

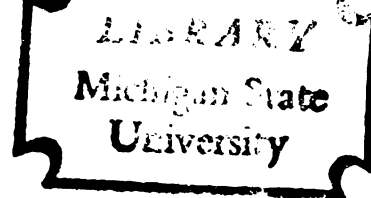
DYNAMICAL EFFECTS OF ISOTROPIC RADIATION

Thesis for the Degree of Ph. D.

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David Francis Savickas

1966



This is to certify that the

thesis entitled

DYNAMICAL EFFECTS OF ISOTROPIC RADIATION

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ABSTRACT

DYNAMICAL EFFECTS OF ISOTROPIC RADIATION

by David Francis Savickas

A small spherical body traveling through an isotropic radiation field will generally experience a force caused by the momentum it absorbs from the radiation. Due to the Doppler shift, the radiation striking the front of the particle will be of higher frequency than the radiation striking the back of the particle. Also, because of diffraction effects, a small particle will absorb amounts of momentum that differ with the radiation's wavelength. These two factors generally combine to exert a force on the particle and change its velocity. The effect of this force on the particle's motion was investigated.

By the use of relativistic mechanics a general expression for the force on the particle due to a plane-parallel beam of radiation was obtained and then integrated over space to obtain the total force of the whole radiation field. The diffraction effects were taken into account by use of either Debye's approximations for the pressure efficiency factors of very small particles, or by the use of graphs of this factor as a function of frequency.

For approximations at low velocities it was found that the ratio of the initial velocity to the velocity at a later time is independent of the initial velocity. Generally the radiation exerts a retarding force on the motion of the particle. However, for special cases where an essentially monochromatic and isotropic field exists, it was found that the particle could be accelerated to higher velocities by the radiation. Numerical estimates for the time it would take a particle in interstellar space to come to $1/e$ its initial velocity were found to be of the order of from 10^{10} to 10^{11} years.

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OF ISOTROPIC RADIATION

By

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INTRODUCTION

A plane wave of energy density u normally incident upon a perfectly absorbing plane surface of area A will exert a force $F = u A$. The radiation's pressure is seen then to be equal to its energy density. The effects of radiation pressure on the motion of particles in space has been previously investigated for places in space where a single source, such as the sun or a star, exerts a strong influence on a particle's motion. However the effects of an isotropic radiation field on a particle's motion have not been previously investigated.

When a particle is at rest in such a field it obviously experiences no radiation force by reason of symmetry. If, however, it is in motion then the radiation incident on the particle in the direction opposite to its velocity will have a higher frequency than the radiation incident on it which is in the direction of its velocity. This difference is due to the Doppler shift and it is the effect of this frequency shift which we wish to investigate. Since the particles in space are small, radiation diffraction effects are important and also will be considered.

CHAPTER I

HISTORICAL BACKGROUND

The existence of radiation pressure was first deduced as a consequence of the electromagnetic theory by Maxwell, and later experimentally confirmed by Lebedew. The effects of radiation pressure exerted by the sun's radiation on spherical particles in space was first considered by J. H. Poynting¹ in 1903. His calculations led to the incorrect result that a body moving through space and receiving no radiation from other sources would suffer a decrease in velocity due to its own emission of radiation as its temperature cooled. This effect was presumed to exist because the radiation was not emitted isotropically about the particle, but rather was concentrated in the forward direction and thinned out in the backward direction. Hence, since radiation carries momentum in the direction emitted, more momentum is lost by the particle in the forward direction than in the backward direction. Thus Poynting's results show that the particle's velocity decreases.

However a detailed investigation by L. Page was made years later showing that, to the order of accuracy questioned, a moving body does not experience a decrease in velocity as a

consequence of its own radiation. Using Page's work J. Larmor² pointed out the error in Poynting's results in 1918. He noted that the loss of momentum did not necessitate a decrease in velocity but rather, by the relation $m = E/c^2$, a decrease in the mass of the particle and concluded that the particle "will move on with constant velocity, but with diminishing momentum so long as it has energy to radiate." However, Larmor's own corrections for the radiation pressure were also incorrect.

It was not until 1937, when H. P. Robertson^{3a} considered the problem, that the matter was cleared up. He made a rigorous relativistic derivation of the radiation pressure caused by a plane-parallel beam of light and then used the simpler form of the classical approximation of this relativistic result to obtain the equation of motion for a particle in the field of the sun. Robertson's results showed the existence of a drag of the same kind which Poynting and Larmor had predicted. His expressions for this force on a spherical particle differed from theirs, but had the same effect of causing the particle to spiral into the sun. He showed the cause of this effect to be the particle's loss of angular momentum.

The relativistic effects of radiation pressure were considered again in 1960 when Richard Schlegel⁴ calculated the force on a plane surface moving through an isotropic radiation field. The results obtained were in error because

although the expression given for energy density was transformed, the expression for the solid angle element $d\Omega$ was not; hence in the transformation from the laboratory frame to the particle frame a factor of $\frac{1 - \frac{v^2}{c^2}}{(1 + \frac{v}{c} \cos \phi')^2}$ was lost.

The correction for this angular transformation was subsequently pointed out by W. Rindler, D. W. Sciama⁵ and J. Terrell.⁶

CHAPTER II

THE RELATIVISTIC FORCE OF ISOTROPIC RADIATION

Calculation by Use of the Stress-Energy Tensor

The purpose of my investigation is to calculate the effects of radiation pressure on a spherical particle moving through an isotropic radiation field. Of particular interest is the radiation field in interstellar space at points distant from any one particular star. According to present day knowledge "the particles responsible for interstellar absorption are definitely tiny solid grains."^{7a} Beyond their existence as small particles not much is known about them. They may be either metallic or dielectric, and their size is probably on the order of the wavelength of light or much smaller. Much of the interpretation of astronomical observation of particles in space has been based on the theories developed by G. Mie⁸ in 1908 and P. Debye^{9a} in 1909. Their theories calculated the scattering of radiation by spherical particles and the pressure exerted by the scattered and absorbed radiation, but with no regard to the effects on the motion of the particles.

The radiation pressure on particles depends strongly on their size and they can be divided into three major groups: first, large sized particles (radius much larger than the wavelength of radiation); second, particles on the order of the wavelength; and third, particles much smaller than the wavelength. For the last two groups diffraction effects are important.

The objective now is to obtain a general expression for the force exerted by an isotropic radiation field on a spherical particle. Consider such a particle moving along the positive direction of the z axis of a coordinate system in an isotropic radiation field. The coordinate system at rest relative to the stars, and in which the radiation is isotropic, will henceforth be referred to as the star system. The coordinate system which moves with the particle will be called the particle system. Because of symmetry no forces perpendicular to the z axis will be exerted on the particle. Hence in using the stress-energy tensor we need only calculate those elements related to the z component of the force. This will be done by dividing the isotropic radiation of energy density u into pencils of radiation (plane waves) of intensity $u \frac{d\Omega}{4\pi}$ where $d\Omega$ is the infinitesimally small solid angle of the pencil of radiation we are considering.

This assumes that the energy density u is composed of an infinite number of plane waves whose directions have an isotropic distribution about any point in space. The amount

of energy density moving in a particular direction, specified by the normal vector to the surface element $d\Omega$ of a unit sphere surrounding a point, is the fractional area $d\Omega/4\pi$ where 4π is the total area of the sphere. The Cartesian components of the electric and magnetic components of a wave in a nonconducting medium satisfy the well known equation $\nabla^2 w - \frac{1}{v^2} \frac{\partial^2 w}{\partial t^2} = 0$. And E. T. Whittaker¹⁰ in 1903 mathematically proved that solutions of this equation can be broken up into plane waves. Thus the isotropic radiation can be represented as the sum of plane waves.

Consider the tensor

$$G^\mu = - \frac{\partial T^{\mu\nu}}{\partial x^\nu} dV \dots \dots \dots (1)$$

where the components of x^μ are defined as

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ct.$$

The quantity dV is interpreted as a four dimensional volume $dV = dx \, dy \, dz \, cdt$. G^μ is a first rank tensor because $\frac{\partial T^{\mu\nu}}{\partial x^\nu}$ is the covariant derivative in flat space-time and dV is an invariant. The invariance of dV can be seen by comparing dV in the star system (x^μ) to $d\bar{V}$ in the particle system ($\bar{x}^\mu$).
Now

$$d\bar{z} = \sqrt{1 - \beta^2} \, dz, \quad d\bar{x} = dx, \quad d\bar{t} = \frac{dt}{\sqrt{1 - \beta^2}}, \text{ and } d\bar{y} = dy.$$

where β is the velocity of the particle in the star system divided by the speed of light. Therefore we see

$$\begin{aligned}
 d\bar{V} &= d\bar{x} d\bar{y} d\bar{z} c d\bar{t} \\
 &= dx dy \sqrt{1 - \beta^2} dz \frac{cdt}{\sqrt{1 - \beta^2}} = dV
 \end{aligned}$$

The components of G^μ physically represent:

$$G^1 = cdP_x, G^2 = cdP_y, G^3 = cdP_z, G^4 = dE,$$

where P and E are momentum and energy respectively. The value of these components follow from the evaluation¹¹ of $-\frac{\partial T^{\mu\nu}}{\partial x^\nu} dV$ for each value of μ . Hence the first three components of G^μ represent c times the amount of momentum contained in the volume dV , and G^4 represents the amount of energy contained in this volume.

The definition of $T^{\mu\nu}$ is as follows: let g_i represent the density of momentum in the direction i . Then

$$T^{14} = cg_1, T^{44} = \text{energy density, and } T^{ij} = \frac{F_i}{A_j} + g_i w_j,$$

where F_i is the force in direction i exerted by the medium (in this case light) through the area A_j normal to the j direction, and w_j is the velocity of the medium in the direction j .

Using these definitions we can write the $T^{\mu\nu}$ components for a pencil of radiation. The relation between the energy and momentum of radiation is

$$E = Pc$$

Therefore energy density and momentum density are related by

$$u_f = gc,$$

where u_f is the fractional amount of energy density u contained

in the solid angle $d\Omega$. This plane wave has the direction cosines $\cos \alpha$, $\cos \beta$, $\cos \gamma$. Hence:

$$g = \frac{u_f}{c} = \frac{u}{c} \frac{d\Omega}{4\pi},$$

$$g_1 = \frac{u}{c} \frac{d\Omega}{4\pi} \cos \alpha, \quad g_2 = \frac{u}{c} \frac{d\Omega}{4\pi} \cos \beta, \quad g_3 = \frac{u}{c} \frac{d\Omega}{4\pi} \cos \gamma,$$

$$w_1 = c \cos \alpha, \quad w_2 = c \cos \beta, \quad \text{and} \quad w_3 = c \cos \gamma.$$

Since we are concerned with radiation in empty space $F_1 = 0$.

