

INTERPOLATION OF STATIONARY RANDOM FIELDS
OVER LOCALLY COMPACT ABELIAN GROUPS

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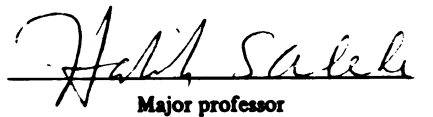
OVER LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT

INTERPOLATION OF STATIONARY RANDOM FIELDS OVER LOCALLY COMPACT ABELIAN GROUPS

By

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Let G be a locally compact abelian group. Let (x_g) be a stationary random process indexed by elements g of G .

A.N. Kolmogorov, P. Masani, and H. Salehi derived numerous results on the minimality and interpolation of random processes indexed by integers. The main efforts of this thesis are to derive similar results for processes indexed by the group G . Although the ideas and concepts used here are similar to the ones used by Salehi in his work, some of the techniques are different, since the integers are ordered and singly generated whereas an arbitrary group need not be.

First, the univariate case is considered. Results comparable to Kolmogorov's Minimality Theorem, the Wold Decomposition Theorem, and the Wold-Cramer Concordance Theorem are obtained. In addition, results similar to the work of H. Salehi on interpolation of stationary random processes are established. This subsumes a correct version of the recent work of L. Bruckner whose main theorem is in error.

Secondly, the multivariate case is considered. Under extra assumptions, most of the results of the univariate case are extended.

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Finally, there is a discussion of some open problems of the multivariate case, as well as infinite dimensional random processes.

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TABLE OF CONTENTS

	Page
1. INTRODUCTION	1
2. $\mathbb{R}^q$ -VALUED c.a.o.s. MEASURES AND STOCHASTIC INTEGRALS	5
3. PRELIMINARY RESULTS ON PREDICTION AND INTERPOLATION OF STATIONARY RANDOM FIELDS	12
4. MINIMALITY AND INTERPOLATION OF UNIVARIATE WSRF's ..	22
5. MINIMALITY AND INTERPOLATION OF q-VARIATE WSRF's ...	63
6. SOME EXAMPLES AND FURTHER REMARKS ON FINITE AND INFINITE DIMENSIONAL STATIONARY RANDOM FIELDS	92
REFERENCES	98

1. INTRODUCTION

The study of stationary stochastic processes was originated by A. Khintchine in 1934 [7]. In subsequent years the theory of stationary stochastic processes has undergone a remarkable development. The basic contribution to the theory of prediction of stationary stochastic processes is due to A.N. Kolmogorov [8], H. Cramer [2], and N. Wiener [31]. Kolmogorov was the first to formulate the basic problems of prediction and minimality of stationary stochastic processes indexed by integers. One of his most famous theorems is on the characterization of minimality for univariate processes in terms of spectral properties. P. Masani [13] extended the concept of minimality to multivariate stationary stochastic processes indexed by integers and obtained a similar characterization of minimality for such processes. H. Salehi [28] extended Masani's work, using generalized inverses, to deal with interpolation of multivariate stationary stochastic processes indexed by integers.

The idea of having processes indexed by elements of a group, instead of the integers, has attracted the attention of several mathematicians. Wang Shou-Jen [30] considered stationary random fields indexed by the lattice points of the plane. He was able to generalize Kolmogorov's minimality theorem for univariate stationary random fields indexed by these lattice points. Later, L. Bruckner [1] studied the question of minimality and interpolation of univariate stationary

stochastic processes indexed by elements of a discrete locally compact abelian group (LCAG). Some of the proofs of Bruckner seem to be in error. An example to justify this claim will be given later (cf. 6.3). Other aspects of the theory of stationary stochastic processes over LCAG's were studied by M. Rosenberg in [22].

In this paper the questions of minimality and interpolation of univariate, as well as multivariate, stationary random fields over LCAG's are systematically studied. Results motivated by those of H. Salehi's on minimality and interpolation of stationary stochastic processes over the integers will be established for stationary random fields over LCAG's. Although many of the ideas and concepts are similar to the ones used in Salehi's paper, some of the techniques used here will be different, since the integers are ordered and singly generated whereas an arbitrary group need not be.

With reference to this background we may now summarize the contents of this thesis and indicate the new results established.

In §2 we first recall the Hilbertian structure of the space $\mathcal{K}^q$, and then introduce the notion of a non-negative, hermitian, $q \times q$ matrix-valued measure $\underline{M}$ over an arbitrary measurable space $(\Omega, \mathcal{B})$. M. Rosenberg [21] defined the integral $\int_{\Omega} \underline{\phi} d\underline{M} \underline{\psi}^*$ for any measure $\underline{M}$ in such a way that the space $L_2(\Omega, \mathcal{B}, \underline{M})$ of $q \times q$ matrix-valued functions $\underline{\phi}$ for which $\int_{\Omega} \underline{\phi} d\underline{M} \underline{\phi}^*$ exists becomes a Hilbert space under the inner product $((\underline{\phi}, \underline{\psi})) = \text{tr}(\int_{\Omega} \underline{\phi} d\underline{M} \underline{\psi}^*)$. We will quote some of his results here. We then, following Rosenberg, introduce the concept of $\mathcal{K}^q$ -valued countably additive orthogonally scattered (c.a.o.s.) measures, and study briefly the theory of integration with respect to such measures.

In §3, we first review the theory of q -variate stationary random fields over LCAG's. We introduce several definitions and notations needed in the following sections. The notion of the rank of a process with respect to a given family of sets, as given here, is a direct generalization of the one given previously by Wiener and Masani [32] and will turn out to be very fruitful in studying several aspects of the theory of q -variate stationary random fields over LCAG's. We then state the Wold Decomposition Theorem, the Wold-Cramer Concordance Theorem, and some basic results due to Kolmogorov, Masani, Salehi, and others in their work on minimality and interpolation of stationary stochastic processes, since we will be primarily interested in these topics.

In §4 we consider a univariate stationary random field over a LCAG G . First, we state the Wold Decomposition Theorem for any family, $\mathcal{J}$, of non-empty Borel sets of G . Under the assumption that G is discrete, we establish several important results, such as Kolmogorov's minimality theorem and the Wold-Cramer Concordance Theorem. We also extend the work of H. Salehi on interpolation of stationary stochastic processes specialized to the univariate case to any stationary random field over any discrete LCAG.

In §5 the same problems as considered in §4 are studied for q -variate ($1 \leq q < \infty$) stationary random fields over LCAG's. Most of the results of the univariate case are extended, though, in some instances, extra assumptions are needed. The results of §5 extend those contained in §4 in the same spirit that Masani and Salehi's work generalized Kolmogorov's work from the univariate case to the multivariate case when the process is indexed by integers.

In §6 we will include several examples which were mentioned in the earlier sections. There will be a brief discussion on the open problems related to minimality and interpolation of q -variate stationary random fields over LCAG's. Also, a few remarks will be made on the minimality and interpolation of infinite-dimensional stationary random fields over LCAG's.

2. $\mathcal{H}^q$ -VALUED c.a.o.s. MEASURES AND STOCHASTIC INTEGRALS

In the first part of this section we review the theory of the spaces $\mathcal{H}^q$, where $\mathcal{H}$ is a Hilbert space. In the second part we shall consider the special cases $\mathcal{H}^q = \underline{L}_2(\Omega, \mathcal{B}, \underline{M})$, where $\underline{M}$ is a non-negative, hermitian, $q \times q$ matrix-valued measure, $\mathcal{H}$ being the space $\bar{L}_2(\Omega, \mathcal{B}, \underline{M})$ of q -dimensional (row) vector-valued functions on Ω . In the third part we define the notion of countably additive orthogonally scattered (c.a.o.s.) measures and study briefly the theory of integration with respect to such measures. These results will be used in later sections of this work.

2.1 Notation. Small underscored letters $\underline{x}$, $\underline{y}$, etc. will denote q -dimensional column vectors with complex components x_i , y_i , etc. Large underscored letters $\underline{A}$, $\underline{B}$, etc. will denote $q \times q$ matrices with complex entries a_{ij} , b_{ij} , etc. and $\underline{F}$, $\underline{G}$, etc. will denote $q \times q$ matrix-valued functions.

2.2 Definition. Let $\mathcal{H}$ be a complex Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$. The Cartesian product $\mathcal{H}^q$ is defined to be the set of all q -dimensional (column) vectors $\underline{x}$ with components in $\mathcal{H}$,

$$\text{i.e.,} \quad \underline{x} = (x_i)_{i=1}^q \in \mathcal{H}^q \Leftrightarrow x_i \in \mathcal{H} \quad i = 1, 2, \dots, q.$$

Addition of vectors in $\mathcal{H}^q$ and multiplication by $q \times q$ complex-valued matrices are defined as usual (cf. [32]).

2.3 Definition. Let $\underline{x} = (x_i)_{i=1}^q$ and $\underline{y} = (y_i)_{i=1}^q \in \mathbb{X}^q$.

Then (a) the Gramian $(\underline{x}, \underline{y})$ of $\underline{x}$ and $\underline{y}$ is defined by

$$(\underline{x}, \underline{y}) = [(x_i, y_j)] \quad 1 \leq i, j \leq q$$

where (x_i, y_j) is the inner product in $\mathbb{X}$. (The Gramian may be thought of as a matrix-valued inner product.)

(b) The inner product of $\underline{x}$ and $\underline{y}$ and the norm of $\underline{x}$ are defined by:

$$((\underline{x}, \underline{y})) = \text{tr}(\underline{x}, \underline{y}) \quad \text{and} \quad \|\underline{x}\| = \sqrt{\text{tr}(\underline{x}, \underline{x})}.$$

(c) We say that $\underline{x} \perp \underline{y} \Leftrightarrow (\underline{x}, \underline{y}) = \underline{0}$

i.e., for all $1 \leq i, j \leq q$, $(x_i, y_j) = 0$

{For the definition given in (c), we refer the reader to [32].}

2.4 Definition. (a) A linear manifold in $\mathbb{X}^q$ is a non-void subset $\mathbb{M}$ such that if $\underline{x}, \underline{y} \in \mathbb{M}$, then $\underline{A} \underline{x} + \underline{B} \underline{y} \in \mathbb{M}$ for all $q \times q$ matrices $\underline{A}, \underline{B}$.

(b) A subspace of $\mathbb{X}^q$ is a linear manifold which is closed in the topology of the norm $\|\cdot\|$.

(c) Let T be an operator on $\mathbb{X}$ into $\mathbb{X}$. Then the inflation $\underline{T}$ of T to $\mathbb{X}^q$ is defined as follows:

$$\text{for all } \underline{x} = (x_i)_{i=1}^q \in \mathbb{X}^q, \quad \underline{T}(\underline{x}) = (T(x_i))_{i=1}^q.$$

(d) For a given $\underline{x} \in \mathbb{X}^q$ and a subspace $\mathbb{M} \subseteq \mathbb{X}^q$, $(\underline{x}|\mathbb{M})$ will denote the orthogonal projection of $\underline{x}$ onto $\mathbb{M}$ (cf. [32], p. 132).

It is easy to see that T is a bounded linear operator on $\mathbb{X} \Leftrightarrow \underline{T}$ is a bounded linear operator on $\mathbb{X}^q$.

We shall now turn to a brief discussion of the Lebesgue integrals for $q \times q$ matrix-valued functions on a space Ω .

2.5 Definition. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space with μ a non-negative measure. Then for all δ , $0 < \delta \leq \infty$, we define $\underline{L}_\delta(\Omega, \mathcal{B}, \mu)$ as follows: $\underline{L}_\delta(\Omega, \mathcal{B}, \mu)$ consists of all $q \times q$ matrix-valued functions $\underline{F} = [f_{ij}]$ on Ω with complex-valued entries $f_{ij} \in L_\delta(\Omega, \mathcal{B}, \mu)$.

2.6 Definition. The integral of a function $\underline{F} = [f_{ij}] \in \underline{L}_1(\Omega, \mathcal{B}, \mu)$ is defined by

$$\int_{\Omega} \underline{F}(\omega) \mu(d\omega) = \left[\int_{\Omega} f_{ij}(\omega) \mu(d\omega) \right].$$

The following is a well-known result (cf. [32]).

2.7 Theorem. (a) $\underline{F} \in \underline{L}_\delta(\Omega, \mathcal{B}, \mu)$, $0 < \delta < \infty \Leftrightarrow \underline{F}$ has measurable entries and $|\underline{F}|_E^{(1)} \in L_\delta(\Omega, \mathcal{B}, \mu)$. $\underline{L}_\delta(\Omega, \mathcal{B}, \mu)$, $1 \leq \delta < \infty$ is a Banach space under the usual algebraic operations and the norm $\|\underline{F}\|_{\delta, \mu} = \left\{ \int_{\Omega} |\underline{F}(\omega)|_E^\delta \mu(d\omega) \right\}^{\frac{1}{\delta}}$.

(b) $\underline{L}_2(\Omega, \mathcal{B}, \mu)$ is a Hilbert space under the same operations and the inner product $((\underline{F}, \underline{G}))_\mu = \text{tr}(\underline{F}, \underline{G})_\mu$, where $(\underline{F}, \underline{G})_\mu = \int_{\Omega} \underline{F}(\omega) \underline{G}^*(\omega) \mu(d\omega)$ is the matricial inner product of $\underline{F}$ and $\underline{G}$.

(c) $\underline{F} \in \underline{L}_\infty(\Omega, \mathcal{B}, \mu) \Leftrightarrow \underline{F}$ has measurable entries and $|\underline{F}|_E$ is essentially bounded. $\underline{L}_\infty(\Omega, \mathcal{B}, \mu)$ is a Banach algebra under the usual algebraic operations and the norm $\|\underline{F}\|_\infty = \text{ess.l.u.b. } |\underline{F}(\omega)|_E$.

¹⁾ If $\underline{A}$ is a $p \times q$ matrix, then the Euclidean norm of $\underline{A}$ is

$$|\underline{A}|_E = \sqrt{\sum_{i=1}^p \sum_{j=1}^q |a_{ij}|^2}.$$

2.8 Remark. It is easy to see that the matricial inner product $(\underline{F}, \underline{G})_{\mu}$ is the Gramian of $\underline{F}$ and $\underline{G}$ (cf. 2.3 (a)), when we look upon $\underline{F}$ and $\underline{G}$ as elements of $\mathcal{K}^q$, where $\mathcal{K}$ is the space $L_2(\Omega, \mathcal{B}, \mu)$ of q -dimensional (row) vector-valued functions on Ω whose entries are in $L_2(\Omega, \mathcal{B}, \mu)$. This remark leads us to the following definition.

2.9 Definition. (a) If $\underline{F}, \underline{G} \in L_2(\Omega, \mathcal{B}, \mu)$, then we say that $\underline{F} \perp \underline{G} \Leftrightarrow (\underline{F}, \underline{G}) = 0$.

(b) A sequence $(\underline{F}_n)_{n=-\infty}^{\infty}$ in $L_2(\Omega, \mathcal{B}, \mu)$ is called orthonormal $\Leftrightarrow (\underline{F}_n, \underline{F}_m) = \delta_{nm} \underline{I}$.

2.10 Remark. In the Hilbert space $L_2(\Omega, \mathcal{B}, \mu)$ considered in 2.7 (b), we took μ to be a non-negative measure over $(\Omega, \mathcal{B})$. For many purposes it is necessary to consider $L_2(\Omega, \mathcal{B}, \underline{M})$, where $\underline{M}$ is a non-negative, hermitian, $q \times q$ matrix-valued measure over $(\Omega, \mathcal{B})$. Rosenberg [21] and Mandrekar and Salehi [9] have studied this question. For ease of reference, we will state the main result here.

2.11 Definition. Let $(\Omega, \mathcal{B})$ be a measurable space. Then $\underline{M} = [M_{ij}]$, $1 \leq i, j \leq q$ is called a (bounded) countably additive, non-negative, hermitian, $q \times q$ matrix-valued measure over $(\Omega, \mathcal{B})$ $\Leftrightarrow$

- (i) $1 \leq i, j \leq q$, M_{ij} is a countably additive (c.a.) (complex-valued) measure on $\mathcal{B}$.
- (ii) $\underline{M}$ is a non-negative, hermitian, $q \times q$ matrix-valued function on $\mathcal{B}$.

2.12 Remark. Let $\underline{M}$ be a non-negative, hermitian, $q \times q$ matrix-valued measure over $(\Omega, \mathcal{B})$, and let $\underline{\phi}, \underline{\psi}$ be $\mathcal{B}$ -measurable

$q \times q$ matrix-valued functions on Ω . Rosenberg has shown [21] that the integral $\int_{\Omega} \underline{\Phi} d\underline{M} \underline{\Psi}^*$ may be so defined that the space $L_2(\Omega, \mathcal{B}, \underline{M})$ of $q \times q$ matrix-valued functions $\underline{\Psi}$ such that $\int_{\Omega} \underline{\Psi} d\underline{M} \underline{\Psi}^*$ exists is a Hilbert space under the usual algebraic operations (cf. [32]), the inner product $((\underline{\Phi}, \underline{\Psi}))_{\underline{M}}$ and the norm $\|\underline{\Phi}\|_{2, \underline{M}}$ where

$$(2.13) \quad ((\underline{\Phi}, \underline{\Psi}))_{\underline{M}} = \text{tr}(\underline{\Phi}, \underline{\Psi})_{\underline{M}}; \quad \|\underline{\Phi}\|_{2, \underline{M}} = ((\underline{\Phi}, \underline{\Phi}))_{\underline{M}};$$

$$(\underline{\Phi}, \underline{\Psi})_{\underline{M}} = \int_{\Omega} \underline{\Phi}(\omega) \underline{M}(d\omega) \underline{\Psi}^*(\omega).$$

He observed that $\underline{M} \ll \text{tr } \underline{M}^{1)}$ and defined

$$(2.14) \quad \int_{\Omega} \underline{\Phi} d\underline{M} \underline{\Psi}^* = \int_{\Omega} \underline{\Phi} \frac{d\underline{M}}{d \text{tr } \underline{M}} \underline{\Psi}^* d \text{tr } \underline{M}$$

where $\frac{d\underline{M}}{d \text{tr } \underline{M}}$ is the matricial Radon-Nikodym derivative of $\underline{M}$ with respect to $\text{tr } \underline{M}$ and the R.H.S. of (2.14) is defined by (2.6).

We now turn to the discussion of $\mathcal{K}^q$ -valued c.a.o.s. measures and study briefly the theory of integration with respect to such measures. Rosenberg has studied this topic in detail. We will state several of his results which will be needed later.

2.15 Definition. Let (i) $\mathcal{K}$ be a complex Hilbert space.

(ii) $\underline{M}$ be a c.a., non-negative, hermitian $q \times q$ matrix-valued measure over $(\Omega, \mathcal{B})$ (cf. 2.11). Then a function $\underline{\xi}$ on $\mathcal{B}$ into $\mathcal{K}^q$ such that for all $B, C \in \mathcal{B}$

$$(\underline{\xi}(B), \underline{\xi}(C)) = \underline{M}(B \cap C)$$

¹⁾The symbol $\ll$ stands for absolute continuity. If $\underline{M}$ is a matrix-valued measure and μ is a scalar measure, then $\underline{M} \ll \mu$ means each entry M_{ij} of $\underline{M}$ is absolutely continuous with respect to μ .

is called an $\mathcal{K}^q$ -valued countably additive orthogonally scattered (c.a.o.s.) measure where $\underline{M}$ is a non-negative, hermitian, matrix-valued measure. When necessary, we shall write $\underline{M}_\xi$ instead of $\underline{M}$. $\underline{M}$ is called the associated measure of $\underline{\xi}$.

It easily follows that $\underline{\xi}(B) \perp \underline{\xi}(C)$ if $B \cap C = \varphi$, and $\underline{\xi}(\bigcup_k B_k) = \sum_k \underline{\xi}(B_k)$ if the B_k 's are disjoint, where the convergence on the right is in the $\mathcal{K}^q$ -norm.

There is a well established theory of integration with respect to such measures for $q \geq 1$. For ease of reference we shall restate the definition of integration and the main theorem.

2.16 Definition. (Step 1). For a simple function

$$\underline{\phi} = \sum_{k=1}^n \underline{A}_k \chi_{E_k}, \text{ where } \underline{A}_k \text{ are } q \times q \text{ matrices,}$$

$$\int_{\Omega} \underline{\phi} d\underline{\xi} = \sum_{k=1}^n \underline{A}_k \underline{\xi}(E_k).$$

A direct computation shows that if $(\underline{\phi}_n)_1^\infty$ is a sequence of simple functions then

$$\left\| \int_{\Omega} \underline{\phi}_m d\underline{\xi} - \int_{\Omega} \underline{\phi}_n d\underline{\xi} \right\| = \left\| \underline{\phi}_m - \underline{\phi}_n \right\|_{2, \underline{M}}.$$

Hence the following definition is unambiguous.

2.17 Definition. (Step 2). Let $\underline{\phi} \in L_2(\Omega, \mathcal{B}, \underline{M}_\xi)$. It is known (cf. [21], p. 296) that there exists a sequence $(\underline{\phi}_n)_1^\infty$ of simple functions such that $\underline{\phi}_n \rightarrow \underline{\phi}$ in $L_2(\Omega, \mathcal{B}, \underline{M}_\xi)$. We define

$$\int_{\Omega} \underline{\phi} d\underline{\xi} = \lim_{n \rightarrow \infty} \int_{\Omega} \underline{\phi}_n d\underline{\xi}.$$

The following is an important theorem on the subject.

2.18 Theorem. Let (i) $\underline{\xi}$ be an $\mathcal{K}^q$ -valued c.a.o.s. measure with associated measure $\underline{M}_{\underline{\xi}}$ over $(\Omega, \mathcal{B})$.

(ii) $S_{\underline{\xi}}$ be the set of all stochastic integrals $\int_{\Omega} \underline{\Phi}(\omega) \underline{\xi}(d\omega)$.

Then (a) $(\int_{\Omega} \underline{\Phi}(\omega) \underline{\xi}(d\omega), \int_{\Omega} \underline{\Psi}(\omega) \underline{\xi}(d\omega)) = (\underline{\Phi}, \underline{\Psi})_{\underline{M}_{\underline{\xi}}}$.

(b) The correspondence $\underline{\Phi} \rightarrow \int_{\Omega} \underline{\Phi}(\omega) \underline{\xi}(d\omega)$ is an isomorphism on

$\underline{L}_2(\Omega, \mathcal{B}, \underline{M}_{\underline{\xi}})$ onto $S_{\underline{\xi}}$. In particular, $S_{\underline{\xi}}$ is a closed subspace of $\mathcal{K}^q$.

3. PRELIMINARY RESULTS ON PREDICTION AND INTERPOLATION OF STATIONARY RANDOM FIELDS

Let G be a locally compact abelian group (LCAG) and G^* the dual group of G . Then G^* is also a LCAG under the compact-open topology. We will denote the elements of G by g and those of G^* by λ . The value of $\lambda \in G^*$ at $g \in G$ will be denoted by (g, λ) . The Borel field $\mathcal{B}$ of G is the σ -field generated by the open subsets of G . Similarly, $\mathcal{B}^*$, the Borel field of G^* , is defined. It is well known that G is discrete if and only if G^* is compact. Furthermore, it is known that there exists a regular Haar measure m defined on $\mathcal{B}^*$. Without loss of generality, when G is discrete, we will assume $m(G^*) = 1$.

We now give the definition of a q -variate stationary random field over G .

3.1 Definition. (a) A q -variate mean continuous weakly stationary random field (WSRF) over a LCAG G (under the operation

$+$) is a function $(\underline{x}_g)_{g \in G}$ such that

(i) $\underline{x}_g \in \mathcal{H}^q$ for each $g \in G$ ($\mathcal{H}$ is a fixed Hilbert space)

(ii) the $q \times q$ Gramian matrix $(\underline{x}_g, \underline{x}_{g'}) = \underline{\Gamma}(g - g')$ depends only on $g - g'$.

(iii) $(\underline{x}_g - \underline{x}_{g'}, \underline{x}_g - \underline{x}_{g'}) \rightarrow 0$ as $g - g' \rightarrow 0$ (mean continuity).

(b) The q -dimensional temporal domain $\mathcal{M}_x$ of a q -variate WSRF $(\underline{x}_g)_{g \in G}$ is the closed subspace of $\mathcal{H}^q$ spanned by the

$\underline{x}_g \in \mathcal{K}^q$, $g \in G$ with $q \times q$ matrix coefficients.

(c) Let $(\underline{x}_g)_{g \in G}$ and $(\underline{y}_g)_{g \in G}$ be $\mathcal{K}^q$ -valued WSRF's over the same G . We say that $(\underline{x}_g)_{g \in G}$ and $(\underline{y}_g)_{g \in G}$ are mutually homogeneously correlated if $(\underline{x}_g, \underline{y}_{g'})$ depends only on $g - g'$.

The following is contained in [22] and will be stated here for completeness.

3.2 Lemma. Let $(\underline{x}_g)_{g \in G}$ be an $\mathcal{K}^q$ -valued WSRF. Then there exists a strongly continuous group of unitary operators $(U_g)_{g \in G}$ on $\mathcal{K}$ such that for each $g \in G$, we have

$$\underline{x}_g = U_g \underline{x}_0 ,$$

where U_g is the inflation of U_g to $\mathcal{K}^q$ (cf. 2.4c).

We shall now recall the generalization of Stone's theorem ([22], Theorem 2.3) and of Bochner's theorem ([22], Theorem 2.4).

3.3 Stone's Theorem. Let $(U_g)_{g \in G}$ be a weakly continuous family of unitary operators on a Hilbert space $\mathcal{K}$ over a LCAG G . Then there exists a unique spectral measure $E(\cdot)$ defined on the Borel subsets of the dual group G^* such that

$$U_g = \int_{G^*} (g, \lambda) E(d\lambda) .$$

3.4 Bochner's Theorem. (a) f is a continuous positive definite complex-valued function on the LCAG G if, and only if, there exists a bounded non-negative regular measure m on the Borel subsets $\mathcal{B}^*$ of the dual group G^* such that for all $g \in G$

$$f(g) = \int_{G^*} (g, \lambda) m(d\lambda) .$$

(b) If for all $g \in G$,

$$\int_{G^*} (g, \lambda) m(d\lambda) = \int_{G^*} (g, \lambda) \mu(d\lambda)$$

where m and μ are bounded complex-valued regular measures on $\mathcal{B}^*$, then $m = \mu$.

