OPTIMAL SINGULAR CONTROL THEORY WITH APPLICATION TO VEHICULAR BRAKING

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This is to certify that the

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ABSTRACT

OPTIMAL SINGULAR CONTROL THEORY WITH APPLICATION TO VEHICULAR BRAKING

by Michael Bishop Scherba

General results from both the Maximum Principle and Green's Theorem are specialized to a class of singular control problems encountered in vehicular braking processes. These problems are in the class of nonlinear problems in which the control appears linearly. These are of the form

$$\dot{x}(t) = f(x,t) + B(x,t)u$$

where the n dimensional vector x(t) is the state of the system at time t and the r dimensional vector u is the control vector. The object is to find a control vector which takes the system from some initial state x_0 at time t_0 to state x_T at time T and minimize the functional

$$J[u] = \int_{t0}^{T} f_0(t,x,u)dt$$

In the vehicular braking problems, the functional J[u] corresponds to stopping distance. This problem is shown to be equivalent to the time optimal problem for the class of functions encountered.

Necessary conditions along singular arcs are established using both the Maximum Principle and Green's Theorem. Algorithms for determining optimal trajectories along both singular and nonsingular arcs are developed using the concept of reachable and controllable sets.

The optimal control as a function of the state variables - the closed loop problem - is solved. Application to vehicular braking processes is

shown by means of both rate and amplitude limited controls. The resulting systems are singular "pang-pang" and singular "bang-bang" systems.

The inability of a single mathematical performance index to encompass all the qualities desired in the vehicular braking system resulted in the development of suboptimal control systems. Favorable comparison with optimal control systems is shown by means of digital and analog techniques. Simulation including real hardware shows application of the theory. Several frequency domain criteria are included to provide insight regarding the effect of time delays in suboptimal vehicular braking systems.

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By

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CHAPTER I

INTRODUCTION

Writers have called attention to the fact that a gap exists between contemporary control theory and control practice [G7]. The gap can be attributed to the fact that theoreticians and designers do not study and solve the same problems in the same order and manner. In the present study - Optimal Control of a Vehicle During Braking - there appear many difficulties. Those of an essentially mathematical nature are of interest to the theoretician. These difficulties are not usually the same as those which concern the designer. The theoretician often finds interesting and worthy of study a simplified version of the designer's problem. On the other hand, the designer often will change the design to bypass a difficulty which he has insufficient time to analyze. This thesis is an attempt to decrease the communication gap due to the divergent interests of theoreticians and designers in the area of optimal vehicular braking.

Nonlinear systems in which the control appears linearly may be singular control problems [H3]. Chapter II presents a unification of singular control theory results found in the literature. The Maximum Principle approach to singular control problems due to Johnson and Gibson [J1], [J2] is presented and extended.

The Green's Theorem approach first presented by Miele [M2] and generalized by Haynes [H1] is quite useful in low order systems. In the first approximation model of the vehicular control system, this method is applied. Studies by Snow [S3] concerning reachable and attainable sets supplement the development of the Green's Theorem approach. Geometric rules for determining optimal trajectories which contain both singular and non-singular arcs are developed and presented.

To establish necessary conditions for minimality of singular arcs, the second variation approach developed by Robbins [R1] and also by Kopp, Kelley, and Moyer [K2], [K3] is presented.

In Chapter III the general theory of singular control as presented in the previous sections is applied to low order time optimal control systems. This specialization is directed toward the vehicular braking control problem.

The first section employs the necessary conditions obtained from extensions of the Maximum Principle to derive the optimal control law. The mathematical model of the friction-slip characteristic used in the vehicular control system is of an exponential nature. This permits further specialization and simplification of the control law. The result of this approach is a closed loop system, in which the control is a function of state variables, operating as a second order system in the singular mode and as a fourth order system in the non-singular mode. The singular "bang-bang" cases concerned are designated Problem 3.1 and Problem 3.2.

Using the Green's Theorem approach, no additional information is obtained for these problems. The necessary conditions for singular arcs to exist are compared and tabulated. It is shown that the necessary conditions of the Maximum Principle approach imply the necessary condition of the Green Theorem approach.

The previous problems dealt with bounded control variables. Problem 3.3 is the singular "pang-pang" time optimal control problem and models a vehicular braking control system when the control pressure is rate limited. It is shown that the control law for both the singular "bang-bang" case and the singular "pang-pang" case are identical on the singular portion of the trajectory. The terminology singular "bang-bang" and singular "pang-pang" is defined in Chapter III.

The "pang-pang" problem increased the order of the system to three, since the control u became a state variable and $\dot{u}$ became the new control signal. The extension of Green's Theorem from two to three dimensions is the traditional Stoke's Theorem. The generalization to higher dimensions will be designated as the n-dimensional Green's Theorem. This generalization is considered and a procedure for applying Green's Theorem to n-dimensional problems is presented. The concluding section of Chapter III applies the second variation approach to Problem 3.1, the singular "bang-bang" time optimal control problem.

The chapters to this point have stressed the theoretical aspects related to the problem of interest. Chapter IV is concerned with the application of the previous material to the vehicular braking control problem. As such, it is of interest to both the theoreticians and the designers. A mathematical model of the one-wheel vehicular braking control model is optimized using the Green's Theorem approach. Reachable and attainable sets are obtained and a realizable control algorithm determined.

A computer program to automate the procedure is discussed and included in Appendix II. By eliminating the constraint on the control signal and using an impulse function in the control, a simple analytical solution is obtained. This provides a design tool for this particular problem and serves to check the digital computer solution.

Implementation of the optimal control based on the Green's Theorem approach shows the feasability of this method. However, the optimal control system model used shows that unavoidable time delays in the IBM 360-65 computer are responsible for the control signal oscillating or chattering about the theoretical value. By means of small signal analysis, relations between the variables are derived so that the magnitudes of the oscillations can be predicted. These are primarily of theoretical interest, since they would have negligible effect on a hardware system. They do indicate that time delays, such as encountered in various transducers, will affect the practical system.

In Chapter IV, the vehicular braking control problem was formulated as a time optimal control problem. By means of Green's Theorem, it is shown that minimizing time to stop the vehicle is equivalent to minimizing vehicle stopping distance. A criterion is also derived which shows the relationship necessary for equivalence.

The one-wheel vehicular braking control model with rate and amplitude limited control is solved by using the n-dimensional Green's Theorem development. The three dimensional trajectory is described.

The problems considered to this point, have been aimed at directly assisting in the development of a vehicular control system. Subject to well defined constraints, the optimal control problem has been solved by both the Maximum Principle and the Green's Theorem approach. The designer, however, is faced with constraints which are not well defined. Diverse facts such as cost, reliability, variability, and noise sensitivity must be considered. The complexity forces the designer - at this stage - to optimize, in some undefined sense, a system which will be called the suboptimal vehicular braking control system. This is done in Chapter V. The basic supoptimal control system developed first is almost indistinguishable from the optimal control system. Since time delays, due to transducers are significant, the final suboptimal system developed takes these into account.

Studies using both the analog and the digital computer are conducted, using the optimal control system as a reference system. It is shown that it is possible to compensate for transducer time delays which are no more than approximately 20 milliseconds. Also, if the time delays are in this range, they may be treated linearly. Representative plots and their comparison with respect to the optimal control system are included.

In order to predict the effect of various system parameters more easily, it is necessary to develop models which reduce the complexity of the system. Several models are developed in Chapter VI, which are useful under various operating conditions. The first model, which is valid for a properly compensated system having time delays, establishes a criterion relating time delay to ripple frequency present in the system response.

Another model developed is appropriate for systems having physically realizable time delays but operating without compensation. This model considers the effects of two nonlinearities, the friction-slip characteristic and the relay characteristic. The use of these models in predicting the effect of system parameters is demonstrated and comparison with computer studies is included.

In summary, this thesis presents a unification of singular control theory and specializes the general results to a class of problems encountered in vehicular braking processes.

It develops necessary conditions along singular and nonsingular trajectories, which are used to develop algorithms necessary to mechanize time optimal models based on both the Maximum Principle and Green's Theorem. The equivalence of minimum time and minimum stopping distance criteria is proved.

Parameter studies of suboptimal systems are made using both analog and digital technique. The suboptimal performance of simulators using real hardware is shown to compare favorably with the optimal control system.

Several frequency criteria are developed to permit evaluation of suboptimal control systems.

CHAPTER II SINGULAR CONTROL THEORY

2.1 Introduction

This chapter is a survey of some of the currently known mathematical techniques applicable to the study of systems described by nonlinear differential equations in which the control appears linearly, i.e.,

$$\dot{x}_i(t) = f_i(x,t) + \sum_{j=1}^{r} b_{ij}(x,t)u_j \qquad i = 1,2,\ldots, n \tag{2.1}$$

It will be shown that when the control appears linearly, a class of solutions which are called singular may appear. In matrix notation, the above equation may be written as

$$\dot{x}(t) = f(x,t) + B(x,t)u \tag{2.2}$$

In the above equation, t is the independent variable (t = time in the practical cases considered). The n dimensional vector x(t) is the state of the system at time t, and the r dimensional vector u is the control vector.

The problems of primary interest are those in which control vectors, i.e., a set of control functions $u_a(t)$, are to be found which will take the system from state x_o at time t_o to state x_T at time T.

By requiring that the control vectors optimize some performance criterion, we have an optimal control problem. This criterion is usually a functional which may depend on time, the state of the system, and the control vector. When expressed in integral form it appears as follows:

$$J[u] = \int_{t_o}^{T} f_o(t,x,u)dt \tag{2.3}$$

In much of the study that follows the scalar f_o will be 1. This is the classical time optimal control problem.

The control vector will be selected from a class of functions U depending on the problem. Bounds on the control and its derivative such as

$$|u_a| \leq 1 \qquad\qquad a = 1,2,\ldots, r$$

and

$$|\dot{u}_a| \leq 1 \qquad \text{are considered.}$$

The system equations may always be written so that the magnitudes of the control components are normalized, i.e., $|u_a| \leq 1$.

For an optimal control problem, a trajectory is said to be <u>singular</u>, if along the trajectory, the necessary conditions for optimality such as provided by the Pontryagin Maximum Principle are <u>satisfied in a trivial manner</u>. Application of the usual necessary conditions here produces no useful information. A definition due to Hermes [H5] states that a control vector is totally singular when the Maximum Principle yields no information in time optimal problem for any components of the optimal control. If a trajectory is nonsingular, it is called normal. The trajectories considered in this study have subarcs which may be normal and other subarcs which may be singular.

2.2 <u>The Maximum Principle Approach</u>

In this section application of the Maximum Principle will show that the usual necessary conditions provide no information regarding the singular control. Hence, other necessary conditions will be developed to provide additional information regarding the control.

The practical cases considered later will be time invariant with a single control variable. Consider,

Problem 2.1

$$\dot{x}_i(t) = f_i[x(t)] + b_i[x(t)]u(t) \qquad i = 1,2,\ldots, n \tag{2.4}$$

or in vector notation

$$\dot{x}(t) = f[x(t)] + b[x(t)]u(t) \tag{2.5}$$

where the state vector is the n dimensional vector x and u(t) is a sectionally continuous scalar control function.

Assume that u(t) is constrained in magnitude by the relation

$$|u(t)| \leq 1 \qquad \text{for all } t \qquad t\varepsilon[0,T] \tag{2.6}$$

The problem is to drive the initial state x(0) = a to x(t) = b while minimizing the functional

$$J[u] = \int_0^T \{f_o[x(t)] + b_o[x(t)]u(t)\}dt \tag{2.7}$$

which can be represented as

$$\dot{x}_o = f_o[x(t)] + b_o[x(t)]u(t) \qquad x\ (0) = 0 \tag{2.8}$$

The Hamiltonian is defined as

$$H(x,u,t,p) = \sum_{i=o}^{n} f_i[x(t)]p_i(t) + u(t) \sum_{i=o}^{n} b_i[x(t)]p_i(t) \tag{2.9}$$

The vector p is the costate or adjoint vector and is given by

$$\dot{p}_i(t) = -\frac{\partial H}{\partial x_i(t)} \qquad i = 1,2,\ldots,n \tag{2.10}$$

or in vector form as

$$\dot{p}(t) = -\frac{\partial H}{\partial x} \tag{2.11}$$

From equation (2.9) and (2.10), we have

$$\dot{p}_i(t) = - \sum_{j=0}^{n} p_j(t) \frac{f_j[x(t)]}{x_i(t)} - u(t) \sum_{j=0}^{n} p_j(t) \frac{\partial b_j[x(t)]}{\partial x_i(t)} \quad (2.12)$$

The following theorem is one form of the Maximum Principle which gives necessary conditions for Problem 2.1 [A1].

Theorem 2.1

If u*(t) is an optimal control and if x*(t) is the corresponding optimal trajectory, then there exists a nonzero absolutely continuous vector valued function $p^*(t)$ and a constant $p_o^* \geq 0$ such that

i) $$\dot{x}_l^*(t) = f_i[x^*(t)] + b_i[x^*(t)]u^*(t) \quad (213)$$

$$\dot{p}_i^*(t) = -\sum_{j=i}^{u} p_j^*(t) \frac{\partial f_j[x^* t)]}{\partial x_i^*(t)} - u^*(t) \sum_{j=i}^{u} p_j^*(t) \frac{\partial b_j[x^*(t)]}{\partial x_j(t)} \quad (2.14)$$

$$x^*(0) = a \qquad x^*(T) = b \qquad i=1,2,\ldots,n \quad (2.15)$$

ii) For $t \in [o,T]$ and all u(t) satisfying the constraint $|u(t)| \leq 1$, the following relation holds

$$u^*(t) \sum_{i=0}^{n} b_i[x^*(t)]p_i^*(t) \leq u(t) \sum_{i=0}^{n} b_i[x^*(t)]p_i^*(t) \quad (2.16)$$

iii) If T is free

$$H(x^*, u^*, t, p^*) = 0 \qquad t \in [0,T] \quad (2.17)$$

iv) If T is fixed

$$H(x^*, u^*, t, p^*) = C = \text{a constant} \qquad t \in [0,T] \quad (2.18)$$

As long as the scalar $\sum_{i=0}^{n} b_i[x(t)]p_i(t)$ is not zero, minimizing equation (2.9) yields the well-defined control law,

$$u^*(t) = - \operatorname{sgn} \sum_{i=0}^{n} b_i[x(t)]p_i(t) \tag{2.19}$$

when $|u| \leq 1$.

Since the signum function is not defined for argument equal to zero, the control u may be any admissible value. Admissible values are those which satisfy the constraint $|u| \leq 1$. This presents no problem as long as the scalar function $\sum\limits_{i=0} b_i[x(t)]p_i(t)$ is not zero over finite time interval. This is the classic "bang-bang" case[L1]. If, however, the scalar $\sum\limits_{t=0} b_i[x(t)]p_i(t)$ is zero over a finite interval then the problem is singular and is no longer "bang-bang". This may be formalized by the following:

<u>Definition 2.1</u> Optimal Singular Control

Problem 2.1 is singular if the optimal control $u^*(t)$, the resulting trajectory $x^*(t)$, and corresponding costate $p^*(t)$ have the following property:

There is at least one half-open interval $(t_1, t_2]$ in $[0,T]$ such that

$$\sum_{i=0} b_i[x^*(t)]p_i^*(t) = 0 \qquad \text{for all } t \in (t_1,t_2] \tag{2.20}$$

$$p_0^*(t) = p_0^* \geq 0$$

The control will be called an extremal control u(t) if it satisfies all the necessary conditions of Theorem 2.1 such that corresponding state $\hat{x}(t)$ and costate $\hat{p}(t)$ have the property that

$$\sum b_i[\hat{x}(t)]\hat{p}_i(t) = 0 \qquad \text{for all } t \in (t_1,t_2] \tag{2.21}$$

$$\hat{p}_0(t) = \hat{p}_0 = 0$$

To avoid nonessential generalization, singular extremal controls in low order systems will be considered using the Maximum Principle. First form the Hamiltonian as given by (2.9) in vector notation

$$H = (f,p) + u\,(b,p) \qquad \text{for all } t \in (t_1,t_2) \tag{2.23}$$

To simplify notation, the x(t) dependence of the variables will be omitted.

In Problem 2.1 consider the free terminal-time case and then minimize H with respect to u, to find

$$\frac{\partial H}{\partial u} = 0 = \sum_{i=0}^{n} b_i p_i = (b,p) \tag{2.24}$$

Here the control u is assumed to be in the interior of its allowed region U so that $\frac{\partial H}{\partial u}$ exists. If u is on the boundary of U, it would then fall into the "bang-bang" category. Transitions from the boundary of U to the interior of U will be considered later.

Problem 2.1 concerns the scalar control case. If u is a vector control (dimension ≥ 2) in the interior of its allowed region, then an extremal arc would be singular if the matrix H_{uu} with typical element $\frac{\partial H}{\partial u_i \, \partial u_j}$ is singular everywhere on the arc. This means that its determinant is identically zero on the arc, assuming the existence of the necessary partial derivatives. Since H = 0 along the arc, another condition obtained from the Hamiltonian is

$$\sum_{i=0}^{n} f_i p_i = 0 \qquad \text{for all } t \in (t_1,t_2) \tag{2.25}$$

Additional necessary conditions may be obtained by differentiating (2.24) and (2.25)

$$\frac{d}{dt} \sum_{i=0}^{n} b_i p_i = \sum_{i=0}^{n} \left(b_i \frac{dp_i}{dt} + \frac{dp_i}{dt} p_i \right) = 0 \tag{2.26}$$

$$\frac{d}{dt}\sum_{i=0}^{n} f_i p_i = \sum_{i=0}^{n}\left(f_i\frac{dp_i}{dt} + \frac{df_i}{dt}p_i\right) = 0 \tag{2.27}$$

Using equation (2.10) and substituting equation (2.25), and (2.26)

$$\sum_{i=0}^{n}\left(-b_i\frac{\partial H}{\partial x_i} + \frac{dp_i}{dt}p_i\right) = 0 \tag{2.28}$$

$$\sum_{i=0}^{n}\left(-f_i\frac{\partial H}{\partial x_i} + \frac{df_i}{dt}p_i\right) = 0 \tag{2.29}$$

Using equation (2.22)

$$\frac{\partial H}{\partial x_i} = \sum_{j=0}^{n} p_j\frac{\partial f_j}{\partial x_i} + u\sum_{j=0}^{n} p_j\frac{\partial b_j}{\partial x_i} \qquad i=0,1,2,\ldots,n \tag{2.30}$$

Substituting the above in (2.28) and (2.29) and making use of (2.4)

$$0 = \sum_{i=0}^{n}\left[-b_i\sum_{j=0}^{n}\left(p_j\frac{\partial f_j}{\partial xi}\right) + up_j\frac{\partial b_j}{\partial xi} + p_i\sum_{j=0}^{n}\frac{\partial b_i}{\partial x_j}(f_j + b_ju)\right] \tag{2.31}$$

$$0 = \sum_{i=0}^{n}\left[-f_i\sum_{j=0}^{n} p_j\frac{\partial f_j}{\partial x_i} + up_j\frac{\partial b_j}{\partial x_i} + p_i\sum_{j=0}^{n}\frac{\partial f_i}{\partial x_j}(f_j + b_ju)\right] \tag{2.32}$$

By interchanging the summation indices in and j in one of the u terms of (2.31), the coefficient of the u term is readily seen to be zero. Hence,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\left(f_jp_i\frac{\partial b_i}{\partial x_j} - b_ip_j\frac{\partial f_j}{\partial x_i}\right) = 0 \tag{2.33}$$

or interchanging indices in the second term, we obtain the necessary condition,

$$\sum_{i=0}^{n}\sum_{j=0}^{n} p_i\left(f_j\frac{\partial b_i}{\partial x_j} - b_j\frac{\partial f_i}{\partial x_j}\right) = 0 \tag{2.34}$$

The interchange of indices in equations (2.31) and (2.33) is permissible since, the indices of summation are dummy indices.

In a similar manner, we obtain from equation (2.32)

$$u \sum_{i=0}^{n} \sum_{j=0}^{n} p_i \left(b_i \frac{\partial f_i}{\partial x_j} - f_j \frac{\partial b_i}{\partial x_j}\right) = 0 \tag{2.35}$$

Equation (2.35) implies equation (2.34) or $u = 0$

Hence, at this stage, using the Maximum Principle Approach, we have the necessary condition for an extremal singular control.

$$\text{NC1} \qquad \sum_{i=0}^{n} b_i p_i = 0 \tag{2.36}$$

$$\text{NC2} \qquad \sum_{i=0}^{n} f_i p_i = 0 \tag{2.37}$$

$$\text{NC3} \qquad \sum_{i=0}^{n} \sum_{j=0}^{n} p_i \left(b_j \frac{\partial f_i}{\partial x_j} - f_j \frac{\partial b_i}{\partial x_j}\right) = 0 \tag{2.38}$$

The differentiation above can be continued resulting in additional necessary conditions. In matrix form these necessary conditions may be represented as

$$\begin{bmatrix} b_0 & b_1 & b_2 \ldots & b_n \\ f_0 & f_1 & f_2 \ldots & f_n \\ a_{10} & a_{11} & a_{12} \ldots & a_{1n} \\ & \ldots\ldots\ldots & & \\ a_{n-1,0} & & & a_{n-1,n} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \ldots \\ p_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix} \tag{2.39}$$

For example,

$$a_{1i} = \sum_{j=0}^{n} \left(b_j \frac{\partial f_i}{\partial x_j} - f_j \frac{\partial b_i}{\partial x_j}\right) \tag{2.40}$$

and higher order terms are obtained from the additional differentiation. In the application these will be obtained when necessary.

2.3.1 The Green's Theorem Approach

The problem to be considered in this section is essentially the same as previously treated in Section 2.2. The Green's Theorem technique discussed in [M2], [H5], [H3] is a powerful tool in resolving the singular control problem, especially when the dimension of the state space is of low order. By this method, it is possible to obtain global conditions for optimality. Consider a system with control appearing linearly, e.g.,

Problem 2.2

$$\dot{x}_i(t) = f_i[x(t)] = b_i[x(t)]u(t) \qquad i = 1,2,\ldots,n \tag{2.41}$$

u(t) is a scalar control function constrained in magnitude by the relation

$$0 \leq u(t) \leq 1 \qquad \text{for all } t \qquad t \in [0,T] \tag{2.42}$$

Drive the initial state x(0)=a to x(T)=b, while minimizing the functional

$$J[u] = \int_0^T f_0[x(t)]dt \tag{2.43}$$

The two dimensional problem will be considered before considering extensions of Green's Theorem to higher dimensions. The basis of the development is the transformation of line integrals into surface integrals. Assume that the integral in (2.43) may be written as

$$J[u] = \int_a^b [P(x_1,x_2)dx_2 + Q(x_1,x_2)dx_1] \tag{2.44}$$

This requires that dt can be expressed in terms of dx_1 and dx_2.

Consider that the class of arcs being investigated is contained in a region bounded by the closed curve $\varepsilon(x_1,x_2) = 0$. The initial point $(x_1(0),x_2(0))$ and the final point $(x_1(T), x_2(T))$ are on the boundary of this domain. Figure 2.1 shows two possible paths, C and D. Note that paths may have corners indicating a discontinuity in the derivative dx/dt.

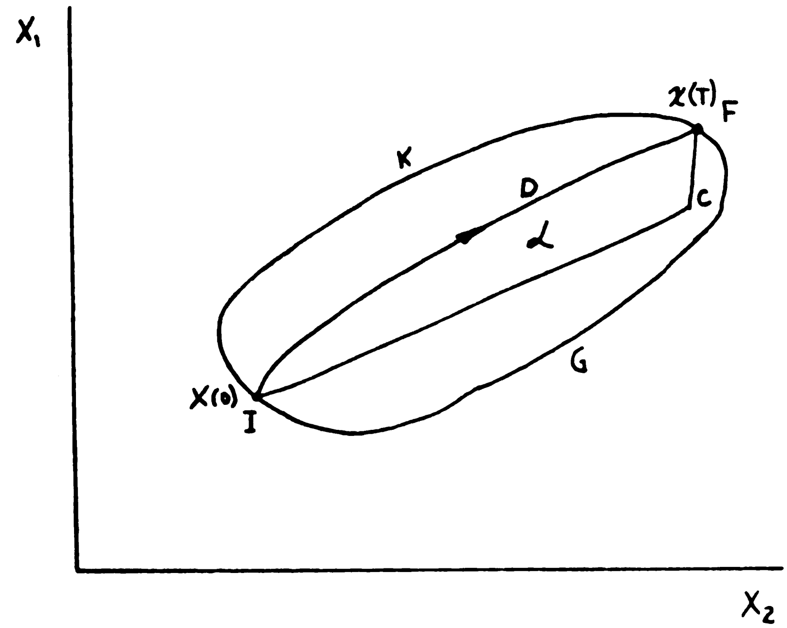


Figure 2.1

Comparison of Trajectories

By comparing the value of the integral (2.44) along all admissible paths, the extremal arc which will be a global minimum (or maximum) can be determined. This is done as follows:

Designate path C by ICF and path D by IDF. Here I and F represent initial and final points on the arc or trajectory. Then subtract the integrals associated with these paths,

$$\Delta J = \int_{ICF} (Pdx_2 + Qdx_1) - \int_{IDF} (Pdx_2 + Qdx_1) \tag{2.45}$$

This is equivalent to the closed contour integral

$$\Delta J = \oint_{ICFDI} (Pdx_2 + Qdx_1) \tag{2.46}$$

Green's Theorem is applicable if the functions P and Q and their partial derivatives are continuous in the region α bounded by the two admissible paths. Assuming that these conditions are met, Green's Theorem is used to transform the line integral into a surface integral. Hence,

$$\oint_{ICFDI} (Pdx_2 + Qdx_1) = \iint_{\alpha} \left(\frac{\partial Q}{\partial x_2} - \frac{\partial P}{\partial x_1}\right) dx_1 dx_2 \tag{2.47}$$

The integration along the trajectories proceeds counterclockwise. A negative sign would be associated with the right hand side of the equation if the integration along the trajectories was clockwise.

The fundamental function, as defined by Miele [M2], is

$$\omega(x_1,x_2) = \frac{\partial Q}{\partial x_2} - \frac{\partial P}{\partial x_1} \tag{2.48}$$

A study of this function in the admissible domain provides the necessary information to evaluate the relative merits of all possible trajectories between I and F. In general, the function $\omega(x_1,x_2)$ will change sign in the admissible domain. However, first consider the case where ω is constant over the entire domain. If ω is positive over the entire domain, then $J_{ICF} > J_{IDF}$. In fact, J_{IGF} is greater than any other admissible trajectory, hence is the maximum arc. Similarly $J_{IDF} < J_{ICF}$ if ω is negative over the entire domain. If ω is zero over the entire domain, then the integrals are independent of path, and $J_{IDF} = J_{ICF}$.

Now, considering the general case where ω may change sign within the admissible domain. It is possible to have several subdomains in which ω is positive and several in which ω is negative. In order to find the trajectory from I to F, corresponding to the maximum or minimum value J, proceed as follows referring to Figure 2.2:

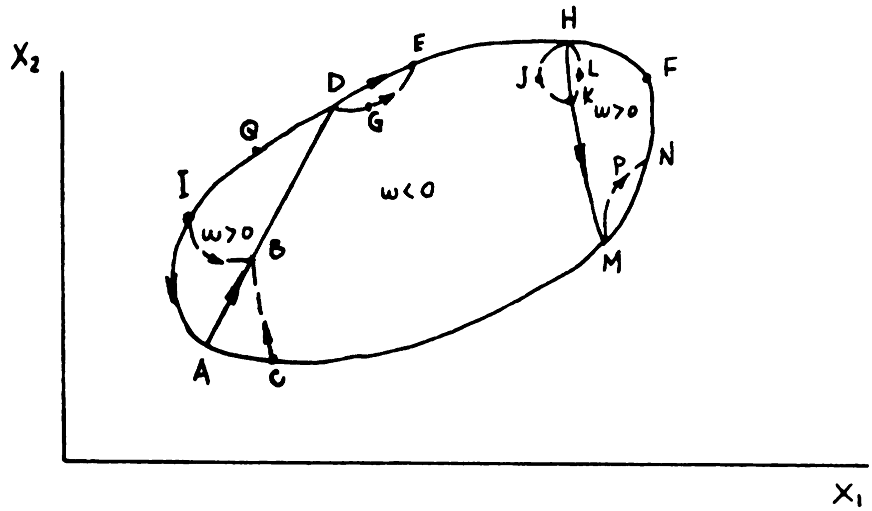


Figure 2.2

Determination of Optimal Trajectory

Starting at I, compare IAB versus IB. Since the domain encircled has $\omega > 0$, and going CCW, $\int_{IAB} > \int_{IB}$. Compare AB vs. ACB. The domain encircled has $\omega > 0$, hence $\int_{AB} < \int_{ACB}$. Likewise $\int_{IAD} \geq \int_{IQD}$

similarly,

$$\int_{DE} > \int_{DGE}$$

$$\int_{DEH} \int_{DGEH}$$

$$\int_{HK} > \int_{HJK}$$

$$\int_{MN} > \int_{MPN}$$

Therefore $\int_{IADHMF}$ is the maximum integral and IADHMF is the corresponding trajectory.

This procedure may be summarized as follows: To determine the trajectory which makes the integral given by equation (2.44) a maximum, start at the initial point I and proceed so that the subdomains $\omega > 0$ are on the left and subdomains where $\omega < 0$ arc on the right. This means the trajectories will be either on the boundary of the domain or on the arc $\omega = 0$. The procedure for minimizing the integral is just the opposite. Hence, the minimizing trajectory is IDAMHF. The singular arcs are those where $\omega' = 0$. The nonsingular arcs are those where $\varepsilon(x_1,x_2) = 0$.

When comparing trajectories using the above procedure, it is assumed that the admissible control functions are able to generate the trajectories, including the singular arc $\omega = 0$. When constructing the domain and its boundary $\varepsilon(x_1,x_2) = 0$, it may not be obvious that the singular arc $\omega = 0$ is not admissible in certain cases.

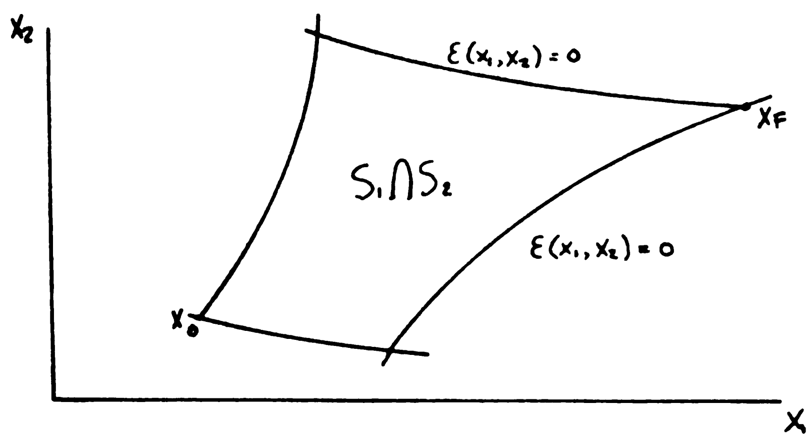


Figure 2.3

Permissible Domain of Operation

The domain of operation is constructed by determining the intersection of two sets, S_1 and S_2. The set S_1 is the set of points attainable by admissible controls starting from the central point x_0. S_2 is the set attainable by admissible controls starting from the final point x_f with time reversed, or equivalently, the set of points from which it is possible to derive x_f using admissible controls. In this domain are found all admissible trajectories.

If, arbitrarily, a trajectory $\omega = 0$ is drawn, it is not obvious that this trajectory can be generated by an admissible control u(t). If it can, then it is a possible candidate for a singular arc. For example, consider the domain shown in Figure 2.4. This domain is associated with the system

$$\dot{x}_1 = x_1 + x_2 u \qquad (2.49)$$

$$\dot{x}_2 = x_2 + x_1 u \qquad (2.50)$$

$$|u| \leq 1 \qquad (2.51)$$

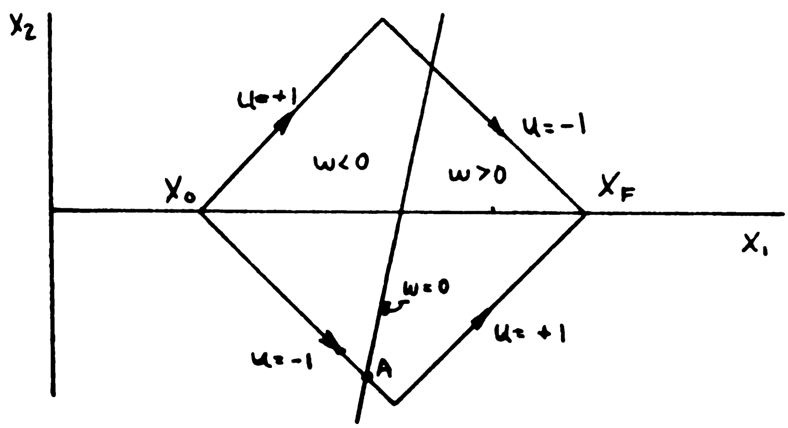


Figure 2.4

Optimal Trajectory Not Along an $\omega = 0$ Arc

The fundamental function $\omega(x_1,x_2)$ associated with various performance criteria can have the form shown. An example is given in [H2]. As is evident from this system, the permissible trajectories emanating from any point in or on the boundary of the domain are confined to angles between -45° and +45°. Hence, when point A is reached, the trajectory continues downward instead of going along $\omega = 0$. If the slope of the line $\omega = 0$ is 1, we have the interesting case where the trajectory is now along $\omega = 0$ and the problem is singular and still "bang-bang". This is because $u = +1$ along $\omega = 0$.

2.3.2 Determination of Reachable Regions

The Green Theorem Approach necessitates determining regions over which control is possible. Two regions are of interest. The first is the set of states that can be reached, given a class of functions U and initial state $x(t_0) = x_0$. The total set will be called the Reachable set. Related to this set is the set of states that can be reached at time T by use of admissible controls. This set was called the T-Reachable set by Snow [S3]. It should be noted that both the Reachable set and the T-Reachable set are independent of any performance criteria.

The second region of interest is the set of states for which there is an admissible control in U that drives the state to a given final state. This set of states will be called the Controllable Set. Related to this set is the set of states for which there is an admissible control U that drives the state to a given final state in time T. This set is defined as T-Controllable by Snow. The name controllable is related to the concept of controllability, which states that a system is controllable if, given any two states, there is a control which will drive the system from one state to the other in finite time.

In the application of the Green's Theorem Approach, only the character of the Reachable Region and the Controllable Region need be known. The intersection of the Reachable Set and the Controllable Set contain all trajectories from the initial point to the final point.

A method of obtaining the Reachable Set was developed by Snow [S3]. His method is based on the solution of three Hamilton-Jacobi partial differential equations. The equations are solved by the method of characteristics. The Reachable region is the region bounded by the surfaces $S(x,t) = S(x_0,t_0)$ where $S(x,t)$ represents the solutions of the Hamilton-Jacobi equations.

The method used here will be based on several theorems developed by Hermes and Haynes [H3]. The theorems are directly applicable to Problem 3.2 which will be considered later.

The system to be considered is two dimensional with the control function appearing linearly.

$$\dot{x}_1 = f_1(x_1,x_2) + b_1(x_1,x_2)u \qquad x_1(0) = x_{10} \tag{2.52}$$

$$\dot{x}_2 = f_2(x_1,x_2) + b_2(x_1,x_2)u \qquad x_2(0) = x_{20} \tag{2.53}$$

The control functional u is a scalar and is in the set of admissible control functions, U.

$$U \equiv \{u: |u(t)| \leq 1 , \quad t \in [0,\infty]\} \tag{2.54}$$

In the development, the solution of equations (2.52) and (2.53) when a constant control $u(t) = \alpha$, $-1 \leq \alpha \leq 1$, is applied is designated as ϕ^α.

It is assumed that f_1, f_2, b_1, and b_2 are once continously differentiable in an open, simply connected set $D \subset R^2$. The initial point x_0 and the final point x_f are always considered to be in D.

The Reachable Set (set of points which can be attained from x_0) is defined as

$$R(x_0) \equiv \{x \in R^2 : x = \phi^u(t,x_0),\ u \in U\} \tag{2.55}$$

The Controllable Set, (set of points from which x_f can be attained in a finite time), is defined as

$$R(x_f) \equiv \{x \in R^2 : x = \phi^u(-t,x_f),\ u \in U\} \tag{2.56}$$

The relationship between the T-Controllable Region for a given system and the T-Reachable Region for the system with time reversed is developed by Snow [S3], who shows that if the system is described by n first order equations, the T-Reachable Region for the forward time equation is precisely the same as the $(T-t_0)$ Controllable Region for the reversed time system. This is not true in general for a system described by a single n^{th}-order differential equation.

If a solution to the Optimal Control problem exists the trajectory connecting x_0 to x_f must lie in $R(x_0) \cap R(x_f)$.

The following lemmas due to Hermes and Haynes [H3] are the basis for the theorem giving sufficient conditions so that the trajectories $\phi^1(\cdot, x_0)$, $\phi^{-1}(\cdot,x^0)$, $\phi^1(\cdot,x_f)$, and $\phi^{-1}(\cdot,x_f)$ determine and bound $R(x_0) \cap R(x_f)$.

The following definitions are used in the theorems and lemmas.

$$\Delta(y) \equiv -b_2(y)\, f_1(y) + b_1(y)\, f_2(y) \qquad y \in D \tag{2.57}$$

$\theta(\alpha,y)$ is the angle traced out by the ray $\varepsilon(\sigma,y)$ as σ varies continuously from -1 to α.

The vector ε is defined as,

$$\varepsilon(\sigma,y) \equiv \begin{bmatrix} f_1(y) + b_1(y)\alpha \\ f_2(y) - b_2(y)\alpha \end{bmatrix} \tag{2.58}$$

The possible directions which solution trajectories can assume at a point y in the two-dimensional space are given by the vector $\varepsilon(\sigma,y)$.

Lemma 2.1

If $\Delta(y) \neq 0$, the set $\{ \varepsilon(\alpha,y) : |\alpha| \leq \}$ of possible directions is bounded by $\varepsilon(-1,y)$ and $\varepsilon(1,y)$ with $0<|\theta(1,y)|<\Pi$.

This lemma implies that the set of possible trajectory directions at X_0 are confined to an angle of less than or equal to Π.

The next lemma shows that if the angle $\theta(1,\phi^1(t,x_0))$ is observed as t increases from zero, the condition $\Delta(\phi^1(t,x\ )) \neq 0$ will not change.

Similarly $\Delta(\phi^{-1}(t,x_0) \neq 0$ implies that the sign of $\phi(1,\phi^{-1}(t,x_0))$ will not change. As expected, all trajectories are confined to a region bounded by $\phi^1(\cdot,x_0)$ and $\phi^{-1}(\cdot,x_0)$.

Lemma 2.2

Let $\gamma(\sigma)$, $\sigma_0 \leq \sigma \leq \sigma_f$, be a continuous curve in D along which $\Delta(\gamma(\sigma)) \neq 0$; then signum $\theta(1, \gamma(\sigma))$ is invariant along the curve. Thus all possible trajectories are bounded by $\phi^1(\cdot,x_0)$ and $\phi^{-1}(\cdot,x_0)$ and are contained in an angle less than or equal to π.

Similar statements can be made about x_f.

In order to insure the existence of solutions to (2.52), (2.53), and (2.54) joining x_0 to x_f, the following conditions are imposed.

<u>Condition 2.1</u> Either $\Gamma(x_0)$ or $\Gamma(x_f)$ properly separates D, where

$$\Gamma(x_0) \equiv \{\phi^1(t,x_0): t \in T(1,x_0)\} \bigcup \{\phi^{-1}(t,x_0): t \in T(-1,x_0),\ t>0\}$$

$$\Gamma(x_f) \equiv \{\phi^{-1}(-t,x_f): t \in T(1,x_f)\} \bigcup \{\phi^{-1}(-t,x_f): t \in T(-1,x_f),\ t>0\}$$

and there exist t_1, t_2, t_3, $t_4 > 0$ such that

i) $\phi^1(t_1,x_0) = \phi^{-1}(-t_2,x_f)$

ii) $\phi^{-1}(t_1,x_0) = \phi^1(-t_4,x_f)$

iii) The trajectory arcs $\phi^1(t,x_0)$, $0 \le t \le t_1$;

$\phi^{-1}(t,x_0)$, $0 \le t \le t_3$; $\phi^1(-t,x_f)$, $0 \le t \le t_4$;

$\phi^{-1}(-t,x_f)$, $0 \le t \le t_2$, all lie in D

iv) $\Delta(x) \neq 0$ in the set $\Gamma(x_0)$ or $\Gamma(x_f)$ which properly separates D

<u>Theorem 2.1</u> If a problem satisfies Condition 2.1 and $\Delta(y) \neq 0$ for

$y \in S$, then $S = R(x_0) \bigcap R(x_f)$

Summarizing, Theorem 2.1, provides a rigorous basis for the determination of the region containing all the admissible trajectories. It also shows that this region is bounded by trajectories resulting from application of "bang-bang" control signals.

2.4 The Second Variation Approach

This section will present equations for minimality of singular arcs over a finite time interval. Special control variations are used to obtain a second variation test for singular arcs. The approach is based on the work of Kopp, Kelly, and Moyer [K2], [K3].

