$\gamma = \left(\frac{ck}{\omega}\right)_p^2\left[\varepsilon_0 - \varepsilon_t\right] \Big/ \left(\frac{ck_\ell}{\omega}\right)_n \qquad \text{(II.24)}$$

Since for the s-polarized case E is always transverse to k, irrotational waves cannot be excited, and the equations of reflection and transmission are the same as II.18.

Equations similar to II.23 have been obtained by A. M. Fedorchenko[26,27] in calculating the conversion of transverse electromagnetic waves into longitudinal waves at a dielectric-plasma plane interface. In Fedorchenko's first paper[26] he pointed out that the usual boundary conditions specifying continuity of tangential E and H are insufficient to solve the problem, and he added the condition that the normal component of all plasma charges' velocity vanishes at the boundary. His results were in general incorrect, however, because this last boundary condition applies only to a plasma-vacuum interface, since the existence of a dielectric presupposes polarization charges which can provide a non-vanishing value for the normal component of current density on the plasma-dielectric interface. But at a plasma-vacuum boundary, i.e. $\varepsilon_0 = 1$, the normal component of the current density or charge motion vanishes and our results applied to homogeneous waves coincide with Fedorchenko's 1962 results.[26] Although he corrected his choice of boundary conditions in the 1967 paper[27] to effectively include the effect of polarization currents in the dielectric, his published results still differ from ours. We ascribe this to a misprint since he claims in the second paper that the results of both papers coincide for a plasma-vacuum boundary, but in fact they do not.

2. Properties of the General Equations

Since the Fresnel equations and the general equations are identical for s-polarized waves, we will be concerned only with the p-polarized case. Comparing the new equations, II.23, with the Fresnel equations, II.15, we see they become identical whenever γ vanishes. From II.24 we find γ is proportional to three factors: the sine squared of the angle of incidence, $(ck/\omega)_p^2 = \varepsilon_0 \sin^2\theta_0$; the "transverse"* conductivity proportional to $(\varepsilon_0 - \varepsilon_t)$; and the inverse of the normal component of the irrotational wave vector in units of the vacuum wave vector of an em wave. The first factor indicates the effect of the irrotational wave increases with the angle of incidence, which we would expect since the normal component of the incident electric field increases. The last factor is the important one. If it is real, i.e. the irrotational waves are undamped, it is approximately equal to the ratio of the phase velocity of the wave to the speed of light, which for plasma waves is usually very small. For damped irrotational waves this last factor becomes small and imaginary,

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* The difference between transverse and longitudinal conductivities will become apparent in the next chapter.

so that in media in which the irrotational waves are highly damped we recover the Fresnel equations.

Energy Conservation. Because physical observations are usually made on the reflected or transmitted intensities or energies rather than amplitudes we next relate the coefficients II.15 to the energies of the waves. For an electromagnetic system the mean energy flow is given by the real part of the complex Poynting vector $\underline{S}$,

$$\underline{S} = \frac{c}{4\pi}\,\frac{1}{2}\,\mathrm{Re}\left[\underline{E} \times \underline{H}^*\right] \qquad \text{(II.25)}$$

where A* indicates the complex conjugate of A. According to the principle of energy conservation the normal component of energy flow across the interface, given by $\underline{n}\cdot\underline{S}$, must be continuous. If we construct the Poynting vectors for the em waves II.9,

$$\underline{S}_0 = \frac{c}{8\pi}\,\varepsilon_0 E_0^2\left(\frac{c\underline{k}_0}{\omega}\right)$$

$$\underline{S}_r = \frac{c}{8\pi}\,\varepsilon_0 E_0^2\,|R_p|^2\left(\frac{c\underline{k}_r}{\omega}\right) \qquad \text{(II.26)}$$

$$\underline{S}_t = \frac{c}{8\pi}\,E_0^2\,|T_p|^2\,\mathrm{Re}\left[\varepsilon_t^*\left(\frac{c\underline{k}_t}{\omega}\right)\right]\,\exp(-2\underline{k}_{t_2}\cdot\underline{r})$$

it is a simple matter to show that the Fresnel equations satisfy the energy principle. But, if we apply the condition

$$\underline{n}\cdot(\underline{S}_0 + \underline{S}_r)_{\underline{n}\cdot\underline{r}=0} = \underline{n}\cdot(\underline{S}_t + \underline{S}_\ell)_{\underline{n}\cdot\underline{r}=0} \qquad \text{(II.27)}$$

to the system with irrotational waves by substituting II.15 into II.26 we find

$$(\underline{n}\cdot\underline{S}_\ell)_{\underline{n}\cdot\underline{r}=0} = \frac{c}{8\pi}\varepsilon_0\left(\frac{ck_0}{\omega}\right)_n E_0^2\left[\frac{4\mathrm{Re}(\gamma_0\alpha_0^*)}{|\alpha_0+\beta_0+\gamma_0|^2}\right], \qquad \text{(II.28)}$$

which we may identify as the energy flux flowing into the irrotational wave (the bracketed expression being the fraction of the incident energy going into irrotational waves). The origin of II.28 may be seen more clearly if we notice that the total electric field in the medium is the sum of E_t and E_ℓ, consequently the Poynting vector consists of two terms: the first is just $\underline{S}_t$ and the second is

$$\frac{c}{8\pi}\,\mathrm{Re}(\underline{E}_\ell \times \underline{H}_t^*) \qquad \text{(II.29)}$$

Designating II.29 as $\underline{S}_\ell$ and substituting the expressions for $\underline{E}_\ell$ and $\underline{E}_t$ (II.20 and II.10) we find

$$(\underline{n}\cdot\underline{S}_\ell)_{\underline{n}\cdot\underline{r}=0} = \frac{c}{8\pi}\,E_0^2\left(\frac{ck}{\omega}\right)_p \mathrm{Re}\left[\varepsilon_t^* L_p T_p^*\right], \qquad \text{(II.30)}$$

which reduces to II.28 with II.15. Because irrotational waves transport energy mechanically not electromagnetically II.29 should not be identified as the Poynting vector of the irrotational wave.

3. Equations of Reflection and Transmission for a Slab

Some of the most interesting and useful phenomena in optics are produced by the interference of em waves in thin dielectric slabs. Because we expect that irrotational waves may exhibit similar interference properties we next calculate the reflection and transmission equations for a slab of material capable of supporting irrotational waves.

Suppose our conducting medium is bounded by two planes; one being $\underline{n}\cdot\underline{r} = 0$ and the other being $\underline{n}\cdot\underline{r} = d$, as in Fig. 3. To the left of this slab of thickness d is a dispersionless dielectric, ε_0, and to the right is another dispersionless dielectric, ε_2. Since no irrotational waves can be excited for s-polarized incidence we will consider only the p-polarized case. The incident and reflected waves in ε_0 are

$$
\begin{aligned}
\underline{E}_0 &= (s \times \frac{c\underline{k}_0}{\omega}) \exp(i\underline{k}_0\cdot\underline{r} - i\omega t) \\
\underline{H}_0 &= \underline{s}\varepsilon_0 \exp(i\underline{k}_0\cdot\underline{r} - i\omega t) \\
\underline{E}_0' &= (\underline{s} \times \frac{c\underline{k}_0'}{\omega})R \exp(i\underline{k}_0'\cdot\underline{r} - i\omega t) \\
\underline{H}_0' &= \underline{s}\varepsilon_0 R \exp(i\underline{k}_0'\cdot\underline{r} - i\omega t)\,,
\end{aligned}
\qquad \text{(II.31)}
$$

where the primed quantities refer to the reflected waves. The conducting medium contains four waves, irrotational and em waves propagating to the right and to the left,

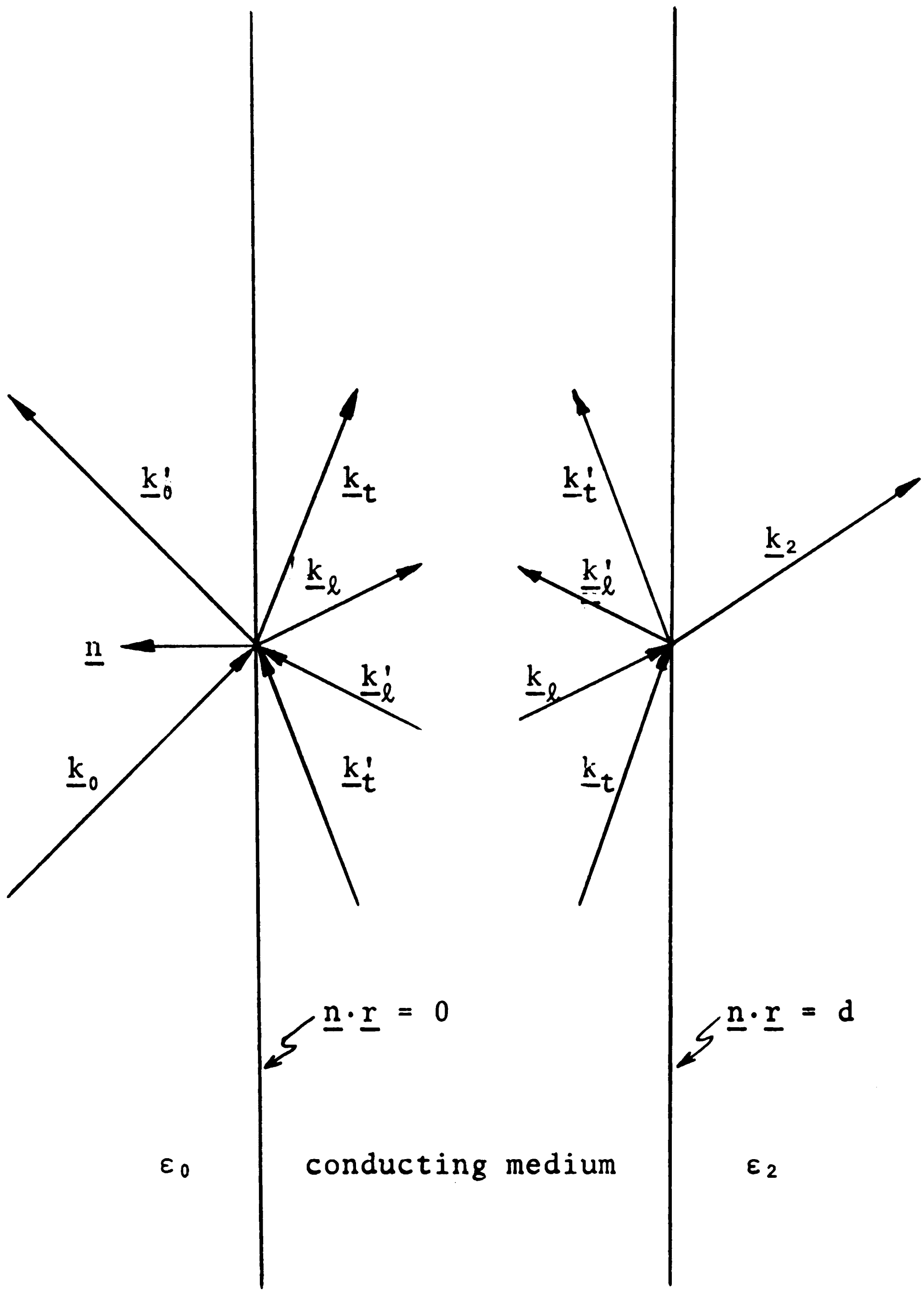

Fig. 3. Reflection and transmission by a conducting slab of thickness d.

$$\underline{E}_t = (\underline{s} \times \frac{c\underline{k}_t}{\omega}) T \exp(i\underline{k}_t \cdot \underline{r} - i\omega t)$$

$$\underline{H}_t = \underline{s}\ \varepsilon_t T \exp(i\underline{k}_t \cdot \underline{r} - i\omega t)$$

$$\underline{E}_t' = (\underline{s} \times \frac{c\underline{k}_t'}{\omega}) T' \exp(i\underline{k}_t' \cdot \underline{r} - i\omega t)$$

$$\underline{H}_t' = \underline{s}\ \varepsilon_t T' \exp(i\underline{k}_t' \cdot \underline{r} - i\omega t) \qquad \text{(II.32)}$$

$$\underline{E}_\ell = \frac{c\underline{k}_\ell}{\omega} L \exp(i\underline{k}_\ell \cdot \underline{r} - i\omega t)$$

$$\underline{E}_\ell' = \frac{c\underline{k}_\ell'}{\omega} L' \exp(i\underline{k}_\ell' \cdot \underline{r} - i\omega t)$$

Finally the transmitted wave in ε_2 is

$$\underline{E}_2 = (\underline{s} \times \frac{c\underline{k}_2}{\omega}) K \exp(i\underline{k}_2 \cdot \underline{r} - i\omega t)$$

$$\underline{H}_2 = \underline{s}\ \varepsilon_2 K \exp(i\underline{k}_2 \cdot \underline{r} - i\omega t) \qquad \text{(II.33)}$$

Snell's law requires the p components of all wave vectors to be equal (law of refraction) and the n components of the primed wave vectors to be equal to the negative n component of the corresponding unprimed wave vector (law of reflection). Applying the continuity condition to E and H at the surface $\underline{n} \cdot \underline{r} = 0$ we obtain

$$\left(\frac{ck_0}{\omega}\right)_n \left[1 - R\right] = \left(\frac{ck_t}{\omega}\right)_n \left[T - T'\right] + \left(\frac{ck}{\omega}\right)_p \left[L + L'\right]$$

$$\left(\frac{ck}{\omega}\right)_p \left[1 + R\right] = \left(\frac{ck}{\omega}\right)_p \left[T + T'\right] - \left(\frac{ck_\ell}{\omega}\right)_n \left[L - L'\right] \qquad \text{(II.34)}$$

$$\varepsilon_0 \left[1 + R\right] = \varepsilon_t \left[T + T'\right]$$

Similarly from the boundary conditions at the surface $\underline{n}\cdot\underline{r} = d$ we have

$$\left(\frac{ck_t}{\omega}\right)_n\left[T\phi_t - T'\phi_t'\right] + \left(\frac{ck}{\omega}\right)_p\left[L\phi_\ell + L'\phi_\ell'\right] = \left(\frac{ck_2}{\omega}\right)_n T$$

$$\left(\frac{ck}{\omega}\right)_p\left[T\phi_t + T'\phi_t'\right] - \left(\frac{ck_\ell}{\omega}\right)_n\left[L\phi_\ell - L'\phi_\ell'\right] = \left(\frac{ck}{\omega}\right)_p T \qquad \text{(II.35)}$$

$$\varepsilon_t\left[T\phi_t + T'\phi_t'\right] = \varepsilon_2 T$$

where

$$\phi_{t/\ell} = \exp(i\underline{n}\cdot\underline{k}_{t/\ell}d) \qquad \text{(II.36)}$$

$$\phi'\phi = 1$$

$$T = K\exp(i\underline{n}\cdot\underline{k}_2 d) \qquad \text{(II.37)}$$

Solving II.34 and II.35 for R and T we find the equations for reflection and transmission for a slab to be

$$R = \frac{N_r}{M}, \quad T = \frac{N_t}{M} \qquad \text{(II.38)}$$

where

$$N_r = (1-\phi_t\phi_\ell)\left[A_0D_2 - D_0A_2\phi_t\phi_\ell\right] - (\phi_t - \phi_\ell)\left[C_0B_2\phi_t - B_0C_2\phi_\ell\right]$$

$$N_t = (A_0+D_0)\phi_t\phi_\ell\left[(\phi_t-\phi_\ell')(A_0-B_0) + (\phi_\ell-\phi_t')(A_0-C_0)\right]$$
$$M = (1-\phi_t\phi_\ell)\left[D_0D_2-A_0A_2\phi_t\phi_\ell\right] - (\phi_t-\phi_\ell)\left[B_0B_2\phi_t-C_0C_2\phi_\ell\right] \qquad \text{(II.39)}$$

and

$$A = \alpha - \beta - \gamma$$
$$B = \alpha - \beta + \gamma$$
$$C = \alpha + \beta - \gamma \qquad \text{(II.40)}$$
$$D = \alpha + \beta + \gamma$$

The subscript 2 on II.40 indicates that ε_0 and k_0 are replaced by ε_2 and k_2 in the corresponding expressions for α, β, and γ; i.e. they correspond to reversing the positions of the dielectrics ε_0 and ε_2.

Although the above expressions are complicated, certain important features are directly evident. Thus for example our suspicion that irrotational waves as well as em waves interfere is supported by the appearance of the phase factor ϕ_ℓ. As we have done for the semi-infinite case, it will be instructive, to compare the above equations with the corresponding Fresnel equations of reflection and transmission by a slab. They are not difficult to derive, and because they may be found in the literature[29] we will write them down directly

$$R = \frac{R_0 - R_2 \phi_t^2}{1 - R_0 R_2 \phi_t^2}$$

(II.41)

$$T = \frac{\alpha_0 + \beta_0}{\alpha_2 + \beta_2} \frac{(1 - R_0^2)\phi_t}{1 - R_0 R_2 \phi_t^2}$$

where R_0 and R_2 are the Fresnel reflection equations II.15 for medium 0 and 2. We would expect II.38 to reduce to II.41 if the irrotational wave is absent or highly damped out. But if $\mathrm{Im}(k_\ell) \to \infty$ then $\phi_\ell \to 0$, $\gamma \to 0$ and II.39 become

$$N_r = (\alpha_0 - \beta_0)(\alpha_2 + \beta_2) - (\alpha_0 + \beta_0)(\alpha_2 - \beta_2)\phi_t^2$$

$$N_t = 4\alpha_0 \beta_0 \phi_t \qquad \text{(II.42)}$$

$$M = (\alpha_0 + \beta_0)(\alpha_2 + \beta_2) - (\alpha_0 - \beta_0)(\alpha_2 - \beta_2)\phi_t^2$$

and we recover II.41. Also we expect to recover the equations for the semi-infinite case if $d \to \infty$. Even if the waves are only slightly damped ϕ_t, $\phi_\ell \to 0$ as $d \to \infty$, so

$$N_r \to A_0 D_2$$

$$N_t \to 0 \qquad \text{(II.43)}$$

$$M \to D_0 D_2$$

and we find

$$R = \frac{A_0}{D_0} = \frac{\alpha_0 - \beta_0 - \gamma_0}{\alpha_0 + \beta_0 + \gamma_0} = R_p \qquad \text{(II.44)}$$

III. DISPERSION RELATIONS FOR A COLLISIONAL PLASMA

1. Theoretical Formulation

Basically, there are two theoretical methods of studying the properties of plasmas; a microscopic, kinetic treatment using the Boltzmann transport equation or some equivalent means[30] to obtain the distribution function for the charged particles, or a macroscopic, hydrodynamic treatment using a closed set of moment equations. Together with Maxwell's equations of electrodynamics, either of the above treatments provides a closed description of a plasma. The microscopic treatment has the advantage of providing a more detailed description of plasma dynamics, but has the disadvantage of being mathematically more difficult. The macroscopic treatment, on the other hand, is relatively simple mathematically and thus provides better physical insight into the various phenomena, but has a narrower range of application and neglects some important processes, for example it cannot predict Landau damping. In the following paragraphs, the two methods will be briefly sketched and their relationship will be illustrated.