Let ϕ be the angle with the z axis and θ be the angle with x axis in spherical coordinates. Then

$$\cos \alpha = \sin \phi \cos \theta,$$

$$\cos \beta = \sin \phi \sin \theta,$$

$$\cos \gamma = \cos \phi.$$

Now using the above expressions the $T^{\mu\nu}$ components may be written as follows:

$$T^{11} = u \frac{d\Omega}{4\pi} \cos^2 \alpha = u \frac{d\Omega}{4\pi} \sin^2 \phi \cos^2 \theta,$$

$$T^{22} = u \frac{d\Omega}{4\pi} \cos^2 \beta = u \frac{d\Omega}{4\pi} \sin^2 \phi \sin^2 \theta,$$

$$T^{33} = u \frac{d\Omega}{4\pi} \cos^2 \gamma = u \frac{d\Omega}{4\pi} \cos^2 \phi,$$

$$T^{44} = u \frac{d\Omega}{4\pi},$$

$$T^{14} = T^{41} = u \frac{d\Omega}{4\pi} \cos \alpha = u \frac{d\Omega}{4\pi} \sin \phi \cos \theta,$$

$$T^{24} = T^{42} = u \frac{d\Omega}{4\pi} \cos \beta = u \frac{d\Omega}{4\pi} \sin \phi \sin \theta,$$

$$T^{34} = T^{43} = u \frac{d\Omega}{4\pi} \cos \gamma = u \frac{d\Omega}{4\pi} \cos \phi,$$

$$T^{13} = T^{31} = u \frac{d\Omega}{4\pi} \cos \alpha \cos \gamma = u \frac{d\Omega}{4\pi} \sin \phi \cos \theta \cos \phi,$$

$$T^{12} = T^{21} = u \frac{d\Omega}{4\pi} \cos \beta \cos \alpha = u \frac{d\Omega}{4\pi} \sin^2 \phi \cos \theta \sin \theta,$$

$$\text{and } T^{23} = T^{32} = u \frac{d\Omega}{4\pi} \cos \beta \cos \gamma = u \frac{d\Omega}{4\pi} \sin \phi \sin \theta \cos \phi.$$

In order to calculate the force acting on the particle we go to the particle system (denoted by bars) and evaluate $\bar{G}^3$:

$$\bar{G}^3 = - \left(\frac{\partial \bar{T}^{31}}{\partial \bar{x}} + \frac{\partial \bar{T}^{32}}{\partial \bar{y}} + \frac{\partial \bar{T}^{33}}{\partial \bar{z}} + \frac{\partial \bar{T}^{34}}{c \partial \bar{t}} \right) d\bar{V}.$$

To find the total value of $\bar{G}^3$ associated with an object of volume $\bar{V}$, the right side of the above equation must be integrated over the object's volume.

$$\bar{G}^3 = - \int_{\bar{V}} \left(\frac{\partial \bar{T}^{31}}{\partial \bar{x}} + \frac{\partial \bar{T}^{32}}{\partial \bar{y}} + \frac{\partial \bar{T}^{33}}{\partial \bar{z}} \right) d\bar{V} \cdot \cdot \cdot \cdot \cdot (2)$$

The last term has dropped out since $\bar{T}^{34}$ is not an explicit function of time. The values of the tensor components $\bar{T}^{31}$, $\bar{T}^{32}$, and $\bar{T}^{33}$ are now needed. To obtain these we use the tensor property

$$\bar{T}^{\mu\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{\partial \bar{x}^\nu}{\partial x^\beta} T^{\alpha\beta}$$

where the relations between $\bar{x}^\mu$ and x^μ are given by

$$\bar{x}^1 = x^1, \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = \gamma (x^3 - \beta x^4), \quad \text{and } \bar{x}^4 = \gamma (x^4 - \beta x^3)$$

$$\text{where } \beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

For the particular values of μ and ν we see that

$$\left. \begin{aligned} \bar{T}^{31} &= \gamma T^{31} - \beta \gamma T^{41}, \\ \bar{T}^{32} &= \gamma T^{32} - \beta \gamma T^{42}, \text{ and} \\ \bar{T}^{33} &= \gamma^2 T^{33} - 2\gamma^2 \beta T^{34} + \beta^2 \gamma^2 T^{44}. \end{aligned} \right\} \dots \dots (3)$$

Substituting the values for T^{31} and T^{41} into the first of these last three expressions, we have

$$\bar{T}^{31} = \gamma \frac{u}{4\pi} d\Omega \sin \phi \cos \theta \cos \phi - \beta \gamma \frac{u}{4\pi} d\Omega \sin \phi \cos \theta.$$

Now use is made of the following relations between the coordinates of the two frames of reference :

$$\cos \phi = \frac{\cos \bar{\phi} + \beta}{(1 + \beta \cos \bar{\phi})},$$

$$\sin \phi = \frac{\sqrt{1 - \beta^2} \sin \bar{\phi}}{(1 + \beta \cos \bar{\phi})},$$

$$d\Omega = \frac{(1 - \beta^2) d\bar{\Omega}}{(1 + \beta \cos \bar{\phi})^2}.$$

The plane angles $\bar{\phi}$ and $\bar{\theta}$ and the solid angle $d\bar{\Omega} = \sin \bar{\phi} d\bar{\phi} d\bar{\theta}$ are measured in the particle system and correspond to the angles ϕ , θ and $d\Omega$ in the star system. The expression for $\cos \phi$ in terms of $\cos \bar{\phi}$ is well known. The expression for $\sin \phi$ can be obtained by simply using the relation $\sin^2 \phi = 1 - \cos^2 \phi$. The expression for $d\Omega$ can be obtained by differentiating $\cos \phi$ and remembering that $\bar{\theta} = \theta$.

Using these expressions relating the angles in the two coordinate systems we can write the tensor components in the star system in terms of the angles $\bar{\theta}$, $\bar{\phi}$, and $d\bar{\Omega}$. Then

substitute them into equation (3) to obtain:

$$\bar{T}^{31} = \frac{u}{4\pi} \frac{(1 - \beta^2)^2 \cos \bar{\theta} \sin \bar{\phi} \cos \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} d\bar{\Omega},$$

$$\bar{T}^{32} = \frac{u}{4\pi} (1 - \beta^2)^2 \frac{\sin \bar{\phi} \sin \bar{\theta} \cos \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} d\bar{\Omega},$$

$$\text{and } \bar{T}^{33} = \frac{u}{4\pi} \frac{(1 - \beta^2)^2 \cos^2 \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} d\bar{\Omega}.$$

These expressions for the components of the stress-energy tensor are those of a pencil of radiation whose direction is specified by the angles $\bar{\theta}$ and $\bar{\phi}$ in the particle system and are independent of any absorbing or scattering surface.

In order to evaluate the integral for $\bar{G}^3$ the derivatives of the tensor components must be specified. Suppose we have a perfectly absorbing particle, i.e. one that absorbs all radiation incident upon it. Consider the first term in equation (2):

$$\int_{\bar{V}} \frac{\partial \bar{T}^{31}}{\partial \bar{x}} d\bar{x} d\bar{y} d\bar{z} c d\bar{t}.$$

Assume that the particle is large compared to the wavelength of radiation and note that the particle is a sphere in the particle system. An incident plane wave in this case illuminates the front half of the sphere and does not illuminate the back half. Assume that there are no electromagnetic waves inside the sphere, so that they do not penetrate the surface but are instead completely absorbed at the surface.

Thus $\bar{T}^{\mu\nu}$ is zero everywhere inside the sphere. Also, since this sphere does not scatter any waves which strike its surface, the stress-energy tensor immediately outside the front half of the sphere is the same as that for the plane wave. The stress-energy tensor is zero in the region just outside the back half of the sphere. Thus:

$$\frac{\partial \bar{T}^{31}}{\partial \bar{x}} d\bar{x} = +\bar{T}^{31} \text{ on the front hemisphere,}$$

$$\frac{\partial \bar{T}^{31}}{\partial \bar{x}} d\bar{x} = 0 \text{ on the back hemisphere.}$$

We are now left with a surface integral over the area S of the front hemisphere

$$\int_S +\bar{T}^{31} d\bar{y} d\bar{z}$$

Since $\bar{T}^{31}$ is independent of $\bar{x}$, $\bar{y}$, and $\bar{z}$ this expression becomes:

$$\bar{T}^{31} \int_S + d\bar{y} d\bar{z}$$

$$\text{But } \int_S + d\bar{y} d\bar{z} = \int_S \vec{i} \cdot d\vec{S}$$

where $\vec{i}$ is a unit vector in the x direction and dS is a portion of the sphere's area. Now consider the volume enclosed by the surface of a hemisphere of radius a and its flat side of area $F = \pi a^2$. By the divergence theorem

$$\int_S \vec{i} \cdot d\vec{S} + \int_F \vec{i} \cdot d\vec{S} = 0.$$

Therefore

$$\int_S \vec{i} \cdot d\vec{S} = - \int_F \vec{i} \cdot d\vec{S} = -\pi a^2 \cos \bar{\alpha}$$

where $\bar{\alpha}$ is the angle between the direction of the incoming radiation and the x axis.

$$\text{Hence } \int_{\bar{V}} \frac{\partial \bar{T}^{31}}{\partial \bar{x}} d\bar{V} = -\pi a^2 \bar{T}^{31} \cos \bar{\alpha} cd\bar{t}.$$

Similarly it can be shown that

$$\int_{\bar{V}} \frac{\partial \bar{T}^{32}}{\partial \bar{y}} d\bar{V} = -\pi a^2 \bar{T}^{32} \cos \bar{\beta} cd\bar{t},$$

$$\text{and } \int_{\bar{V}} \frac{\partial \bar{T}^{33}}{\partial \bar{z}} d\bar{V} = -\pi a^2 \bar{T}^{33} \cos \bar{\gamma} cd\bar{t}.$$

Substituting these values into equation (2) we have

$$\bar{G}^3 = \pi a^2 cd\bar{t} (\cos \bar{\alpha} \bar{T}^{31} + \cos \bar{\beta} \bar{T}^{32} + \cos \bar{\gamma} \bar{T}^{33}).$$

Remembering that the value of $\bar{G}^3$ is $cd\bar{P}_z$ we can now write

$$\bar{P}_z = \frac{cd\bar{P}_z}{cd\bar{t}} = \pi a^2 (\cos \bar{\alpha} \bar{T}^{31} + \cos \bar{\beta} \bar{T}^{32} + \cos \bar{\gamma} \bar{T}^{33}).$$

Substituting into the above equation the values for the cosine angles and the tensor components in terms of $\bar{\phi}$ and $\bar{\theta}$ we have

$$\bar{P}_z = \pi a^2 \frac{(1 - \beta^2)^2}{(1 + \beta \cos \bar{\phi})^4} \frac{d\bar{\Omega}}{4\pi} \frac{u}{4\pi} (\sin^2 \bar{\phi} \cos^2 \bar{\theta} + \sin^2 \bar{\phi} \sin^2 \bar{\theta} + \cos^2 \bar{\phi}) \cos \bar{\phi}.$$

Since the velocity of the particle is in the z direction

$F_z = \bar{F}_z$ and therefore

$$F_z = \bar{F}_z = \pi a^2 u \frac{(1 - \beta^2)^2 \cos \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} \frac{d\bar{\Omega}}{4\pi} \dots (4)$$

This expression gives the z component of force in the star system exerted on the particle by the portion of radiation contained in the solid angle $d\bar{\Omega}$. One of the factors $(1 - \beta^2)/(1 + \beta \cos \bar{\phi})^2$ comes from the transformation of the energy density and the other comes from the transformation for the solid angle. Since u is the magnitude of the pressure, $\pi a^2 u$ is the total magnitude of force which would be exerted on the particle if β were zero. Thus the effective absorbing area of the sphere is equal to the cross-sectional area of the sphere as expected. An alternative way of looking at this result is to consider the absorption of momentum. The sphere can be replaced by a perfectly absorbing disk of area πa^2 whose normal is parallel to the path of the incident radiation. This disk then absorbs the same amount of momentum in the same direction per unit time as does the sphere, since both objects absorb all radiation incident on their surface. Hence they both experience the same force.