Consider the system of differential equations and boundary conditions

$$\dot{x}_i = f_i(x_i,\ldots,x_n,u_1,\ldots,u_r,t) \qquad i = 1,2,\ldots,n \tag{2.59}$$

$$x_i(t_0) = x_{i0} \qquad i = 1,\ldots,n \tag{2.60}$$

$$x_i(T) = x_{if} \qquad i = 1,\ldots,m(m \leq n) \tag{2.61}$$

The cost functional to be minimized will be formulated in the Mayer form of the calculus of variation, i.e., minimize

$$J(x_{m+1}(T),\ldots x_n(T),\ T) \tag{2.62}$$

Although the minimization is also subject to constraints on the controls, i.e., $|u| \leq 1$, these constraints will present no difficulty since the control corresponding to a singular arc is usually interior to the boundary of U.

Therefore, in the development that follows, u will be considered to be in the interior of the class of admissible controls U.

The Hamiltonian is defined as

$$H = \sum_{i=1}^{n} p_i f_i \tag{2.63}$$

Introduce the auxiliary vector $p_i(t)$. This turns out to be the adjoint or costate vector and is defined by the following differential equations and boundary conditions,

$$\dot{p}_i = -\sum_{j=i}^{n} p_i \frac{\partial f_j}{\partial i} = -\frac{\partial H}{\partial x_i} \qquad i = 1,\ldots,n \qquad (2.64)$$

$$p_i(T) = \frac{\partial J}{\partial x_{if}} \qquad i = m+1,\ldots,n \qquad (2.65)$$

Necessary conditions for P to be a minimum are that the Hamiltonian be a minimum for all admissible controls,

$$H(u_1^* + \Delta u_1,\ldots,u_r^* + \Delta u_r) \geq H(u_1^*,\ldots,u_r^*) \qquad (2.66)$$

Asterix denotes the optimal controls.

Singular subarcs occur when the matrix H_{uu}, with typical element $\frac{\partial H}{\partial u_i\ \partial u_j}$, is singular over a finite interval of time.

Emphasis will be on the case in which a single variable appears linearly in the system equation as in the vehicle braking problem.

The total variation in the cost functional $J[u]$ due to a variation in the vector u is

$$\Delta J = \sum_{i=m+1}^{n} \frac{\partial J}{\partial x_{if}} \Delta x_{if} + \tfrac{1}{2} \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \frac{\partial^2 J(x_f + \theta \Delta x_f)}{\partial x_{if}\ \partial x_{jf}} \Delta x_{if}\ \Delta x_{jb} \qquad (2.67)$$

where $0 \leq \theta \leq 1$.

Consider the case where the end points are fixed. Then, due to a change Δu away from the optimal u^*,

$$\Delta \dot{x}_i = f_i(x^* + \Delta x, u^* + \Delta u, t) - f_i(x^*, u^*, t) \qquad i=1,\ldots,n \qquad (2.68)$$

$$\Delta x_i(t_0) = 0 \qquad i=1,\ldots,n \qquad (2.69)$$

$$\Delta x_i(t_f) = 0 \qquad i=1,\ldots,n \qquad (2.70)$$

Consider the equation

$$\sum_{i=1}^{n} p_i \Delta\dot{x}_i = \sum_{i=1}^{n} p_i [f_i(x^*+\Delta x, u^*+\Delta u, t) - f_i(x^*, u^*, t)] \tag{2.71}$$

Using the Hamiltonian in equation (2.71),

$$\sum_{i=1}^{n} p_i \Delta\dot{x}_i = H(x^*+\Delta x, u^*+\Delta u, t) - H(x^*, u^*, t) \tag{2.72}$$

Now consider

$$\frac{d}{dt} \sum_{i=1}^{n} p_i \Delta x_i = \sum_{i=1}^{n} p_i \Delta\dot{x}_i + \sum_{i=1}^{n} p_i \Delta x_i \tag{2.73}$$

Multiplying through by dt and integrating,

$$\sum_{i=1}^{n} p_i \Delta x_i \Big|_{t_0}^{t_f} = \int_{t_0}^{t_f} [\sum_{i=1}^{n} p_i \Delta\dot{x}_i + \sum_{i=1}^{n} p_i \Delta x_i]\, dt \tag{2.74}$$

Due to the boundary conditions

$$\sum_{i=m+1}^{n} \frac{\partial J}{\partial x_{if}} = \int_{t_0}^{t_f} \{ \sum_{i=1}^{n} p_i [f_i(x^*+\Delta x, u^*+\Delta u, t) - f_i(x^*, u^*, t)] - \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}(x^*, u^*, t)\Delta x_i \}\, dt \tag{2.75}$$

Using Taylor's expansion, and substituting the Hamiltonian,

$$\sum_{i=m+1}^{n} \frac{\partial J}{\partial x_{if}} \Delta x_{if} = \int_{t_0}^{t_f} [H(p, x^*, u^*+\Delta u, t) + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}(p, x^*, u^*+\Delta u, t)\Delta x_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j}(p, x^*, +\theta\Delta x, u^*+\Delta u, t)\, \Delta x_i \Delta x_j - H(p, x^*, u^*, t) - \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}(p, x^*, u^*, t)\Delta x_i]\, dt \tag{2.76}$$

Substituting (2.76) into (2.67) the total variation due to a variation in control vector u is as follows:

$$\Delta J = \int_{t_0}^{t_f} [H(x^*, u^* + \Delta u, t) - H(x^*, u^*, t)]\, dt$$

$$+ \int_{t_0}^{t_f} [\frac{\partial H}{\partial x_i}(p, x^*, u^* + u, t) - \frac{\partial H}{\partial x_i}(p, x^*, u^*, t)]\Delta x_i dt$$

$$+ \tfrac{1}{2} \int \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} (p, x^* + \theta \Delta x, u^* + \Delta u, t) \Delta x_i \Delta x_j dt$$

$$+ \tfrac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 J(x_f + \theta \Delta x_f)}{\partial x_{if} \partial x_{jf}} \Delta x_{if} \Delta x_{jf} \tag{2.77}$$

At this point, assume that the control U appears linearly in the system equation (2.59). The control will then also appear linearly in the Hamiltonian. When the first integral of (2.77) is expanded using the Taylor expansion, the $\frac{\partial^2 H}{\partial u^2}$ term will vanish because the Hamiltonian is linear in u.

Hence, minimality cannot be established. Therefore, to obtain additional necessary conditions, further inspection of the second order terms is required. The classical derivation of the Legendre necessary condition was obtained by employing a special derivation in conjunction with the second variation [G1]. The second variation for the present case is obtained from Kelley. Letting $\Delta u = K\delta x$, the second variation is

$$\Delta J_2 = K^2\int_{t_0}^{t_f} \frac{\partial^2 H}{\partial x_i \partial u}(x^*, u^*, t)\ \delta x_i \delta u\ dt +$$

$$\frac{K^2}{2}\int_{t_0}^{t_f} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial H}{\partial x_i \partial x_j}(x^*, u^*, t)\ \Delta x_i\ \Delta x_j\ dt +$$

$$\frac{K^2}{2}\int \sum_{J=1}^{n} \sum_{i=1}^{n} \frac{\partial J(x^*, t)}{\partial x_{if}\ \partial x_{jf}}\ \delta x_{if}\ \delta x_{jf} \tag{2.78}$$

The first and second control variations used by Kelly, Kopp, and Moyer are shown in Figure 2.5.

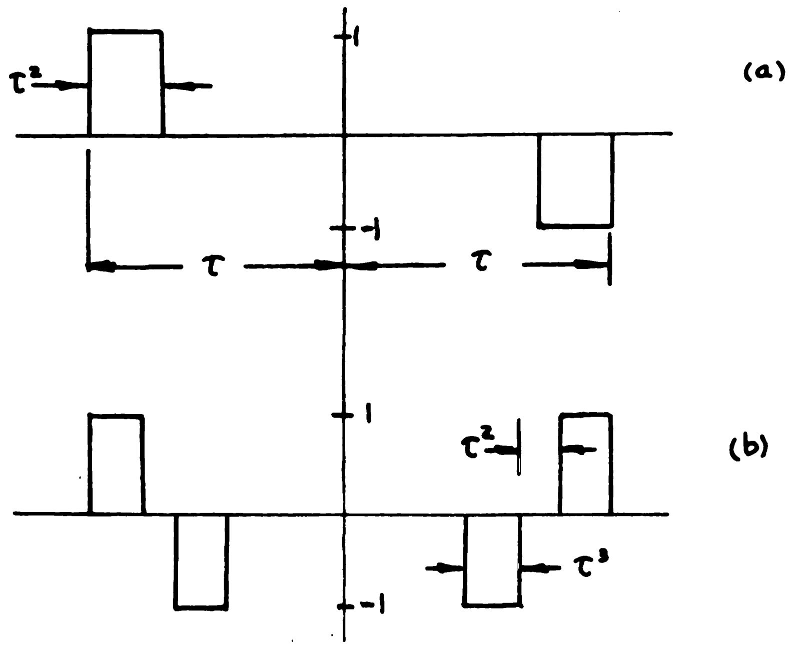


Figure 2.5

a) First special control variation b) Second control variation

The first special control variation is designated as $\phi_0'(t,\tau)$. The time $t=0$ is the center of an interval 2τ, and may occur at any interior point of the singular subarc. The parameter τ will approach zero in the limit.

In minimizing J_2, the constraint equations are

$$\delta \dot{x}_i = \sum_{j=1}^{n} \frac{\partial f_i(x,u,t)}{\partial x_j} \delta x_j + \frac{\partial f_i(x,u,t)}{\partial u} \delta u \tag{2.79}$$

$$\delta x_i(t_0) = 0 \qquad i=1,\ldots,n \tag{2.80}$$

$$\delta x_i(t_f) = 0 \qquad i=1,\ldots,m \tag{2.81}$$

Letting

$$A_{i,1} = \frac{\partial f_i}{\partial u} = \frac{\partial^2 H}{\partial p_i \partial u} \tag{2.82}$$

$$A_{i,2} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} A_{j,1} - \dot{A}_{i,1} \tag{2.83}$$

or

$$A_{i,2} = \sum_{j=1}^{n} \frac{\partial^2 H}{\partial p_i \partial x_j} A_{j,1} - \dot{A}_{i,1} \tag{2.84}$$

the necessary condition obtained by Kelly, Kopp, and Moyer for the singular arc to be minimizing is

$$\tfrac{1}{2} \frac{d}{dt} \left[\sum_{i=1}^{n} \frac{\partial^2 H}{\partial u \partial x_i} A_{i,2} \right] + \sum_{i=1}^{n} \frac{\partial^2 H}{\partial u \partial x_i} A_{i,2} - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} A_{i,1} A_{j,1} \leq 0 \tag{2.85}$$

An equivalent and more compact form is due to Robbins [R1]

$$\frac{\partial}{\partial u} \left(\frac{d^2}{dt^2} \frac{\partial H}{\partial u} \right) \leq 0 \tag{2.86}$$

The equality part of the sign in the conditions (2.85) and (2.86) means that the conditions are met marginally and the nature of the extremal is still undetermined. Hence, it is necessary to proceed to the second special variation, and so on. Using the Robbins form, the second special variation leads to

$$\frac{\partial}{\partial u} \left(\frac{d^4}{dt^4} \frac{\partial H}{\partial u} \right) \geq 0 \tag{2.87}$$

The general form of the necessary condition is

$$(-1)^{k} \frac{\partial}{\partial u}\left(\frac{d^{2k}}{dt^{2k}} \frac{\partial H}{\partial u}\right) \geq 0 \tag{2.88}$$

where k is a positive integer.

CHAPTER III TIME OPTIMAL SINGULAR CONTROL

3.1 Maximum Principle Approach

The problem to be considered is a time optimal problem where it is desired to drive x_0 to 0 in minimum time. Consider the system equations of Problem 3.1.

Singular Bang-Bang Time Optimal Control Problem

$$\dot{x}_1(t) = f_1[x_1(t), x_2(t)] + b_1 u \qquad x_1(0) = x_{10} \tag{3.1}$$

$$\dot{x}_2(t) = f_2[x_1(t), x_2(T)] + b_2 u \qquad x_2(0) = x_{20} \tag{3.2}$$

b_1 and b_2 are constant and u is a scalar

This is the Mayer form H[6], if we minimize $x_0(T)$ where

$$\dot{x}_0(t) = 1 \qquad x_0(0) = 0 \tag{3.3}$$

The control will be constrained to

$$0 \leq u \leq 1 \tag{3.4}$$

Again dropping the arguments for notational convenience, the Hamiltonian is

$$H = p_0 + p_1 f_1 + p_1 b_1 u + p_2 f_2 + p_2 b_2 u \tag{3.5}$$

The costate equations are

$$\dot{p}_0 = 0 \tag{3.6}$$

$$-\dot{p}_1 = p_1 \frac{\partial f_1}{\partial x_1} + p_2 \frac{\partial f_2}{\partial x_1} \tag{3.7}$$

$$-\dot{p}_2 = p_2 \frac{\partial f_1}{\partial x_1} + p_2 \frac{\partial f_2}{\partial x_2} \tag{3.8}$$

Minimizing H with respect to u yields

$$\frac{\partial H}{\partial u} = p_1 b_1 + p_2 b_2 = 0 \qquad \text{(NC1)} \tag{3.9}$$

Again, since we are interested in singular arcs, u is assumed to be in the interior of its allowed region and $\partial H/\partial u$ is assumed to exist. Since $H = 0$, and $p_0 = 1$, (This is permissible since p_0 functions only as a scale factor).

$$1 + p_1 f_1 + p_2 f_2 = 0 \qquad \text{(NC2)} \quad (3.10)$$

Differentiating (3.9)

$$\dot{p}_1 b_1 + \dot{p}_2 b_2 = 0 \qquad \text{(NC3)} \quad (3.11)$$

$$b_1 \left(p_1 \frac{\partial f_1}{\partial x_1} + p_2 \frac{\partial f_2}{\partial x_1}\right) + b_2 \left(p_1 \frac{\partial f_1}{\partial x_2} + p_2 \frac{\partial f_2}{\partial x_2}\right) = 0 \qquad (3.12)$$

Differentiating (3.10)

$$p_1 \left(\frac{\partial f_1}{\partial x_1} \dot{x}_1 + \frac{\partial f_1}{\partial x_2} \dot{x}_2 \right) + f_1 \dot{p}_1 + p_2 \left(\frac{\partial f_2}{\partial x_1} \dot{x}_1 + \frac{\partial f_2}{\partial x_2} \dot{x}_2 \right) + f_2 \dot{p}_2 = 0 \qquad (3.13)$$

Replacing $\dot{x}$ and $\dot{p}$,

$$p_1 \left[\frac{\partial f_1}{\partial x_1} (f_1 + b_1 u) + \frac{\partial f_1}{\partial x_2} (f_2 + b_2 u)\right] - f_1 \left(p_1 \frac{\partial f_1}{\partial x_1} + p_2 \frac{\partial f_2}{\partial x_1}\right) +$$

$$p_2 \left[\frac{\partial f}{\partial x_1} (f_1 + b_1 u) + \frac{\partial f_2}{\partial x_2} (f_2 + b_2 u)\right] - f_2 \left(p_1 \frac{\partial f_1}{\partial x_1} + p_2 \frac{\partial f_2}{\partial x_2}\right) = 0 \qquad (3.14)$$

From (3.11) the coefficient of u is zero while the remaining terms cancel out. Hence (3.11) is the third necessary condition.

$$p_1 \left(b_1 \frac{\partial f_1}{\partial x_1} + b_2 \frac{\partial f_1}{\partial x_2}\right) + p_2 \left(b_1 \frac{\partial f_2}{\partial x_1} + b_2 \frac{\partial f_2}{\partial x_2}\right) = 0 \qquad \text{(NC4)} \quad (3.15)$$

This necessary condition appears as row 3 of equation(2.39), i.e.,

$$p_1 a_{11} + p_2 a_{12} = 0 \qquad (a_{10} = 0) \qquad (3.16)$$

where

$$a_{11} = b_1 \frac{\partial f_1}{\partial x_1} + b_2 \frac{\partial f_1}{\partial x_2} \qquad (3.17)$$

$$a_{12} = b_1 \frac{\partial f_2}{\partial x_1} + b_2 \frac{\partial f_2}{\partial x_2} \tag{3.18}$$

Differentiation of this equation will yield additional information.

$$p_1 \dot{a}_{11} + \dot{p}_1 a_{11} + p_2 \dot{a}_{12} + \dot{p}_2 a_{12} = 0 \tag{3.19}$$

$$\dot{a}_{11} = b_1 \left(\frac{\partial^2 f_1}{\partial x_1{}^2} \dot{x}_1 + \frac{\partial^2 f_1}{\partial x_1 \partial x_2} \dot{x}_2 \right. + b_2 \left(\frac{\partial^2 f_1}{\partial x_2 \partial x_1} \dot{x}_1 + \frac{\partial^2 f_1}{\partial x_2{}^2} \dot{x}_2 \right) \tag{3.20}$$

$$\dot{a}_{11} = \left(b_1 \frac{\partial^2 f_1}{\partial x_1{}^2} + b_2 \frac{\partial^2 f_1}{\partial x_2 \partial x_1}\right)(f_1 + b_1 u) + \left(b_1 \frac{\partial^2 f_1}{\partial x_1 \partial x_2} + b_2 \frac{\partial^2 f_1}{\partial x_1{}^2}\right)(f_2 + b_2 u) \tag{3.21}$$

$$\dot{a}_{12} = \left(b_1 \frac{\partial^2 f_2}{\partial x_1{}^2} + b_2 \frac{\partial^2 f_2}{\partial x_2 \partial x_1}\right)(f_1 + b_1 u) + \left(b_1 \frac{\partial^2 f_2}{\partial x_1 \partial x_1} + b_2 \frac{\partial^2 f_2}{\partial x_2{}^2}\right)(f_2 + b_2 u) \tag{3.22}$$

Substituting in (3.19) the following result is obtained,

$$\sum_{i=0}^{2} \sum_{j=0}^{2} \sum_{k=0}^{2} \left(- b_j p_i \frac{\partial f_k}{\partial x_j} \frac{\partial f_i}{\partial x_k} + b_j f_k \frac{\partial^2 f_i}{x_k \partial x_j} + u b_j b_k \frac{\partial^2 f_i}{x_k \partial x_j} \right) = 0 \tag{3.23}$$

In the general case the upper summation limit would be n.

It is possible to solve this equation for the control u. To be more specific consider the determination of the singular arc for,

<u>Problem 3.2 Singular Bang-Bang Time Optimal Control</u>

$$\dot{x}_1 = f_1 \qquad x_1(0) = x_{10} \tag{3.24}$$

$$\dot{x}_2 = f_2 + b_2 u \qquad x_2(0) = x_{20} \qquad b_2 = \text{constant} \tag{3.25}$$

$$\dot{x}_0 = 1 \qquad x_3(0) = 0 \tag{3.26}$$

Since $b_1 = 0$,

NC1 implies $p_2 = 0$ on the singular.

NC3 implies $\dot{p}_2 = 0$

NC2 implies $\quad p_1 f_1 = -1$

NC4 implies $\quad p_1 \dfrac{\partial f_1}{\partial x_2} = 0$

This implies that $p_1 = 0$ or $\dfrac{\partial f_1}{\partial x_2} = 0$

The condition $p_1 = 0$ would contradict NC2 which requires that $H = 0$ on the optimal trajectory. Hence NC4 implies that $\dfrac{\partial f_1}{\partial x_2} = 0$. Substituting necessary conditions in (3.19) rather than the more formidable (3.23),

$$f_1 \frac{\partial^2 f_1}{\partial x_2 \partial x_1} + (f_2 + b_2 u) \frac{\partial^2 f_1}{\partial x_2^{\,2}} = 0 \tag{3.27}$$

on the singular arc. The optimal control is synthesized as

$$u = - \frac{f_1 \dfrac{\partial^2 f_1}{\partial x_2 \partial x_1} + f_2 \dfrac{\partial^2 f_1}{\partial x_2^{\,2}}}{b_2 \dfrac{\partial^2 f_1}{\partial x_2^{\,2}}} \tag{3.28}$$

For a practical case to be considered later, it will be convenient to consider

$$\frac{\partial f_1}{\partial x_1} = - \frac{x_2}{x_1} \frac{\partial f_1}{\partial x_2} \tag{3.29}$$

and

$$\frac{\partial f_2}{\partial x_1} = - \frac{x_2}{x_1} \frac{\partial f_2}{\partial x_2} \tag{3.30}$$

This modified version of Problem 3.2 shall be designated as Problem 3.2M.

Differentiating with respect to x_2 and taking advantage of the necessary condition $\dfrac{\partial f_1}{\partial x_2} = 0$ on the singular arc,

$$\frac{\partial^2 f_1}{\partial x_1 \partial x_2} = - \frac{x_2}{x_1} \frac{\partial^2 f_1}{\partial x_2^{\,2}} \tag{3.31}$$

Substituting this in (3.28) gives the optimal control function for this case as

$$u = \left(\frac{x_2}{x_1} f_1 - f_2\right)/b_2 \tag{3.32}$$

This control function can be shown to be a constant. Differentiate (3.32) with respect to time.

$$b_2\frac{du}{dt} = \frac{x_2}{x_1}\left[\frac{\partial f_1}{\partial x_1}f_1 + \frac{\partial f_2}{\partial x_2}(f_2 + b_2u)\right] + \frac{f_1}{x_1^{\,2}}\left[x_1(f_2 + b_2u) - x_2f_1\right] -$$

$$\left[\frac{\partial f_2}{\partial x_1}f_1 + \frac{\partial f_2}{\partial x_2}(f_2 + b_2u)\right] \tag{3.33}$$

From (3.29) and the necessary condition $\frac{\partial f_1}{\partial x_2} = 0$, and assuming x , $\neq$ 0,

$$b_2\frac{du}{dt} = \frac{f_1}{x_1^{\,2}}\left(x_1 \cdot \frac{x_2}{x_1}f_1 - x_2f_1\right) - \left(\frac{\partial f_2}{\partial x_1}f_1 + \frac{\partial f_2}{\partial x_2} \cdot \frac{x_2}{x_1}f_1\right) \tag{3.34}$$

Finally using (3.30)

$$\frac{du}{dt} = 0 \qquad b_2 \neq 0 \tag{3.35}$$

Therefore the singular control for this problem is a constant.

3.1.2 Closed Loop System Control

The problem of determining the optimal control as a function of the state of the system is called the closed loop problem. On the singular arc, (3.32) provides this information for the modified problem 3.2M. Equation (3.28) would be required for Problem 3.2. A block diagram for Problem 3.2M will be shown. As (3.28) indicates, three additional function generators would be required to implement Problem 3.2.

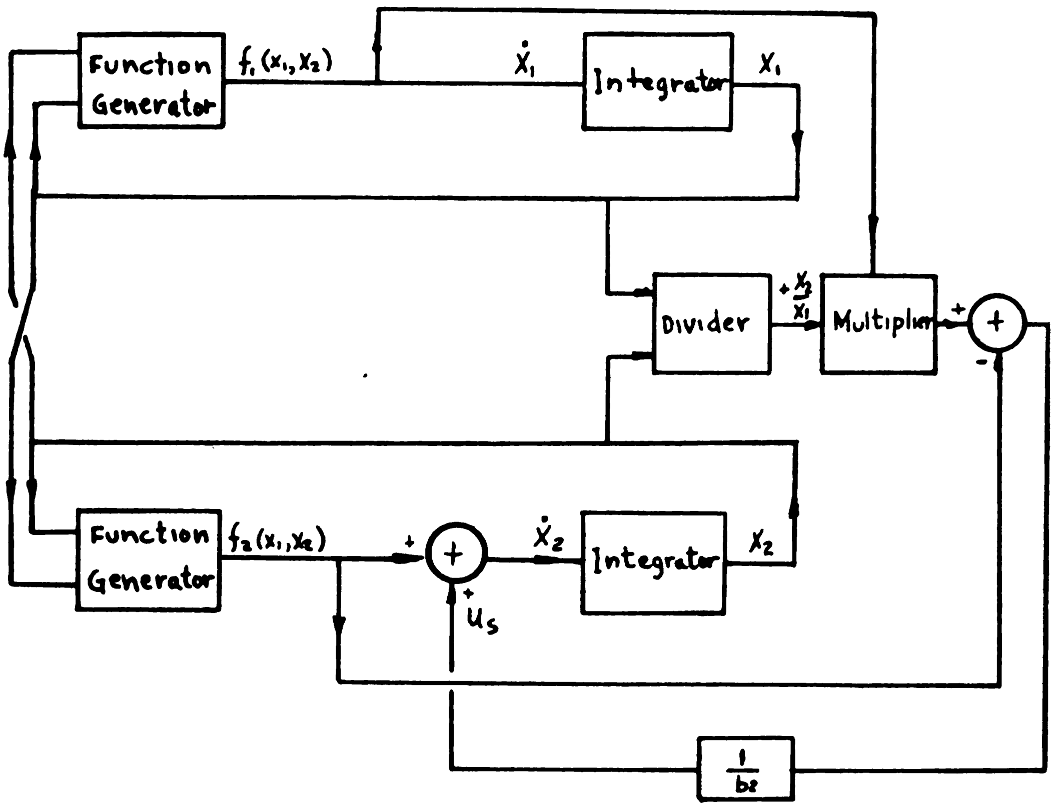


Figure 3.1

Closed Loop System On Singular Arc

When operating on a nonsingular arc, the control as given by (2.19) and applied to Problem 3.2M is,

$$u^*(t) = -\operatorname{sgn} b_2 p_2(t) \tag{3.36}$$

The costate equations for Problem 3.1M are

$$\dot{p} = 0 \tag{3.37}$$

$$-\dot{p}_1 = p_1 \frac{\partial f_1}{\partial x_1} + p_2 \frac{\partial f_2}{\partial x_1} \tag{3.38}$$

$$-\dot{p}_2 = p_1 \frac{\partial f_1}{\partial x_2} + p_2 \frac{\partial f_2}{\partial x_2} \tag{3.39}$$

The costate equations require four function generators for Problem 3.1. In Problem 3.1M,

since,

$$\dot{p}_1 = \frac{\partial f_1}{\partial x_2}\left(p_1 \frac{x_2}{x_1}\right) + \frac{\partial f_2}{\partial x_2}\left(p_2 \frac{x_2}{x_1}\right) \tag{3.40}$$

$$-\dot{p}_2 = p_1 \frac{\partial f_1}{\partial x_2} + p_2 \frac{\partial f_2}{\partial x_2} \tag{3.41}$$

The block diagram which applies for Problem 3.1M when operating on the nonsingular arc is then given by Figure 3.2.

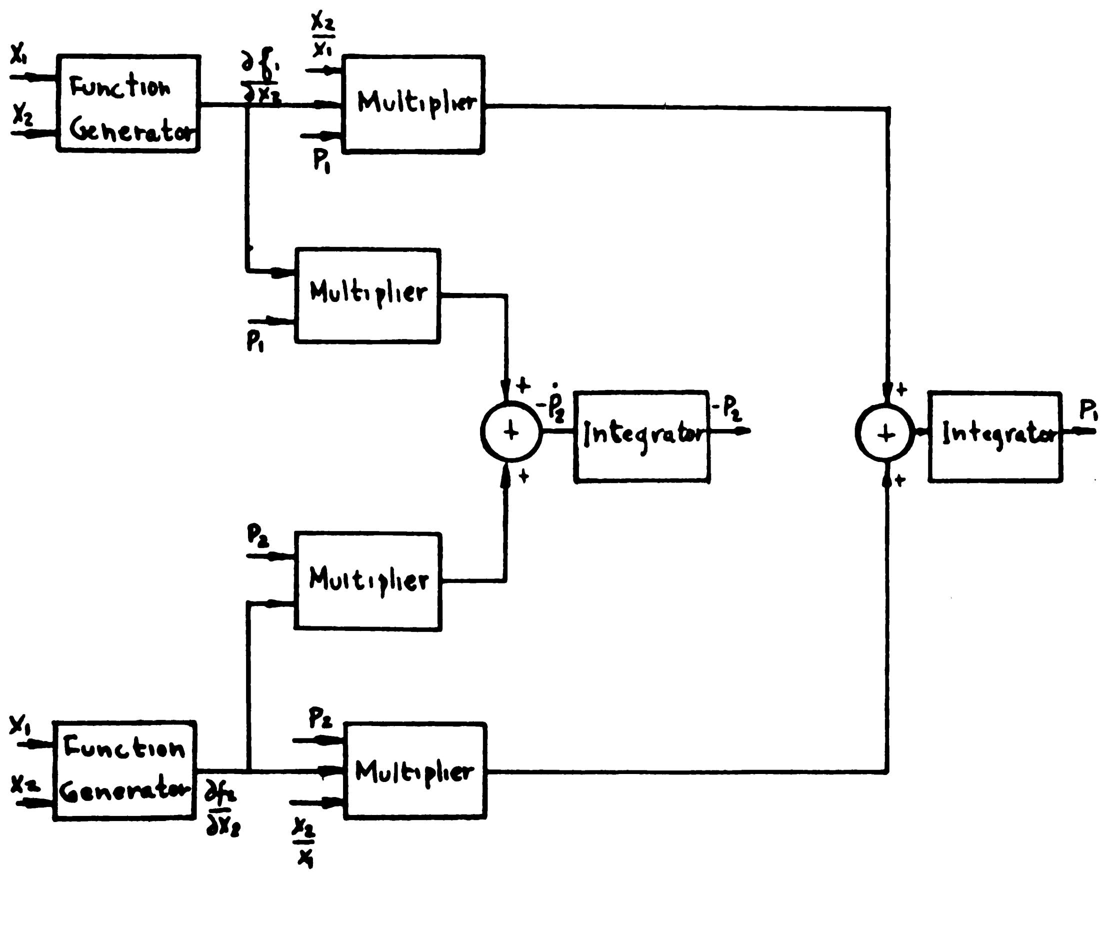
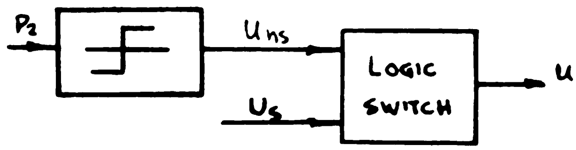


Figure 3.2

Portion of Closed Loop System on Nonsingular Arc

The complete closed loop system which functions on the complete trajectory requires a decision element to switch u to either the singular control u or the nonsingular control u .



The decision element must operate according to the logic demanded by the necessary conditions. If the system is on a nonsingular arc, then it shold transfer to the singular arc.

$$u = u_s \quad \text{if } p_2 = 0 \quad \dot{p}_2 = 0 \tag{3.42}$$

$$u = u_{ns} \quad \text{otherwise} \tag{3.43}$$

3.2.1 Green's Theorem Approach

Determination of singular and nonsingular arcs associated with the time optimal problem is considered. The system constraints are:

$$\dot{x}_1 = f_1(x_1,x_2) + b_1u \tag{3.44}$$

$$\dot{x}_2 = f_2(x_1,x_2) + b_2u \tag{3.45}$$

$$0 \leq u(t) \leq 1 \qquad 0 \leq t \leq T \tag{3.46}$$

Since the functional to be minimized is,

$$J[u] = \int_0^T dt \tag{3.47}$$

it will be necessary to put this in the form of equation (2.44).

Since the procedure basically eliminates u(t) from the equations, it is only necessary to find a vector orthogonal to the column vector $b = [b_1 \ b_2]^T$. In this case, we may use $[-b_2 \ b_1]^T$ as the orthogonal column vector.

Multiplying through by $[-b_2 \; b_1]$ and solving for dt, equation (3.47) becomes

$$J[u] = \int_{x_{10},x_{20}}^{x_1\;,x_2} \frac{b_1 dx_2 - b_2 dx_1}{b_1 f_2 - b_2 f_1} \tag{3.48}$$

The fundamental function $\omega(x_1,x_2)$ is

$$\omega(x_1,x_2) = \frac{\partial}{\partial x_2}\left(-\frac{b_2}{b_1 f_2 - b_2 f_1}\right) - \frac{\partial}{\partial x_1}\left(\frac{b_1}{b_1 f_2 - b_2 f_1}\right) \tag{3.49}$$

$$\omega(x_1,x_2) = \frac{b_2\left(b_1 \frac{\partial f_2}{\partial x_2} - b_2 \frac{\partial f_1}{\partial x_2}\right) + b_1\left(b_1 \frac{\partial f_2}{\partial x_1} - b_2 \frac{\partial f_1}{\partial x_1}\right)}{(b_1 f_2 - b_2 f_1)^2} \tag{3.50}$$

Since $\omega(x_1,x_2) = 0$ is of primary interest, we obtain the condition on the singular arc as,

$$\begin{gathered} b_1 b_2 \frac{\partial f_2}{\partial x_2} - b_2^{\,2} \frac{\partial f_2}{\partial x_2} + b_1^{\,2} \frac{\partial f_2}{\partial x_1} - b_1 b_2 \frac{\partial f_1}{\partial x_2} = 0 \\ (b_2 f_2 - b_2 f_1) \neq 0 \end{gathered} \tag{3.51}$$

In Problem 3.2, $b_1 = 0$, therefore a necessary condition for singular arc is

$$\frac{\partial f_1}{\partial x_2} = 0 \tag{3.52}$$

This same result was also obtained by the Maximum Principle approach. The necessary conditions involving the costate variables will not appear, since they are not present in the Green Theorem approach. A comparison of the necessary conditions for Problem 3.1 using both approaches is shown as follows:

Necessary condition for singular arcs for Problem 3.1

Maximum Principle Approach

NC1 $\quad p_1 b_1 + p_2 b_2 = 0$

NC2 $\quad 1 + p_1 f_1 + p_2 f_2 = 0$

NC3 $\quad \dot{p}_1 b_1 + \dot{p}_2 b_2 = 0$

NC4 $\quad p_1 \left(b_1 \dfrac{\partial f_1}{\partial x_1} + b_2 \dfrac{\partial f_1}{\partial x_2}\right) + p_2 \left(b_1 \dfrac{\partial f_2}{\partial x_1} + b_2 \dfrac{\partial f_2}{\partial x_2}\right) = 0$

Necessary condition for singular arcs for Problem 3.1

Green's Theorem Approach

NC1 $\quad b_1 b_2 \dfrac{\partial f_2}{\partial x_2} - b_2^2 \dfrac{\partial f_1}{\partial x_2} + b_1^2 \dfrac{\partial f_2}{\partial x_1} - b_1 b_2 \dfrac{\partial f_1}{\partial x_1} = 0$

It is evident that NC1 of the Green's Theorem Approach is obtainable by eliminating the costate variables from NC1 and NC4 for the Maximum Principle Approach. Hence the necessary conditions of the Maximum Principle Approach imply the necessary condition of the Green Theorem Approach.

3.2.2 Problem 3.3 Singular Pang-Pang Time Optimal Control

The previous problems involved controls that were bounded. In the next case the control signal will be rate limited in addition to being magnitude limited. Consider the following system equations,

$$\dot{x}_1(t) = f_1(x_1, x_2) \tag{3.53}$$

$$\dot{x}_2(t) = f_2(x_1, x_2) + g_2 u, \qquad 0 \leq |u| \leq 1 \tag{3.54}$$

$$\dot{x}_3(t) = \dot{u} = v \qquad |\dot{u}| \leq 1 \tag{3.55}$$

By letting $u = x_3$, the order of the state equations has been increased to three. Also x_3 may now be a bounded state variable. These complications are compensated for by having a control signal which is only magnitude limited. In a practical case to be considered later there is only rate limiting. This will be designated in Problem 3.3M and will be considered using both the Maximum Principle Approach and the Green Theorem Approach.

Since the general expression is long, the necessary conditions for singular control will be obtained by going to the basic equations directly. The process will be as before; to repeatedly differentiate until no further information is obtainable.

For the time optimal program we again introduce

$$\dot{x}_0 = 1 \qquad x_0(0) = 0 \tag{3.56}$$

The Hamiltonian is, again letting $p_0 = 1$,

$$H = 1 + p_1 f_1 + p_2 f_2 + p_2 g_2 x_3 + p_3 v \tag{3.57}$$

For the nonsingular trajectories, $v = -\text{sgn}\ p_3$ minimizes H. For the singular case, we obtain the first necessary condition

$$p_3 = 0 \qquad \text{(NC1)} \tag{3.58}$$

If we write the costate equations,

$$\dot{p}_0 = 0 \qquad p_0 = 1 \tag{3.59}$$

$$-\dot{p}_1 = p_1 \frac{\partial f_1}{\partial x_1} + p_2 \frac{\partial f_2}{\partial x_1} \tag{3.60}$$

$$-\dot{p}_2 = p_1 \frac{\partial f_1}{\partial x_2} + p_2 \frac{\partial f_2}{\partial x_2} \tag{3.61}$$

$$-\dot{p}_3 = p_2 g_2 \tag{3.62}$$

Necessary condition, $p_3 = 0$ on the singular arc, implies $\dot{p}_3 = 0$, or equivalently,

$$p_2 = 0, \qquad g_2 \neq 0 \qquad \text{(NC2)} \qquad (3.63)$$

This in turn implies, $\dot{p}_2 = 0$. Therefore, from equation (3.61),

$$p_1 \frac{\partial f_1}{\partial x_2} = 0 \qquad \text{(NC3)} \qquad (3.64)$$

Using equation (3.57), $p_2 = 0$ and $p_3 = 0$, we have on the singular arc,

$$H = 1 + p_1 f_1 \qquad (3.65)$$

Since $H = 0$ on the optimal trajectory,

$$p_1 = -\frac{1}{f_1} \qquad \text{(NC4)} \qquad (3.66)$$

Assuming $p \neq 0$, necessary condition 3 becomes

$$\frac{\partial f_1}{\partial x_2} = 0 \qquad \text{(NC3)} \qquad (3.67)$$

This was expected since the projection of the singular arc in the x_1-x_2 plane for this problem should be the same as Problem 3.2.

The value of the control function u on the singular arc is to be determined next.

Differentiating equation (3.65), with respect to time,

$$p_1 \left(\frac{\partial f_1}{\partial x_1} \dot{x}_1 + \frac{\partial f_1}{\partial x_2} \dot{x}_2\right) + \dot{p}_1 f_1 = 0 \qquad (3.68)$$

Substituting for $\dot{x}_1$, and $\dot{x}_2$ and $\dot{p}_1$,

$$p_1 \left(\frac{\partial f_1}{\partial x_1} f + \frac{\partial f_1}{\partial x_2} f_2 + \frac{\partial f_1}{\partial x_2} g_2 u\right) - \left(f_1 p_1 \frac{\partial f_1}{\partial x_1} + f_1 p_2 \frac{\partial f_2}{\partial x_1}\right) = 0 \qquad (3.69)$$

Using equation (3.71) and (3.63), 0=0.

Thus, no new information was obtained by that approach. Differentiate equation (3.67) with respect to time.

$$\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\dot{x}_1 + \frac{\partial^2 f_1}{\partial x_2^2}\dot{x}_2 = 0 \tag{3.70}$$

Substituting for $\dot{x}_1$ and $\dot{x}_2$,

$$\frac{\partial^2 f_1}{\partial x_1 \partial x_2} f_1 + \frac{\partial^2 f_1}{\partial x_2^2}(f_2 + g_2 u) = 0 \tag{3.71}$$

This equation may be solved for u,

$$u = -\frac{\dfrac{\partial^2 f_1}{\partial x_1 \partial x_2} f_1 + \dfrac{\partial^2 f_1}{\partial x_2^2} f_2}{g_2 \dfrac{\partial^2 f_1}{\partial x_2^2}} \tag{3.72}$$

The control signal u (t) for both the singular "bang-bang" case and the singular "pang-pang" control are identical on the singular arc. Assume that $u(t) = x_3(t)$ is in the interior of its allowed region, U. Hence, there is no magnitude limiting of u(t). Equivalently, the state variable x_3 is not bounded.

3.2.3 N-Dimensional Singular Control Problems

In the case of Problems where the dimension of the state space is greater than two, the Green's Theorem Approach must be extended. The extension of Green's Theorem from two to three dimensions is the traditional Stoke's Theorem. The generalization to higher dimensions is designated as both the generalized Green's Theorem [H1] and the generalized Stoke's Theorem. The developments are via exterior calculus and differential forms. A brief treatment of differential forms is given in Appendix III.

Consider the differential constraint equation,

$$\dot{x}_i(t) = f_i(x) + \sum_{j=1}^{r} b_{ij}(x)u_j(t) \qquad i = 1,2,\ldots,n \quad j \leq n-1 \tag{3.73}$$

or,

$$\dot{x}(t) = f(x) + B(x)u(t) \tag{3.74}$$

In order to transform equation (3.73) into the proper form for application of Green's Theorem, it is necessary to eliminate the $u_j(t)$. Since $j \leq n-1$, an n-dimensional vector, $\Psi(x)$, orthogonal to the columns of B can be found. Hence the inner product

$$(\Psi(x),\dot{x}(t)) = (\Psi(x),f(x)) \tag{3.75}$$

since

$$(\Psi(x),B(x)) = 0 \tag{3.76}$$

Equation (3.75) permits the determination of dt in the functional J[u], where

$$J[u] = \int_{t_0}^{t_f} f_0 dt \tag{3.77}$$

Hence,

$$J[u] = \int_{x_0}^{x_f} f_0 \frac{(\Psi(x),dx)}{(\Psi(x),f(x))} \tag{3.78}$$

In the notation and nomenclature forms, equation (3.77) may be written as

$$J[u] = \int_{\Gamma} \pi \tag{3.79}$$

where, π is the pfaffian or one-form

$$\pi = f_0 \frac{(\Psi(x),dx)}{(\Psi(x),f(x))} = \sum_{i}^{n} \alpha(x)dx_i \tag{3.80}$$

The generalized Green's Theorem is (see Appendix III)

$$\int_s d\pi = \int_\Gamma \pi \tag{3.81}$$

As in the two dimensional form of Green's Theorem, Γ is a curve from x_0 and x_f and s is a surface containing the points x_0 and x_f. The term $d\pi$ is called the exterior derivative of π and is the differential two-form defined as,

$$d\pi = \sum_{i,j=1}^{n} \frac{\partial \alpha i}{\partial x j} dx_j \wedge dx_i \tag{3.82}$$

The exterior multiplication sign $\wedge$ is often omitted. An alternate useful form may be obtained by using the rules from differential form theory.