As in other many-body problems, one is interested only in the knowledge of macroscopic properties of the plasma, such as the mean particle density, mean velocity or temperature, conductivity tensor, etc., rather than detailed description of individual particle motion. It is sufficient, therefore, to describe a given class of particles by a distribution function $f(\underline{r}, \underline{v}, t)$ such that fd^3vd^3r represents at time t the probable number of particles with velocities between $\underline{v}$ and $\underline{v} + d\underline{v}$, and positions between $\underline{r}$ and $\underline{r} + d\underline{r}$. All macroscopic properties may be determined from the knowledge of f, which for each species of particles is governed by the Boltzmann transport equation

$$\frac{\delta f}{\delta t} + \underline{v}\cdot\nabla_r f + \frac{1}{m}\underline{F}\cdot\nabla_v f = \left(\frac{\delta f}{\delta t}\right)_{coll.}, \qquad \text{(III.1)}$$

where ∇_r and ∇_v represent the gradient operators in coordinate and velocity space, $\underline{F}$ is the force field at point $(\underline{r}, \underline{v})$, and $(\delta f/\delta t)_{coll.}$ represents the time rate of change of f due to collisions. The essential feature of III.1 is the collision term. Because charged particles interact primarily through their electromagnetic fields, predominantly by Coulomb interactions, these collisions conveniently divide into three classes.[31]

The first corresponds to collisions with impact parameters less than the mean parameter for 90^o deflection, sometimes called the distance of closest approach. These large angle deflections, usually described by the familiar two-body collision integrals of the kinetic theory of gases, are negligible in most cases of interest. The second class consists of a succession of uncorrelated small angle Coulomb deflections with impact parameters greater than the distance of minimum approach but less than the Debye length, λ_D. These encounters are best treated by the Fokker-Planck collision integral. The third class of encounters with impact parameters greater than the Debye length describe a plasma. With many particles in a sphere of radius λ_D; i.e. $N\lambda_D^3 > 1$, N being the particle density, macroscopic electric forces suppress density fluctuations over distances greater than λ_D, hence the collision term is replaced by a macroscopic em force field due to the correlated effect of many particles. Because Vlasov[8] first used this fluid-like treatment of a plasma, the resulting collisionless Boltzmann equation usually bears his name.

The macroscopic treatment begins with the hydrodynamic equations of motion, which presents a problem since they can be an infinite set of coupled equations. For the sake of illustration only the equations of motion in the low temperature approximation (LTA) will be considered. Let N be the particle density, $\underline{F}$ the force on the particles of mass m, $\underline{J} = N\underline{V}$ the net particle current, and $\underline{\underline{P}}$ the pressure tensor resulting from the particle motion. Then the hydrodynamic equation of motion of the particles is

$$\frac{\delta}{\delta t}\underline{J} = \frac{1}{m}\underline{F}N - \frac{1}{m}\nabla_r \cdot \underline{\underline{P}} \ . \qquad \text{(III.2)}$$

In the LTA some assumption must be made about the pressure tensor, for example

$$P = mNV^2 \qquad \text{(III.3)}$$

The above equations along with the continuity equation

$$\frac{\delta}{\delta t}N + \nabla_r \cdot \underline{J} = 0 \ , \qquad \text{(III.4)}$$

and Maxwell's equations determine the system of charged particles.

To show how the microscopic and macroscopic treatments are related, we start with the following moment integrals

$$\begin{aligned} N(\underline{r},t) &= \int f(\underline{r},\underline{v},t)d^3v \ , \\ \underline{J}(\underline{r},t) &= \int \underline{v}f(\underline{r},\underline{v},t)d^3v \ , \\ \underline{\underline{P}}(\underline{r},t) &= \int m\underline{v}\underline{v}f(\underline{r},\underline{v},t)d^3v . \end{aligned} \qquad \text{(III.5)}$$

Obviously N, $\underline{J}$, and $\underline{\underline{P}}$ are respectively the mean particle density, current, and pressure. Note that III.3 is just the trace of the expression for $\underline{\underline{P}}$, III.5. If we now take the velocity integral of the Boltzmann equation III.1, we arrive at the continuity Eq. III.4. Similarly multiplying the Boltzmann equations by the velocity $\underline{v}$ and integrating we arrive at the hydrodynamic Eq. III.2. In performing the integrals we have assumed that the particle number and momentum or current are each conserved under collisions so that the collision terms vanish. We have also assumed that f is well behaved in velocity space and its surface integrals vanish. Higher moment equations are formed by integrating the Boltzmann equation with additional velocity factors; e.g. the next moment equation gives the equation of motion of the pressure tensor $\underline{\underline{P}}$ in terms of the heat flow tensor $\underline{\underline{\underline{Q}}}$, etc.. Each higher moment recovers more information about the system that is completely contained in f. This infinite set of coupled equations must be truncated and some approximation given to the last term.

Because the macroscopic approach excludes certain detailed information such as the effect of collisions due to other mechanisms, e.g. scattering by lattice, we will use the microscopic approach.

2. Calculation of the Distribution Function

Let us consider a homogeneous, isotropic, and unbounded gas of charged particles, whose net charge is zero. This gas may contain different species of charged particles; each species being characterized by a set of parameters including mass, charge, temperature, Fermi energy, etc.. In addition to any external force fields, $\underline{F}$ in III.1 contains the interaction between all the charged particles, while the collision term contains all other scattering mechanisms such as impurities, phonons, etc.. Suppose the system is disturbed from equilibrium such that the force field is harmonic in space and time, i.e. proportional to $\exp(i\underline{k}\cdot\underline{r} - i\omega t)$. Or, to put it in another equally valid way, since any disturbance in an unbounded steady state system can be Fourier decomposed into harmonic components, we choose one. Then the differential operators in space and time become algebraic:

$$\frac{\delta}{\delta t} = -i\omega \ , \quad \nabla_r = i\underline{k} \ . \qquad \text{(III.6)}$$

With no static external electric or magnetic fields the force field is

$$\underline{F} = -e(\underline{E} + \frac{1}{c}\,\underline{v} \times \underline{B}) \qquad \text{(III.7)}$$

Although the following calculations will be made for electrons, the results are quite general and can be applied to any "free" charged particles by changing the parameters. III.7 may be rewritten in terms of the E field only,

$$\underline{F} = -e\left[\frac{\underline{\underline{kv}}}{\omega} + (1 + \frac{\underline{k}\cdot\underline{v}}{\omega})\underline{\underline{I}}\right]\cdot\underline{E} \qquad \text{(III.8)}$$

by the use of Maxwell's equation

$$\nabla_r \times \underline{E} = -\frac{1}{c}\frac{\delta}{\delta t}\underline{B} , \qquad \text{(III.9)}$$

where $\underline{\underline{I}}$ in III.8 represents the unit identity tensor. We will treat the collision term by the relaxation time ansatz

$$\left.\frac{\delta f}{\delta t}\right|_{\text{coll.}} = -\gamma\left(f(\underline{r},\underline{v},t) - f_s(\underline{r},\underline{v},t)\right) , \qquad \text{(III.10)}$$

where γ is the collision frequency, f is the distribution function before scattering, and f_s is the distribution function after scattering, i.e. the local equilibrium function. The total distribution function for the electrons is

$$f_0 = \frac{3N_0}{8\pi\varepsilon_F^{3/2}}\left(\frac{2m}{h^2}\right)^{3/2}\left[1 + \exp\left(\frac{\varepsilon_F - \varepsilon}{kT}\right)\right]^{-1} \qquad \text{(III.11)}$$

where N_0 is the average density, ε the energy per particle, ε_F the Fermi energy, and kT the thermal energy. Because the Fermi energy is a function of density, at T = 0 being

$$\varepsilon_F^0 = \frac{\hbar}{2m}(3\pi^2 N)^{2/3} , \qquad \text{(III.12)}$$

and because the density varies in space and time,

$$N(\underline{r},t) = N_0 + N_1 \exp(i\underline{k}\cdot\underline{r} - i\omega t) \qquad \text{(III.13)}$$

the local equilibrium distribution also varies. Assuming the density variations are small compared with the equilibrium value ($N_0 >> N_1$), we can make a linear expansion of f_s about f_0,

$$f_s = f_0 + \left(\frac{\delta f}{\delta \varepsilon_F}\right)\left(\frac{\delta \varepsilon_F}{\delta N}\right)_{N_0} N_1(\underline{r},t). \qquad \text{(III.14)}$$

From now on quantities subscripted with 1 are small compared to the equilibrium value (subscript 0) and their space-time dependence is $\exp(i\underline{k}\cdot\underline{r} - i\omega t)$, which will not be written explicitly.

Before proceeding any further something must be said about the temperature of the electrons and their equilibrium distribution. In general III.11 is difficult to handle except in the limit of very low or very high temperatures. But because the Fermi energy of electrons at room temperature is so much greater than the thermal energy, III.11 may be approximated to be at a temperature of absolute zero. On the other hand, charged carriers in a semiconductor or gas plasma have high temperatures and III.11 may be approximated by the Boltzmann distribution. Because we are interested in a low temperature solid state plasma we shall use the zero temperature approximation. Thus III.14 may be rewritten as

$$f_s = f_0 - \left(\frac{\delta f_0}{\delta \varepsilon}\right)\frac{2}{3}\,\frac{\varepsilon_F^0}{N_0}\,N_1 \qquad \text{(III.15)}$$

and III.1 becomes

$$\frac{\delta}{\delta t} f + \underline{v}\cdot\nabla_r f - \frac{e}{m}\underline{E}_1\cdot\left[\frac{\underline{v}\underline{k}}{\omega} + (1-\frac{\underline{k}\cdot\underline{v}}{\omega})\underline{\underline{I}}\right]\cdot\nabla_v f = -\gamma\left[f-f_s\right] \quad (III.16)$$

If we assume III.16 has a solution linear in harmonic quantities.

$$f(\underline{r},\underline{v},t) = f_0 + f_1\exp(i\underline{k}\cdot\underline{r} - i\omega t) \qquad (III.17)$$

we find

$$f_1 = e\left(\frac{\delta f_0}{\delta\varepsilon}\right)\frac{\underline{v}\cdot\underline{E}_1}{\gamma - i\omega + i\underline{k}\cdot\underline{v}} - \frac{2}{3}\frac{\overset{0}{\varepsilon}_F}{N_0}\left(\frac{\delta f_0}{\delta\varepsilon}\right)\frac{\gamma N_1}{\gamma - i\omega + i\underline{k}\cdot\underline{v}} \qquad (III.18)$$

3. The Conductivity Tensor

The conductivity tensor may be best described as the linear response function of a system relating the components of an electric field to the components of a current density produced by this field. The most general linear relation possible between two vector quantities that vary in space and time is

$$\underline{J}(\underline{r},t) = \int d^3r'\int_{-\infty}^{\infty} dt'\underline{\underline{\sigma}}(\underline{r},t,\underline{r}',t')\cdot\underline{E}(\underline{r}',t'), \qquad (III.19)$$

i.e. the current density $\underline{J}$ at the position $\underline{r}$ and time t depends on the electric field $\underline{E}$ at all points in space and all times through some tensorial relation $\underline{\underline{\sigma}}$. But for a homogeneous causally related system $\underline{\underline{\sigma}}$ is a function of relative position and time only, and its Fourier expansion is

$$\underline{\underline{\sigma}}(\underline{r}-\underline{r}',t-t') = (2\pi)^{-4}\int d^3k'\int d\omega'\underline{\underline{\sigma}}(k',\omega') \times \exp\left\{i\underline{k}'\cdot(\underline{r}-\underline{r}') - i\omega'(t-t')\right\} \quad (III.20)$$

Since we are interested in fields with harmonic space time dependence,

$$\underline{E}_1(\underline{r}',t') = \underline{E}(\underline{k},\omega)\exp(i\underline{k}\cdot\underline{r}' - i\omega t') \quad (III.21)$$

III.19 may be reduced to

$$\underline{J}_1(\underline{r},t) = \underline{\underline{\sigma}}(\underline{k},\omega)\cdot\underline{E}_1(\underline{r},t) \quad (III.22)$$

by the use of the orthogonality property of the exponential function.

The current density $\underline{J}$ and the distribution function f are related by the moment equation

$$\underline{J} = \sum_i q_i\int d^3v_i\underline{v}_i f_i, \quad (III.23)$$

where q_i is the charge and $\underline{v}_i$ is the velocity of the ith species of charged particles. The current for the electrons whose distribution function we have just calculated is

$$\underline{J}\,(\underline{r},t) = -e\int d^3v\underline{v}f_1(\underline{r},t) \tag{III.24}$$

Only f_1 contributes to the current since f_0 is symmetric in v. Because f_1 is a function of $\underline{E}_1$ and N_1, III.24 has two parts: The conductivity current which is proportional to $\underline{E}_1$; and the diffusion current, which is due to the variation of the local equilibrium density and is proportional to N_1.

$$\underline{J}_1(\underline{r},t) = \underline{\underline{\sigma}}'(\underline{k},\omega)\cdot\underline{E}_1 - e\omega\underline{R}(\underline{k},\omega)N_1 \tag{III.25}$$

where

$$\underline{\underline{\sigma}}'(\underline{k},\omega) = e^2\int d^3v\left(-\frac{\delta f_0}{\delta\varepsilon}\right)\frac{\underline{v}\,\underline{v}}{\gamma-i\omega+i\underline{k}\cdot v} \tag{III.26}$$

$$\underline{R} = \int d^3v\frac{2}{3}\,\frac{\gamma}{\omega}\,\frac{\varepsilon_F}{N_0}\left(-\frac{\delta f_0}{\delta\varepsilon}\right)\frac{\underline{v}}{\gamma-i\omega+i\underline{k}\cdot\underline{v}} \tag{III.27}$$

To reduce III.25 to the form III.22 we define the tensor $\underline{\underline{R}}$ as

$$\underline{\underline{R}}(\underline{k},\omega) = \underline{R}(\underline{k},\omega)\underline{k}\ , \tag{III.28}$$

so that the equation of continuity

$$\underline{k}\cdot\underline{J}_1 + e\omega N_1 = 0 \tag{III.29}$$

may be rewritten as

$$\underline{\underline{R}} \cdot \underline{J}_1 = -e\omega \underline{R} N_1 . \tag{III.30}$$

Substituting III.30 into III.25, the generalized conductivity tensor becomes

$$\underline{\underline{\sigma}}(\underline{k},\omega) = \left(\underline{\underline{I}} - \underline{R}(\underline{k},\omega)\underline{k}\right)^{-1} \cdot \underline{\underline{\sigma}}'(\underline{k},\omega) , \tag{III.31}$$

where $\underline{\underline{A}}^{-1}$ represents the inverse of tensor $\underline{\underline{A}}$. Note, if more than one charged species is present $\underline{R}$ and $\underline{\underline{\sigma}}'$ are replaced by $\sum_i \underline{R}_i$ and $\sum_i \underline{\underline{\sigma}}_i'$.

<u>Evaluation of $\underline{\underline{\sigma}}'$ and $\underline{\underline{R}}$.</u> At absolute zero the Fermi-Dirac distribution function III.11 becomes a step function in energy or velocity which permits us to reduce velocity integrations to solid angle integrations by the identity

$$\int d^3v G(\underline{v}) \left(-\frac{\delta f_0}{\delta \varepsilon}\right) = \frac{3N_0}{4\pi m v_F^{\,2}} \int d\Omega G(\underline{v}_F) , \tag{III.32}$$

Since III.32 is a good approximation for temperatures less than the Fermi temperature we apply it to our metallic plasma and obtain

$$\underline{\underline{\sigma}}' = \frac{\omega_p^2}{4\pi} \frac{3}{4\pi} \frac{1}{\gamma - i\omega} \underline{\underline{T}} , \tag{III.33}$$

$$\underline{R} = \frac{\gamma}{\omega} \frac{v_F}{\gamma - i\omega} \frac{1}{4\pi} \underline{V} . \tag{III.34}$$

The integrals

$$\underline{\underline{T}} = \int \frac{\hat{\underline{r}}\hat{\underline{r}}d\Omega}{1+i\underline{a}\cdot\hat{\underline{r}}} \qquad \text{(III.35)}$$

and

$$\underline{V} = \int \frac{\hat{\underline{r}}d\Omega}{1+i\underline{a}\cdot\hat{\underline{r}}} , \qquad \text{(III.36)}$$

where $\hat{\underline{r}}$ is a unit radial vector in spherical coordinates and

$$\underline{a} \equiv \frac{\underline{k}v_F}{\gamma - i\omega} , \qquad \text{(III.37)}$$

are evaluated in Appendix A.

So far in this chapter, $\underline{k}$ has been a general vector, but now we will choose our coordinate system by requiring the real part of $\underline{k}$, i.e. the direction of phase propagation, to be in the positive z direction. Since, as was demonstrated in Chapter II, for inhomogeneous waves the imaginary part of k has a different direction, we choose the x direction so that the most general $\underline{k}$ lies in the x-z plane. In other words $\underline{k}$ and $\underline{a}$ possess only x and z components.

In this coordinate system we find (Appendix A) for homogeneous waves with $\underline{a} = a_z\hat{\underline{3}}$,

$$\underline{\underline{\sigma}}' = \begin{vmatrix} \sigma_t & 0 & 0 \\ 0 & \sigma_t & 0 \\ 0 & 0 & \sigma_\ell' \end{vmatrix} \qquad \text{(III.38)}$$

where

$$\sigma_t = \frac{3\omega_p}{8\pi(\gamma - i\omega)} \frac{1}{a^2} \left[\frac{1+a^2}{a} \arctan a - 1 \right] \qquad \text{(III.39)}$$

$$\sigma_\ell' = \frac{3\omega_p^2}{4\pi(\gamma - i\omega)} \frac{1}{a^2} \left[1 - \frac{\arctan a}{a} \right] \qquad \text{(III.40)}$$

and R has only the z-z component

$$R_{zz} = R = \frac{\gamma}{i\omega} \left[1 - \frac{\arctan a}{a} \right], \qquad \text{(III.41)}$$

where a represents the scalar magnitude of $\underline{a}$;

$$a^2 = \underline{a} \cdot \underline{a} \ , \qquad \text{(III.42)}$$

and for the homogeneous wave is just a_z.