Equation (4) can be generalized to include diffraction effects. In the particle system the incident wave is still a plane wave carrying momentum in its direction of motion. In this reference frame, which is electromagnetically

equivalent to any other frame, we have a plane wave incident upon a stationary spherical particle and classical electromagnetic theory is applicable. When diffraction occurs the amount of momentum absorbed may be more or less than that of the perfectly absorbing sphere. The increase or decrease of absorbed momentum due to this diffraction effect must be in the direction of the incoming wave since the wave and sphere are symmetrical about this direction in the particle system. This increase or decrease in absorbed momentum can thus be accounted for by simply multiplying equation (4) by factor Q . Since we are considering the particle in its rest frame, Q in that frame must be independent of β and can only be a function of the particle's radius a and the incident wavelength $\bar{\lambda}$. Hence $Q = Q(a, \bar{\lambda})$. The ratio of force exerted on a spherical particle to the force on a perfectly absorbing disk of the same radius was calculated by Debye in 1909 using classical electromagnetic theory. This ratio is the Q factor and is usually designated Q_{pr} .

Hence

$$F_{zr} = a^2 \frac{u}{4} Q_{pr}(a, \bar{\lambda}) \frac{(1 - \beta^2)^2 \cos \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} d\bar{\Omega} \dots (5)$$

The "r" subscript serves to remind us that this is the force due to a single ray.

Agreement with the Calculation from
Robertson's Equation

This result can also be arrived at through use of H. P. Robertson's paper "Dynamical Effects of Radiation in the Solar System." He studied the mechanism by which a particle, orbiting the sun, would eventually fall into the sun by the loss of angular momentum. These particles absorb radiation from the sun but re-radiate it with a loss of their own angular momentum. At the beginning of his paper Robertson derived a relativistic expression for the force exerted on a spherical particle by a pencil of radiation. He then made a classical approximation and derived results for the motion of a particle orbiting the sun. His relativistic expression for the force was derived by transforming the force vector, a first rank tensor. He^{3b} wrote it in its final form, in the star system, as

$$\frac{dm_0}{ds} u^\mu = \frac{fw}{c} (\ell^\mu - wu^\mu). \dots \dots \dots (6)$$

where m_0 is the proper mass, s the proper time, u^μ is a component of unit velocity, and c is the speed of light. For $\mu = 1, 2, \text{ or } 3$ the value of ℓ^μ is c times the cosine angle of incident radiation in the star system and ℓ^0 is equal to unity. Also w is equal to λ_0 , the transformed component of ℓ^0 in the particle system. The expression f is the product of the energy density d of the incoming ray in the star system and the particle's effective cross sectional area A .

To change equation (6) to an expression for our case of interest let $d = u \frac{d\Omega}{4\pi}$. Evaluate λ^0 by the relation

$$\lambda^0 = \frac{\partial \bar{x}^0}{\partial x^\beta} \ell^\beta,$$

where $\bar{x}^0 = \bar{t}$, $x^0 = t$, $\bar{x}^3 = \bar{z}$, $x^3 = z$.

Since $\ell^0 = 1$, $\ell^3 = c \cos \phi$, and $\bar{t} = (t - \beta z)\gamma$,

we find $\lambda^0 = \gamma(1 - \beta \cos \phi)$.

The particle's velocity v is in the z direction only, so

$u^\mu = v\gamma$. Substituting these relations into equation (6)

we have

$$\begin{aligned} \frac{d(m_0 \gamma v)}{dt} &= A\gamma (1 - \beta \cos \phi) \cos \phi \frac{u}{4\pi} d\Omega \\ &\quad - A\gamma^2 (1 - \beta \cos \phi)^2 \frac{\gamma v}{c} \frac{u}{4\pi} d\Omega. \end{aligned}$$

let $m = m_0 \gamma$, divide both sides of the above equation by γ ,

and make the substitution for $\cos \phi$ and $d\Omega$ in terms of

$\cos \bar{\phi}$ and $d\bar{\Omega}$. The result is

$$\frac{d(mv)}{dt} = \frac{Au}{4\pi} \frac{(1 - \beta^2)^2}{(1 + \beta \cos \bar{\phi})^4} \cos \bar{\phi} d\bar{\Omega}.$$

Since A is the effective area $A = \pi a^2 Q_{pr}$. We have then

$$F_{zr} = a^2 \frac{u}{4} Q_{pr} \frac{(1 - \beta^2)^2 \cos \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} d\bar{\Omega} \dots \dots (7)$$

which agrees with expression (5).

The Exact Expression for Q_{pr}

Robertson used the classical approximation of equation (6) to study the effect that radiation from the sun would have on the motion of a particle under the gravitational influence of the sun. No calculations have been made of the effect on the motion of particles resulting from a Doppler shift of frequency.

My purpose is to determine the effect of this frequency shift on a particle in an isotropic radiation field. Equation (5) gives the force exerted by a single pencil of light. In order to obtain the total force due to the whole radiation field, equation (5) must be integrated over the unit sphere. To do this $Q_{pr}(a, \bar{\lambda})$ must be known. Defining Q_{pr} by the equation $F = d Q_{pr} \pi a^2$ where d is the energy density, πa^2 the particle's cross-sectional area and F the force exerted on the particle, the value for Q_{pr} was derived by Debye^{9b} in rationalized Heaviside units and was found to be:

$$Q_{pr} = \frac{\bar{\lambda}^2 H^2}{\pi^2 a^2} \operatorname{Re} \left[\sum_{n=1}^{\infty} (2n+1) \frac{\alpha_n^1 + \alpha_n^2}{2} - \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \alpha_n^{*1} \alpha_n^2 - \sum_{n=1}^{\infty} \frac{n(n+2)}{n+1} (\alpha_n^{*1} \alpha_{n+1}^1 + \alpha_n^{*2} \alpha_{n+1}^2) \right] \quad (8)$$

The amplitude of the wave is represented by H and its wavelength is $\bar{\lambda}$ in the particle system. The alpha terms

are complicated functions defined as follows:

$$\left. \begin{aligned} \alpha_n^1 &= \frac{N_1^a \psi_n(k_a a) k_1 \psi_n'(k_1 a) - k_a \psi_n'(k_a a) N_1^1 \psi_n(k_1 a)}{N_1^a \zeta_n(k_a a) k_1 \psi_n'(k_1 a) - k_a \zeta_n'(k_a a) N_1^1 \psi_n(k_1 a)}, \\ \text{and} \\ \alpha_n^2 &= \frac{N_2^a \psi_n(k_a a) k_1 \psi_n'(k_1 a) - k_a \psi_n'(k_a a) N_2^1 \psi_n(k_1 a)}{N_2^a \zeta_n(k_a a) k_1 \psi_n'(k_1 a) - k_a \zeta_n'(k_a a) N_2^1 \psi_n(k_1 a)}. \end{aligned} \right\} \dots (9)$$

The functions of ka are defined by

$$\psi_n(ka) = \left(\frac{\pi ka}{2} \right)^{1/2} J_{n+1/2}(ka)$$

and

$$\zeta_n(ka) = \left(\frac{\pi ka}{2} \right)^{1/2} H_{n+1/2}^2(ka).$$

The functions $J_{n+1/2}(ka)$ and $H_{n+1/2}^2(ka)$ are respectively Bessel and Hankel functions. The constants in equations (9) are defined as

$$N_1^a = 1 \frac{\bar{\omega}}{c}, \quad N_2^a = 1 \frac{\bar{\omega}}{c}, \quad k_a^2 = \frac{\bar{\omega}^2}{c^2},$$

$$N_1^1 = 1 \frac{\epsilon \bar{\omega}}{c} + \frac{\sigma}{c}, \quad N_2^1 = 1 \frac{\mu \bar{\omega}}{c},$$

$$k_1^2 = \frac{\epsilon \mu \bar{\omega}^2}{c^2} - 1 \frac{\mu \sigma \bar{\omega}}{c^2}.$$

The angular velocity, speed of light, dielectric constant, magnetic permeability, and conductivity are respectively represented by $\bar{\omega}$, c , ϵ , μ , and σ . The bars placed on ω and λ again denote quantities measured in the particle system.

CHAPTER III

DYNAMICAL EFFECTS OF THE RADIATION FIELD

Exact Solutions and Useful Approximations

The dependence of the frequency $\bar{\nu}$ on the angle $\bar{\phi}$ is given by the well known relativistic expression

$$\bar{\nu} = \frac{\nu \sqrt{1 - \beta^2}}{1 + \beta \cos \bar{\phi}} \quad (10)$$

where ν is the frequency of radiation in the star system. Since the terms in equation (8) are functions of $\bar{\nu}$ equation (5) is seen to be difficult to integrate because of the complicated dependence on $\bar{\phi}$.

Because of the complexity of the integration I had tried an alternate approach in order to obtain an exact solution for F_z . This method is rather long and complicated and was not completed. The basic idea of it is to avoid complicated integrations by the use of boundary conditions. The radiation pressure is independent of the direction of polarization of the waves so any convenient direction of polarization can be assumed. This makes it possible to write an expression for the electric and magnetic components

of the whole isotropic and monochromatic radiation field and not just a single ray of radiation. The components are then transformed relativistically from the star system to the particle system. My experience with this method ends at this point. I have obtained by use of a method developed by E. T. Whittaker in a paper cited above, the expressions for the electromagnetic components in the particle system. At this point these components must be matched with a general expression for the scattered waves by using boundary conditions on the surface of the spherical particle. This is difficult since the frequency of the incident radiation varies with the angle $\bar{\phi}$. If the scattered waves could be found a stress-energy tensor could be written and probably integrated over any arbitrary surface containing the particle.

For many situations however, approximations are easy to obtain, and they simplify Q_{pr} to make equation (5) possible to integrate by the use of restrictions on the values of $\bar{\lambda}$ or a . Also computed graphs of Q_{pr} as a function of $2\pi a/\bar{\lambda}$ can be used to make approximations.

The Relativistic Motion of a Large Perfectly Absorbing Sphere

Consider the case of a perfectly absorbing sphere whose radius a is large compared to the wavelength of incident radiation. The value for Q_{pr} is then unity as it is also in the case of a large completely reflecting sphere.