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \tag{3.83}$$

and

$$dx_i \wedge dx_i = 0 \tag{3.84}$$

then

$$d\pi = \omega_{ij} dx_i dx_j \qquad i=1,\dots,n-1 \qquad j=(i+1),\dots,n \tag{3.85}$$

where

$$\omega_{ij} = \frac{\partial \alpha_i}{\partial x_j} - \frac{\partial \alpha_j}{\partial x_i} \tag{3.86}$$

In a 3-dimensional case, the exterior derivative given by (3.80) would be written as

$$d\pi = \omega_{12} dx_1 dx_2 + \omega_{13} dx_1 dx_3 + \omega_{23} dx_2 dx_3 \tag{3.87}$$

and

$$\omega_{12} = \frac{\partial \alpha_1}{\partial x_2} - \frac{\partial \alpha_2}{\partial x_1} \tag{3.88}$$

$$\omega_{13} = \frac{\partial \alpha_1}{\partial x_3} - \frac{\partial \alpha_3}{\partial x_1} \tag{3.89}$$

$$\omega_{23} = \frac{\partial\alpha_2}{\partial x_3} - \frac{\partial\alpha_3}{\partial x_2} \tag{3.90}$$

These will be useful in the 3-dimensional case to be considered later.

The procedure for applying Green's Theorem to n-dimensional may be summarized as follows:

1) Convert the functional to be minimized to the line integral form by equation (3.79).
2) Use equation (3.86) and $\omega_{ij} = 0$ to determine singular hypersurfaces. There are no more than (n-1) independent hypersurfaces. The intersection of hypersurfaces, if it exists, is a singular arc.
3) Compare trajectories by using the generalized Green's Theorem as given by equation (3.81). The possibilities of singular arcs must be investigated.

3.3 The Second Variation Approach

The necessary condition of Kelly, Kopp, and Moyer will be applied in Problem 3.1, the singular "bang-bang" time optimal control problem. Using equations (2.82), (2.83) and (2.84)

$$A_{1,1} = \frac{\partial f_1}{\partial u} = b_1 \tag{3.91}$$

$$A_{2,1} = \frac{\partial f_2}{\partial u} = b_2 \tag{3.92}$$

$$A_{1,2} = \frac{\partial f_1}{\partial x_1}A_{1,1} + \frac{\partial f_1}{\partial x_2}A_{2,1} - \dot{A}_{1,1} \tag{3.93}$$

$$A_{2,2} = \frac{\partial f_2}{\partial x_1}A_{1,1} + \frac{\partial f_2}{\partial x_2}A_{2,1} - \dot{A}_{2,1} \tag{3.94}$$

Simplifying equations (3.92) and (3.93)

$$A_{1,2} = b_1\frac{\partial f_1}{\partial x_1} + b_2\frac{\partial f_2}{\partial x_2} \tag{3.95}$$

$$A_{2,2} = b_1\frac{\partial f_2}{\partial x_1} = b_2\frac{\partial f_2}{\partial x} \tag{3.96}$$

The necessary condition for minimality is, using equation (2.85)

$$\frac{\partial^2 H}{\partial x_1^2}A_{1,1}{}^2 + \frac{\partial^2 H}{\partial x_1 \partial x_2}(A_{1,1}A_{1,2} + A_{2,1}A_{1,1}) + \frac{\partial^2 H}{\partial x_2^2}A_{2,1}{}^2 \geq 0 \tag{3.97}$$

Substituting equations (3.95) and (3.96)

$$b_1{}^2\frac{\partial^2 H}{\partial x_1{}^2} + 2b_1b_2\frac{\partial^2 H}{\partial x_1 \partial x_2} + b_2{}^2\frac{\partial^2 H}{\partial x_2{}^2} \geq 0 \tag{3.98}$$

The Hamiltonian for Problem 3.1 was

$$H = 1 + p_1f_1 + p_1b_1u + p_2f_2 + p_2b_2u \tag{3.99}$$

Substituting partial derivatives of H into equation (3.98)

$$b_1{}^2(p_1\frac{\partial^2 f_1}{\partial x_1{}^2} + p_2\frac{\partial^2 f_2}{\partial x_1{}^2}) + 2b_1b_2(p_1\frac{\partial^2 f_1}{\partial x_1 \partial x_2} + p_2\frac{\partial^2 f_2}{\partial x_1 \partial x_2}) +$$

$$b_2{}^2(p_1\frac{\partial^2 f_1}{\partial x_2{}^2} + p_2\frac{\partial^2 f_2}{\partial x_2{}^2}) \geq 0 \tag{3.100}$$

This is one form of the necessary condition for minimality along the singular arc. Using the alternate form,

$$\frac{\partial}{\partial u}(\frac{d^2}{dt^2}\frac{\partial H}{\partial u}) \leq 0 \tag{3.101}$$

since

$$\frac{\partial H}{\partial u} = p_1b_1 + p_2b_2 \tag{3.102}$$

The first necessary condition is obtained from equation (3.102) and

$$p_1 b_1 + p_2 b_2 = 0 \tag{3.103}$$

and

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = b_1 \frac{d^2 p_1}{dt^2} + b_2 \frac{d^2 p_2}{dt^2} \tag{3.104}$$

then substituting expressions for $\dot{p}_1$ and $\dot{p}_2$,

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = - b_1 \frac{d}{dt}\left(p_1 \frac{\partial f_1}{\partial x_1} + p_2 \frac{\partial f_2}{\partial x_1}\right) - b_2 \frac{d}{dt}\left(p_1 \frac{\partial f_1}{\partial x_2} + p_2 \frac{\partial f_2}{\partial x_2}\right) \tag{3.105}$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial u} \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = \frac{\partial}{\partial u} \quad & - b_1 p_1 \left[\frac{\partial^2 f_1}{\partial x_1{}^2}(f_1 + b_1 u) + \frac{\partial^2 f_1}{\partial x_1 \partial x_2}(f_2 + b_2 u)\right] - \\ & b_1 p_2 \left[\frac{\partial^2 f_2}{\partial x_1{}^2}(f_1 + b_1 u) + \frac{\partial^2 f_2}{\partial x_1 \partial x^2}(f_2 + b_2 u)\right] - \\ & b_2 p_1 \left[\frac{\partial^2 f_1}{\partial x_1 \partial x_2}(f_1 + b_1 u) + \frac{\partial^2 f_1}{\partial x_2{}^2}(p_2 + b_2 u)\right] - \\ & b_2 p_2 \left[\frac{\partial^2 f_2}{\partial x_1 \partial x_2}(f_1 + b_1 u) + \frac{\partial^2 f_2}{\partial x_2{}^2}(f_2 + b_2 u)\right]\end{aligned} \tag{3.106}$$

After taking the partial derivative with respect to u, and using equation (3.101), equation (3.100) is obtained. Although the form $\frac{\partial}{\partial u}\left(\frac{d^2}{dt^2} \frac{\partial H}{\partial u}\right)$ is quite compact it nevertheless involves quite a few manipulations. If as indicated previously, the equality applies, it is then necessary to proceed further using equation (2.25).

Using the somewhat simpler Problem 3.2, the necessary condition for minimality along the singular arc is obtained by substituting $b_1 = 0$ in equation (3.99) and (3.103), then

$$p_2 = 0 \qquad \text{NC1} \qquad (3.107)$$

$$p_1 \frac{\partial^2 f_1}{\partial {x_2}^2} \quad 0 \qquad \text{NC2} \qquad (3.108)$$

Using H = 0, equation (26) gives

$$p_1 = -\frac{1}{f_1} \qquad \text{NC3} \qquad (3.109)$$

Hence, NC2 can be written as

$$\frac{1}{f_1} \frac{\partial^2 f_1}{\partial {x_2}^2} \leq 0 \qquad \text{NC2'} \qquad (3.110)$$

CHAPTER IV OPTIMAL CONTROL OF THE VEHICULAR BRAKING PROCESS

4.1 Introduction

The vehicular braking process to be considered first will consist of a single wheel carrying a body on a flat horizontal surface.

Figure 4.1 and 4.2 show the model used and the pertinent parameters.

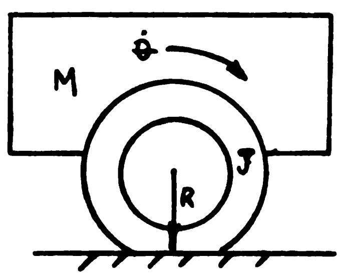


Figure 4.1

Model of One-Wheel System

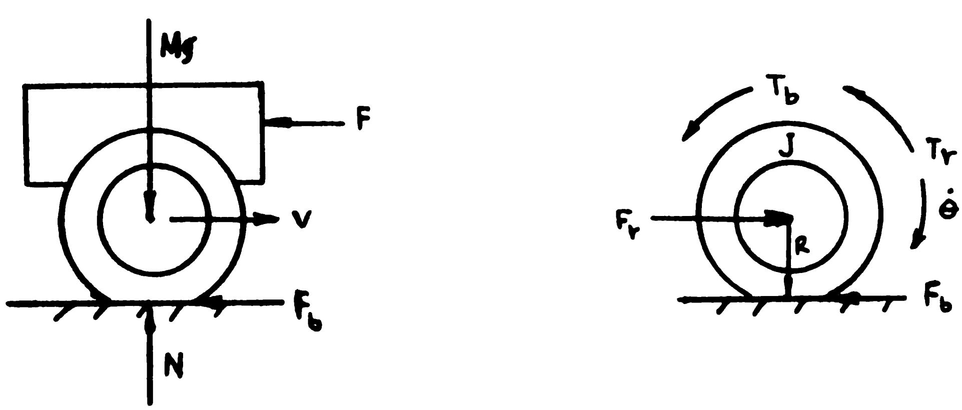


Figure 4.2

Force and Torque Diagrams

The symbols in the figures are defined as follows:

F_b	braking force developed at the tire-surface interface
F_e	external force on the vehicle
F_r	reaction force between the body and axle
g	acceleration due to gravity
J	Polar mass moment of inertia of the wheel and associated rotating members
M	mass of the vehicle
N	normal force at the tire-surface interface
R	rolling radius of the wheel
T_b	torque exerted on the wheel by the brake
T_r	rolling resistance and bearing friction torque
v	vehicle velocity
μ	coefficient of friction
η	slip
$\dot{\theta}$	angular velocity of wheel

As the vehicle moves with velocity v, the wheel runs under slip as it transmits driving, braking, or cornering forces to the surface. Slip is defined as the ratio of effective slip velocity in a specified direction to the forward ground speed of the vehicle. Since braking will be the main concern, slip during braking is defined as

$$\eta = \frac{v-R\dot{\theta}}{v} \quad \text{and } 0 \leq \eta \leq 1 \tag{4.1}$$

The so-called "panic stop" usually results in a slip of 1.0, corresponding to zero wheel velocity. The braking force F_b developed at the tire surface interface is due to the friction coefficient μ and is defined as

$$F_b = \mu N \tag{4.2}$$

where N is the normal force at the tire-surface interface.

Investigators in the area of tire friction [K13] [N6], have found that the friction characteristic depends on factors such as vehicle velocity, normal load, tire tread pattern, tire inflation pressure, tire temperature, and surface composition. The friction-slip model used in this study is shown in figure 4.3

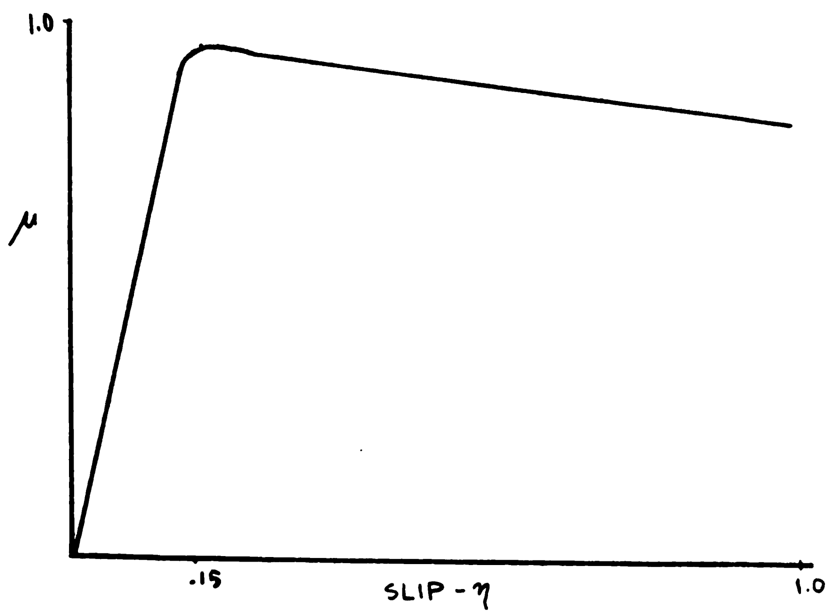


Figure 4.3

Model of Friction-Slip Characteristic

The selection of this model is based on investigations [F6], [K14], which show that regardless of surface composition, the friction coefficient μ, usually has a peak value and this peak value occurs when the slip is in a range about the 0.15 point.

The optimal control for the one wheel model will use minimum time as the optimum criterion. As will be shown in section 4.3.3 this control will also minimize stopping distance. Optimal control theory will show that the control should bring the state of the system to the peak of friction-slip curve and then keep it there. It should be noted that due to the low value of slip, the wheel velocity will be an appreciable fraction of the vehicle velocity. This will also benefit more complex models such as two wheel and four wheel models which are concerned with lateral stability. When one tire of the vehicle is subjected to a different friction characteristic than the opposite tire, a torque tending to rotate or spin the vehicle is developed. When the wheels of the vehicle are rotating, the tendency to spin is reduced and the vehicle has more lateral stability. This problem will be considered in more detail later.

4.2 Development of the System Equations

The differential constraint equations for the optimal control problem are obtained by referring to Figure 4.2.

The normal force N is obtained from the summation of the vertical forces,

$$N = Mg \tag{4.3}$$

The horizontal forces are summed, obtaining,

$$F_b + F_e = -M \frac{dv}{dt} \tag{4.4}$$

The external force F_e is neglible with respect to the braking force. Then using the relation (4.2), equation (4.4) becomes

$$M \frac{dv}{dt} = -\mu Mg \tag{4.5}$$

or

$$\dot{v} = -\mu g \tag{4.6}$$

Now considering the torques associated with the wheel,

$$F_b R - T_b - T_r = J \frac{d\dot{\theta}}{dt} \tag{4.7}$$

Here the rolling resistance and bearing friction torque are assumed negligible with respect to the brake torque. To make the units of wheel velocity the same as the units of vehicle velocity equation (4.7) is written as

$$R \frac{d\dot{\theta}}{dt} = (\mu MgR^2 - RKP_b)/J \tag{4.8}$$

In the above equation F_b is replaced by μMg and brake torque is assumed to be linearly related to brake pressure P_b, i.e.,

$$T_b = KP_b \tag{4.9}$$

Typical values for an equivalent one wheel model are

$R = 1.1$ ft.

$Mg = 5000$ lbs.

$J = 5.0$ ft-lbs/sec.^{2}

$K = 6.0$ ft-lbs/p.s.i.

Letting

$$x_1 = v \tag{4.10}$$

$$x_2 = R\Theta \tag{4.11}$$

Here the first state vector corresponds to vehicle velocities and the second state vector corresponds to wheel velocity. The differential constraint equation then becomes

$$\dot{x}_1 = -32\mu \tag{4.12}$$

$$\dot{x}_2 = 1210\mu - 1.32P_b \tag{4.13}$$

Constraining the brake pressure to $0 \leq P_b \leq 1200$ psi and introducing the control u, $0 \leq u \leq 1$ equation (4.12) becomes

$$\dot{x}_2 = 1210\mu - 1584u \tag{4.14}$$

The friction coefficient μ is a function of x_1 and x_2 which will be designated state variables.

$$\mu = \mu(x_1, x_2) \tag{4.15}$$

As indicated previously, the friction-slip characteristic has the general shape as shown in Figure 4.3.

This shape will be generated by

$$\mu(x_1, x_2) = \mu_o \, [\varepsilon^{-a(1-x_2/x_1)} - \varepsilon^{-b(1-x_2/x_1)}] \tag{4.16}$$

Recall that the slip is given by

$$\eta = 1 - x_2/x_1 \tag{4.17}$$

The factor μ_o will take into account the surface-tire interface. For example, $\mu_o = 1$ will correspond an interface having the highest friction coefficient such as concrete, while a low friction surface such as ice may have a value of $\mu_o = .06$. This function will have a peak of approximately 0.947 μ_o at a slip of 0.2 when a = 0.225 and b = 23.5. These are values that are used in most of the later computations.

4.3.1 Optimization of the One-Wheel Vehicular Braking Control Model

The problem to be considered is the time optimal regular problem, i.e., take the vehicle from an initial state to the origin in minimum time, subject to the constraints

$$\dot{x}_1 = -32\mu(x_1, x_2) \qquad x_1(0) = 60. \tag{4.18}$$

$$\dot{x}_2 = 1210\mu(x_1, x_2) - 1584u \qquad x_2(0) = 60. \tag{4.19}$$

$$0 \leq u \leq 1 \tag{4.20}$$

The function $\mu(x_1,x_2)$ is given by equation (4.15) with $\mu_0 + 1$, $a = .225$, and $b = 23.5$. This problem when stated in real physical terms is: Find the control pressure P(t), constrained to be between zero and 1200 psi, which will stop a given vehicle initially travelling at 60 ft/sec (approximately 40 MPH) in minimum time. It should be noted that there is no limitation on pressure rate in this case. This problem will be labeled Problem 4.1 and will be called the singular "bang-bang" control problem.

This problem fits the format of Problem 3.2, where

$$f_1 = -32\mu\ (x_1,x_2) \tag{4.21}$$

$$f_2 = 1210\mu\ (x_1,x_2) \tag{4.22}$$

$$\mu = \varepsilon^{-a(1-x_2/x_1)} - \varepsilon^{-b(1-x_2/x_1)} \tag{4.23}$$

Any of the methods in Chapter III provide the necessary condition for a singular arc

$$\frac{\partial\mu}{\partial x_2} = 0 \tag{4.24}$$

or,

$$\omega = \frac{a}{x_1}\varepsilon^{-a(1-x_2/x_1)} - \frac{b}{x_1}\varepsilon^{-b(1-x_2/x_1)} = 0 \tag{4.25}$$

Solution of this equation gives an equation relating x_1 and x_2 on the singular arc. For the values of the constants used, a linear relation exists,

$$x_2 = kx_1 \qquad k \text{ is approximately } 0.8 \tag{4.26}$$

This problem is most readily solved using the Green Theorem Approach. The regions which are reachable and controllable are found and their intersection provides the region containing all permissible trajectories from the initial point x_0 to the final point x_f. In order to find the region containing all the trajectories, Theorem 2.1 is used. If the conditions of Theorem 2.1 are satisfied then the region containing all trajectories from x_o to x_f are bounded by the four trajectories obtained by solving (4.19) and (4.18) as follows:

a) starting at x_o, use u=1 and obtain ϕ^1.

b) starting at x_o, use $u=0^+$ and obtain ϕ_o^{o+}.

c) starting x_f, integrate backwards (reverse time) use u=1, obtain ϕ_f^1.

d) starting at x_f, integrate backwards use u>0, obtain ϕ_f^{o+}.

To satisfy the conditions of Theorem 2.1, it was necessary to use u>0 in (b) since u=0 leads to $\dot{x}_1=0$ and $\dot{x}_2=0$ and the Δ condition of Theorem 2.1 is not satisfied. Also it is convenient in this study to use $0 \leq u \leq 1$ instead of $|u|\leq|1$. Theorem 2.1 is applicable in either case, since it is only necessary to change (4.19) to

$$\dot{x}_2 = 1210\ \mu(x_1,x_2) - 752 - 752u \qquad |u|\leq 1 \qquad (4.27)$$

The results appear in Figure 4.4

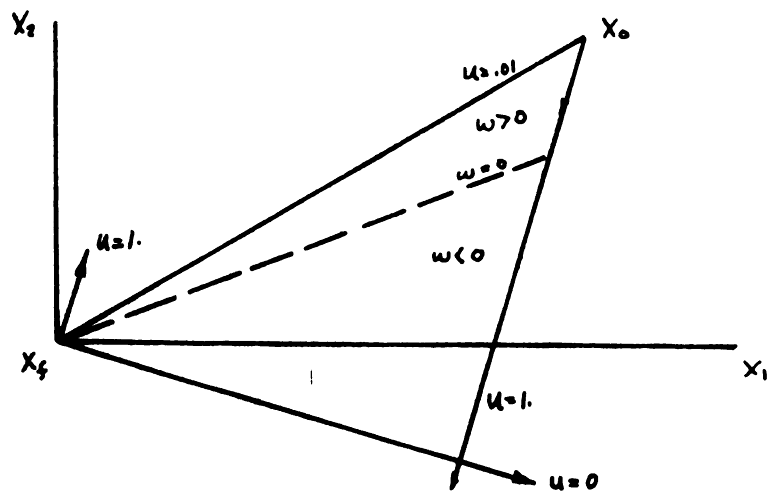


Figure 4.4

Reachable and Controllable Regions

The singular arc, $\omega=0$, obtained from the necessary condition is also shown in Figure 4.1. We are now ready to apply the Green Theorem Approach. In this problem, ω is positive in the region above the curve $\omega=0$, which is a straight line in this case.

The global optimal trajectory is obtained as follows:

a) Start at x_o, keeping the region $\omega<0$ on the right, proceed to the point where the trajectory intersects the $\omega=0$ line.

b) Now proceed along the $\omega=0$ line until x_f is reached.

Note that along the $\omega=0$, the region $\omega>0$ is on the right and the region $\omega<0$ is on the left. Also, if the final point x_f was not on the curve $\omega=0$ the procedure would be the same except that a boundary of the region containing the admissible trajectories would be reached before x_f is reached. Traversing the boundary keeping $\omega>0$ on the right or $\omega<0$ on the left would then ultimately terminate the trajectory at x_f. Figure 4.5 shows a case where $x_f=(a,0)$. The optimal trajectory is x_o-a-b-x_f.

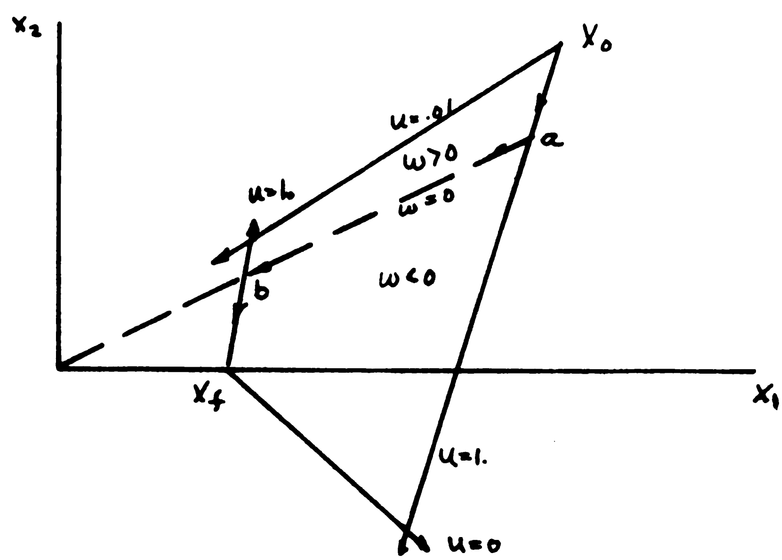


Figure 4.5

Optimal Trajectory

Having found the optimal trajectory, it is now possible to describe the Optimal Control. The boundaries establish the "bang-bang" values of u while the value of control on the singular arc is determined from (3.3.) and (3.34). The approximate value of u=.735. Figure 4.6 shows the Optimum Control function as a function of time.

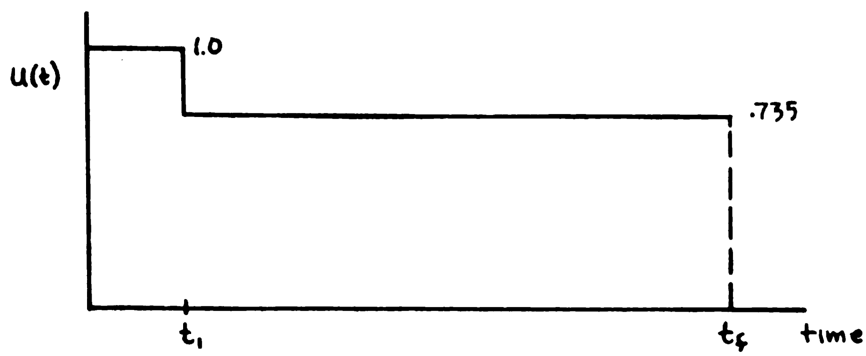


Figure 4.6

Optimal Control for Vehicular Braking System

Constraint on Control Signal

A program P2 was written to automate the above procedures. This program does the following:

1) scans the ω region of all permissible trajcctories
2) finds the proper boundary
3) finds the singular arc
4) finds the optimal control on all parts of the traiectory including the singular arc

The listing of P2 and a typical output is shown in Appendix II.

4.3.2 Analytical Verification of the Optimal Control

In the previous section, the Green's Theorem Approach showed that the optimal control is piecewise constant. As shown in Figure 4.3, maximim control of u=1 is applied for time t_1; then, reduced control is applied until the state vectors both reach zero. Recall that the two state vectors correspond to vehicle velocity and wheel velocity.

In this case, a very simple solution is possible if the constraint on u(t) is removed. The optimal control will consist of an impulse at t=0 and then a value of less than 1.0 for the remaining time. See Figure 4.7

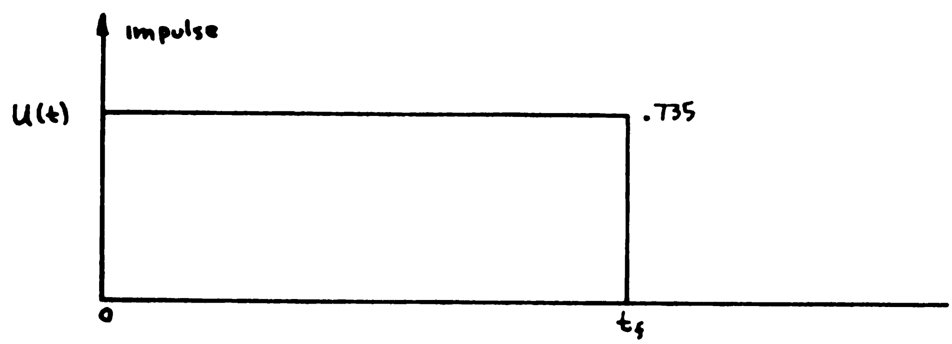


Figure 4.7

Optimal Control for Vehicular Braking System
No Constraint on Control Signal

The impulse of Figure 4.7 drives the system to the peak of the friction curve in zero time while the pulse of Figure 4.6 drives the system to the peak of the μ-curve in minimum time t_1. As will be shown t_1 is much less than t_f, the time to drive the system to zero. Hence, the impulse method leads to negligible error.

From (4.23), the maximum value of the friction coefficient μ is

0.947, corresponding to $x_2/x_1=0.8$.

From (4.18), if an impulse is applied at t=0, no change in x_1 takes place.

Applying an impulse of strength δ and using $x_2=0.8x_1$, (4.18) becomes,

$$\int_{60}^{48} dx_2 = -1584 \int_{o}^{o+} \delta dt \tag{4.28}$$

The strength of the impulse is

$$\delta = .00757 \tag{4.29}$$

If it is assumed that the area of the pulse of Figure 4.3 is equal to the strength of the impulse, then time t_1 would be equal to .00757 seconds. This also assumes that u is limited to 1.0. Although this is not accurate, it is sufficient to show that t_1 is much less than t_f.

From (4.18) and using $\mu=0.947$,

$$\int_{60}^{o} dx_1 = -32 \times .947 \int_{o}^{t_f} dt \tag{4.30}$$

Solving for t_f,

$$t_f = 1.98 \text{ seconds} \tag{4.31}$$

Corresponding to this minimum stopping time, the minimum stopping distance is 60 feet.

The optimal control signal during this interval may be found by eliminating μ from (4.19) and (4.18).

Integrating the resulting equation,

$$\int_{48}^{o} dx_2 = -37.81 \int_{60}^{o} dx_1 - 1584 \int_{o}^{1.98} u_2 dt \tag{4.32}$$

Solving,

$$u_2 = 0.735 \tag{4.33}$$

The impulse solution will differ only from the actual solution because of t_1. By assuming $\mu = t/t_1$ during interval t_1, equation (4.19) may be solved to yield,

$$t_1 = .0123 \text{ seconds} \tag{4.34}$$

This agrees with the digital computer solution shown in Appendix II. The results of the analytical approach are summarized in Figure 4.8

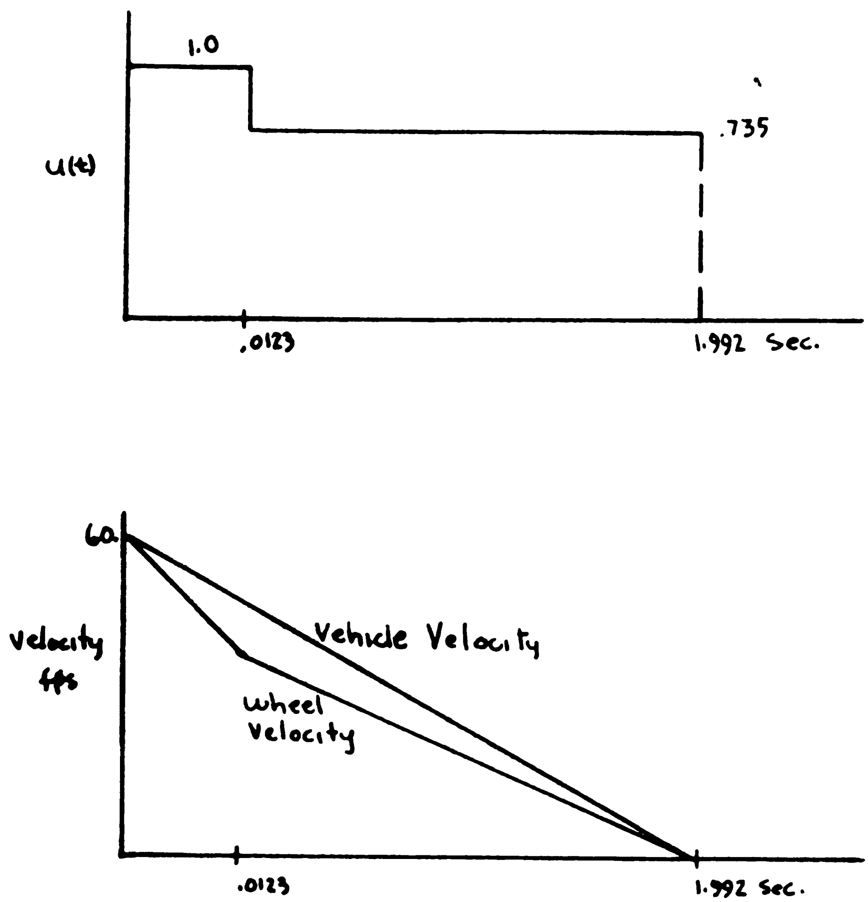


Figure 4.8

Optimal Control and State Vectors vs Time

4.3.3 Equivalence of Minimum Time and Minimum Stopping Distance Criteria

In this section it will be shown that minimizing the time to go from x_o to x_f is equivalent to minimizing the vehicle stopping distance.

It has been shown that the intersection of the reachable region $R(x_o)$ and the controllable region $R(x_f)$ contains the set of all possible trajectories from x_o to x_f. Also, the construction of this set is independent of the performance criteria imposed by the functional $J[u]$.

Using the Green Theorem Approach, it has been shown that the optimal trajectory is contained in the boundary of this set unless singular arcs exist. Then the optimal trajectory will contain portions of the boundary and portions of the singular arc. Hence, the solution is no longer "bang-bang". Two criteria will be equivalent if the singular arcs generated by these criteria are the same.

Consider

$$J_1[u] = \int_{t_o}^{t_f} dt \tag{4.35}$$

$$J_2[u] = \int_{t_o}^{t_f} x_1 dt \tag{4.36}$$

Since x_1 corresponds to vehicle velocity in problem(4.1), $J_2[u]$ is the stopping distance while $J_1[u]$ corresponds to the time required to stop.

To show that these criteria are equivalent, it will be shown that the critical function, $\omega = 0$, which determines the singular arc is the same for both criteria.

From equation (4.18), determine dt and substitute in

$$J_1[u] = \int_{x_0}^{x_f} \frac{dx_1}{-32\mu(x_1,x_2)} \tag{4.37}$$

$$J_2[u] = \int_{x_0}^{x_f} \frac{x\ dx}{-32_\mu(x_1,x_2)} \tag{4.38}$$

Since, the general form for the cost function is

$$J = \int Pdx_1 + Qdx_2 = \iint (\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2})\ dx_1 dx_2 \tag{4.39}$$

and

$$P_1 = -\frac{1}{32\mu(x_1,x_2)} \qquad Q_1 = 0 \tag{4.40}$$

$$P_2 = -\frac{x}{32\mu(x_1,x_2)} \qquad Q_2 = 0 \tag{4.41}$$

Solving for the partial derivatives

$$\frac{\partial P_1}{\partial x_2} = \frac{\partial_\mu(x_1,x_2)}{\partial x_2} \frac{1}{32_\mu{}^2} \tag{4.42}$$

$$\frac{\partial P_2}{\partial x_2} = \frac{x_1\partial_\mu(x_1,x_2)}{\partial x_2} \frac{1}{32_\mu{}^2} \tag{4.43}$$

We are interested in the condition $\omega = 0$, where

$$\omega = (\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}) \tag{4.44}$$

If $x_1 \neq 0$ and $\mu \neq 0$, then referring back to equation (4.16)

$$\omega_1 = \omega_2 = \frac{\partial \mu(x_1,x_2)}{\partial x_2} \tag{4.45}$$

Typically, as indicated in Figure 4.1, the $x \neq 0$ and $\mu \neq 0$ conditions are satisfied except at the origin. The origin presents no problem since $\mu(0,0) = 0$.

Therefore minimizing stopping time is equivalent to minimizing stopping distance in Problem 4.1.

Any criteria which results in $P = \dfrac{f(x_1)}{\mu(x_1,x_2)}$ would be equivalent since the partial derivative is taken with respect to x .

4.3.4 An Optimal Digital Control System

In principle, the implementation of the optimal control is straight forward. Based on the Green's Theorem Approach the steps are as follows:

1) Apply maximum permissible control.
2) Continually solve for ω. The condition $\omega=0$ indicates that the singular arc has been reached.
3) Reduce the control signal u(t) in order to hold the condition, $\omega=0$.

Due to time delays, the desired control u(t) is not obtained in zero time. As a result the control oscillates about the predicted value of 0.735.

The program shown in Appendix II shows this variation in u(t). The state vectors corresponding to vehicle and wheel velocity are essentially ideal in the digital computer system.

The Digital Optimal Control System in block form appears in Figure 4.9

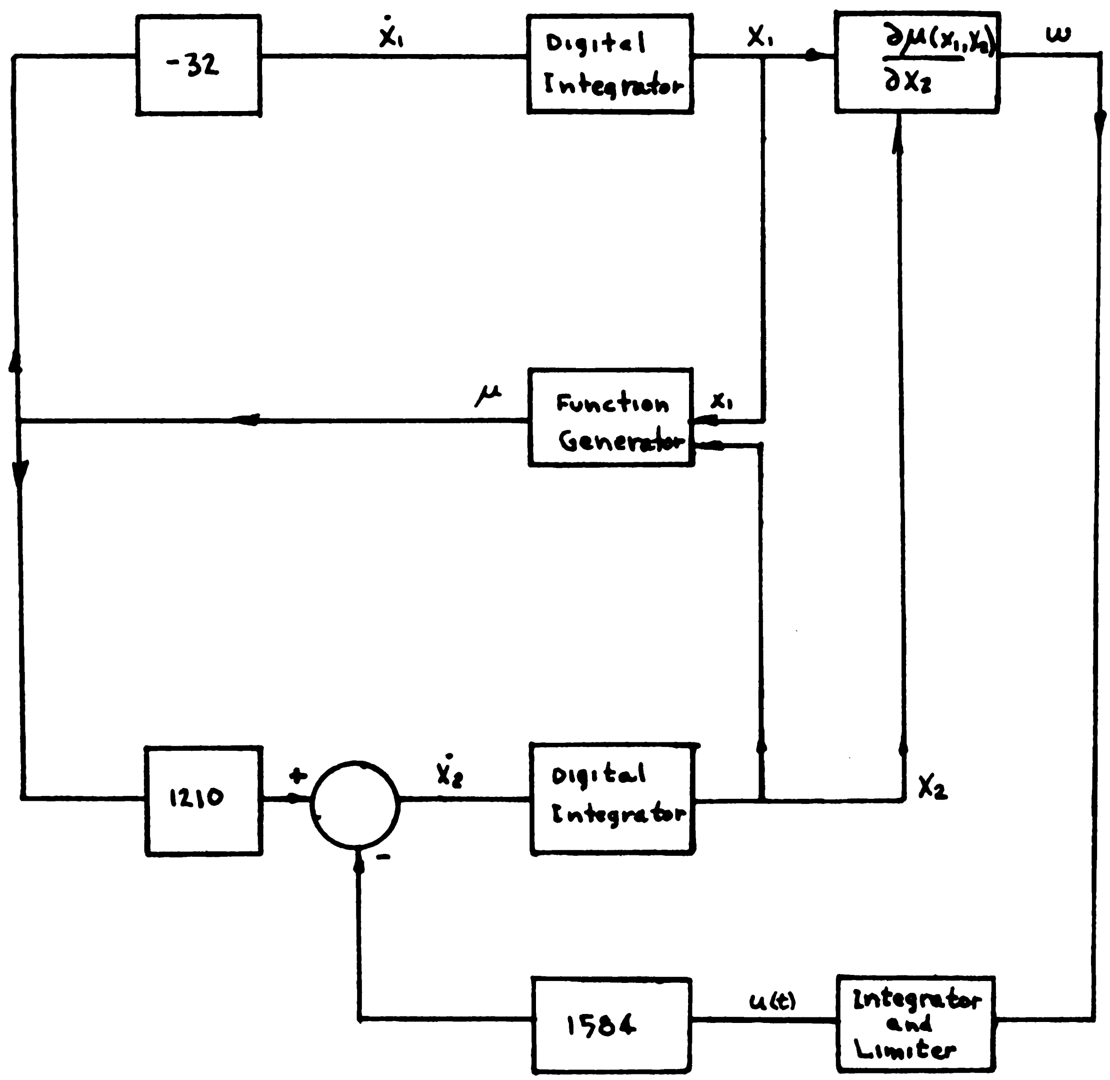


Figure 4.9

Digital Optimal Control System

Based on Green's Theorem Approach

The significant blocks are the blocks which determine the control signal u(t). If ω is positive the digital integration increases u unless u is at its limiting value of 1.0. When ω is zero, u would remain constant except for the fact that time delays cause ω to overshoot. In the digital solution, u changes by 0.01 per integration time interval of 0.000002. The results of this simulation can be summarized by Figure 4.10. The velocity signals which are the state variables x_1 and x_2 are close to the ideal values. There is a ripple frequency of approximately 300cps. This is a function of the digital integration gain. The peak wheel velocity ripple is approximately 0.020. The ω signal has a peak value of approximately 0.012, while the peak value of the Δu signal is 0.25.