Defining σ_ℓ as

$$\sigma_\ell = \frac{\sigma_\ell'}{1-R} \ , \qquad \text{(III.43)}$$

III.38 takes the simple form

$$\underline{\sigma} = \begin{vmatrix} \sigma_t & 0 & 0 \\ 0 & \sigma_t & 0 \\ 0 & 0 & \sigma_\ell \end{vmatrix} \qquad \text{(III.44)}$$

The tensor $\underline{R}$ for the inhomogeneous case, as evaluated in Appendix A, may be put in the following form:

$$\underline{R} = \frac{\underline{k}\,\underline{k}}{\underline{k}\cdot\underline{k}} \frac{\gamma}{i\omega} \left[1 - \frac{\arctan a}{a}\right] \qquad \text{(III.45)}$$

Similarly, the non-zero components of $\underline{\sigma}'$ may be written as

$$\begin{aligned} \sigma'_{xx} &= \sigma_t + \frac{k_x^2}{\underline{k}\cdot\underline{k}} \sigma_\mu' \\ \sigma'_{xz} &= \sigma'_{zx} = \frac{k_x k_z}{\underline{k}\cdot\underline{k}} \sigma_\mu' \\ \sigma'_{yy} &= \sigma_t \\ \sigma'_{zz} &= \sigma'_\ell - \frac{k_x^2}{\underline{k}\cdot\underline{k}} \sigma_\mu' \ , \end{aligned} \qquad \text{(III.46)}$$

where σ_ℓ' and σ_t are given by III.39 and III.40 and

$$\sigma_\mu' = \sigma_\ell' - \sigma_t = \frac{3}{8\pi} \frac{\omega_p^2}{\gamma - i\omega} \frac{1}{a^2} \left[3 - \frac{3+a}{a} \arctan a\right] \qquad \text{(III.47)}$$

After some simple but tedious algebra, III.45 and III.46 may be combined to give the components of $\underline{\sigma}$

$$\sigma_{xx} = \sigma_t + \frac{k_x^2}{\underline{k}\cdot\underline{k}}\,\sigma_\mu$$

$$\sigma_{yy} = \sigma_t$$

$$\sigma_{zz} = \sigma_\ell - \frac{k_x^2}{k}\,\sigma_\mu \qquad \text{(III.48)}$$

$$\sigma_{xz} = \sigma_{zx} = \frac{k_x k_z}{\underline{k}\cdot\underline{k}}\,\sigma_\mu ,$$

where σ_t and σ_ℓ are given by III.39 and III.43 and

$$\sigma_\mu = \sigma_\mu' + R\sigma_\ell \qquad \text{(III.49)}$$

so that

$$\sigma_\ell = \sigma_\mu + \sigma_t \qquad \text{(III.50)}$$

4. Dispersion Relations

A dispersion relation, in the sense that we employ the term, gives the wave vector k, which in general may be complex, as a function of the frequency ω, which we assume to be real. Because of our sign convention, a positive imaginary part of k corresponds to a wave growing in space, while a negative value corresponds to a wave attenuating in space.

The dispersion relation is derived by requiring the fields of the wave to satisfy Maxwell's equations

$$\nabla \times \underline{H}_1 = \frac{4\pi}{c} J_1 - \frac{1}{c}\frac{\delta \underline{E}_1}{\delta t}$$

$$\nabla \times \underline{E}_1 = -\frac{1}{c}\frac{\delta \underline{H}_1}{\delta t} \qquad \text{(III.51)}$$

Assuming $\underline{J}$ and $\underline{E}$ are related by III.22, we solve for $\underline{E}$

$$\left[(c/\omega)^2 \underline{k}\cdot\underline{k}\underline{\underline{I}} - (c/\omega)^2 \underline{k}\;\underline{k} - \underline{\underline{\varepsilon}}(\underline{k},\omega)\right]\cdot \underline{E}_1 = 0 \qquad \text{(III.52)}$$

where

$$\underline{\underline{\varepsilon}} = \underline{\underline{I}} + \frac{4\pi i}{\omega}\underline{\underline{\sigma}} \qquad \text{(III.53)}$$

is the dielectric tensor. For a non-trivial solution to III.52 we demand that the secular equation

$$\left|\left|(c/\omega)^2 \underline{k}\cdot\underline{k}\underline{\underline{I}} - (c/\omega)^2 \underline{k}\;\underline{k} - \underline{\underline{\varepsilon}}\right|\right| = 0 \qquad \text{(III.54)}$$

be satisfied, which gives us the desired dispersion relations.

Using the conductivity tensor we have just calculated and III.53 we may write down the components of the dielectric tensor for an inhomogeneous wave:

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_t + \frac{k_x^2}{k^2}\,\varepsilon_\mu \\
\varepsilon_{yy} &= \varepsilon_t \\
\varepsilon_{xz} &= \varepsilon_{zx} = \frac{k_x k_z}{k^2}\,\varepsilon_\mu \\
\varepsilon_{zz} &= \varepsilon_\ell - \frac{k_x^2}{k^2}\,\varepsilon_\mu ,
\end{aligned} \qquad \text{(III.55)}$$

where

$$\varepsilon_t = 1 - \sum_j^n \frac{\omega_{pj}^2}{\omega(\omega+i\gamma_z)}\,\frac{3}{2a_j^2}\left[\frac{1+a_j^2}{a_j}\arctan a_j \quad -1\right] \qquad \text{(III.56)}$$

$$\varepsilon_\ell = 1 - \frac{\displaystyle\sum_j^n \frac{\omega_{pj}^2}{\omega(\omega+i\gamma_j)}\,\frac{3}{a_j{}^2}\left[1 - \frac{\arctan \quad a_j}{a_j}\right]}{\displaystyle 1 + i\sum_j^n \frac{\gamma_j}{\omega}\left[1 - \frac{\arctan \quad a_j}{a_j}\right]} \qquad \text{(III.57)}$$

$$\varepsilon_\mu = \varepsilon_\ell - \varepsilon_t \,. \qquad \text{(III.58)}$$

At this point we have generalized our results to an n-component plasma and that is the reason for the summation in above expressions.

For homogeneous waves, i.e. for $k_x = 0$, the secular equation yields three solutions; two degenerate dispersion relations

$$(c/\omega)^2 \underline{k}_t \cdot \underline{k}_t = \varepsilon_t(\underline{k}_t, \omega) \qquad \text{(III.59)}$$

associated with electric fields polarized in the x and y direction; i.e. transverse waves, and the dispersion relation

$$\varepsilon_\ell(\underline{k}_\ell, \omega) = 0 \qquad \text{(III.60)}$$

associated with the electric field in the z-direction; i.e. a longitudinal wave. These results are well known, but it should be pointed out that usually no distinction is made between the longitudinal dielectric function ε_ℓ and the transverse dielectric function ε_t.

For inhomogeneous waves, however, we would expect the dispersion relation for y polarized waves to be III.59, but the x and z polarized waves to be mixed. Indeed the y-polarized solution factors out and we are left with

$$\left[(c/\omega)^2 k_z^2 - \varepsilon_{xx}\right]\left[(c/\omega)^2 k_x^2 - \varepsilon_{zz}\right] - \left[(c/\omega)^2 k_x k_z + \varepsilon_{xz}\right]\left[(c/\omega)^2 k_z k_x + \varepsilon_{zx}\right] = 0 . \qquad \text{(III.61)}$$

But the above equation may be simplified with III.58 into

$$\left[(c/\omega)^2 k^2 - \varepsilon_t\right] \varepsilon_\ell = 0 , \qquad \text{(III.62)}$$

which shows the remaining two waves to be independent with "transverse" and "longitudinal" like dispersion relations. But they are not transverse and longitudinally polarized waves as in the homogeneous case. To see how the fields are related to the wave vectors let us solve III.52 explicitly for the fields:

$$\left[(c/\omega)^2 k_z^2 - \varepsilon_{xx}\right] E_x - \left[(c/\omega)^2 k_x k_z + \varepsilon_{xz}\right] E_z = 0$$

$$\left[(c/\omega)^2 k_z k_x + \varepsilon_{zx}\right] E_x - \left[(c/\omega)^2 k_x^2 - \varepsilon_{zz}\right] E_z = 0 \qquad \text{(III.63)}$$

Substituting the "transverse" dispersion relation III.59 into either Eq. III.63 we find

$$k_x E_x + k_z E_z = 0$$

or (III.64)

$$\underline{k}_t \cdot \underline{E}_t = 0$$

Similarly the "longitudinal" dispersion relation III.60 substituted into III.63 gives

$$k_x E_z - k_z E_x = 0$$

or (III.65)

$$\underline{k}_\ell \times \underline{E}_\ell = 0$$

In other words, the dispersion relation III.59 corresponds to irrotational wave fields or polarization waves. Thus in Chapter II, Eq. II.11 which defines ε is just the dispersion relation for the em waves. Furthermore, we have proved that in a homogeneous, isotropic plasma with collisions, the wave fields can be separated into two independent parts, one irrotational and the other divergence free. In general this is not true, for example in a drifted plasma where a static external electric field is present isotropy is destroyed, and the wave fields can no longer be separated in this manner.

Discussion of Dispersion Relations. We will now examine some of the general properties of the waves in a plasma as given by the dispersion relations III.59 and III.60. From the definition of a (Eq. III.37) and the fact that $c >> v_F$, the absolute value of a is always less than 1 for the em wave. Thus we may approximate ε_t by expanding III.56 as a power series in a. Since

$$\arctan a = \sum_{n=0} (-1)^n \frac{a^{2n-1}}{2n+1} \qquad \text{(III.66)}$$

we find

$$\varepsilon_t = 1 - \sum_i \frac{\omega_{pi}^2}{\omega(\omega+i\gamma)} \left[1 + \sum_n \frac{(-1)^n 3a_i^{2n}}{(2n+3)(2n+1)} \right] \qquad \text{(III.67)}$$

Or to zero and first order in a^2 we have

$$\varepsilon_t = 1 - \frac{\omega_p^2}{\omega(\omega+i\gamma)} \tag{III.68}$$

$$\varepsilon_t = 1 - \frac{\omega_p^2}{\omega(\omega+i\gamma)}\left[1 + \frac{1}{5}\frac{\underline{k}\cdot\underline{k}v_F^2}{(\omega+i\gamma)^2}\right] . \tag{III.69}$$

For the sake of simplicity we wrote III.68 and III.69 for a one component plasma, but the results may be easily generalized to a multi-component plasma. Solving III.69 explicitly for k_t, we have

$$\left(\frac{ck_t}{\omega}\right)^2 = \frac{1 - \frac{\omega_p^2}{\omega(\omega+i\gamma)}}{1 + \frac{1}{5}\left(\frac{v_F}{c}\right)^2\frac{\omega\omega_p^2}{(\omega+i\gamma)^3}} . \tag{III.70}$$

Since $(v_F/c)^2$ is about 10^{-5} or less, Eq. III.70 indicates that the commonly used expression III.68, for ε_T , is good for frequencies $\omega/\omega_p > 10^{-2}$.

We next examine the behavior of the irrotational polarization wave in a one component plasma. For frequencies near ω_p, $|a| < 1$ provided $\gamma < \omega$, and ε_ℓ, expanded to first order in a^2, is

$$\varepsilon_\ell = 1 - \frac{\frac{\omega_p^2}{\omega(\omega+i\gamma)}\left(1 + \frac{3}{5}\frac{k_L^2 v_F^2}{(\omega+i\gamma)^2}\right)}{1 - i\frac{\gamma}{\omega}\frac{k_L^2 v_F^2}{3(\omega+i\gamma)^2}} . \tag{III.71}$$

Note, that to zero order in a^2, ε_ℓ and ε_t are equal, but because this approximation neglects all except the infinite wavelength dispersion of the irrotational, for our problem we cannot equate ε_ℓ with ε_t. For a collisionless plasma ($\gamma = 0$), the dispersion relation becomes

$$\left(\frac{ck_\ell}{\omega}\right)^2 = \frac{5}{3}\left(\frac{c}{v_F}\right)^2\left[\left(\frac{\omega}{\omega_p}\right)^2 - 1\right] \qquad \text{(III.72)}$$

As for the em wave, k_ℓ is primarily imaginary for $\omega < \omega_p$, real for $\omega > \omega_p$ and vanishes near ω_p. The primary difference is that k_ℓ is greater than k_t by approximately $\frac{c}{v_F}$ or inversely the em wavelengths are always larger than the irrotational wavelength by the factor $\frac{c}{v_F}$. This fact, we shall later see, has important consequences.

IV. RESULTS

1. Plasma Slab of Infinite Thickness

Taking the dispersion relations calculated in the last chapter, we shall evaluate the expressions for transmission and reflection of em waves by a plasma. Since the size of the effect due to the irrotational plasma wave depends on the ratios of γ to α and β (Eqs. II.16 and II.24) we shall consider approximate values of these quantities and determine if the effect is measurable. Neglecting the collision frequency in III.68 and III.72 we find

$$\alpha = \left(1 - x^{-2}\right) \cos \theta$$

$$\beta = \sqrt{\cos^2 \theta - x^{-2}} \qquad \text{(IV.1)}$$

$$\gamma = \frac{\sqrt{3/5}\,\sin^2 \theta}{x^2 \sqrt{x^2 - 1}} \left(\frac{v_F}{c}\right),$$

where $x = \omega/\omega_p$, θ is the angle of incidence, and the first medium is chosen to be vacuum, so $\varepsilon_0 = 1$. Since the fraction of the energy flowing into the irrotational wave (Eq. II.29) has a maximum when $\beta = \left(\frac{ck_t}{\omega}\right)_n$ vanishes, the largest effect of the irrotational wave should occur when the em wave is refracted into a surface wave along the interface. This condition is achieved in IV.1 when

$$\cos\theta = x^{-1} = (\omega_p/\omega), \tag{IV.2}$$

so that

$$\alpha = \sin^2\theta\cos\theta,$$
$$\gamma = \sqrt{3/5}\,(v_F/c)\sin\theta\cos^3\theta. \tag{IV.3}$$

Since there is no transmitted em wave the condition for all the energy of the incident wave to go into the longitudinal plasma wave is $\alpha = \gamma$, or

$$\tan\theta\sec\theta = \sqrt{3/5}\,(v_F/c)$$
$$\sec\theta = \omega/\omega_p \tag{IV.4}$$

For typical solid state plasmas with a Fermi velocity of about 10^8cm/sec, this critical angle is only a few minutes, i.e. almost normal incidence, and the frequency exceeds the plasma frequency by a few tenths of a percent. But because the usual Brewster angle, corresponding to $\alpha = \beta$, is also a few minutes near the plasma frequency, the dip in the reflection due to the longitudinal plasma wave would be indistinguishable from the dip due to the usual Brewster condition.

2. Plasma Slabs of Finite Thickness

Although the experimental conditions for observing irrotational polarization waves in the reflection spectrum of a very thick plasma slab appear difficult to realize, they may be met for thin slabs. Since the problem lies with the fact that at the frequency and angle at which the polarization wave has a measurable effect, the reflection changes drastically because the plasma becomes transparent to em waves, let us consider plasma slabs or films thin enough to be transparent to these em waves. In other words, we can exploit the fact that the wavelengths of the em waves in plasmas exceed the wavelengths of the polarization waves by the factor c/v_F, by choosing plasma slabs thin compared to the wavelength of the em wave; $\mathrm{Re}(\underline{n}\cdot\underline{k}_t)d \ll \pi$, but thick compared to the wavelength of the polarization wave; $\mathrm{Re}(\underline{n}\cdot\underline{k}_l)d > \pi$. Such a slab would be almost entirely transparent to em waves permitting the effects of the irrotational waves to be observable.

<u>Numerical Results.</u> Defining the transmittance, T, and reflectance, R, in the usual way; as the fraction of energy transmitted and reflected by the entire slab:

$$T = \frac{\underline{n}\cdot\underline{S}_t}{\underline{n}\cdot\underline{S}_0}, \quad R = \frac{\underline{n}\cdot\underline{S}_r}{\underline{n}\cdot\underline{S}_0}, \qquad \text{(IV.5)}$$

we calculate them by taking the absolute value squared of the expressions II.38:

$$T = \frac{|N_t|^2}{|M|^2} \quad , \quad R = \frac{|N_r|^2}{|M|^2} \; . \tag{IV.6}$$

N_t, N_r, and M are given by II.39, and for a plasma slab bounded by a vacuum on both sides they simplify to:

$$N_t = 4\alpha\left[(1 - \phi_t^2)\phi_\ell\gamma + (1 - \phi_\ell^2)\phi_t\beta\right]$$

$$N_r = (1 - \phi_t\phi_2)^2 AD - (\phi_t - \phi_\ell)^2 CB \tag{IV.7}$$

$$M = (1 - \phi_t\phi_\ell)\left[D^2 - A^2\phi_t\phi_\ell\right] - (\phi_t - \phi_\ell)\left[B^2\phi_t - C^2\phi_\ell\right]$$

These expressions have been computed numerically for frequencies between 0.6 ω_p and 1.5 ω_p in steps of 0.005 ω_p, for three slab thicknesses, and four angles of incidence. The values of slab thickness chosen were d/λ_p = 0.0305, 0.0915, and 0.1525, where λ_p is the wavelength of an em wave in vacuum at the plasma frequency and is defined by

$$\lambda_p = \frac{2\pi C}{\omega_p} \; . \tag{IV.8}$$

Angles of incidence of 10, 30, 60, and 80 degrees were used in the computation.

The exact, collision dependent, dispersion relations (Eqs. III.56 through III.60) were solved numerically for a one component plasma described by a Fermi velocity $v_F = 1.4 \times 10^8$ cm/sec. and a constant collision frequency equal to $10^{-2}\omega_p$ ($\omega_p\tau = 100$). The results were plotted directly by the computer, along with the absorptance A, defined as

$$A = 1 - R - T, \tag{IV.9}$$

and are presented in Figs. 4 through 10.