3.5 Remark. It is known (cf. [22]) that if $E(\cdot)$ is a spectral measure defined on the Borel field $\mathcal{B}^*$ for a Hilbert space $\mathcal{H}$ and $\underline{E}(\cdot)$ is the inflation of $E(\cdot)$ to $\mathcal{H}^q$, then $\underline{E}(\cdot) \equiv E(\cdot)x_0$ is an $\mathcal{H}^q$ -valued c.a.o.s. measure. The non-negative, hermitian, $q \times q$ matrix-valued measure $\underline{F}$ defined by:

$$\underline{F}(B) = (\underline{E}(B), \underline{E}(B))$$

where $B \in \mathcal{B}^*$, is called the spectral distribution of the WSRF $(x_g)_{g \in G}$.

With this in mind, we state the following lemma (cf. [22], p. 339).

3.6 Lemma. Let $(U_g)_{g \in G}$ be the shift group of the $\mathcal{H}^q$ -valued WSRF $(x_g)_{g \in G}$ and let $E(\cdot)$ be the associated spectral measure. Let $\underline{S}$ be defined by:

$$\underline{S} = \{y \in \mathcal{H}^q : y = \int_{G^*} \underline{\Phi}(\lambda) \underline{E}(d\lambda) x_0, \underline{\Phi} \in L_2(G^*, \mathcal{B}^*, \underline{F})\}.$$

Then

$$\underline{M}_x = \underline{S}.$$

Now, applying (2.18) together with this lemma, we obtain the following important theorem.

3.7 Isomorphism Theorem. With the above notation, we have:

(a) For $\underline{\Phi}, \underline{\Psi} \in \underline{L}_2(G^*, \underline{\mathcal{B}}^*, \underline{F})$,

$$\left(\int_{G^*} \underline{\Phi}(\lambda) \underline{E}(d\lambda) \underline{x}_0, \int_{G^*} \underline{\Psi}(\lambda) \underline{E}(d\lambda) \underline{x}_0 \right) = (\underline{\Phi}, \underline{\Psi})_{\underline{F}} = \int_{G^*} \underline{\Phi}(\lambda) \underline{F}(d\lambda) \underline{\Psi}^*(\lambda).$$

(b) The correspondence $\underline{\Phi} \rightarrow \int_{G^*} \underline{\Phi}(\lambda) \underline{E}(d\lambda) \underline{x}_0$ is an isomorphism on $\underline{L}_2(G^*, \underline{\mathcal{B}}^*, \underline{F})$ onto $\underline{\mathcal{M}}_x$.

We will now introduce some new notations.

3.8 Notations. Let $\mathcal{J}$ be any family of Borel subsets of $\mathcal{B}$ closed under translations (i.e., if $I \in \mathcal{J}$, then $I + g \in \mathcal{J}$ for all $g \in G$). Let I be an arbitrary element of $\mathcal{J}$ and $(\underline{x}_g)_{g \in G}$ be an $\mathcal{H}^q$ -valued WSRF over G .

(i) We will let $\underline{\mathcal{M}}_{I,x}$ denote $\mathcal{S}(\underline{x}_g, g \in I)$; i.e., the closed subspace of $\mathcal{H}^q$ spanned by $\underline{x}_g, g \in I$.

(ii) We will let $\underline{\mathcal{M}}_{I^c,x}^\perp$ denote $\underline{\mathcal{M}}_{I,x}^\perp \cap \underline{\mathcal{M}}_x$; i.e., $\underline{\mathcal{M}}_{I^c,x}^\perp$ is the closed subspace of $\underline{\mathcal{M}}_x$ orthogonal to $\underline{\mathcal{M}}_{I,x}$.

(iii) We will let $\underline{\mathcal{M}}_{\mathcal{J},x} = \bigcap_{I \in \mathcal{J}} \underline{\mathcal{M}}_{I,x}$.

The following definition is a generalization of the concept of rank given by Wiener and Masani (cf. [32], p. 136).

3.9 Definition. Let $I \in \mathcal{B}$ and $g \in G$. Then the rank of the $\mathcal{H}^q$ -valued WSRF $(\underline{x}_g)_{g \in G}$ with respect to I and g , denoted by $\rho_{I,g}$, is defined to be the rank of the Gramian matrix of $\underline{x}_g - (\underline{x}_g | \underline{\mathcal{M}}_{I,x})$ with itself; i.e.,

$$\rho_{I,g} = \text{rank}(\underline{x}_g - (\underline{x}_g | \underline{\mathcal{M}}_{I,x}), \underline{x}_g - (\underline{x}_g | \underline{\mathcal{M}}_{I,x})).$$

3.10 Remark. If G is a discrete LCAG, the families $\mathcal{J}$ of Borel subsets of $\mathcal{B}$ with which we will be concerned are:

(i) $\mathcal{J}_0$ is the family of complements of singletons of G .
(ii) $J = \{g_0, g_1, \dots, g_n\} \subseteq G$. $\mathcal{J}_{g_0, g_1, \dots, g_n}$ is the family of the complements of the translates of J ; i.e., $\mathcal{J}_{g_0, \dots, g_n} = \{J^c + g, g \in G\}$, where J^c is the complement of J in G . For simplicity, when there is no danger of confusion, $\mathcal{J}_{g_0, \dots, g_n}$ will be denoted by $\mathcal{J}_n$.

(iii) $\mathcal{J}_\infty$ is the family of complements of finite subsets of G .

(iv) For $G = \mathbb{Z}$, the integers, $\mathcal{J}_p$ is the family of I_n 's where $I_n = \{k : k \leq n\}$.

We introduce here the following definitions which arose in this study.

3.11 Definition. Let $\mathcal{J}$ be a family of Borel subsets of G . Then

(i) An $\mathbb{R}^d$ -valued WSRF $(\underline{x}_g, g \in G)$ is called $\mathcal{J}$ -singular if for all $I \in \mathcal{J}$, $\mathbb{M}_{I, \underline{x}} = \mathbb{M}_{\underline{x}}$; i.e. $\mathbb{M}_{\mathcal{J}, \underline{x}} = \mathbb{M}_{\underline{x}}$.

(ii) An $\mathbb{R}^d$ -valued WSRF $(\underline{x}_g)_{g \in G}$ is called $\mathcal{J}$ -regular if $\mathbb{M}_{\mathcal{J}, \underline{x}} = \{\underline{0}\}$.

For $G = \mathbb{Z}$, the integers, Masani, Salehi, and others have introduced some of these notions under somewhat different terminology. To make the relation between their work and ours clear, we will state some of their results and make the appropriate comparisons.

3.12 Definition (Kolmogorov, Masani). An $\mathbb{R}^d$ -valued WSRF $(\underline{x}_n)_{n=-\infty}^{\infty}$ is said to be minimal if, and only if, $\underline{x}_0 \notin \mathbb{M}_{I, \underline{x}}$, where

$I = \{\dots -2, -1, 1, 2, \dots\}$.

3.13 Remark. It is easy to see that if $G = \mathbb{Z}$, an $\mathcal{H}^q$ -valued WSRF $(x_n)_{-\infty}^{\infty}$ is not minimal if, and only if, $(x_n)_{-\infty}^{\infty}$ is $\mathcal{J}_0$ -singular. Obviously, a WSRF is either minimal or not minimal.

However, in general, it is not true that a WSRF is either regular or singular, as the example in §6 (cf. (6.3)) shows. The statement of the main theorem (4.1) of L. Bruckner, which he considers his extension of Kolmogorov's minimality theorem, as well as its proof, is in error. The error stems from the fact that he claims that a WSRF is either $\mathcal{J}_0$ -regular or $\mathcal{J}_0$ -singular. The exact relationship between the two concepts of regularity and minimality for a WSRF will be given in Theorem 4.8.

In §4, we will define the concept of minimality for any discrete LCAG, and extend Kolmogorov's minimality theorem.

To give the flavor of the types of theorems proved in §5, we give some existing results for integers. The following, Kolmogorov's minimality theorem (cf. [13], Theorem 2.8) is one of the most fundamental results of this theory.

3.14 Theorem (Masani). Let $(x_n)_{-\infty}^{\infty}$ be an $\mathcal{H}^q$ -valued WSRF with spectral distribution $\underline{F}$. Let $\rho_{I,0}$ be the rank of $(x_n)_{-\infty}^{\infty}$ with respect to $I = \{\dots, -2, -1, 1, 2, \dots\}$ and 0. Then $(x_n)_{-\infty}^{\infty}$ is minimal and $\rho_{I,0} = q$ if, and only if, $\underline{F}'$ is invertible a.e. on $\mathbb{C}$, and $\underline{F}'^{-1} \in \underline{L}_1$.

Also, in [28], H. Salehi introduces the notion of interpolation for $\mathcal{H}^q$ -valued WSRF's over the integers. He proves several theorems on the interpolability of a given WSRF in terms of the spectral

distribution of the random field. From one of his theorems (cf. [28], Theorem 2) he deduces the following, from which Masani's multivariate extension of the minimality theorem follows.

3.15 Theorem (Salehi). Let $(x_k)_{-\infty}^{\infty}$ be an $\mathbb{K}^q$ -valued WSRF. Let $I = \{\dots, -2, -1, 1, 2, \dots\}$ and let z_0 be the orthogonal projection of x_0 onto the subspace $\mathcal{M}_{I,x}^{\perp}$. Let $z_k = U^k z_0$ where $(U^k)_{-\infty}^{\infty}$ is the associated shift group of $(x_k)_{-\infty}^{\infty}$. Let $y_k = (z_0, z_0)^{\#} z_k$ where $(z_0, z_0)^{\#}$ is the generalized inverse of (z_0, z_0) (cf. [20], p. 355). Then

$$(a) \quad (z_0, z_0)^{\#} = (y_0, y_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\underline{J}(d\lambda)^2 \underline{J}}{d\underline{F}} \quad 1)$$

where $\underline{J}$ is the projection matrix on the space C^q of q -tuples of complex numbers onto the range of (z_0, z_0) in the privileged basis of C^q .

$$(b) \quad (x_k)_{-\infty}^{\infty} \text{ is minimal iff } \int_{-\pi}^{\pi} \frac{\underline{J}(d\lambda)^2 \underline{J}}{d\underline{F}} \neq \underline{0} .$$

In §5, we will extend Salehi's theorems of interpolation of WSRF's with respect to the integers to $\mathbb{K}^q$ -valued WSRF's over any LCAG. Our extension will yield a generalization of Masani's (Kolmogorov's) minimality theorem from integers to any LCAG.

We may add that our notion of $\mathcal{J}_n$ -singularity coincides with the concept of interpolation of a finite set of integers introduced by Salehi [28]. Similarly, our notion of $\mathcal{J}_{\infty}$ -singularity and Salehi's concept of interpolability of the entire random field are the same.

1) For the definition of $\int_{-\pi}^{\pi} \frac{\underline{J}(d\lambda)^2 \underline{J}}{d\underline{F}}$, see ([28], p. 308).

We will now state some known results (most of which concern WSRF's over integers) which we will use in the later sections in connection with our results about the concordance of the Wold and Cramer decompositions. We start with Cramer's decomposition.

3.16 Remark. Let $\underline{N}$ be a non-negative, hermitian, $q \times q$ matrix-valued measure defined on the family of Borel subsets of $(-\infty, \infty)$. Let μ be a σ -finite, non-negative, scalar-valued measure on the same family. Then there exist unique matricial measures $\underline{N}^a$ and $\underline{N}^s$ such that $\underline{N} = \underline{N}^a + \underline{N}^s$, $\underline{N}^a \ll \mu$, $\underline{N}^s \perp \mu$ ¹⁾ and $\underline{N}^a$ and $\underline{N}^s$ are non-negative, hermitian measures. This was proved by Cramer [2] and goes by his name.

The following, a finite dimensional Cramer decomposition theorem, can be derived from Mandrekar and Salehi's result ([12], Theorem 3.15).

3.17 Theorem (Cramer's decomposition). Let $\underline{F}$ be the spectral distribution associated with the $\mathbb{K}^q$ -valued WSRF $(\underline{x}_g)_{g \in G}$ where G is a LCAG. Let m be the Haar measure on $\mathcal{B}^*$, the family of Borel sets of G^* . Then

$$\underline{F} = \underline{F}^a + \underline{F}^s$$

where $\underline{F}^a \ll m$, $\underline{F}^s \perp m$, and both $\underline{F}^a$ and $\underline{F}^s$ are non-negative, hermitian. For simplicity, the Radon-Nikodym derivative of $\underline{F}^a$ with respect to the Haar measure m is called the spectral density of the WSRF $(\underline{x}_g)_{g \in G}$.

1) $\underline{N}^s \perp \mu$ means each component N_{ij}^s of $\underline{N}^s$ is singular with respect to the measure μ .

On the other hand, for multivariate processes indexed by integers, the Wold decomposition theorem was proved by Wiener and Masani ([32], Theorem 6.11).

3.18 Theorem. Let $(\underline{x}_n)_{-\infty}^{\infty}$ be an $\mathbb{R}^q$ -valued WSRF. If $(\underline{x}_n)_{-\infty}^{\infty}$ is non-deterministic; i.e., for some n , $\underline{x}_n \notin \mathbb{M}_{I_{n-1}, \underline{x}}$, $I_n = \{k : k \leq n\}$, then

$$\underline{x}_n = \underline{u}_n + \underline{v}_n$$

where (i) $\underline{u}_n \perp \underline{v}_n$ for all n ,

(ii) $(\underline{u}_n)_{-\infty}^{\infty}$ is purely non-deterministic; i.e., $\mathbb{M}_{\mathcal{J}, \underline{u}} = \{\underline{0}\}$,

(iii) $(\underline{v}_n)_{-\infty}^{\infty}$ is deterministic; i.e., $\mathbb{M}_{\mathcal{J}, \underline{v}} = \mathbb{M}_{\underline{v}}$.

In §5 we will prove a Wold decomposition theorem with respect to a given family $\mathcal{J}$, closed under translations, of Borel subsets of G , a LCAG.

Under certain assumptions, there is concordance between the Wold decomposition in the time domain and the Cramer decomposition in the spectral domain. The following theorem is due to Wiener and Masani ([32], Theorem 7.11).

3.19 Theorem. Let (i) $(\underline{x}_n)_{-\infty}^{\infty}$ be an $\mathbb{R}^q$ -valued WSRF and $I_0 = \{\dots -2, -1, 0\}$; (ii) $\rho_{I_0, 1} = q$; (iii) $\underline{x}_n = \underline{u}_n + \underline{v}_n$ be its Wold decomposition; (iv) $\underline{F}, \underline{F}_{\underline{u}}, \underline{F}_{\underline{v}}$ be the spectral distributions of the random fields $(\underline{x}_n)_{-\infty}^{\infty}, (\underline{u}_n)_{-\infty}^{\infty}, (\underline{v}_n)_{-\infty}^{\infty}$ respectively; (v) $\underline{F}^a, \underline{F}^s$ be the absolutely continuous and singular parts of $\underline{F}$ respectively. Then

$$\underline{F}_{\underline{u}} = \underline{F}^a ; \quad \underline{F}_{\underline{v}} = \underline{F}^s .$$

Later, Robertson ([19], Theorem 5.2) obtained the concordance result under a weaker assumption.

3.20 Theorem. For any $\mathcal{K}^q$ -valued WSRF $(\underline{x}_n)_{-\infty}^{\infty}$ with $\rho_{I_0,1} = \rho$ ($0 \leq \rho \leq q$), there is concordance between the Wold decomposition and the Cramer decomposition if, and only if, $\text{rank } \underline{F}'(\theta) = \rho$ a.e. (Leb.), where $\underline{F}$ is the spectral distribution of the random field $(\underline{x}_n)_{-\infty}^{\infty}$ and $\underline{F}'$ is the derivative of $\underline{F}$ with respect to the Haar measure on the circle.

In §4 and §5, we will prove a Wold-Cramer concordance theorem for the cases $\mathcal{J} = \mathcal{J}_0$, $\mathcal{J} = \mathcal{J}_n$, and $\mathcal{J} = \mathcal{J}_{\infty}$.

3.21 Remark. As we already mentioned in the first paragraph of this section, the group G is discrete if, and only if, its dual G^* is compact. In this case, the Haar measure of G^* will be finite. These facts allow us to stay in the framework of the theory of Fourier series rather than Fourier transforms.

For a non-discrete group G the notion of minimality of a WSRF over G has not been treated in the literature. This may be due to the fact that in a discrete group every point is a neighborhood of itself while in a general group this may not be the case. However, one may consider the problem of interpolation for WSRF's over a group which is not necessarily discrete (See [25], [29] and [35]). The results for the non-discrete case do not follow from those of the discrete case and need a separate discussion involving the theory of Fourier transforms.

For these reasons and in order to keep this thesis within its previously stated confines we are not including the theory of minimality and interpolation of WSRF's over a non-discrete group.

4. MINIMALITY AND INTERPOLATION OF UNIVARIATE WSRF's

In this section we will extend Kolmogorov's minimality theorem (Theorem 4.7) for univariate WSRF's over the integers to LCAG's. L. Bruckner has considered this case. As we mentioned in Remark 3.13, his main theorem is in error. (The earlier work of Rozanov on WSRF's indexed by integers [24] also indicates Bruckner's mistake.) Among other things we will give a corrected version of his theorem (Corollary 4.9) and give the exact relationship between the concepts of singularity and regularity defined by Bruckner and the notion of minimality given by Kolmogorov and Masani (Theorem 4.8).

We will state the Wold decomposition theorem (Theorem 4.2) for univariate WSRF's with respect to a family $\mathcal{J}$, closed under translations, of nonempty Borel subsets of a LCAG G . This provides an extension of the usual Wold decomposition theorem given for WSRF's over the integers. Using our result on minimality we will then establish the concordance relation between the Cramer decomposition and the Wold decomposition theorems (Theorem 4.13). This will constitute a natural extension for the univariate case of the same result given by Wiener and Masani ([32], p. 146) and Doob ([3], p. 576) for WSRF's over the integers.

We will specialize our result on minimality to the case where the random field is over the integers. In this case, the notions of past, future, and past & future are well defined. Using our results, we will examine the relationship between the past and the past &

future of such a random field.

We finally give an extension of Salehi's results on interpolation of random processes over the integers to univariate WSRF's over LCAG's. This will provide a natural extension of earlier results of this section in the same way that Salehi's work on interpolation provided a generalization of Kolmogorov and Masani's work on minimality. Although the ideas and concepts used here are similar to the ones used by Salehi in his work, some of our techniques are different, since the integers are ordered and singly generated whereas an arbitrary group need not be.

The main reason we have considered the univariate case separately is that Kolmogorov's minimality theorem and most of the results of Salehi on interpolation theory can be extended without any further assumptions. To get the corresponding results in the multivariate case we have to make certain assumptions, under which we are able to carry out our work.

Throughout this section only univariate WSRF's will be considered. We first state the Wold decomposition theorem whose proof is given in L. Bruckner's paper for the univariate case. First, though, we will need the following definition.

4.1 Definition. Let $(x_g)_{g \in G}$ and $(y_g)_{g \in G}$ be univariate WSRF's over a LCAG G . Let $\mathcal{J}$ be any family of non-empty Borel sets of G . Then $(y_g)_{g \in G}$ is said to be $\mathcal{J}$ -subordinate to $(x_g)_{g \in G}$ if

- (i) $\mathcal{M}_y \subseteq \mathcal{M}_x$;
- (ii) $\mathcal{M}_{I,y} \subseteq \mathcal{M}_{I,x}$ for all $I \in \mathcal{J}$;
- (iii) $(x_g)_{g \in G}$ and $(y_g)_{g \in G}$ are mutually homogeneously correlated.

We will now state the Wold decomposition theorem for a univariate WSRF over a LCAG G .

4.2 Theorem (Wold decomposition). Let $\mathcal{J}$ be any family of non-empty Borel sets of G closed under translations. Let $(x_g)_{g \in G}$ be a univariate WSRF with values in $\mathcal{X}$. Then there exists a unique decomposition of $(x_g)_{g \in G}$ with respect to $\mathcal{J}$ in the form

$$x_g = y_g + w_g$$

where

- (i) $(y_g)_{g \in G}$ and $(w_g)_{g \in G}$ are $\mathcal{X}$ -valued WSRF's on G ;
- (ii) $(y_g)_{g \in G}$ and $(w_g)_{g \in G}$ are $\mathcal{J}$ -subordinate to $(x_g)_{g \in G}$;
- (iii) $(y_g)_{g \in G}$ and $(w_g)_{g \in G}$ are orthogonal; i.e.,
 $(y_g, w_{g'}) = 0$ for any $g, g' \in G$;
- (iv) $(y_g)_{g \in G}$ is $\mathcal{J}$ -regular; $(w_g)_{g \in G}$ is $\mathcal{J}$ -singular.

We will now state the definition of minimality for a univariate WSRF over a discrete LCAG.

4.3 Definition. Let G be a discrete LCAG. Then the univariate $\mathcal{X}$ -valued WSRF $(x_g)_{g \in G}$ is minimal if, and only if, $x_0 \notin \mathcal{M}_{I, x}$ where $I = \{0\}^c$ ¹⁾.

The proof of the minimality theorem for WSRF's over a discrete LCAG will depend on the following lemmas.

4.4 Lemma. Let $(x_g)_{g \in G}$ be an $\mathcal{X}$ -valued WSRF over a discrete LCAG G . Let $\hat{x}_g$ denote $x_g - (x_g | \mathcal{M}_{I+g, x})$, where $I = \{0\}^c$. Then $(\hat{x}_g)_{g \in G}$ is an $\mathcal{X}$ -valued WSRF over G . Also, $(x_g)_{g \in G}$ and

¹⁾ $\{0\}^c$ will stand for the complement of the zero element of G .

$(\hat{x}_g)_{g \in G}$ have the same shift.

Proof. Let $(U_g)_{g \in G}$ be the unitary group of shift operators associated with $(x_g)_{g \in G}$. Then $U_g \hat{x}_g = U_g x_g - U_g (x_g | \mathcal{M}_{I+g, x}) = x_{g+g} - U_g (x_g | \mathcal{M}_{I+g, x})$. In particular, $U_g \hat{x}_0 = x_g - U_g (x_0 | \mathcal{M}_{I, x})$. Since $U_g \mathcal{M}_{I, x} = \mathcal{M}_{I+g, x}$, it follows that $U_g (x_0 | \mathcal{M}_{I, x}) = (x_g | \mathcal{M}_{I+g, x})$. Hence $((x_g | \mathcal{M}_{I+g, x}))_{g \in G}$ is a WSRF, which implies that $(\hat{x}_g)_{g \in G}$ is also a WSRF. Q.E.D.

4.5 Lemma. Let $(x_g)_{g \in G}$ be an $\mathcal{M}$ -valued WSRF. Then $(x_g)_{g \in G}$ is minimal if, and only if, for all $g \in G$, $x_g \notin \mathcal{M}_{I+g, x}$, $I = \{0\}^c$.

Proof. The proof of sufficiency follows trivially. If $(x_g)_{g \in G}$ is minimal, then by (4.3) $x_0 \notin \mathcal{M}_{I, x}$. But $U_g \mathcal{M}_{I, x} = \mathcal{M}_{I+g, x}$ implies that $x_g \notin \mathcal{M}_{I+g, x}$. Therefore $x_g \notin \mathcal{M}_{I+g, x}$ for all $g \in G$. Q.E.D.

The next lemma plays an important role in the theory of minimality of WSRF's.

4.6 Lemma (Main Lemma I). Let $(x_g)_{g \in G}$ be an $\mathcal{M}$ -valued WSRF over a discrete LCAG G with the shift group of unitary operators $(U_g)_{g \in G}$ and E be the spectral measure of $(U_g)_{g \in G}$. Let $\hat{x}_g = x_g - (x_g | \mathcal{M}_{I+g, x})$, $I = \{0\}^c$ and $\sigma = |\hat{x}_0|^2$. Let F be the spectral distribution of $(x_g)_{g \in G}$ and f be its spectral density; i.e., f is the Radon Nikodym derivative of F^a {absolutely continuous component of F as in 3.17} with respect to the Haar measure. Then

$$\hat{x}_g = \int_{G^*} (g, \lambda) \varphi_0(\lambda) E(d\lambda) x_0$$

where φ_0 is defined by

$$\varphi_0 = \begin{cases} \sigma/f, & \text{on the carrier of } F^a \\ 0, & \text{on the carrier of } F^s. \end{cases}$$

Proof. Without any loss of generality, it suffices to prove that $\hat{x}_0$ is given by

$$\hat{x}_0 = \int_{G^*} \varphi_0(\lambda) E(d\lambda) x_0.$$

Since $\hat{x}_0 \in \mathcal{M}_x$, by the Isomorphism Theorem (3.7)

$$\hat{x}_0 = \int_{G^*} \varphi_0(\lambda) E(d\lambda) x_0$$

for some $\varphi_0 \in L_2(G^*, \mathcal{B}^*, F)$. Also, (cf. 3.2),

$$x_g = \int_{G^*} (g, \lambda) E(d\lambda) x_0.$$

Therefore,

$$\begin{aligned} (I) \quad (x_g, \hat{x}_0) &= \int_{G^*} (g, \lambda) \overline{\varphi_0}(\lambda) F(d\lambda) \\ &= \int_{G^*} (g, \lambda) \overline{\varphi_0}(\lambda) f(\lambda) m(d\lambda) + \int_{G^*} (g, \lambda) \overline{\varphi_0}(\lambda) F^s(d\lambda). \end{aligned}$$

Recalling that $\hat{x}_0 = x_0 - (x_0|_{\mathcal{M}_{I,x}})$, we see that $(x_g, \hat{x}_0) = \delta_{g,0} \sigma$ where $\delta_{g,0}$ is the usual Kronecker delta and σ is as above. From [26], p. 10, we know that $\int_{G^*} (g, \lambda) m(d\lambda) = \delta_{g,0}$. Therefore,

$$(II) \quad (x_g, \hat{x}_0) = \sigma \int_{G^*} (g, \lambda) m(d\lambda).$$

Combining (I) and (II), we get the following equation:

$$\sigma \int_{G^*} (g, \lambda) m(d\lambda) = \int_{G^*} (g, \lambda) \overline{\varphi_0}(\lambda) f(\lambda) m(d\lambda) + \int_{G^*} (g, \lambda) \overline{\varphi_0}(\lambda) F^s(d\lambda)$$

which in turn is equivalent to:

$$(III) \quad \int_{G^*} (g, \lambda) [\sigma - \bar{\varphi}_0(\lambda)] m(d\lambda) = \int_{G^*} (g, \lambda) \bar{\varphi}_0(\lambda) F^s(d\lambda) .$$

By Bochner's Theorem (cf. Theorem 3.4), we get that

$$(IV) \quad \int_{B^*} (\sigma - \bar{\varphi}_0(\lambda) f(\lambda)) m(d\lambda) = \int_{B^*} \bar{\varphi}_0(\lambda) F^s(d\lambda), \quad B^* \in \mathcal{B}^* .$$

Since F^s is singular with respect to m , it follows that each of the integrals in (IV) is equal to zero. Therefore,

$$(V) \quad 0 = \int_{B^*} (\sigma - \bar{\varphi}_0(\lambda) f(\lambda)) m(d\lambda) = \int_{B^*} \bar{\varphi}_0(\lambda) F^s(d\lambda) \quad \text{for all } B^* \in \mathcal{B}^* .$$

From (V) and [5], p. 105, it follows that

$$\sigma - \bar{\varphi}_0 f = 0 \quad \text{a.e. } m$$

$$\bar{\varphi}_0 = 0 \quad \text{a.e. } F^s .$$

Therefore, if $\sigma \neq 0$, we have

$$\varphi_0 = \begin{cases} \sigma/f, & \text{on the carrier of } F^a \\ 0, & \text{on the carrier of } F^s . \end{cases}$$

If $\sigma = 0$, then $|\hat{x}_0|^2 = 0$ and, hence, $\int_{G^*} |\varphi_0|^2 dF = 0$. Therefore, $\varphi = 0$ a.e. F . Q.E.D.