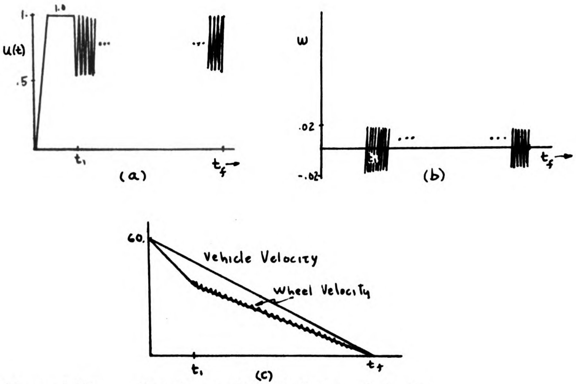


Figure 4.10 Wave Forms of Digital Control System

(a) Control Signal (b) Singular Function (c) Wheel and Vehicle Velocity

A sinusoidal analysis based on small signal follows. Consider simplified diagram,

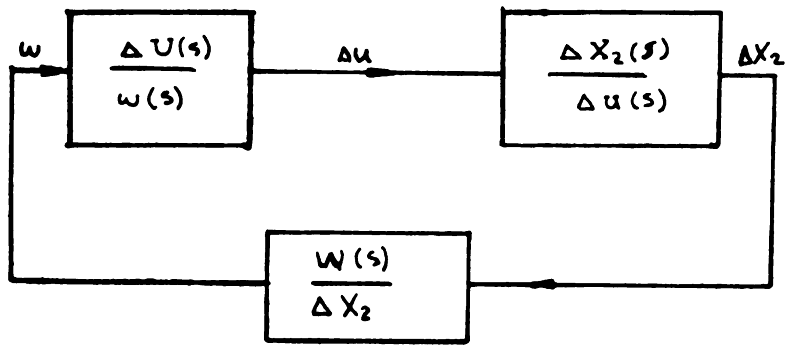


Figure 4.11

Diagram for Small Signal Sinusoidal Analysis

The velocities x_1 and x_2 tend toward zero slowly when measured on the ripple frequency time scale. Hence, at a given point (x_1, x_2) the following equations may be considered to apply.

$$x_1 = x_{10} \tag{4.46}$$

$$x_2 = x_{20} + \ x\,\Delta\sin\omega t \tag{4.47}$$

also,

$$x_{20} = 0.8x_{10} \tag{4.48}$$

Since ω is given by 4.25, and letting $k = \Delta x_2/x_{10}$

$$\omega = \frac{1}{x_{10}}\left(a\varepsilon^{-a(1-.8-k\sin\omega t)} - b\varepsilon^{-b(1-.8-k\sin\omega t)}\right) \tag{4.49}$$

Assuming that $k<<.2$, and using $\varepsilon^{x} \approx 1 + x$,

$$\omega = \frac{1}{x_{10}}\left(a\varepsilon^{-.2a}(1 + ak\sin\omega t) - b\varepsilon^{-.2b}(1 + bk\sin\omega t)\right) \tag{4.50}$$

Since $a\varepsilon^{-.2a} = b\varepsilon^{-.2b}$

$$\omega = (ak-bk)\ \sin \omega t/x_{10} \tag{4.51}$$

$$\omega = -5.25\ k\ \sin \omega t/x_{10} \tag{4.52}$$

$$\omega = -5.25\ \Delta x_2\ \sin \omega t/x_{10}^{2} \tag{4.53}$$

Using (4.19), a relation between Δx_2 and Δu can be found. The μ term has a constant term which cancels the steady state term of u. The sinusoidal variation in μ is small relative to the cosinusoidal variation of x_2. Therefore,

$$u \simeq u_0 + \Delta u \cos \omega t \tag{4.54}$$

and

$$2\pi f \Delta x_2 \cos 2\pi ft = -1584\ \Delta u \cos 2\pi ft \tag{4.55}$$

This yields the relationship,

$$2\pi f x_{10}^{2}/5.25 = 1584 \Delta u \tag{4.56}$$

or

$$\omega = 1320\ \Delta u/x_{10}^{2} f \tag{4.57}$$

The use of the above equation in conjunction with Figure 4.11, permits determination of the small signal variations in the system.

4.4.1 The One-Wheel Vehicular Braking Control Model with Rate and Amplitude Limited Control

This problem is an extension of Problem 4.1 and fits the format of Problem 3.3. It will be designated Problem 4.2.

The differential constraint equations are

$$\dot{x}_1 = -32\mu(x_1,x_2) \qquad x_1\ (0) = 60 \tag{4.58}$$

$$\dot{x}_2 = 1210\mu(x_1,x_2) - 1584x_3 \qquad x_2\ (0) = 60 \tag{4.59}$$

$$\dot{x}_3 = v \qquad x_3\ (0) = 0 \tag{4.60}$$

$$|\ v\ | \leq 1 \tag{4.61}$$

$$x_3 \geq 0 \tag{4.62}$$

The function $\mu(x_1,x_2)$ is given in Problem 4.1. The control u(t) of Problem 4.1 has been made a state variable with a constraint.

It is desired to minimize the time necessary to drive the state from x_0 to x_f. The row vector x_f is [0 0 x_{3f} is not specified.

This problem when stated in real physical terms is: Find the control pressure as a function of time which will stop a given vehicle initially traveling at 60 ft./sec. in minimum time. The pressure is rate limited to 12000 psi/sec.

In the physical problem under consideration the pressure would normally be amplitude limited (e.g.$0 \leq P(t) \leq$ 1200psi). This would result in a singular problem which has a bounded state variable (x_3). In this problem the upper and lower bounds are not penetrated by the optimal control and hence the additional complication due to bounded state variables is not encountered. The approach necessary when bounded state variables occur will not be considered at this time.

The development of section 3.2.3 will be applied to Problem 4.2 of section 4.4.4. As may be seen from equations (4.58) to (4.60), the state vector is 3-dimensional while the control in 1-dimensional. Hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v \tag{4.63}$$

also

$$J]u] = \int_{t_o}^{t_f} dt \tag{4.64}$$

Since there are two independent vectors orthogonal to $b = [0\ 0\ 1]^T$, dt may be expressed as

$$dt = dx_1/f_1 = dx_1/-32\mu \tag{4.65}$$

or

$$dt = dx_2/f_2 = dx_2/(1210\mu - 1584x_3 \tag{4.66}$$

Applying equations (3.80) and (3.88) - (3.90) to (4.65)

$$a_1 = -1/32\mu \tag{4.67}$$

$$\omega_{12} = \partial(-1/32\mu)/\partial x_2 \tag{4.68}$$

$\omega_{12} = 0$ implies a singular surface exists and is given by

$$\partial\mu(x_1,x_2)/\partial x_2 = 0 \tag{4.69}$$

This surface will contain part of the optimal trajectory as shown in Figure 4.13. It corresponds to the arc B-C. Its projections in the x_1-x_2 plane is D-0 and is the same as in Problem 4.1. Applying (3.80) and (3.88) - (3.90) to (4.66),

$$a_2 = \frac{1}{1210\mu-1584x_3} \tag{4.70}$$

Now

$$\omega_{12} = -\frac{\partial}{\partial x_1}\left(\frac{1}{1210\mu-1584x_3}\right) \tag{4.71}$$

$$\omega_{13} = 0 \tag{4.72}$$

$$\omega_{23} = \frac{\partial}{\partial x_3}\left(\frac{1}{1210\mu-1584x_3}\right) \tag{4.73}$$

Only $\omega_{12} = 0$ implies a singular surface.

$$\frac{\partial\mu}{\partial x_1} = -\frac{x_2}{x_1}\frac{\partial\mu}{\partial x_2} \tag{4.74}$$

If $x_1 \neq 0$, this produces the same singular condition given by (4.69).

The net result is that two equivalent forms are available to evaluate J[u].

$$J[u] = \int_{x_0}^{x_f} \frac{dx_1}{-32\mu(x_1,x_2)} \tag{4.75}$$

and

$$J[u] = \int_{x_0}^{x_f} \frac{dx_2}{1210\mu(x_1,x_2)-1584x_3} \tag{4.76}$$

4.4.2 Solution of the Pang-Pang Singular Control Problem by the Green Theorem Approach

Normally problems of the type formulated in the previous section result in the bang-bang behavior of $\dot{u}(t)$. This is referred to pang-pang operation with respect to u(t).

Several modifications with respect to the procedure used in Problem 4.2 will be necessary. First, no theorem is available for the general case which determined the reachable region in terms of special trajectories. For example, in a 3-dimensional problem with a 2-dimensional control, the reachable region would intuitively appear as shown in Figure 4.12.

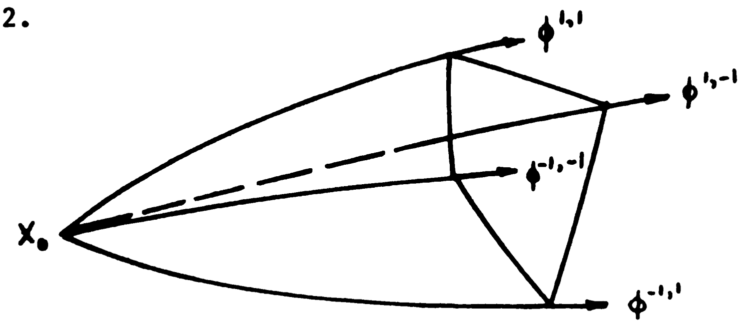


Figure 4.12

Reachable Region for 3-dimensional Problem

In the figure, four edges determined by trajectories $\phi^{1,1}$, $\phi^{1,-1}$, $\phi^{-1,-1}$, $\phi^{-1,1}$ might be expected. The trajectory $\phi^{1,1}$ is defined as the solution x(t) with control $u_1=1$, $u_2=1$, $x(0)=x_0$. The other trajectories would be defined in a similar manner.

In the case under consideration only one control signal is available. This reduces the reachable region to a surface. Hence, two dimensional theory as given by Theorem 2.1 may be applied. In Figure 4.5, x_0-A-B-D-x_0 shows part of the reachable region. The curve x_0-A is part of the trajectory due to the application of $v(t)=1$. The curve x_0-D is in the x_1-x_2 plane and is due to the constraint which requires that $x_3 \geq 0$. This curve cannot be generated by any control $v(t)$ but can be approached as close as desired by using a sufficiently small $v(t)$.

The trajectory of the pang-pang problem is shown in Figure 4.13. The trajectory starts at x_0, which lies in the x_1-x_2 plane. The surface $\omega=0$ is a plane shown passing through points C, O, and D. Physically the system acts as follows:

The pressure is increased at its maximum permissible rate until point A is reached. Then, the pressure is decreased at its maximum permissible rate until the singular surface is reached. The trajectory then continues from point B to the final point C. Along this trajectory, which is singular, the pressure is maintained at a constant value. The value of pressure required is determined by the singular control law.

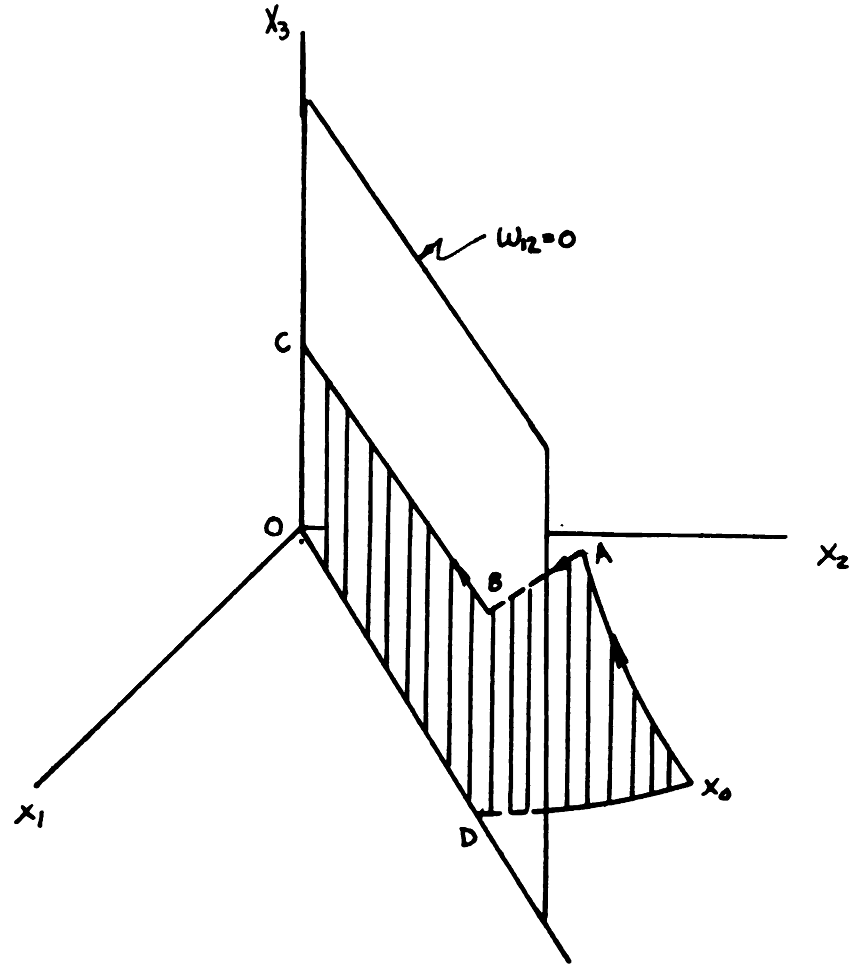


Figure 4.13

Optimal Trajectory for 3-Dimensional Problem

CHAPTER V SUBOPTIMAL VEHICULAR BRAKING CONTROL

5.1 Introduction

Theory indicates that the optimal control drives the state vector to the peak of the friction-slip curve as fast as possible and then holds it there. From a practical point of view the Maximum Principle is not feasible. Figures 3.1 and 3.2 show the block diagram necessary to mechanize the Maximum Principle solution. Since the initial costate vector must be determined on-line, subject to various initial conditions on the state vector, a rather complex and fast system would be needed to satisfy this requirement. This in itself would present a formidable optimal control problem. The approach using Green's Theorem is more feasible from a practical point of view. Figure 4.6 shows the block diagram using this approach. The difficulties associated with this method are due to the block which computes the ω function which determines the singular condition. Again this component must operate with fast response on-line. As indicated in section 4.3.4, the digital solution using the IBM 360-65 produced the wave forms shown in Figure 4.10. The results with respect to optimal stopping time were essentially optimal. The velocity waveforms were also essentially optimal. A slight ripple frequency appeared on the wheel velocity output. This ripple had a peak amplitude of 0.020 fps while the vehicle and wheel velocities were as high as 60fps. The control waveform deviated considerably from the ideal waveform. The ideal control signal appears in Figure 4.8. It should be noted that the digital solution oscillated at 300cps about the ideal value of 0.735. Hence from a practical point of view, this approach can be mechanized. However, in the interest of simplicity and economy the following suboptimal control was designed. The basic idea is illustrated in Figure 5.1.

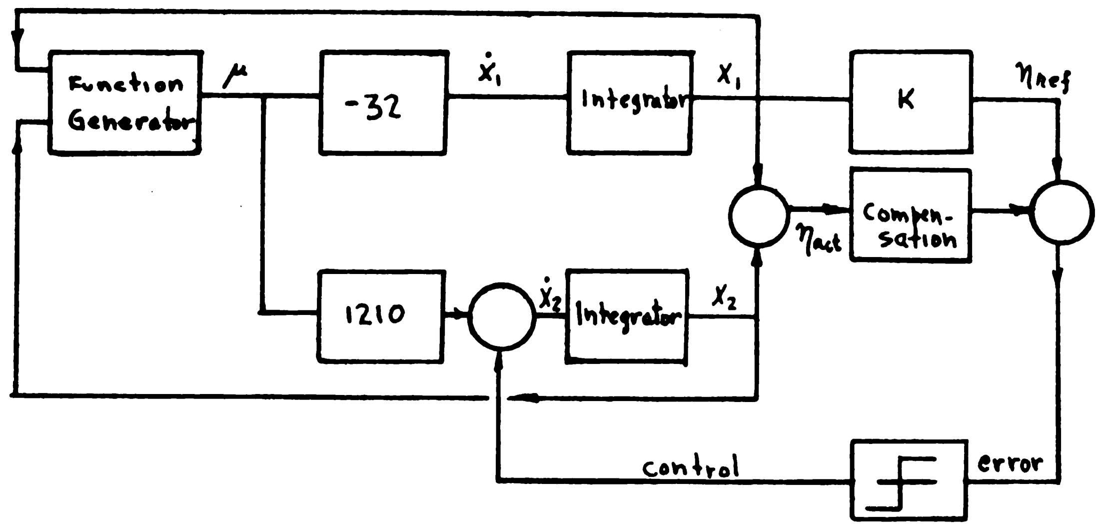


Figure 5.1

Basic Suboptimal Control

The basic premise is that the peak point on the friction-slip curve occurs at essentially the same relative slip regardless of vehicle velocity. Reference to Figure 4.3 shows that this point corresponds to a relative slip of 0.15, i.e., $1 - x_2/x_1 = 0.15$. This implies that x_2 which corresponds to wheel velocity is 85 percent of the vehicle velocity x_1 at the peak point. In addition, it is to be noted that the friction coefficient decreases slowly as the relative slip increases. At a slip of 1.0, the friction coefficient is typically 0.8 of its maximum value. This is the value achieved in the so-called "panic" stop. Therefore, based on this curve the stopping distance can be reduced to 0.8 of the "panic" value. Minimizing stopping distance is however only one factor in making a safe stop. A factor that is equally important is maintaining lateral stability. Due to small variations such as road surface unbalance, brake torque, and wind gusts, yawing may and usually does occur. These effects may be minimized if the slip is kept small. The net result implies that operation should be close to the peak of the friction-slip curve. The most effective operation will be shown to occur at slip values slightly greater than 0.15.

5.2 Block Diagram of Suboptimal Control System

The real system must contain transducers which are not ideal. The principal characteristic can be approximated by a time delay. For example, the wheel signal transducer generates an alternating voltage with frequency being proportional to wheel velocity. The electronic processing generates a voltage proportional to wheel velocity. This processing results in a delayed wheel velocity signal. Similar delays are encountered from the other transducers that must be used in the system. Hence, a realistic model of the vehicular braking control system appears as shown in Figure 5.2b.

This system was programmed on both analog and digital computers. The analog computer studies permitted interconnecting real components with simulated components. The diagram shown in Figure 5.2a shows the four basic elements, the road characteristics, vehicle dynamics, electronic control module, and the braking pressure actuator. All four elements were simulated on both analog and digital computers. In the analog setup, the simulated electronic pressure actuator could be readily replaced by real components. This feature permitted evaluation of various module and actuator designs.

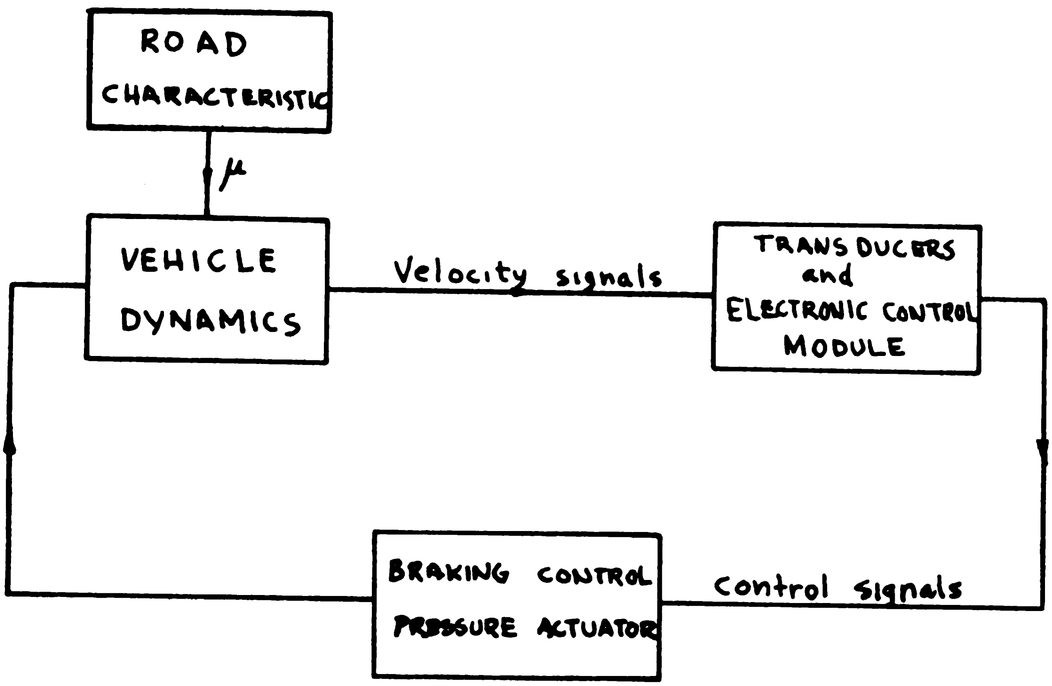


Figure 5.2a

Basic Block Diagram

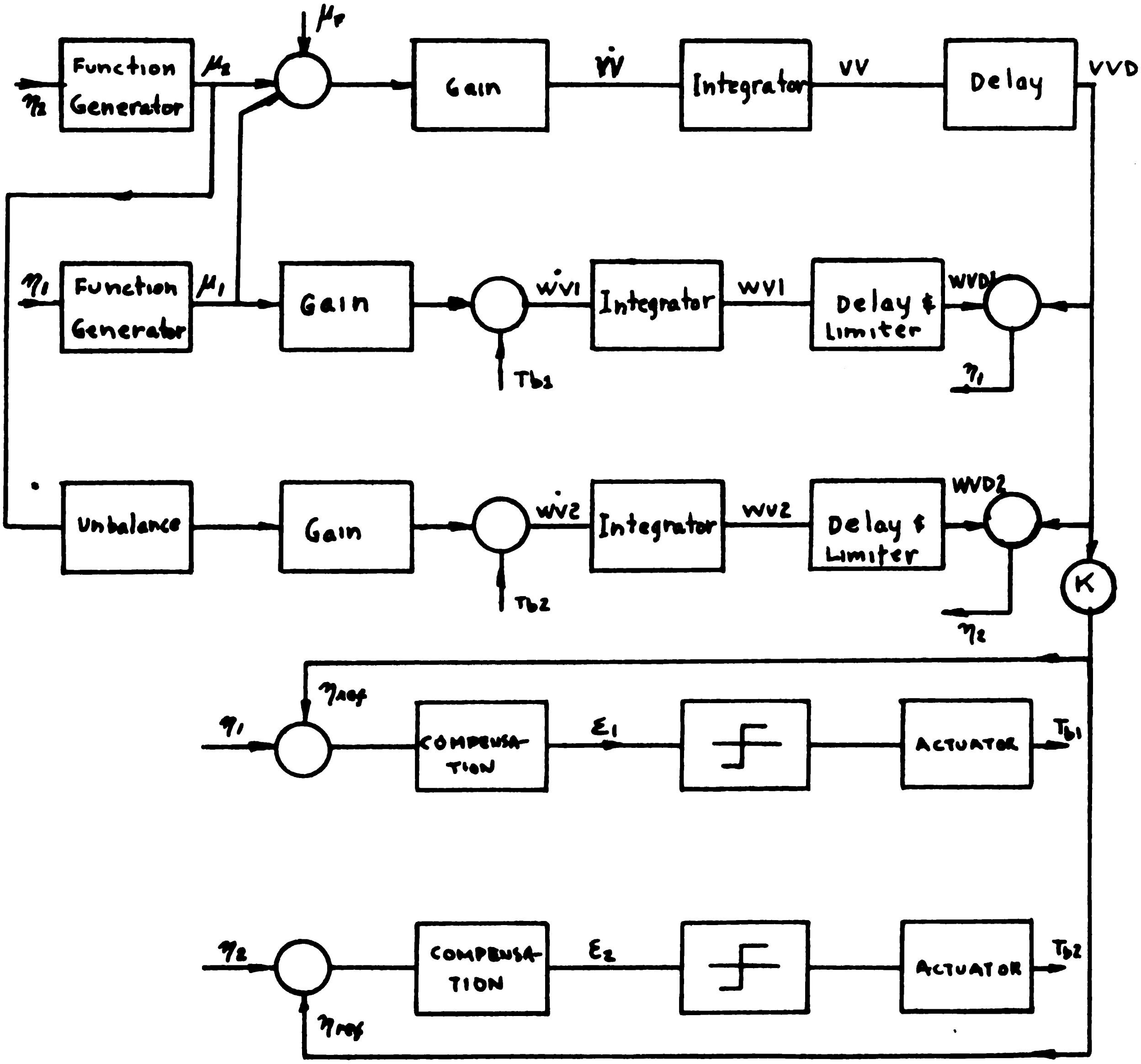


Figure 5.2b

Suboptimal Control System With Time Delays

5.3 Analog Computer Studies

The analog equivalent of the system shown in Figure 5.2 was studied extensively. In these studies, real components such as pressure actuators and electronic control modules would be compared with their simulated models. The primary object was to determine factors which adversely affected performance. The optimal control system was used as the reference system. Figure 5.3 shows response characteristics of a suboptimal system in which all components are simulated except the actuator which is designated X2. As may be observed from the wheel velocity trace, this response is, from a practical stand point, essentially optimal. This is a very low friction case, having a panic coefficient of nominally 0.09. The percentage error due to potentiometer settings is greater at the lower values of friction. To remove this source of error, stopping distances are compared with the skid control on and off. The panic stopping distance, for an initial vehicle velocity of 60 fps, assuming that the vehicle is a point mass, is given by the following equation,

$$\text{Panic Stopping Distance} = 56/\mu \tag{5.1}$$

In this case, the nominal value of 0.09 would predict a value of 622 feet instead of 583 feet. This implies that μ is actually 0.096 instead of 0.09. The significant fact is the reduced value of 507 feet. It should be noted that an ideal wheel velocity transducer is used. The erratic operation that occurred was due to vacuum pressure going below the design minimum. In summary, this test shows nearly optimal response even with a real actuator in the system.

Figure 5.4 shows the effect on the same system by using a transducer that can be approximated by a time delay of 0.020 seconds. Stopping distance is increased slightly, although considerably better than 186 feet, the value corresponding to the panic value of $\mu = 0.3$.

Actuator X1 has an initial pressure rise characteristic which has adverse effects. This causes the wheel velocity to initially drop to lower than desired values. This can cause lock-up under some road friction conditions. Figures 5.5a and 5.5b show these characteristics. The initial part of this response is magnified in Figures 5.6a and 5.6b. This actuator was one which had been considered satisfactory for systems which did not employ anti-skid controls. An interesting nonlinear phenomenon occurs in this set of traces. The operating point is unstable. The right wheel velocity approaches the vehicle velocity or free wheels, while the left wheel velocity goes to the lock-up condition. This condition will be considered in Chapter VI.

Figures 5.7a and 5.7b show a typical response characteristic of the system without compensation. As may be noted the wheels alternately free wheel and lock-up. These conditions are also clearly seen on the friction curve. The panic value in this case is 0.8. Since the stopping distance is the same whether the control is on or off, the average coefficient of friction is 0.8. Also, it is to be noted that the pulsing frequency is considerably lower. In this case, it is approximately 3 cps. With compensation, pulsing frequencies as high as 20 cps have been encountered.

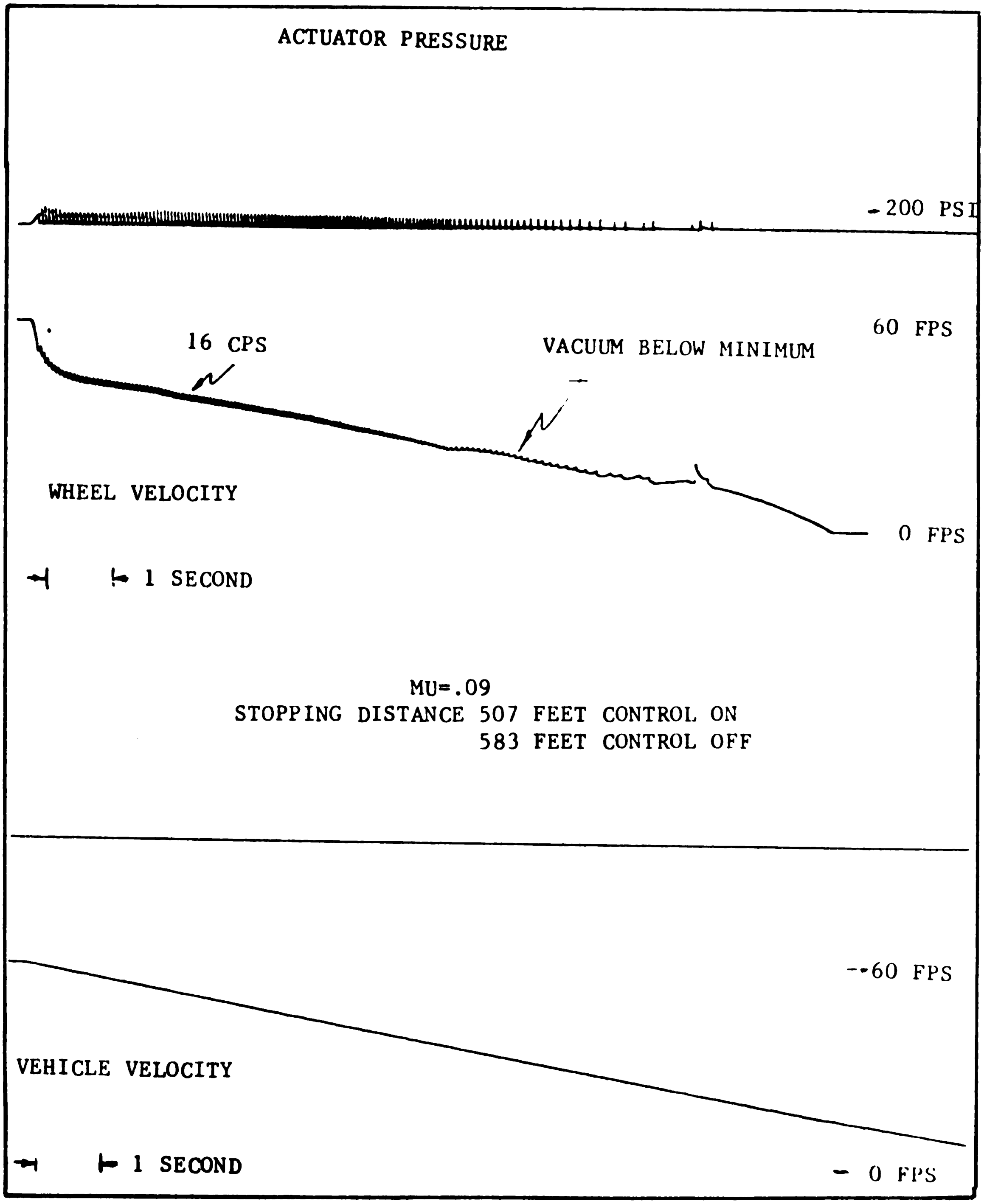


Figure 5.3

Velocity Response-Real Actuator X2
Ideal Wheel Velocity Transducer

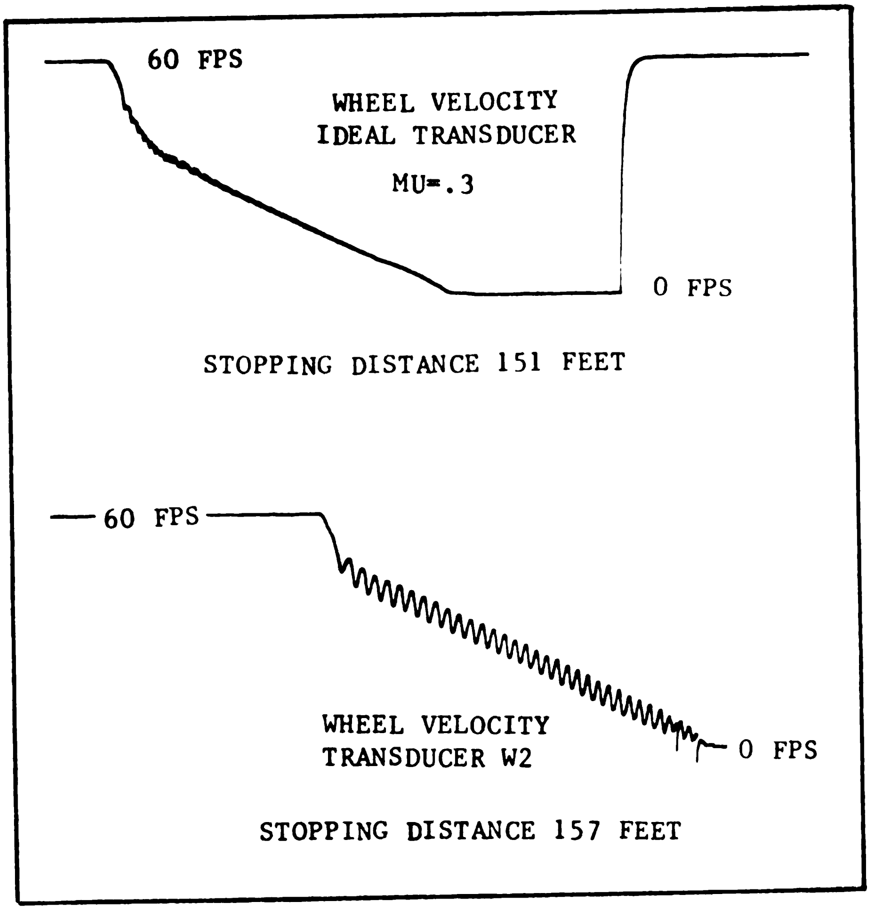


Figure 5.4

Effect of Wheel Velocity Transducer

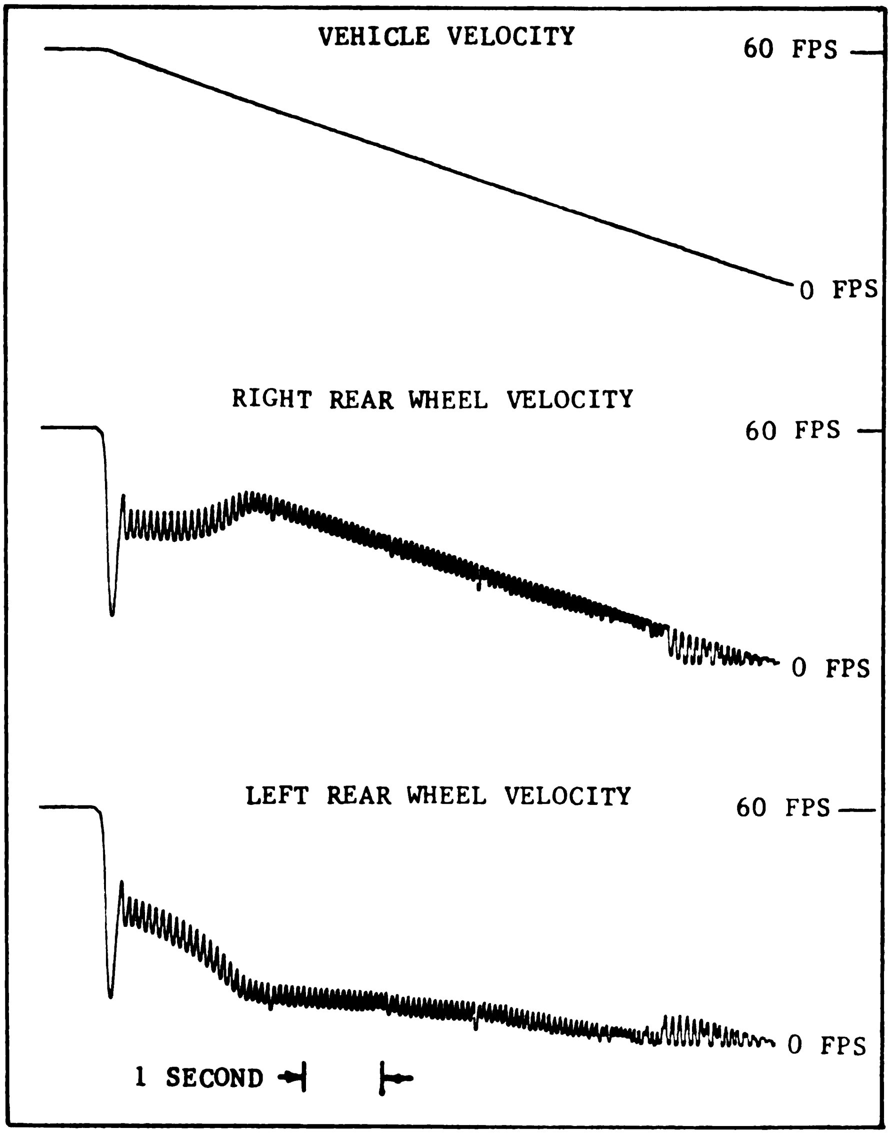


Figure 5.5a

Effect of Initial Delay in Actuator X1

Velocity Response

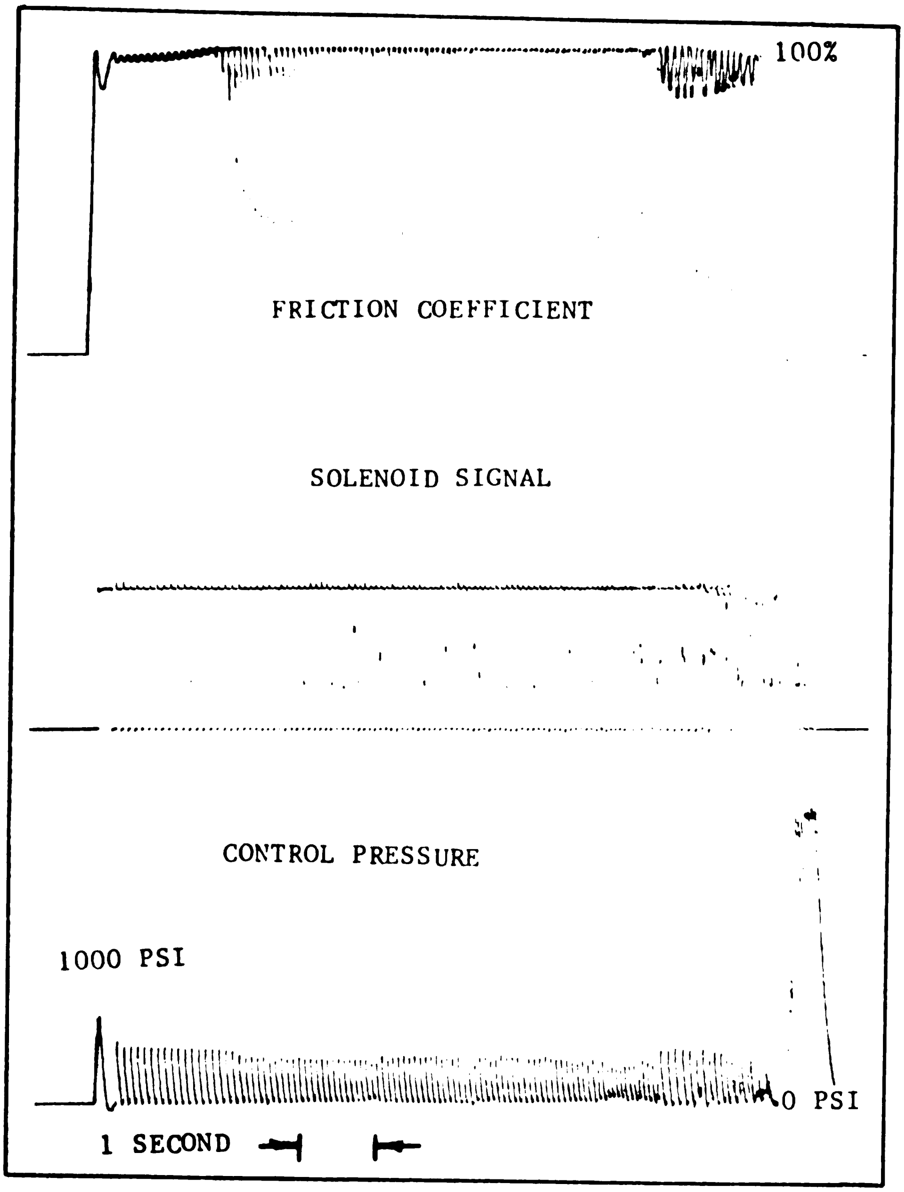


Figure 5.5b

Effect of Initial Delay in Actuator X1 Solenoid, Control Pressure, and Friction Response