When Q_{pr} is unity we have from equation (5)

$$F_{zr} = \frac{a^2 u}{4} \frac{(1 - \beta^2)^2 \cos \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} d\bar{\Omega} \dots \dots \dots (11)$$

The total force is then

$$F_z = \frac{a^2 u}{4} \int_0^\pi \int_0^{2\pi} \frac{(1 - \beta^2) \cos \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} \sin \bar{\phi} d\bar{\theta} d\bar{\phi}.$$

Let $s = \cos \bar{\phi}$, then $ds = -\sin \bar{\phi} d\bar{\phi}$.

Therefore

$$F_z = 2\pi \frac{a^2}{4} u \int_{-1}^1 (1 - \beta^2)^2 \frac{s ds}{(1 + \beta s)^4}.$$

The value under the integral sign is

$$\frac{(1 - \beta^2)^2}{2} \left[\frac{-1}{2(1 - \beta)^2} + \frac{1}{3(1 - \beta)^3} + \frac{1}{2(1 + \beta)^2} - \frac{1}{3(1 + \beta)^3} \right],$$

which simplifies to $\frac{8}{3} \frac{\beta}{1 - \beta^2}$.

$$\text{Hence } F_z = -\pi a^2 u \frac{4}{3} \frac{\beta}{1 - \beta^2} \dots \dots \dots (12)$$

The force exerted on the particle of rest mass m_0 is

$$F_z = c \frac{d}{dt} \left(\frac{m_0 \beta}{\sqrt{1 - \beta^2}} \right) \dots \dots \dots (13)$$

Equating the right sides of equations (12) and (13) we write

$$c \frac{d}{dt} \left(\frac{m_0 \beta}{\sqrt{1 - \beta^2}} \right) = -\pi a^2 u \frac{4}{3} \frac{\beta}{1 - \beta^2} \dots \dots \dots (14)$$

It is important to note at this point that equation (14) assumes that all absorbed radiation is re-radiated isotropically in the particle system. If it were not re-radiated the increase of absorbed energy would cause an increase in the mass of the particle other than that due to the relativistic increase of mass with velocity. It might be thought that an expression should be added to the right side of equation (14) since it was derived on the basis of absorption and scattering only and no stress-energy tensor terms were added to account for re-radiation. But these additional terms are not necessary. The sum of any such additional terms is zero since the net loss of re-radiated momentum in the particle system is zero. Now using equation (14) the particle's velocity can be found as a function of time.

Differentiating the left hand side of equation (14) yields

$$cm_0 \frac{d}{dt} \left(\frac{\beta}{\sqrt{1 - \beta^2}} \right) = m_0 c \frac{1}{(1 - \beta^2)^{3/2}} \frac{d\beta}{dt} \dots (15)$$

Substituting this expression into equation (14) and rearranging terms we have

$$\frac{4}{3} \frac{\pi a^2}{m_0 c} u dt = - \frac{d\beta}{\beta \sqrt{1 - \beta^2}} .$$

Now integrate the left side of the above equation with respect to t from $t = 0$ to $t = t$ and integrate the right side

with respect to β from β_0 to β where β_0 is the initial velocity v_0 divided by c and β is v/c at a later time t . We obtain:

$$\frac{4}{3} \frac{\pi a^2}{m_0 c} u t = \ln \left(\frac{1 + \sqrt{1 - \beta^2}}{\beta} \cdot \frac{\beta_0}{1 + \sqrt{1 - \beta_0^2}} \right)$$

Rearranging terms we find

$$\frac{\beta}{1 + \sqrt{1 - \beta^2}} = \frac{\beta_0}{1 + \sqrt{1 - \beta_0^2}} e^{-\frac{4}{3} \frac{\pi a^2 u t}{m_0 c}} \dots \dots \dots (16)$$

When β is small (low velocities), we have a classical approximation:

$$\frac{v}{v_0} = e^{-\frac{4}{3} \frac{\pi a^2 u t}{m_0 c}}$$

Thus when $v = \frac{1}{e} v_0$, the time constant t_e is found to be

$$t_e = \frac{3}{4} \frac{m_0 c}{\pi a^2 u} \dots \dots \dots (17)$$

This result is interesting because it shows that the time taken for the particle to slow down to $\frac{1}{e}$ its initial velocity is independent of its initial velocity, but is proportional to its rest mass and inversely proportional to both the energy density u and its cross-sectional area πa^2 .

Solving equation (16) for β we have

$$\beta = \frac{\frac{2\beta_0}{1 + \sqrt{1 - \beta_0^2}} e^{-\frac{4}{3} \frac{\pi a^2 u t}{m_0 c}}}{1 + \frac{\beta_0^2}{(1 + \sqrt{1 - \beta_0^2})^2} e^{-\frac{8}{3} \frac{\pi a^2 u t}{m_0 c}}} \dots \dots \dots (18)$$

Consider equation (18) when t is very large. This expression reduces to

$$\beta = \frac{2\beta_0}{1 + \sqrt{1 - \beta_0^2}} e^{-\frac{4}{3} \frac{\pi a^2 u t}{m_0 c}}.$$

If β_0 is large then the maximum effect of the relativistic correction is that β is almost twice as large as the classical approximation. We see the retarding effect is not as great in the relativistic case. Looking at equation (12) this might seem paradoxical. If β^2 is small equation (12) reduced to the classical result

$$F_z = -\pi a^2 u \frac{4}{3} \beta \dots \dots \dots (19)$$

as opposed to its relativistic form in which F_z is greater than the classical value by a factor of $1/(1 - \beta^2)$. Hence the relativistic force is greater than the classical force. But now look at equation (15). It shows that the relativistic mass increase changes the expression $\frac{d}{dt} \frac{mv}{dt}$ by a factor of $1/(1 - \beta^2)^{3/2}$. The relativistic increase of force due to radiation is not as great as the relativistic increase

in mass. The net result is that the particle's velocity is retarded at a slower rate at higher relativistic velocities.

The cause of the retardation of velocity for the perfectly absorbing sphere is similar to the cause of the loss of velocity of a particle in the neighborhood of the sun. The only difference is that instead of losing angular momentum the particle is losing linear momentum by re-radiating the absorbed radiation. The radiation striking the particle on the front side has an increased energy density due to the Doppler shift. For the same reason radiation striking the particle on the back has smaller energy density. This effect combined with the crowding of radiation in front and the thinning out in back exerts a net retarding force on the particle.

In the derivation of equation (18) it was assumed that the particle was large compared to the radiation's wavelength. This assumption breaks down when velocities close to that of light occur. This is because at large enough velocities the wavelength of radiation striking the back of the particle is increased. Hence for certain velocities it can take on values of the same order as the radius of the particle and thus diffraction effects can become important. At this point Q_{pr} is no longer equal to unity. The value for β_0 at which Q_{pr} is no longer equal to unity depends on the ratio of the radius a to the wavelength λ . If a/λ is large then the values of β_0 which are valid in

equation (18) are large. Similarly if a/λ is small the values of β_0 in this equation must also be small.

Linear Approximations for Low Velocity Particles

Cases will now be considered in which Q_{pr} is not a constant but varies as a function of the wavelength of incident radiation. If we consider dielectrics where $\sigma = 0$, and look at the results of Debye quoted earlier (equation (8)) we see that Q_{pr} varies only as a function of the ratio a/λ . However, for conducting particles this is not the case.

A comprehensive study of Mie scattering theory and Debye's radiation pressure is presented by H. C. Van de Hulst¹² in his book Light Scattering by Small Particles. He states that if d is the energy density of incident radiation then the total energy of scattered radiation E_{sca} is defined by use of the cross section A_{sca} defined by the relation

$$E_{sca} = d A_{sca}.$$

Similar definitions are used to define the cross sections A_{abs} and A_{ext} for the absorption and extinction cross sections. Conservation of energy requires that $A_{ext} = A_{sca} + A_{abs}$. Non-absorbing particles will have $A_{ext} = A_{sca}$. If $\overline{\cos \phi}$ defines the average cosine angle at which radiation is scattered then the radiation pressure cross section A_{pr} can be defined by the relation

$$A_{pr} = A_{ext} - \overline{\cos \phi} A_{sca}.$$

Discussions in light scattering theory use efficiency factors Q rather than cross sections and are defined by dividing the cross sections by the cross sectional area of the particle πa^2 . Then the above equation becomes

$$Q_{pr} = Q_{ext} - \overline{\cos \phi} Q_{sca} \dots \dots \dots (20)$$

It is by the use of this definition of the efficiency factor for the radiation pressure and Debye's expression for Q_{pr} that calculations have been made.

Since the efficiency factors are complicated, much use has been made of computers to calculate them. One such case of particular interest is given by W. M. Irvine^{13a} who computed Q_{pr} by use of equation (20) for different values of the index of refraction for both dielectric and absorbing spheres. Since the index of refraction for metals varies as an explicit function of frequency, Q_{pr} is for them a more complicated function than it is for a dielectric, for which the index of refraction is a function of ϵ and μ only. Since Irvine's results show graphs of Q_{pr} as functions of the wavelength for fixed indexes of refraction, they can be used for computing the force on dielectrics. The dielectric constant ϵ will be assumed to be essentially constant for dielectrics when a small range of frequencies are involved. Since the temperature of particles in interstellar space are estimated^{7b} to be from 10^0K to 30^0K , or up to 100^0K , the

variation of ϵ with frequency will be less than the usually small variation at room temperatures.¹⁴

In order to understand what kind of approximations of Q_{pr} are important, the extreme values of frequency in the particle system must be considered. The frequency $\bar{\nu}$ seen by the particle is given in equation (10). The two extreme values of frequency which it sees are the higher frequency $\bar{\nu}_f$ of the radiation in front at angle $\bar{\phi} = \pi$ and the lower frequency $\bar{\nu}_b$ of radiation in back at an angle $\bar{\phi} = 0$. Thus

$$\left. \begin{aligned} \bar{\nu}_f &= \nu \sqrt{\frac{1+\beta}{1-\beta}} = \nu \frac{1+\beta}{\sqrt{1-\beta^2}} \\ \bar{\nu}_b &= \nu \sqrt{\frac{1-\beta}{1+\beta}} = \nu \frac{1-\beta}{\sqrt{1-\beta^2}} \end{aligned} \right\} \dots \dots \dots (21)$$

Notice that for small β , $\bar{\nu}_f$ and $\bar{\nu}_b$ differ from ν by the same amount $\nu\beta$ but the values are shifted in opposite directions. The difference between these frequencies is generally

$$\Delta\bar{\nu} = \bar{\nu}_f - \bar{\nu}_b = \nu \frac{2\beta}{\sqrt{1-\beta^2}} \dots \dots \dots (22)$$

Now square both sides of this equation and solve for β .