We are now ready to state and prove the minimality theorem for a univariate WSRF $(x_g)_{g \in G}$ over a LCAG G .

4.7 Theorem (Kolmogorov minimality theorem). Let G be a discrete LCAG and $(x_g)_{g \in G}$ a univariate $\mathcal{X}$ -valued WSRF over G with spectral distribution F . Then $(x_g)_{g \in G}$ is minimal if, and only if,

$\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m)$ where f is the spectral density of $(x_g)_{g \in G}$.

Proof. Sufficiency. Set $\varphi(\lambda) = \begin{cases} \frac{1}{f}(\lambda), & \text{on the carrier of } F^a \\ 0 & , \text{ on the carrier of } F^s. \end{cases}$

Then

$$\begin{aligned} \int_{G^*} |\varphi(\lambda)|^2 F(d\lambda) &= \int_{G^*} \left[\frac{1}{f(\lambda)} \right]^2 f(\lambda) m(d\lambda) + \int_{G^*} 0^2 F^s(d\lambda) \\ &= \int_{G^*} \frac{1}{f(\lambda)} m(d\lambda) < \infty \quad (\text{by assumption}). \end{aligned}$$

Hence, $\varphi \in L_2(G^*, \mathcal{B}^*, F)$. Now, by the Isomorphism Theorem (3.7), there exists a $y \in \mathcal{M}_x$ such that

$$y = \int_{G^*} \varphi(\lambda) E(d\lambda) x_0.$$

Note that $(y, y) = \int_{G^*} |\varphi(\lambda)|^2 F(d\lambda) = \int_{G^*} \frac{1}{f(\lambda)} m(d\lambda) \neq 0$. (Otherwise, $\frac{1}{f} = 0$ on every set of positive Haar measure which is impossible because F is a finite measure.)

We will next show that $y \perp x_g, g \neq 0$. Let x_g be an arbitrary element of the WSRF $(x_g)_{g \in G}$. Then

$$(x_g, y) = \int_{G^*} (g, \lambda) \overline{\varphi(\lambda)} F(d\lambda) = \int_{G^*} (g, \lambda) m(d\lambda) = \delta_{g,0}.$$

Since $\{x_g, g \neq 0\}$ is dense in $\mathcal{M}_{I,x}$, $I = \{0\}^c$, we have that $y = cx_0$ where c is a non-zero constant. Hence, $x_0 \notin \mathcal{M}_{I,x}$ which implies that $(x_g)_{g \in G}$ is minimal.

Necessity. Suppose $(x_g)_{g \in G}$ is minimal. Then $\hat{x}_0 \neq 0$ (cf. Lemma 4.5). Now, by Main Lemma 4.6,

$$\hat{x}_0 = \int_{G^*} \varphi_0(\lambda) E(d\lambda) x_0$$

where

$$\varphi_0 = \begin{cases} (\hat{x}_0, \hat{x}_0)/f, & \text{on the carrier of } F^a \\ 0 & , \text{ on the carrier of } F^s . \end{cases}$$

Hence,

$$(\hat{x}_0, \hat{x}_0) = (\hat{x}_0, \hat{x}_0)^2 \int_{\star}^G \frac{1}{f(\lambda)} m(d\lambda).$$

Therefore,

$$(\hat{x}_0, \hat{x}_0)^{-1} = \int_{\star}^G \frac{1}{f(\lambda)} m(d\lambda).$$

But, since $\hat{x}_0 \neq 0$, we have $(\hat{x}_0, \hat{x}_0)^{-1}$ is finite and hence

$$\int_{\star}^G \frac{1}{f(\lambda)} m(d\lambda) < \infty ,$$

or $\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m).$ Q.E.D.

Next, we will establish a theorem on the relationship between the concept of minimality and that of $\mathcal{J}_0$ -regularity introduced in 3.11.

4.8 Theorem. Let $(x_g)_{g \in G}$ be a (non-trivial)¹⁾ univariate $\mathcal{K}$ -valued WSRF over a discrete LCAG G . Then $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -regular if, and only if, $(x_g)_{g \in G}$ is minimal and F , the spectral distribution of $(x_g)_{g \in G}$ is absolutely continuous with respect to the Haar measure m .

Proof. Necessity. Since $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -regular,

$$\mathcal{M}_{\mathcal{J}_0, x} = \bigcap_{I+g \in \mathcal{J}_0} \mathcal{M}_{I+g, x} = \{0\}, \text{ where } I = \{0\}^c. \text{ Hence, } x_g \notin \mathcal{M}_{I+g, x}$$

¹⁾ i.e. $\mathcal{M}_x \neq \{0\}$.

for $g \in G$, and so $(x_g)_{g \in G}$ is minimal.

We now wish to show that F is absolutely continuous.

Define the function ψ on G^* as follows:

$$\psi(\lambda) = \begin{cases} 0, & \text{on the carrier } F^a \\ 1, & \text{on the carrier } F^s. \end{cases}$$

Then

$$\begin{aligned} \int_{G^*} |\psi(\lambda)|^2 F(d\lambda) &= \int_{G^*} (0) f(\lambda) m(d\lambda) + \int_{G^*} 1 F^s(d\lambda) \\ &= F^s(G^*) < \infty. \end{aligned}$$

Hence $\psi \in L_2(G^*, \mathcal{B}^*, F)$. By the Isomorphism Theorem (3.7) there exists

$z \in \mathcal{M}_x$ such that $z = \int_{G^*} \psi(\lambda) E(d\lambda) x_0$. From Lemma 4.6 (Main Lemma I),
 $\hat{x}_g = \int_{G^*} (g, \lambda) \varphi_0(\lambda) E(d\lambda) x_0$ where

$$\varphi_0 = \begin{cases} (\hat{x}_0, \hat{x}_0)/f, & \text{on carrier of } F^a \\ 0, & \text{on carrier of } F^s. \end{cases}$$

Hence,

$$\begin{aligned} \text{(I)} \quad (\hat{x}_g, z) &= \int_{G^*} (g, \lambda) \varphi_0(\lambda) \bar{\psi}(\lambda) F(d\lambda) \\ &= \int_{G^*} (g, \lambda) \frac{(\hat{x}_0, \hat{x}_0)}{f(\lambda)} \cdot 0 \cdot f(\lambda) m(d\lambda) + \int_{G^*} (g, \lambda) \cdot 0 \cdot 1 F^s(d\lambda) \\ &= 0 \quad \text{for all } g \in G. \end{aligned}$$

Note that $\mathcal{M}_{\mathcal{J}_0, x} = \{0\}$ iff $\mathcal{M}_{\mathcal{J}_0, x}^\perp = \mathcal{M}_x$. Since $\mathcal{M}_{\mathcal{J}_0, x}^\perp = \text{closure}$
 $\bigcup_{I+g} \mathcal{M}_{I+g, x}^\perp = \text{closure} \bigcup_g \sigma(\hat{x}_g) = \sigma(\hat{x}_g, g \in G)$, we have

$$\text{(II)} \quad \mathcal{E}(\hat{x}_g, g \in G) = \mathcal{M}_x.$$

From (I) and (II) it follows that $z \perp \mathcal{M}_x$. But $z \in \mathcal{M}_x$, and hence $z = 0$. Therefore, $0 = |z|^2 = \int_{G^*} |\psi(\lambda)|^2 F(d\lambda) = \int_{G^*} F^S(d\lambda) = F^S(G^*)$, which shows $F^S = 0$. Hence, F is absolutely continuous.

Sufficiency. Let $z \in \mathcal{M}_x$. If we can show that $z \perp \hat{x}_g$ for all $g \in G$ implies that $z = 0$, then we will have shown that $\{\hat{x}_g, g \in G\}$ is dense in $\mathcal{M}_x$, and hence that $\mathcal{C}(\hat{x}_g, g \in G) = \mathcal{M}_x$. But, as above, $\mathcal{C}(\hat{x}_g, g \in G) = \mathcal{M}_x$ iff $\mathcal{M}_{\mathcal{J}_0, x} = \{0\}$; i.e., $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -regular.

From Main Lemma I, we get that

$$\hat{x}_g = \int_{G^*} (g, \lambda) \varphi_0(\lambda) E(d\lambda) x_0 \quad \text{for all } g \in G,$$

where φ_0 is as in the lemma. Since $z \in \mathcal{M}_x$, $z = \int_{G^*} \psi(\lambda) E(d\lambda) x_0$ where $\psi \in L_2(G^*, \mathcal{B}^*, F)$. Hence, if we assume that $G(z, \hat{x}_g) = 0$ for all $g \in G$, we get that

$$\begin{aligned} 0 = (z, \hat{x}_g) &= \int_{G^*} \psi(\lambda) (-g, \lambda) \varphi_0(\lambda) F(d\lambda) \\ &= (\hat{x}_0, \hat{x}_0) \int_{G^*} \psi(\lambda) (-g, \lambda) m(d\lambda) \quad \text{for all } g \in G. \end{aligned}$$

From minimality, we have $\hat{x}_0 \neq 0$. Therefore, we obtain the following:

$$\int_{G^*} \psi(\lambda) (-g, \lambda) m(d\lambda) = 0 \quad \text{for all } g \in G.$$

This implies that all the Fourier coefficients of ψ are zero, and hence, by Bochner's Theorem 3.4, $\psi = 0$ a.e. m. Hence, since F is absolutely continuous,

$$(z, z) = \int_{G^*} |\psi(\lambda)|^2 F(d\lambda) = \int_{G^*} 0^2 f(\lambda) m(d\lambda) = 0,$$

which shows that $z = 0$.

Q.E.D.

From the above theorem and Theorem 4.7, we obtain the following corollary.

4.9 Corollary. Let $(x_g)_{g \in G}$ be as in 4.8. Then $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -regular if, and only if, F , the spectral distribution of $(x_g)_{g \in G}$, is absolutely continuous and $\int_G \frac{1}{f(\lambda)} m(d\lambda) < \infty$, where f is the spectral density of $(x_g)_{g \in G}$.

As in the case where $G = \mathbb{Z}$, there is a definite relation between the concept of $\mathcal{J}_0$ -singularity and non-minimality, as the following remark shows.

4.10 Remark. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{M}$ -valued WSRF over G , a discrete LCAG. Then $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -singular if, and only if, $\frac{1}{f} \notin L_1(G^*, \mathcal{B}^*, m)$ where f is the spectral density of $(x_g)_{g \in G}$ (or, equivalently, $(x_g)_{g \in G}$ is not minimal).

Proof. Let $I = \{0\}^c$. Then $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -singular iff $\mathcal{M}_{I+g, x} = \mathcal{M}_x$, $g \in G$ iff $x_g \in \mathcal{M}_{I+g, x}$, $g \in G$ iff $(x_g)_{g \in G}$ is not minimal iff $\frac{1}{f} \notin L_1(G^*, \mathcal{B}^*, m)$, by Theorem 4.7. Q.E.D.

As we have mentioned earlier (cf. Remark 3.13), the main theorem 4.1 of L. Bruckner [1] and the proof of this theorem are in error. Using results of 4.7 - 4.9, we first give a characterization of a WSRF over a discrete LCAG which is neither $\mathcal{J}_0$ -regular nor $\mathcal{J}_0$ -singular in terms of its spectral distribution. An example of such a random field will be given in §6. We will then give a condition under which a WSRF $(x_g)_{g \in G}$ over a discrete LCAG must be either $\mathcal{J}_0$ -singular or $\mathcal{J}_0$ -regular, as Bruckner claims.

4.11 Theorem. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{M}$ -valued WSRF over G , a discrete LCAG. Let F be the spectral distribution of $(x_g)_{g \in G}$ and f be its spectral density. Then $\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m)$

and $F^S \neq 0$ if, and only if, $\{0\} \subsetneq \mathcal{M}_{\mathcal{J}_0, x} \subsetneq \mathcal{M}_x$.

Proof. Necessity. Suppose $\mathcal{M}_{\mathcal{J}_0, x} = \mathcal{M}_x$. Then $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -singular and, hence, by Remark 4.10, $\frac{1}{f} \notin L_1(G^*, \mathcal{B}^*, m)$, which is a contradiction. Now, suppose $\mathcal{M}_{\mathcal{J}_0, x} = \{0\}$. Then $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -regular and, hence, by Theorem 4.8, F is absolutely continuous, which is a contradiction. Hence, $\{0\} \subsetneq \mathcal{M}_{\mathcal{J}_0, x} \subsetneq \mathcal{M}_x$.

Sufficiency. If $\mathcal{M}_{\mathcal{J}_0, x} \neq \mathcal{M}_x$, then $(x_g)_{g \in G}$ is not $\mathcal{J}_0$ -singular, and, hence, by Remark 4.10, $\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m)$. If $\mathcal{M}_{\mathcal{J}_0, x} \neq \{0\}$, then $(x_g)_{g \in G}$ is not $\mathcal{J}_0$ -regular and, hence, by Corollary 4.9, either $\frac{1}{f} \notin L_1(G^*, \mathcal{B}^*, m)$ or F is not absolutely continuous. But, above, we showed that $\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m)$. Therefore, F is not absolutely continuous; i.e., $F^S \neq 0$. Q.E.D.

4.12 Theorem. Let $(x_g)_{g \in G}$ be as in 4.11. Let F , the spectral distribution of $(x_g)_{g \in G}$, be absolutely continuous with respect to the Haar measure m . Then $(x_g)_{g \in G}$ is either $\mathcal{J}_0$ -singular or $\mathcal{J}_0$ -regular.

Proof. If $(x_g)_{g \in G}$ is not $\mathcal{J}_0$ -singular, then $\mathcal{M}_{\mathcal{J}_0, x} \neq \mathcal{M}_x$. Then there exists an x_g , $g \in G$, such that $x_g \notin \mathcal{M}_{I+g, x}$, $I = \{0\}^c$; i.e., $(x_g)_{g \in G}$ is minimal. By Theorem 4.7, $\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m)$. This fact, together with the assumption that F is absolutely continuous, imply, by Theorem 4.8, that $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -regular. Q.E.D.

Our next objective will be to establish the Wold-Cramer concordance relation for a univariate $\mathcal{M}$ -valued WSRF $(x_g)_{g \in G}$ over G , a discrete LCAG, with respect to $\mathcal{J}_0$. The proof of this theorem will depend on results on minimality and regularity of the WSRF $(x_g)_{g \in G}$ over G , a discrete LCAG, as established in this section.

Our proof will resemble the proof given by Wiener and Masani ([32], p. 146) and Doob ([3], p. 576) for the Wold-Cramer Concordance Theorem with respect to the past of a process. The following is our Wold-Cramer Concordance Theorem.

4.13 Theorem (Wold-Cramer concordance for $\mathcal{J}_0$). Let

- (i) $(x_g)_{g \in G}$ be a univariate $\mathcal{X}$ -valued WSRF over G , a discrete LCAG;
- (ii) $(w_g)_{g \in G}$ and $(y_g)_{g \in G}$ be the components of $(x_g)_{g \in G}$ as occurred in the Wold Decomposition Theorem with respect to $\mathcal{J}_0$;
- (iii) F , F_y , and F_w be the spectral distributions of $(x_g)_{g \in G}$, $(y_g)_{g \in G}$, and $(w_g)_{g \in G}$ respectively and f , f_y , and f_w their corresponding spectral densities;
- (iv) F^a , F^s be the absolutely continuous and singular components of F with respect to the Haar measure m , as in the Cramer Decomposition Theorem ;
- (v) $\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m)$.

Then

$$F_y = F^a ; F_w = F^s .$$

Proof. From Lemma 4.6 (Main Lemma I), $\hat{x}_0 = \int_{\star}^G \varphi_0(\lambda E(d\lambda))x_0$ where

$$\varphi_0 = \begin{cases} (\hat{x}_0, \hat{x}_0)/f, & \text{on the carrier of } F^a \\ 0 & , \text{ on the carrier of } F^s . \end{cases}$$

Since $\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m)$, by Theorem 4.7, $(x_g)_{g \in G}$ is minimal and, hence, $(\hat{x}_0, \hat{x}_0) > 0$. We will need this fact later in the proof of the theorem.

Now, $(x_g, x_0) = (w_g + y_g, w_0 + y_0) = (w_g, w_0) + (y_g, y_0)$. Also,

$$(x_g, x_0) = \int_{G^*} (g, \lambda) F(d\lambda); \quad (w_g, w_0) = \int_{G^*} (g, \lambda) F_w(d\lambda);$$

$$(y_g, y_0) = \int_{G^*} (g, \lambda) F_y(d\lambda).$$

Hence,

$$\int_{G^*} (g, \lambda) F(d\lambda) = \int_{G^*} (g, \lambda) (F_w + F_y)(d\lambda) \quad \text{for all } g \in G.$$

Therefore, by Bochner's Theorem 3.4, we get that

$$(I) \quad dF = dF_w + dF_y.$$

Since $(y_g)_{g \in G}$ is non-trivial by (v) and is $\mathcal{J}_0$ -regular, by Theorem 4.8, we see that F_y is absolutely continuous, and hence, from (I), that

$$(II) \quad dF = dF_w + f_y dm.$$

From the Cramer decomposition theorem 3.15, we have

$$(III) \quad dF = f dm + dF^S.$$

Combining (II) and (III), we obtain

$$(IV) \quad f_w dm + dF_w^S + f_y dm = f dm + dF^S$$

which is equivalent to

$$(V) \quad (f - f_y - f_w) dm = dF_w^S - dF^S.$$

Since the left-hand side of (V) is absolutely continuous and the right-hand side is singular with respect to the Haar measure m ,

it follows that

$$f = f_y + f_w \quad \text{a.e. } m,$$

$$dF^S = dF_w^S.$$

We now wish to show that $f = f_y$ a.e. m , which will complete our proof. Since $\hat{x}_g = x_g - (x_g | \mathcal{M}_{I+g,x})$, $I = \{0\}^c$, we see that $\hat{x}_g \perp \mathcal{M}_{I+g,x}$ for all $g \in G$ and, hence, $\hat{x}_g \perp \bigcap_{g \in G} \mathcal{M}_{I+g,x} = \mathcal{M}_{J_0,x}$ for all $g \in G$. However, from the Wold Decomposition Theorem it follows that $\mathcal{M}_{J_0,x}^\perp = \mathcal{M}_y$. Hence, $\hat{x}_g \in \mathcal{M}_y$ for all $g \in G$. Since $\varphi_0 \in L_2(G^*, \mathcal{B}^*, F)$, by (I) it follows that $\varphi_0 \in L_2(G^*, \mathcal{B}^*, F_y) \cap L_2(G^*, \varphi^*, F_w)$. Hence, the integrals in the following equations are well defined.

$$\begin{aligned} \text{(VI)} \quad \hat{x}_0 &= \int_{G^*} \varphi_0(\lambda) E(d\lambda) x_0 = \int_{G^*} \varphi_0(\lambda) E(d\lambda) (y_0 + w_0) \\ &= \int_{G^*} \varphi_0(\lambda) E(d\lambda) y_0 + \int_{G^*} \varphi_0(\lambda) E(d\lambda) w_0. \end{aligned}$$

But by Lemma 3.6 $\int_{G^*} \varphi_0(\lambda) E(d\lambda) y_0 \in \mathcal{M}_y$ and $\int_{G^*} \varphi_0(\lambda) E(d\lambda) w_0 \in \mathcal{M}_w$.

Also, $\hat{x}_0 \in \mathcal{M}_y$. Therefore, by (VI), $\int_{G^*} \varphi_0(\lambda) E(d\lambda) w_0 \in \mathcal{M}_y = \mathcal{M}_w^\perp$.

Hence, $\int_{G^*} \varphi_0(\lambda) E(d\lambda) w_0 = 0$ and

$$\text{(VII)} \quad \hat{x}_0 = \int_{G^*} \varphi_0(\lambda) E(d\lambda) y_0.$$

Combining (VII) with the result in Main Lemma I, we get

$$\text{(VIII)} \quad \hat{x}_0 = \int_{G^*} \varphi_0(\lambda) E(d\lambda) x_0 = \int_{G^*} \varphi_0(\lambda) E(d\lambda) y_0.$$

Using the first equality in (VIII), we get

$$(IX) \quad (\hat{x}_0, \hat{x}_0) = \int_{G^*} |\varphi_0(\lambda)|^2 F(d\lambda) = (\hat{x}_0, \hat{x}_0)^2 \int_{G^*} \frac{1}{f(\lambda)} m(d\lambda) .$$

Using the second equality in (VIII), we obtain

$$(X) \quad (\hat{x}_0, \hat{x}_0) = \int_{G^*} |\varphi_0(\lambda)|^2 F_y(d\lambda) = (\hat{x}_0, \hat{x}_0)^2 \int_{G^*} \frac{f_y(\lambda)}{f^2(\lambda)} m(d\lambda) .$$

Combining (IX) and (X) and recalling that $(\hat{x}_0, \hat{x}_0) > 0$, we get

$$(XI) \quad \int_{G^*} \frac{1}{f(\lambda)} m(d\lambda) = \int_{G^*} \frac{f_y(\lambda)}{f^2(\lambda)} m(d\lambda) .$$

Hence,

$$(XII) \quad \int_{G^*} \frac{1}{f(\lambda)} [1 - f_y(\lambda)/f(\lambda)] m(d\lambda) = 0 .$$

But, since $f = f_y + f_w$ a.e. m , $f \geq f_y$ a.e. m and, hence,
 $1 - f_y/f \geq 0$ a.e. m . Therefore, by (XII)

$$1/f [1 - f_y/f] = 0 \text{ a.e. } m .$$

But $1/f > 0$ a.e. m . Therefore, $1 - f_y/f = 0$ a.e. m , and hence,
 $f_y = f$ a.e. m . Q.E.D.

4.14 Remark. If $1/f \notin L_1(G^*, \mathcal{B}^*, m)$, we note that Remark 4.10 implies that $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -singular, which shows that $x_g = w_g$ for all g , and, hence, that $F = F_w$. In this case F_w could be absolutely continuous with respect to the Haar measure.

Using Theorem 4.8, we can easily show that if $(y_g)_{g \in G}$ is non-trivial and the Wold-Cramer Concordance Theorem holds (i.e., $F^a = F_y$; $F^s = F_w$), then $1/f$ must be in $L_1(G^*, \mathcal{B}^*, m)$.

In general, any analytic condition on the spectral distribution of a WSRF gives rise to certain geometric properties of the random

field itself. As we have seen, the analytic condition

$1/f \in L_1(G^*, \mathcal{B}^*, m)$ is equivalent to minimality for the process.

For a WSRF over the integers, the analytic condition

$\log f \in L_1(G^*, \mathcal{B}^*, m)$ implies, among other things, that the process is non-deterministic (for the definition of a non-deterministic processes, cf. Theorem 3.18).

From our work one may suspect that the analytic condition $1/f \in L_1(G^*, \mathcal{B}^*, m)$ will have a definite relation to the past & future of the process in the same way that the weaker analytic condition $\log f \in L_1(G^*, \mathcal{B}^*, m)$ had a close tie with the past of the process. For this reason we will temporarily digress to a short discussion of WSRF's over the integers where these notions of past and past & future are meaningful. Using our results on minimality we will then make appropriate comparisons between the past and the past & future of a WSRF over the integers.

We will first set up some notations.

4.15 Notations. Let Z be the integers. Let $(x_n)_{n=-\infty}^{\infty}$ be a univariate $\mathcal{X}$ -valued WSRF over Z . Let $J_k = \{k\}^c$ and $I_k = \{n: n \leq k\}$. Then

- (i) $\mathcal{M}_{J_k, x}$ will denote $\mathcal{G}(x_n, n \neq k)$
- (ii) $\mathcal{M}_{I_k, x}$ will denote $\mathcal{G}(x_n, n \leq k)$ (cf. Remark 3.10, (iv)).

With these notations, we see that $\mathcal{M}_{I_{k-1}, x} \subseteq \mathcal{M}_{J_k, x}$ for all k . Hence $\mathcal{M}_{\mathcal{J}_p, x} = \bigcap_{k=-\infty}^{\infty} \mathcal{M}_{I_k, x} \subseteq \bigcap_{k=-\infty}^{\infty} \mathcal{M}_{J_k, x} = \mathcal{M}_{\mathcal{J}_0, x}$.

In the following theorems we will examine conditions under which $\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_0, x}$ or $\mathcal{M}_{\mathcal{J}_p, x} \subsetneq \mathcal{M}_{\mathcal{J}_0, x}$.

4.16 Theorem. Let $(x_n)_{-\infty}^{\infty}$ be a univariate $\mathcal{K}$ -valued WSRF over the integers. Let $\frac{1}{f} \in L_1(G^*, \mathcal{B}^*, m)$, where f is the spectral density of $(x_n)_{-\infty}^{\infty}$. Then $\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_0, x}$.