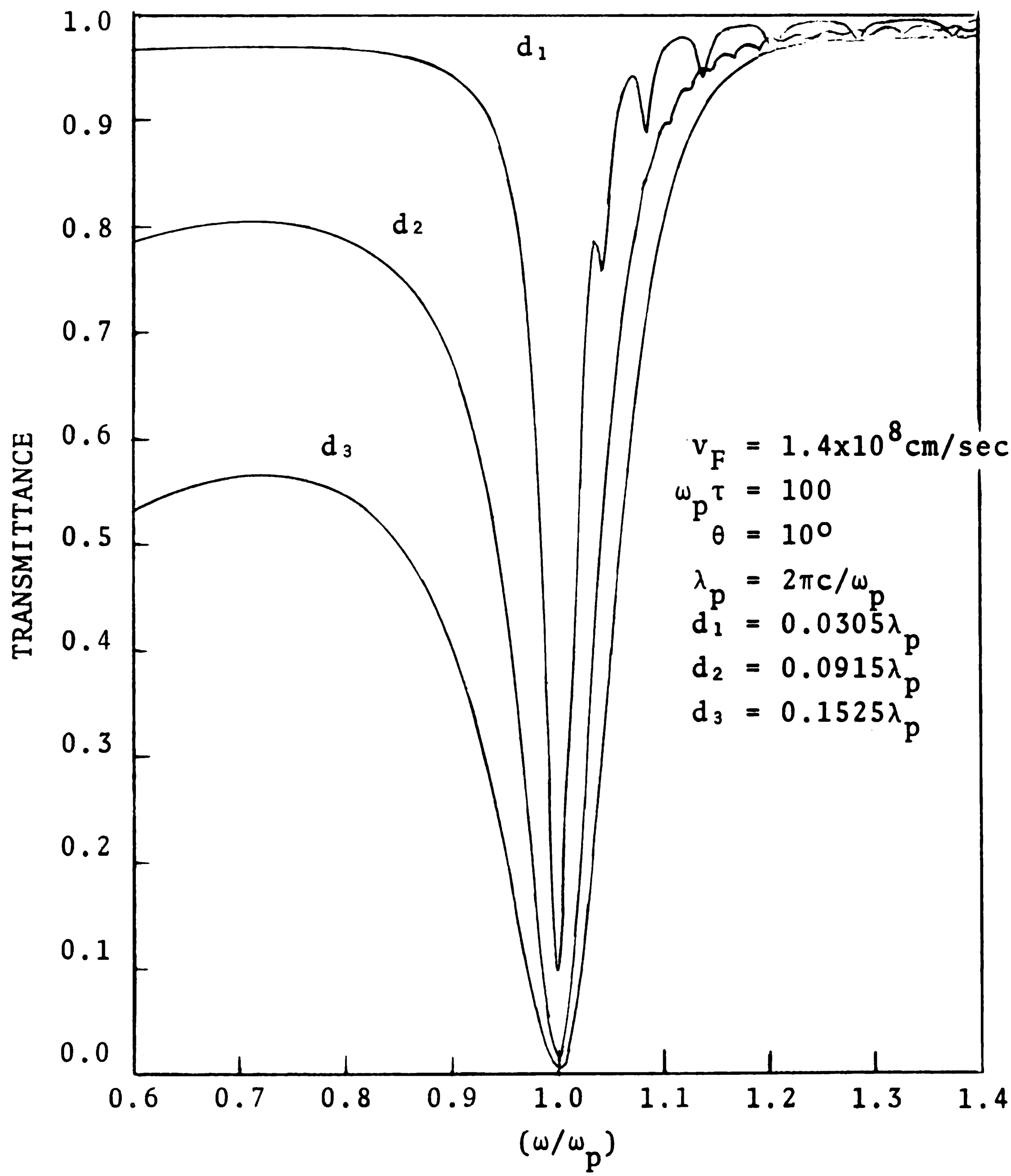

Fig. 4a. Transmittance versus frequency for p-polarized radiation incident at an angle of 30 degrees on plasma slabs of thicknesses d/λ_p = 0.0305, 0.0915, 0.1545, where λ_p is the vacuum wavelength at the plasma frequency, $\lambda_p = 2\pi c/\omega_p$. The plasma is described by a Fermi velocity $v_F = 1.4x10^8$ cm/sec and a collision frequency corresponding to $\omega_p\tau$=100.

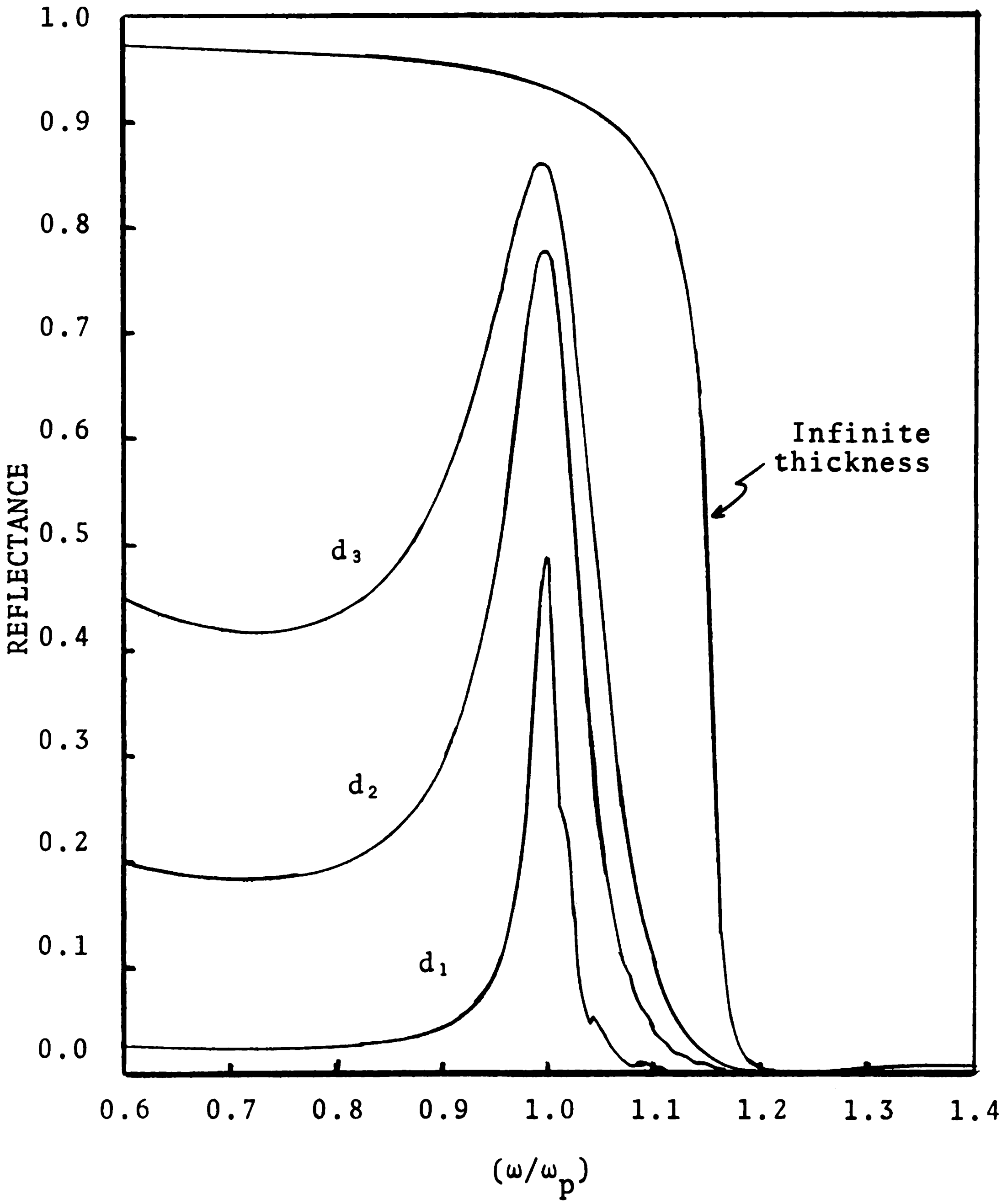

Fig. 4b. Reflectance versus frequency for 30 degree incidence and various thicknesses.

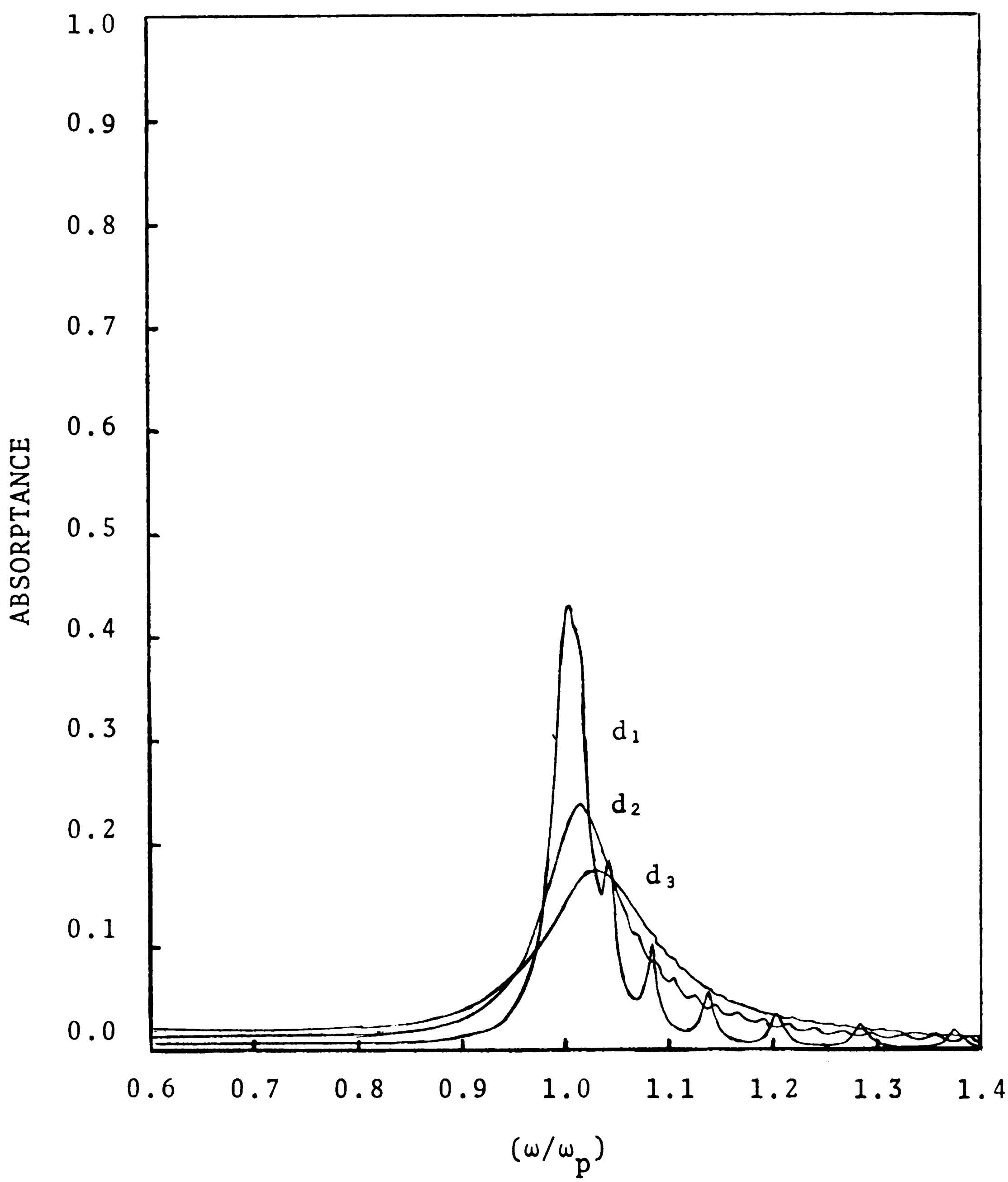

Fig. 4c. Absorptance versus frequency for 30 degree incidence, and various thicknesses.

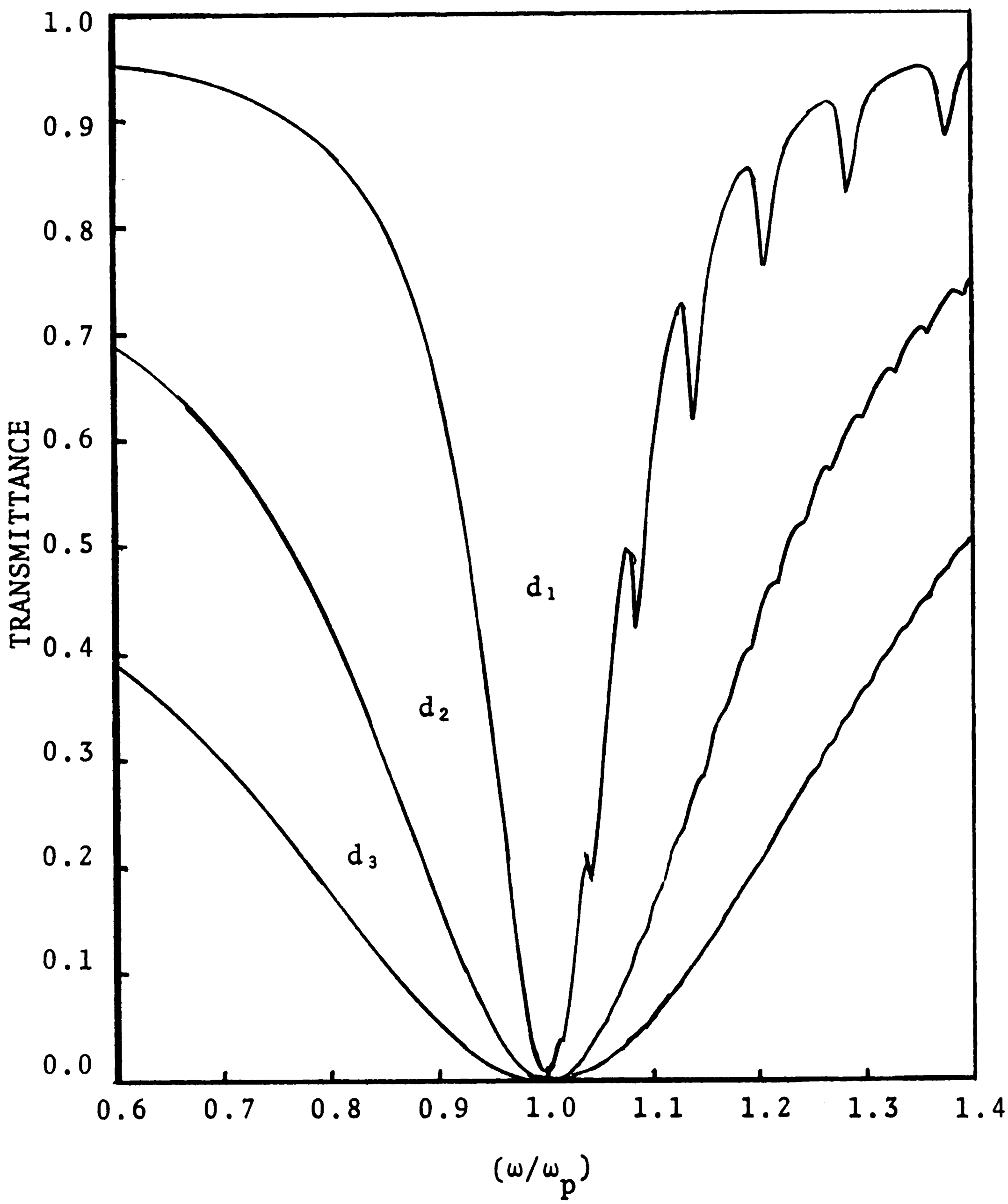

Fig. 5a. Transmittance versus frequency for 60 degree incidence and various thicknesses.

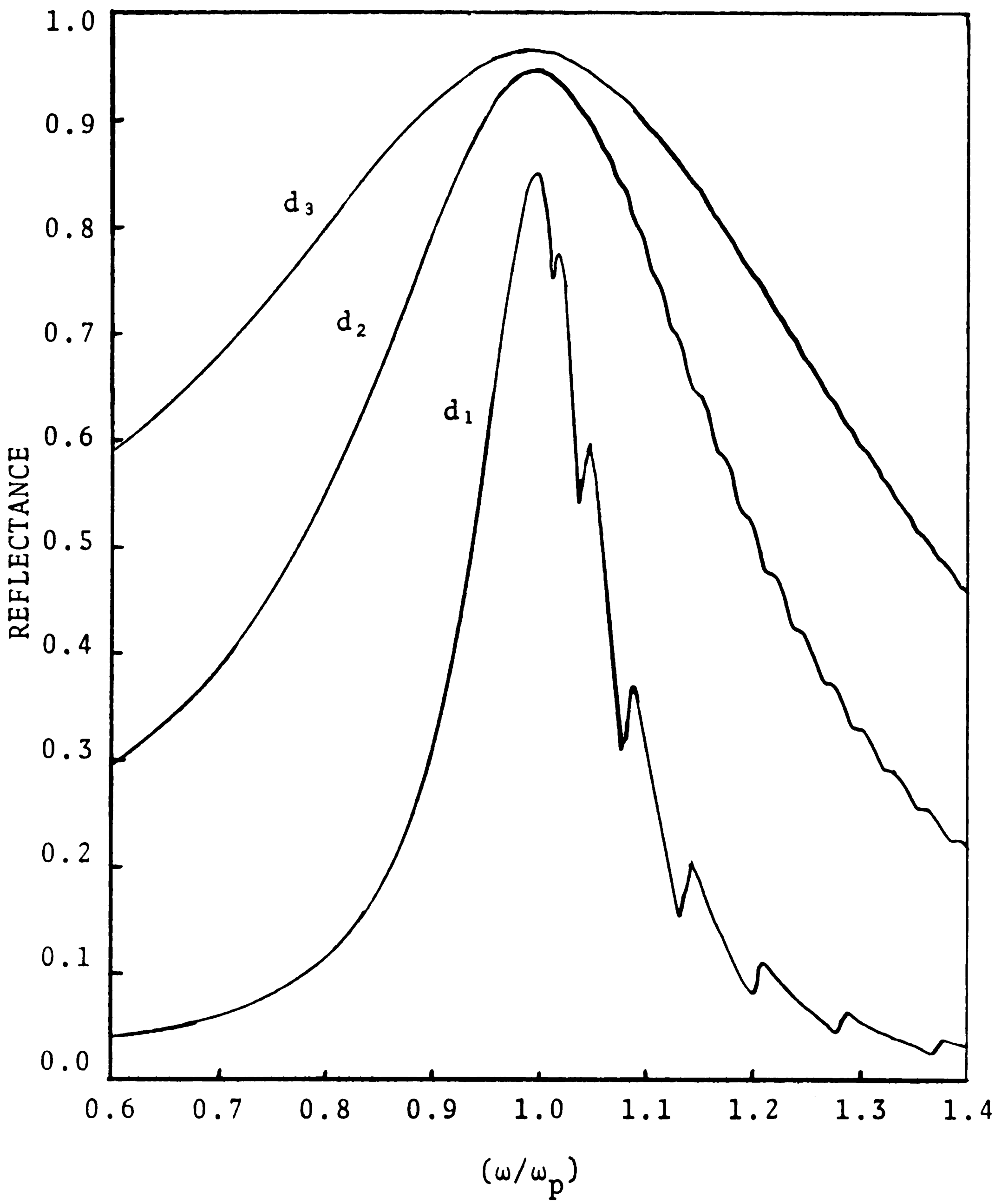

Fig. 5b. Reflectance versus frequency for 60 degree incidence and various thicknesses.