We find

$$\beta = \frac{\Delta\bar{\nu}}{\sqrt{(\Delta\bar{\nu})^2 + 4\nu^2}} \dots \dots \dots (23)$$

It is seen from equation (23) that β can be small and still give rise to large frequency shifts if ν is large. Since

figure (5) gives Q_{pr} as a function of $x = 2\pi a/\bar{\lambda}$, we write

$$\Delta x = \frac{2\pi a}{c} \Delta \bar{\nu}.$$

Equation (23) can then be written

$$\beta = \frac{\frac{\Delta x}{x}}{\sqrt{\left(\frac{\Delta x}{x}\right)^2 + 4}} \dots \dots \dots (24)$$

Now we look at Figure (4). The function^{13b} Q_{ext} is seen to be made up of major oscillations with minor oscillations, usually called "ripples," superimposed on them. The major oscillations are caused by the interference between diffracted radiation and transmitted radiation. When the diffracted radiation and transmitted radiation constructively interfere a maxima occurs. When they destructively interfere a minima occurs. Since there is no transmitted radiation in the case of the perfect reflector no such maxima or minima occur in that case. This is seen in Figure (5).^{9c} Finer features of these curves are not so easily explained. Because of the complementary nature of the $\overline{\cos \phi} Q_{sca}$ term in equation (20) the major oscillations do not occur as strongly in the graph of Q_{pr} as they do in the graph of Q_{ext} .

To make an approximation for the Q_{pr} function we replace the curves in the graphs by simpler curves. There are two ways of doing this. The whole range of values of x from zero to infinity can be considered and a curve fit can be made which ignores the minor features of Q_{pr} (such as

"ripples"), and yet still has the general form of the graph of Q_{pr} . But if the minor features are important an alternate approach must be used. A small section of the curve will have to be considered and a curve fit made for it. For practical considerations it is these minor features that are important. To illustrate this we use equation (24). This equation gives the velocity for a particle which encounters values for x extending over an interval Δx .

Consider in Figure (4) the region around $x = 14$. If Δx is measured from the peak of the ripple occurring at this point to the first trough on the right, then $\Delta x \cong \frac{1}{2}$. The result of substituting these values into equation (24) is that $\beta \cong \frac{1}{60}$. Since many dust particles in space have velocities^{15a} averaging about 7 km/sec, these particles travel at speeds much smaller than one-sixtieth the speed of light and hence intervals smaller than one-half are important.

Suppose a particle moves in a monochromatic and isotropic radiation field and has a radius a such that $x_0 = 2\pi a/\lambda_0$ where λ_0 is the wavelength in the star system. Assume also that x_0 falls half way between a peak and trough of a "ripple". As the particle's velocity increases the value of the frequencies of radiation striking it on the front and back diverge. For small β it is seen from equation (21) that the values for x for radiation incident on the front and back diverge equal amounts from the value x_0 . The divergence of values of Q_{pr} reaches a maximum when the

difference between x for the radiation incident on the front and back differ by Δx equal to the distance between a peak and adjacent trough. At this velocity the maximum difference occurs between the Q_{pr} value for the radiation on the front and the radiation on the back. As the velocity increases beyond this point the difference between the front and back values of Q_{pr} will generally go to zero but increase again at still higher velocities.

Because the curve between the peaks and troughs of the "ripples" are very close to straight lines we will approximate the Q_{pr} function as a straight line in these regions. Therefore we write

$$Q_{pr} = m\bar{x} + b = m \, 2\pi a \, \frac{\bar{v}}{c} + b,$$

where m and b are constants whose values can be taken from the graph. Now using equation (10) the above expression for Q_{pr} can be written

$$Q_{pr} = m \frac{2\pi a}{c} v \frac{(1 - \beta^2)^{1/2}}{(1 + \beta \cos \bar{\phi})} + b \dots \dots \dots (25)$$

Substituting equation (25) into equation (5) we obtain under integration

$$F_z = \iint \frac{a^3 u \pi}{2c} m v \frac{(1 - \beta^2)^{5/2}}{(1 + \beta \cos \bar{\phi})^5} \cos \bar{\phi} \, d\bar{\Omega} + \iint \frac{a^2 u}{4} b \frac{(1 - \beta^2)^2 \cos \bar{\phi}}{(1 + \beta \cos \bar{\phi})^4} \, d\bar{\Omega}.$$

Designate the first integral on the right hand side of the above equation I_1 and the second integral I_2 . Since I_2 is the same as b times the integral in equation (11), we have

$$I_2 = -b a^2 \pi u \frac{4}{3} \frac{\beta}{(1 - \beta^2)} \dots \dots \dots (26)$$

As before, using the relations $s = \cos \bar{\phi}$ and $ds = -\sin \bar{\phi} d\bar{\phi}$ the first integral I_1 can be integrated with respect to $\bar{\theta}$ and written

$$I_1 = \frac{a^3 u \pi^2 m v}{c} (1 - \beta^2)^{5/2} \int_{-1}^1 \frac{s ds}{(1 + \beta s)^5}.$$

Evaluating the integral we find

$$I_1 = \frac{a^3 u \pi^2 m v}{c} \frac{(1 - \beta^2)^{5/2}}{\beta^2} \left[\frac{-1}{3(1 + \beta)^3} + \frac{1}{4(1 + \beta)^4} + \frac{1}{3(1 - \beta)^3} - \frac{1}{4(1 - \beta)^4} \right].$$

This simplifies to

$$I_1 = -\frac{a^3 u \pi^2 m v}{3c} \frac{(2\beta^3 + 10\beta)}{(1 - \beta^2)^{3/2}} \dots \dots \dots (27)$$

The expression for force exerted on the particle is obtained by adding equations (26) and (27):

$$F_z = -\frac{u}{3} \pi a^2 \left[\frac{m \pi a (2\beta^3 + 10\beta)}{\lambda_o (1 - \beta^2)^{3/2}} + \frac{4b\beta}{(1 - \beta^2)} \right]$$

$$\text{or } F_z = -\frac{u}{3} \pi a^2 \left[\frac{m x_o (\beta^3 + 5\beta)}{(1 - \beta^2)^{3/2}} + \frac{4b\beta}{(1 - \beta^2)} \right] \dots \dots \dots (28)$$

Retarding and Accelerating Forces

Now to find the force exerted on a particle in a region where the linear approximation holds we need only to substitute in the values for m and b and evaluate F_z from equation (28). It is seen that as β approaches zero, F_z also approaches zero as expected. The force experienced by equation (28) is the result of two effects. The first effect is due to the Doppler frequency shift which makes the frequency of radiation incident on the front higher than the frequency of radiation incident on the back of the sphere. The second effect, which we shall call the Q_{pr} effect, is due to the different values of radiation pressure efficiency factor Q_{pr} . The value of Q_{pr} for radiation incident on the front may be higher or lower than the value of Q_{pr} for radiation incident on the back. If it is higher, then the Q_{pr} effect adds to the Doppler effect to give a larger retarding force to the particle. If it is lower, then the retarding force will be lower than the force it would have experienced due to the Doppler effect alone. The strength of the Doppler effect increases as the particle's velocity increases. The strength of the Q_{pr} effect depends on the value of the slope m and the constant b . If the slope m is positive it is seen from equation (28) that the Doppler and Q_{pr} effects add to each other. This is simply due to the fact that when m is positive the higher frequency radiation (in front) has a larger Q_{pr} and therefore larger amounts of it are absorbed than the lower

frequency radiation (in back) which has a smaller Q_{pr} value. If however, m is negative, the higher frequency radiation will be less absorbed than the lower frequency radiation. When m is negative the magnitude of the force is smaller and the effect of the Doppler shift is diminished.

Looking at equation (28) it is seen also that if m is a large negative number it is possible for F_z to be zero, and for still larger negative values F_z will become positive. In these cases the Q_{pr} effect outweighs the Doppler effect. To find this critical value for m set equation (28) equal to zero and solve for m . For practical reasons we are most interested in small values of β so terms with β of higher order than one can be ignored. Thus equation (28) becomes

$$F_z = -\frac{u}{3} \pi a^2 (5mx_0 + 4b)\beta. \quad (29)$$

If $F_z = 0$ then $5mx_0 + 4b = 0$ and we find

$$m = -\frac{4}{5} \frac{b}{x_0} \quad (30)$$

where, as before, $x_0 = \frac{2\pi a}{\lambda_0}$ and λ_0 is the wavelength in the star system. It is seen that at low velocities if the value m is given by equation (30), the Q_{pr} effect will balance out the Doppler shift and the total force on the particle will be zero. It also follows that if m is more negative than the value given in equation (30), the total force on the particle will be positive. Thus the particle will experience an acceleration in the forward direction. Of course the

physical cause for this is that the particle absorbs more momentum from the back where the Q_{pr} function is larger than from the front, and consequently its velocity increases.

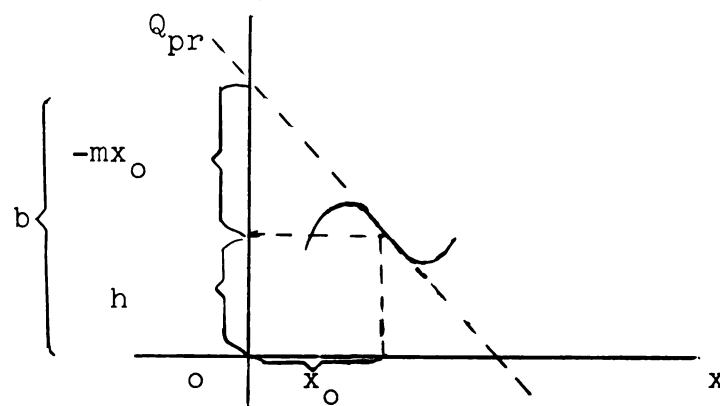
In order to make equation (30) more immediately understandable, make the substitution

$$b = h - mx_0 \dots \dots \dots (31)$$

This is illustrated in Figure (1). The value for Q_{pr} when $x = x_0$ is h . Equation (30) can be rewritten as

$$mx_0 = -\frac{4}{5} (h - mx_0).$$

FIGURE 1. Linear approximation for Q_{pr} .



$$\text{Therefore } mx_0 = -4h \text{ for } F_z = 0, \dots \dots \dots (32)$$

$$mx_0 < -4h \text{ for } F_z > 0, \dots \dots \dots (33)$$

$$mx_0 > -4h \text{ for } F_z < 0, \dots \dots \dots (34)$$

The interesting point of equations (32), (33) and (34) is that they are independent of velocity (at low velocities). The factor determining whether the force a particle experiences is positive, negative, or zero is determined by

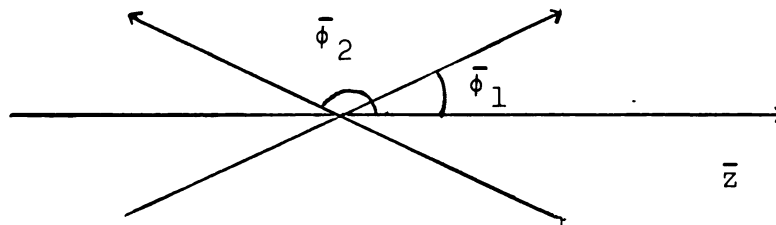
the relative size of mx_0 and h . Imagine if h were to increase significantly while the value of mx_0 remained the same. The fractional difference of the value of Q_{pr} for the radiation in front from that in the back of the particle would diminish. Thus the Doppler shift would dominate and the force would be negative for a large enough value for h . On the other hand if h is held constant and we imagine that mx_0 becomes more negative, then the values of Q_{pr} will differ more widely for the incident rays. This can be seen by re-writing equation (25) in the form

$$Q_{pr} = mx_0 \left(\frac{(1 - \beta^2)^{1/2}}{(1 + \beta \cos \bar{\phi})} - 1 \right) + h.$$

As mx_0 increases the fractional difference of values for Q_{pr} for the two angles $\bar{\phi}_1$ and $\bar{\phi}_2$ increases and enables the Q_{pr} effect to dominate.