Proof. From Theorem 4.13, we see that $F_w = F^s$; $F_y = F^a$, where F_w and F_y are the spectral distributions of the components $(w_n)_{-\infty}^{\infty}$ and $(y_n)_{-\infty}^{\infty}$ of $(x_n)_{-\infty}^{\infty}$ given by the Wold decomposition theorem with respect to the family $\{J_k\}_{-\infty}^{\infty}$ (cf. Theorem 4.2). In addition, since $1/f$ and f are in $L_1(G^*, \mathcal{B}^*, m)$, then $\log f$ is also in $L_1(G^*, \mathcal{B}^*, m)$. Hence, by Theorem 3.17, $F_v = F^s$; $F_u = F^a$, where F_v and F_u are the spectral distributions of the components $(v_n)_{-\infty}^{\infty}$ and $(u_n)_{-\infty}^{\infty}$ given by the usual Wold decomposition theorem with respect to the family $\{I_k\}_{-\infty}^{\infty}$ (cf. Theorem 3.16). Combining these two results, we obtain

$$(I) \quad F_w = F_v; \quad F_y = F_u.$$

From (I) and the fact that $\mathcal{M}_{\mathcal{J}_0, x} \supseteq \mathcal{M}_{\mathcal{J}_p, x}$, we will show that $w_n = v_n$ for all n . We observe that

$$(II) \quad (x_n | \mathcal{M}_{\mathcal{J}_p, x}) = ((x_n | \mathcal{M}_{\mathcal{J}_0, x}) | \mathcal{M}_{\mathcal{J}_p, x}).$$

Hence, since $v_n = (x_n | \mathcal{M}_{\mathcal{J}_p, x})$ and $w_n = (x_n | \mathcal{M}_{\mathcal{J}_0, x})$, it follows that $v_n = (w_n | \mathcal{M}_{\mathcal{J}_p, x})$. Using this last relation, we can easily show that $v_n \perp (w_n - v_n)$ for each n , from which it follows that

$$(III) \quad |w_n|^2 = |u_n|^2 + |w_n - v_n|^2 \quad \text{for all } n.$$

Now, $(w_n, w_n) = \int_{G^*} (g, \lambda) \overline{(g, \lambda)} F_w(d\lambda) = F_w(G^*)$ and

$$(v_n, v_n) = \int_{G^*} (g, \lambda) \overline{(g, \lambda)} F_v(d\lambda) = F_v(G^*) .$$

Combining (I) and (III), we get that $|w_n - v_n|^2 = 0$ for all n and, hence, $w_n = v_n$ for all n , which implies that $\mathcal{M}_w = \mathcal{M}_v$. Since $\mathcal{M}_v = \mathcal{M}_{\mathcal{J}_p, x}$ and $\mathcal{M}_w = \mathcal{M}_{\mathcal{J}_0, x}$, we see that $\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_0, x}$. Q.E.D.

In the above theorem we assumed that the analytic condition $1/f \in L_1(G^*, \mathcal{B}^*, m)$ held and saw that $\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_0, x}$. The natural question to ask is what happens when we assume that the weaker analytic condition of $\log f \in L_1(G^*, \mathcal{B}^*, m)$ holds, but not the condition $1/f \in L_1(G^*, \mathcal{B}^*, m)$. The answer is given in the next theorem.

4.17 Theorem. Let $(x_n)_{-\infty}^{\infty}$ be as in Theorem 4.16. Let $1/f \notin L_1(G^*, \mathcal{B}^*, m)$. Let $\log f \in L_1(G^*, \mathcal{B}^*, m)$. Then $\mathcal{M}_{\mathcal{J}_p, x} \subsetneq \mathcal{M}_{\mathcal{J}_0, x} = \mathcal{M}_x$.

Proof. Since $1/f \notin L_1(G^*, \mathcal{B}^*, m)$, by Remark 4.10, $(x_n)_{-\infty}^{\infty}$ is $\mathcal{J}_0$ -singular, and, hence $\mathcal{M}_{\mathcal{J}_0, x} = \mathcal{M}_x$. It is well known ([3], p. 577) that if $\log f \in L_1(G^*, \mathcal{B}^*, m)$, then $(x_n)_{-\infty}^{\infty}$ is non-deterministic and, hence, $\mathcal{M}_{\mathcal{J}_p, x} \subsetneq \mathcal{M}_x$. Q.E.D.

In §6, we will give an example of a process for which $\log f \in L_1(G^*, \mathcal{B}^*, m)$, but $\frac{|p|^2}{f} \notin L_1(G^*, \mathcal{B}^*, n)$, where p is any given polynomial. In particular, $1/f$ will not be in $L_1(G^*, \mathcal{B}^*, m)$. (cf. 6.4).

4.18 Remark. If we assume $\log f \notin L_1(G^*, \mathcal{B}^*, m)$, then $1/f$ will not be in $L_1(G^*, \mathcal{B}^*, m)$. By Remark 4.10 and ([3], p. 577), we have $\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_0, x} = \mathcal{M}_x$. We also note that when $1/f \in L_1(G^{*p}, \mathcal{B}^*, m)$ then $\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_0, x} = \{0\}$ iff $F^S = 0$. Furthermore, in case $1/f \notin L_1(G^{*p}, \mathcal{B}^*, m)$, but $\log f \in L_1(G^*, \mathcal{B}^*, m)$, then $\mathcal{M}_{\mathcal{J}_p, x} = \{0\}$ iff $F^S = 0$.

4.19 Remark. The rest of this section will be devoted to an extension of Salehi's work on the interpolation of WSRF's indexed by integers to univariate WSRF's over discrete LCAG's. (This concept

for WSRF's indexed by integers or real numbers was first studied by A.M. Yaglom [35] and later by Y.A. Rozanov [24].) This will provide a natural extension of earlier results of this section in the same way that Salehi's work on interpolation provided a generalization of Kolmogorov and Masani's work on minimality. Although the ideas and concepts used here are similar to the ones used by Salehi in his work, some of the techniques are different, since the integers are ordered and singly generated whereas an arbitrary group need not be.

Our main reason, besides the historical one, for treating the minimality problem separately is that in this case the results are obtained in a more closed, compact, clear, and simplified form. In addition some of the results on interpolation are obtained under the added assumption that the group G is endowed with an order relation compatible with the structure of G .

We will now recall some of our notations and introduce some new ones needed in the rest of this section.

4.20 Notation. Let $(x_g)_{g \in G}$ be a univariate $\mathbb{K}$ -valued WSRF over the discrete LCAG G with spectral density f . Let $J = \{g_0, g_1, \dots, g_n\}$ be a fixed set of $n+1$ elements of G . Then, we will denote by:

- (i) $\mathcal{H}_{J,x} = \mathcal{M}_{J^c,x}^\perp \cap \mathcal{M}_x$ (cf. 3.8);
- (ii) $\mathcal{P}_J = \{P : P(\lambda) = \sum_{k=0}^n c_k(g_k, \lambda), c_0, c_1, \dots, c_n \text{ arbitrary complex numbers, and } |P(\lambda)|^2 / f(\lambda) \in L_1(G^*, \mathcal{B}^*, m)\};$
- (iii) $\mathcal{J}_n = \{J^c + g, g \in G\}.$

4.21 Remark. The set of polynomials $\mathcal{P}_J$ and the subspace $\mathcal{H}_{J,x}$ were introduced in Salehi's work and will play an important

role in the theory of interpolation of WSRF's. It is obvious that

$z \in \eta_{J,x}$ if, and only if, $z \perp x_g$ for all $g \in J^c$ and that

$\eta_{J,x} = \mathcal{S}(\hat{x}_{g_0}, \dots, \hat{x}_{g_n})$ where $\hat{x}_{g_i}$ is defined by

$$\hat{x}_{g_i} = x_{g_i} - (x_{g_i} | \mathcal{M}_{J^c, x}).$$

We will now make the following definition which is an extension of non-minimality for a WSRF over a discrete LCAG.

4.22 Definition. Let J be as above and $(x_g)_{g \in G}$ be a univariate $\mathcal{K}$ -valued WSRF over G , a discrete LCAG. We say that

$(x_g)_{g \in G}$ is interpolable with respect to J if

$$\mathcal{M}_{J,x} \subseteq \mathcal{M}_{J^c, x}$$

or, equivalently,

$$\eta_{J,x} = \{0\}.$$

It is clear that $\eta_{J,x}$ is a subspace of $\mathcal{M}_x$. It is also obvious that the set θ_J is a linear subset of all polynomials. We introduce an inner product in θ_J in the following manner.

$$(P_1, P_2)_{1/f} = \int_{\star}^G \frac{P_1(\lambda) \overline{P_2(\lambda)}}{f(\lambda)} m(d\lambda), \quad P_1, P_2 \in \theta_J.$$

The proof of the following lemma is straightforward and thus will be omitted.

4.23 Lemma. With the above notation, θ_J is an inner product space over the complex numbers with the inner product

$$(P_1, P_2)_{1/f} = \int_{\star}^G \frac{P_1(\lambda) \overline{P_2(\lambda)}}{f(\lambda)} m(d\lambda), \quad P_1, P_2 \in \theta_J.$$

The fact that this inner product space is finite-dimensional and, hence, complete will follow from the following important lemma. This lemma will be used repeatedly in the interpolation of WSRF's over G .

4.24 Lemma (Main Lemma II). With the above setting the finite-dimensional subspace $\eta_{J,x}$ (cf. Remark 4.21) and the inner product space θ_J are isometric; i.e., there exists a linear operator T on $\eta_{J,x}$ onto θ_J such that

$$(z_1, z_2) = (Tz_1, Tz_2)_{1/f}, \quad z_1, z_2 \in \eta_{J,x}.$$

Proof. Let $z \in \eta_{J,x}$. We define the polynomial P_z by

$$(I) \quad P_z(\lambda) = \sum_{k=0}^n (z, x_{g_k}) (g_k, \lambda).$$

We claim that P_z is an element of θ_J . In view of the fact that the subspace $\eta_{J,x}$ is spanned by $\{\hat{x}_{g_0}, \hat{x}_{g_1}, \dots, \hat{x}_{g_n}\}$, it suffices to prove that $P_{\hat{x}_{g_i}} \in \theta_J$, $0 \leq i \leq n$. For simplicity, $P_i(\lambda)$ will denote $P_{\hat{x}_{g_i}}(\lambda) = \sum_{k=0}^n (\hat{x}_{g_i}, x_{g_k}) (g_k, \lambda)$. Since $\hat{x}_{g_i} \in \eta_x$, by the Isomorphism Theorem (cf. Theorem 3.7), there exists

$\varphi_i \in L_2(G^*, \mathcal{B}^*, F)$ such that $\hat{x}_{g_i} = \int_{\star}^G \varphi_i(\lambda) E(d\lambda) x_0$. From the fact that $\hat{x}_{g_i} \perp x_g$, $g \in J^c$, we get the following:

$$(II) \quad (\hat{x}_{g_i}, x_g) = \int_{\star}^G \varphi_i(\lambda) (-g, \lambda) F(d\lambda) = \begin{cases} 0 & , g \in J^c \\ (\hat{x}_{g_i}, x_g) & , g \in J \end{cases}.$$

Let $c_k = (\hat{x}_{g_i}, x_{g_k})$, $0 \leq k \leq n$. Then we have the following equations.

$$\begin{aligned}
\text{(III)} \quad \int_{G^*} P_i(\lambda) (-g, \lambda) m(d\lambda) &= \sum_{k=0}^n c_k \int_{G^*} (g_k, \lambda) (-g, \lambda) m(d\lambda) \\
&= \sum_{k=0}^n c_k \int_{G^*} (g_k - g, \lambda) m(d\lambda) \\
&= \begin{cases} 0 & , g \in J^c \\ c_k & , g \in J . \end{cases}
\end{aligned}$$

From (II) and (III) we see

$$\text{(IV)} \quad \int_{G^*} P_i(\lambda) (-g, \lambda) m(d\lambda) = \int_{G^*} \varphi_i(\lambda) (-g, \lambda) F(d\lambda)$$

which is equivalent to

$$\text{(V)} \quad \int_{G^*} (-g, \lambda) [P_i(\lambda) - \varphi_i(\lambda) f(\lambda)] m(d\lambda) = \int_{G^*} (-g, \lambda) \varphi_i(\lambda) F^S(d\lambda) .$$

But F^S is singular with respect to m . Using measure theoretical arguments, we get

$$\begin{aligned}
\text{(VI)} \quad \int_{G^*} (-g, \lambda) [P_i(\lambda) - \varphi_i(\lambda) f(\lambda)] m(d\lambda) &= 0 \\
\int_{G^*} (-g, \lambda) \varphi_i(\lambda) F^S(d\lambda) &= 0
\end{aligned}$$

which imply by Bochner's theorem 3.4

$$\begin{aligned}
\text{(VII)} \quad P_i(\lambda) &= \varphi_i(\lambda) f(\lambda), \text{ on the carrier of } F^a \\
\varphi_i(\lambda) &= 0, \text{ on the carrier of } F^S.
\end{aligned}$$

But $\int_{G^*} |\varphi_i(\lambda)|^2 F(d\lambda) < \infty$ and, hence, $\int_{G^*} \frac{|P_i(\lambda)|^2}{f(\lambda)} m(d\lambda) < \infty$. Therefore $P_i \in \mathcal{O}_J$.

We now define the operator T on $\mathcal{H}_{J,x}$ into $\mathcal{O}_J$ by

$$\text{(VIII)} \quad Tz = P_z, \quad z \in \mathcal{H}_{J,x} .$$

Clearly, T is linear and it is not hard to show that it preserves the inner product.

It remains to prove that T is onto. To do that, we show that for any given $P \in \mathcal{P}_J$, $P = \sum_{k=0}^n c_k(g_k, \lambda)$, there exists a $z \in \mathcal{M}_{J,x}$ such that $P = P_z$. We remark that the function

$$\varphi(\lambda) = \begin{cases} P(\lambda)/f(\lambda), & \text{on the carrier of } F^a \\ 0 & , \text{ on the carrier of } F^s \end{cases}$$

is in $L_2(G^*, \mathcal{B}^*, F)$. Define $z \in \mathcal{M}_x$ by

$$z = \int_{G^*}^* \varphi(\lambda) E(d\lambda) x_0 .$$

We now examine Tz :

$$\begin{aligned} (Tz)(\lambda) &= T\left(\int_{G^*}^* \varphi(\lambda) E(d\lambda) x_0\right)(\lambda) \\ &= \sum_{k=0}^n \left(\int_{G^*}^* \varphi(\lambda) E(d\lambda) x_0, \int_{G^*}^* (g_k, \lambda) E(d\lambda) x_0\right) \cdot (g_k, \lambda) \\ &= \sum_{k=0}^n \int_{G^*}^* \varphi(\lambda) (-g_k, \lambda) F(d\lambda) \cdot (g_k, \lambda) \\ &= \sum_{k=0}^n \int_{G^*}^* \frac{P(\lambda)}{f(\lambda)} (-g_k, \lambda) f(\lambda) m(d\lambda) \cdot (g_k, \lambda) \\ &= \sum_{k=0}^n \sum_{j=0}^n c_j \int_{G^*}^* (g_j - g_k, \lambda) m(d\lambda) \cdot (g_k, \lambda) \\ &= \sum_{k=0}^n c_k (g_k, \lambda) = P(\lambda) . \end{aligned}$$

Therefore, $P_z = P$.

Q.E.D.

The following is the analogue of Kolmogorov's minimality theorem (cf. Theorem 4.7) for the case when J has $n+1$ elements.

4.25 Theorem. Let $J = \{g_0, \dots, g_n\}$ be a fixed set of $n+1$ elements of G , a discrete LCAG. Let $(x_g)_{g \in G}$ be a univariate $\mathbb{K}$ -valued WSRF over G with f its spectral density. Then $(x_g)_{g \in G}$ is not interpolable with respect to J if, and only if, there exists a non-zero trigonometric polynomial $P(\lambda)$ on G^* of the form $P(\lambda) = \sum_{k=0}^n c_k(g_k, \lambda)$ such that $|P|^2/f \in L_1(G^*, \mathcal{B}^*, m)$; i.e., $\vartheta_J \neq \{0\}$.

Proof. Necessity. Since $(x_g)_{g \in G}$ is non-interpolable, $\mathcal{M}_{J^c, x} \neq \mathcal{M}_x$. Hence, there exists a $g \in J$ such that $x_g \notin \mathcal{M}_{J^c, x}$. Without loss of generality, let $g = g_0$. Since $x_{g_0} \notin \mathcal{M}_{J^c, x}$, it follows that $\hat{x}_{g_0} \neq 0$. By Main Lemma II, the function $P(\lambda) = \sum_{k=0}^n (\hat{x}_{g_0}, x_{g_k}) (g_k, x)$ is such that $|P|^2/f \in L_1(G^*, \mathcal{B}^*, m)$. Clearly, P is a non-zero polynomial since $(\hat{x}_{g_0}, x_{g_0}) = |\hat{x}_{g_0}|^2 > 0$.

Sufficiency. Now suppose there exists a non-zero polynomial P of the form $P(\lambda) = \sum_{k=0}^n c_k(g_k, \lambda)$ such that $|P|^2/f \in L_1(G^*, \mathcal{B}^*, m)$. Then $P \in \vartheta_J$ and, hence, by Lemma 4.24, there exists a $z \in \mathcal{M}_{J, x}$ such that $z = \int_{\star}^G \varphi(\lambda) E(d\lambda) x_0$ where

$$\varphi(\lambda) = \begin{cases} P(\lambda)/f(\lambda), & \text{on the carrier of } F^a \\ 0, & \text{on the carrier of } F^s. \end{cases}$$

Hence, $|z|^2 = \int_{\star}^G |P(\lambda)|^2/f(\lambda) m(d\lambda)$. Since P is a non-zero polynomial

and f is finite-valued a.e. m , it easily follows that

$\int_{\star}^G |P(\lambda)|^2/f(\lambda) m(d\lambda) > 0$. Hence, $|z|^2 > 0$, and so $z \neq 0$. Therefore,

we have exhibited a non-zero element in $\mathcal{M}_{J, x}$; namely, z , and, hence,

$(x_g)_{g \in G}$ is not interpolable with respect to J .

Q.E.D.

The proof of the following corollary is immediate.

4.26 Corollary. With the same setting as in the above theorem, we have: $(x_g)_{g \in G}$ is interpolable with respect to J if, and only if, $|P|^2/f \notin L_1(G^*, \mathcal{G}^*, m)$ for any non-zero trigonometric polynomial P on G^* of the form $P(\lambda) = \sum_{k=0}^n c_k(g_k, \lambda)$; i.e., $\theta_J = \{0\}$.

The following lemma will be used in the proof of the next theorem. Its proof is very similar to the proofs used in Lemmas 4.4 and 4.5 and, hence, will be omitted.

4.27 Lemma. Let $J = \{g_0, g_1, \dots, g_n\}$. Let $\hat{x}_{g_i} \in \eta_{J,x}$ be such that $\hat{x}_{g_i} \neq 0$ for some fixed $g_i \in J$. Then $\hat{x}_{g+g_i} = x_{g+g_i} - (x_{g+g_i} | \mathcal{M}_{J^c+g,x}) \neq 0$ for all $g \in G$.

Next, we will establish a theorem on the relationship between the concept of non-interpolability and that of $\mathcal{J}_n$ -regularity introduced in 3.11.

4.28 Theorem. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{X}$ -valued WSRF over G , a discrete LCAG, F be its spectral distribution, and f be its spectral density. Let $J = \{g_0, g_1, \dots, g_n\}$, $\mathcal{J}_n = \{J^c + g, g \in G\}$.

(a) If $(x_g)_{g \in G}$ is non-trivial and is $\mathcal{J}_n$ -regular, then $(x_g)_{g \in G}$ is not interpolable with respect to J , {hence with respect to $J + g$, for all $g \in G$ }, and F is absolutely continuous.

(b) Let G be ordered. If F is absolutely continuous and $(x_g)_{g \in G}$ is not interpolable with respect to J , then $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -regular.

Proof (a). Since $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -regular, then $\mathcal{M}_{\mathcal{J}_n,x} = \{0\}$. In particular, for some $g \in G$, $\mathcal{M}_{J^c+g,x} \neq \mathcal{M}_x$. Then, by Lemma 4.27, $\eta_{J,x} \neq \{0\}$; i.e., $(x_g)_{g \in G}$ is not interpolable with respect to J .

We now wish to show that F is absolutely continuous.

Since $\mathcal{M}_{\mathcal{J}_n, x} = \{0\}$, it follows that $\mathcal{M}_{\mathcal{J}_n, x}^\perp = \mathcal{M}_x$. But

$$\mathcal{M}_{\mathcal{J}_n, x}^\perp = \text{closure} \bigcup_{g \in G} \mathcal{J}_n^c + g, x = \text{closure} \bigcup_{g \in G} \mathcal{J}_n + g, x = \mathcal{E}(\hat{x}_{g_i + g}, 0 \leq i \leq n, g \in G). \text{ Hence,}$$

$$(I) \quad \mathcal{M}_x = \mathcal{E}(\hat{x}_{g_i + g}, 0 \leq i \leq n, g \in G) .$$

In the proof of Main Lemma II (Lemma 4.24), we saw that

$$\hat{x}_{g_i} = \int_{G^*} \varphi_i(\lambda) E(d\lambda) x_0 \quad \text{where}$$

$$\varphi_i = \begin{cases} P_i/f, & \text{on the carrier of } F^a \\ 0, & \text{on the carrier of } F^s, \end{cases}$$

$0 \leq i \leq n$. Hence, from $\hat{x}_{g_i + g} = U_g \hat{x}_{g_i}$, it follows that

$$(II) \quad \hat{x}_{g_i + g} = \int_{G^*} (g, \lambda) \varphi_i(\lambda) E(d\lambda) x_0, \quad 0 \leq i \leq n, g \in G .$$

Now, as in the proof of the necessity part of Theorem 4.8, define the function ψ on G^* as follows:

$$(III) \quad \psi = \begin{cases} 0, & \text{on the carrier of } F^a \\ 1, & \text{on the carrier of } F^s . \end{cases}$$

Then $\psi \in L_2(G^*, \mathcal{B}^*, F)$ and, hence, by the Isomorphism Theorem, there

exists a $z \in \mathcal{M}_x$ such that $z = \int_{G^*} \psi(\lambda) E(d\lambda) x_0$. Hence,

$$\begin{aligned} (IV) \quad (\hat{x}_{g_i + g}, z) &= \int_{G^*} (g, \lambda) \varphi_i(\lambda) \overline{\psi(\lambda)} F(d\lambda) \\ &= \int_{G^*} (g, \lambda) \frac{P_i(\lambda)}{f(\lambda)} \cdot 0 \cdot f(\lambda) m(d\lambda) + \int_{G^*} (g, \lambda) \cdot 0 \cdot 1 F^s(d\lambda) \\ &= 0 . \end{aligned}$$

From (I) and (IV), we see that $z \perp \mathcal{M}_x$. But $z \in \mathcal{M}_x$ and hence $z = 0$. Therefore, $0 = |z|^2 = \int_{G^*} |\psi(\lambda)|^2 F(d\lambda) = \int_{G^*} F^S(d\lambda) = F^S(G^*)$, which shows $F^S = 0$. Hence, F is absolutely continuous.

(b). The proof of this part of the theorem will closely resemble the proof of the sufficiency part of Theorem 4.8. Let $z \in \mathcal{M}_x$. If we can show that $z \perp \hat{x}_{g_i+g}$ for all $i \in \{0, 1, \dots, n\}$ and all $g \in G$ implies that $z = 0$, then we will have shown that $\{\hat{x}_{g_i+g}, 0 \leq i \leq n, g \in G\}$ is dense in $\mathcal{M}_x$ and, hence, that $\mathcal{C}(\hat{x}_{g_i+g}, 0 \leq i \leq n, g \in G) = \mathcal{M}_x$. But, $\mathcal{C}(\hat{x}_{g_i+g}, 0 \leq i \leq n, g \in G) = \mathcal{M}_x$ if, and only if, $\mathcal{M}_{\mathcal{J}_n, x} = \{0\}$; i.e., $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -regular.

In the proof of Main Lemma II (Lemma 4.24), we showed that

$$\hat{x}_{g_i} = \int_{G^*} \varphi_i(\lambda) E(d\lambda) x_0 \quad \text{where}$$

$$\varphi_i = \begin{cases} P_i/f, & \text{on the carrier of } F^a \\ 0 & , \text{ on the carrier of } F^S, \end{cases}$$

$0 \leq i \leq n$. Since $\hat{x}_{g_i+g} = U_g \hat{x}_{g_i}$, we get

$$(I) \quad \hat{x}_{g_i+g} = \int_{G^*} (g, \lambda) \varphi_i(\lambda) E(d\lambda) x_0, \quad 0 \leq i \leq n, g \in G.$$

Since $z \in \mathcal{M}_x$, $z = \int_{G^*} \psi(\lambda) E(d\lambda) x_0$, where $\psi \in L_2(G^*, \mathcal{B}^*, F)$. Hence, if we assume that $G(z, \hat{x}_{g_i+g}) = 0$, $0 \leq i \leq n, g \in G$, we get

$$(II) \quad 0 = (\hat{x}_{g_i+g}, z) = \int_{G^*} \varphi_i(\lambda) (-g, \lambda) \overline{\psi(\lambda)} F(d\lambda)$$

$$= \int_{G^*} P_i(\lambda) (-g, \lambda) \overline{\psi(\lambda)} m(d\lambda), \quad 0 \leq i \leq n, g \in G.$$

This implies that all the Fourier coefficients of $P_i \cdot \overline{\psi}$ are zero, $0 \leq i \leq n$, and, hence, by Bochner's Theorem (3.4),

$$(III) \quad P_i \cdot \bar{\psi} = 0 \quad \text{a.e. } m, \quad 0 \leq i \leq n.$$

Since $(x_g)_{g \in G}$ is not interpolable with respect to J , by Theorem 4.25, there exists some $g_i \in J$ such that $\hat{x}_{g_i} \neq 0$. This implies that the corresponding polynomial P_i is non-zero. Since G is ordered and P_i is a non-zero polynomial, it will be shown in the following lemma (Lemma 4.29) that P_i cannot vanish on a set of positive Haar measure. Hence, by (III), $\psi = 0$ a.e. m . Therefore,

$$|z|^2 = \int_G |\psi(\lambda)|^2 F(d\lambda).$$

But, F is absolutely continuous and, hence,

$$|z|^2 = \int_{G^*} |\psi(\lambda)|^2 f(\lambda) m(d\lambda) = 0, \quad \text{which shows } z = 0. \quad \text{Q.E.D.}$$

4.29 Lemma. Let G be endowed with an order relation compatible with its structure. Let P be a non-zero trigonometric polynomial on G^* . Then P cannot vanish on a set of positive Haar measure.

Proof. Because G is ordered one can show that there exists some $g_i \in \{g_0, \dots, g_n\}$ such that

$$(I) \quad P(\lambda) = (g_i, \lambda) P_1(\lambda),$$

where $P_1(\lambda) = \sum_{j=0}^n d_j (g_j, \lambda)$, with $g_0 = 0$, $d_0 \neq 0$, and $g_j \geq 0$, $1 \leq j \leq n$. It follows (cf. [26], Theorem 8.4.1) that $\log |P_1| \in L_1(G^*, \mathcal{B}^*, m)$ and, from (I), $\log |P| \in L_1(G^*, \mathcal{B}^*, m)$. But this implies $P \neq 0$ on every set of positive Haar measure. **Q.E.D.**

An immediate consequence of Theorem 4.25 and Theorem 4.28 is the following corollary.