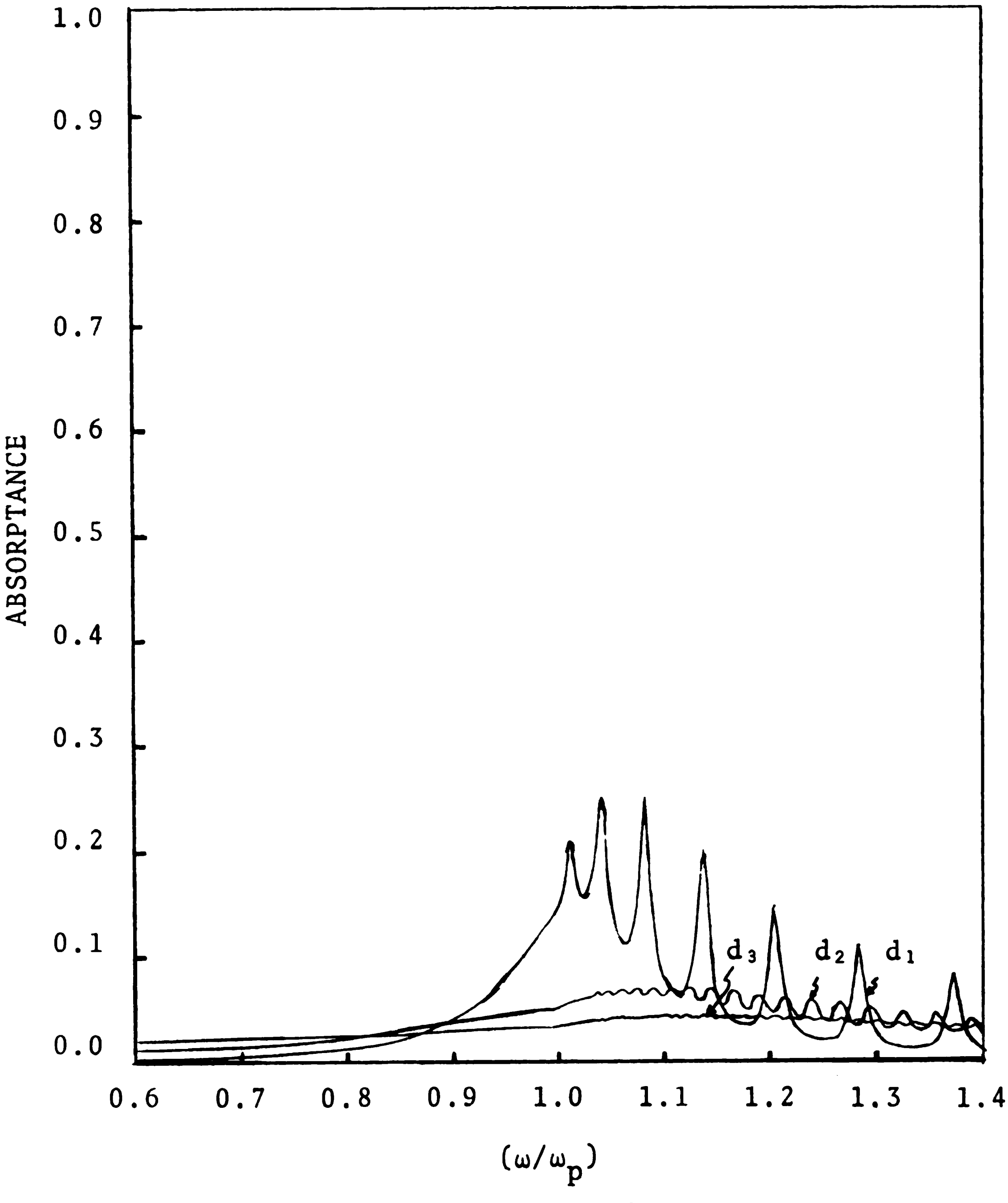

Fig. 5c. Absorptance versus frequency for 60 degree incidence and various thicknesses.

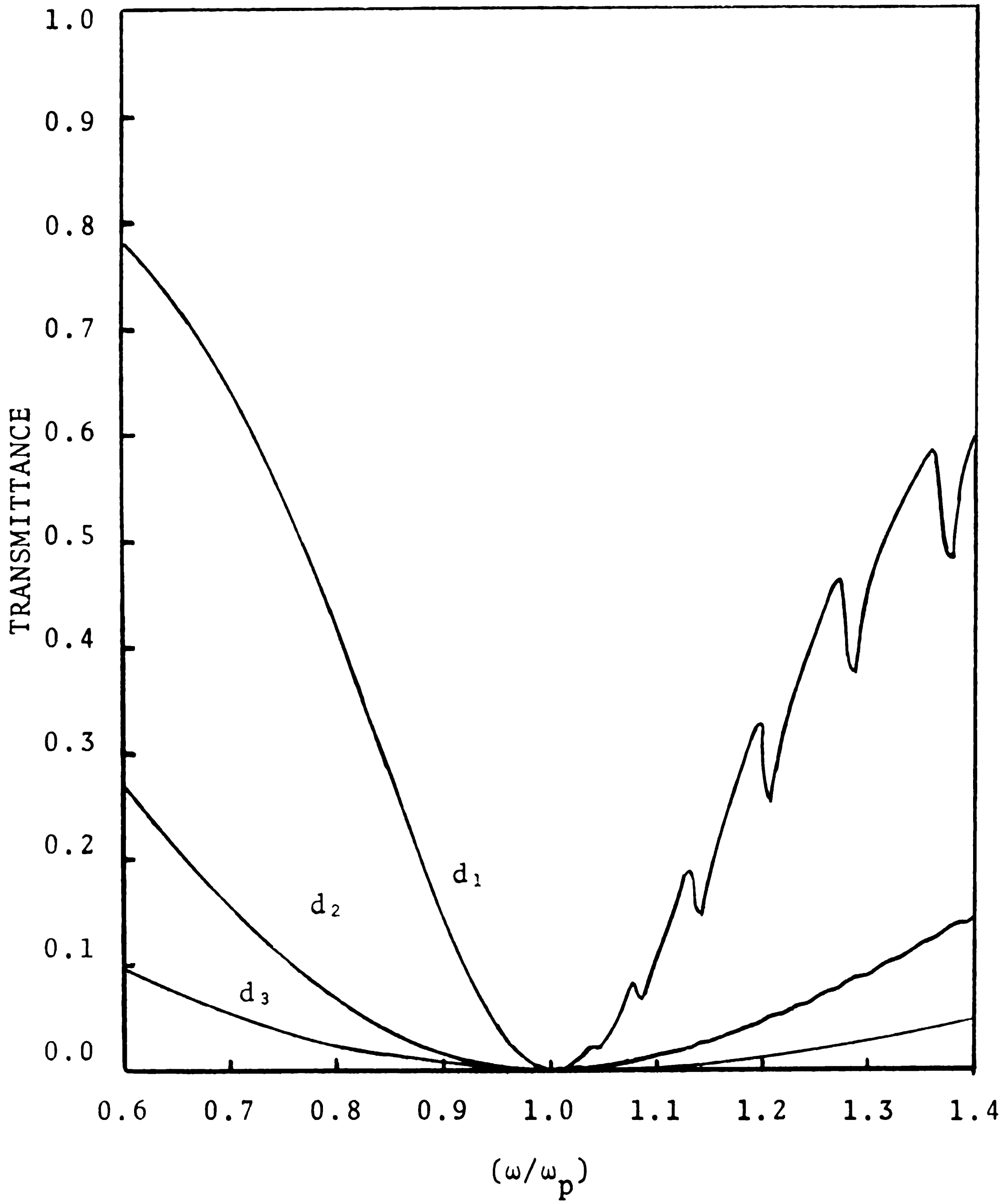

Fig. 6a. Transmittance versus frequency for 80 degree incidence and various thicknesses.

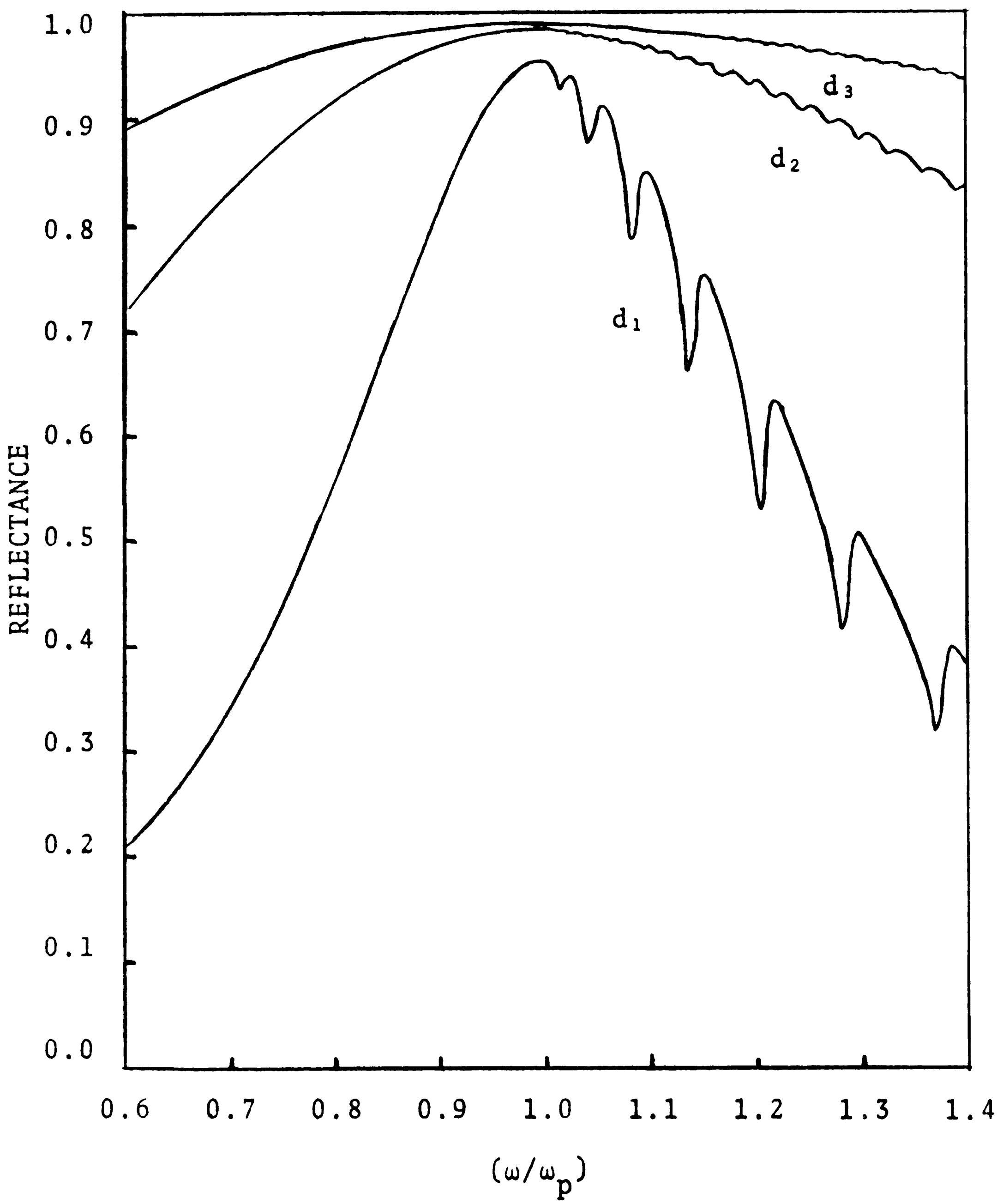

Fig. 6b. Reflectance versus frequency for 60 degree incidence and various thicknesses.

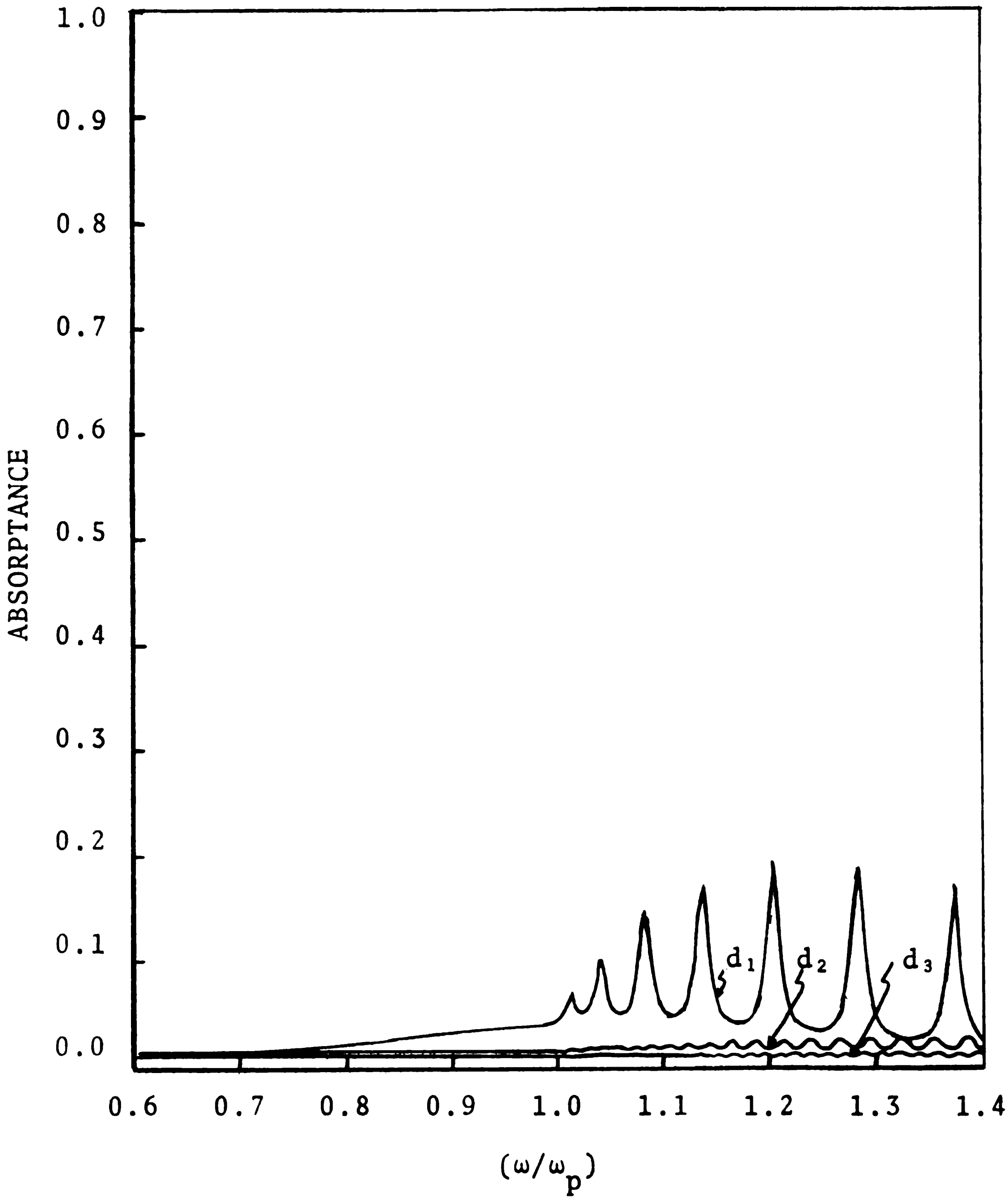

Fig. 6c. Absorptance versus frequency for 80 degree incidence and various thicknesses.

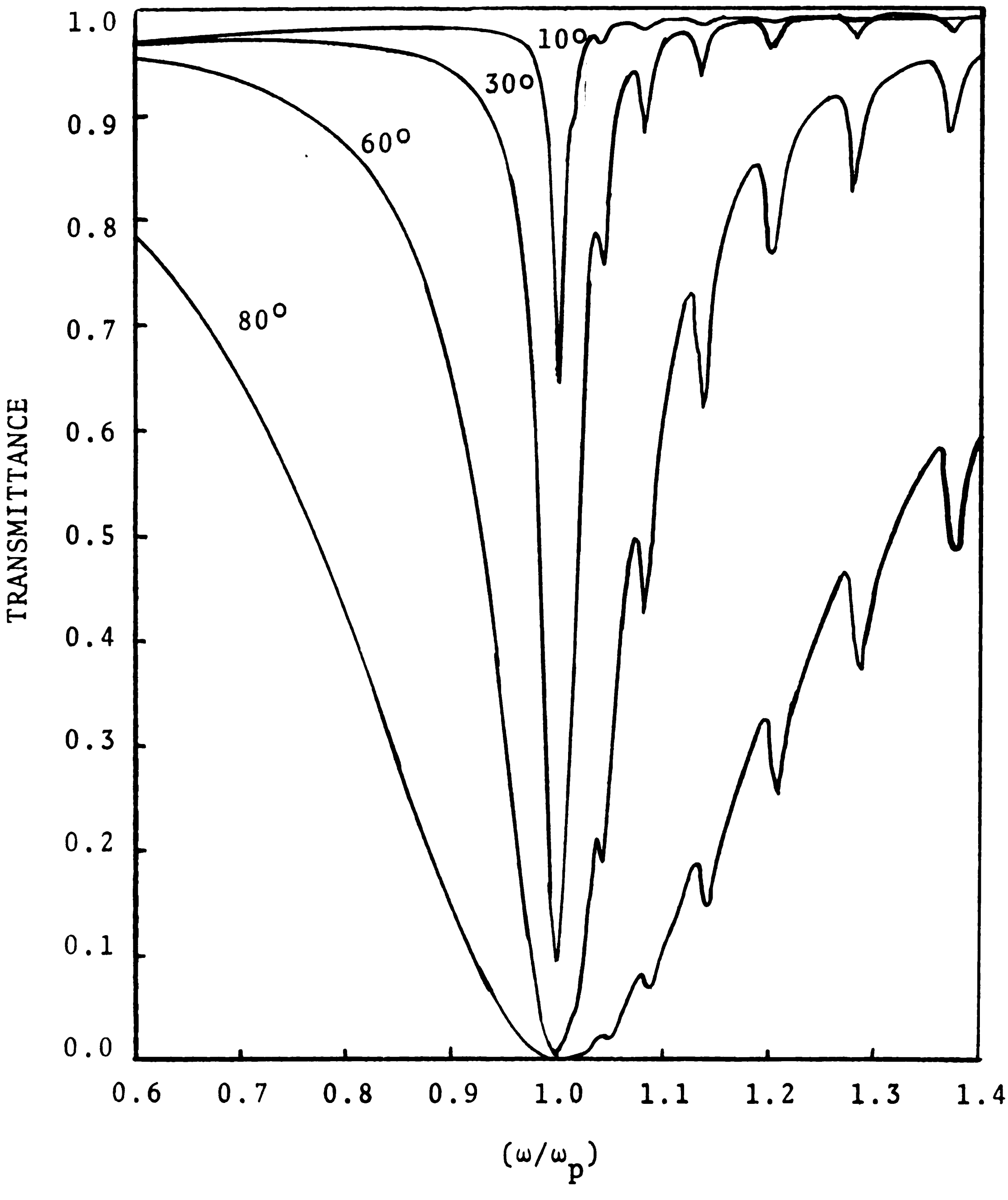

Fig. 7a. Transmittance versus frequency for slab thickness $d = 0.0305\lambda_p$ at various angles of incidence.