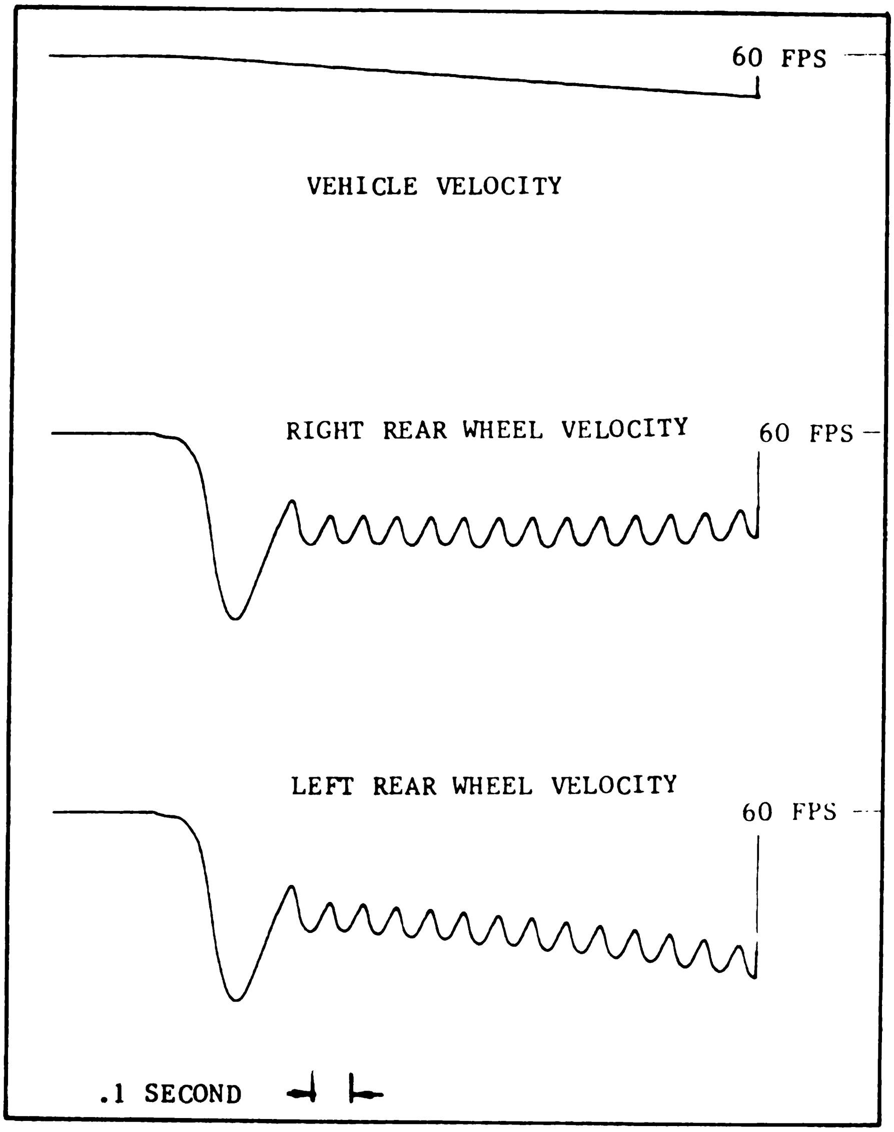


Figure 5.6a

Initial Velocity Response

Effect of Initial Delay in Actuator X1

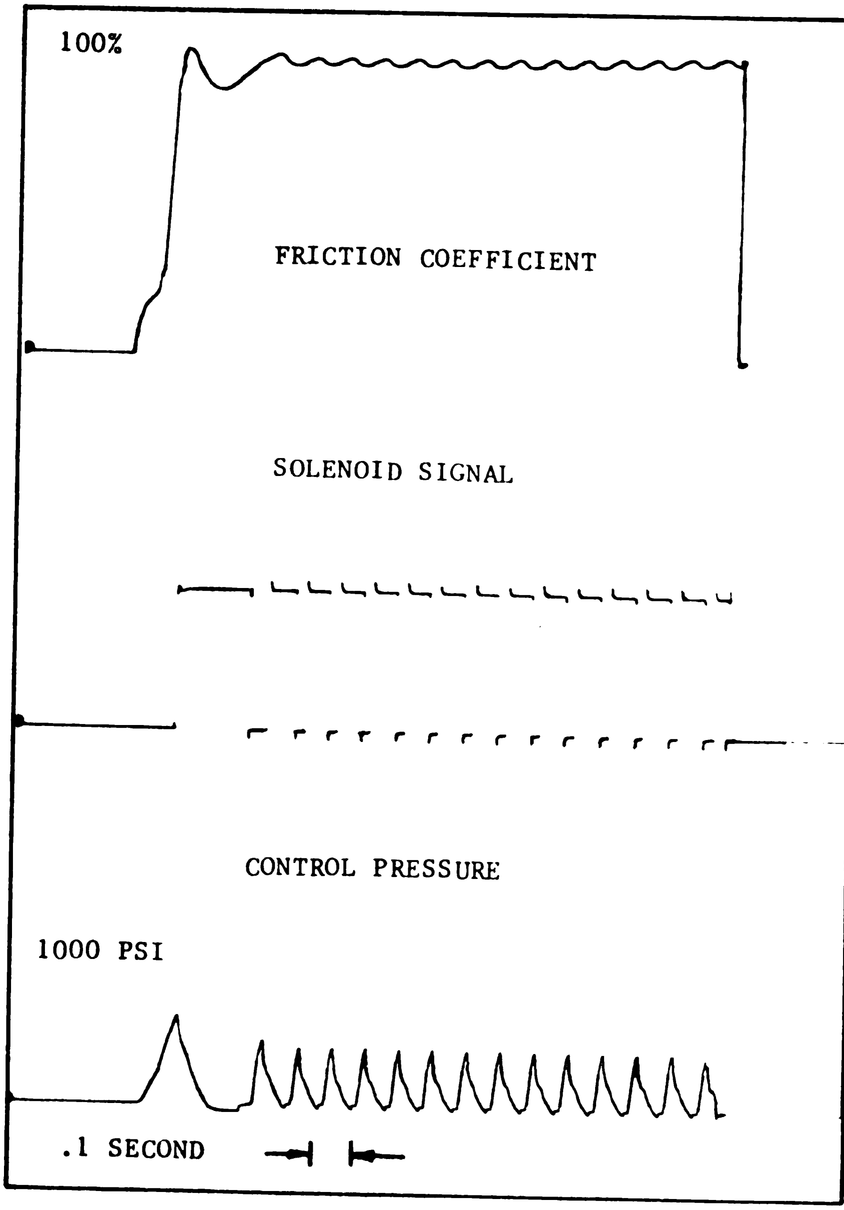


Figure 5.6b

Initial Response

Effect of Initial Delay in Actuator X1

Solenoid, Control Pressure, and Friction

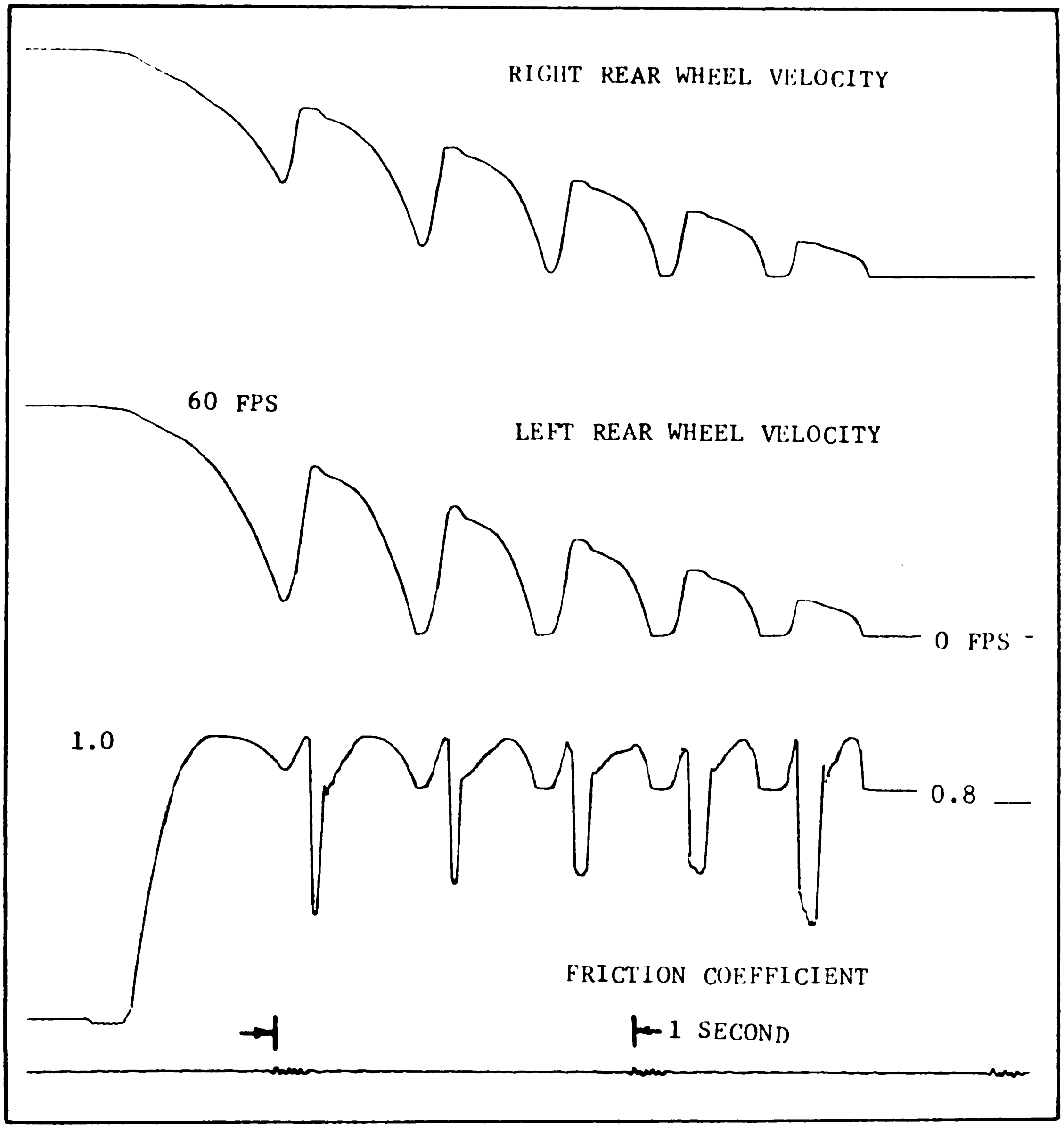


Figure 5.7a

Response Without Compensation

Stopping Distance Control On 72 Feet
Control Off 72 Feet

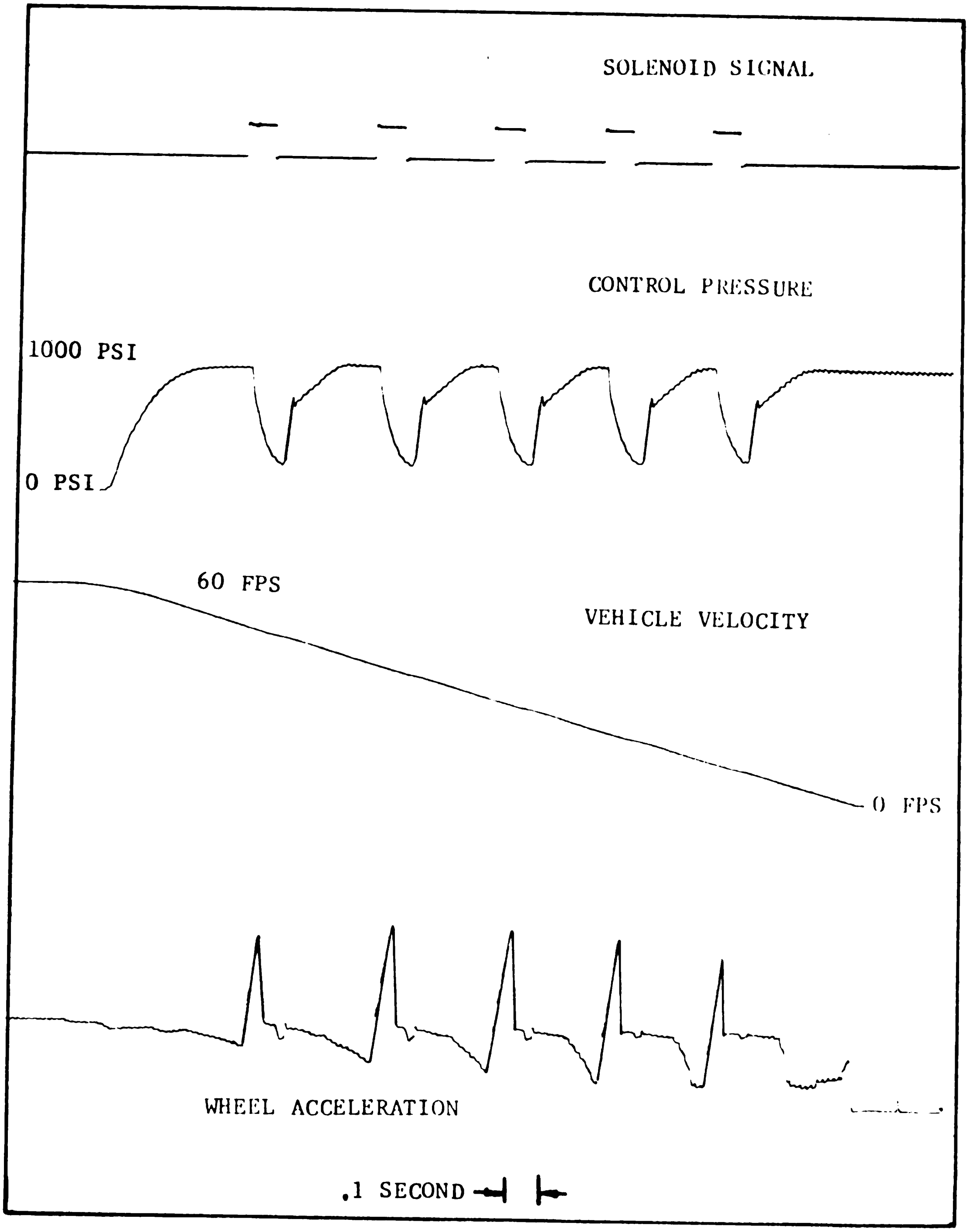


Figure 5.7b

Response Without Compensation

5.4 Digital Computer Studies

The Continuous Systems Modeling Program (CSMP) on the IBM 360-365 was used to supplement the studies made on the analog computer. The listing of the simple one-wheel model is shown in Figure 5.8. This is designated PROGRAM I. Listings of other models are shown in Appendix IV.

Sample outputs and histograms of PROGRAM I are shown in Figures 5.9 to 5.15. The model described by PROGRAM I is almost ideal with respect to transducers. There is no delay in the pressure transducer and a negligible delay (1 millisecond) in the wheel velocity transducer. No compensation is used.

In Figure 5.10, the first portion of the wheel velocity of suboptimal vehicular control system is shown. In this suboptimal system, the wheel velocity has a peak ripple velocity of approximately 7 feet per second, whereas, the optimal control would have no ripple. More significant is the friction coefficient, shown in Figure 5.11. After it passes the peak value of 1.0, it varies from between 0.9 and 1.0. Hence, the average value is approximately 0.95 for suboptimal system and just under 1.0 for the optimal system. It should be noted that this is not a realistic suboptimal control system, since the transducers are essentially ideal.

Figures 5.13 and 5.14 show similar results when the friction coefficient is 0.3. The supoptimal μ averages 92 percent of the optimal value of 0.3. The low μ case is, with respect to ripple magnitudes, more nearly optimal than the high μ case. Stopping distances show that the suboptimal system requires 59 feet at a μ of 1.0 and 202 feet at a μ of 0.3. This compares with 56 feet and 186 feet for the optimal control system. These values are for an initial velocity of 60 feet per second or approximately 40 miles per hour.

Of interest is the ripple frequency, since this may be used as a measure of optimality. The suboptimal system with essentially zero delay has a ripple frequency of approximately 11 cps at $\mu = 1.0$ and 18 cps when $\mu = 0.3$. When transducers, which have time delays are used, these frequencies go down. To compensate for the delays, compensation networks are employed. In the suboptimal control system, the effectiveness of the compensation can be judged by the ripple frequency generated. The higher ripple frequencies imply that the system is more nearly optimal. A criterion, based on ripple frequency, for estimating time delays associated with transducers is established in Chapter VI.

The last two figures in this set, Figures 5.12 and 5.15 show the control pressure for the $\mu = 1$ and $\mu = 0.3$ cases. As generated on the digital computer, the rise and fall rates are 15000 and 45000 psi/second for this case.

In the digital studies, use was also made of the CSMP plotting features. Typical plots are shown in Figures 5.16 to 5.23.

The effect of having unbalanced time delays was studied. Figure 5.16 and 5.17 show the effect of having 20 milliseconds delay in one of the wheel velocity transducers and no delay in the other. Vehicle velocity and pressure transducers also had no delay. The reference velocity in the suboptimal control system was set for 0.5 of the vehicle velocity and the system used the average of the wheel velocity signals to establish the error signal. These results show that, by using proper compensation, this amount of delay may be tolerated. The principal disadvantage is that the wheels lock up at a vehicle velocity of approximately 5 feet per second. In other respects, the supoptimal control operates as desired. This is clearly shown in Figure 5.17. The control pressure quickly brings μ to its maximum value. Then, due to the slip reference setting of 0.5, keeps μ at a relatively high value for almost the entire stopping period. Earlier lockup is very clearly shown in this figure. The value of μ is seen to drop to the panic value of 0.8 while the control pressure rises rapidly to its maximum of 1200 psi. The stopping distance for this suboptimal system is 60.21 feet as compared to 56 feet for the optimal control system. The listing of this program, P 252, is shown in Appendix IV.

The next set, Figures 5.18 to 5.21, show the effects of having 20 milliseconds delay in both wheel velocity signals. There are no other changes from the previous system. The ripple frequency variation is now quite prominent. The relatively high frequency of 12 cps indicates satisfactory operation at this value of μ. The stopping distance has increased slightly to 60.76 feet.

The effect of subjecting one side of the vehicle to a peak μ 0.95 and the other side to a peak μ of 1.0 is shown in Figures 5.22 and 5.23. All other components of the system are as in Program 252. The side of the vehicle which is subjected to a μ 0.95 and has a 20 millisecond delay in the transducer locks up early as seen by the WV2D trace in Figure 5.22. Stopping distance was 63.4 feet as compared to the optimal value of 57.2 feet. This is a two wheel model which used the average of the two wheel velocity signal to produce a one-dimensional control signal. This example points out the disadvantage of the one dimensional control versus the two dimensional control system. Figure 5.23 clearly indicates the changes in the μ characteristic and control pressure that occur after early lock-up.

The final figures of this section show the response of the system which is designated as the reference suboptimal control system. As may be noted from Figures 5.24 and 5.25, the wheel velocity response and the friction coefficient response are almost undistinguishable from those determined from the theoretical optimal control system. In Figure 5.23, trajectory O-A-B is nonsingular, while B-C is the singular trajectory. The control pressure response, in Figure 5.26, does not peak as expected in the optimal response. The variation in the flat portion of the control pressure is due to the anticipatory nature of the compensation network. Stopping distance for this system is 57.75 feet.

These figures are representative of the digital studies which show that the suboptimal control system considered performs satisfactorily and compares favorably with the reference optimal control system which, it should be noted, does not contain time delays.

```
         ****CONTINUOUS SYSTEM MODELING PROGRAM****

     ***PROBLEM INPUT STATEMENTS***

TITLE      ASKC
PARAM      B=1.,C=1.,D=1.
FUNCT      MUET1=(-.15,-1.),(0.,0.0),(.075,.6),(.15,1.),(.57
METHOD     RKSFX
           MU1=B*AFGEN(MUET1,ETA1)
           VVDOT=-16.*(MU1+D)
           VV=INTGRL(60.0,VVDOT)
           VVD=DELAY(5,.015,VV)
           WV1DOT=1210.*MU1-0.79*T1
           SDIST=INTGRL(0.0,VV)
           WV1=INTGRL(60.0,WV1DOT)
           WV1D=DELAY(5,.001,WV11)
           ETA1A=VVD-WV1D
           T1=1.66*P11
           ETA1=1.0-WV11/VV
           WV11=LIMIT(0.,60.0,WV1)
           ETA1R=0.5*VVD
           Y1=DERIV(0.0,ETA1A)
           ERRIN1=ETA1A-ETA1R
           ERROT1=INSW(ERRIN1, 1.0,-3.0)
           E1=15000.*ERROT1
           P1=INTGRL(0.0,X1)
           X1=E1*X11
PROCED     X11=DU1(P1,E1)
           IF(E1)1,1,2
         1 IF(P1)3,4,4
         3 X11=0.0
           GO TO 7
         4 X11=1.0
           GO TO 7
         2 IF(P1-1200.)5,5,6
         5 X11=1.0
           GO TO 7
         6 X11=0.0
         7 CONTINUE
ENDPRO
           P11=DELAY(15,.0,P1)
TIMER      FINTIM=4.,DELT=.0004,                      PRDEL=.02
FINISH     VV=1.0
LABEL      ANSKC*JGG*SCHERBA
PRINT      ETA1R,ETA1A,MU1,ERRIN1,ERROT1,P11,        VV,P1,SDI
PRTPLT     ETA1R,ETA1A,MU1,ERRIN1,ERROT1,P11,        VV,P1,SDI
END
PARAM      B=0.3,C=0.3,D=.3
RESET LABEL
LABEL        ANSKC  (MU=.3)
END
STOP
```

Figure 5.8   PROGRAM I   CSMP Listing of One-Wheel Model

```
ASKC                                                                        RKSFX    INTEGRA

TIME  =  0.0              ETAIR =  0.0              ETAIA =  0.0             MU1   =  0.0
                          ERROT1= -3.0000E 00       P11   =  0.0             VV    =  6.00
                          SDIST =  0.0              WV1D  =  0.0             X1    = -4.50

TIME  =  2.0000E-02       ETAIR =  2.9961E 01       ETAIA =  1.5459E 00      MU1   =  1.89
                          ERROT1=  1.0000E 00       P11   =  2.7600E 02      VV    =  5.96
                          SDIST =  1.1967E OC       WV1D  =  5.8375E 01      X1    =  1.50

TIME  =  4.0000E-02       ETAIR =  2.9782E 01       ETAIA =  3.9497E 00      MU1   =  5.07
                          ERROT1=  1.0000E 00       P11   =  5.7599E 02      VV    =  5.92
                          SDIST =  2.3857E CC       WV1D  =  5.5615E 01      X1    =  1.50

TIME  =  6.0000E-02       ETAIR =  2.9552E 01       ETAIA =  6.7228E 00      MU1   =  7.91
                          ERROT1=  1.0000E 00       P11   =  8.7597E 02      VV    =  5.86
                          SDIST =  3.5651E 0C       WV1D  =  5.2381E 01      X1    =  1.50

TIME  =  8.0000E-02       ETAIR =  2.9277E 01       ETAIA =  1.0470E 01      MU1   =  9.93
                          ERROT1=  1.0000E 00       P11   =  1.1760E 03      VV    =  5.80
                          SDIST =  4.7330E OC       WV1D  =  4.8084E 01      X1    =  1.50

TIME  =  1.0000E-01       ETAIR =  2.8961E 01       ETAIA =  1.7594E 01      MU1   =  9.63
                          ERROT1=  1.0000E 00       P11   =  1.2030E 03      VV    =  5.74
                          SDIST =  5.8882E CO       WV1D  =  4.0328E 01      X1    =  0.0

TIME  =  1.2000E-01       ETAIR =  2.8646E 01       ETAIA =  2.5575E 01      MU1   =  9.28
                          ERROT1=  1.0000E 00       P11   =  1.2030E 03      VV    =  5.68
                          SDIST =  7.0309E OC       WV1D  =  3.1716E 01      X1    =  0.0

TIME  =  1.4000E-01       ETAIR =  2.8335E 01       ETAIA =  3.0110E 01      MU1   =  9.11
                          ERROT1= -3.0000E 00       P11   =  6.1796E 02      VV    =  5.62
                          SDIST =  8.1612E OC       WV1D  =  2.6561E 01      X1    = -4.50
```

Figure 5.9 Output of PROGRAM I

MINIMUM 0.0 — WVID VERSUS TIME — MAXIMUM 6.0000E 01

TIME	WVID
0.0	0.0
2.0000E-02	5.8375E 01
4.0000E-02	5.5615E 01
6.0000E-02	5.2381E 01
8.0000E-02	4.8084E 01
1.0000E-01	4.0328E 01
1.2000E-01	3.1716E 01
1.4000E-01	2.6561E 01
1.6000E-01	3.5312E 01
1.8000E-01	3.8127E 01
2.0000E-01	3.3029E 01
2.2000E-01	2.4893E 01
2.4000E-01	3.1824E 01
2.6000E-01	3.7258E 01
2.8000E-01	3.5055E 01
3.0000E-01	2.6534E 01
3.2000E-01	2.5978E 01
3.4000E-01	3.4480E 01
3.6000E-01	3.5740E 01
3.8000E-01	2.9255E 01
4.0000E-01	2.1988E 01
4.2000E-01	3.0044E 01
4.4000E-01	3.4839E 01
4.6000E-01	3.1947E 01
4.8000E-01	2.3317E 01
5.0000E-01	2.3628E 01
5.2000E-01	3.2008E 01
5.4000E-01	3.3206E 01
5.6000E-01	2.6700E 01
5.8000E-01	1.9169E 01
6.0000E-01	2.6956E 01
6.2000E-01	3.2101E 01
6.4000E-01	2.9642E 01

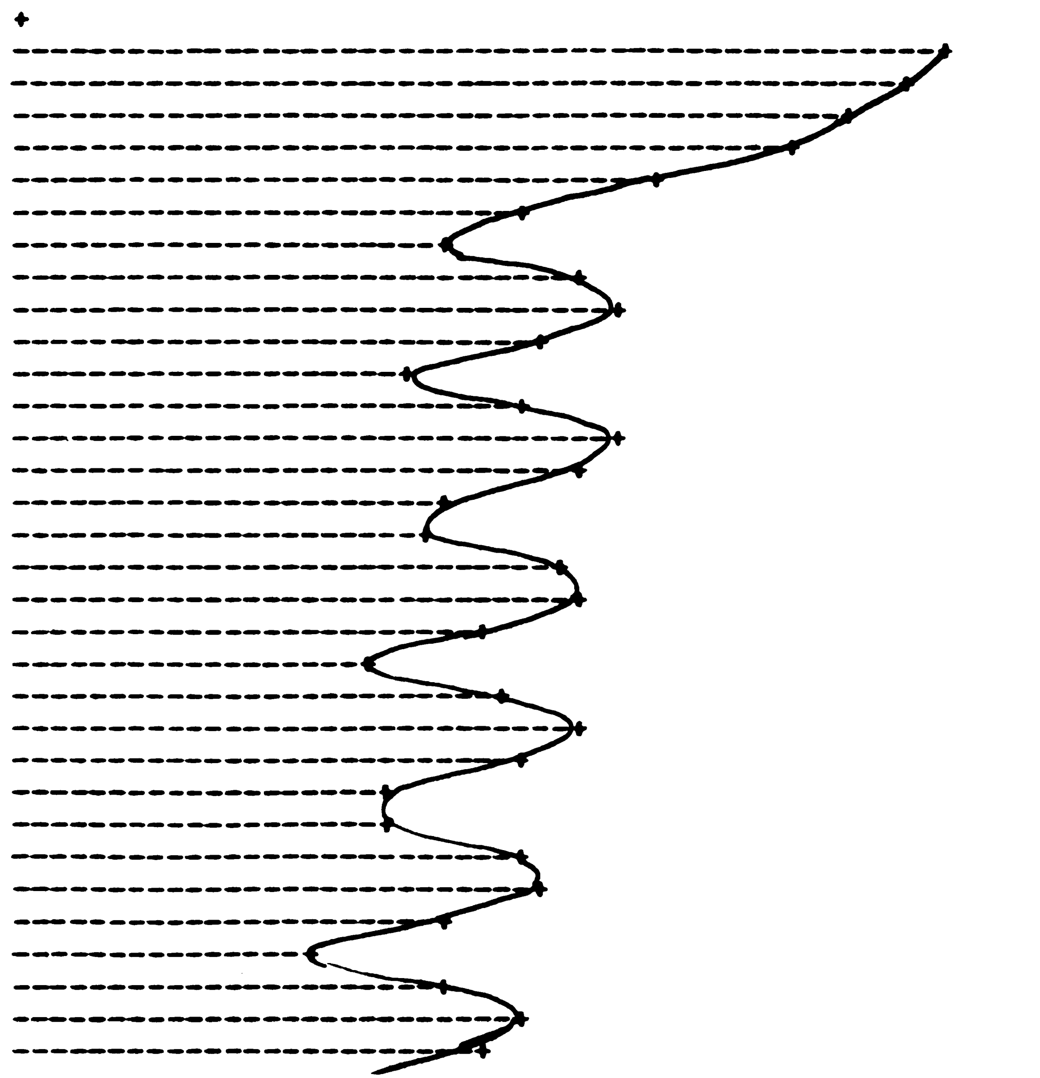


Figure 5.10 Histogram of Wheel Velocity

TIME	MU1
0.0	0.0
2.0000E-02	1.8980E-01
4.0000E-02	5.0710E-01
6.0000E-02	7.9133E-01
8.0000E-02	9.9338E-01
1.0000E-01	9.6306E-01
1.2000E-01	9.2861E-01
1.4000E-01	9.1122E-01
1.6000E-01	9.5021E-01
1.8000E-01	9.6251E-01
2.0000E-01	9.4042E-01
2.2000E-01	9.0751E-01
2.4000E-01	9.4226E-01
2.6000E-01	9.6700E-01
2.8000E-01	9.5708E-01
3.0000E-01	9.1877E-01
3.2000E-01	9.2252E-01
3.4000E-01	9.6297E-01
3.6000E-01	9.6930E-01
3.8000E-01	9.3838E-01
4.0000E-01	9.0656E-01
4.2000E-01	9.5023E-01
4.4000E-01	9.7479E-01
4.6000E-01	9.6021E-01
4.8000E-01	9.1686E-01
5.0000E-01	9.2563E-01
5.2000E-01	9.7057E-01
5.4000E-01	9.7750E-01
5.6000E-01	9.4260E-01
5.8000E-01	9.0494E-01
6.0000E-01	9.5319E-01
6.2000E-01	9.8321E-01
6.4000E-01	9.6930E-01

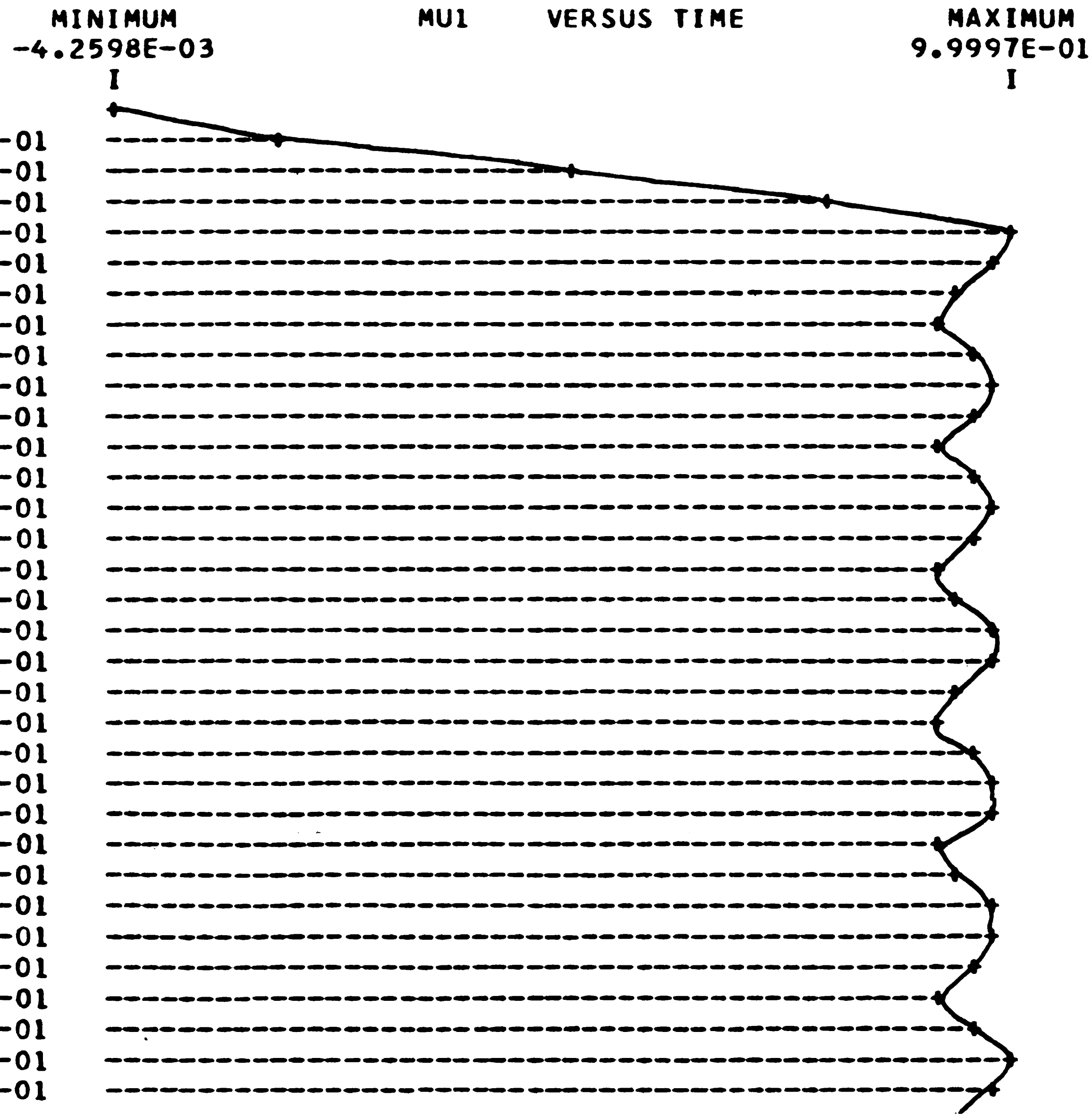
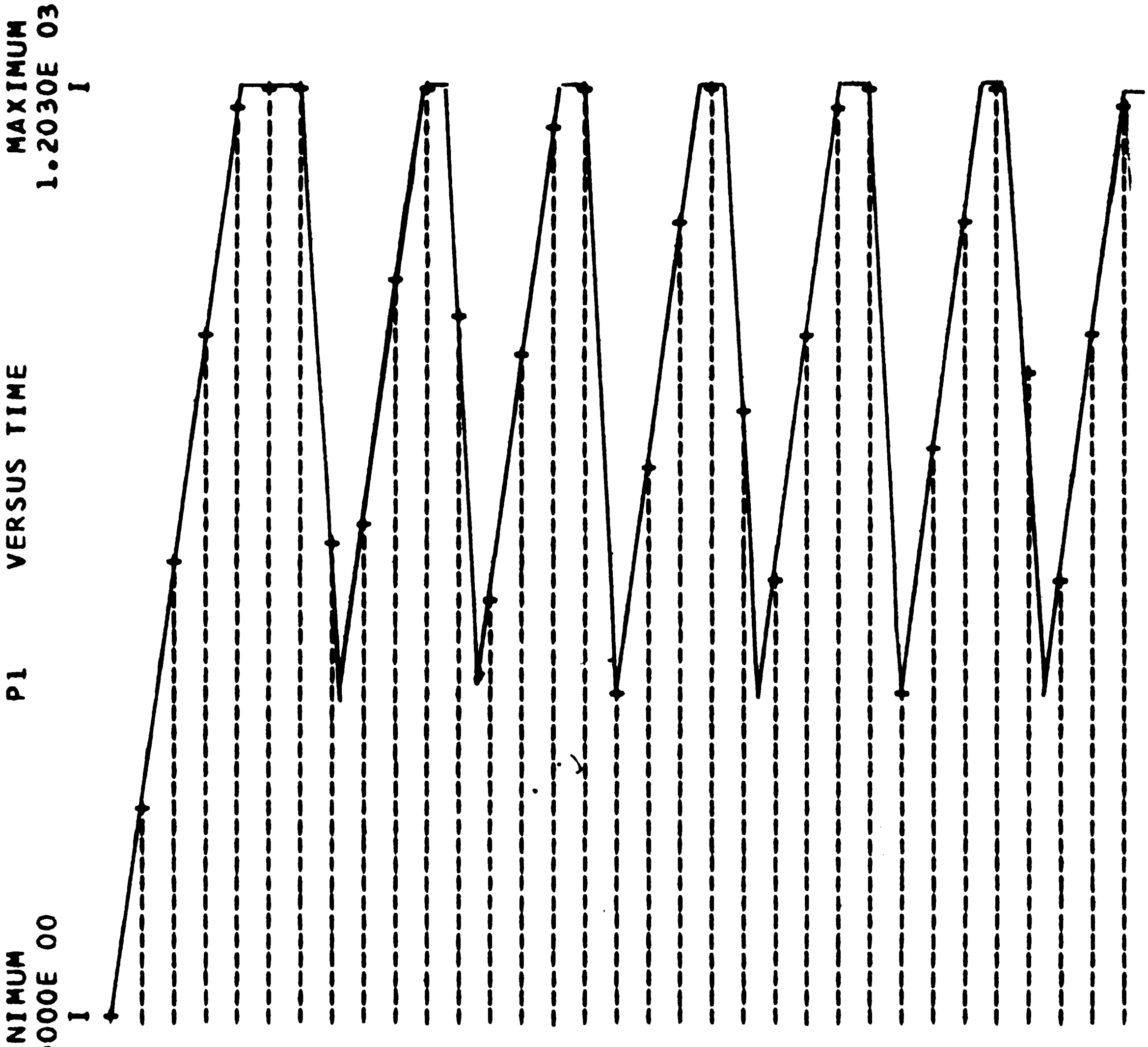
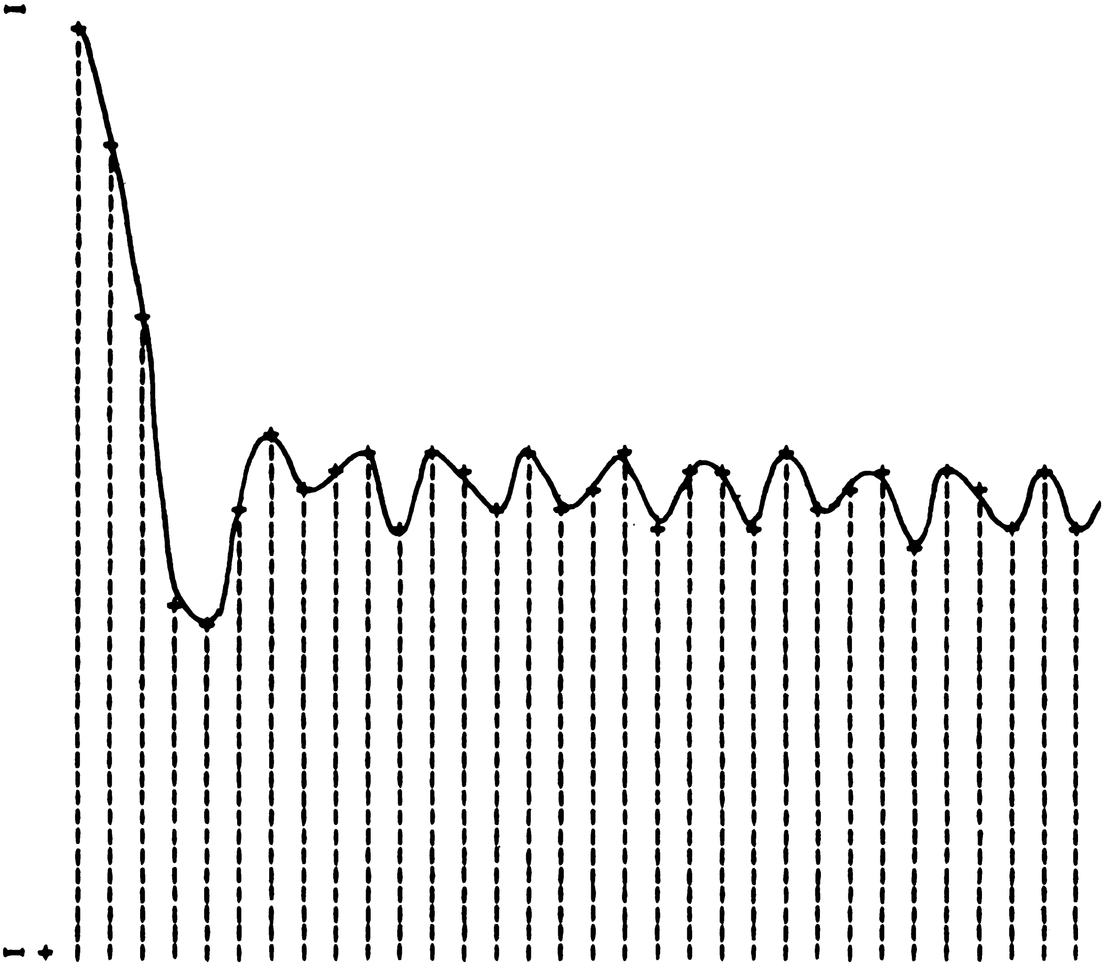


Figure 5.11 Histogram of Friction Coefficient



TIME	P1
0.0	0.0
2.0000E-02	2.7600E 02
4.0000E-02	5.7599E 02
6.0000E-02	8.7597E 02
8.0000E-02	1.1760E 03
1.0000E-01	1.2030E 03
1.2000E-01	1.2030E 03
1.4000E-01	6.1796E 02
1.6000E-01	6.4195E 02
1.8000E-01	9.4194E 02
2.0000E-01	1.2029E 03
2.2000E-01	8.8793E 02
2.4000E-01	5.3992E 02
2.6000E-01	8.3991E 02
2.8000E-01	1.1399E 03
3.0000E-01	1.2029E 03
3.2000E-01	4.1090E 02
3.4000E-01	7.1088E 02
3.6000E-01	1.0109E 03
3.8000E-01	1.2019E 03
4.0000E-01	7.8786E 02
4.2000E-01	5.7186E 02
4.4000E-01	8.7184E 02
4.6000E-01	1.1718E 03
4.8000E-01	1.2008E 03
5.0000E-01	4.2083E 02
5.2000E-01	7.2082E 02
5.4000E-01	1.0208E 03
5.6000E-01	1.2028E 03
5.8000E-01	8.1580E 02
6.0000E-01	5.6379E 02
6.2000E-01	8.6378E 02
6.4000E-01	1.1638E 03

Figure 5.12 Histogram of Control Pressure



TIME	WV1D
0.0	0.0
2.0000E-02	5.7681E 01
4.0000E-02	5.1427E 01
6.0000E-02	3.9894E 01
8.0000E-02	2.2344E 01
1.0000E-01	2.1501E 01
1.2000E-01	2.8096E 01
1.4000E-01	3.2778E 01
1.6000E-01	2.9912E 01
1.8000E-01	3.0207E 01
2.0000E-01	3.2381E C1
2.2000E-01	2.7473E 01
2.4000E-01	3.1606E 01
2.6000E-01	3.0957E 01
2.8000E-01	2.8005E 01
3.0000E-01	3.2226E 01
3.2000E-01	2.8730E 01
3.4000E-01	2.9748E 01
3.6000E-01	3.1694E 01
3.8000E-01	2.6701E 01
4.0000E-01	3.1014E 01
4.2000E-01	3.0373E 01
4.4000E-01	2.7051E 01
4.6000E-01	3.1287E 01
4.8000E-01	2.7867E 01
5.0000E-01	2.9265E 01
5.2000E-01	3.0823E 01
5.4000E-01	2.5864E 01
5.6000E-01	3.0341E 01
5.8000E-01	2.9226E 01
6.0000E-01	2.6969E 01
6.2000E-01	3.0739E 01
6.4000E-01	2.6692E 01