It is interesting to look at the force exerted on the particle by two rays whose angles with the z axis are supplementary in the particle system as shown in Figure (2).

FIGURE 2. Supplementary ray diagram.



When β is small, higher orders of β can be neglected and the Doppler effect is a linear function of $\cos \bar{\phi}$. Since the Q_{pr} function is also linear over a small range of frequencies, and since both the Doppler function and the Q_{pr} function are anti-symmetric about $\bar{\phi} = \pi/2$ it is expected that the conditions stated by equations (32), (33), and (34) can be gotten by splitting up the isotropic radiation into pairs of rays which are anti-symmetric about $\bar{\phi} = \pi/2$. For the pair of rays in Figure (2) then

$$F_z = F_{zr}(\bar{\phi}_1) + F_{zr}(\bar{\phi}_2).$$

Substituting the force values given by equations (5) and (25), we have

$$F_z = a^2 \frac{u}{4} d\bar{\Omega} \left[mx_o \frac{\cos \bar{\phi}_1}{(1 + \beta \cos \bar{\phi}_1)^5} + \frac{b \cos \bar{\phi}_1}{(1 + \beta \cos \bar{\phi}_1)^4} + \frac{mx_o \cos \bar{\phi}_2}{(1 + \beta \cos \bar{\phi}_2)^5} + \frac{b \cos \bar{\phi}_2}{(1 + \beta \cos \bar{\phi}_2)^4} \right].$$

From Figure (2) $\bar{\phi}_1 = \pi - \bar{\phi}_2$ so $\cos \bar{\phi}_2 = -\cos \bar{\phi}_1$. Using this relation and expanding the denominators of the above equation to exclude terms of higher order than β we have

$$F_z = a^2 \frac{u}{4} d\bar{\Omega} \left[mx_o \cos \bar{\phi}_1 (1 - 5\beta \cos \bar{\phi}_1) + b \cos \bar{\phi}_1 (1 - 4\beta \cos \bar{\phi}_1) - mx_o \cos \bar{\phi}_1 (1 + 5\beta \cos \bar{\phi}_1) - b \cos \bar{\phi}_1 (1 + 4\beta \cos \bar{\phi}_1) \right]$$

or
$$F_z = -a^2 \frac{u}{4} d\bar{\Omega} \cos^2 \bar{\phi}_1 (5mx_o + 4b) \beta,$$

Therefore $F_z = 0$ when $m = -\frac{4}{5} \frac{b}{x_o}$.

This result agrees with equation (30). Equations (32), (33), and (34) also obviously agree. Thus when the net force on the particle is zero the particle absorbs equal amounts of momentum from the radiation at angles $\bar{\phi}_1$ and $\bar{\phi}_2$. The radiation at angle $\bar{\phi}_1$ has a larger frequency and higher density than the radiation at angle $\bar{\phi}_2$, but it has a smaller Q_{pr} factor which cancels out the increased momentum effect of the higher frequency and energy density.

Consider a dielectric particle with an index of refraction $n = 1.33$ as shown in Figure (4). If it is moving in a monochromatic and isotropic radiation field, and the ratio of its size to the wavelength of radiation is such that x_0 occurs at a point on its Q_{pr} graph where the slope has a large negative value, then at low velocities it will experience an accelerating force. However, as its velocity increases, the range of wavelengths of radiation striking it also increases, and at a large enough velocity the linear approximation for $Q_{pr}(x)$ will no longer hold. Then the force on the particle will decrease and eventually reach zero and the velocity will remain constant. By making a non-linear approximation for $Q_{pr}(x)$ this velocity can be calculated.

As an example we will assume $Q_{pr} = fd^3 + md + h$, where $d = x - x_0$, $h = Q_{pr}(x_0)$ and f and m are constants. Approximating equation (5) to the first order in β we find

$$F_{zr} = a^2 \frac{u}{4} Q_{pr} (1 - 4\beta \cos \bar{\phi}) \cos \bar{\phi} d\bar{\Omega}.$$

Using the same approximation for equation (10) we find

$$\bar{v} = v (1 - \beta \cos \bar{\phi}).$$

Therefore $d = x - x_0 = -x_0 \beta s$ where $s = \cos \bar{\phi}$. Substituting this expression for d into the above expression for Q_{pr} and then substituting this Q_{pr} function into the expression for F_{zr} yields under integration

$$F_z = a^2 u \frac{\pi}{2} \int_{-1}^1 (-fx_0^3 \beta^3 s^4 - m\beta x_0 s^2 + hs + 4fx_0^3 \beta^4 s^5 - 4mx_0 \beta^2 s^3 - 4\beta h s^2) ds$$

Carrying out the integration we find

$$F_z = -a^2 u \pi \left(\frac{1}{5} f x_0^3 \beta^3 + \frac{1}{3} m \beta x_0 + \frac{4}{3} \beta h \right).$$

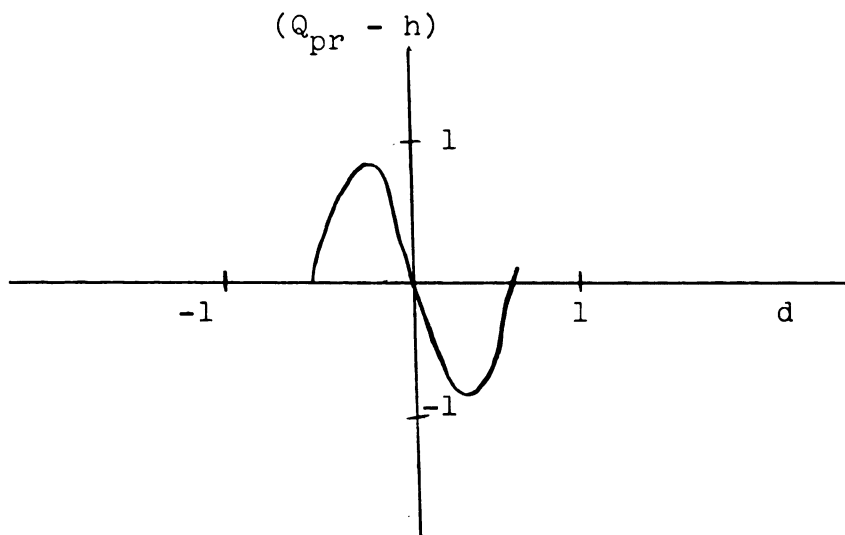
As expected the above equation reduces to equation (28) when the term containing β^3 is neglected. Setting $F_z = 0$ and solving for β we have

$$\beta = \sqrt{\frac{-mx_0 - 4h}{\frac{3}{5} f x_0^3}} \dots \dots \dots (35)$$

Consider a particle of a size such that x_0 lies half way between a peak and a trough in the Q_{pr} graph. One such value for x_0 would be approximately 12.5 in Figure (4). Q_{pr} can be approximated in this region by setting $f = 1.5$, $m = -.40$ and $h = .6$. A sketch of $(Q_{pr} - h)$ vs. d is contained in Figure (3). Setting these parameters into equation (35) we find $\beta = .04 = \frac{1}{25}$. At this velocity the radiation

exerts no force. The radiation exerts an accelerating force on the particle for velocities lower than $\beta = .04$ and a retarding force for larger velocities. This phenomenon depends on monochromatic radiation and can be expected to occur only in cases where the energy density of radiation is concentrated about a particular wavelength. Since interstellar radiation energy density is generally small a large amount of time would be necessary for the particle to achieve this velocity as a result of radiation pressure alone.

FIGURE 3.
Graph of Cubic Approximation for Q_{pr} .



Very Small Particle Approximations

Now we turn our attention to particles small compared to the wavelength of radiation. For these cases Debye^{9d} made approximations by considering three different sets of values for the conductivity and the dielectric

constant. For the dielectric with $\epsilon = \infty$ and $\sigma = 0$, Debye found that Q_{pr} is proportional to $\left(\frac{2\pi a}{\bar{\lambda}}\right)^4$. For finite values of ϵ , Q_{pr} is also proportional to $\left(\frac{2\pi a}{\bar{\lambda}}\right)^4$ but has a different proportionality constant. If σ is not zero then Q_{pr} is proportional to $\frac{2\pi a}{\bar{\lambda}}$. In all cases $\mu = 1$.

Now we are interested in calculating the effects of the radiation forces on very small particles. First consider the case where $\sigma \neq 0$. Debye gives the pressure efficiency factor on the particle due to a plane wave as

$$Q_{pr} = 12 \frac{\left(\frac{\sigma}{\omega}\right)}{(\epsilon + 2)^2 + \left(\frac{\sigma}{\omega}\right)^2} \cdot \frac{2\pi a}{\bar{\lambda}} \dots \dots \dots (36)$$

Rewrite this in the form

$$Q_{pr} = 4\bar{x}F(\bar{\lambda}) \dots \dots \dots (37)$$

$$\text{where } F(\bar{\lambda}) = \frac{3 \left(\frac{\sigma}{\omega}\right)}{(\epsilon + 2)^2 + \left(\frac{\sigma}{\omega}\right)^2},$$

In calculating the total force due to the radiation field a difficulty arises because for metals both ϵ and σ show a strong dependence on wavelength and therefore cannot be treated as constants. However the values of ϵ and σ have been experimentally determined for various metals and wavelengths, and from these values $F(\bar{\lambda})$ can be obtained.

J. L. Greenstein¹⁶ lists values for $F(\bar{\lambda})$ for iron and nickel at various wavelengths. It can be seen from his table that

for wavelengths near 5000A, $F(\bar{\lambda})$ can be approximated for these metals by the linear function

$$F(\bar{\lambda}) = r\bar{\lambda} + t \dots \dots \dots (38)$$

where r and t are constants. For iron Greenstein states the values of $F(\bar{\lambda})$ to be .68, .62, and .55 for the wavelengths of 4410A, 5080A, and 5890A respectively. For nickel he has $F(\bar{\lambda})$ equal to .33, .24, and .19 for the same respective wavelengths. These values also agree with computations made from the indexes of refraction given by Van de Hulst.^{12a} The function $F(\bar{\lambda})$ is closely approximated over this region by

$$\left. \begin{aligned} F(\bar{\lambda}) &= -8.8 \times 10^3 \bar{\lambda} + 1.07 \\ \text{and } F(\bar{\lambda}) &= -10^4 \bar{\lambda} + .76 \end{aligned} \right\} \dots \dots \dots (39)$$

for iron and nickel respectively. The wavelength is measured in centimeters.