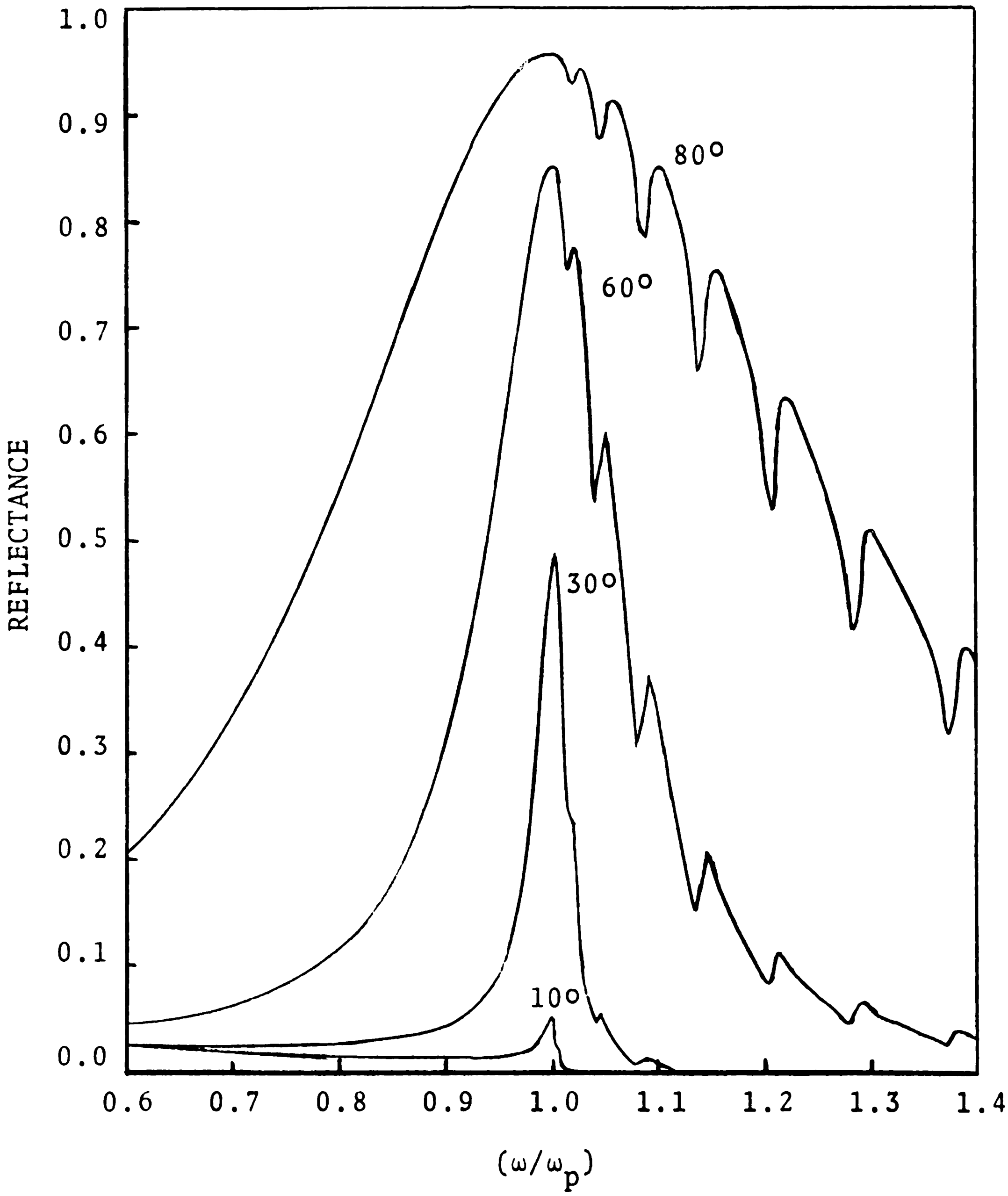

Fig. 7b. Reflectance versus frequency for slab thickness $d = 0.0305\lambda_p$ at various angles of incidence.

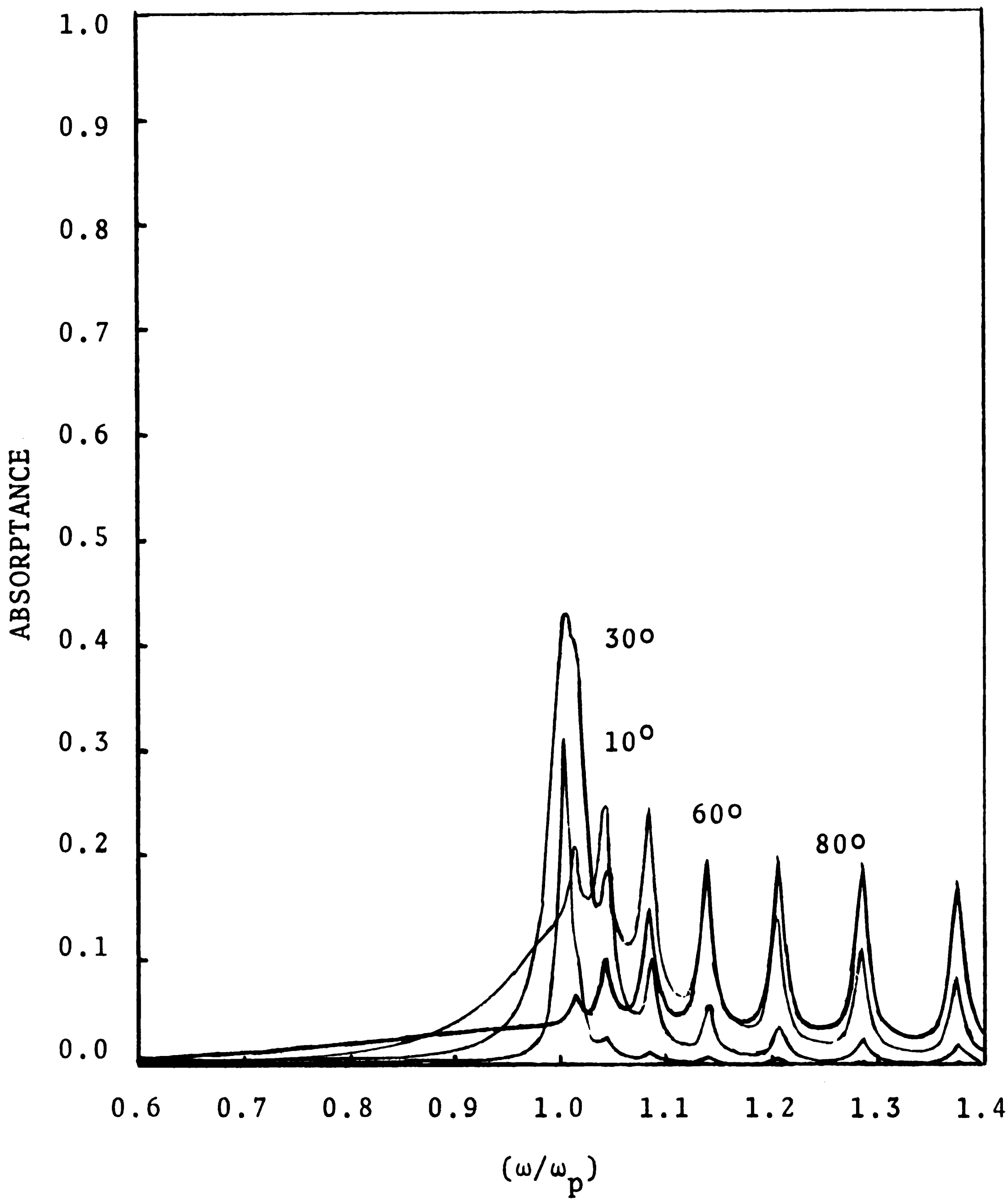

Fig. 7c. Absorptance versus frequency for slab thickness $d = 0.0305\lambda_p$ at various angles of incidence.

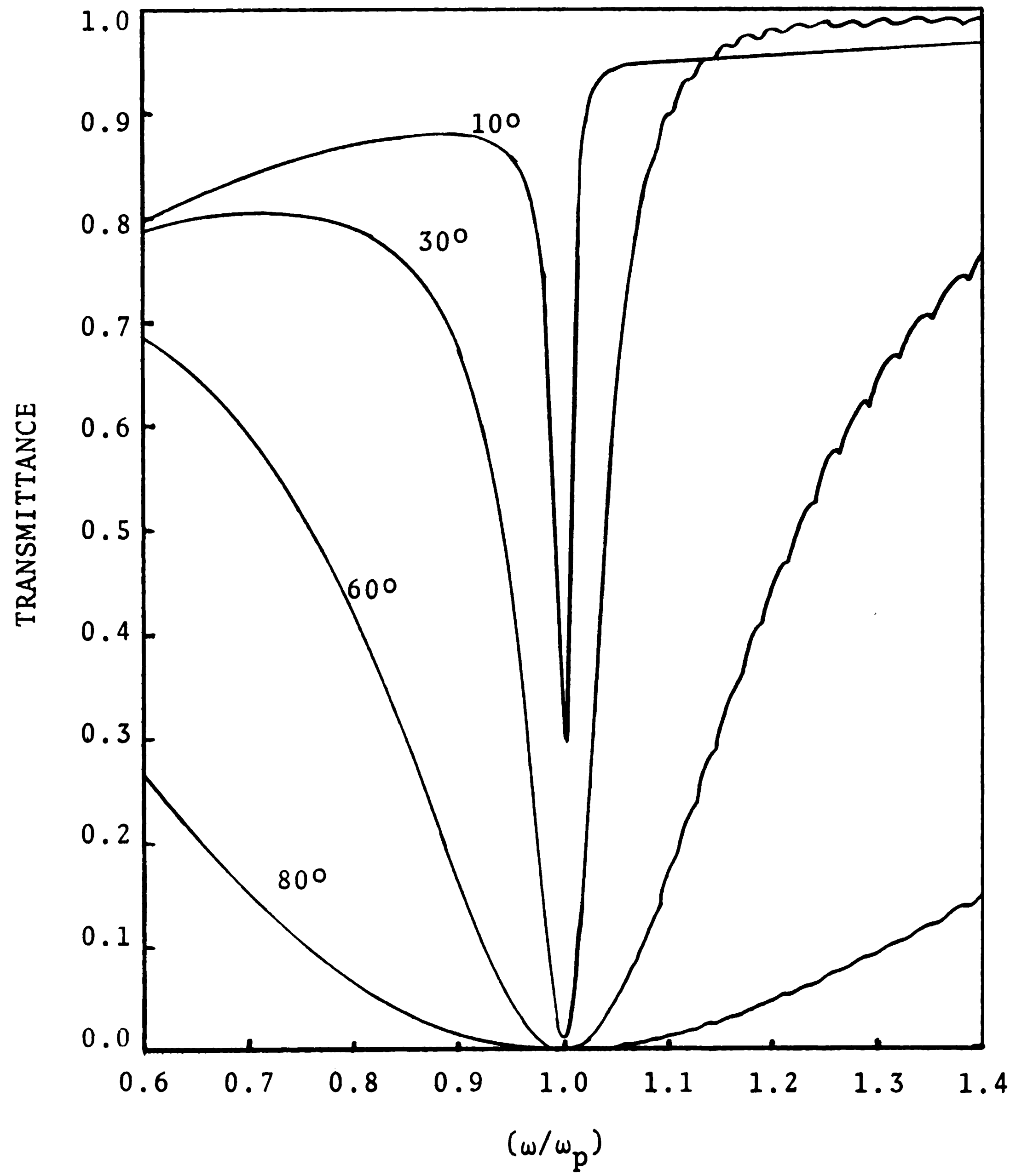

Fig. 8a. Transmittance versus frequency for slab thickness $d = 0.0915\lambda_p$ at various angles of incidence.

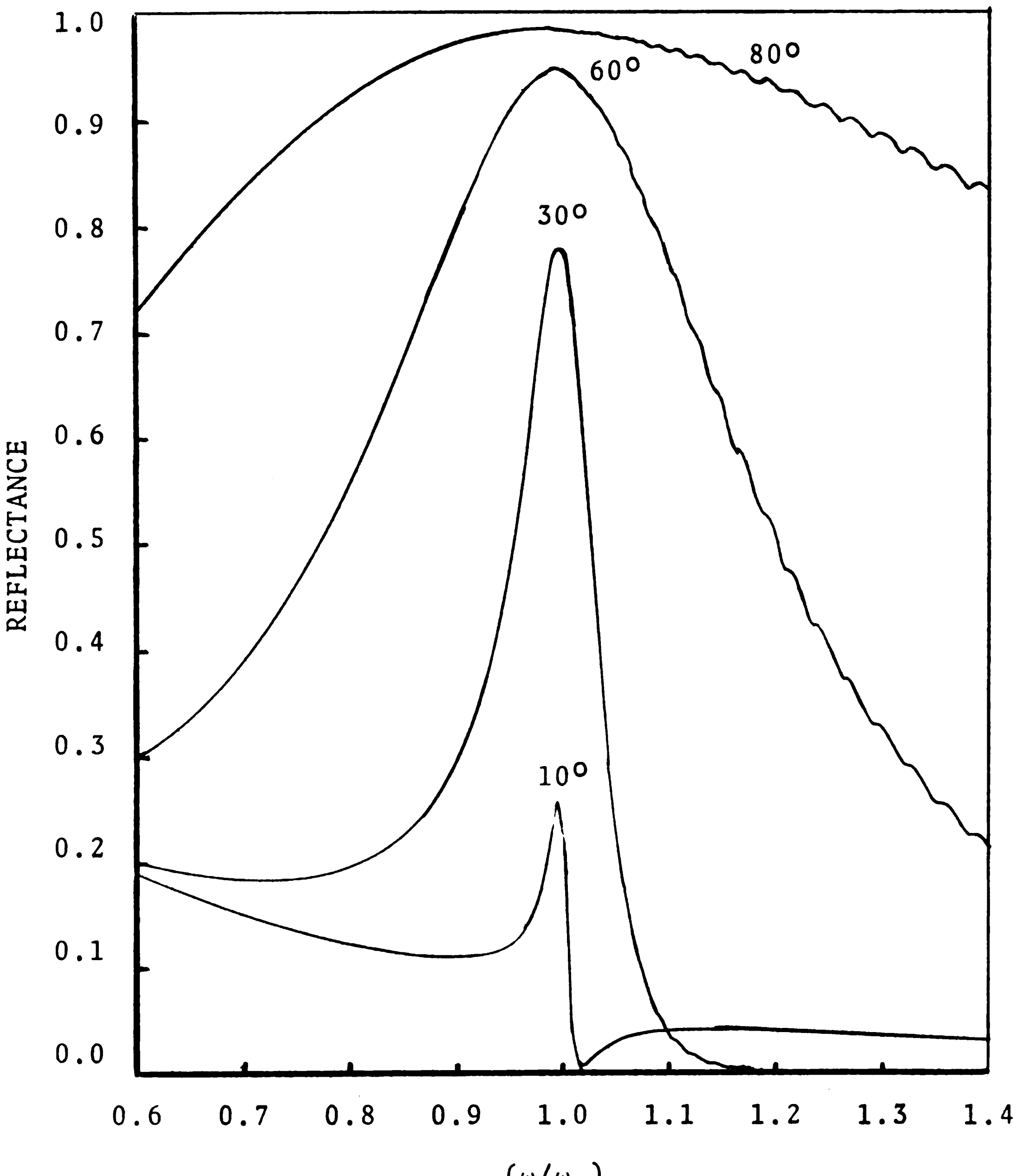

Fig. 8b. Reflectance versus frequency for slab thicknesses $d = 0.0915\lambda_p$ at various angles of incidence.

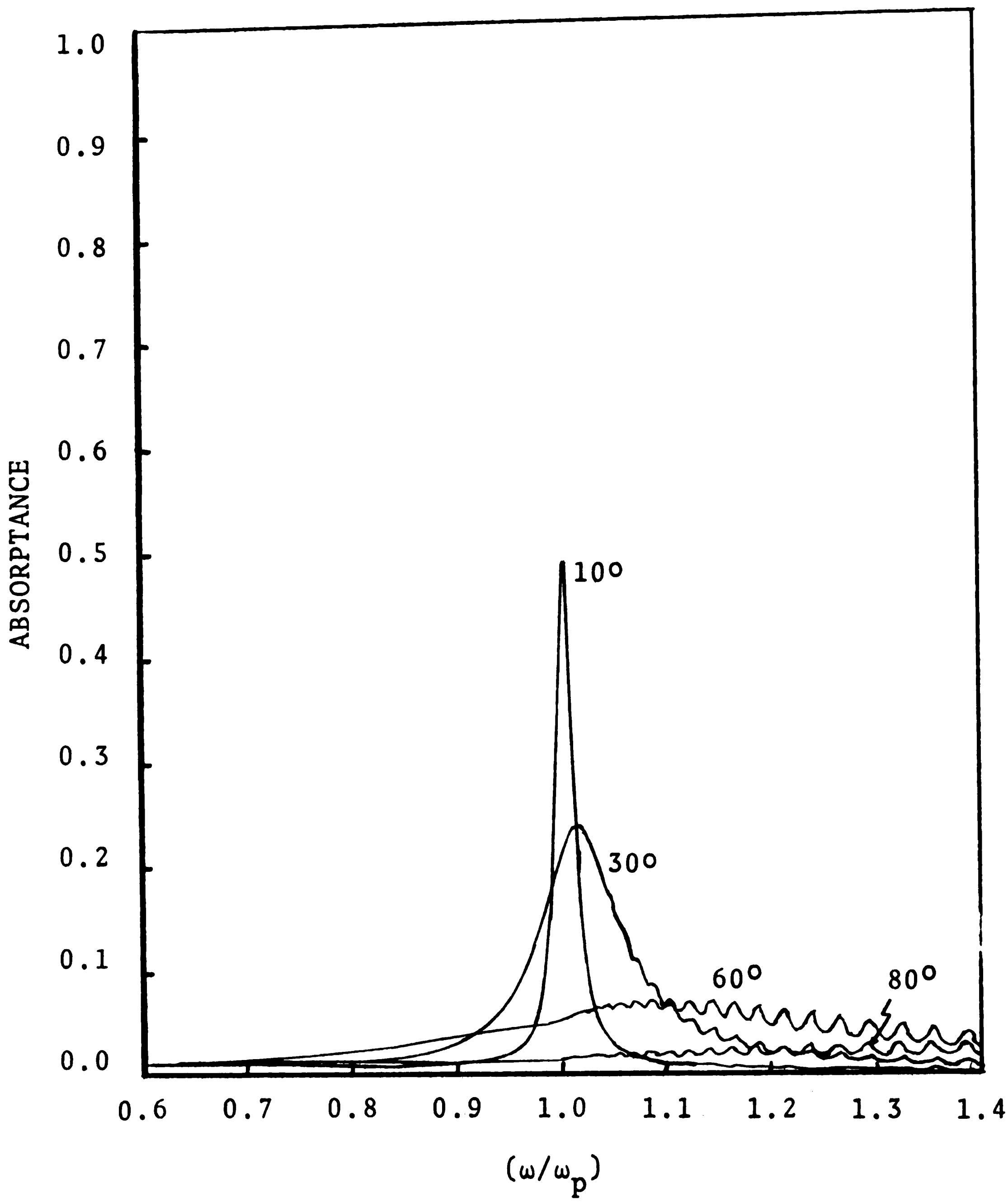

Fig. 8c. Absorptance versus frequency for slab thickness $d = 0.0915\lambda_p$ at various angles of incidence.