Figure 5.13 Histogram of Wheel Velocity

TIME	MU1
0.0	0.0
2.0000E-02	9.8216E-02
4.0000E-02	2.9315E-01
6.0000E-02	2.8619E-01
8.0000E-02	2.6559E-01
1.0000E-01	2.6573E-01
1.2000E-01	2.7371E-01
1.4000E-01	2.7915E-01
1.6000E-01	2.7530E-01
1.8000E-01	2.7656E-01
2.0000E-01	2.7887E-01
2.2000E-01	2.7296E-01
2.4000E-01	2.7848E-01
2.6000E-01	2.7731E-01
2.8000E-01	2.7448E-01
3.0000E-01	2.7945E-01
3.2000E-01	2.7471E-01
3.4000E-01	2.7694E-01
3.6000E-01	2.7902E-01
3.8000E-01	2.7289E-01
4.0000E-01	2.7875E-01
4.2000E-01	2.7757E-01
4.4000E-01	2.7416E-01
4.6000E-01	2.7932E-01
4.8000E-01	2.7455E-01
5.0000E-01	2.7730E-01
5.2000E-01	2.7893E-01
5.4000E-01	2.7278E-01
5.6000E-01	2.7891E-01
5.8000E-01	2.7707E-01
6.0000E-01	2.7504E-01
6.2000E-01	2.7965E-01
6.4000E-01	2.7399E-01

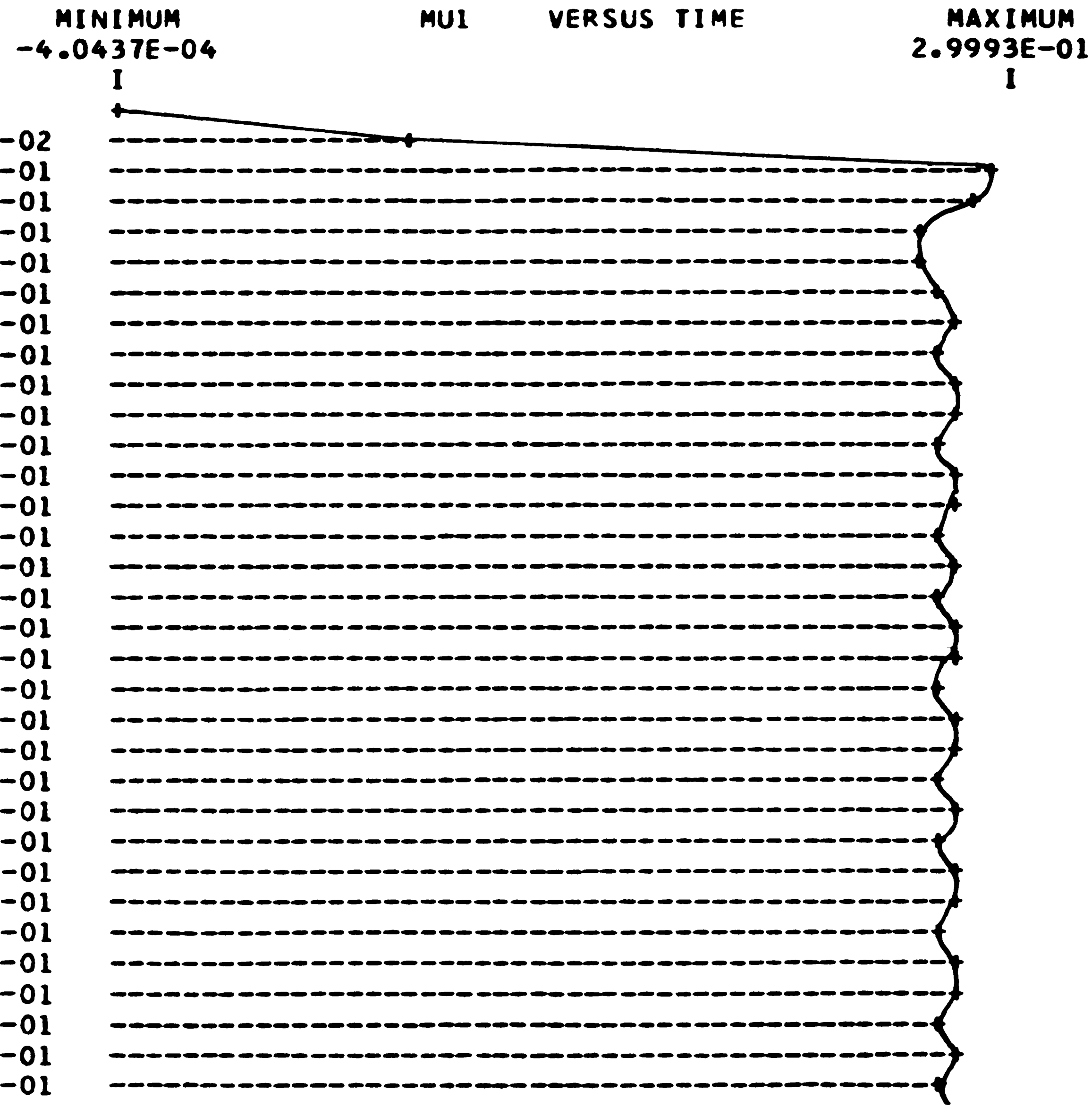


Figure 5.14 Histogram of Friction Coefficient

TIME	P1
0.0	0.0
2.0000E-02	2.7600E 02
4.0000E-02	5.7599E 02
6.0000E-02	8.7597E 02
8.0000E-02	6.5997E 02
1.0000E-01	-3.0323E 00
1.2000E-01	-3.0323E 00
1.4000E-01	2.3097E 02
1.6000E-01	5.3096E 02
1.8000E-01	3.8955E 01
2.0000E-01	3.3895E 02
2.2000E-01	3.0294E 02
2.4000E-01	1.4694E 02
2.6000E-01	4.4694E 02
2.8000E-01	-6.0686E 00
3.0000E-01	2.5493E 02
3.2000E-01	5.5492E 02
3.4000E-01	4.7919E 01
3.6000E-01	3.4791E 02
3.8000E-01	2.8791E 02
4.0000E-01	1.4691E 02
4.2000E-01	4.4690E 02
4.4000E-01	-1.0576E-01
4.6000E-01	2.5189E 02
4.8000E-01	5.5188E 02
5.0000E-01	6.2882E 01
5.2000E-01	3.6288E 02
5.4000E-01	2.4287E 02
5.6000E-01	1.6487E 02
5.8000E-01	4.6486E 02
6.0000E-01	-6.1421E 00
6.2000E-01	2.7586E 02
6.4000E-01	5.0384E 02

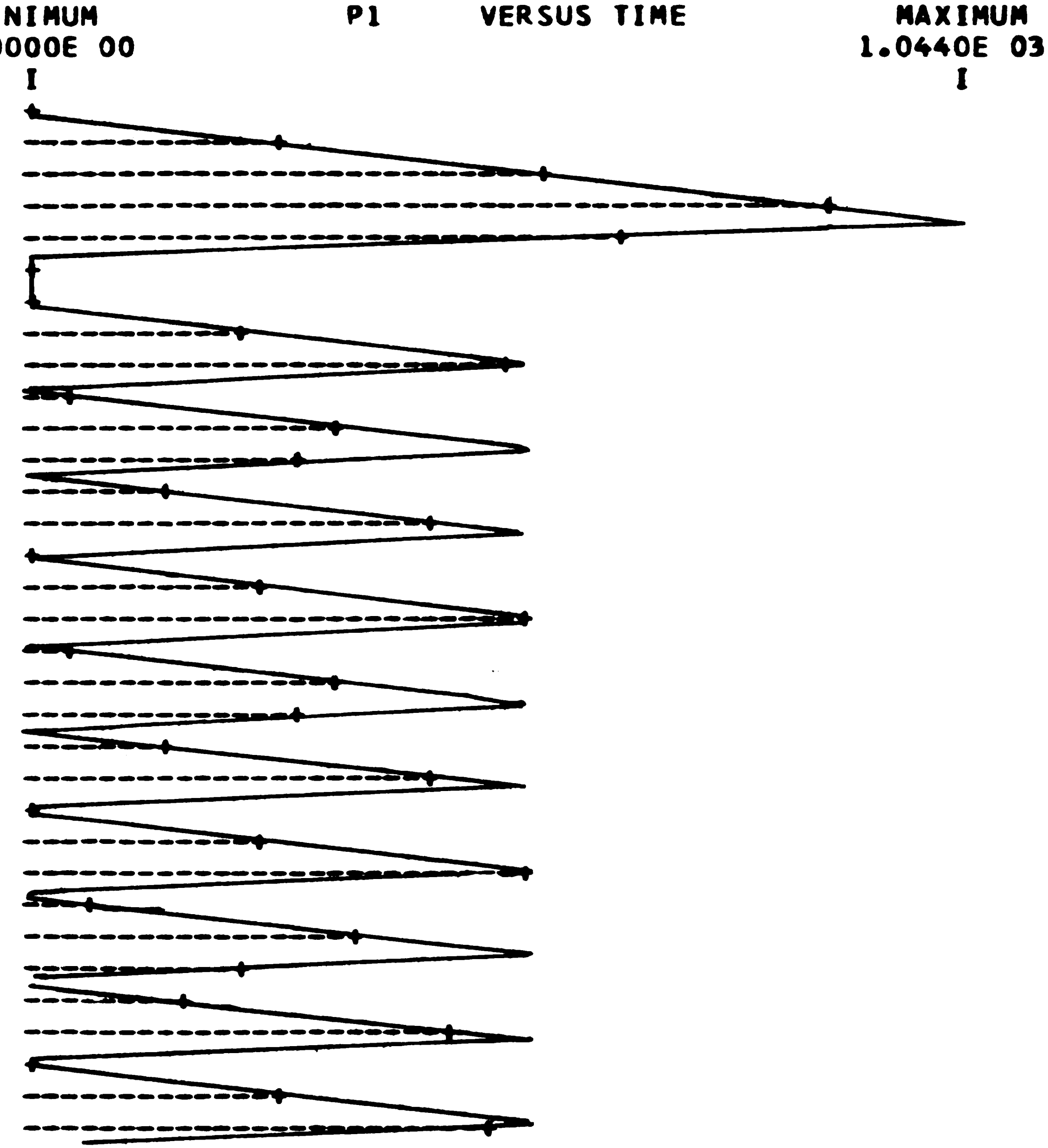


Figure 5.15 Histogram of Control Pressure

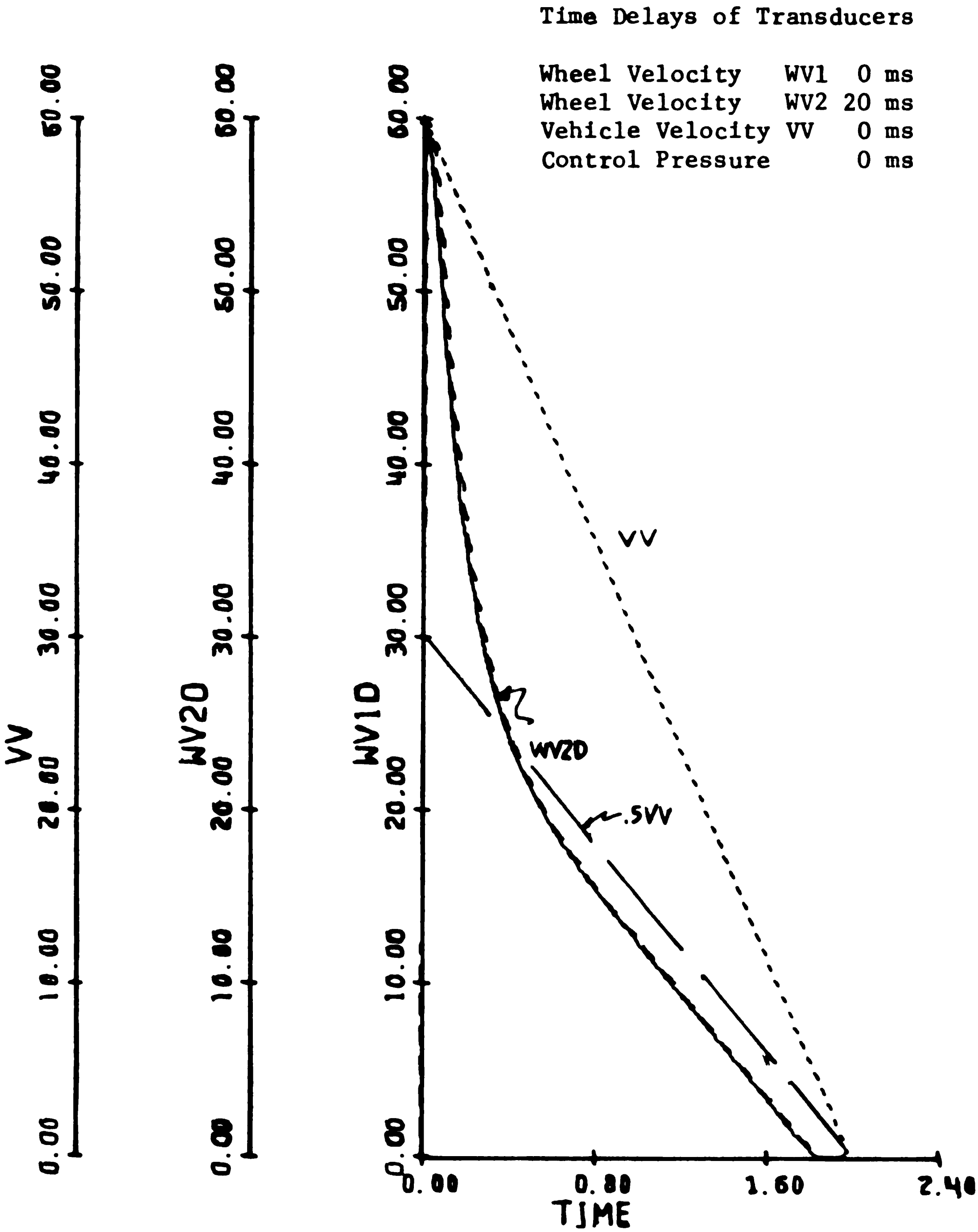


Figure 5.16

Velocity Response
Suboptimal System Employing a Single Control Pressure Signal

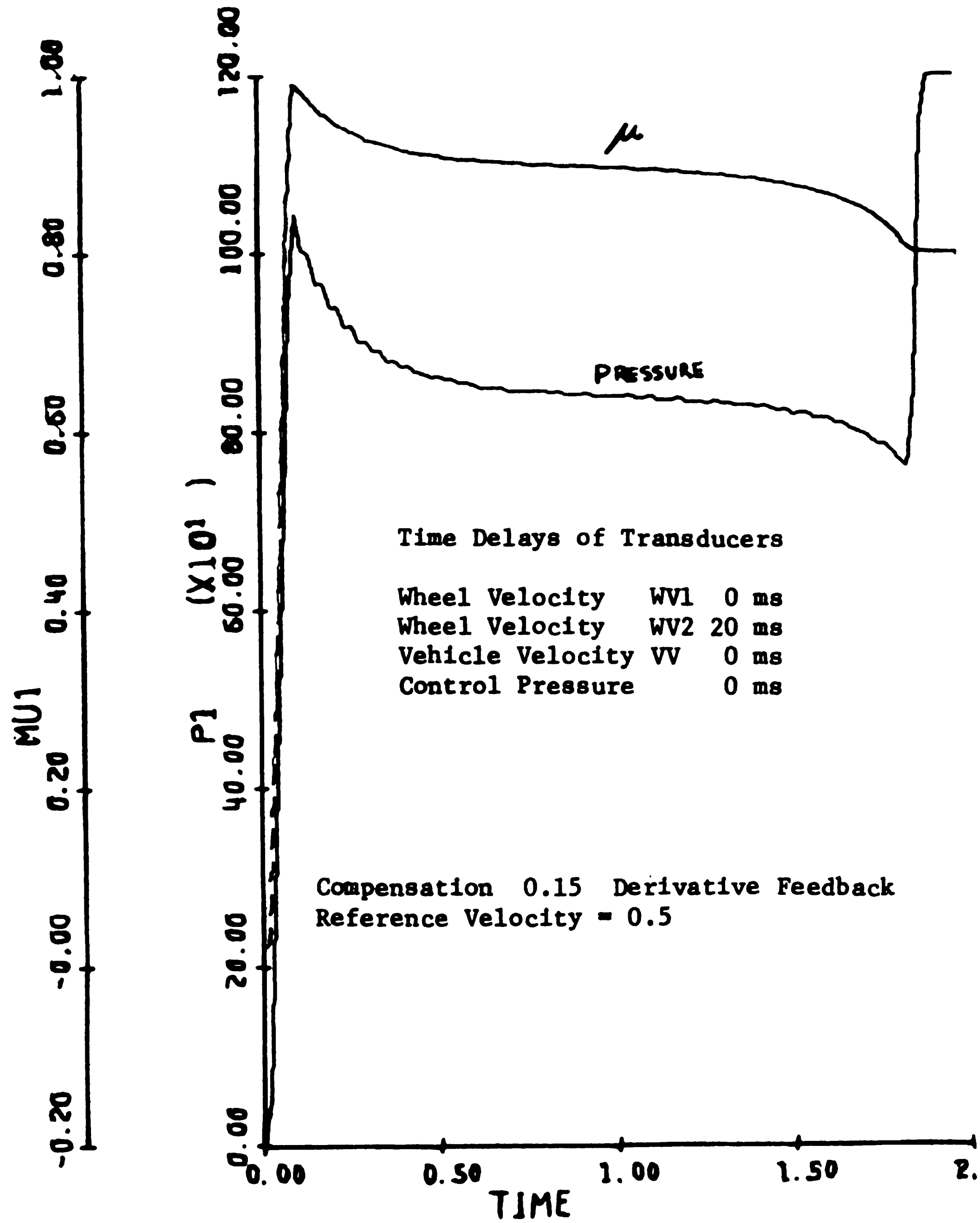


Figure 5.17

Friction and Control Pressure Response
Suboptimal System Employing a Single Control Pressure Signal

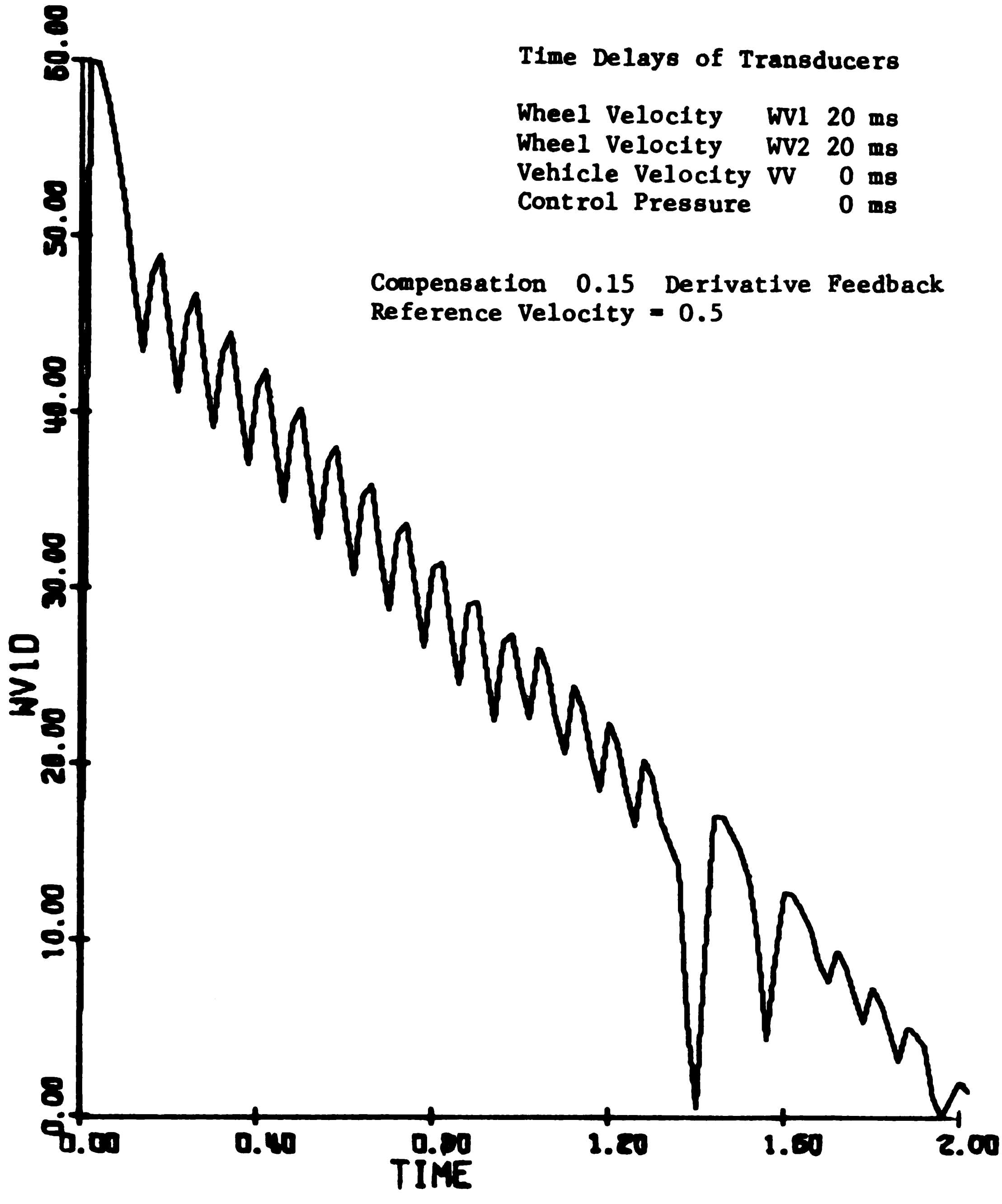


Figure 5.18

Wheel Velocity Response
Suboptimal System Employing a Single Control Pressure Signal

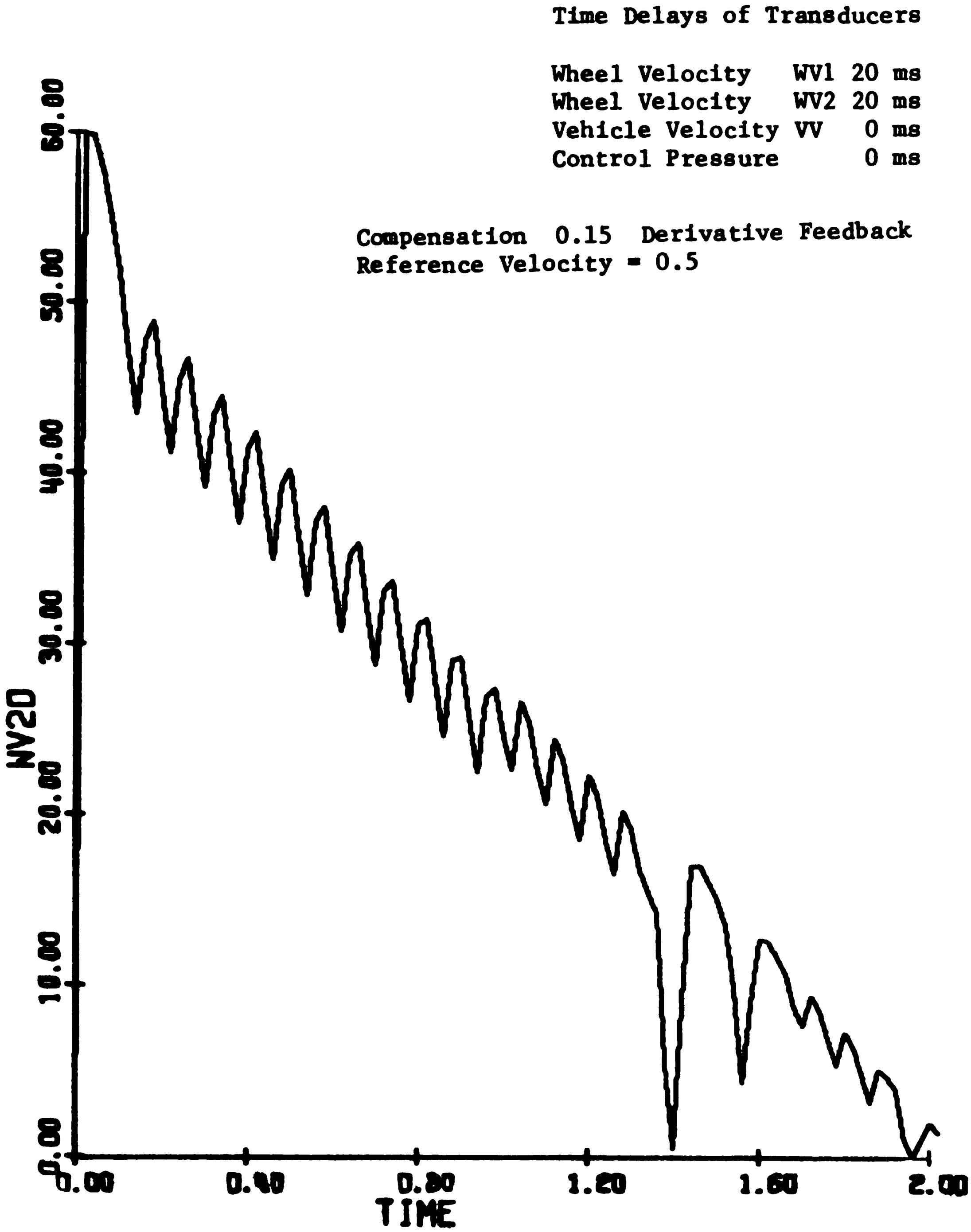


Figure 5.19

Wheel Velocity Response
Suboptimal System Employing a Single Control Pressure Signal

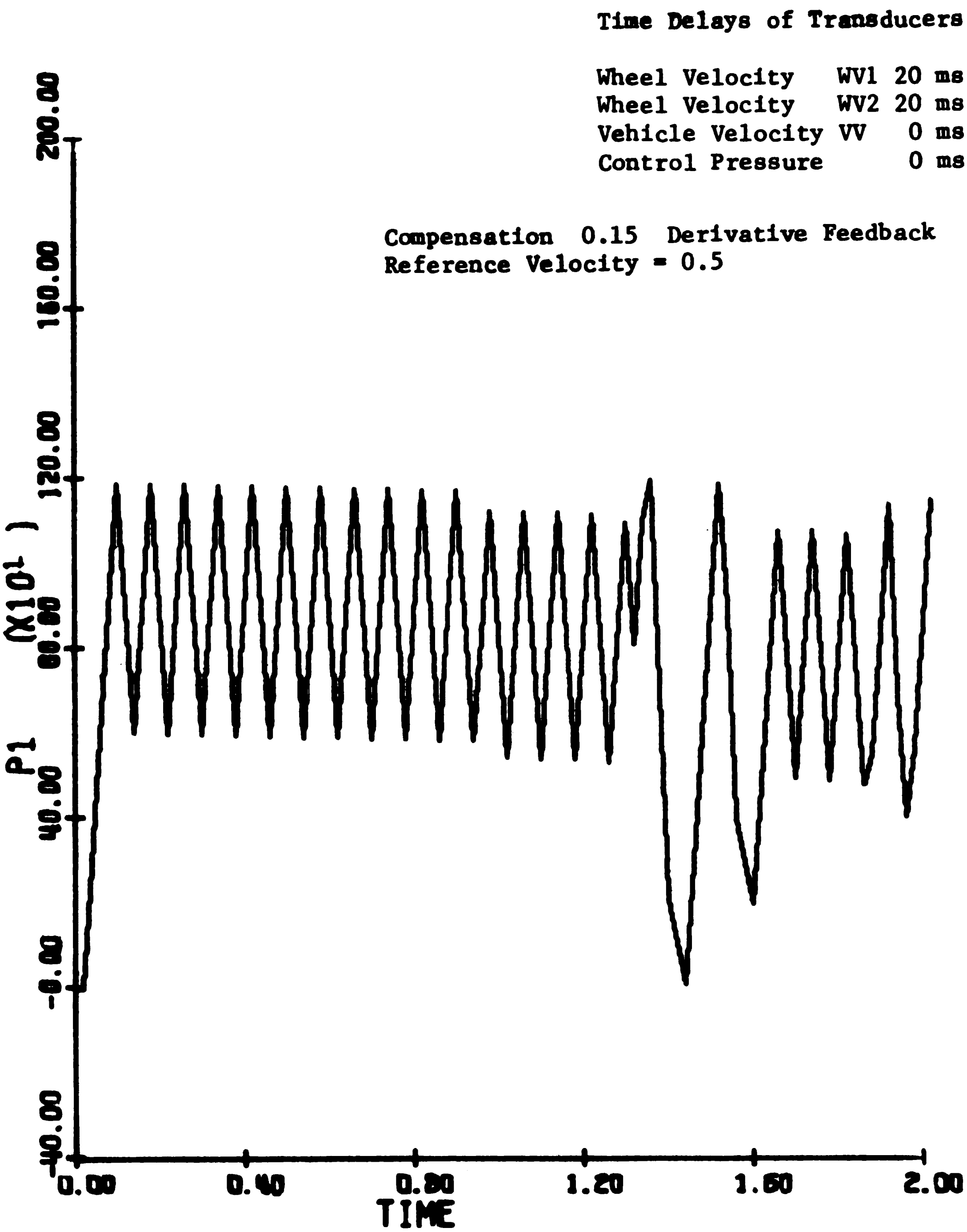


Figure 5.20

Control Pressure Response
Suboptimal System Employing a Single Control Pressure Signal

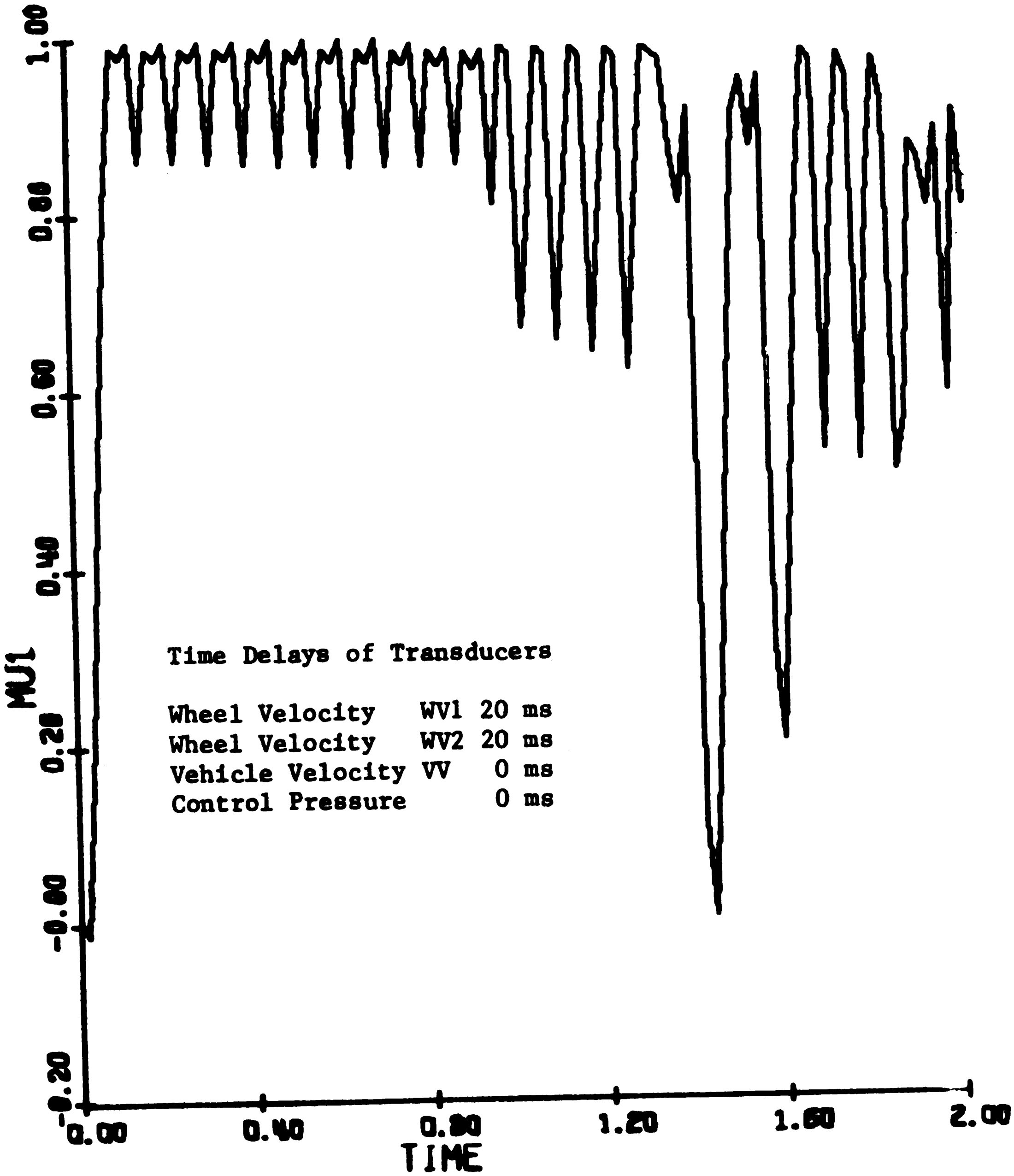


Figure 5.21

Friction Characteristic
Suboptimal System Employing a Single Control Pressure Signal

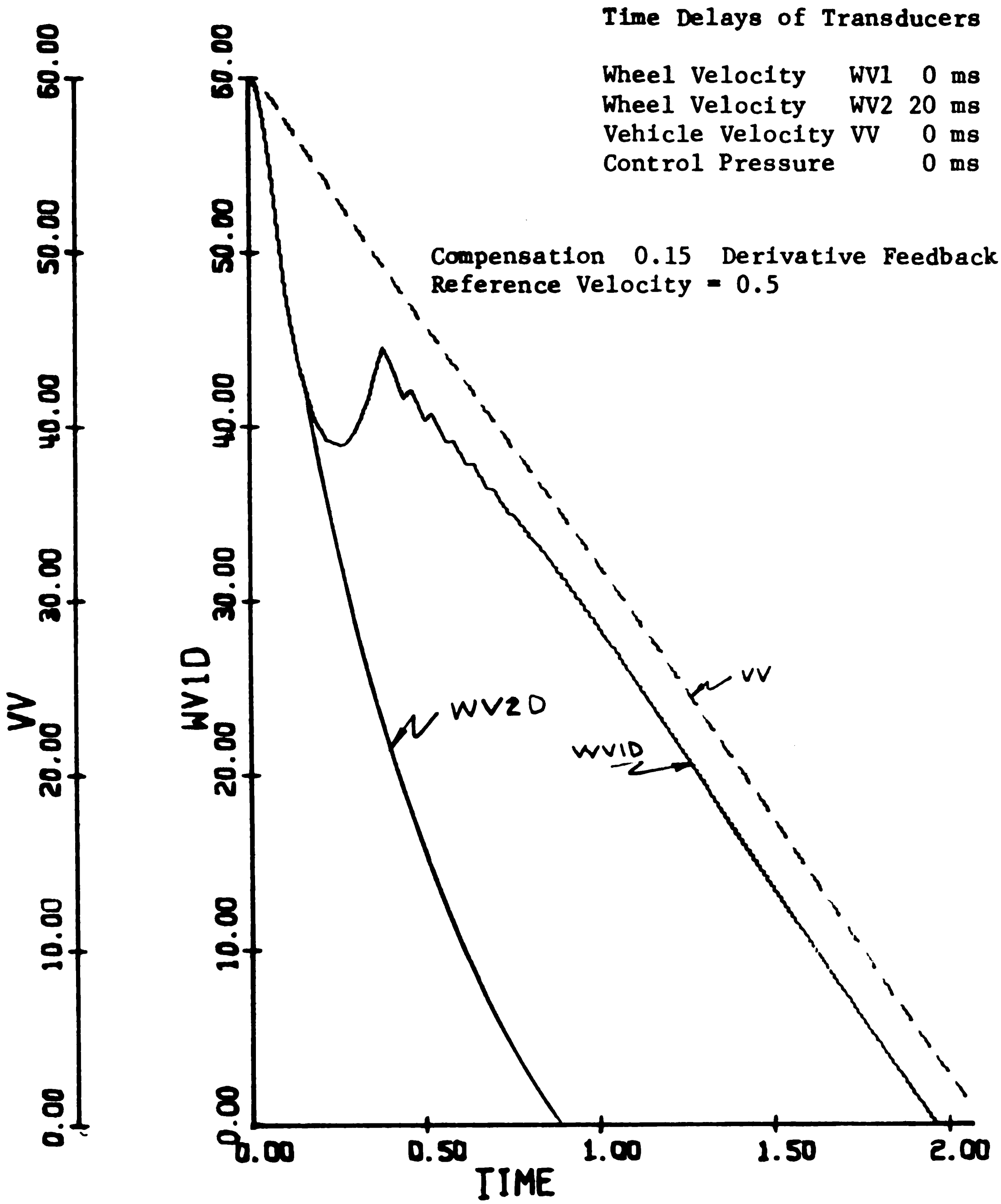


Figure 5.22

Wheel Velocity Response - Unbalanced Friction
Suboptimal System Employing a Single Control Pressure Signal

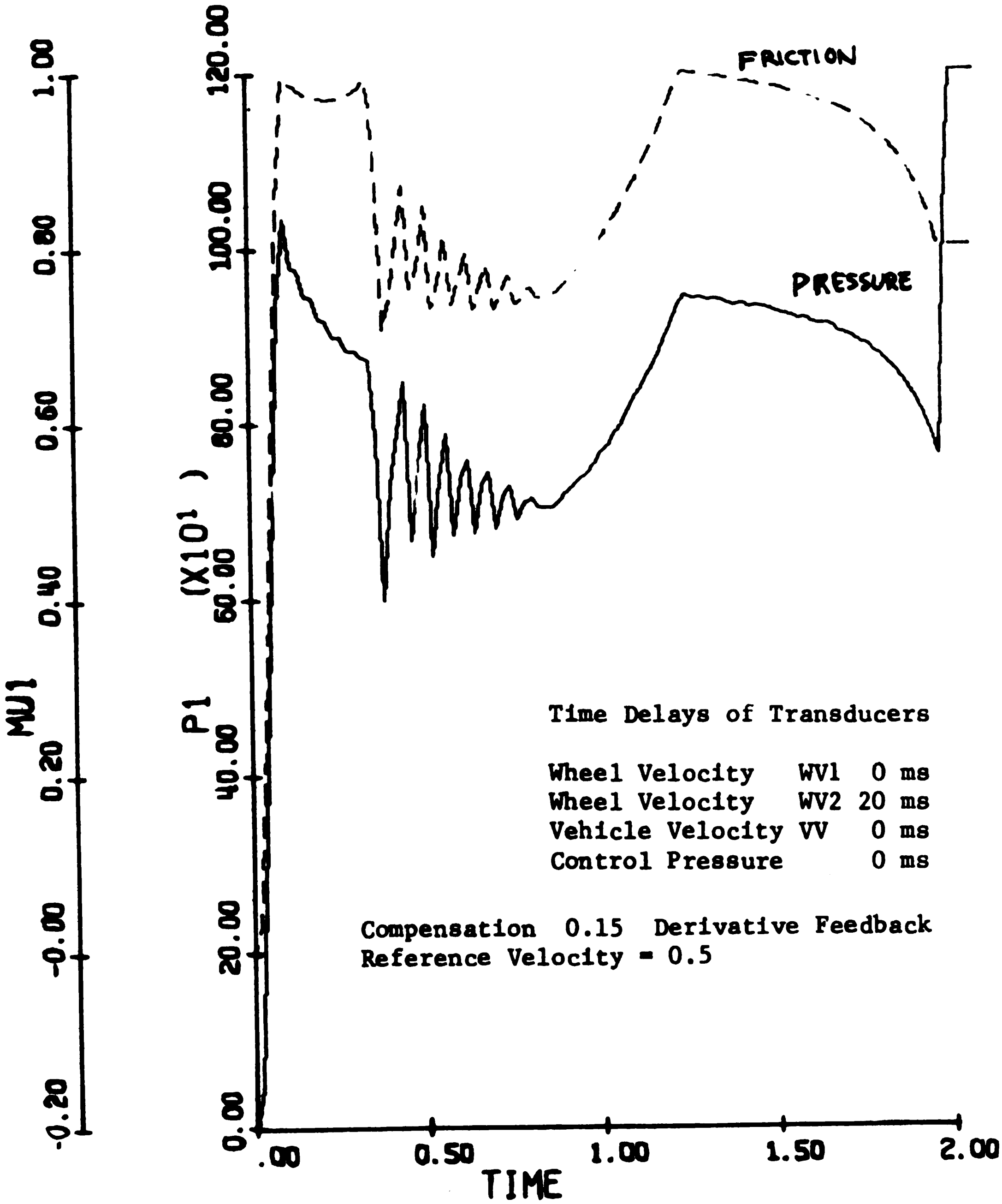


Figure 5.23

Friction Coefficient and Control Pressure Response
Suboptimal System Employing a Single Control Pressure Signal