Combining equations (37) and (38) we find

$$Q_{pr} = 4t\bar{x} + 8\pi ar \dots \dots \dots (40)$$

A linear approximation of $F(\bar{\lambda})$ in $\bar{\lambda}$ leads to a linear approximation of Q_{pr} as a function of $\bar{x}$. Thus from the values for r and t in equations (39) we have for iron

$$Q_{pr} = 4.3 \bar{x} - 2.2 \times 10^5 a, \dots \dots \dots (41)$$

and for nickel

$$Q_{pr} = 3.0 \bar{x} - 2.5 \times 10^5 a. \dots \dots \dots (42)$$

Since iron and nickel are thought to be relatively abundant in space, the effect of radiation pressure on them is of

interest. An estimate of this effect will be made later by use of the above Q_{pr} functions.

Now we consider the two other radiation pressure efficiency factors given by Debye.

For $\sigma = 0$ and $\epsilon = \infty$

$$Q_{pr_1} \cong \frac{14}{3} \left(\frac{2\pi a}{\bar{\lambda}} \right)^4 \dots \dots \dots (43)$$

For $\sigma = 0$ and finite ϵ

$$Q_{pr_2} \cong \frac{8}{3} \left(\frac{\epsilon - 1}{\epsilon + 1} \right)^2 \left(\frac{2\pi a}{\bar{\lambda}} \right)^4 \dots \dots \dots (44)$$

Both functions have the same dependence on $\bar{\lambda}$, they only differ in their constant coefficients. Writing these functions in terms of $\bar{v}$ and using equation (10), the resulting functions Q_{pr_1} and Q_{pr_2} are substituted into equation (5) and yield

$$F_z = K \iint \frac{4u\pi^4 a^6}{c^4} \frac{v^4 (1 - \beta^2)^4}{(1 + \beta \cos \bar{\phi})^8} \cos \bar{\phi} d\bar{\Omega}$$

where K represents the constant coefficients in either Q_{pr_1} or Q_{pr_2} . Now as before, using the substitution $s = \cos \bar{\phi}$, the above equation becomes

$$F_z = K \frac{4u\pi^4 a^6}{c^4} v^4 (1 - \beta^2)^4 \int_{-1}^1 \frac{2\pi s ds}{(1 + \beta s)^8}.$$

Integration yields

$$F_z = K \frac{8u\pi^5 a^6}{c^4} v^4 \frac{(1 - \beta^2)^4}{\beta^2} \left[-\frac{1}{6(1 + \beta)^6} + \frac{1}{7(1 + \beta)^7} + \frac{1}{6(1 - \beta)^6} - \frac{1}{7(1 - \beta)^7} \right].$$

By multiplying and rearranging the terms, this equation becomes

$$F_z = -128Ku\pi^5 \frac{a^6 v^4}{c^4} \frac{\left(\frac{1}{3}\beta + \frac{2}{3}\beta^3 + \frac{1}{7}\beta^5\right)}{(1 - \beta^2)^3}.$$

Hence for a particle with $\sigma = 0$ and $v = \infty$ the constant K has the value $\frac{14}{3}$ so

$$F_{z_1} = -7 \times 2^8 \frac{u}{3} \pi^5 a^6 \frac{v^4}{c^4} \frac{\left(\frac{1}{3}\beta + \frac{2}{3}\beta^3 + \frac{1}{7}\beta^5\right)}{(1 - \beta^2)^3}. \quad \dots (45)$$

For $\sigma = 0$ and ϵ finite, $K = \frac{8}{3} \left(\frac{\epsilon + 1}{\epsilon - 1}\right)^2$ so

$$F_{z_2} = -2^{10} \frac{u}{3} \pi^5 a^6 \frac{v^4}{c^4} \left(\frac{\epsilon - 1}{\epsilon + 1}\right) \frac{\left(\frac{1}{3}\beta + \frac{2}{3}\beta^3 + \frac{1}{7}\beta^5\right)}{(1 - \beta^2)^3} \dots (46)$$

Both F_{z_1} and F_{z_2} are proportional to a^6 and therefore become smaller much faster than does the mass. These forces are good approximations when the wavelengths are large compared to the particle's size. At higher relativistic velocities this approximation will eventually break down since the wavelength of radiation incident on the front of the particle is shortened.

At low velocities higher order terms may be ignored so equations (45) and (46) reduce to

$$F_{z_1} = -7 \times 2^8 \frac{u}{3} \pi^5 a^6 \frac{v^4}{c^4} \frac{\beta}{3} \cdot \cdot \cdot \cdot \cdot \cdot (47)$$

$$\text{and } F_{z_2} = -2^{10} \frac{u}{3} \pi^5 a^6 \frac{v^4}{c^4} \left(\frac{\epsilon - 1}{\epsilon + 1} \right) \frac{\beta}{3} \cdot \cdot \cdot \cdot \cdot \cdot (48)$$

It might seem that these forces are large because of the large value of v^4 . But it must be remembered that these forces are based on the approximation that $a/\lambda = \frac{av}{c}$ is much less than unity. Hence the radius of the particle will be very small and consequently both forces will be small.

CHAPTER IV

EFFECTS OF RADIATION ON PARTICLES IN INTERSTELLAR SPACE

The Time Constant t_e

In the preceding chapters we obtained expressions for the force exerted by the radiation on particles in terms of the following parameters: the radiation energy density, frequency, the particles radius, dielectric constant, and conductivity. However the actual magnitude of the forces are by themselves not very informative. The forces are small, but another parameter, the particle's mass, is involved and since the mass is also small the effect of the force on the particle's motion may be significant. In this chapter we will make numerical estimates of the effect of the radiation force by computing the time constant t_e for each case.

For practical reasons we are interested in particles traveling at low velocities compared to the speed of light. For this reason we use the first order approximations for the forces which are proportional to β . In general we have expressed all forces in the form

$$F = k\beta,$$

where k is a constant which can be either positive (accelerating forces) or negative (retarding forces). In most cases k is negative. Since we neglect relativistic mass changes at low velocities, we have

$$m_o \frac{dv}{dt} = k\beta.$$

The integral of this expression yields

$$v = v_o e^{\frac{kt}{m_o c}},$$

where v_o is the velocity at $t = 0$.

The value of its time constant is

$$t_e = \frac{cm_o}{|k|} \dots \dots \dots (49)$$

If k is negative then t_e is the time it takes a particle to reach $\frac{1}{e}$ times its velocity at $t = 0$. This result is independent of the initial velocity v_o . If k is positive t_e is the time it takes to increase its velocity to e times its initial velocity. The time constant is a reasonable indication of the effect of the force on the particle's motion. We will now evaluate it for the various forces previously obtained.

Numerical Results

If the force on the large perfect absorbing sphere is approximated for small β , then we have from equation (17)

$$t_e = \frac{3 m_o c}{4\pi a^2 u}.$$

This expression, in terms of the particle's density ρ , is written

$$t_e = \frac{\rho a c}{u} \dots \dots \dots (50)$$

The time constant is seen to be proportional to the particle's density and radius, and inversely proportional to the energy density. In interstellar space it is estimated^{15b} that

$$u = 12 \times 10^{-13} \text{ erg/cm}^3.$$

Consider a metal such as copper, iron or nickel with a density of approximately 8 gm/cm^3 . We choose a particle of radius $a = 2 \times 10^{-4} \text{ cm}$ which, for many cases, is large enough to exclude significant diffraction effects. For these values equation (50) yields

$$t_e \approx 4 \times 10^{19} \text{ seconds} \approx 10^{12} \text{ years}.$$

This is a long time indeed, even by astronomical standards. Astronomers¹⁷ have recently estimated the age of the galaxies to be about 10^{10} years. This is much too short a time for isotropic radiation to affect the motion of larger interstellar particles. The time constant above can be made smaller by decreasing the radius a , but any appreciably smaller value for a will necessitate the consideration of diffraction effects.

The above value for t_e was calculated for particles large compared to the largest wavelength making an important contribution to the energy density u . Now let us go to the

opposite extreme and consider particles small compared to the smallest wavelength contributing significantly to the radiation energy density. From equations (49) and (29) we find

$$t_e = \frac{c}{u} \frac{4a\rho}{(5mx_0 + 4b)} \quad (51)$$

Since b is seen in equations (41) and (42) to be proportional to a , the time constants for small metal spheres are independent of the particle's size. If we choose $\lambda = 5 \times 10^{-5}$ cm and consider iron of density $\rho = 8$ gm/cm³, the time constant can be evaluated from equation (51) using the values for m and b in equation (41). At these values we have for iron

$$t_e \approx 4 \times 10^{17} \text{ seconds} \approx 10^{10} \text{ years.}$$

For nickel, using the values from equation (42) and a density $\rho = 9$ gm/cm³, equation (51) yields

$$t_e \approx 10^{18} \text{ seconds} \approx 3 \times 10^{10} \text{ years}$$

Radiation pressure is generally larger for metal particles than for dielectric particles because metal both scatters and absorbs radiation. However, since the density of metal is also larger, this partly counterbalances the effect of the larger radiation pressure on the particle's motion.

Debye's approximation given in equation (36) holds when the ratio $2\pi a/\bar{\lambda}$ is on the order of .8 or smaller. It breaks down of course when the particle is so small that it does not have macroscopic qualities. The effect of radiation

pressure on metal particles larger than those considered here is complex and I have not made any estimates of t_e for those cases.

For small dielectric particles with $\epsilon = \infty$ (perfect reflector) we have, using equations (47) and (49)

$$t_e = \frac{9m_o c^5}{7 \times 2^8 u \pi^5 a^6 v^4} = \frac{3\rho a c}{28ux^4} \dots \dots \dots (52)$$

To find an order of magnitude for this time constant let

$x = \frac{1}{10}$, $\rho = 1 \text{ gm/cm}^3$, and $a = 10^{-6} \text{ cm}$ for $\lambda = 6000\text{\AA}$.

Therefore

$$t_e \cong 3 \times 10^{19} \text{ seconds} \cong 10^{12} \text{ years.}$$

However if $x = \frac{1}{3}$ and $a = 3 \times 10^{-6} \text{ cm}$, all other parameters remaining the same as above, we find

$$t_e \cong 7 \times 10^{17} \text{ seconds} \cong 2 \times 10^{10} \text{ years.}$$

Again the time constant is large but it is close to the estimated age of the galaxies.

For the case of a small dielectric particle with a dielectric constant ϵ we have from equation (48)

$$t_e = \left(\frac{\epsilon + 1}{\epsilon - 1} \right) \frac{9m_o c^5}{2^{10} \pi^5 u a^6 v^4} = \left(\frac{\epsilon + 1}{\epsilon - 1} \right) \frac{3\rho a c}{64ux^4} \dots \dots (53)$$

Comparing this to equation (52) we see that for $\epsilon = 1.5$ the values for t_e are about twice those given by equation (52) for the perfect reflector. Hence for the same two sets of parameters chosen for equation (52), equation (53) yields

$t_e = 2 \times 10^{12}$ years for the first case and

$t_e = 4 \times 10^{10}$ years for the second case.