4.30 Corollary. Let $(x_g)_{g \in G}$ be as in 4.28. Then

(a) If $(x_g)_{g \in G}$ is non-trivial and is $\mathcal{J}_n$ -regular, then F , the spectral distribution of $(x_g)_{g \in G}$, is absolutely continuous and

for all $J + g \in \mathcal{J}_n$, $J = \{g_0, \dots, g_n\}$, $\vartheta_{J+g} \neq \{0\}$.

(b) Let G be ordered. If F is absolutely continuous and for $J \in \mathcal{J}_n$, $J = \{g_0, g_1, \dots, g_n\}$, $\vartheta_J \neq \{0\}$, then $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -regular.

4.31 Remark. If G is not ordered, one can easily construct a non-zero polynomial P on G^* such that $P = 0$ on some set of positive Haar measure. An example of such a polynomial will be provided in §6, Example 6.5.

We see that the assumption that G is ordered was used in the proof of Theorem 4.28(b). As we saw earlier (cf. Theorem 4.8), this assumption is not needed when J consists of a single point. It may be that the conclusion of part (b) of Theorem 4.28 is true even without the assumption that G is ordered. However, our proof does not demonstrate this.

Just as there was a definite relation between the concepts of $\mathcal{J}_0$ -singularity and non-minimality, there is also a relation between the concepts of $\mathcal{J}_n$ -singularity and interpolability, as the following remark shows.

4.32 Remark. Let $(x_g)_{g \in G}$ be a univariate WSRF over G , a discrete LCAG. Let $J = \{g_0, g_1, \dots, g_n\}$ be a fixed set of $n+1$ elements in G . Then $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -singular if, and only if, for all $g \in G$, $\eta_{J+g, x} = \{0\}$, or, equivalently, for all $g \in G$, $\vartheta_{J+g} = \{0\}$.

Proof. $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -singular iff $\mathcal{M}_{J+g, x}^c = \mathcal{M}_x$, $g \in G$,
iff $\eta_{J+g, x} = \{0\}$, $g \in G$, or, equivalently $\vartheta_{J+g} = \{0\}$, $g \in G$. Q.E.D.

Now, using Theorem 4.28, Corollary 4.30 and Remark 4.32, we will first give a characterization of a WSRF over a discrete LCAG

which is neither $\mathcal{J}_n$ -singular nor $\mathcal{J}_n$ -regular in terms of its spectral distribution. We will then give conditions under which a WSRF $(x_g)_{g \in G}$ over a discrete LCAG must be either $\mathcal{J}_n$ -singular or $\mathcal{J}_n$ -regular.

4.33 Theorem. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{K}$ -valued WSRF over G , a discrete LCAG. Let F be the spectral distribution of $(x_g)_{g \in G}$ and f be its spectral density. Let $J = \{g_0, g_1, \dots, g_n\}$ be a fixed set of elements in G and $\mathcal{J}_n = \{J^c + g, g \in G\}$.

- (a) If $\theta_J \neq \{0\}$ and $F^S \neq 0$, then $\{0\} \subsetneq \mathcal{M}_{\mathcal{J}_n, x} \subsetneq \mathcal{M}_x$.
- (b) Let G be ordered. If $\{0\} \subsetneq \mathcal{M}_{\mathcal{J}_n, x} \subsetneq \mathcal{M}_x$, then $\theta_{J+g} \neq \{0\}$ for all $g \in G$ and $F^S \neq 0$.

Proof (a). Suppose $\mathcal{M}_{\mathcal{J}_n, x} = \mathcal{M}_x$. Then $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -singular and, hence, by Remark 4.32, $\theta_J = \{0\}$, which is a contradiction. Now, suppose $\mathcal{M}_{\mathcal{J}_n, x} = \{0\}$. Then $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -regular and, hence, by Corollary 4.30, F is absolutely continuous, which is a contradiction. Hence, $\{0\} \subsetneq \mathcal{M}_{\mathcal{J}_n, x} \subsetneq \mathcal{M}_x$.

(b) If $\mathcal{M}_{\mathcal{J}_n, x} \neq \mathcal{M}_x$, then $(x_g)_{g \in G}$ is not $\mathcal{J}_n$ -singular and, hence, by Remark 4.32, $\theta_{J+g} \neq \{0\}$ for all $g \in G$. If $\mathcal{M}_{\mathcal{J}_n, x} \neq \{0\}$, then $(x_g)_{g \in G}$ is not $\mathcal{J}_n$ -regular. By Corollary 4.30(b), this implies that either F is not absolutely continuous or for all $g \in G$, or $\theta_{J+g} = \{0\}$. But the latter cannot happen. Hence, F is not absolutely continuous; i.e., $F^S \neq 0$. Q.E.D.

4.34 Theorem. Let G be a discrete LCAG which is ordered. Let J and $\mathcal{J}_n$ be as in 4.33. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{K}$ -valued WSRF over G and F , its spectral distribution, be absolutely continuous with respect to m . Then either $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -singular or $\mathcal{J}_n$ -regular.

Proof. If $(x_g)_{g \in G}$ is not $\mathcal{J}_n$ -singular, then $\mathcal{M}_{\mathcal{J}_n, x} \neq \mathcal{M}_x$. Hence, $\mathcal{M}_{J^c, x} \neq \mathcal{M}_x$, which implies that $(x_g)_{g \in G}$ is not interpolable with respect to J . Based on this and the fact that F is absolutely continuous and G is ordered, Theorem 4.28(b) implies that $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -regular. Q.E.D.

Our next objective will be to establish the Wold-Cramer concordance relation for a univariate $\mathcal{X}$ -valued WSRF $(x_g)_{g \in G}$ over G , a discrete LCAG with respect to $\mathcal{J}_n$. The proof of this theorem will depend on results on interpolation and $\mathcal{J}_n$ -regularity of $(x_g)_{g \in G}$ that were just established.

4.35 Theorem (Wold-Cramer concordance for $\mathcal{J}_n$). Let

- (i) $(x_g)_{g \in G}$ be a univariate $\mathcal{X}$ -valued WSRF over G , a discrete LCAG, which is ordered; $J = \{g_0, \dots, g_n\}$ and $\mathcal{J}_n = \{J^c + g, g \in G\}$;
- (ii) $(w_g)_{g \in G}$ and $(y_g)_{g \in G}$ be the components of $(x_g)_{g \in G}$ as occurred in the Wold decomposition theorem with respect to $\mathcal{J}_n$;
- (iii) F, F_y , and F_w be the spectral distributions of $(x_g)_{g \in G}$, $(y_g)_{g \in G}$ and $(w_g)_{g \in G}$ respectively and f, f_y , and f_w their corresponding spectral densities;
- (iv) F^a, F^s the absolutely continuous and singular components of F with respect to the Haar measure m , as in the Cramer decomposition theorem;
- (v) $\theta_J \neq \{0\}$.

Then

$$F_y = F^a ; F_w = F^s .$$

Proof. By Main Lemma II, 4.24, $\eta_J \neq \{0\}$. Without loss of generality we may assume that $\hat{x}_{g_0} \neq 0$. In the proof of Lemma 4.24 (Main Lemma II), we saw that $\hat{x}_{g_0} = \int_{G^*} \varphi_0(\lambda) E(d\lambda) x_0$ where

$$\varphi_0 = \begin{cases} P_0/f, & \text{on the carrier of } F^a \\ 0, & \text{on the carrier of } F^s, \end{cases}$$

$P_0(\lambda) = \sum_{k=0}^n (\hat{x}_{g_0}, x_{g_k})(g_k, \lambda)$. Also, since $\hat{x}_{g_0} \perp \mathcal{M}_{J^c, g}$ and $\mathcal{M}_{J_n, x} = \bigcap_{g \in G} \mathcal{M}_{J^c, g}$, we see that $\hat{x}_{g_0} \perp \mathcal{M}_{J_n, x}$; i.e., $\hat{x}_{g_0} \in \mathcal{M}_{J_n, x}^\perp$. However, from the Wold decomposition theorem (Theorem 4.2), it follows that $\mathcal{M}_y = \mathcal{M}_{J_n, x}^\perp$. Hence, $\hat{x}_{g_0} \in \mathcal{M}_y$.

Since $(y_g)_{g \in G}$ is non-trivial and is J_n -regular, by Theorem 4.28(b), it follows that F_y is absolutely continuous with respect to m . Then, using the same technique as in the proof of the Wold-Cramer concordance theorem with respect to J_0 (Theorem 4.13), we obtain

$$(I) \quad f = f_y + f_w \quad \text{a.e. } m,$$

$$dF^s = dF_w^s.$$

Hence, if we can show that $f = f_y$ a.e. m , our proof will be finished. In the same manner as in Theorem 4.13, we get

$$(II) \quad \hat{x}_{g_0} = \int_{G^*} \varphi_0(\lambda) E(d\lambda) x_0 = \int_{G^*} \varphi_0(\lambda) E(d\lambda) y_0.$$

Using the first equality in (II), we obtain

$$(III) \quad (\hat{x}_{g_0}, \hat{x}_{g_0}) = \int_{G^*} |\varphi_0(\lambda)|^2 F(d\lambda) = \int_{G^*} |P_0(\lambda)|^2 / f(\lambda) m(d\lambda).$$

Using the second equality in (II), we get

$$(IV) \quad (\hat{x}_{g_0}, \hat{x}_{g_0}) = \int_{G^*} |\varphi_0(\lambda)|^2 f_y(d\lambda) = \int_{G^*} \frac{|P_0(\lambda)|^2}{f^2(\lambda)} f_y(\lambda) m(d\lambda) .$$

Combining (III) and (IV), we get

$$(V) \quad \int_{G^*} |P_0(\lambda)|^2 / f(\lambda) m(d\lambda) = \int_{G^*} \frac{|P_0(\lambda)|^2}{f^2(\lambda)} f_y(\lambda) m(d\lambda),$$

which is equivalent to

$$(VI) \quad \int_{G^*} \frac{|P_0(\lambda)|^2}{f(\lambda)} \left[1 - \frac{f_y(\lambda)}{f(\lambda)}\right] m(d\lambda) = 0 .$$

But (I) implies that $f \geq f_y$ a.e. m and, hence, $1 - f_y/f \geq 0$ a.e. m .

Then, by (VI), we get

$$(VII) \quad \frac{|P_0|^2}{f} \left[1 - \frac{f_y}{f}\right] = 0 \text{ a.e. } m.$$

Since G is ordered, Lemma 4.28 implies that $|P_0|^2 > 0$ a.e. m .

Since $1/f > 0$ a.e. m , it follows from (VII) that $1 - f_y/f = 0$ a.e. m ,

and, thus, $f = f_y$ a.e. m .

Q.E.D.

4.36 Remark. If any non-zero trigonometric polynomial P on G^* of the form $P(\lambda) = \sum_{k=0}^n c_k(g_k, \lambda)$ satisfies the condition $|P|^2/f \notin L_1(G^*, \mathcal{B}^*, m)$, then, by Remark 4.32, $(x_g)_{g \in G}$ is $\mathcal{J}_n$ -singular, which shows that $x_g = w_g$ for all g and, hence, that $F = F_w$. In this case, F_w could be absolutely continuous with respect to m .

We will now specialize our results on interpolation of WSRF's to processes indexed by the integers. As in the case of minimality, under suitable analytic conditions we will make appropriate comparisons between the subspaces in the time domain (i.e., $\mathcal{M}_{p,x}$ and $\mathcal{M}_{n,x}$).

First, we will recall some notation and introduce some new ones.

4.37 Notation. Let Z be the integers. Let $(x_n)_{-\infty}^{\infty}$ be a univariate $\mathcal{K}$ -valued WSRF over Z . Let $J = \{k_0, k_1, \dots, k_n\}$; $k_0 < k_1 < \dots < k_n$ be a fixed set of $n+1$ integers and $I_k = \{j : j \leq k\}$. Let

$$(i) \quad \mathcal{M}_{J+K, x}^c = \mathcal{G}(x_j, j \neq k_i + k, 0 \leq i \leq n).$$

$$(ii) \quad \mathcal{M}_{I_k, x} = \mathcal{G}(x_j, j \leq k).$$

Obviously, $\mathcal{M}_{J, x} = \bigcap_{k=-\infty}^{\infty} \mathcal{M}_{I_k, x} \subseteq \bigcap_{k=-\infty}^{\infty} \mathcal{M}_{J+K, x}^c = \mathcal{M}_{J, x}$.

In the following theorems, we will examine conditions under which $\mathcal{M}_{J, x} = \mathcal{M}_{J_n, x}$ or $\mathcal{M}_{J, x} \subsetneq \mathcal{M}_{J_n, x}$.

4.38 Theorem. Let $(x_n)_{-\infty}^{\infty}$ be a univariate $\mathcal{K}$ -valued WSRF over Z , the integers. Suppose there exists a non-zero trigonometric polynomial P on G^* of the form $P(\lambda) = \sum_{j=0}^n c_j(k_j, \lambda)$ such that $|P|^2/f \in L_1(G^*, \mathcal{B}^*, m)$, where f is the spectral density of $(x_n)_{-\infty}^{\infty}$.

Then $\mathcal{M}_{J, x} = \mathcal{M}_{J_n, x}$.

Proof. By Theorem 4.35, we see that $F_w = F^s$; $F_y = F^a$, where F_w and F_y are the spectral distributions of the components $(w_n)_{-\infty}^{\infty}$ and $(y_n)_{-\infty}^{\infty}$ of $(x_n)_{-\infty}^{\infty}$ given by the Wold decomposition theorem with respect to the family $J_n = \{J + k, k \in Z\}$ (cf. Theorem 4.2). Also, since there exists a non-zero P on G^* of the form $P(\lambda) = \sum_{j=0}^n c_j(k_j, \lambda)$ such that $|P|^2/f \in L_1(G^*, \mathcal{B}^*, m)$, Theorem 4.25 tells us that $(x_g)_{g \in G}$ is not interpolable with respect to $J = \{k_0, \dots, k_n\}$, which implies that there exists a $k \in J$ such that $x_k \notin \mathcal{G}(x_j, j \in J^c)$. But this implies $x_k \notin \mathcal{G}(x_j, j < k_0)$ and, hence, by stationarity, $x_{k_0} \notin \mathcal{G}(x_j, j < k_0)$. Thus, $(x_k)_{-\infty}^{\infty}$

is non-deterministic and so $\log f \in L_1(G^*, \mathcal{B}^*, m)$. Thus, by Theorem 3.17, $F_v = F^s$; $F_u = F^a$ where F_v and F_u are the spectral distributions of the components $(v_n)_{-\infty}^\infty$ and $(u_n)_{-\infty}^\infty$ given by the usual Wold decomposition theorem with respect to the family $\{I_k\}_{-\infty}^\infty$. Hence,

$$(I) \quad F_w = F_v; F_y = F_u.$$

By (I) and the fact that $\mathcal{M}_{\mathcal{J}_n, x} \supset \mathcal{M}_{\mathcal{J}_p, x}$ we can use an argument similar to the one given in the proof of Theorem 4.16 to show that

$$\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_n, x}. \quad \text{Q.E.D.}$$

4.39 Theorem. Let $(x_n)_{-\infty}^\infty$ be as in Theorem 4.38. Let $\log f \in L_1(G^*, \mathcal{B}^*, m)$ where f is the spectral density of $(x_n)_{-\infty}^\infty$. Suppose that for any non-zero trigonometric polynomial P on G^* of the form $P(\lambda) = \sum_{j=0}^n c_j(k_j, \lambda)$, we have $|P|^2/f \notin L_1(G^*, \mathcal{B}^*, m)$. Then $\mathcal{M}_{\mathcal{J}_p, x} \subsetneq \mathcal{M}_{\mathcal{J}_n, x} = \mathcal{M}_x$.

Proof. By Remark 4.32, we see that $(x_n)_{-\infty}^\infty$ is $\mathcal{J}_n$ -singular and hence, $\mathcal{M}_{\mathcal{J}_n, x} = \mathcal{M}_x$. It is well-known ([3], p. 577) that if $\log f \in L_1(G^*, \mathcal{B}^*, m)$, then $(x_n)_{-\infty}^\infty$ is non-deterministic and, hence, $\mathcal{M}_{\mathcal{J}_p, x} \subsetneq \mathcal{M}_x$.

4.40 Remark. If we assume $\log f \notin L_1(G^*, \mathcal{B}^*, m)$, then $\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_x$ and, hence, $\mathcal{M}_{\mathcal{J}_n, x} = \mathcal{M}_x$.

This concludes our discussion on the problem of interpolation with respect to $\mathcal{J}_n = \{J^c + g\}$, where J is a fixed set of $n+1$ elements of G , a discrete LCAG. We will devote the rest of §4 to interpolation theory with respect to $\mathcal{J}_\infty$, the family of complements of finite sets of elements in G . First, though, we will recall some notation, introduced earlier, which is relevant to what we will be doing.

4.41 Notation. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{X}$ -valued WSRF over G , a discrete LCAG. Let f be the spectral density of $(x_g)_{g \in G}$ and J be any finite set of elements of G . As in 4.20 we will set

- (i) $\mathcal{M}_{J,x} = \mathcal{M}_{J^c,x}^\perp \cap \mathcal{M}_x$;
- (ii) $\mathcal{O}_J = \{P: P(\lambda) = \sum_{g \in J} c_g(g, \lambda), c_g \text{'s are arbitrary complex numbers, and } |P|^2/f \in L_1(G^*, \mathcal{B}^*, m)\}$;
- (iii) $\mathcal{J}_\infty =$ family of complements of finite sets of G .

We are now able to give the following definition of interpolability.

4.42 Definition. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{X}$ -valued WSRF over G , a discrete LCAG. We say that $(x_g)_{g \in G}$ is interpolable if $(x_g)_{g \in G}$ is interpolable with respect to every finite set of elements of G .

The following remark follows immediately from Theorem 4.25.

4.43 Remark. Let $(x_g)_{g \in G}$ be as in 4.42. Then $(x_g)_{g \in G}$ is interpolable if, and only if, $\mathcal{O}_J = \{0\}$ for any finite set $J \subseteq G$.

As we mentioned earlier (cf. Remark 3.13), there was an error in the main Theorem 4.1 of L. Bruckner [1]. A similar type of error regarding the relation between $\mathcal{J}_\infty$ -regularity and its characterization in terms of the spectral density of the WSRF is contained in Theorem 5.2 of Bruckner. The following establishes a relationship between the concept of non-interpolability and that of $\mathcal{J}_\infty$ -regularity and includes a corrected version of Bruckner's result. Here, again, part (b) may be true without the assumption that G is ordered, but at this point we are not able to dispense with

this assumption.

4.44 Theorem. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{X}$ -valued WSRF over G , a discrete LCAG. Let F denote the spectral distribution of $(x_g)_{g \in G}$ and f its spectral density.

(a) If $(x_g)_{g \in G}$ is non-trivial and is $\mathcal{J}_\infty$ -regular, then there exists a finite subset J of G such that $\vartheta_J \neq \{0\}$ and F is absolutely continuous with respect to m ;

(b) Let G be ordered. If F is absolutely continuous and there exists a finite set J of G such that $\vartheta_J \neq \{0\}$, then $(x_g)_{g \in G}$ is $\mathcal{J}_\infty$ -regular.

Proof (a). Trivially, there exists a finite set J of G such that $\vartheta_J \neq \{0\}$.

$$\text{Let } \psi = \begin{cases} 0, & \text{on the carrier of } F^a \\ 1, & \text{on the carrier of } F^s. \end{cases}$$

It is obvious that $\psi \in L_2(G^*, \mathcal{B}^*, F)$. Let $z \in \mathcal{M}_X$ correspond to ψ . Using techniques similar to those in the proof of Theorem 4.28(a), we can show that $z \perp \mathcal{M}_{J,x}$ for all finite subsets J of G . But this implies, because $(x_g)_{g \in G}$ is $\mathcal{J}_\infty$ -regular, that $z \perp \mathcal{M}_X$. Hence, $z = 0$, which implies $(z, z) = F^s(G^*) = 0$.

(b) Trivially,

$$(I) \quad \mathcal{M}_{\mathcal{J}_\infty, x} \subseteq \mathcal{M}_{\mathcal{J}_n, x}, \quad \mathcal{J}_n = \{J^c + g, g \in G\}.$$

By (I) and Theorem 4.28(b), we have $\mathcal{M}_{\mathcal{J}_\infty, x} = \{0\}$. Thus, $(x_g)_{g \in G}$ is $\mathcal{J}_\infty$ -regular. Q.E.D.

In the following remark we will state a characterization of $\mathcal{J}_\infty$ -singularity for a WSRF over a discrete LCAG.

4.45 Remark. Let $(x_g)_{g \in G}$ be as in 4.44. Then $(x_g)_{g \in G}$ is $\mathcal{J}_\infty$ -singular if, and only if, any non-zero trigonometric polynomial P on G^* satisfies the condition that $|P|^2/f \notin L_1(G^*, \mathcal{B}^*, m)$; i.e., for all finite sets, J , of elements of G , $\vartheta_J = \{0\}$.

Next, we will give conditions under which a WSRF over a discrete LCAG is neither $\mathcal{J}_\infty$ -singular nor $\mathcal{J}_\infty$ -regular. We will then give conditions under which a process must be either $\mathcal{J}_\infty$ -singular or $\mathcal{J}_\infty$ -regular. The proofs of these results follow from 4.43 and 4.44 in the same manner that the proofs of 4.33 and 4.34 were derived from 4.28 and 4.32.

4.46 Theorem. Let $(x_g)_{g \in G}$ be a univariate $\mathcal{X}$ -valued WSRF over G , a discrete LCAG. Let F be the spectral distribution of $(x_g)_{g \in G}$ and f be its spectral density.

(a) If there exists a finite set J of G such that $\vartheta_J \neq \{0\}$ and $F^S \neq 0$, then $\{0\} \subsetneq \mathcal{M}_{\mathcal{J}_\infty, x} \subsetneq \mathcal{M}_x$.

(b) Let G be ordered. If $\{0\} \subsetneq \mathcal{M}_{\mathcal{J}_\infty, x} \subsetneq \mathcal{M}_x$, then there exists a finite set J of G such that $\vartheta_J \neq \{0\}$ and $F^S \neq 0$.

4.47 Theorem. Let $(x_g)_{g \in G}$ be as in 4.46. Let G be ordered and F , the spectral distribution of $(x_g)_{g \in G}$, be absolutely continuous with respect to m . Then either $(x_g)_{g \in G}$ is $\mathcal{J}_\infty$ -singular or $\mathcal{J}_\infty$ -regular.

4.48 Remark. Let G be ordered. We have proved that if $F^S = 0$, then we have either $\mathcal{J}$ -regularity or $\mathcal{J}$ -singularity for the cases $\mathcal{J} = \mathcal{J}_0$, $\mathcal{J}_n$, and $\mathcal{J}_\infty$ (cf. Theorems 4.12, 4.34, and 4.47). We remark that if the WSRF is $\mathcal{J}_0$ -regular, then it is also $\mathcal{J}_n$ and $\mathcal{J}_\infty$ -regular; or, equivalently, if the WSRF is $\mathcal{J}_\infty$ -singular, then it is also $\mathcal{J}_n$ and $\mathcal{J}_0$ -singular. Other cases of interest may happen; e.g., a WSRF

may be $\mathcal{J}_0$ -singular and yet $\mathcal{J}_n$ and hence $\mathcal{J}_\infty$ -regular.

We will now establish the Wold-Cramer concordance theorem for $\mathcal{J}_\infty$. Since the proof of this theorem is very similar to the proof of Wold-Cramer concordance theorem for $\mathcal{J}_n$ (Theorem 4.35), we will only sketch it.

4.49 Theorem (Wold-Cramer concordance for $\mathcal{J}_\infty$). Let

(i) $(x_g)_{g \in G}$ be a univariate $\mathcal{K}$ -valued WSRF over G , a discrete LCAG, which is ordered; $\mathcal{J}_\infty$ = family of complements of finite sets of G ;

(ii) $(w_g)_{g \in G}$ and $(y_g)_{g \in G}$ be the components of $(x_g)_{g \in G}$ as occurred in the Wold decomposition theorem with respect to $\mathcal{J}_\infty$;

(iii) F, F_y , and F_w be the spectral distributions of $(x_g)_{g \in G}$, $(y_g)_{g \in G}$, and $(w_g)_{g \in G}$ respectively and f, f_y , and f_w their corresponding spectral densities;

(iv) F^a, F^s the absolutely continuous and singular components of F with respect to the Haar measure m , as in the Cramer decomposition theorem;

(v) there exist a finite set J in G such that $\vartheta_J \neq \{0\}$.

Then

$$F_y = F^a; F_w = F^s.$$

Proof. Let z be a non-zero element in $\mathcal{M}_{J,x}$. Trivially, $z \perp \mathcal{M}_{\mathcal{J}_\infty, x}$ and, hence, from the Wold decomposition theorem with respect to $\mathcal{J}_\infty$, $z \in \mathcal{M}_y$. A similar proof to that of Theorem 4.35 shows that $f_y = f$ a.e. m . Hence, $F_y = F^a$; $F_w = F^s$. Q.E.D.

4.50 Remark. If any non-zero trigonometric polynomial P on G^* satisfies the condition that $|P|^2/f \notin L_1(G^*, \mathcal{P}^*, m)$, then,

by Remark 4.45, $(x_g)_{g \in G}$ is $\mathcal{J}_\infty$ -singular. Hence, $x_g = w_g$ for all g , and, thus, $F = F_w$. In this case, F_w could be absolutely continuous with respect to m .

In specializing our results for $\mathcal{J}_\infty$ to the case when $G = \mathbb{Z}$, the integers, we will simply state the results comparing $\mathcal{M}_{\mathcal{J}_p, x}$ and $\mathcal{M}_{\mathcal{J}_\infty, x}$, since their proofs follow closely the corresponding proofs for $\mathcal{J}_n$.

4.51 Theorem. Let $(x_n)_{n=-\infty}^\infty$ be a univariate $\mathcal{K}$ -valued WSRF over $\mathbb{Z}$, with spectral distribution F and spectral density f . If, for all $n \in \mathbb{Z}$, there exists a finite set J containing n such that $\theta_J \neq \{0\}$, then $\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_\infty, x}$.

4.52 Theorem. Let $(x_n)_{n=-\infty}^\infty$ be as in 4.51. Let $\log f \in L_1(G^*, \mathcal{B}^*, m)$. Suppose for all finite sets, J , of elements of $\mathbb{Z}$, $\theta_J = \{0\}$. Then $\mathcal{M}_{\mathcal{J}_p, x} \subsetneq \mathcal{M}_{\mathcal{J}_\infty, x} = \mathcal{M}_x$ (see Example 6.4).