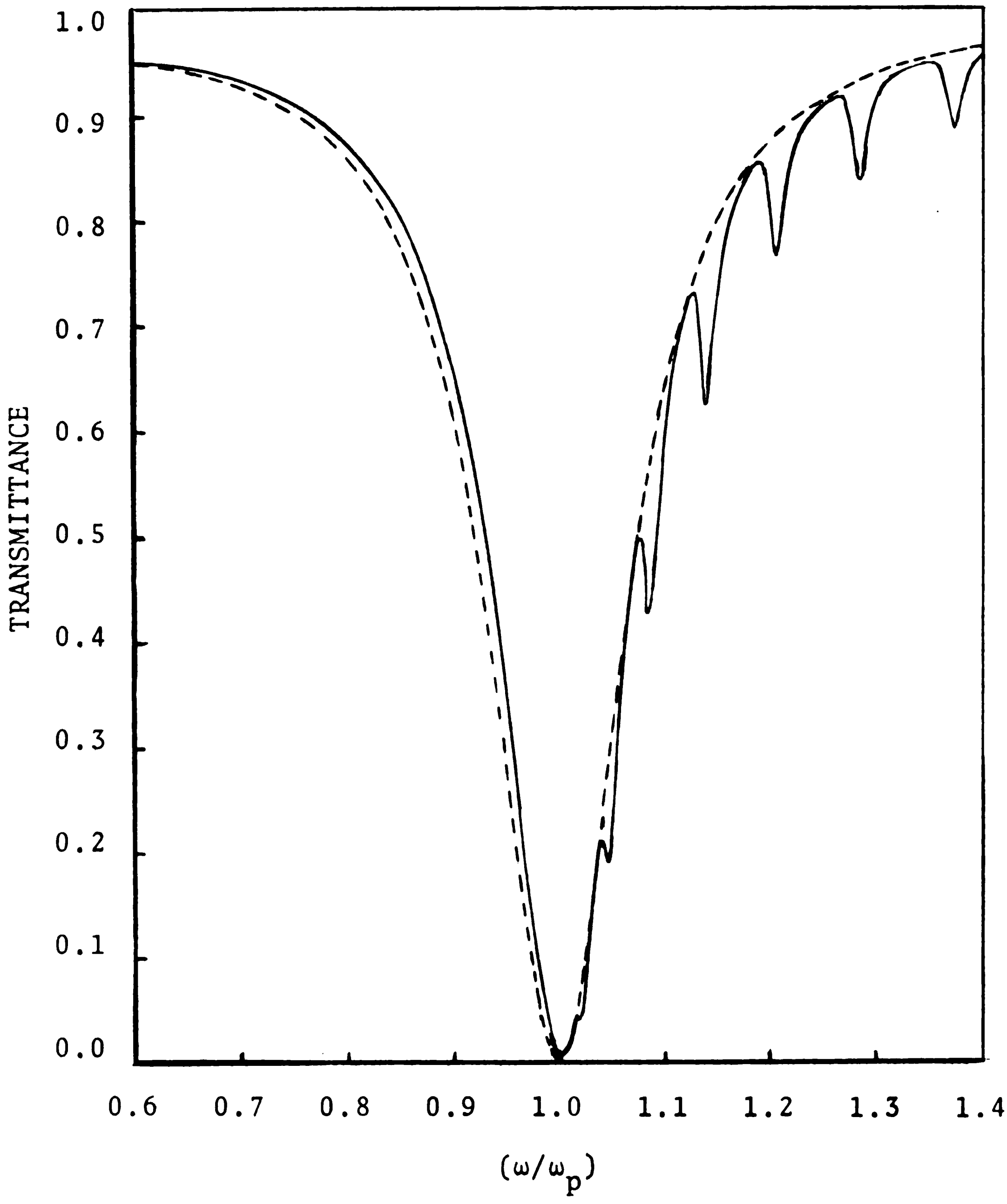

Fig. 9a. Comparison of the transmittance versus frequency calculated using the Fresnel equations (II.41), dashed curve, and the general equations, solid curve, for a thickness d = $0.0305\lambda_p$ and for 60 degree angle of incidence.

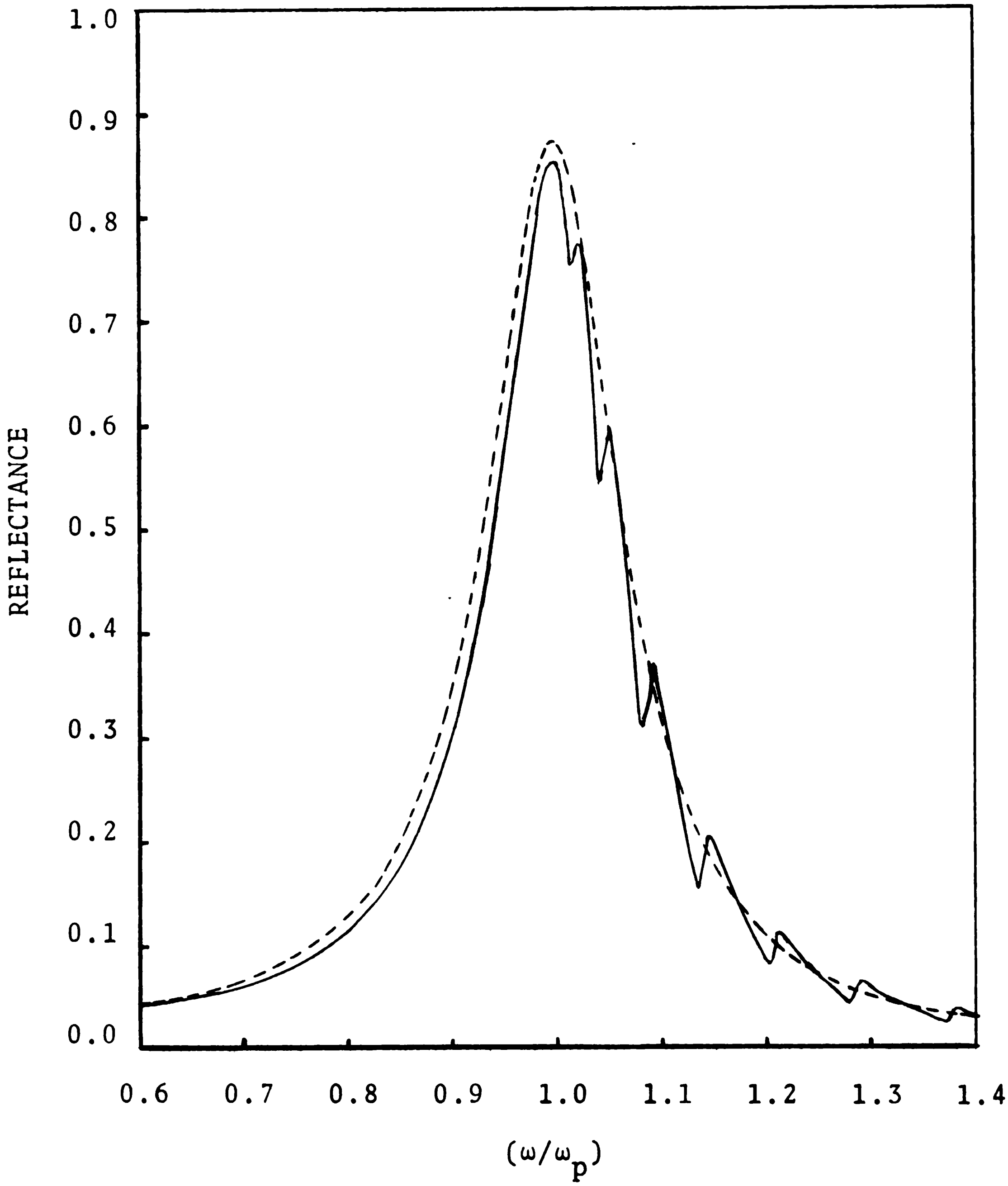

Fig. 9b. Comparison of the reflectance versus frequency calculated using the Fresnel equations (II.41), dashed curve, and the general equations, solid curve, for thickness $d = 0.0305\lambda_p$ and for 60 degree angle of incidence.

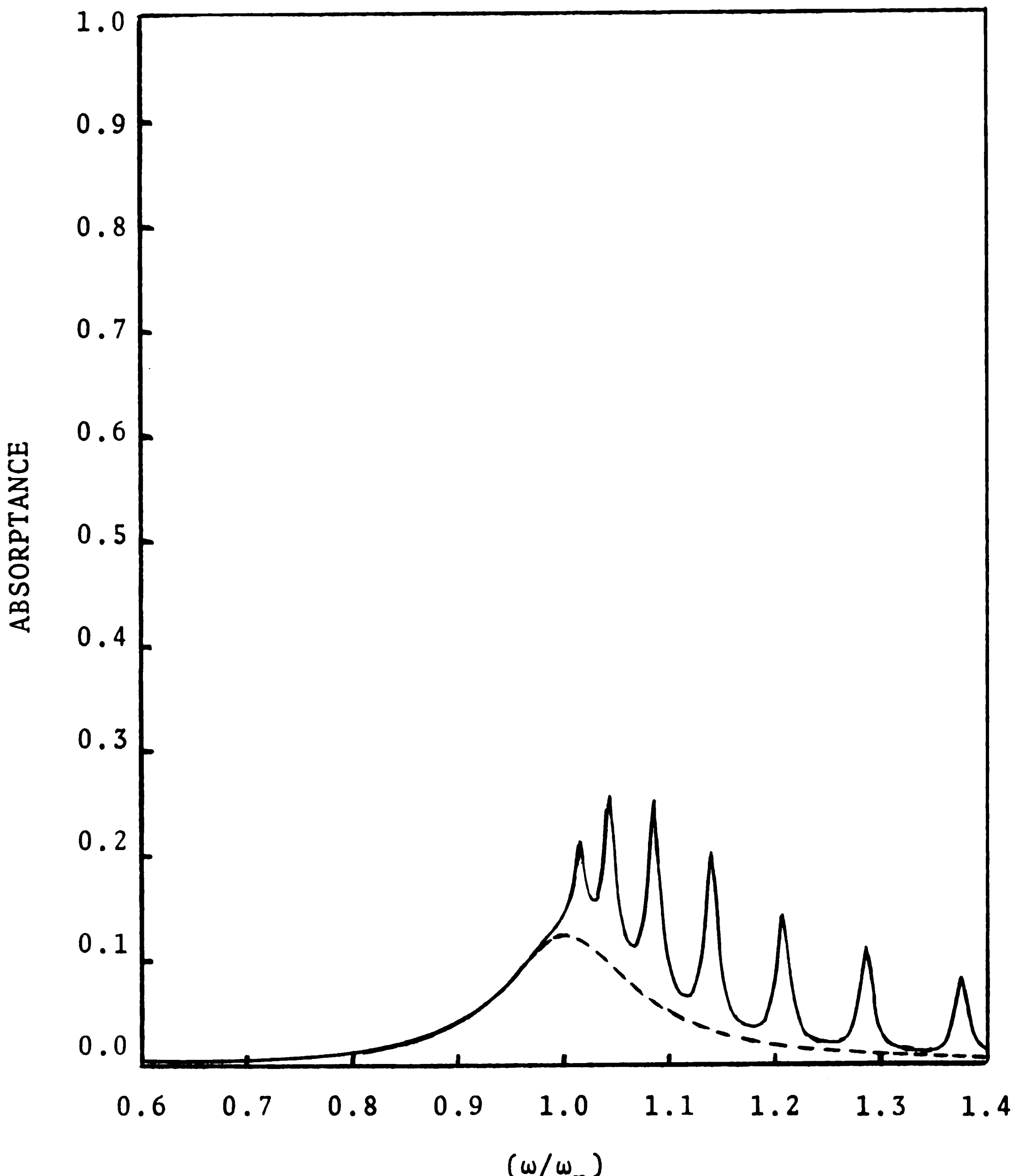

Fig. 9c. Comparison of the absorptance versus frequency calculated using the Fresnel equations (II.41), dashed curve, and the general equations, solid curve, for thickness $d = 0.0305\lambda_p$ and for 60 degree angle of incidence.

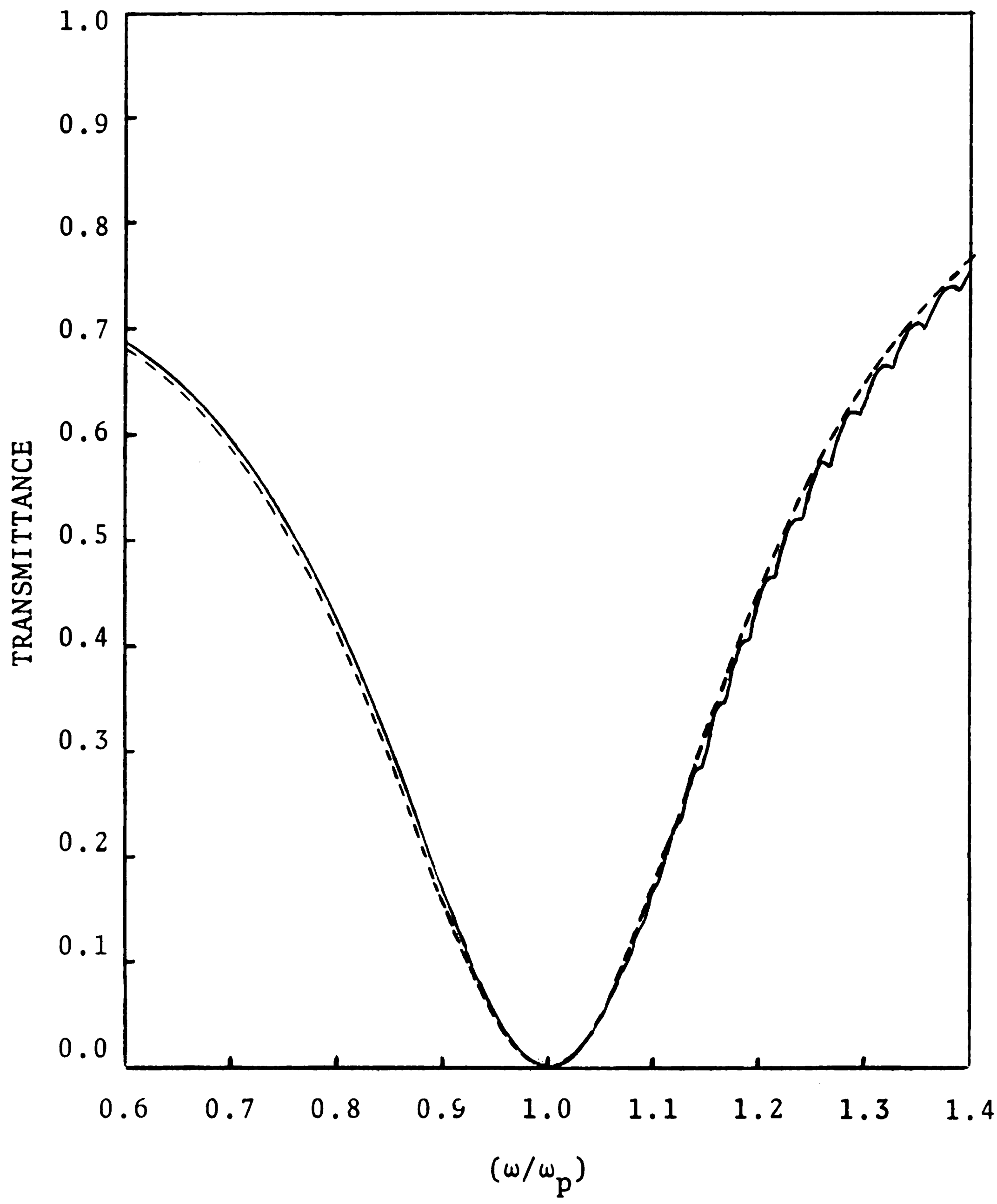

Fig. 10a. Comparison of the transmittance versus frequency calculated using the Fresnel equations (II.41), dashed curve, and the general equations, solid curve, for thickness d = $0.0915\lambda_p$ and for 60 degree angle of incidence.

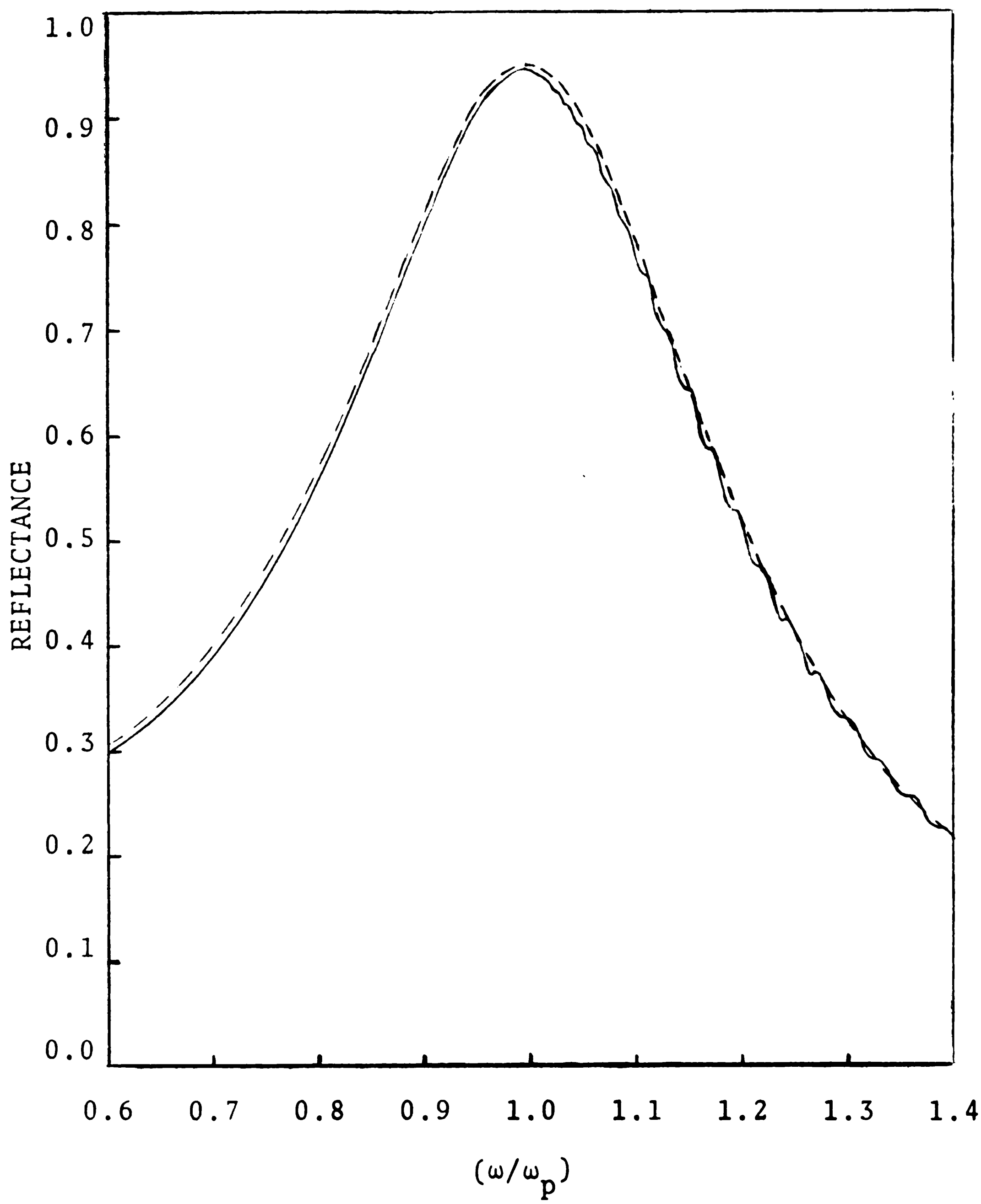

Fig. 10b. Comparison of the reflectance versus frequency calculated using the Fresnel equations (II.41), dashed curve and the general equations, solid curve, for thickness $d = 0.0915\lambda_p$ and for 60 degree incidence.