It should be remembered that particles in interstellar space generally have low temperatures and this could be an important factor in determining the values for the dielectric constant and conductivity of a particle.

Particles with a size on the order of the radiation's wavelength are also important to consider. For such cases equation (29) is useful in finding the effects that the radiation has on the particle's motion. Since we are interested in low velocities, higher order term of β have been ignored in equation (29) and using equation (31) we write

$$F_z = -\frac{u}{3} \pi a^2 (mx_0 + 4h) \beta \dots \dots \dots (54)$$

Now consider Figures (4) and (5). Figure (4) is an example of Q_{pr} varying as a function of x for a dielectric ($n = 1.33$). Figure (5) shows the Q_{pr} for a perfect reflector. From these graphs we can measure the slopes at various values of x and substitute them into equation (54) to determine the force on the particle. In Figure (5) the greatest force exerted on a particle will occur where $x < 1$ since the slope there is very large. A representative point on this part of the graph can be taken as the point where $Q_{pr} = 1$. Thus $x = .7$ and $m = 5$. Equation (54) then yields

$$F_z = -\frac{8}{3} u \pi a^2 \beta.$$

If we choose $\lambda = 5 \times 10^{-5}$ cm and assume all energy density u has this wavelength we have then $a = 5 \times 10^{-6}$ cm and $F_z = -3 \times 10^{-22}$. To determine the effect of this force on the particle's motion we again compute the time constant. Substitute the coefficient of β from equation (54) into the denominator of equation (49) and write the mass in terms of its density ρ and radius a . The result is

$$t_e = \frac{4cap}{u|mx_0 + 4h|} \quad (55)$$

Assume $\rho = 1$ and the values of the parameters in the force above, we have then from equation (55)

$$t_e \cong 5 \times 10^{16} \text{ seconds} \cong 2 \times 10^9 \text{ years.}$$

In interstellar space however, the radiation density u does not consist of monochromatic radiation, but is made up of a continuum of different wavelengths. If this radiation is concentrated over a region of about $\frac{1}{5}\lambda$ then it can be seen from Figure (5) that the slopes for values of x in this region are of about the same size as the slope chosen above at $x = .7$. For such a case t_e would be expected to be of the same order of magnitude as computed above. Its exact value would, of course, depend on the distribution of energy density. For cases where $x > 1$ we see from Figure (5) that the slopes are negative. For a particle of radius $a = 5 \times 10^{-5}$ cm, again setting $\lambda = 5 \times 10^{-5}$ cm, we find $x \cong 6$. Therefore $m \cong 0$ and $h \cong 1$. Here we see that equation (55) reduces to equation (50) as expected for larger

particles. It is also seen that for dielectric material of density $\rho = 1$ equation (55) yields for these values

$$t_e = \frac{5}{4} \times 10^{18} \text{ seconds} \approx 4 \times 10^{10} \text{ years.}$$

For smaller values of x , where x is still greater than unity, the slope is negative but the value of h increases. In this region we see then that the value for t_e does not vary appreciably.

Figure (4) shows the behavior of Q_{pr} for $n = 1.33$ to be quite different from that for $n = \infty$ shown in Figure (5). The ripples in Figure (4) make the behavior of the particle's motion much more sensitive to the wavelength than in the case of the perfect reflector. For values of x ranging from about 2 to 5, values of t_e will be on the same order of magnitude as for the perfect reflector. If we choose $x = 4$ to represent this region we find that $t_e \approx 5 \times 10^{10}$ years when $\lambda = 5 \times 10^{-5}$ and $\rho = 1$. Of particular interest are the ripples occurring at values of x larger than 10. As was shown before, a particle may experience either a retarding or an accelerating force in this region. If the slope is positive it is a retarding force. If the slope has a large enough negative value the force accelerates the particle. Consider a typical value for the larger slopes of a particle with a value x_0 between 10 and 15. If the slope is positive then in the linear regions of the graph $m = 1$. If it is negative $m \approx -\frac{1}{2}$. Hence for a retarding force in the middle

of this region

$$t_e \approx 2 \times 10^{10} \text{ years}$$

when $\lambda = 5 \times 10^{-5}$ cm and $\rho = 1$. An accelerating force yields $t_e \approx 8 \times 10^{10}$ years in the same region, where now the time constant is the time necessary for the velocity to increase by a factor of e .

Again these results hold for monochromatic radiation. Unless the energy density is concentrated at certain wavelengths an averaging effect would be expected which would reduce the effects of the "ripples". In such a situation some wavelengths would exert a retarding force, while others would exert an accelerating force. If we assume a distribution among the wavelengths such that the value for Q_{pr} can be approximated by a straight line of slope zero, and $h \approx \frac{1}{2}$; then if $\rho = 1$ the time constant has a value between 10^{11} and 10^{12} years in this region.

CHAPTER V

SUMMARY

The derivation of the expression for radiation force was based on relativistic mechanics. The stress-energy tensor for a pencil of radiation was transformed from the star system to the particle system. Then by integrating the partial derivatives of its components the radiation force was obtained. This result was also shown to be in agreement with a result calculated from Robertson's equation.

The effects of the Doppler shift and the dependence of Q_{pr} on wavelength in an isotropic, and often monochromatic, radiation field have been investigated. We have shown that the total resulting force on a particle moving through the field will generally retard the motion of the particle; although it can in special cases accelerate a particle to a velocity where the total radiation force will become zero. It has also been shown that at non-relativistic speeds, the ratio of the initial velocity to the velocity at a later time will be independent of the initial velocity.

Finally, numerical estimates were made to determine the order of magnitude of the time constants for this effect

on particles in interstellar space. The results showed that the time constant has a value on the order of from 10^{10} to 10^{11} years for many cases. This value is comparable to some present day estimates of the age of the galaxies.

FIGURE (4)

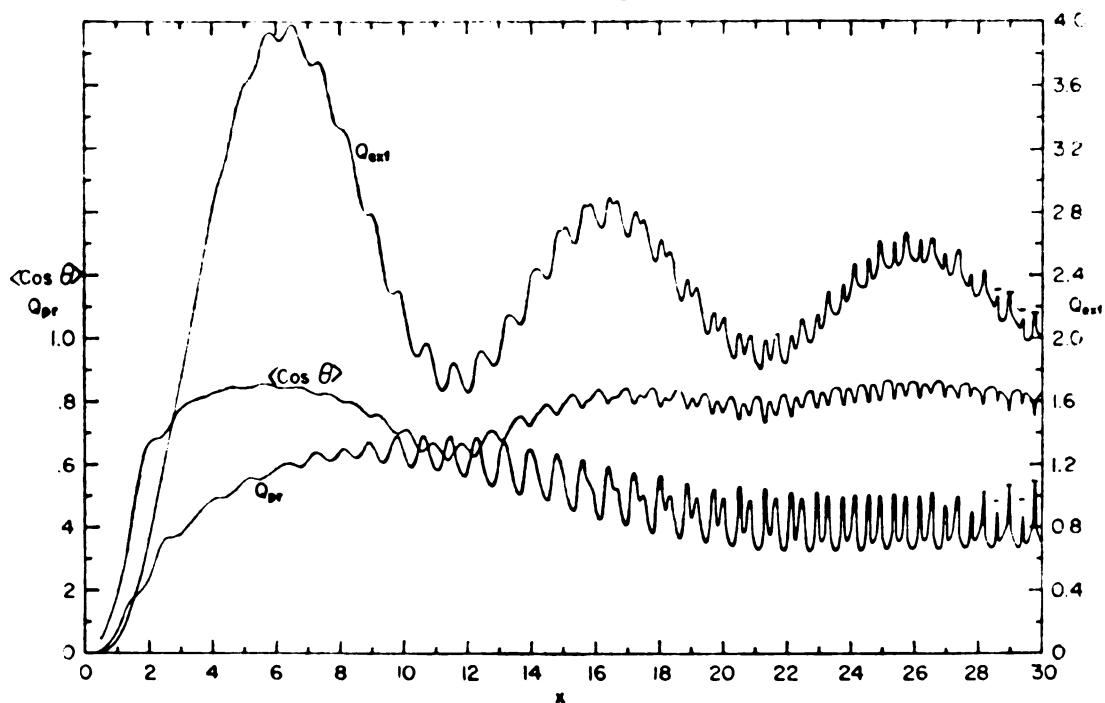
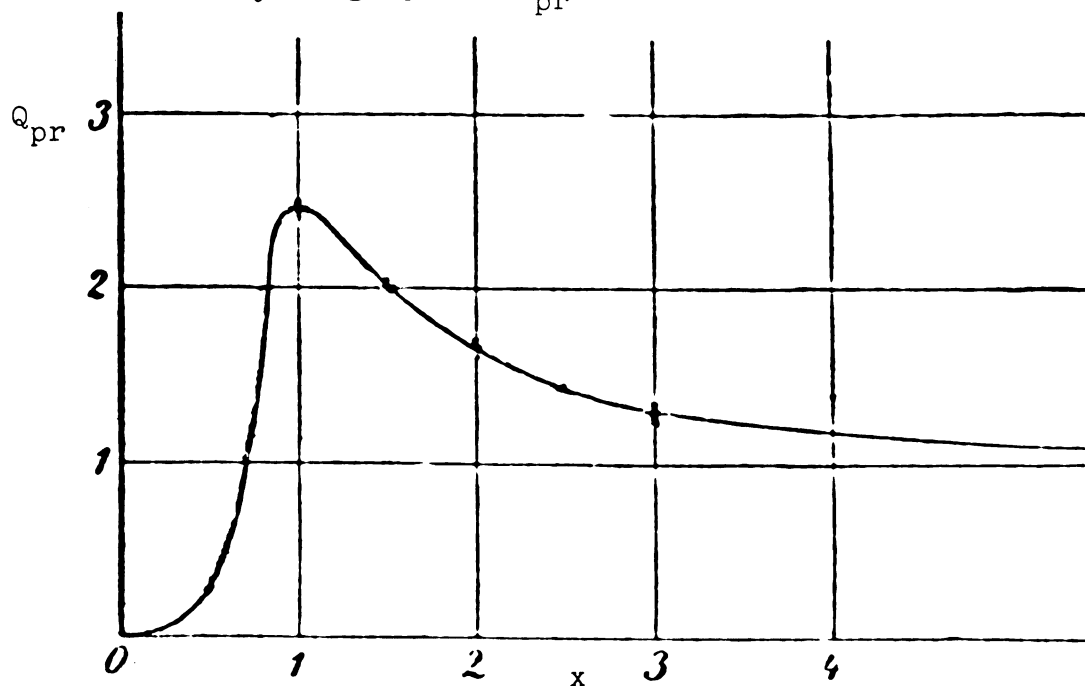
Irvine's graph of Q_{pr} for $n = 1.33$.

FIGURE (5)

Debye's graph of Q_{pr} for the perfect reflector

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