4.53 Remark. If $\log f \notin L_1(G^*, \mathcal{B}^*, m)$, then

$$\mathcal{M}_{\mathcal{J}_p, x} = \mathcal{M}_{\mathcal{J}_\infty, x} = \mathcal{M}_x.$$

5. MINIMALITY AND INTERPOLATION OF q -VARIATE WSRF's

In this section we will consider the problems of minimality and interpolation for q -variate ($\mathbb{R}^q$ -valued) WSRF's over a discrete LCAG. In the univariate case the fact that the spectral density is a non-zero scalar a.e. m and, hence, has a well-defined inverse simplifies the work considerably. Since in the multivariate case the spectral density is matrix-valued and, hence, does not have an inverse in general, the results on minimality and interpolation become harder to handle.

By employing the notion of the generalized inverse of a matrix, we can extend several of our results on the univariate case to the multivariate case. The notion of a generalized inverse in connection with the minimality and interpolation of a WSRF indexed by integers was first introduced and exploited by H. Salehi [28]. We shall use his ideas. However, in some cases, it will be necessary to use actual inverses. In these cases, we will make the assumption that certain matrices have full rank.

To avoid any duplication between our work on the univariate case (as presented in §4) and on the multivariate case (as presented in this section) we will omit the proofs of those results in the multivariate case which are analogous to the proofs of the corresponding works in the univariate case. Hence, we will provide proofs only for those statements which involve new techniques or new ideas.

Our first objective will be to establish the Wold decomposition theorem for the multivariate case. Since the proof is similar to the proof for the univariate case, we will only outline the main steps of the proof for the benefit of the reader. First, we introduce the following definition.

5.1 Definition. Let $(\underline{x}_g)_{g \in G}$ and $(\underline{y}_g)_{g \in G}$ be $\mathbb{K}^q$ -valued WSRF's over G , a LCAG. Let $\mathcal{J}$ be any family of non-empty Borel sets of G . Then $(\underline{y}_g)_{g \in G}$ is said to be $\mathcal{J}$ -subordinate to $(\underline{x}_g)_{g \in G}$ if

- (i) $\mathcal{M}_y \subseteq \mathcal{M}_x$;
- (ii) $\mathcal{M}_{I,y} \subseteq \mathcal{M}_{I,x}$ for all $I \in \mathcal{J}$;
- (iii) $(\underline{x}_g)_{g \in G}$ and $(\underline{y}_g)_{g \in G}$ are mutually homogeneously correlated.

5.2 Theorem (Wold decomposition). Let $\mathcal{J}$ be any family of non-empty Borel sets of G closed under translations. Let $(\underline{x}_g)_{g \in G}$ be an $\mathbb{K}^q$ -valued WSRF over G , a LCAG. Then there exists a unique decomposition of $(\underline{x}_g)_{g \in G}$ with respect to $\mathcal{J}$ in the form

$$\underline{x}_g = \underline{y}_g + \underline{w}_g$$

where

- (i) $(\underline{y}_g)_{g \in G}$ and $(\underline{w}_g)_{g \in G}$ are $\mathbb{K}^q$ -valued WSRF's on G ;
- (ii) $(\underline{y}_g)_{g \in G}$ and $(\underline{w}_g)_{g \in G}$ are $\mathcal{J}$ -subordinate to $(\underline{x}_g)_{g \in G}$;
- (iii) $(\underline{y}_g)_{g \in G}$ and $(\underline{w}_g)_{g \in G}$ are orthogonal; i.e., $(\underline{y}_g, \underline{w}_{g'}) = 0$ for any $g, g' \in G$;
- (iv) $(\underline{y}_g)_{g \in G}$ is $\mathcal{J}$ -regular; $(\underline{w}_g)_{g \in G}$ is $\mathcal{J}$ -singular.

Proof. Let $(U_g)_{g \in G}$ be the group of unitary operators on $\mathbb{K}^q$ associated with $(x_g)_{g \in G}$. It can be shown that $U_g \mathbb{M}_{J,x} = \mathbb{M}_{J,x}$ and hence $U_g (\mathbb{M}_{J,x}^\perp) = \mathbb{M}_{J,x}^\perp$ for all $g \in G$. Let $w_g = (x_g | \mathbb{M}_{J,x})$ and $y_g = x_g - w_g$ for all $g \in G$. Then it is easy to see that $\mathbb{M}_w = \mathbb{M}_{J,x}$ and $\mathbb{M}_y = \mathbb{M}_{J,x}^\perp$. The argument that $(y_g)_{g \in G}$ and $(w_g)_{g \in G}$ are WSRF's and the proofs that $(w_g)_{g \in G}$ and $(y_g)_{g \in G}$ are J -subordinate to $(x_g)_{g \in G}$ are straightforward.

To prove that $(y_g)_{g \in G}$ is J -regular, we observe that $\mathbb{M}_{J,y}$ is both perpendicular to $\mathbb{M}_{J,x}$ and contained in it, so that $\mathbb{M}_{J,y} = \{0\}$. Therefore, $(y_g)_{g \in G}$ is J -regular. An argument similar to the classical one (cf. [32], p. 137) shows that $\mathbb{M}_{J,x} = \mathbb{M}_{J,y} \oplus \mathbb{M}_{J,w}$ and hence $\mathbb{M}_{J,w} = \mathbb{M}_w$. Therefore, $(w_g)_{g \in G}$ is J -singular.

We remark that for any decomposition of $(x_g)_{g \in G}$ into $(y_g)_{g \in G}$ and $(w_g)_{g \in G}$ satisfying conditions (i)-(iv), we have that $\mathbb{M}_{J,w} = \mathbb{M}_{J,x}$. This important relation makes the decomposition unique.

We will now state the definition of minimality for a q -variate WSRF over a discrete LCAG.

5.3 Definition. Let G be a discrete LCAG. Then the $\mathbb{K}^q$ -valued WSRF $(x_g)_{g \in G}$ is minimal if, and only if, $x_0 \notin \mathbb{M}_{I,x}$, where $I = \{0\}^c$.

The proof of the minimality theorem for q -variate WSRF's over a discrete LCAG will depend on the following lemmas. The proofs of these lemmas are analogous to the proofs of Lemmas 4.4 and 4.5 and hence will be omitted.

5.4 Lemma. Let $(x_g)_{g \in G}$ be an $\mathbb{K}^q$ -valued WSRF over a discrete LCAG G . Let $\hat{x}_g$ denote $x_g - (x_g | \mathbb{M}_{I+g,x})$, where $I = \{0\}^c$. Then $(\hat{x}_g)_{g \in G}$ is a $\mathbb{K}^q$ -valued WSRF over G . In addition, $(x_g)_{g \in G}$

and $(\hat{x}_g)_{g \in G}$ have the same shift.

5.5 Lemma. Let $(x_g)_{g \in G}$ be an H^q -valued WSRF over G , a discrete LCAG. Then $(x_g)_{g \in G}$ is minimal if, and only if, for all $g \in G$, $x_g \notin \mathcal{M}_{I+g, x}$, $I = \{0\}^c$.

The next lemma plays an important role in the theory of minimality of H^q -valued WSRF's.

5.6 Lemma (Main Lemma I). Let $(x_g)_{g \in G}$ be an H^q -valued WSRF over G , a discrete LCAG, with the shift group of unitary operators $(U_g)_{g \in G}$ and E be the spectral measure of $(U_g)_{g \in G}$. Let $\hat{x}_g = x_g - (x_g | \mathcal{M}_{I+g, x})$, $I = \{0\}^c$. Let $\underline{F}$ be the spectral distribution of $(x_g)_{g \in G}$, $\underline{F}'$ its spectral density, and $\underline{F}'^\#$ the generalized inverse of $\underline{F}'$ (cf. [17] and [20]). Then

$$\hat{x}_g = \int_{\star, G} (g, \lambda) \underline{\Phi}_0(\lambda) \underline{E}(d\lambda) x_0$$

where $\underline{\Phi}_0$ is defined by

$$\underline{\Phi}_0 = \begin{cases} (\hat{x}_0, \hat{x}_0) \underline{F}'^\# & , \text{ on the carrier of } \underline{F}^a \\ \underline{0} & , \text{ on the carrier of } \underline{F}^s. \end{cases}$$

Proof. Without loss of generality, it suffices to prove that $\hat{x}_0$ is given by

$$\hat{x}_0 = \int_{\star, G} \underline{\Phi}_0(\lambda) \underline{E}(d\lambda) x_0.$$

Since $\hat{x}_0 \in \mathcal{M}_x$, by the Isomorphism Theorem (3.7),

$$\hat{x}_0 = \int_{\star, G} \underline{\Phi}_0(\lambda) \underline{E}(d\lambda) x_0$$

for some $\underline{\Phi}_0 \in \underline{L}_2(G^*, \underline{\mathcal{B}}^*, \underline{F})$. Also (cf. 3.2),

$$\underline{x}_g = \int_{G^*} (g, \lambda) \underline{E}(d\lambda) \underline{x}_0 \quad \text{for all } g \in G.$$

Using the same techniques as in the proof of the univariate case (Lemma 4.6), we arrive at the following

$$(I) \quad (\hat{x}_0, \hat{x}_0) = \underline{\Phi}_0 \underline{F}', \text{ on the carrier of } \underline{F}^a$$

$$\underline{\Phi}_0 = \underline{0} \quad , \text{ on the carrier of } \underline{F}^s .$$

Letting $\underline{\Psi} = \underline{\Phi}_0 \underline{P}_{\underline{\mathcal{R}}_{\underline{F}'}}$, where $\underline{P}_{\underline{\mathcal{R}}_{\underline{F}'}}$ denotes the projection operator onto $\underline{\mathcal{R}}_{\underline{F}'}$, the range of $\underline{F}'$, we can easily show that $\underline{\Psi} = \underline{\Phi}_0$ in $\underline{L}_2(G^*, \underline{\mathcal{B}}^*, \underline{F})$. Therefore, (I) can be written as

$$(II) \quad \underline{\Psi} \underline{F}' = (\hat{x}_0, \hat{x}_0) \underline{F}'^{\#} \text{ , on the carrier of } \underline{F}^a$$

$$\underline{\Psi} = \underline{0} \quad , \text{ on the carrier of } \underline{F}^s .$$

Hence, by (II), it is clear that

$$\underline{\Psi} = \begin{cases} (\hat{x}_0, \hat{x}_0) \underline{F}'^{\#} \text{ , on the carrier of } \underline{F}^a \\ \underline{0} \text{ , on the carrier of } \underline{F}^s . \end{cases} \quad \text{Q.E.D.}$$

For the proof of Kolmogorov's minimality theorem, we need the following lemma, whose proof is found in [20].

5.7 Lemma. Let $\underline{\mathcal{B}}$ be a σ -algebra of subsets of a space Ω and μ be a non-negative σ -finite measure on $\underline{\mathcal{B}}$. Let $\underline{\Phi}$ be a non-negative, hermitian, $q \times q$ matrix-valued function on Ω such that $\underline{\Phi} \in \underline{L}_1(\Omega, \underline{\mathcal{B}}, \mu)$. Then

$$\text{rank} \left(\int_{\Omega} \underline{\Phi} d\mu \right) \geq \text{rank } \underline{\Phi} \text{ a.e. } \mu .$$

The following theorem on the minimality of a WSRF over a discrete LCAG is an analogue of a theorem by Masani (cf. 3.14) for processes indexed by integers.

5.8 Theorem (Kolmogorov minimality theorem). Let $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over G , a discrete LCAG, $\underline{F}$ its spectral distribution, and $\underline{F}'$ its spectral density. Then $(\underline{x}_g)_{g \in G}$ is minimal and $\rho_{I,0} = q$, $I = \{0\}^c$ if, and only if, $\underline{F}'^{-1}$ exists a.e. m and $\underline{F}'^{-1} \in \underline{L}_1(G^*, \mathcal{B}^*, m)$.

Proof. Sufficiency. Set

$$\underline{\Phi} = \begin{cases} \underline{F}'^{-1}, & \text{on the carrier of } \underline{F}^a \\ \underline{0} & , \text{ on the carrier of } \underline{F}^s . \end{cases}$$

Then

$$\begin{aligned} \text{(I)} \quad \int_{G^*} \underline{\Phi} d\underline{F} \underline{\Phi}^* &= \int_{G^*} \underline{\Phi} \underline{F}' \underline{\Phi}^* dm + \int_{G^*} \underline{\Phi} d\underline{F}^s \underline{\Phi}^* \\ &= \int_{G^*} \underline{F}'^{-1} dm . \end{aligned}$$

Hence, $\underline{\Phi} \in \underline{L}_2(G^*, \mathcal{B}^*, \underline{F})$. Now, by the Isomorphism Theorem (3.7) there exists $\underline{y} \in \mathcal{M}_x$ such that

$$\text{(II)} \quad \underline{y} = \int_{G^*} \underline{\Phi}(\lambda) \underline{E}(d\lambda) \underline{x}_0 .$$

Following the proof of Lemma 4.6, we can show that

$$\text{(III)} \quad (\underline{x}_g, \underline{y}) = \delta_{g,0} \underline{1} .$$

Hence, $\underline{y} = \underline{A} \hat{\underline{x}}_0$. Note that by (I), $\underline{y} \neq \underline{0}$ and thus $\hat{\underline{x}}_0 \neq \underline{0}$.

Therefore, $\underline{x}_0 \notin \mathcal{M}_{I,x}$, which implies $(\underline{x}_g)_{g \in G}$ is minimal.

By (II) and the fact that $\underline{y} = \underline{A} \hat{\underline{x}}_0$, we get

$$(IV) \quad (\underline{y}, \underline{y}) = \int_{\underline{G}} \underline{F}'^{-1}(\lambda) m(d\lambda) = \underline{A}(\hat{\underline{x}}_0, \hat{\underline{x}}_0) \underline{A}^*.$$

By assumption, $\text{rank}(\underline{F}'^{-1}) = q$ a.e. m and hence, by Lemma 5.6, $\text{rank}(\underline{y}, \underline{y}) = q$. Thus, $\text{rank}(\hat{\underline{x}}_0, \hat{\underline{x}}_0) = q$. Hence, $\rho_{I,0} = q$.

Necessity. Since $\rho_{I,0} = q$, then $\text{rank}(\hat{\underline{x}}_0, \hat{\underline{x}}_0) = q$. But, in the proof of Main Lemma I (5.6), we had

$$\hat{\underline{x}}_0 \underline{F}' = (\hat{\underline{x}}_0, \hat{\underline{x}}_0) \text{ a.e. } m.$$

Hence, $\text{rank}(\underline{F}') = q$ a.e. m , which implies $\underline{F}'^{-1}$ exists a.e. m .

Since $(\hat{\underline{x}}_0, \hat{\underline{x}}_0) = \int_{\underline{G}} \hat{\underline{x}}_0 d\underline{F} \hat{\underline{x}}_0^* = \int_{\underline{G}} (\hat{\underline{x}}_0, \hat{\underline{x}}_0) \underline{F}'^{-1}(\hat{\underline{x}}_0, \hat{\underline{x}}_0) dm$, and $\text{rank}(\hat{\underline{x}}_0, \hat{\underline{x}}_0) = q$, we get

$$(\hat{\underline{x}}_0, \hat{\underline{x}}_0)^{-1} = \int_{\underline{G}} \underline{F}'^{-1} dm$$

and hence $\underline{F}'^{-1} \in \underline{L}_1(\underline{G}^*, \mathcal{B}^*, m)$.

Q.E.D.

The following is a partial analogue of Theorem 4.8. We note that in part (b), we need a stronger assumption than minimality; namely, $\rho_{I,0} = q$, $I = \{0\}^c$.

5.9 Theorem. Let $(\underline{x}_{\underline{g}})_{\underline{g} \in \underline{G}}$ be a non-trivial $\mathbb{K}^q$ -valued WSRF over a discrete LCAG $\underline{G}$.

(a) If $(\underline{x}_{\underline{g}})_{\underline{g} \in \underline{G}}$ is $\mathcal{J}_0$ -regular, then $(\underline{x}_{\underline{g}})_{\underline{g} \in \underline{G}}$ is minimal and $\underline{F}$, the spectral distribution of $(\underline{x}_{\underline{g}})_{\underline{g} \in \underline{G}}$ is absolutely continuous with respect to m ;

(b) If $\underline{F}$ is absolutely continuous and $\rho_{I,0} = q$, $I = \{0\}^c$, then $(\underline{x}_{\underline{g}})_{\underline{g} \in \underline{G}}$ is $\mathcal{J}_0$ -regular.

Proof (a). Since $\mathcal{M}_{\mathcal{J}_0, x} = \{0\}$, $\mathcal{M}_{\mathcal{J}_0, x}^\perp = \mathcal{M}_x$. But, as in the proof of Theorem 4.8, we have $\mathcal{M}_{\mathcal{J}_0, x}^\perp = \mathcal{C}(\hat{x}_g, g \in G)$. Let

$$\underline{\Psi} = \begin{cases} \underline{0}, & \text{on the carrier of } \underline{F}^a \\ \underline{I}, & \text{on the carrier of } \underline{F}^s. \end{cases}$$

As in the proof of Theorem 4.8 we can show $\underline{\Psi} \in L_2(G^*, \mathcal{B}^*, \underline{F})$. By the Isomorphism Theorem (3.7), there exists a $\underline{y} \in \mathcal{M}_x$ such that $(\underline{y}, \underline{y}) = \int_{G^*} \underline{\Psi} d\underline{F} \underline{\Psi}^*$. It is not hard to show that $(\underline{y}, \hat{x}_g) = 0$ for all $g \in G$, and hence $\underline{y} = 0$. Thus, since $(\underline{y}, \underline{y}) = \underline{F}^s(G^*)$, $\underline{F}^s = 0$.

Trivially, since $\mathcal{M}_{\mathcal{J}_0, x} = \{0\}$, $\underline{x}_g \notin \mathcal{M}_{I+g, x}$ for all g , and hence $(\underline{x}_g)_{g \in G}$ is minimal.

(b) Let $\underline{z} \in \mathcal{M}_x$. If we can show that $\underline{z} \perp \hat{x}_g$ for all $g \in G$ implies that $\underline{z} = 0$, then we will have shown that $\{\hat{x}_g, g \in G\}$ is dense in $\mathcal{M}_x$, and hence that $\mathcal{C}(\hat{x}_g, g \in G) = \mathcal{M}_x$. But $\mathcal{C}(\hat{x}_g, g \in G) = \mathcal{M}_x$ iff $\mathcal{M}_{\mathcal{J}_0, x} = \{0\}$; i.e., $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_0$ -regular.

From Main Lemma I (5.6), we get that

$$\hat{x}_g = \int_{G^*} (g, \lambda) \hat{\Phi}_0(\lambda) \underline{E}(d\lambda) \underline{x}_0 \quad \text{for all } g \in G,$$

where $\hat{\Phi}_0$ is as in 5.6. Since $\underline{z} \in \mathcal{M}_x$, $\underline{z} = \int_{G^*} \underline{\Psi}(\lambda) \underline{E}(d\lambda) \underline{x}_0$ where $\underline{\Psi} \in L_2(G^*, \mathcal{B}^*, \underline{F})$. As in the univariate case,

$$0 = (\underline{z}, \hat{x}_g) = (\hat{x}_0, \hat{x}_0) \int_{G^*} \underline{\Psi}(-g, \lambda) m(d\lambda) \quad \text{for all } g \in G.$$

Since $\rho_{I,0} = q$; i.e., $\text{rank}(\hat{x}_0, \hat{x}_0) = q$, hence,

$$0 = \int_{G^*} \underline{\Psi}(\lambda) (-g, \lambda) m(d\lambda) \quad \text{for all } g \in G.$$

As in the proof of Theorem 4.8, we get $\underline{\Psi} = \underline{0}$ a.e. m . Therefore, since $\underline{F}$ is absolutely continuous,

$$(\underline{z}, \underline{z}) = \int_G \underline{\Psi}^* d\underline{F} \underline{\Psi} = \int_G \underline{\Psi}^* \underline{F}' \underline{\Psi} dm = 0$$

which shows that $\underline{z} = \underline{0}$.

Q.E.D.

From the above theorem and Theorem 5.8, we obtain the following corollary.

5.10 Corollary. Let $(\underline{x}_g)_{g \in G}$ be as in 5.9.

(a) If $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_0$ -regular, then $(\underline{x}_g)_{g \in G}$ is minimal and $\underline{F}$ is absolutely continuous;

(b) If $\underline{F}$ is absolutely continuous, $\underline{F}'^{-1}$ exists a.e. m , and $\underline{F}'^{-1} \in \underline{L}_1(G^*, \beta^*, m)$, then $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_0$ -regular.

5.11 Remark. We can easily give a characterization of $\mathcal{J}_0$ -singularity in terms of $\rho_{I,0}$. In fact, $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_0$ -singular if, and only if, $\rho_{I,0} = 0$. However, in terms of the spectral domain, we are, at this time, only partially able to extend the characterization of $\mathcal{J}_0$ -singularity for the univariate case to the multivariate case, as the following remark indicates. The proof is straightforward and, hence, will be omitted.

5.12 Remark. Let $(\underline{x}_g)_{g \in G}$ be an $\mathbb{R}^q$ -valued WSRF over G , a discrete LCAG. If $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_0$ -singular, then either $\underline{F}'^{-1}$ doesn't exist a.e. m or else $\underline{F}'^{-1} \notin \underline{L}_1(G^*, \beta^*, m)$.

As in the univariate case, there exist conditions under which a q -variate WSRF over a discrete LCAG is neither $\mathcal{J}_0$ -singular nor $\mathcal{J}_0$ -regular. Once again the proof is not difficult and thus will be omitted.

5.13 Theorem. Let $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over a discrete LCAG G . If $\rho_{I,0} \neq 0$, $I = \{0\}^c$, and $\underline{F}^s \neq \underline{0}$, then $\{0\} \subsetneq \mathcal{M}_{\mathcal{J}_0, x} \subsetneq \mathcal{M}_x$.

In the following we prove a theorem on the concordance of the Wold decomposition with respect to $\mathcal{J}_0$ and the Cramer decomposition for a q -variate WSRF over a discrete LCAG under the assumption that the process has full rank. The problem remains open when this condition is not satisfied. As one can see from Theorem 3.20, for q -variate processes indexed by the integers Robertson has given a necessary and sufficient condition involving the rank of the spectral density for concordance of the Wold decomposition with respect to the past of the process and the Cramer decomposition. For q -variate WSRF's over a discrete LCAG, it would be interesting to give a necessary and sufficient condition involving the rank of the spectral density for concordance between the Wold decomposition with respect to $\mathcal{J}_0$ and the Cramer decomposition.

5.14 Theorem (Wold-Cramer concordance for $\mathcal{J}_0$). Let

- (i) $(\underline{x}_g)_{g \in G}$ be an $\mathbb{R}^q$ -valued WSRF over G , a discrete LCAG;
- (ii) $(\underline{w}_g)_{g \in G}$ and $(\underline{y}_g)_{g \in G}$ be the components of $(\underline{x}_g)_{g \in G}$ as occurred in the Wold decomposition theorem with respect to $\mathcal{J}_0$;
- (iii) $\underline{F}$, $\underline{F}_y$, and $\underline{F}_w$ be the spectral distributions of $(\underline{x}_g)_{g \in G}$, $(\underline{y}_g)_{g \in G}$ and $(\underline{w}_g)_{g \in G}$ respectively and $\underline{F}^a$, $\underline{F}_y^a$, and $\underline{F}_w^a$ their corresponding spectral densities;
- (iv) $\underline{F}^a$, $\underline{F}^s$ be the absolutely continuous and singular components of $\underline{F}$ with respect to the Haar measure, as in the Cramer decomposition theorem;

(v) $\underline{F}'^{-1}$ exist a.e. m and $\underline{F}'^{-1} \in \underline{L}_1(G^*, \mathcal{G}^*, m)$.

Then

$$\underline{F}_y = \underline{F}^a ; \underline{F}_w = \underline{F}^s .$$

Proof. By assumption (v) and Main Lemma I, 5.6, we get

$$(I) \quad \hat{\underline{x}}_0 = \int_{\underline{G}} \underline{\Phi}_0(\lambda) \underline{E}(d\lambda) \underline{x}_0$$

where

$$\underline{\Phi}_0 = \begin{cases} (\hat{\underline{x}}_0, \underline{x}_0) \underline{F}'^{-1}, & \text{on the carrier of } \underline{F}^a \\ \underline{0} & , \text{ on the carrier of } \underline{F}^s . \end{cases}$$

By Theorem 5.9(b), $\underline{F}_y$ is absolutely continuous with respect to m .

Now, using the same type of proof used in Theorem 4.13, we obtain the following relations:

$$(II) \quad \underline{F}' = \underline{F}'_y + \underline{F}'_w \quad \text{a.e. } m$$

$$d\underline{F}^s = d\underline{F}_w^s .$$

Since we can easily show that $\hat{\underline{x}}_0 \in \mathcal{M}_y$, again imitating the proof of Theorem 4.13, we obtain

$$(III) \quad \hat{\underline{x}}_0 = \int_{\underline{G}} \underline{\Phi}_0(\lambda) \underline{E}(d\lambda) \underline{y}_0 .$$

By (I), it follows that

$$(IV) \quad (\hat{\underline{x}}_0, \hat{\underline{x}}_0) = \int_{\underline{G}} \underline{\Phi}_0 d\underline{F} \underline{\Phi}_0^* = \int_{\underline{G}} (\hat{\underline{x}}_0, \underline{x}_0) \underline{F}'^{-1} (\hat{\underline{x}}_0, \underline{x}_0) dm .$$

By (III), we get

$$(V) \quad (\hat{x}_0, \hat{x}_0) = \int_{G^*} \hat{\Phi}_0 dF_y \hat{\Phi}_0^* = \int_{G^*} (\hat{x}_0, \hat{x}_0) \underline{F}'^{-1} \underline{F}' \underline{F}'^{-1} (\hat{x}_0, \hat{x}_0) dm.$$

Combining (IV) and (V), we have

$$(VI) \quad \int_{G^*} (\hat{x}_0, \hat{x}_0) \underline{F}'^{-1} (\hat{x}_0, \hat{x}_0) dm = \int_{G^*} (\hat{x}_0, \hat{x}_0) \underline{F}'^{-1} \underline{F}' \underline{F}'^{-1} (\hat{x}_0, \hat{x}_0) dm.$$

But, by assumption (v) and Theorem 5.8, rank $(\hat{x}_0, \hat{x}_0) = q$, and, hence,

(VI) is equivalent to

$$(VII) \quad \int_{G^*} (\underline{F}'^{-1} - \underline{F}'^{-1} \underline{F}' \underline{F}'^{-1}) dm = 0.$$

By (II), $\underline{F}' \geq \underline{F}'_y$ a.e. m and, hence, $\underline{F}'^{-1} \geq \underline{F}'^{-1} \underline{F}'_y \underline{F}'^{-1}$ a.e. m .