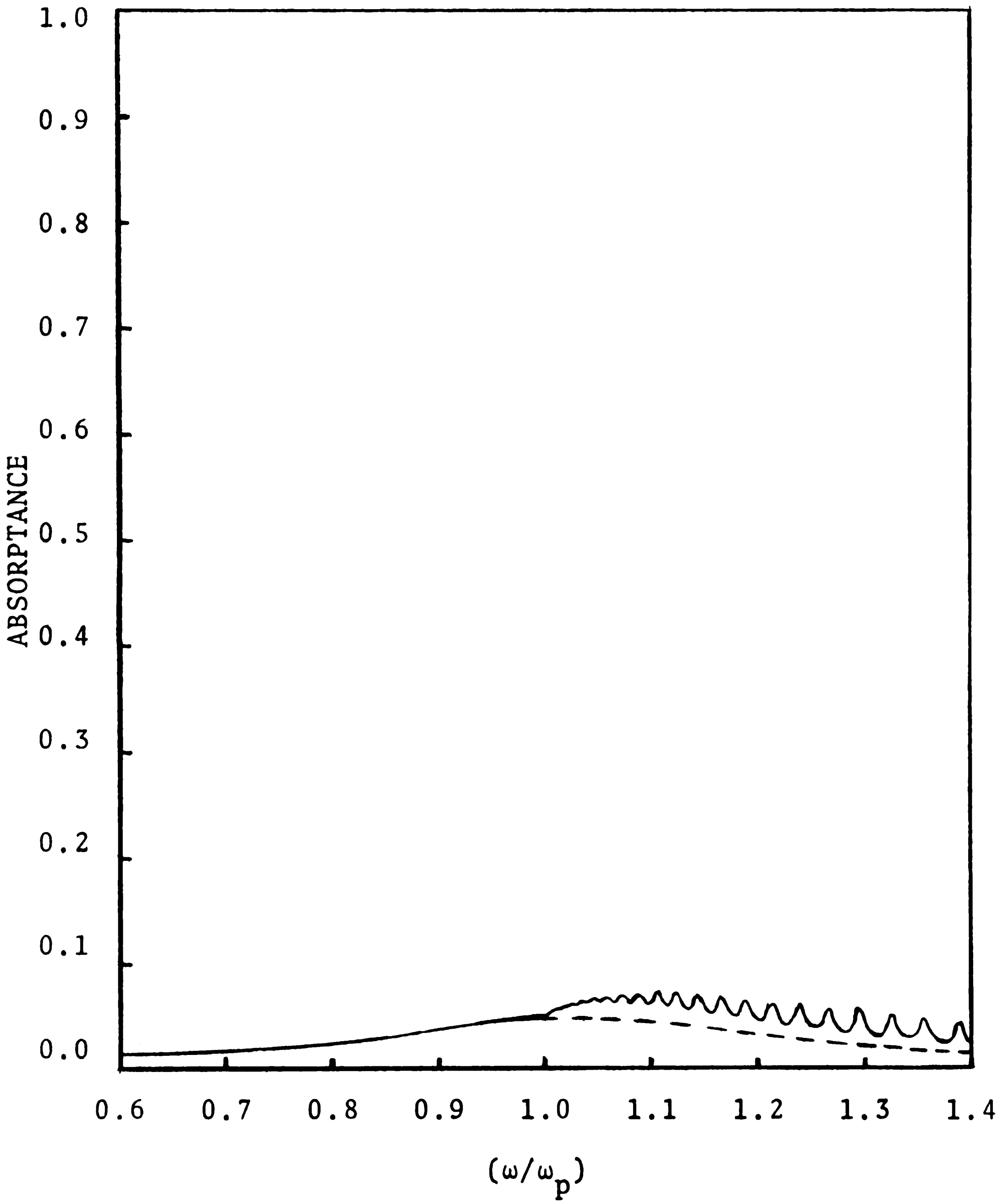

Fig. 10c. Comparison of the absorptance versus frequency calculated using the Fresnel equation (II.41), dashed curve, and the general equations, solid curve, for thickness $d = 0.0915\lambda_p$ and for 60 degree incidence.

V. DISCUSSION AND CONCLUSION

The maxima and minima structure in the reflectance, transmittance, and absorptance spectra appearing in Figs. 4 through 10 arises from spatial resonances of longitudinal plasma waves of finite wavelength propagating in a slab of finite thickness. The source and the condition for the occurrence of this structure become apparent if we approximate the equations of reflection and transmission IV.7 under the condition that the wavelength of the em wave is larger than the slab thickness. Then $\phi_t = \exp(i\underline{n}\cdot\underline{k}_t d)$ is nearly unity, and

$$N_r \simeq |AD - BC|(1 - \phi_\ell)^2$$

$$= -4\gamma\beta(1 - \phi_\ell)^2$$

$$N_t \simeq 4\alpha\beta(1 - \phi_\ell^{\,2}) \qquad (V.1)$$

$$M \simeq \left(D^2 - A^2\phi_\ell - B^2 - C\phi_\ell\right)(1 - \phi_\ell)$$

$$= 4\beta\left(\alpha(1 + \phi_\ell) + \gamma(1 - \phi_\ell)\right)(1 - \phi_\ell)$$

Neglecting damping, so that the waves are homogeneous and α and γ (Eqs. II.16 and II.24) are real, we find R and T

(Eqs. IV.6):

$$R = \frac{\gamma^2(1 - \cos\psi)}{\Delta}$$

$$T = \frac{\alpha^2(1 + \cos\psi)}{\Delta} \qquad (V.2)$$

where

$$\Delta = \alpha^2(1 + \cos\psi) + \gamma^2(1 - \cos\psi) \qquad (V.3)$$

and

$$\psi = \mathrm{Re}(\underline{n}\cdot\underline{k}_\ell)d \qquad (V.4)$$

Clearly, even if $\gamma << d$, the reflectance (transmittance) will have relative maxima (minima) whenever

$$\psi = n\pi; \quad n = 1, 3, 5, \ldots \qquad (V.5)$$

i.e., whenever the slab thickness equals an odd number of half wavelengths of the longitudinal wave. The resonance condition can be visualized as a standing polarization wave, with the volume charge density developed in the region of the boundary coupling the incident em wave to the polarization wave.

Superimposed in these relative maxima (minima) is one large maximum (minimum) due to the em wave being refracted into a surface wave. This effect increases with angle of incidence as seen in Figs. 7 and 8.

It is interesting to note that our expressions for the reflectance and transmittance V.2, reduce to the corresponding equations of McAlister and Stern[25] near the plasma frequency. Near ω_p, ψ becomes small so

$$\begin{aligned} \phi_\ell &\simeq 1 + i\psi \\ &= 1 + i\left(\frac{ck_\ell}{\omega}\right)_n \frac{2\pi d}{\lambda} , \end{aligned} \tag{V.6}$$

and the expressions for R and T become

$$\begin{aligned} R &\simeq \left|\frac{i\gamma\psi}{2\alpha - i\gamma\psi}\right|^2 \\ &= \left|\frac{i\frac{2\pi d}{\lambda}\frac{\sin^2\theta}{\cos\theta}(1-\varepsilon)}{\Delta}\right|^2 \\ T &\simeq \left|\frac{2\alpha}{2\alpha - i\gamma\psi}\right|^2 \\ &= \left|\frac{2\varepsilon}{\Delta}\right|^2 \end{aligned} \tag{V.7}$$

where

$$\Delta = 2\varepsilon - i\,\frac{2\pi d}{\lambda}\,\frac{\sin^2\theta}{\cos\theta}(1-\varepsilon) \tag{V.8}$$

Eqs. V.7 differ from the corresponding expressions of McAlister and Stern (Eqs. (3) and (4) in their paper) by the factor $1 - \varepsilon \approx (\omega_p/\omega)^2$, which approaches unity near the plasma frequency.

Our results indicate that the wavelength of the longitudinal high frequency plasma wave in a metallic plasma such as an alkaline or noble metal may be measured by an optical experiment, provided the metallic foil is quite thin. Silver, for example, whose conduction electrons have a Fermi velocity of 1.4×10^8 cm/sec, has a plasma frequency of 5.8×10^{15} sec^{-1}, corresponding to a plasma wavelength λ_p of 3280 Angstrom units in thickness. For comparison, the screening length for the silver plasma, $\lambda_p \approx v_F/\omega_p$, is approximately 50 Angstrom units. Thus the spectrum of the thinnest foil, which provides the most resolved resonance peaks, may not be applicable since the wavelengths are approaching the limiting screening length. On the other hand such an experiment may provide information concerning the critical cut off wavelength. Another experimental difficulty may be the collision frequency. Although the value $\omega_p\tau = 100$ is well within the value $\omega_p t = 233$ obtained from mobility measurements in silver, these measurements were made in large, pure samples and may not apply to evaporated films.

A calculation was also made of the reflectance and transmittance by the acoustic wave in a two component plasma. But because the phase velocity of the acoustic wave is much smaller than that of the high frequency mode no measurable effects were obtained. Furthermore, the acoustic waves are strongly Landau damped even without collisions. Thus experiments, such as those of McWhorter and May,[32] to excite acoustic plasma waves by electromagnetic radiation seem very unlikely. The situation may be improved by applying static electric fields to the plasma to amplify the acoustic wave.[33] But as a result of the anisotropy produced by such a field the irrotational and divergence free waves are no longer independent, so that the present theory would no longer be applicable.

Using the method described in Chapter II, we have also calculated the case of longitudinal plasma waves incident on a boundary and emitting em waves. This problem was treated by Field[15] and our equation for the ratio of the em wave electric field to the plasma wave electric field agreed with the corresponding equation in Field's paper. This is not surprising, since Field also describes the real boundary as having a finite transition region of the order of the screening length and he uses the continuity of the normal component of the electric field for a boundary condition. Our results, however, are more general since we consider a dielectric plasma boundary and inhomogeneous waves, which are more realistic for a lossy system such as a plasma.

In summary, we have studied the linear coupling between longitudinal polarization waves and transverse electromagnetic waves introduced by the existence of plane boundaries separating dielectric media or vacuum from conducting media capable of supporting polarization waves. We have investigated this coupling by extending the usual Fresnel equations to include the effects of irrotational fields. Because the theory was developed in terms of the plasma dispersion relations a means has been suggested for studying experimentally the wavelength dependence of plasma waves. We have also shown that the dispersion relations for inhomogeneous waves in the plasma are still separable into electromagnetic-like and polarization waves. Finally, absorptance, reflectance, and transmittance spectra were numerically calculated and the results indicate that the dispersion relations for plasma waves in thin metallic films may be optically measurable.

APPENDIX A.

EVALUATION OF $\underline{\underline{T}}$ AND $\underline{V}$

In the spherical coordinate system, Fig. 11, the integrals

$$\underline{\underline{T}} = \int \frac{d\Omega \hat{\underline{r}}\,\hat{\underline{r}}}{1+i\underline{a}\cdot\hat{\underline{r}}} \tag{A.1}$$

$$\underline{V} = \int \frac{d\Omega \hat{\underline{r}}}{1+i\underline{a}\cdot\hat{\underline{r}}} \tag{A.2}$$

become

$$\underline{\underline{T}} = \int_0^{2\pi} d\phi \int_{-1}^{+1} \frac{du\,\hat{\underline{r}}\,\hat{\underline{r}}}{1 + ia_z u + ia_x\sqrt{1-u^2}\cos\phi} \tag{A.3}$$

$$\underline{V} = \int_0^{2\pi} d\phi \int_{-1}^{+1} \frac{du\,\hat{\underline{r}}}{1 + ia_z u + ia_x\sqrt{1-u^2}\cos\phi} \tag{A.4}$$

For homogeneous waves, $a_x = 0$ and A.3 and A.4 take on a simple form which is easily evaluated:

$$T_{xx} = T_{yy} = \frac{2\pi}{a_z^{\,2}}\left[\frac{1+a_z^2}{a_z}\arctan a_z - 1\right]$$

$$T_{zz} = \frac{4\pi}{a_z^{\,2}}\left[1 - \frac{\arctan a_z}{a_z}\right] \tag{A.5}$$

and

$$V_z = \frac{4\pi}{ia_z}\left[1 - \frac{\arctan a_z}{a_z}\right] \tag{A.6}$$

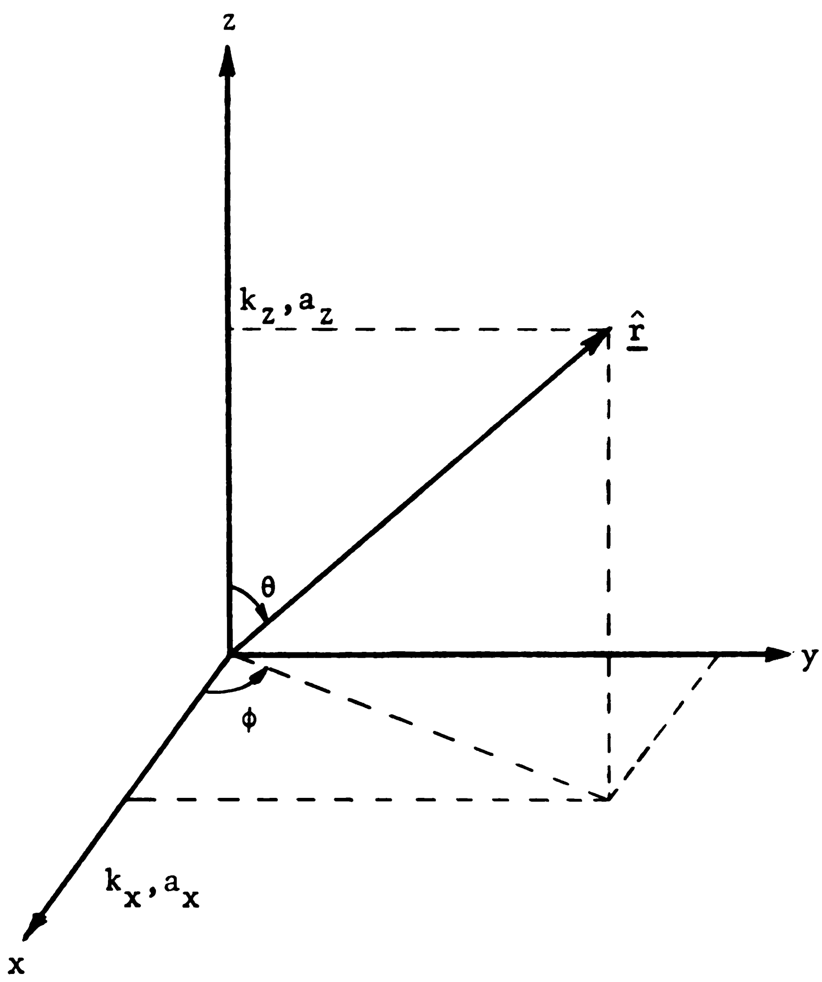

Fig. 11. Coordinate system defining direction of tensor components.

all other components being zero.

To evaluate A.3 and A.4 for the general inhomogeneous wave case we need the value of the integral

$$I = \int_0^{\pi} \frac{dx}{v + w\cos x} \qquad \text{(A.7)}$$

The indefinite integral of A.7 has the following values[34]

$$(v^2 - w^2)^{-\frac{1}{2}} \arccos\left(\frac{w + v\cos x}{v + w\cos x}\right) \quad v^2 > w^2$$

$$v^{-1} \tan(x/2) \qquad v = w$$

$$-v^{-1} \cot(x/2) \qquad v = -w$$

$$(w^2 - v^2)^{-\frac{1}{2}} \ln\left[\frac{w + v\cos x + \sqrt{w^2 - v^2}\sin x}{v + w\cos x}\right] \quad w^2 > v^2$$

Therefore the integral A.7 is defined as long as v^2 is unequal to w^2

$$I = \frac{\pi}{\sqrt{v^2 - w^2}}, \quad v^2 \neq w^2 \qquad \text{(A.8)}$$

From A.8 we find

$$\int_0^{2\pi} \frac{d\phi}{v + w\cos\phi} = \frac{2\pi}{\sqrt{v^2 - w^2}}$$

$$\int_0^{2\pi} \frac{\cos\phi d\phi}{v+w\cos\phi} = \frac{2\pi}{w}\left[1 - \frac{v}{\sqrt{v^2-w^2}}\right]$$

$$\int_0^{2\pi} \frac{\cos^2\phi d\phi}{v+w\cos\phi} = \frac{2\pi}{w^2}\left[\frac{v^2}{\sqrt{v^2-w^2}} - v\right] \tag{A.9}$$

$$\int_0^{2\pi} \frac{\sin\phi \cos^n\phi d\phi}{v+w\cos\phi} = 0$$

$$\int_0^{2\pi} \frac{\sin^2\phi d\phi}{v+w\cos\phi} = \frac{2\pi}{w^2}\left\{v - \sqrt{v^2-w^2}\right\}$$

We now extend the integrals A.9 to include complex v and w and state without proof that A.9 exists as long as the absolute value of U does not vanish, where

$$U = v^2-w^2 = 1 + a_x{}^2 + 2ia_z u - (a_x{}^2 + a_z{}^2)u \tag{A.10}$$

The value of the first three integrals of the form

$$I_n = \int_{-1}^{+1} \frac{u^n du}{\sqrt{U}}$$

can be expressed as

$$I_0 = \frac{2\arctan a}{a}$$

$$I_1 = -2i\frac{a_z}{a^2}\left[1 - \frac{\arctan a}{a}\right] \tag{A.11}$$

$$I_3 = \frac{2a_z^2 - a_x^2}{a^4} - \frac{2a_z^2 - a_x^2(1+a^2)}{a^5}\arctan a$$

where

$$a^2 = \underline{a}\cdot\underline{a} = a_x^2 + a_z^2 \tag{A.12}$$

With A.9 and A.11 the desired integrals may now be evaluated:

$$T_{xx} = 2\pi\left[\frac{a_z^2(1+a^2) - 2a_x^2}{a^5}\arctan a - \frac{a_z^2 - 2a_x^2}{a^4}\right]$$

$$T_{xz} = T_{zx} = 2\pi\frac{a_x a_z}{a^4}\left[3 - \frac{3+a^2}{a}\arctan a\right]$$

(A.13)

$$T_{yy} = \frac{2\pi}{a^2}\left[\frac{1+a^2}{a}\arctan a - 1\right]$$

$$T_{zz} = 2\pi\left[\frac{2a_z^2 - a_x^2}{a^4} - \frac{2a_z^2 - a_x^2(1+a^2)}{a^5}\arctan a\right]$$

$$T_{xy} = T_{yx} = T_{yz} = T_{zy} = 0$$

and

$$V_x = \frac{4\pi}{i} \frac{a_x}{a^2} \left[1 - \frac{\arctan a}{a}\right]$$

$$V_y = 0 \tag{A.14}$$

$$V_z = \frac{4\pi}{i} \frac{a_z}{a^2} \left[1 - \frac{\arctan a}{a}\right]$$

with the condition that

$$|a| \neq 1 \tag{A.15}$$

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