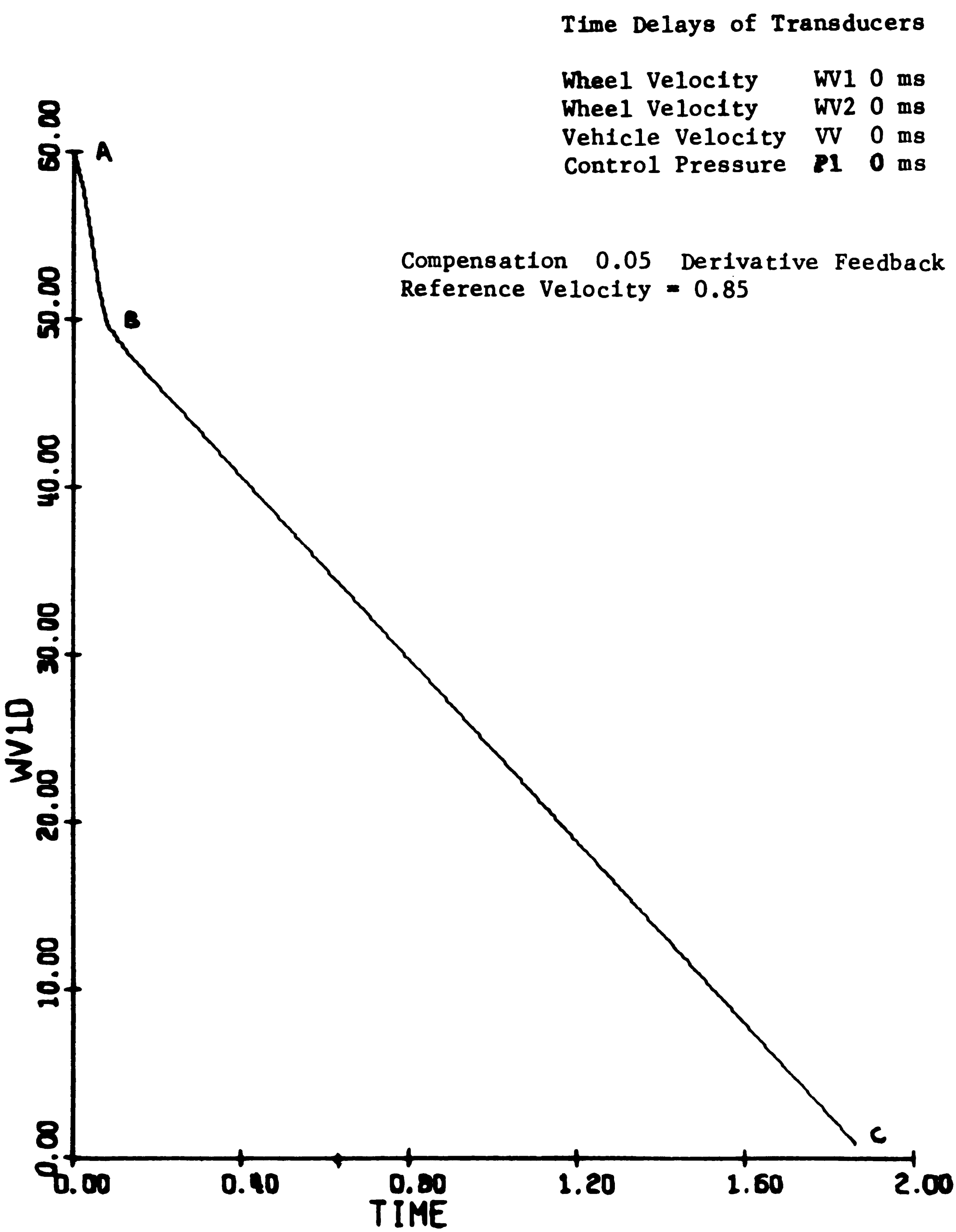


Figure 5.24

Wheel Velocity Response
Reference Suboptimal Control System

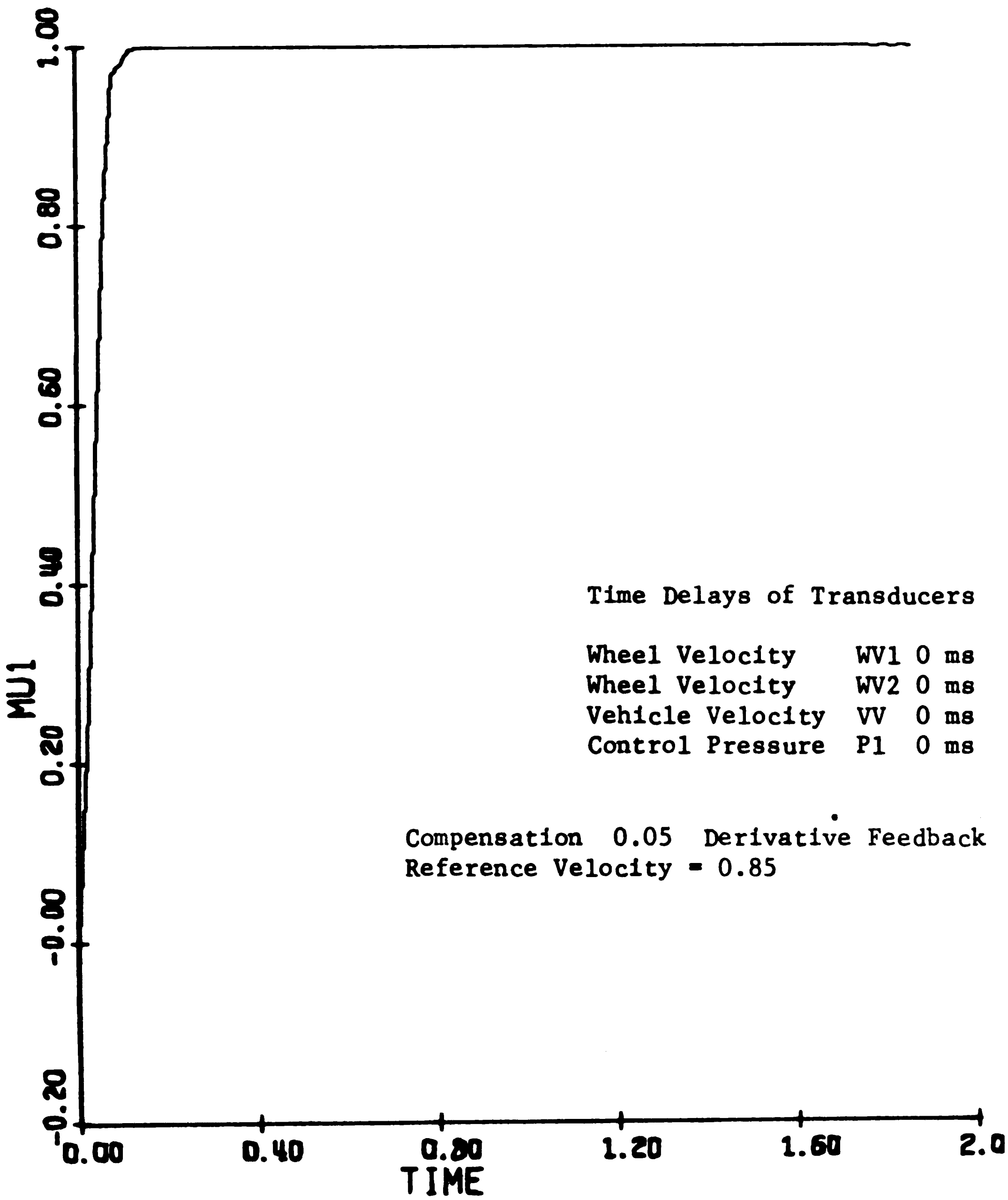


Figure 5.25

Friction Coefficient Response
Reference Suboptimal Control System

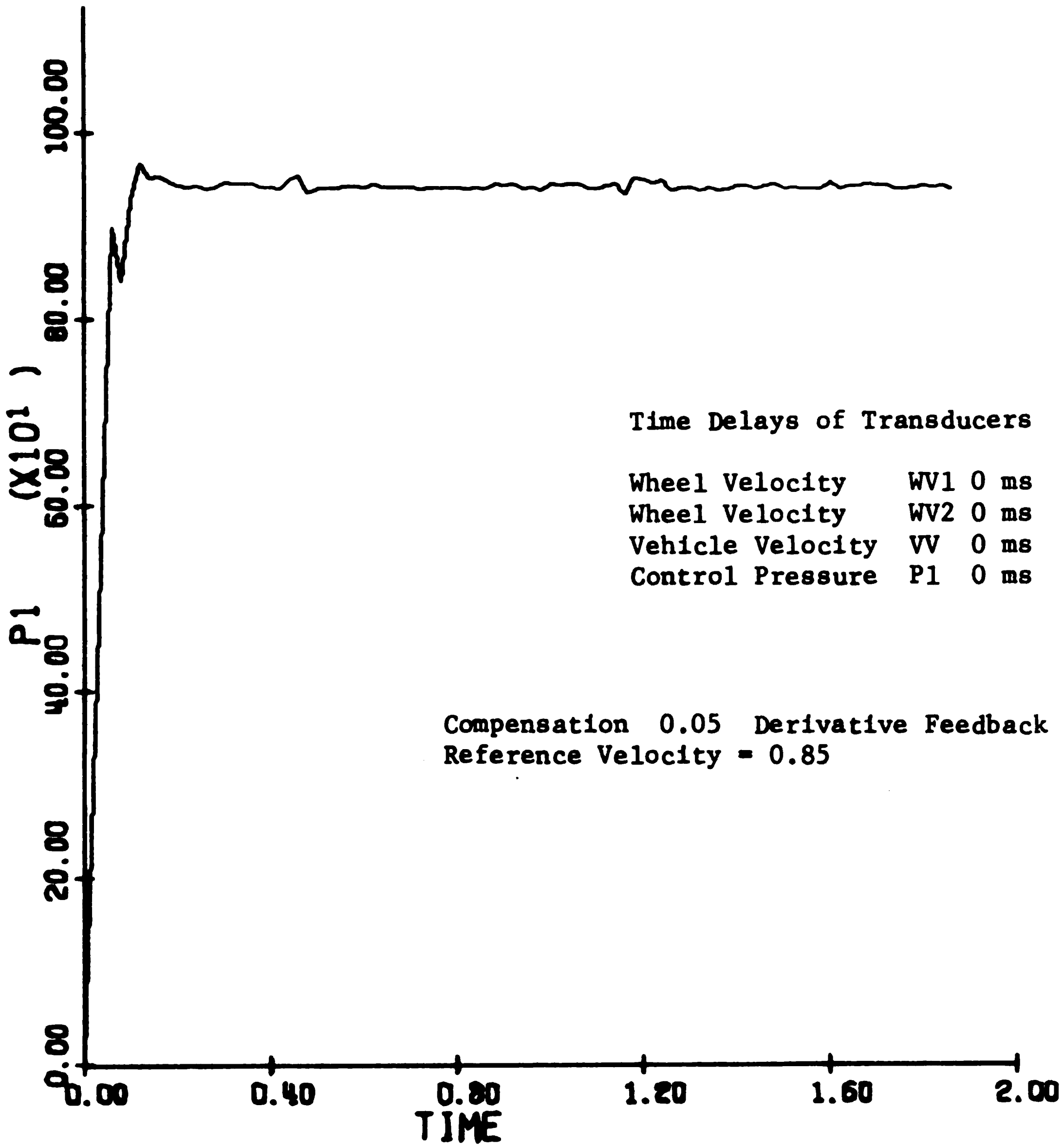


Figure 5.26

Control Pressure Response
Reference Suboptimal Control System

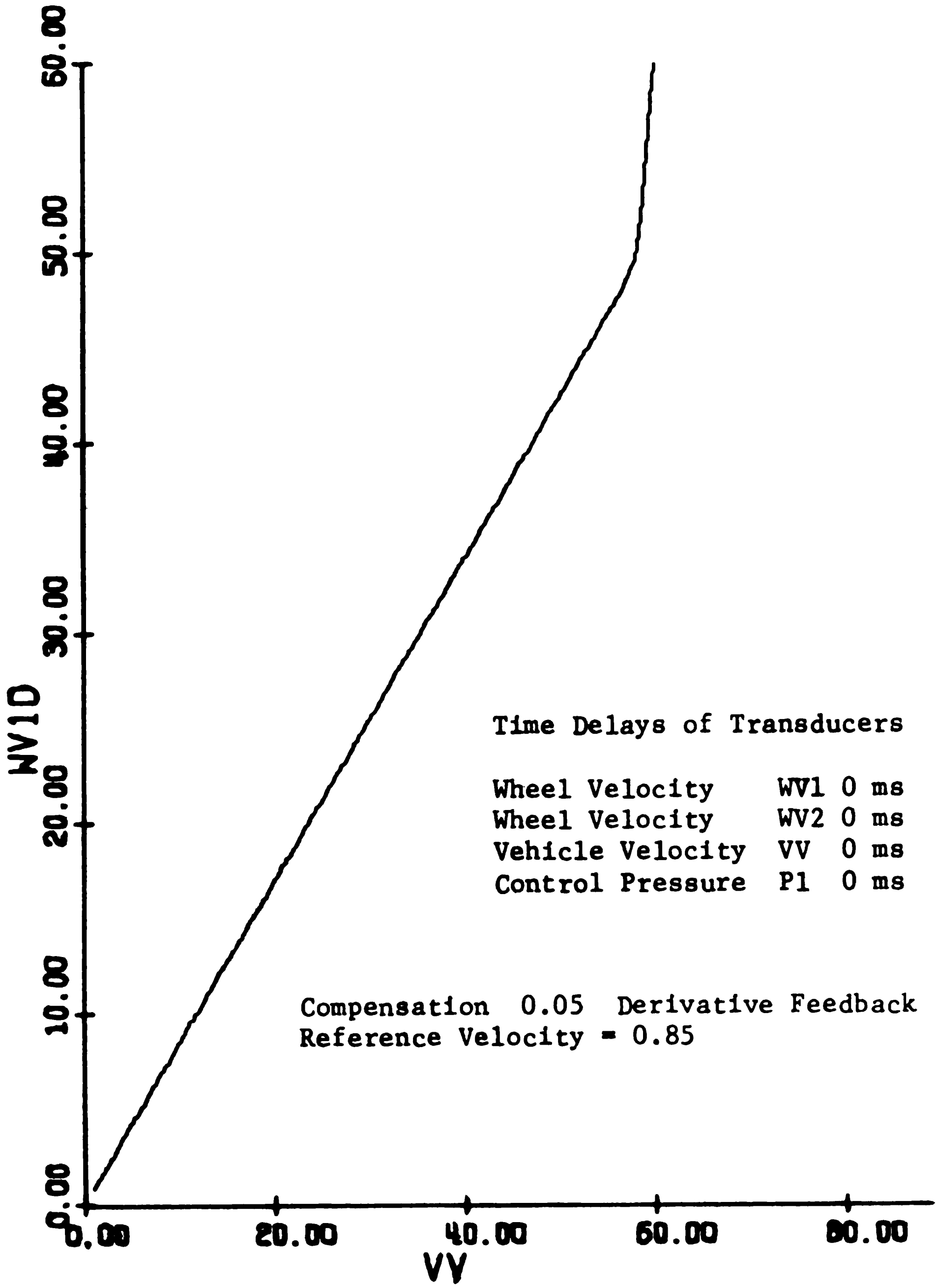


Figure 5.27

Phase Plane Plot
Reference Suboptimal Control System

CHAPTER VI NONLINEAR PHENOMENON IN VEHICULAR BRAKING PROCESSES

6.1 Introduction

As the complexity of systems increases, it becomes more difficult to predict the effects of the various system parameters. Also, in the vehicular system considered, the wheel velocities are tightly coupled. This results in the generation of frequencies vastly different than those obtained in the loosely coupled case. The presence of nonlinearities is responsible for the generation of additional frequencies. In this chapter, attempts to improve the intuitive feeling for some aspects of the system will be made.

6.2 Time Delay Criteria by Describing Function Technique

The suboptimal control system generates prominent variations in wheel velocity, friction coefficient, and control pressure. These would not be present in the optimal control system. These ripple frequencies are primarily due to the time delay constraints present in the suboptimal control system with minimal time delays. It will be shown that the ripple frequencies may be used as a measure indicating the degree of optimality achieved. Describing function techniques will be used to develop criteria.

The one-wheel model using s-plane representations is shown in Figure 6.1. The friction-slip curve, shown in Figure 6.2, will be linearized about a typical operating point. The gain characteristic will normally be negative; but, as evident from the friction characteristic, may be positive. This gain term will be defined as,

$$k_\mu = \Delta\mu/\Delta\eta \tag{6.1}$$

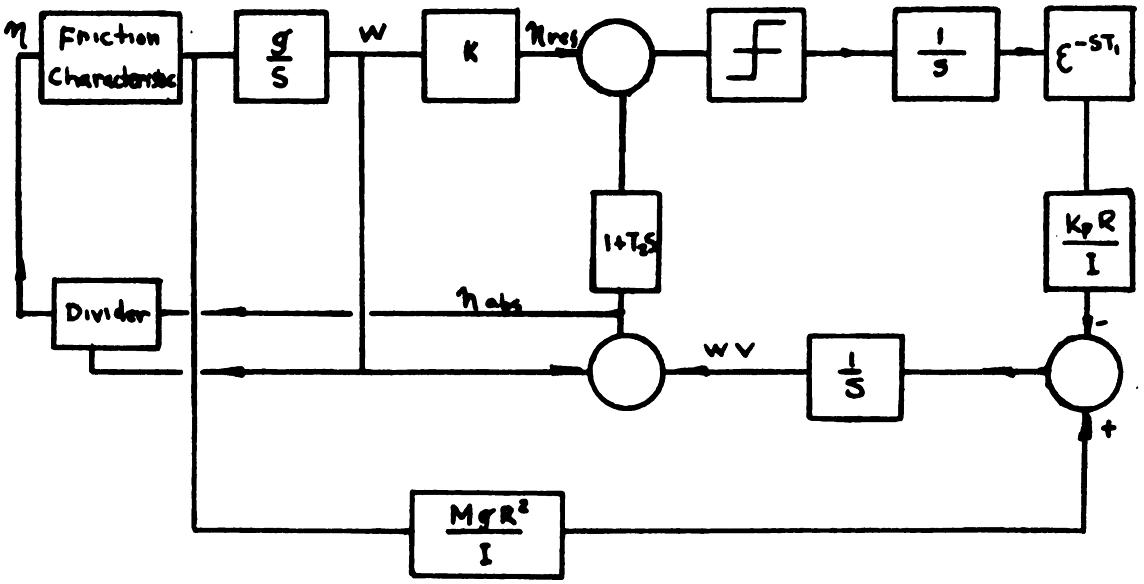


Figure 6.1

S-plane Model of One-wheel Vehicular Control System

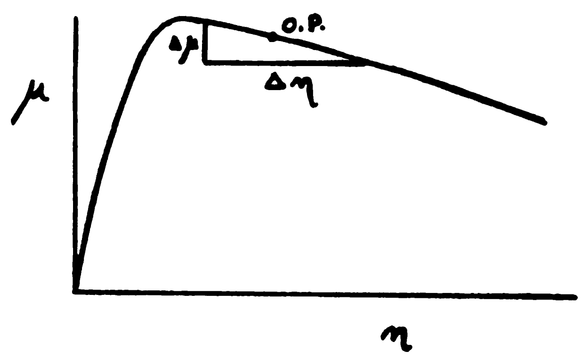


Figure 6.2

Friction-slip Curve Linearized about Operating Point

Since the vehicle velocity VV changes with respect to the wheel velocity very slowly, it will be assumed constant. The effect of a changing vehicle velocity will be discussed later. On an incremental basis, the circuit may be represented as shown in Figure
The notation is the same as previously defined in Chapter IV.

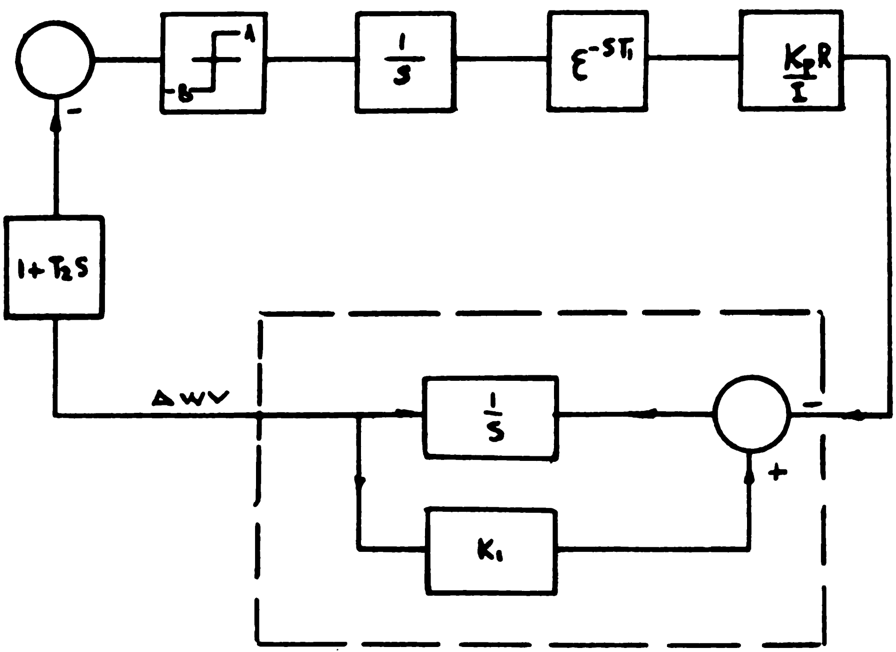


Figure 6.3

Simplified Incremental Block Diagram
One-wheel Vehicular Control System

The block involving the μ characteristic has a transfer function given by

$$T.\ F. = \frac{1}{s - k_1} \tag{6.2}$$

where,

$$k_1 = k_\mu M_g R^2 / I_{VV} \tag{6.3}$$

For application of the Describing Function Technique, the final form shown in Figure 6.4 is desirable.

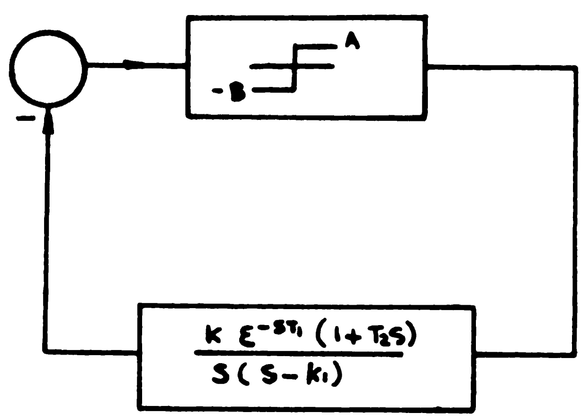


Figure 6.4

Nonlinear System used to Develop Time Delay Criterion

where,

$$K = K_p R/I$$

T_2 is the time constant associated with the lead network.

T_1 is the total time delay present in the system.

Study of this block diagram reveals several significant characteristics of the vehicular control system. First, if both time delays are neglected, the system stability depends on the sign of k_1, which in turn depends on the friction-slip gain k_μ. Since the operating point is usually on the negative slope portion of the friction-slip curve, the system is inherently unstable. This instability, however, has no adverse affect on the optimality of the system. Some of the unusual phenomenon observed is however due to this characteristic.

The second characteristic of interest is the chattering of the wheel velocity as it is driven to zero. The chattering is readily explained by means of the Describing Function Technique. Designating the nonlinear element as $N(e, \omega)$ and the linear element as $G(\omega)$, the oscillatory of chattering condition is

$$G(\omega) = - 1/N(e, \omega) \qquad (6.4)$$

Here e is the amplitude of the input to the relay element. The input signal is assumed to be sinusoidal. A sketch of a typical Nyquist Plot appears as shown in Figure 6.5

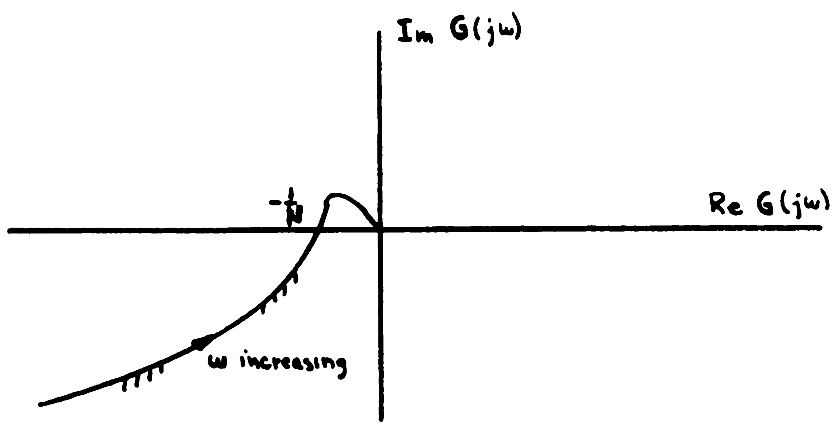


Figure 6.5

Typical Nyquist Plot

For typical values of $T_1 = .010$ seconds and $T_2 = .05$, the critical point is reached when ω is approximately 142 radians per second and $N = 2860$. A computer study was conducted to establish the accuracy that could be expected. Using a symmetrical relay characteristic which switches between -5000 and +5000, the input was approximately 3. Since the fundamental component of the square wave is $4/\pi \cdot 5000$, the gain N of the relay element is 2120. This, with the loop gain of 1.32 gives a total gain of 2800. The frequency was 22 cps or 138 radians per second. These values are very close even though the generated waveforms in the system are square, triangular and finally, approximately sinusoidal at the input of the relay element.

A study of the Nyquist Plots shows that a simple criterion may be established to evaluate time delays in this system. By neglecting the effect of the gain factor k_1, the angle criterion at the critical point $-1/N$ is satisfied by the following condition:

$$\text{Angle of } \varepsilon^{-j\omega T_1}(1+j\omega T_2) = 0 \tag{6.4}$$

This is equivalent to the condition,

$$\text{Tan } \omega T_1 = \omega T_2, \qquad \omega T_1 \leq \pi/2 \tag{6.5}$$

Solution of this transcendental equation gives the chatter frequency in terms of the two time constants. Observation of various solutions shows that ωT_1 is almost $\pi/2$ radians for all cases of interest. Thus, the simplified form below may be used,

$$T_1 = \frac{\pi}{2\omega} \text{ seconds} \tag{6.6}$$

Thus if $\omega = 142$ radians per second, the time delay in the system is approximately .011 seconds. This simplified criterion is quite useful in establishing the time delay in the system.

Concerning the effect of the neglected term k_1, it is readily shown that at the frequency of chatter, this term is insignificant except at very low velocities where it tends to reduce the chatter frequency.

Due to the asymmetric character of the relay element, a Dual Input Describing Function Technique was also investigated. For suboptimal operation, due to the small variations which are essentially sinusoidal, no significant additional information was obtained by this method.

6.3 Effect of Friction-slip Nonlinearity

In the previous section, the Describing Function Technique was shown to be useful in establishing a criterion for estimating time delays of the suboptimal control system. If compensation is not used, the system performance is adversely affected. The criterion established in the previous section is no longer valid and it is necessary to include the effect of the friction-slip nonlinearity shown in Figure 6.2. For time delays which are in the realizable range - 10 milliseconds to 20 milliseconds - operation will be on the positive slope portion of the friction-slip curve. On this portion of the curve, the gain factor k_1 is significant and to a first approximation the following criterion will establish the dominant frequency of the variation:

$$\text{Angle of } \varepsilon^{-j\omega T_1}/(k_1 + j\omega) = -\Pi/2 \text{ radians} \tag{6.7}$$

Referring to Figure 6.4, it is to be noted that the loss of the lead term due to the compensation network causes the system to make up this phase change by finding a suitable gain factor k_1 which reduces the phase of the $(s + k_1)$ term in the denominator of the transfer function.

Several cases where no compensation was used were investigated. The results of a system having no compensation and 10 ms delay is shown in Figures 6.6 to 6.9.

Detailed study of the waveforms in these figures indicates that the criterion given by (6.7) accurately predicts system performance. The block diagram shown in Figure 6.10 will be used to illustrate the procedure. Except for the time delays, the diagram is based on the program listing shown in Figure 5.8.

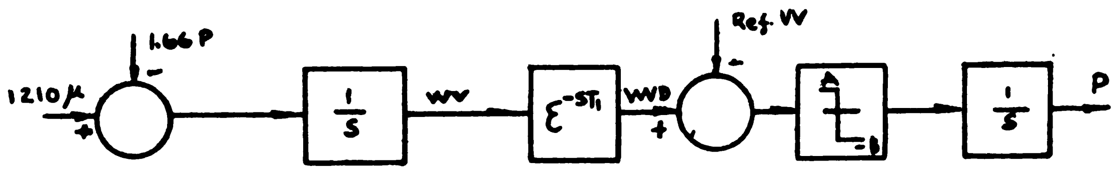


Figure 6.10

Diagram Used to Evaluate Variational Frequency
and Amplitudes of Variables

The procedure is as follows for the system having 10 milliseconds delay.

Since operation is on the positive slope portion of the friction-slip curve, k_1 is estimated as 40. The transcendental angle criterion (6.7) is then solved for ω. The result is approximately 60 radians per second, comparing favorably with the measured frequency of 9.75 cps.

The friction-slip coefficient response should lag the pressure response by 34.5 degrees. Detailed analysis of the response curves in Figure 6.7 and 6.8 shows this to be the case.

The pressure amplitude is found by finding the fundamental component of the asymmetrical relay output and dividing by ω. The peak to peak fundamental is $\frac{4}{\Pi} \cdot .78 \cdot 60{,}000/60$ or 995 psi. This is essentially the same as the measured value.

From the transfer function $1/(s + k_1)$ which relates pressure and wheel velocity, the wheel velocity amplitude is calculated as 13.8 fps peak to peak. This is higher than the measured value which is approximately 10 fps peak to peak.

The magnitude of the friction-slip coefficient variation is found from

$$1210\mu = WV \cdot k_1 \tag{6.8}$$

This results in a predicted value of 0.46, which is condiderably lower than the measured value of approximately 0.9.

Considering the large amplitudes, the results are not unsatisfactory. The Dual Input Describing Function Technique was not used here, but would probably improve the accuracy. The presence of other frequencies is clearly evident from the pressure response in Figure 6.7.

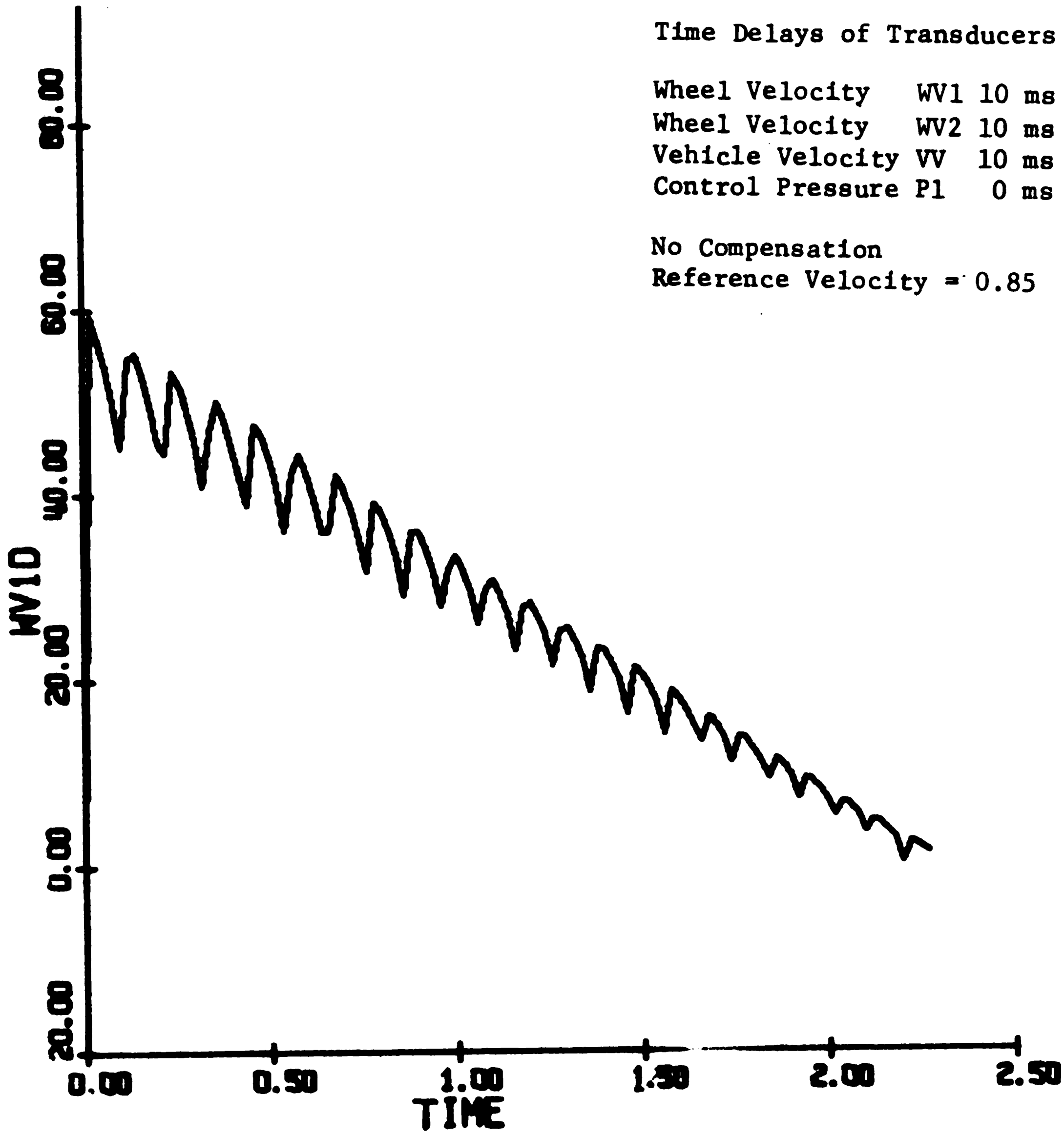


Figure 6.6

Wheel Velocity Response

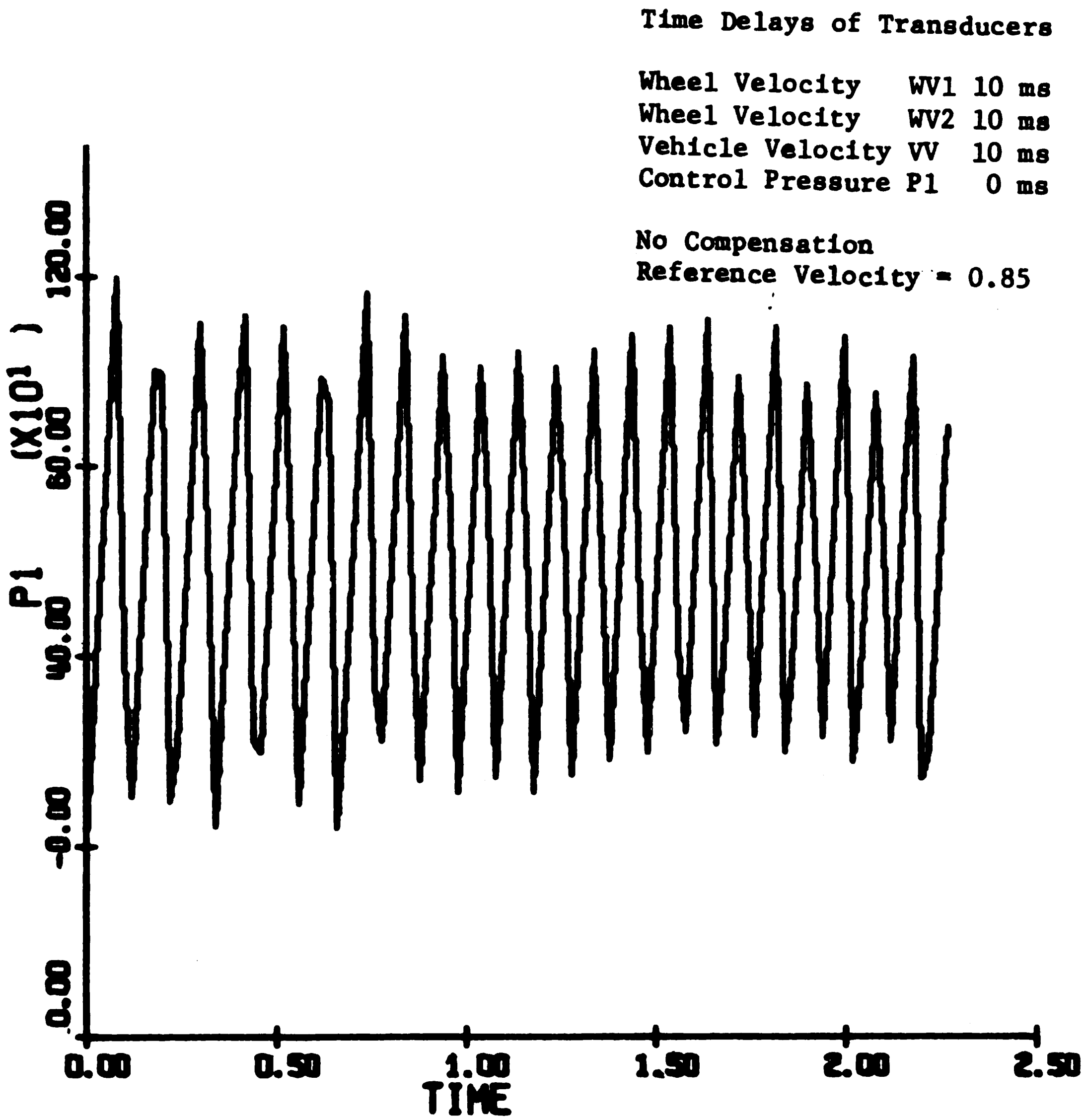


Figure 6.7

Control Pressure Response

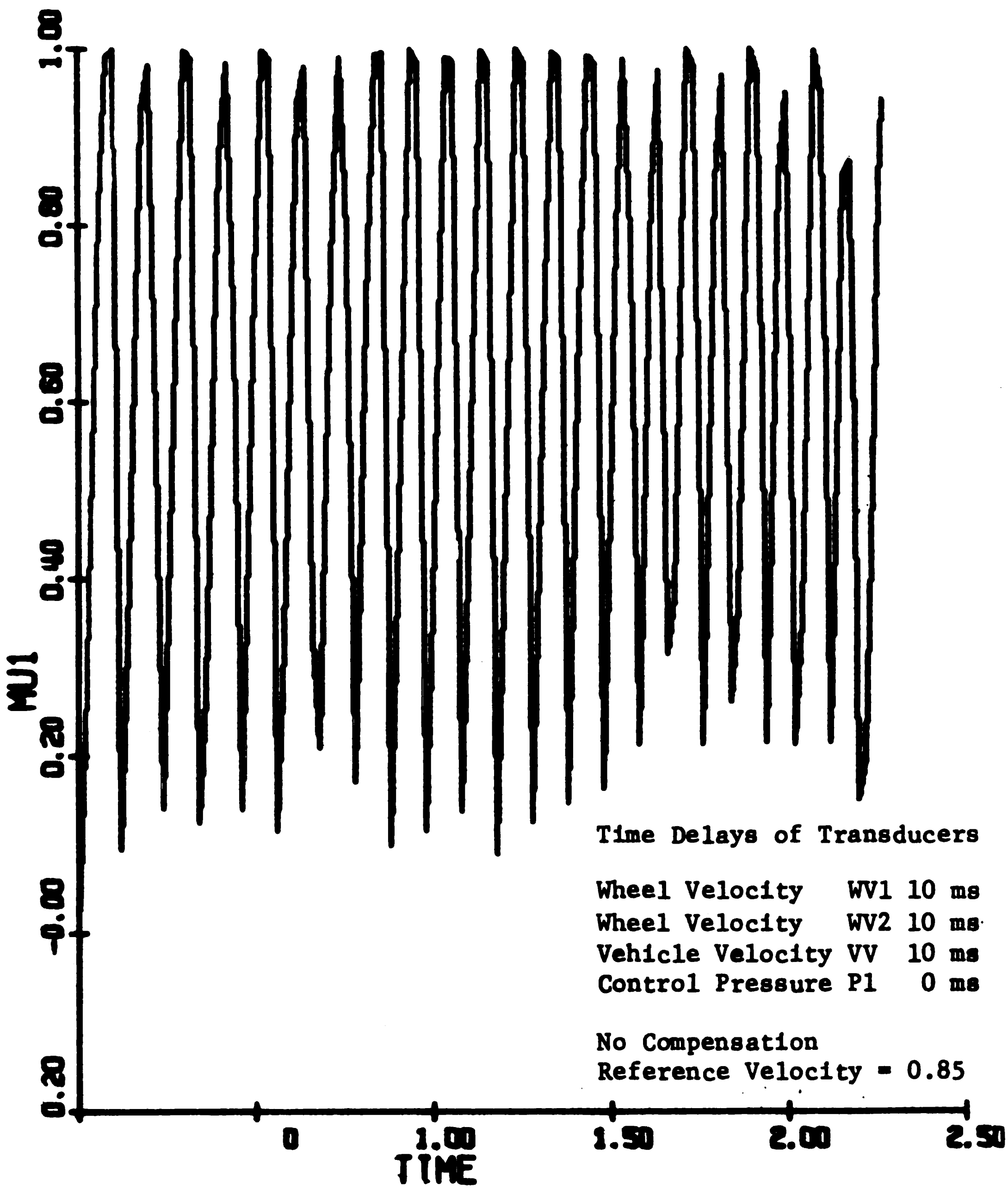


Figure 6.8

Friction Coefficient Response

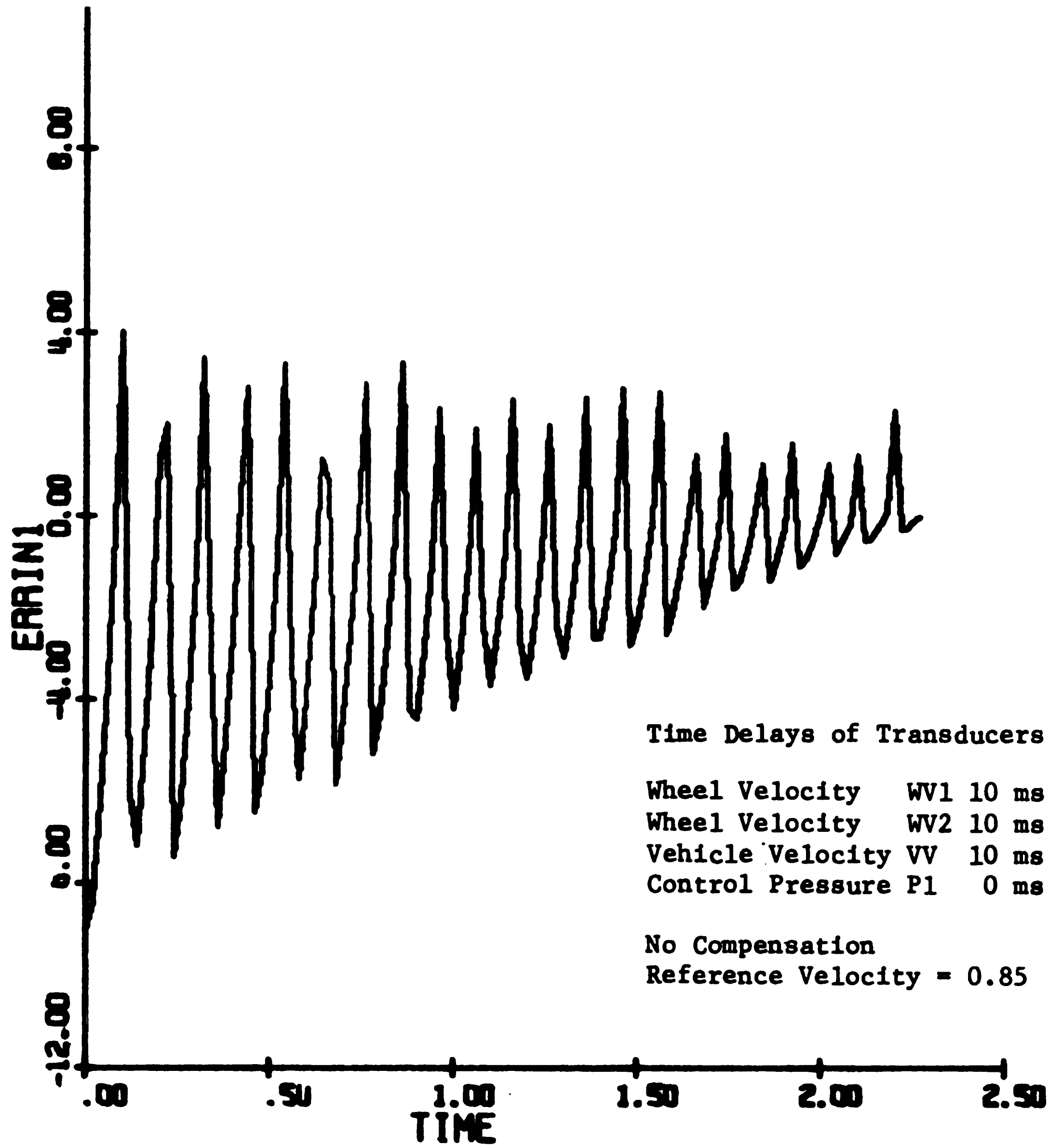


Figure 6.9

Output Response of Asymmetrical Relay

CHAPTER VII CONCLUSION

The development of the system equation for the vehicular braking control system shows that the control signal appears linearly. This implies, since the system is nonlinear, the possibility of singular controls. From the unified singular control theory presented, necessary conditions which the time optimal control must satisfy are developed. The class of functions encountered in the vehicular braking control system are such that the minimum stopping time problem is equivalent to the minimum stopping distance problem.