This fact and (VII) imply, by Lemma 5.7, that $\underline{F}'^{-1} = \underline{F}'^{-1} \underline{F}'_y \underline{F}'^{-1}$

a.e. m and, hence, $\underline{F}' = \underline{F}'_y$ a.e. m . Q.E.D.

5.15 Remark. If $\rho_{I,0} = 0$, $I = \{0\}^c$, then $(x_g)_{g \in G}$ is $\mathcal{J}_0$ -singular and hence, $x_g = w_g$ for all $g \in G$. Thus $\underline{F} = \underline{F}_w$. In this case, $\underline{F}_w$ could be absolutely continuous with respect to m .

We will now specialize our results to the discrete group Z , the integers. We will first recall some notations (cf. 4.15).

5.16 Notation. Let Z be the integers. Let $(x_k)_{k=-\infty}^{\infty}$ be a q -variate WSRF over Z . Let $J_k = \{k\}^c$; $I_k = \{n: n \leq k\}$. Then

$$(i) \quad \mathbb{M}_{J_k, x} = \mathcal{G}(x_n, n \neq k)$$

$$(ii) \quad \mathbb{M}_{I_k, x} = \mathcal{G}(x_n, n \leq k).$$

$$\text{Clearly, } \mathbb{M}_{\mathcal{J}_p, x} = \bigcap_{k=-\infty}^{\infty} \mathbb{M}_{I_k, x} \subseteq \bigcap_{k=-\infty}^{\infty} \mathbb{M}_{J_k, x} = \mathbb{M}_{\mathcal{J}_0, x}.$$

In the following theorem we will give a condition under

which $\mathbb{M}_{\mathcal{J}_p, x} = \mathbb{M}_{\mathcal{J}_0, x}$.

5.17 Theorem. Let $(x_n)_{n=-\infty}^{\infty}$ be an $\mathcal{K}^q$ -valued WSRF over Z .

Let $\underline{F}'^{-1}$ exist a.e. m and $\underline{F}'^{-1} \in \underline{L}_1(G^*, \mathcal{B}^*, m)$. Then $\mathbb{M}_{\mathcal{J}_p, x} = \mathbb{M}_{\mathcal{J}_0, x}$.

Proof. By Theorem 5.14, $\underline{F}_y = \underline{F}^a$; $\underline{F}_w = \underline{F}^s$. Since $\mathbb{M}_{I_{n-1},x} \subseteq \mathbb{M}_{J_n,x}$, it follows that $\rho_{I_0,1} \geq \rho_{J_0,0}$. But, by Theorem 5.8, $\rho_{J_0,0} = q$ and, hence, $\rho_{I_0,1} = q$. Then by Theorem 3.19, $\underline{F}_v = \underline{F}^s$; $\underline{F}_u = \underline{F}^a$. $\underline{F}_v, \underline{F}_u$ are the spectral distributions of the components $(\underline{v}_n)_{-\infty}^\infty, (\underline{u}_n)_{-\infty}^\infty$ in the usual Wold decomposition of $(\underline{x}_n)_{-\infty}^\infty$ with respect to the past. Hence, we get

$$(I) \quad \underline{F}_u = \underline{F}_y ; \underline{F}_v = \underline{F}_w .$$

Using the same techniques as in the proof of Theorem 4.16, by (I) and the fact that $\mathbb{M}_{J_0,x} \supseteq \mathbb{M}_{J_p,x}$, we get $\mathbb{M}_{J_p,x} = \mathbb{M}_{J_0,x}$. Q.E.D.

5.18 Remark. In Theorem 4.17 we gave analytic conditions in terms of the spectral density under which $\mathbb{M}_{J_p,x} \stackrel{\subseteq}{=} \mathbb{M}_{J_0,x} = \mathbb{M}_x$. For the q -variate case, in general, a reasonable analytic condition is not available. However, in terms of the rank of the process we make the observation that if $\rho_{J_0,0} = 0$ and $\rho_{I_0,1} > 0$, then $\mathbb{M}_{J_p,x} \stackrel{\subseteq}{=} \mathbb{M}_{J_0,x} = \mathbb{M}_x$. On the other hand, if $\rho_{I_0,0} = 0$, then $\mathbb{M}_{J_p,x} = \mathbb{M}_{J_0,x} = \mathbb{M}_x$.

The rest of this section will be devoted to an extension of Salehi's work on the interpolation of q -variate WSRF's indexed by integers to q -variate WSRF's over discrete LCAG's. In this connection we may add that comments similar to the ones made in Remark 4.19 regarding the minimality and interpolation of a univariate WSRF can also be made for the multivariate case. To avoid duplication we will not repeat these comments and will refer the interested reader to Remark 4.19.

As we saw in the theory on minimality, some of the results in the univariate case could only be partially extended to the

multivariate case. We will find that the same thing happens in our theory on interpolation. Whenever necessary, we will make the assumption that certain matrices have full rank.

We will now recall some old notation and introduce some new ones needed in the rest of this section.

5.19 Notation. Let $(\underline{x}_g)_{g \in G}$ be an $\mathbb{K}^q$ -valued WSRF over a LCAG G . Let $\underline{F}$ be the spectral distribution of $(\underline{x}_g)_{g \in G}$, $\underline{F}'$ its spectral density, $\underline{F}'^{\#}$ the generalized inverse of $\underline{F}'$, and $\underline{Q} = \underline{F}'^{\#} \underline{F}'^{(1)}$. Let $J = \{g_0, g_1, \dots, g_n\}$ be a fixed set of $n+1$ elements of G . Then

$$(i) \quad \mathcal{M}_{J,x} = \mathcal{M}_{J^c,x}^{\perp} \cap \mathcal{M}_x;$$

$$(ii) \quad \mathcal{Q}_J = \{ \underline{P}: \underline{P}(\lambda) = \sum_{k=0}^n (g_k, \lambda) \underline{A}_k, \underline{A}_k \text{'s are arbitrary } q \times q \text{ complex-valued matrices; } \underline{P} \underline{Q} = \underline{P} \text{ a.e. } m \text{ and } \underline{P} \underline{F}'^{\#} \underline{P}^* \in \underline{L}_1(G^*, \mathcal{B}^*, m) \};$$

$$(iii) \quad \mathcal{J}_n = \{J^c + g, g \in G\}.$$

5.20 Remark. It is easy to see that $\underline{z} \in \mathcal{M}_{J,x}$ if, and only if, $\underline{z} \perp \underline{x}_g$ for all $g \in J^c$ and that $\mathcal{M}_{J,x} = \mathcal{G}(\hat{\underline{x}}_{g_0}, \hat{\underline{x}}_{g_1}, \dots, \hat{\underline{x}}_{g_n})$, where $\hat{\underline{x}}_{g_i}$ is defined by $\hat{\underline{x}}_{g_i} = \underline{x}_{g_i} - (\underline{x}_{g_i} | \mathcal{M}_{J^c,x}^{\perp})$.

We now make the following definition which is an extension of non-minimality for a q -variate WSRF over a discrete LCAG.

5.21 Definition. Let J be as above and $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over G , a discrete LCAG. We say that $(\underline{x}_g)_{g \in G}$ is interpolable with respect to J if

$$\mathcal{M}_{J,x} \subseteq \mathcal{M}_{J^c,x}$$

¹⁾ $\underline{Q}$ is the orthogonal projection onto the range of $\underline{F}'$.

or, equivalently,

$$\mathcal{M}_{J,x} = \{0\}.$$

It is clear that $\mathcal{M}_{J,x}$ is a subspace of $\mathcal{M}_x$. It is also obvious that the set $\mathcal{Q}_J$ is a linear subset of all the matrix-valued polynomials on G^* . We introduce a matricial inner product in $\mathcal{Q}_J$ in the following manner:

$$(\underline{P}_1, \underline{P}_2)_{\underline{F}, \#} = \int_{G^*} \underline{P}_1(\lambda) \underline{F}^{\#}(\lambda) \underline{P}_2^*(\lambda) m(d\lambda), \quad \underline{P}_1, \underline{P}_2 \in \mathcal{Q}_J.$$

The proof of the following lemma is straightforward and thus will be omitted.

5.22 Lemma. With the above notation, $\mathcal{Q}_J$ is an inner product space over the ring of $q \times q$ matrices with the inner product

$$((\underline{P}_1, \underline{P}_2))_{\underline{F}, \#} = \text{tr} \int_{G^*} \underline{P}_1(\lambda) \underline{F}^{\#}(\lambda) \underline{P}_2^*(\lambda) m(d\lambda), \quad \underline{P}_1, \underline{P}_2 \in \mathcal{Q}_J.$$

The fact that the inner product space is finite-dimensional and, hence, complete will follow from the following important lemma. This lemma will be used repeatedly in the interpolation of $\mathbb{K}^q$ -valued WSRF's over G , a discrete LCAG.

5.23 Lemma (Main Lemma II). With the above setting the finite-dimensional subspace $\mathcal{M}_{J,x}$ and the inner product space $\mathcal{Q}_J$ are isometric; i.e., there exists a linear operator $\underline{T}$ on $\mathcal{M}_{J,x}$ onto $\mathcal{Q}_J$ such that

$$(\underline{z}_1, \underline{z}_2) = (\underline{T} \underline{z}_1, \underline{T} \underline{z}_2)_{\underline{F}, \#}, \quad \underline{z}_1, \underline{z}_2 \in \mathcal{M}_{J,x}.$$

Proof. Let $\underline{z} \in \mathcal{M}_{J,x}$. We define the polynomial $\underline{P}_z$ by

$$(I) \quad \underline{P}_z(\lambda) = \sum_{k=0}^n (g_k, \lambda)(\underline{z}, \underline{x}_{g_k}) .$$

We claim that $\underline{P}_z$ is an element of $\mathcal{Q}_J$. In view of the fact that the subspace $\mathcal{M}_{J,x}$ is spanned by $\{\hat{x}_{g_0}, \hat{x}_{g_1}, \dots, \hat{x}_{g_n}\}$, it suffices to prove that $\underline{P}_{\hat{x}_{g_i}} \in \mathcal{Q}_J$, $0 \leq i \leq n$. For simplicity, $\underline{P}_i(\lambda)$ will denote $\underline{P}_{\hat{x}_{g_i}}(\lambda) = \sum_{k=0}^n (g_k, \lambda)(\hat{x}_{g_i}, \underline{x}_{g_k})$. Since $\hat{x}_{g_i} \in \mathcal{M}_x$, by the Isomorphism Theorem (3.7), there exists $\underline{\Phi}_i \in L_2(G^*, \mathcal{B}^*, \underline{F})$ such that $\hat{x}_{g_i} = \int_G^* \underline{\Phi}_i(\lambda) \underline{E}(d\lambda) \underline{x}_0$. Using a similar proof to that of Lemma 4.24, we get

$$(II) \quad \begin{aligned} \underline{P}_i &= \underline{\Phi}_i \underline{F}' , \text{ on the carrier of } \underline{F}^a \\ \underline{\Phi}_i &= \underline{0} , \text{ on the carrier of } \underline{F}^s . \end{aligned}$$

$$\begin{aligned} \text{Thus } \int_G^* \underline{P}_i \underline{F}'^{\#} \underline{P}_i^* dm &= \int_G^* \underline{\Phi}_i \underline{F}' \underline{F}'^{\#} \underline{\Phi}_i^* dm \\ &= \int_G^* \underline{\Phi}_i \underline{F}' \underline{\Phi}_i^* dm \\ &= \int_G^* \underline{\Phi}_i d\underline{F} \underline{\Phi}_i^* . \end{aligned}$$

Hence, $\underline{P}_i \underline{F}'^{\#} \underline{P}_i^* \in L_1(G^*, \mathcal{B}^*, m)$. Also, by (II) it follows that

$$\underline{P}_i \underline{Q} = \underline{P}_i \text{ a.e. } m \text{ and thus } \underline{P}_i \in \mathcal{Q}_J .$$

We now define the operator $\underline{T}$ on $\mathcal{M}_{J,x}$ into $\mathcal{Q}_J$ by

$$(III) \quad \underline{T} \underline{z} = \underline{P}_z , \underline{z} \in \mathcal{M}_{J,x} .$$

Clearly, $\underline{T}$ is linear and it is not hard to show that it preserves the matricial inner product.

It remains to show that $\underline{T}$ is onto $\mathcal{Q}_J$. To do that, we show that for any given $\underline{P} \in \mathcal{Q}_J$, $\underline{P}(\lambda) = \sum_{k=0}^n (g_k, \lambda) \underline{A}_k$, there exists a $\underline{z} \in \mathcal{M}_{J,x}$ such that $\underline{P} = \underline{P}_z$. We remark that the function

$$\underline{\Phi} = \begin{cases} \underline{P} \underline{F}'^{\#}, & \text{on the carrier of } \underline{F}^a \\ \underline{0} & \text{, on the carrier of } \underline{F}^s \end{cases}$$

is in $L_2(G^*, \mathcal{B}^*, \underline{F})$. Define $\underline{z} \in \mathcal{M}_x$ by

$$\underline{z} = \int_{G^*} \underline{\Phi}(\lambda) \underline{E}(d\lambda) \underline{x}_0.$$

As in the proof of 4.24, by examining $\underline{T} \underline{z}$, we get:

$$\begin{aligned} \underline{T} \underline{z}(\lambda) &= \sum_{k=0}^n (g_k, \lambda) \int_{G^*} (-g_k, \lambda) \underline{P}(\lambda) \underline{F}'^{\#}(\lambda) \underline{F}'(\lambda) m(d\lambda) \\ &= \sum_{k=0}^n (g_k, \lambda) \int_{G^*} (-g_k, \lambda) \underline{P}(\lambda) m(d\lambda) \\ &= \sum_{k=0}^n (g_k, \lambda) \underline{A}_k = \underline{P}(\lambda), \end{aligned}$$

where the second equality follows because $\underline{P} \underline{Q} = \underline{P}$ a.e. m. Q.E.D.

5.24 Remark. If $\underline{F}'^{-1}$ exists a.e. m, then $\underline{Q} = \underline{I}$ a.e. m and, hence, the condition $\underline{P} \underline{Q} = \underline{P}$ a.e. m is automatically satisfied. In particular, this is true in the univariate case since f has an inverse a.e. m.

The following is the analogue of Kolmogorov's minimality theorem (cf. Theorem 5.8) for the case when J has $n+1$ elements. The proof follows directly from Main Lemma II (5.23), and thus will be omitted.

5.25 Theorem. Let $J = \{g_0, g_1, \dots, g_n\}$ be a fixed set of $n+1$ elements of G , a discrete LCAG. Let $(\underline{x}_g)_{g \in G}$ be an $\mathcal{A}^J$ -valued

WSRF over G . Then $(\underline{x}_g)_{g \in G}$ is not interpolable with respect to J if, and only if, $\varrho_J \neq \{0\}$.

The following corollary is immediate.

5.26 Corollary. With the same setting as in the above theorem, $(\underline{x}_g)_{g \in G}$ is interpolable with respect to J if, and only if, $\varrho_J = \{0\}$.

We will need the next two lemmas in the proof of Theorem 5.29. The proof of the first is easy and is omitted.

5.27 Lemma. Let $J = \{g_0, g_1, \dots, g_n\}$ and $g_i \in J$. Then $\hat{x}_{g_i} \neq 0$ if, and only if, $\hat{x}_{g_i+g} = \underline{x}_{g_i+g} - (\underline{x}_{g_i+g} | \mathcal{M}_{J^c+g, x}) \neq 0$.

5.28 Lemma. Let G be an ordered, discrete LCAG and K be any finite subset of G . Let $\underline{P}$ be a non-zero trigonometric polynomial of the form $\underline{P}(\lambda) = \sum_{g \in K} (g, \lambda) \underline{A}_g$, where each $\underline{A}_g$ is a $q \times q$ complex-valued matrix. Then $\text{rank } \underline{P} = \text{constant a.e. m.}$

Proof. By examining the minors of $\underline{P}$ of various orders, one can show there exists a minor of order r , say $\underline{A}_r$, $1 \leq r \leq q$, such that $\underline{A}_r$ is a non-zero polynomial and all minors of higher order are identically zero. As in the proof of Lemma 4.29, we can prove that $\underline{A}_r \neq 0$ a.e. m. Therefore, $\text{rank } \underline{P} = r$ a.e. m. Q.E.D.

The following theorem is an analogue of Theorem 4.28. At this stage we have only been able to prove it under a full rank condition.

5.29 Theorem. Let $(\underline{x}_g)_{g \in G}$ be an $\mathbb{M}^d$ -valued WSRF over G , a discrete LCAG. Let $J = \{g_0, g_1, \dots, g_n\}$ be a fixed set of $n+1$ elements in G ; $\mathcal{J}_n = \{J^c + g, g \in G\}$.

(a) If $(\underline{x}_g)_{g \in G}$ is non-trivial and is $\mathcal{J}_n$ -regular, then $(\underline{x}_g)_{g \in G}$ is not interpolable with respect to $J + g$ for all $g \in G$,

and $\underline{F}$ is absolutely continuous with respect to the Haar measure m .

(b) Let G be ordered. If $\underline{F}$ is absolutely continuous and there exists a polynomial $\underline{P}$ in $\mathcal{Q}_J$ such that $\text{rank } \underline{P} = q$ on a set of positive Haar measure, then $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_n$ -regular.

Proof (a). The proof that $(\underline{x}_g)_{g \in G}$ is not interpolable with respect to $J + g$ for all $g \in G$ follows immediately. As in the proof of Theorem 4.28(a), we can show that $\mathcal{J}_n$ -regularity implies

$$(I) \quad \mathcal{M}_x = \mathcal{G}(\hat{x}_{\underline{g}_i + g}, 0 \leq i \leq n, g \in G).$$

In the proof of Main Lemma II (5.23), we saw that

$$\hat{x}_{\underline{g}_i} = \int_{\star} \underline{\Phi}_i(\lambda) \underline{E}(d\lambda) \underline{x}_0 \quad \text{where} \quad \underline{\Phi}_i = \begin{cases} \underline{P}_i \underline{F}'^{\#}, & \text{on the carrier of } \underline{F}^a \\ \underline{0} & , \text{ on the carrier of } \underline{F}^s, \end{cases}$$

$\underline{P}_i(\lambda) = \sum_{k=0}^n (g_k, \lambda) (\hat{x}_{\underline{g}_i}, \underline{x}_{\underline{g}_k})$, $0 \leq i \leq n$. By stationarity it follows that

$$(II) \quad \hat{x}_{\underline{g}_i + g} = \int_{\star} (g, \lambda) \underline{\Phi}_i(\lambda) \underline{E}(d\lambda) \underline{x}_0, \quad 0 \leq i \leq n, g \in G.$$

Using the same techniques as in Theorem 4.28(a), by (I) and (II) we obtain $\underline{F}^s(G^*) = \underline{0}$ which implies that $\underline{F}$ is absolutely continuous.

(b) Let $\underline{z} \in \mathcal{M}_x$. If we can show that $\underline{z} \perp \hat{x}_{\underline{g}_i + g}$ for all $i \in \{0, 1, \dots, n\}$ and all $g \in G$ implies that $\underline{z} = \underline{0}$, then we will have shown that $\mathcal{G}(\hat{x}_{\underline{g}_i + g}, 0 \leq i \leq n, g \in G) = \mathcal{M}_x$. But

$\mathcal{M}_{\mathcal{J}_n, x}^{\perp} = \mathcal{G}(\hat{x}_{\underline{g}_i + g}, 0 \leq i \leq n, g \in G)$ and hence $\mathcal{M}_{\mathcal{J}_n, x} = \{\underline{0}\}$ which shows that $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_n$ -regular.

By Lemma 5.28 and the fact that $\text{rank } \underline{P} = q$ on a set of positive Haar measure, we get $\text{rank } \underline{P} = q$ a.e. m . Since $\underline{P} \underline{Q} = \underline{P}$ a.e. m , it follows that $\text{rank } \underline{F}' = q$ a.e. m and, hence,

$\text{rank } \underline{P} \underline{F}'^{-1} \underline{P}^* = q \text{ a.e. m.}$ Then, by Lemma 5.7,

$$(I) \quad \int_{\star}^{\underline{P} \underline{F}'^{-1} \underline{P}^*} dm \neq \underline{0}.$$

Let $\underline{y}$ be the element in $\mathcal{M}_{j,x}$ corresponding to $\underline{P}$ (cf. Lemma 5.23). By (I), $\underline{y}$ is non-zero. Then, similar to the proof of Lemma 5.6, we can show that

$$(II) \quad \underline{U}_g(\underline{y}) = \int_{\star}^{\underline{P} \underline{F}'^{-1} \underline{P}^*} (g, \lambda) \underline{\Phi}(\lambda) \underline{E}(d\lambda) \underline{x}_0$$

where

$$\underline{\Phi} = \begin{cases} \underline{P} \underline{F}'^{-1}, & \text{on the carrier of } \underline{F}^a \\ \underline{0} & , \text{ on the carrier of } \underline{F}^s. \end{cases}$$

Let $\underline{z} = \int_{\star}^{\underline{\Psi}(\lambda) \underline{E}(d\lambda) \underline{x}_0} \in \mathcal{M}_x$ such that $(\underline{z}, \hat{\underline{x}}_{g_i + g}) = \underline{0}$, $0 \leq i \leq n$, $g \in G$. It then follows that $(\underline{z}, \underline{U}_g \underline{y}) = \underline{0}$ for all $g \in G$. But

$$(III) \quad \underline{0} = (\underline{z}, \underline{U}_g \underline{y}) = \int_{\star}^{\underline{\Psi} d\underline{F}(-g, \lambda) \underline{\Phi}^*} \text{ for all } g \in G.$$

By (II) and (III), we get

$$(IV) \quad \underline{0} = \int_{\star}^{\underline{\Psi}(\lambda) \underline{P}^*(\lambda) (-g, \lambda) m(d\lambda)} \text{ for all } g \in G.$$

Hence, $\underline{\Psi} \underline{P}^* = \underline{0}$ a.e. m. Since $\underline{P}$ has full rank a.e. m, we conclude that $\underline{\Psi} = \underline{0}$ a.e. m. Now, since $\underline{F}$ is absolutely continuous, it follows that

$$(V) \quad (\underline{z}, \underline{z}) = \int_{\star}^{\underline{\Psi} d\underline{F} \underline{\Psi}^*} = \int_{\star}^{\underline{\Psi} \underline{F}' \underline{\Psi}^*} dm = \underline{0}.$$

Hence, $\underline{z} = \underline{0}$.

Q.E.D.

¹⁾ $(\underline{U}_g)_{g \in G}$ is the shift group of $(\underline{x}_g)_{g \in G}$.

Results similar to Corollary 4.30 and Remark 4.31 hold for the multivariate case, but will not be stated.

The following shows a relation between the concept of $\mathcal{J}_n$ -singularity and the notion of interpolability.

5.30 Remark. Let $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over G , a discrete LCAG. Let $J = \{g_0, \dots, g_n\}$ be a fixed subset of G . Then $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_n$ -singular if, and only if, for all $g \in G$, $\mathcal{Q}_{J+g} = \{\underline{0}\}$.

Now, we will give a characterization for an $\mathcal{K}^q$ -valued WSRF over a discrete LCAG which is neither $\mathcal{J}_n$ -singular nor $\mathcal{J}_n$ -regular in terms of its spectral distribution.

5.31 Theorem. Let $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over G , a discrete LCAG and $\underline{F}$ be its spectral distribution. Let $J = \{g_0, g_1, \dots, g_n\}$ be a fixed subset of G and $\mathcal{J}_n = \{J^c + g, g \in G\}$.

(a) If $\mathcal{Q}_J \neq \{\underline{0}\}$ and $\underline{F}^s \neq 0$, then $\{\underline{0}\} \subsetneq \mathcal{M}_{\mathcal{J}_n, x} \subsetneq \mathcal{M}_x$.

(b) (i) If $\mathcal{M}_{\mathcal{J}_n, x} \neq \mathcal{M}_x$, then $\mathcal{Q}_{J+g} \neq \{\underline{0}\}$ for all $g \in G$.

(ii) Let G be ordered. If $\{\underline{0}\} \neq \mathcal{M}_{\mathcal{J}_n, x}$ and there exists a polynomial $\underline{P}$ in $\mathcal{Q}_J$ such that $\text{rank } \underline{P} = q$ on a set of positive Haar measure then $\underline{F}^s \neq \underline{0}$.

Proof (a). The proof is analogous to the proof of 4.33(a).

(b) (i) If $\mathcal{M}_{\mathcal{J}_n, x} \neq \mathcal{M}_x$, then $(\underline{x}_g)_{g \in G}$ is not $\mathcal{J}_n$ -singular and, hence, by Remark 5.30 and stationarity, $\mathcal{Q}_{J+g} \neq \{\underline{0}\}$ for all $g \in G$.

(ii) If $\mathcal{M}_{\mathcal{J}_n, x} \neq \{\underline{0}\}$, then $(\underline{x}_g)_{g \in G}$ is not $\mathcal{J}_n$ -regular. Hence, by 5.29(b), either $\underline{F}$ is not absolutely continuous or else for all $\underline{P} \in \mathcal{Q}_J$, $\text{rank } \underline{P} < q$ a.e. m. But the latter is not true by assumption. Hence, $\underline{F}^s \neq \underline{0}$. Q.E.D.

Our next goal will be to establish the Wold-Cramer concordance relation with respect to $\mathcal{J}_n$ for a q -variate WSRF over a discrete LCAG.

5.32 Theorem (Wold-Cramer concordance for $\mathcal{J}_n$). Let

- (i) $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over G , a discrete LCAG, which is ordered; $J = \{g_0, g_1, \dots, g_n\}$ and $\mathcal{J}_n = \{J^c + g, g \in G\}$.
- (ii) $(\underline{w}_g)_{g \in G}$ and $(\underline{y}_g)_{g \in G}$ be the components of $(\underline{x}_g)_{g \in G}$ as occurred in the Wold decomposition theorem with respect to $\mathcal{J}_n$;
- (iii) $\underline{F}$, $\underline{F}_y$, and $\underline{F}_w$ be the spectral distributions of $(\underline{x}_g)_{g \in G}$, $(\underline{y}_g)_{g \in G}$ and $(\underline{w}_g)_{g \in G}$ respectively;
- (iv) $\underline{F}^a$, $\underline{F}^s$ be the absolutely continuous and singular components of $\underline{F}$ with respect to the Haar measure, as in the Cramer decomposition theorem;
- (v) $\underline{P}$ be a polynomial in $\mathcal{Q}_J$ such that $\text{rank } \underline{P} = q$ on a set of positive Haar measure.

Then

$$\underline{F}_y = \underline{F}^a ; \underline{F}_w = \underline{F}^s .$$

Proof. By assumption (v) and the fact that G is ordered, as in the proof of Theorem 5.29, $\text{rank } \underline{F}' = q$ a.e. m , $\int_{\mathcal{G}_0^*} \underline{P} \underline{F}'^{-1} \underline{P}^* dm \neq \underline{0}$, and if $\underline{z}$ is the element in $\mathcal{M}_{J,x}$ corresponding to $\underline{P}$,

$$(I) \quad \underline{z} = \int_{\mathcal{G}^*} \underline{\Phi}(\lambda) \underline{E}(d\lambda) \underline{x}_0$$

where

$$\underline{\Phi} = \begin{cases} \underline{P} \underline{F}'^{-1} , & \text{on the carrier of } \underline{F}^a \\ \underline{0} & , \text{on the carrier of } \underline{F}^s . \end{cases}$$

Also, since $\hat{\underline{x}}_{g_i} \in \mathcal{M}_{\mathcal{J}_n, x}^\perp$ for $0 \leq i \leq n$, it follows that $\underline{z} \in \mathcal{M}_{\mathcal{J}_n, x}^\perp$. But in the proof of the Wold decomposition theorem, we observed that

$\mathcal{M}_y = \mathcal{M}_{\mathcal{J}_n, x}^\perp$ and, hence, $\underline{z} \in \mathcal{M}_y$.

By Theorem 5.29(a), $\underline{F}_y$ is absolutely continuous. Thus, using the same type of proof as used in Theorem 4.13, we obtain

$$(II) \quad \underline{F}' = \underline{F}'_y + \underline{F}'_w \quad \text{a.e. } m$$

$$d\underline{F}^s = d\underline{F}_w^s$$

where $\underline{F}'_y$ and $\underline{F}'_w$ are the spectral densities of $\underline{F}_y$ and $\underline{F}_w$ respectively. Since $\underline{z} \in \mathcal{M}_y$, again imitating the proof of Theorem 4.13, we get

$$(III) \quad \underline{z} = \int_{G^*} \underline{\Phi}(\lambda) \underline{E}(d\lambda) \underline{z}_0.$$

By (I), it follows that

$$(IV) \quad (\underline{z}, \underline{z}) = \int_{G^*} \underline{\Phi}(\lambda) \underline{E}(d\lambda) \underline{\Phi}^*(\lambda) = \int_{G^*} \underline{P}(\lambda) \underline{F}'^{-1}(\lambda) \underline{P}^*(\lambda) m(d\lambda).$$

By (III), we get

$$(V) \quad (\underline{z}, \underline{z}) = \int_{G^*} \underline{\Phi}(\lambda) \underline{F}_y(d\lambda) \underline{\Phi}^*(\lambda) = \int_{G^*} \underline{P}(\lambda) \underline{F}'^{-1}(\lambda) \underline{F}'_y(\lambda) \underline{F}'^{-1}(\lambda) \underline{P}^*(\lambda) m(d\lambda).$$

Combining (IV) and (V) and rearranging, we have

$$(VI) \quad \int_{G^*} (\underline{P} \underline{F}'^{-1} \underline{P}^* - \underline{P} \underline{F}'^{-1} \underline{F}'_y \underline{F}'^{-1} \underline{P}^*) dm = 0.$$

By (II), $\underline{F}' \geq \underline{F}'_y$ a.e. m and, hence, $\underline{P} \underline{F}'^{-1} \underline{P}^* - \underline{P} \underline{F}'^{-1} \underline{F}'_y \underline{F}'^{-1} \underline{P}^* \geq 0$ a.e. m .

This fact and (VI) imply, by Lemma 5.7, that $\underline{P} \underline{F}'^{-1} \underline{P}^* = \underline{P} \underline{F}'^{-1} \underline{F}'_y \underline{F}'^{-1} \underline{P}^*$ a.e. m . Since $\text{rank } \underline{P} = q$ a.e. m and $\text{rank } \underline{F}' = q$ a.e. m , we get $\underline{F}'_y = \underline{F}'$ a.e. m . Q.E.D.

5.33 Remark. If $\mathcal{Q}_J = \{0\}$, then, by Remark 5.30, $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_n$ -singular, and, hence, $\underline{F} = \underline{F}_w$. In this case, $\underline{F}_w$ could be

absolutely continuous with respect to m .

We will now specialize our results on interpolation of q -variate WSRF's to processes indexed by the integers. Under suitable analytic conditions, we will make appropriate comparisons between the subspaces $\mathbb{M}_{\mathcal{J}_p, x}$ and $\mathbb{M}_{\mathcal{J}_n, x}$ in the time domain. First, though, we will need the following notation.

5.34 Notation. Let Z be the integers. Let $(\underline{x}_n)_{-\infty}^{\infty}$ be an $\mathbb{K}^q$ -valued WSRF over Z . Let $J = \{k_0, k_1, \dots, k_n; k_0 < k_1 < \dots < k_n\}$ be a fixed set of $n+1$ integers and $I_k = \{j \in Z : j \leq k\}$. Let

$$(i) \quad \mathbb{M}_{J^c+k, x} = \mathcal{G}(\underline{x}_j, j \neq k_i + k, 0 \leq i \leq n);$$

$$(ii) \quad \mathbb{M}_{I_k, x} = \mathcal{G}(\underline{x}_j, j \leq k).$$

Obviously, $\mathbb{M}_{\mathcal{J}_p, x} = \bigcap_{-\infty}^{\infty} \mathbb{M}_{I_k, x} \subseteq \bigcap_{-\infty}^{\infty} \mathbb{M}_{J^c+k, x} = \mathbb{M}_{\mathcal{J}_n, x}$. In the following theorems, we will examine conditions under which $\mathbb{M}_{\mathcal{J}_p, x} = \mathbb{M}_{\mathcal{J}_n, x}$ or $\mathbb{M}_{\mathcal{J}_p, x} \subsetneq \mathbb{M}_{\mathcal{J}_n, x}$.

5.35 Theorem. Let $(\underline{x}_n)_{-\infty}^{\infty}$ be a q -variate WSRF over Z . Suppose there exists a polynomial $\underline{P}$ in $\mathcal{Q}_J$ such that $\text{rank } \underline{P} = q$ on a set of positive Haar measure. Then $\mathbb{M}_{\mathcal{J}_p, x} = \mathbb{M}_{\mathcal{J}_n, x}$.

Proof. By Theorem 5.32, we see that $\underline{F}_y = \underline{F}^a$; $\underline{F}_w = \underline{F}^s$.

Let $\underline{F}_u$ and $\underline{F}_v$ be the spectral distributions of the components $(\underline{u}_n)_{-\infty}^{\infty}$ and $(\underline{v}_n)_{-\infty}^{\infty}$ given by the usual Wold decomposition theorem with respect to the family $\{I_k\}_{-\infty}^{\infty}$. It can be shown without much difficulty that $\underline{F}_y \leq \underline{F}_u$ and, hence, $\text{rank } \underline{F}_y' \leq \text{rank } \underline{F}_u'$ a.e. m . Since $\text{rank } \underline{P} = q$ a.e. m and $\underline{F}_y = \underline{F}^a$, it follows that $\underline{F}_u'$ has full rank a.e. m . Therefore, by Lemma 5.1, [19], we have $\rho_{I_0, 1} = q$. Hence, by Theorem 3.17, $\underline{F}_u = \underline{F}^a$; $\underline{F}_v = \underline{F}^s$. Now, using a similar

proof to the one used in Theorem 4.16, we get $\mathbb{M}_{\mathcal{J}_p, x} = \mathbb{M}_{\mathcal{J}_n, x}$.

Q.E.D.

5.36 Theorem. Let $(x_n)_{-\infty}^{\infty}$ be as above. Let $\log \det \underline{F}' \in \underline{L}_1(G^*, \mathcal{B}^*, m)$ and $\mathcal{Q}_J = \{0\}$. Then $\mathbb{M}_{\mathcal{J}_p, x} \subsetneq \mathbb{M}_{\mathcal{J}_n, x} = \mathbb{M}_x$.

Proof. If $\mathcal{Q}_J = \{0\}$, then, by Remark 5.30, $(x_n)_{-\infty}^{\infty}$ is $\mathcal{J}_n$ -singular and, hence, $\mathbb{M}_{\mathcal{J}_n, x} = \mathbb{M}_x$. It is well known ([32], p. 145) that if $\log \det \underline{F}' \in \underline{L}_1(G^*, \mathcal{B}^*, m)$, then $(x_n)_{-\infty}^{\infty}$ is non-deterministic and, thus, $\mathbb{M}_{\mathcal{J}_p, x} \subsetneq \mathbb{M}_x$. Q.E.D.

5.37 Remark. If we assume $\rho_{I_0, 1} = 0$, then $\mathbb{M}_{\mathcal{J}_p, x} = \mathbb{M}_x$ and, hence, $\mathbb{M}_{\mathcal{J}_n, x} = \mathbb{M}_x$.

This concludes our discussion on the problems of interpolation with respect to $\mathcal{J}_n = \{J^c + g, g \in G\}$, where J is a fixed set of $n+1$ elements of G , a discrete LCAG. We will devote the rest of §5 to interpolation theory with respect to $\mathcal{J}_{\infty}$, the family of complements of finite sets of G . First, though, we will recall some notation.

5.38 Notation. Let $(x_g)_{g \in G}$ be an $\mathcal{K}^q$ -valued WSRF over G , a discrete LCAG. Let $\underline{F}'$ be the spectral density of $(x_g)_{g \in G}$, $\underline{F}'^{\#}$ the generalized inverse of $\underline{F}'$, and $\underline{Q} = \underline{F}'^{\#} \underline{F}'$. Let J be any finite subset of G . Then, similar to the notation in 5.19, we set

$$(i) \quad \mathbb{M}_{J, x} = \mathbb{M}_{J^c, x}^{\perp} \cap \mathbb{M}_x;$$

$$(ii) \quad \mathcal{Q}_J = \{ \underline{P}: \underline{P}(\lambda) = \sum_{g \in J} (g, \lambda) \underline{A}_g, \underline{A}_g \text{'s are arbitrary } q \times q \text{ complex-valued matrices; } \underline{P} \underline{Q} = \underline{P} \text{ a.e. } m \text{ and } \underline{P} \underline{F}'^{\#} \underline{P}^* \in \underline{L}_1(G^*, \mathcal{B}^*, m) \}.$$

We are now able to give the following definition of interpolability.

5.39 Definition. Let $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over G , a discrete LCAG. We say that $(\underline{x}_g)_{g \in G}$ is interpolable if $(\underline{x}_g)_{g \in G}$ is interpolable with respect to every finite set of elements of G .

The following remark follows immediately from Corollary 5.26.

5.40 Remark. Let $(\underline{x}_g)_{g \in G}$ be as in 5.39. Then $(\underline{x}_g)_{g \in G}$ is interpolable if, and only if, $\mathcal{Q}_J = \{0\}$ for any finite subset J of G .

The following establishes a relationship between the concept of non-interpolability and that of $\mathcal{J}_\infty$ -regularity.

5.41 Theorem. Let $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over G , a discrete LCAG, and $\underline{F}$ be the spectral distribution of $(\underline{x}_g)_{g \in G}$.

(a) If $(\underline{x}_g)_{g \in G}$ is non-trivial and is $\mathcal{J}_\infty$ -regular, then there exists a finite subset J in G such that $\mathcal{Q}_J \neq \{0\}$ and $\underline{F}$ is absolutely continuous with respect to the Haar measure m ;

(b) Let G be ordered. If $\underline{F}$ is absolutely continuous and for some finite set J in G there exists a polynomial $\underline{P} \in \mathcal{Q}_J$ such that the rank $\underline{P} = q$ on a set of positive Haar measure, then $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_\infty$ -regular.

Proof (a). The proof that there exists a finite set J in G such that $\mathcal{Q}_J \neq \{0\}$ is trivial. Using similar techniques as in the proofs of Theorem 4.44(a) and Theorem 5.29(a), we can easily show that $\underline{F}$ is absolutely continuous.

(b) Trivially,

$$(I) \quad \mathcal{M}_{\mathcal{J}_\infty, x} \subseteq \mathcal{M}_{\mathcal{J}_n, x}, \quad \mathcal{J}_n = \{J^c + g, g \in G\}.$$

Therefore, our assumptions satisfy those of Theorem 5.29(b) and, thus, $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_n$ -regular. Hence, in view of (I), $(\underline{x}_g)_{g \in G}$ is

$\mathcal{J}_\infty$ -regular.

Q.E.D.

In the following remark, we will state a characterization of $\mathcal{J}_\infty$ -singularity for an $\mathcal{K}^q$ -valued WSRF over a discrete LCAG.

5.42 Remark. Let $(\underline{x}_g)_{g \in G}$ be as in 5.40. Then $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_\infty$ -singular if, and only if, $\mathcal{Q}_J = \{\underline{0}\}$ for any finite subset J of G .

Next, we will give a characterization for a q -variate WSRF over a discrete LCAG which is neither $\mathcal{J}_\infty$ -singular nor $\mathcal{J}_\infty$ -regular. Its proof is similar to that of Theorem 5.31 and thus is not given.

5.43 Theorem. Let $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over G , a discrete LCAG and $\underline{F}$ be its spectral distribution.

- (a) If $\mathcal{Q}_J \neq \{\underline{0}\}$ for some finite subset J of G and $\underline{F}^S \neq \underline{0}$, then $\{\underline{0}\} \subsetneq \mathcal{M}_{\mathcal{J}_\infty, x} \subsetneq \mathcal{M}_x$.
- (b) (i) If $\mathcal{M}_{\mathcal{J}_\infty, x} \neq \mathcal{M}_x$, then $\mathcal{Q}_J \neq \{\underline{0}\}$ for some finite subset J of G .
- (ii) Let G be ordered. If $\underline{0} \neq \mathcal{M}_{\mathcal{J}_\infty, x}$ and there exists a polynomial $\underline{p}$ in $\mathcal{Q}_J$ for some finite subset J of G such that $\text{rank } \underline{p} = q$ on a set of positive Haar measure, then $\underline{F}^S \neq \underline{0}$.

We will now state the Wold-Cramer concordance theorem for the multivariate case with respect to the family $\mathcal{J}_\infty$. Its proof follows from the proof of Theorem 5.32 in the same way that the proof of Theorem 4.49 follows from the proof of Theorem 4.35, and, hence, is omitted.

5.44 Theorem (Wold-Cramer concordance for $\mathcal{J}_\infty$). Let

- (i) $(\underline{x}_g)_{g \in G}$ be a q -variate WSRF over a discrete LCAG G , which is ordered; $\mathcal{J}_\infty$ = family of complements of finite sets of G ;

(ii) $(\underline{w}_g)_{g \in G}$ and $(\underline{y}_g)_{g \in G}$ be the components of $(\underline{x}_g)_{g \in G}$ as occurred in the Wold decomposition theorem with respect to $\mathcal{J}_\infty$;

(iii) $\underline{F}$, $\underline{F}_y$, and $\underline{F}_w$ the spectral distributions of $(\underline{x}_g)_{g \in G}$, $(\underline{y}_g)_{g \in G}$, and $(\underline{w}_g)_{g \in G}$ respectively;

(iv) $\underline{F}^a$, $\underline{F}^s$ the absolutely continuous and singular components of $\underline{F}$ with respect to m , as in the Cramer decomposition theorem;

(v) For some finite subset J , $\mathcal{Q}_J$ contain a polynomial $\underline{P}$ such that $\text{rank } \underline{P} = q$ on a set of positive Haar measure.

Then

$$\underline{F}_y = \underline{F}^a ; \underline{F}_w = \underline{F}^s .$$

5.45 Remark. If $\mathcal{Q}_J = \{0\}$ for every finite subset J of G , then, by Remark 5.41, $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_\infty$ -singular and, hence, $\underline{F} = \underline{F}_w$. In this case, $\underline{F}_w$ may be absolutely continuous with respect to m .

In specializing our results for $\mathcal{J}_\infty$ to the case when $G = \mathbb{Z}$, the integers, we will simply state the results comparing $\mathbb{M}_{\mathcal{J}_p, x}$ and $\mathbb{M}_{\mathcal{J}_\infty, x}$, since their proofs follow closely the corresponding proofs for $\mathcal{J}_n$.

5.46 Theorem. Let $(\underline{x}_n)_{n=-\infty}^\infty$ be a q -variate WSRF over $\mathbb{Z}$. If there exists a polynomial $\underline{P} \in \mathcal{Q}_J$ for some finite set J of integers such that $\text{rank } \underline{P} = q$ a.e. m , then $\mathbb{M}_{\mathcal{J}_\infty, x} = \mathbb{M}_{\mathcal{J}_p, x}$.

5.47 Theorem. Let $(\underline{x}_n)_{n=-\infty}^\infty$ be as in 5.45. Let $\log \det \underline{F}' \in \underline{L}_1(G^*, \mathcal{G}^*, m)$. Suppose $\mathcal{Q}_J = \{0\}$ for every finite set J of integers. Then $\mathbb{M}_{\mathcal{J}_p, x} \subsetneq \mathbb{M}_{\mathcal{J}_\infty, x} = \mathbb{M}_x$.

5.48 Remark. If $\rho_{I_0,1} = 0$, then $\mathcal{M}_{\rho,x} = \mathcal{M}_{\rho_\infty,x} = \mathcal{M}_x$.

6. SOME EXAMPLES AND FURTHER REMARKS ON FINITE AND INFINITE DIMENSIONAL STATIONARY RANDOM FIELDS

As we pointed out earlier, this section will be devoted to the construction of some examples and to the discussion of some open problems on q -variate WSRF's over LCAG's. We will also remark briefly on the problems of minimality and interpolation of infinite dimensional WSRF's over LCAG's.

Our discussions in the preceding sections have been mainly on processes over discrete LCAG's. Concrete examples of such groups are as follows:

- 6.1 Examples. (i) $G = \mathbb{Z}$, the set of all integers;
(ii) $G = \mathbb{R}$, the set of all real numbers;
(iii) $G = \mathbb{R}^n$, n -dimensional Euclidean space;
(iv) $G = \mathbb{Z}^n$, the set of all lattice points in n -dimensional Euclidean spaces.

The following discrete LCAG should be of interest in the study of WSRF's. As far as we know, in connection with stochastic processes, this group has not been considered.

6.2 Example. Let T denote the unit circle. Let T_∞ stand for the infinite (countable) Cartesian product of T with itself. Since T is compact, T_∞ is also compact under the usual product topology. Let $\mathbb{Z}_\infty^f$ denote the set of all infinite (countable) sequences of integers only finitely many of which are different from

zero. It is clear that Z_{∞}^f is a discrete LCAG.

By Theorem 2.2.3, [26], it follows that the dual of Z_{∞}^f is T_{∞} . Since Z_{∞}^f is a discrete LCAG and T_{∞} is compact, the usual Bochner theorem, 3.4, holds. Hence, our results in §4 and §5 on minimality and interpolation of WSRF's indexed by elements of Z_{∞}^f are valid.

The following is a counterexample to L. Bruckner's claim, Theorem 4.1 of [1], that a process must be either $\mathcal{J}_0$ -singular or $\mathcal{J}_0$ -regular. Because of Theorem 4.11, it suffices to find a process whose spectral distribution F has the properties that $1/f \in L_1(G^*, \mathcal{B}^*, m)$ and $F^S \neq 0$. The example is as follows:

6.3 Example. Let $G = \mathbb{Z}$, the integers. Then $G^* = [0, 2\pi]$.

Define dF in the following manner:

- (i) $f = 1$ on $[0, 2\pi]$;
- (ii) μ be the singleton measure with mass 1 at π ;
- (iii) $dF = f dx + d\mu$.

Clearly, $F^S \neq 0$ and $\int_{G^*} 1/f(x) dx = 2\pi < \infty$.

Any WSRF over the integers with spectral distribution F will constitute a counterexample to L. Bruckner's claim.

Our next example will be to construct a process over the integers whose spectral density f is such that $\log f \in L_1(G^*, \mathcal{B}^*, m)$, but, for any polynomial P on G^* , $|P|^2/f \notin L_1(G^*, \mathcal{B}^*, m)$. This example will show, among other things, that the assumptions in Theorems 4.17, 4.39, and 4.52 are not vacuous; i.e., there do exist processes such that $\mathcal{M}_{\mathcal{J}_p, x} \subsetneq \mathcal{M}_{\mathcal{J}_{\infty}, x} = \mathcal{M}_{\mathcal{J}_n, x} = \mathcal{M}_{\mathcal{J}_0, x} = \mathcal{M}_x$.

6.4 Example. Let $G = \mathbb{Z}$, the integers. Let $(x_n)_{n=-\infty}^{\infty}$ be any WSRF over $\mathbb{Z}$, whose spectral distribution is absolutely continuous and whose spectral density is given by $f(\lambda) = e^{1/\sqrt{\lambda}}$. Then $\log f = -1/\sqrt{\lambda} \in L_1(G^*, \mathcal{B}^*, m)$. Since $e^{1/\sqrt{\lambda}} = \sum_{n=0}^{\infty} \frac{1}{n! \lambda^{n/2}}$, by simple manipulations one can show that $|P|^2/f \notin L_1(G^*, \mathcal{B}^*, m)$ for any non-zero polynomial P .

Our last example will show that if G is not an ordered group, then it is possible to construct a non-zero polynomial P on G^* such that $P = 0$ on a set of positive Haar measure, as Remark 4.31 claims.

6.5 Example. Let $G = \{0,1\}$ and its binary operation "+" be defined in the following way:

$$0 + 0 = 0; 0 + 1 = 1 + 0 = 1; 1 + 1 = 0.$$

It is easy to see that G cannot be ordered compatible with its structure and that G^* contains only two elements, λ_1 and λ_2 , defined in the following manner:

$$\lambda_1(0) = 1; \lambda_1(1) = 1; \lambda_2(0) = 1; \lambda_2(1) = -1.$$

Define P by $P(\lambda) = (0, \lambda) + (1, \lambda)$. Then $P(\lambda_1) = \lambda_1(0) + \lambda_1(1) = 2$; $P(\lambda_2) = \lambda_2(0) + \lambda_2(1) = 0$. Note that $P = 0$ on the set $\{\lambda_2\}$ where $m(\{\lambda_2\}) = 1/2$ and $P \neq 0$ on the set $\{\lambda_1\}$ where $m(\{\lambda_1\}) = 1/2$.

Next, we will mention some open problems that arose from our study of q -variate WSRF's over LCAG's.

6.6 Open problems.

(I) Let $(x_n)_{n=-\infty}^{\infty}$ be a q -variate WSRF over Z , the integers. Wiener and Masani, [32], showed that $\log \det \underline{F}' \in \underline{L}_1([0, 2\pi], \mathcal{S}^*, \text{Leb.})$ if, and only if, the rank of the process with respect to the past is full. Later, Wiener and Masani, [34], extended this result to cover bivariate processes, not necessarily of full rank. The most interesting result, in connection with this area, is due to Matveev, [16]. He gave a necessary and sufficient condition in terms of the spectral density of the process for the process to have any rank between zero and q .

In Theorem 5.8, a characterization for full rank of a WSRF over a discrete LCAG with respect to the "past & future" was given in terms of the spectral density. It would be very interesting to extend this result, in the same spirit that Matveev extended Wiener and Masani's result, and obtain a characterization in terms of the spectral density for the rank of a WSRF over a LCAG with respect to the "past & future" to assume any value between zero and q .

(II) For the integers, Robertson (Theorem 3.20) gave a complete characterization for concordance between the Wold decomposition with respect to the past and the Cramer decomposition in terms of the rank of the spectral density. For a WSRF over any discrete LCAG, we believe a characterization for concordance between the Wold decomposition with respect to the "past & future" and the Cramer decomposition in terms of the spectral density is possible. In fact, Theorem 5.14 tells us that our conjecture is true when we assume that the rank of the spectral density is full a.e. m and its inverse is in $\underline{L}_1(G^*, \mathcal{S}^*, m)$. Similarly, in general, the concordance between

the Wold decomposition with respect to $\mathcal{J}_n$ and $\mathcal{J}_\infty$ and the Cramer decomposition remain open.

(III) In the univariate case, for a non-trivial WSRF over a discrete LCAG, we saw, by Corollary 4.9, that $\mathcal{J}_0$ -regularity implies that $1/f \in L_1(G^*, \mathcal{B}^*, m)$. In the multivariate case, for a non-trivial WSRF over a discrete LCAG, Theorem 5.9(a) implies that $\underline{F}$ is absolutely continuous. If the rank (with respect to the "past & future") of the WSRF is full, then, by Theorem 5.8, $\underline{F}'^{-1}$ exists a.e. m and is in $L_1(G^*, \mathcal{B}^*, m)$. In general, when the rank is not full, it seems reasonable to assume that perhaps a similar implication holds; i.e., if $(\underline{x}_g)_{g \in G}$ is $\mathcal{J}_0$ -regular, then maybe $\underline{F}'^\# \in L_1(G^*, \mathcal{B}^*, m)$.

(IV) In the statements of some of our results in both the univariate and multivariate cases involving $\mathcal{J}_n$ - and $\mathcal{J}_\infty$ -regularity; e.g., Theorem 4.28(b), Theorem 4.35, Theorem 5.29(b), Theorem 5.32, and Theorem 5.44, we assumed that the group G was endowed with an order relation compatible with its structure. We feel that one should be able to dispense with this assumption to carry out the work.

(V) In several of our theorems in the multivariate case, such as 5.9(b), 5.17 and 5.35, we have assumed that certain matrices have full rank. It may be possible to obtain these results under weaker assumptions.

We now direct our attention to a short discussion on infinite dimensional stationary random fields.

6.7 Remark. Based on the isomorphism, Theorem 3.7, between the time and spectral domain of $(\underline{x}_g)_{g \in G}$, a q -variate (q finite) WSRF over a LCAG G , we were able to obtain analytic characterizations

of the notions of minimality and interpolation for $(x_g)_{g \in G}$. This work enabled us to establish various interesting results concerning the time and spectral domain of $(x_g)_{g \in G}$, as presented in sections 4 and 5.

Recently, V. Mandrekar and H. Salehi [9] have studied the structure of the space of square-integrable operator-valued functions with respect to a non-negative operator-valued measure. They established [11] an isomorphism theorem between the time and spectral domains of a WSRF over a LCAG. Based on this, they settled some questions on subordination of an infinite-dimensional WSRF with respect to another infinite-dimensional WSRF [11]. They also used this isomorphism in connection with infinite-dimensional linear differential systems driven by white noise [10].

The same way that $L_2(G^*, \mathcal{G}^*, F)$ was used to study various results on minimality and interpolation of a q -variate WSRF over a discrete LCAG, one can use the space of square-integrable operator-valued functions with respect to a non-negative operator-valued measure (cf. [9]) to study the problems of minimality and interpolation for infinite-dimensional WSRF's over a discrete LCAG. These questions are under study by us and the results will be announced elsewhere.

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