Based on the necessary conditions developed, the closed loop problem is solved and a block diagram showing the mechanization using the Maximum Principle approach is presented. Since the initial costate vector must be determined on-line, subject to various initial conditions on the state vector, any cost functional which takes into account factors such as, cost and simplicity would eliminate this method as a possible candidate.

A more practical approach is the mechanization developed by applying the Green Theorem approach. The critical component in this method is the ω function block which determined the singular condition. This method is quite possible in applications which are relatively slow. An algorithm for determining the optimal control is presented. For the vehicular braking control system, where significant changes occur in milliseconds, the method becomes costly.

At this stage of design, the gap between theory and practice is apparent.

The mathematical models which have been developed are not sufficiently sophisticated to include noise, variability, cost, reliability and other realistic factors. Inclusion of these factors would subject the models to further constraints and adversely affect the performance.

The mathematically expedient models function as reference models, indicating the ultimate that can be expected, and also giving clues as to how the optimal control should function. As a result, a system called the suboptimal vehicular braking control system was developed. This system is optimal in the sense that it heuristically considers cost and simplicity and is suboptimal since minimum stopping distance is slightly greater than the optimal control system subject to a simple cost criterion. The advantages gained far outweigh the effect of slightly greater stopping distances. In zero time delay case, the stopping distance for the suboptimal control system was 57.27 feet as against 57.2 feet for the optimal control system. For systems with time delays, it would be desirable to have optimal control models which include time delays. However, by employing proper compensation, the suboptimal control systems with realistic time delays compare favorably with optimal control systems having no time delays.

Whereas, most of the effort was devoted to one-wheel models, studies of two-wheel models indicate that coupling effects will introduce several new problems. This is especially true if the system is constrained to use one control signal to control two wheels under different friction conditions. Several criteria were developed to assist in the understanding of the nonlinear phenomena which take place. The criterion which evaluates time delays present in the system is particularly useful.

From the viewpoint of the builder of vehicular braking systems - in particular, the automobile manufacturer - cost is a heavily weighted factor in the performance functional. Elimination of a costly transducer is desirable. At the present time, the vehicle velocity transducer is in this category. This leads to a very significant vehicular braking control problem - the optimal control with inaccessible state variables.

Based on analog and digital studies already conducted, suboptimal control systems with inaccessible state variables compare favorably with the optimal control system having accessible state variables. Hence, the solution to the inaccessible state variable problem is of interest. With the addition of time delays, these significant problems are left for future development.

APPENDIX I

SUBOPTIMAL ONE-WHEEL MODEL CONTROL SYSTEM

```
        ****CONTINUOUS SYSTEM MODELING PROGRAM****

    ***PROBLEM INPUT STATEMENTS***

TITLE      ASKC
FUNCT      MUET1=(-.15,-1.),(0.,0.0),(.075,.6),(.15,1.),(.57
FUNCT      MUET2=(-.15,-1.),(0.,0.0),(.075,.6),(.15,1.),(.57
METHOD     RKSFX
PARAM      A=1.,B=.95
           MU1=A*AFGEN(MUET1,ETA1)
           MU2=B*AFGEN(MUET2,ETA2)
           VVDOT=-8.*(MU1+MU2+A+B)
           VV=INTGRL(60.0,VVDOT)
           VVD=VV
           SDIST=INTGRL(0.0,VV)
           WV1=INTGRL(60.0,WV1DOT)
           WV1DOT=1210.*MU1-0.79*T1
           WV2DOT=1210.*MU2-0.79*T2
           WV2=INTGRL(60.0,WV2DOT)
           WV11=LIMIT(0.,60.0,WV1)
           WV1D=DELAY(1,.020,WV11)
           ETA1=1.0-WV11/VV
           WV22=LIMIT(0.,60.0,WV2)
           WV2D=DELAY(1,.020,WV22)
            ETA2=1.0-WV22/VV
           T1=1.66*P1
           T2=1.66*P2
           ETA1R=0.5*VVD
           ETA1A=VVD-(WV1D+WV2D)*.5
           Y1=DERIV(0.0,ETA1A)
           ERRIN1=(ETA1A+.15*Y1)-ETA1R
           ERRIN2=(ETA2A+.15*Y2)-ETA2R
           ERROT1=INSW(ERRIN1, 1.0,-3.0)
           E1=15000.*ERROT1
           P1=INTGRL(0.0,X1)
           P2=P1
           X1=X11*E1
           X11=IOR(Y111,Y112)
           Y111=AND(Y11,P1)
           Y112=AND(E1,Y12)
           Y11=NOT(E1)
           Y12=NOT(P1-1200.)
TIMER      FINTIM=4.,DELT=.0002,DELMIN=.000000001,PRDEL=.02
FINISH     VV=1.0
PRTPLT  SIZE(4.,6.),TIME,WV1D,VV
LABEL   WHEEL VELOCITY, VEHICLE VELOCITY VS TIME
PARAM SIZE=0.
PRINT      ETA1R,ETA1A,MU1,ERRIN1,ERROT1,     WV2D,VV,P1,SDI
PRTPLT   SIZE(4.,6.),TIME,P1,MU1
END
STOP
```

Typical Output of Suboptimal One-Wheel Control System

TIME - SEC.				
0.0	ETA1R =	3.0000E 01	ETA1A =	6.0000E 01
	ERROT1=	-3.0000E 00	WV2D =	0.0
	SDIST =	0.0	WV1D =	0.0
2.0000E-02	ETA1R =	2.9845E 01	ETA1A =	-3.1007E-01
	ERROT1=	1.0000E 00	WV2D =	6.0000E 01
	SDIST =	1.1969E 00	WV1D =	6.0000E 01
4.0000E-02	ETA1R =	2.9677E 01	ETA1A =	-4.4492E-01
	ERROT1=	1.0000E 00	WV2D =	5.9801E 01
	SDIST =	2.3874E 00	WV1D =	5.9797E 01
6.0000E-02	ETA1R =	2.9462E 01	ETA1A =	1.1416E 00
	ERROT1=	1.0000E 00	WV2D =	5.7759E 01
	SDIST =	3.5703E 00	WV1D =	5.7807E 01
8.0000E-02	ETA1R =	2.9200E 01	ETA1A =	3.4402E 00
	ERROT1=	1.0000E 00	WV2D =	5.4877E 01
	SDIST =	4.7436E 00	WV1D =	5.5042E 01
1.0000E-01	ETA1R =	2.8895E 01	ETA1A =	6.2622E 00
	ERROT1=	1.0000E 00	WV2D =	5.1340E 01
	SDIST =	5.9056E 00	WV1D =	5.1716E 01
1.2000E-01	ETA1R =	2.8586E 01	ETA1A =	1.0605E 01
	ERROT1=	-3.0000E 00	WV2D =	4.6034E 01
	SDIST =	7.0551E 00	WV1D =	4.7100E 01
1.4000E-01	ETA1R =	2.8277E 01	ETA1A =	1.4292E 01
	ERROT1=	-3.0000E 00	WV2D =	4.1052E 01
	SDIST =	8.1924E 00	WV1D =	4.3471E 01
1.6000E-01	ETA1R =	2.7977E 01	ETA1A =	1.0123E 01
	ERROT1=	1.0000E 00	WV2D =	4.3872E 01
	SDIST =	9.3173E 00	WV1D =	4.7789E 01
1.8000E-01	ETA1R =	2.7668E 01	ETA1A =	7.2314E 00
	ERROT1=	1.0000E 00	WV2D =	4.7189E 01
	SDIST =	1.0430E 01	WV1D =	4.9021E 01
2.0000E-01	ETA1R =	2.7360E 01	ETA1A =	1.0994E 01
	ERROT1=	-3.0000E 00	WV2D =	4.2632E 01
	SDIST =	1.1531E 01	WV1D =	4.4818E 01
2.2000E-01	ETA1R =	2.7051E 01	ETA1A =	1.4730E 01
	ERROT1=	-3.0000E 00	WV2D =	3.7541E 01
	SDIST =	1.2619E 01	WV1D =	4.1204E 01
2.4000E-01	ETA1R =	2.6751E 01	ETA1A =	1.0613E 01
	ERROT1=	1.0000E 00	WV2D =	4.0238E 01
	SDIST =	1.3695E 01	WV1D =	4.5541E 01
2.6000E-01	ETA1R =	2.6444E 01	ETA1A =	7.7312E 00
	ERROT1=	1.0000E 00	WV2D =	4.3437E 01
	SDIST =	1.4759E 01	WV1D =	4.6875E 01

APPENDIX II

PROGRAM USED IN OPTIMAL DIGITAL CONTROL SYSTEM

BASED ON GREEN'S THEOREM APPROACH

```
      DIMENSION  XY(10000), YY(10000), BUFFER(1000)
      CALL PLCTS(BUFFER(1), 4000)
      CALL PLOT(0.0,-12.0, 3)
      CALL PLCT(2.0,-11.5,2)
      EXTERNAL EVAL, OUT
      DIMENSION P(5),AUX(8,2),Y(2),DY(2)
      I=1
      COMMON K,U,F1,W
      COMMON/AREA1/XY,YY,I
      U=.8
      KLOCP=1
    2 K=99
      N=2
      P(1)=0.
      P(3)=.000002
      P(2)=2.
      P(4)=1.
      Y(1)=60.
      Y(2)=60.
      DY(1)=1.
      CY(2)=0.0
      WRITE(25,2000)
 2000 FORMAT(6X,'T'12X'VV'12X'WV'12X'U'12X'F1'12X'W'12X'IB
    7 CALL RKGS (P,Y,DY,N,IBIS,EVAL,OUT,AUX)
      IF(Y(1).LE.0.11190   ) GC TO 5
      KLOCP=KLOOP+1
      IF(KLCOP-15) 2,5,5
    5 CALL SCALE(YY,5.0,10000,1,10.0)
      CALL SCALE(XY,5.0,10000,1,10.0)
      CALL AXIS(0.0,0.0,'Y',1,6.0,90.0,YY(10001),YY(10002)
      CALL AXIS(0.0,0.0,'X',-1,6.0,0.0,XY(10001),XY(10002)
      CALL LINE(XY,YY,10000,1,0,0)
      CALL PLCT(0.0,0.0,999)
      STOP
      END
```

```
      SUBROUTINE EVAL (T,Y,DY)
      DIMENSION Y(2),DY(2),P(5)
      COMMON K,U,F1,W
      W=.225*EXP(.225*(Y(2)/Y(1)-1.))-23.5*EXP(23.5*(Y(2)/Y
    4 F1=EXP(-.225)*EXP(.225*Y(2)/Y(1))-EXP(-23.5)*EXP(23.5
      V=U
      DY(1)=-32.*F1
      DY(2)=1210.*F1-1584.*V
   11 RETURN
      END
```

```
      SUBROUTINE OUT (T,Y,DY,IBIS,N,P)
      DIMENSION Y(2),DY(2),P(5)
      DIMENSION XY(10000), YY(10000)
      COMMON K,U,F1,W
      COMMON/AREA1/XY,YY,I
      IF(NN.NE.1) GO TO 301
      CALL FIND2 (U,W,F1,T,Y,IBIS,P)
      RETURN
  301 CALL FIND1(U,W,F1,T,Y,DY,IBIS,N,P,NN,K)
      RETURN
      END
```

```
      SUBROUTINE FIND2(U,W,F1,T,Y,IBIS,P)
      DIMENSION Y(2),P(5)
      DIMENSION XY(10000), YY(10000)
      COMMON/AREA1/XY,YY,I
      IF(T.GE.0.15608) P(5) = 1.0
      IF(W.GT.0.0) GO TO 1
      IF(W.EQ.0.0) GO TO 3
      GO TO 2
    1 IF(U.GT.0.0) U=U-.01
      GO TO 3
    2 IF(U.LT.0.9999) U=U+.01
    3 K=K+1
      IF(K-100) 4,5,5
    5 WRITE(25,2001) T,Y,U,F1,W,IBIS
      XY(I)=Y(1)
      YY(I)=Y(2)
      I=I+1
 2001 FORMAT(6E13.5,I5,'YES')
  202 K=0
    4 RETURN
      END
```

```
      SUBROUTINE FIND1(U,W,F1,T,Y,DY,IBIS,N,P,NN,K)
      DIMENSION Y(2),DY(2),P(5)
      DIMENSION XY(10000), YY(10000)
      COMMON/AREA1/XY,YY,I
      IF(U.GT.0.9999) GO TO 200
      IF(ABS(W).GT.0.03) U=U+.01
  200 IF((T.GT.0.00005).AND.(U.LT.0.9999)) P(5)=1.
      K=K+1
      IF(K-100) 1,2,2
    2 WRITE(25,2001) T,Y,U,F1,W,IBIS
      XY(I)=Y(1)
      YY(I)=Y(2)
      I=I+1
      IF(ABS(W).GT.0.01) GO TO 202
      NN=1
 2001 FORMAT(6E13.5,I5)
  202 K=0
    1 RETURN
      END
```

T	VV	WV	U	F1	W
0.0	0.60000E 02	0.60000E 02	0.81000E 00	-0.35763E-06	-0.23275E 02
0.20000E-03	0.59998E 02	0.59701E 02	0.10000E 01	0.10849E 00	-0.20700E 02
0.39994E-03	0.59995E 02	0.59421E 02	0.10000E 01	0.19919E 00	-0.18544E 02
0.59988E-03	0.59992E 02	0.59160E 02	0.10000E 01	0.27486E 00	-0.16743E 02
0.79981E-03	0.59989E 02	0.58917E 02	0.10000E 01	0.33893E 00	-0.15217E 02
0.99974E-03	0.59986E 02	0.58687E 02	0.10000E 01	0.39383E 00	-0.13907E 02
0.11997E-02	0.59983E 02	0.58470E 02	0.10000E 01	0.44138E 00	-0.12771E 02
0.13996E-02	0.59979E 02	0.58264E 02	0.10000E 01	0.48294E 00	-0.11777E 02
0.15995E-02	0.59975E 02	0.58067E 02	0.10000E 01	0.51935E 00	-0.10904E 02
0.17995E-02	0.59969E 02	0.57879E 02	0.10000E 01	0.55140E 00	-0.10135E 02
0.19994E-02	0.59963E 02	0.57698E 02	0.10000E 01	0.57999E 00	-0.94482E 01
0.21993E-02	0.59957E 02	0.57523E 02	0.10000E 01	0.60570E 00	-0.88295E 01
0.23993E-02	0.59951E 02	0.57354E 02	0.10000E 01	0.62893E 00	-0.82696E 01
0.25992E-02	0.59945E 02	0.57191E 02	0.10000E 01	0.64998E 00	-0.77613E 01
0.27991E-02	0.59939E 02	0.57033E 02	0.10000E 01	0.66914E 00	-0.72977E 01
0.29991E-02	0.59933E 02	0.56878E 02	0.10000E 01	0.68666E 00	-0.68731E 01
0.31990E-02	0.59926E 02	0.56728E 02	0.10000E 01	0.70275E 00	-0.64825E 01
0.33989E-02	0.59920E 02	0.56582E 02	0.10000E 01	0.71751E 00	-0.61236E 01
0.35989E-02	0.59914E 02	0.56439E 02	0.10000E 01	0.73113E 00	-0.57917E 01
0.37988E-02	0.59908E 02	0.56300E 02	0.10000E 01	0.74373E 00	-0.54841E 01
0.39984E-02	0.59902E 02	0.56163E 02	0.10000E 01	0.75542E 00	-0.51982E 01
0.41977E-02	0.59896E 02	0.56029E 02	0.10000E 01	0.76628E 00	-0.49319E 01
0.43970E-02	0.59890E 02	0.55897E 02	0.10000E 01	0.77635E 00	-0.46843E 01
0.45963E-02	0.59884E 02	0.55769E 02	0.10000E 01	0.78575E 00	-0.44529E 01
0.47956E-02	0.59878E 02	0.55642E 02	0.10000E 01	0.79455E 00	-0.42356E 01
0.49949E-02	0.59872E 02	0.55516E 02	0.10000E 01	0.80279E 00	-0.40316E 01

TABLE A 2.1 SAMPLE OUTPUT OF FIRST PART OF OPTIMAL DIGITAL PROGRAM-NONSINGULAR TRAJECTORY

T	VV	WV	U	F1	W
0.20740E-01	0.59389E 02	0.47872E 02	0.10000E 01	0.94681E 00	-0.31150E-01
0.20939E-01	0.59383E 02	0.47784E 02	0.10000E 01	0.94685E 00	-0.23197E-01
0.21138E-01	0.59377E 02	0.47695E 02	0.10000E 01	0.94688E 00	-0.15503E-01
0.21338E-01	0.59371E 02	0.47607E 02	0.10000E 01	0.94690E 00	-0.80605E-02
0.21537E-01	0.59365E 02	0.47518E 02	0.10000E 01	0.94690E 00	-0.86105E-03
0.21736E-01	0.59359E 02	0.47514E 02	0.80000E 00	0.94690E 00	-0.91559E-03
0.21936E-01	0.59353E 02	0.47510E 02	0.71999E 00	0.94690E 00	-0.10061E-02
0.22135E-01	0.59347E 02	0.47502E 02	0.67999E 00	0.94690E 00	-0.73475E-03
0.22334E-01	0.59341E 02	0.47498E 02	0.67999E 00	0.94690E 00	-0.78434E-03
0.22534E-01	0.59334E 02	0.47494E 02	0.67998E 00	0.94690E 00	-0.83369E-03
0.22733E-01	0.59328E 02	0.47487E 02	0.71998E 00	0.94690E 00	-0.66608E-03
0.22932E-01	0.59322E 02	0.47482E 02	0.79998E 00	0.94690E 00	-0.67383E-03
0.23132E-01	0.59316E 02	0.47475E 02	0.87998E 00	0.94690E 00	-0.50998E-03
0.23331E-01	0.59310E 02	0.47471E 02	0.87997E 00	0.94690E 00	-0.55903E-03
0.23530E-01	0.59304E 02	0.47466E 02	0.87997E 00	0.94690E 00	-0.60862E-03
0.23729E-01	0.59298E 02	0.47462E 02	0.87997E 00	0.94690E 00	-0.65792E-03
0.23929E-01	0.59292E 02	0.47460E 02	0.79996E 00	0.94690E 00	-0.92059E-03
0.24128E-01	0.59286E 02	0.47457E 02	0.71996E 00	0.94690E 00	-0.10114E-02
0.24327E-01	0.59279E 02	0.47449E 02	0.67996E 00	0.94690E 00	-0.73987E-03
0.24527E-01	0.59273E 02	0.47444E 02	0.67995E 00	0.94690E 00	-0.78946E-03
0.24726E-01	0.59267E 02	0.47440E 02	0.67995E 00	0.94690E 00	-0.83882E-03
0.24925E-01	0.59261E 02	0.47433E 02	0.59995E 00	0.94690E 00	-0.68104E-03
0.25125E-01	0.59255E 02	0.47424E 02	0.51995E 00	0.94690E 00	-0.34404E-03
0.25324E-01	0.59249E 02	0.47414E 02	0.47994E 00	0.94690E 00	0.14341E-03
0.25523E-01	0.59243E 02	0.47402E 02	0.57994E 00	0.94690E 00	0.73785E-03
0.25722E-01	0.59237E 02	0.47395E 02	0.65994E 00	0.94690E 00	0.89455E-03
0.25922E-01	0.59231E 02	0.47389E 02	0.75993E 00	0.94690E 00	0.10138E-02
0.26121E-01	0.59225E 02	0.47386E 02	0.83993E 00	0.94690E 00	0.79167E-03
0.26320E-01	0.59218E 02	0.47386E 02	0.91993E 00	0.94690E 00	0.39291E-03
0.26520E-01	0.59212E 02	0.47388E 02	0.99993E 00	0.94690E 00	-0.17202E-03
0.26719E-01	0.59206E 02	0.47389E 02	0.89992E 00	0.94690E 00	-0.64689E-03
0.26918E-01	0.59200E 02	0.47368E 02	0.81992E 00	0.94690E 00	-0.95654E-03
0.27118E-01	0.59194E 02	0.47382E 02	0.71992E 00	0.94690E 00	-0.94002E-03

TABLE A 2.2 SAMPLE OUTPUT OF OPTIMAL DIGITAL PROGRAM AT START OF SINGULAR TRAJECTORY

APPENDIX III

Differential Forms

A2.1 Introduction

Knowledge of differential forms is useful in the analysis and synthesis of engineering systems. A more complete treatment may be found in Flanders [F7].

The objects that occur under integral signs are called exterior differential forms. For example, the line integral, surface integral, and volume integral lead to the following differential forms in 3-dimensional Euclidean space:

$$\omega = A\,dx + B\,dy + C\,dz \qquad \text{(one-form)} \qquad (A2.1)$$

$$\alpha = P\,dydz + Q\,dzdx + R\,dxdy \qquad \text{(two-form)} \qquad (A2.2)$$

$$\lambda = H\,dxdydz \qquad \text{(three-form)} \qquad (A2.3)$$

In the n-dimensional space, the quantities are called r-forms in n variables.

A2.2 Exterior Algebra

In the algebra of differential forms the operations of addition and multiplication obey the usual associative and distributive laws. Multiplication, however, is not commutative but anticommutative, i.e.,

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \qquad (A2.4)$$

The exterior product is sometimes called the wedge product. Often the product symbol $\wedge$ is omitted. Hence,

$$dx_i\,dx_j = -dx_j\,dx_i \qquad (A2.4)$$

This implies

$$dx_i \wedge dx_i = 0 \qquad (A2.5)$$

In the two-form given (A2.2), use of (A2.4) eliminates terms like dzdy.

The exterior product has the following properties:

(1) $(\omega+\zeta)\wedge\eta = (\omega\wedge\eta) + (\zeta\wedge\eta)$ (A2.5)

(2) $(c\omega)\wedge\zeta = c(\omega\wedge\zeta)$ (A2.6)

(3) $\zeta \wedge \omega = (-1)^{rs} \omega \wedge \zeta$, (A2.7)

if ω has degree r and ζ has degree s

(4) $(\zeta\wedge\omega)\wedge\eta = \zeta\wedge(\omega\wedge\eta)$ (A2.8)

A2.3 The Exterior Derivative

The exterior derivative of a p-form ω is a (p+1)-form $d\omega$ obtained by applying an operator d to transform ω to $d\omega$.

For example if ω is a three-form in four variables

$$\omega = \sum_{i<j<k} \omega_{ijk}\, dx_i \wedge dx_j \wedge dx_k \quad \text{(A2.9)}$$

or, omitting the product symbol

$$\omega = \omega_{123}\, dx_1 dx_2 dx_3 + \omega_{124}\, dx_1 dx_2 dx_4 + \omega_{234}\, dx_2 dx_3 dx_4 \quad (2.10)$$

The exterior differential $d\omega$ is defined as

$$d\omega = \sum_{i<j<k} d\omega_{ijk} \quad dx_i \quad dx_j \quad dx_k \quad (2.11)$$

Where ω_{ijk} is a function of x_1, x_2, x_3, and x_4 and is assumed to be differentiable.

This definition is readily generalized.

The exterior differential has the following properties:

(1) $d(\omega+\eta) = d\omega + d\eta$ (A2.12)

(2) $d(\omega\wedge\eta) = d\omega \wedge \eta + (-1)^{r} \omega \wedge d\eta$, (A2.13)

if ω is an r-form and η is an s-form

(3) $d(d\omega) = 0$ (A2.14)

ω and η are assumed to be differentiable.

Property 3 is called the Poincare' lemma. It implies the equality of mixed second partial derivatives. The general case is proved by induction.

For simplicity only the 0-form is 3 variables is considered,

$$\omega = f(x) \quad \text{(A2.15)}$$

Then there results the 1-form

$$d\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad \text{(2.16)}$$

Then

$$d(d\omega) = d(\frac{\partial f}{\partial x}) \wedge dx + d(\frac{\partial f}{\partial y}) \wedge dy + d(\frac{\partial f}{\partial z}) \wedge dz \quad \text{(A2.17)}$$

Carrying out the differentiation, and using the properties of exterior multiplication,

$$d(d\omega) = 0 \quad \text{(A2.18)}$$

In 3-space, the Poincare' lemma $d(d\omega) = 0$ interprets as

$$\text{curl (grad f)} = 0 \quad \text{(A2.19)}$$

$$\text{div (curl V)} = 0 \quad \text{(A2.20)}$$

A2.4 Integration of Forms

The primary purpose of this section is to present the n-dimensional Green's Theorem, also called the n-dimensional Stoke's Theorem. What the classical theorems state for curves and surfaces, these theorems state for the higher-dimensional analogs called manifolds.

An n-dimensional manifold consists of a space M and a collection of local coordinates neighborhoods N_1, N_2, ... such that each point of M lies in at least one of the neighborhoods. Whereas, an n-dimensional manifold may not be a Euclidean space, it appears to be Euclidean to a short-sighted observer in the manifold.

The proof of the n-dimensional Green's Theorem is simplified if the concepts of chains and Euclidean simplices are introduced. This is done to eliminate the need to chop up manifolds into small pieces. Instead of working with manifolds where things are more difficult, Euclidean spaces may be used where things are relatively more simple.

Euclidean simplices are defined as follows:

A 0-simplex is a single point (p_0).

A 1-simplex is a directed closed segment on a straight line. It is completely determined by its ordered pair of vertices (P_0, P_1).

A 2-simplex is a closed triangle with vertices taken in some definite order. It is determined by the ordered triple (P_0, P_1, P_2).

A 3-simplex is similarly the ordered quadruple (P_0, P_1, P_2, P_3).

In general, an n-simplex is the closed convex hull (P_0, ..., P_n) of (n+1) independent points taken in a definite order. Independent points means that the n vectors (P_1-P_0), (P_2-P_0), ... (P_n-P_0) are linearly independent. The convexity condition implies that the n-simplex is the set of points.

$$P = t_oP_o + \dots + t_nP_n \qquad t_i \geq 0, \sum t_i = 1 \tag{A2.21}$$

The boundary ∂S of a simplex S is a formal sum of one lower dimension with integer coefficients defined as follows:

$$\partial\,(P_0, P_1, \dots, P_n) = \sum_{i=0}^{n} (-1)^i (P_0, \dots P_{i-1}; P_{i+1}, \dots, P_n) \tag{A2.22}$$

For example, the 3-simplex is bounded by four faces, i.e.

$$\partial(P_0, P_1, P_2, P_3,) = (P_1, P_2, P_3)-(P_0, P_2, P_3)+(P_0, P_1, P_3)- (P_0, P_1, P_2) \qquad \text{(A2.23)}$$

The terms having positive signs correspond to orientations which may be associated with an outward normal if the points are traversed in a counter clockwise direction. See Figure A2-1.

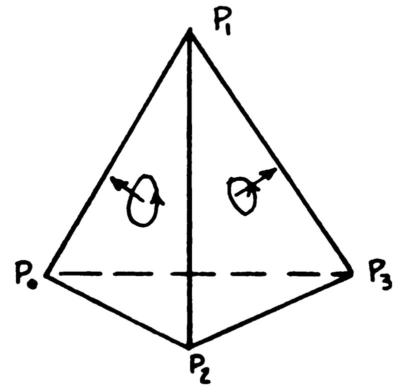


Figure A 2.1 3-Simplex with Orientation

An <u>n-chain</u> is a formal sum

$$C = \sum a^i S_i \qquad \text{(A2.24)}$$

where a^i are constants and S_i are n-simplices.

The boundary of the chain is defined as

$$\partial C = \sum a^i \partial S_i \qquad \text{(A2.25)}$$

As a result, the boundary of each chain has zero boundary.

$$\partial(\partial C) = 0 \qquad \text{(A2.26)}$$

For example consider the boundary of the 2-simplexS where

$$S = (P_0, P_1, P_2) \qquad \text{(A2.27)}$$

Then

$$\partial S = \partial(P_0, P_1, P_2) = (P_1, P_2)-(P_0, P_2)+(P_0, P_1) \quad \text{(A2.28)}$$

and

$$\partial(\partial S) = (P_2-P_1)-(P_2-P_0)+(P_1-P_0) \quad \text{(A2.29)}$$

Hence

$$\partial(\partial S) = 0 \quad \text{(A2.30)}$$

and

$$\partial(\partial C) = 0 \quad \text{(A2.31)}$$

It is convenient to have standard models of the simplices. The standard n-simplex is defined as

$$S^{-n} = (R_0, \ldots R_n) \quad \text{(A2.32)}$$

The points $R_0, \ldots, R_n$ in n-dimensional space are taken as

$$R_0 = (0 \ldots 0)$$

$$R_1 = (10 \ldots 0)$$

$$R_2 = (010 \ldots 0)$$

$$\vdots$$

$$R_n = (00 \ldots 01)$$

Integration of a n-form defined on a domain N of E^n which includes S^{-n} is written as

$$\int_{s^{-n}} \omega = \int_{s^{-n}} A(x_1, \ldots, X_n)\ dx_1 dx_2 \ldots dx_n \quad \text{(A2.33)}$$

The right side is standard ordinary n-fold integration over the standard n-simplex.

Since we wish to integrate a n-form on a manifold M, it is necessary to relate the standard n-simplex to the n-simplex in M (denoted by σ^n).

Hence

$$\int_{\sigma} \omega = \int_{s^{-n}} \phi^{*}\omega \qquad \text{(A2.34)}$$

where ϕ is a smooth mapping of the neighborhood N of s^{-n} into M.

It can be shown that

$$\int_{\partial\sigma} \omega = \int_{\sigma} d\omega \qquad \text{(A2.35)}$$

Also since $C = \sum a_i \sigma_i^{\,n}$,

$$\int_{\partial C} \omega = \int_{C} d\omega \qquad \text{(A2.36)}$$

This is Stoke's Theorem in its most general form. Recall that C is a chain and ∂C is its boundary.

APPENDIX IV

ALTERNATE DIGITAL PROGRAM

```
****CONTINUOUS SYSTEM MODELING PROGRAM****

***PROBLEM INPUT STATEMENTS***

TITLE     ASKC
FUNCT     MUET1=(-.15,-1.),(C.,0.0),(.C75,.6),(.15,1.),(.57
FUNCT     MUET2=(-.15,-1.),(0.,0.0),(.075,.6),(.15,1.),(.57
METHOC    RKSFX
PARAM     A=1.,B=1.
          MU1=A*AFGEN(MUET1,ETA1)
          MU2=B*AFGEN(MUET2,ETA2)
          VVDOT=-8.*(MU1+MU2+A+B)
          VV=INTGRL(60.0,VVDOT)
          VVD=VV
          SDIST=INTGRL(0.0,VV)
          WV1=INTGRL(60.0,WV1DOT)
          WV1COT=1210.*MU1-C.79*T1
          WV2DOT=1210.*MU2-0.79*T2
          WV2=INTGRL(60.0,WV2DOT)
          WV11=LIMIT(0.,60.0,WV1)
          WV1C=DELAY(1,.010,WV11)
          ETA1=1.0-WV11/VV
          WV22=LIMIT(0.,60.0,WV2)
          WV2D=DELAY(1,.010,WV22)
           ETA2=1.0-WV22/VV
          T1=1.66*P1
          T2=1.66*P2
          P2=P1
          ETA1R=.15*VV
          ETA1A=VVC-(WV1D+WV2D)*.5
          Y1=DERIV(0.0,ETA1A)
          ERRIN1=(ETA1A+.15*Y1)-ETA1R
          ERRIN2=(ETA2A+.15*Y2)-ETA2R
          ERROT1=INSW(ERRIN1, 1.0,-3.0)
          E1=15000.*ERROT1
          P1=INTGRL(C.0,X1)
          X1=X11*E1
          X11=IOR(Y111,Y112)
          Y111=AND(Y11,P1)
          Y112=AND(E1,Y12)
          Y11=NOT(E1)
          Y12=NOT(P1-1200.)
TIMER     FINTIM=4.,DELT=.COC2,DELMIN=.000000001,PRDEL=.02,O
FINISH    VV=1.0
PRINT     ETA1R,ETA1A,MU1,ERRIN1,ERRCT1,     WV2D,VV,P1,SDIST
PRTPLT SIZE(5.,6.)VV,WV1D
PARAM SIZE=0.
```

****CONTINUOUS SYSTEM MODELING PROGRAM****

PROBLEM INPUT STATEMENTS

```
TITLE    ASKC
METHOD   RKSFX
PARAM    B=1.,C=1.,D=2.
FUNCT    MUET1=(-.15,-1.),(0.,0.0),(.075,.6),(.15,1.),(.57
FUNCT    MUET2=(-.15,-1.),(0.,0.0),(.075,.6),(.15,1.),(.57
         MU1=B*AFGEN(MUET1,ETA1)
         MU2=C*AFGEN(MUET2,ETA2)
         VVDOT=-8.*(MU1+MU2+D)
         VV=INTGRL(60.0,VVDOT)
         VVD=DELAY(5,.015,VV)
         WV1DOT=1210.*MU1-0.79*T1
         WV2DOT=1210.*MU2-0.79*T2
         SDIST=INTGRL(0.0,VV)
         WV1=INTGRL(60.0,WV1DOT)
         WV2=INTGRL(60.,WV2DOT)
         ETA1A=VVD-WV1D
         ETA2A=VVD-WV2D
         T1=1.66*P11
         T2=1.66*P22
         ETA1=1.0-WV11/VV
          ETA2=1.0-WV22/VV
         WV11=LIMIT(0.,60.0,WV1)
         WV22=LIMIT(0.,60.0,WV2)
         WV2D=DELAY(5,.015,WV22)
         WV1D=DELAY(5,.015,WV11)
         ETA1R=.85*VVD
         ETA2R=.85*VVD
         Y1=DERIV(0.0,WV1D)
         Y2=DERIV(0.0,WV2D)
         ERRIN1=ETA1R-WV1D-.10*Y1
         ERRIN2=ETA2R-WV2D-.10*Y2
         ERROT1=INSW(ERRIN1, 1.0,-3.0)
         ERROT2=INSW(ERRIN2,1.0,-3.0)
         E1=15000.*ERROT1
         E2=15000.*ERROT2
         P1=INTGRL(0.0,E1)
         P2=INTGRL(0.0,E2)
         P22=LIMIT(0.,1200.,P2)
         P11=LIMIT(0.,1200.,P1)
TIMER    FINTIM=4.,DELT=.0002,DELMIN=.000000001,PRDEL=.02,(
FINISH   VV=1.0
PRTPLT SIZE(4.,6.),TIME,WV1D,VV
LABEL  WHEEL VELOCITY VS TIME
PARAM SIZE=0.
PRTPLT SIZE(5.,6.),TIME,P1
PRTPLT SIZE(5.,6.),TIME,MU1
PRTPLT SIZE(5.,6.),TIME,ERRIN1
PRTPLT SIZE(5.,6.),TIME,WV2D
PRINT    ETA1R,MU1,ERRIN1,ERROT1,WV1D,VVD,VV,P1,SDIST
END
```

```
                  ****CONTINUOUS SYSTEM MODELING PROGRAM****

     ***PROBLEM INPUT STATEMENTS***

TITLE      BRAKING DYNAMICS
FUNCT      MUET1=(-.15,-1.),(0.,0.0),(.075,.6),(.15,1.),(.57,.9),
METHOD   RKSFX
PARAM      A1=1.,A2=.9,A3=.9,A4=1.
PARAM      MA=156.,A=4.,B=6.,IV=1326.,W=3.,IWR=1.375,W1=1250.,W2=
PARAM      W3=1250.,W4=1250.
           MU1=A1*AFGEN(MUET1,ETA1)
           MU2=A2*AFGEN(MUET1,ETA2)
           MU3=A3*AFGEN(MUET1,ETA3)
           MU4=A4*AFGEN(MUET1,ETA4)
           UDOT=V*R-(1./MA)*(B1+B2+B3+B4)
           U=INTGRL(60.,UDOT)
           VDOT=-U*R-(1./MA)*(L1+L2+L3+L4)
           V=INTGRL(0.,VDOT)
           RDOT=(1./IV)*(B*(L3+L4)-A*(L1+L2)+W*(B1+B4-B2-B3))
           R=INTGRL(0.,RDOT)
           ALPH1=(V+A*R)/U1
           ALPH2=(V+A*R)/U2
           ALPH3=(V-B*R)/U3
           ALPH4=(V-B*R)/U4
           U1=U-W*R
           U2=U+W*R
           U3=U+W*R
           U4=U-W*R
           SDIST=INTGRL(0.0,U)
           WV1DOT=B1/IWR-0.79*T1
           WV2DOT=B2/IWR-0.79*T2
           WV3DOT=B3/IWR-0.79*T3
           WV4DOT=B4/IWR-0.79*T4
           WV1=INTGRL(60.,WV1DOT)
           WV2=INTGRL(60.,WV2DOT)
           WV3=INTGRL(60.,WV3DOT)
           WV4=INTGRL(60.,WV4DOT)
           WV11=LIMIT(0.,60.,WV1)
           WV22=LIMIT(0.,60.,WV2)
           WV33=LIMIT(0.,60.,WV3)
           WV44=LIMIT(0.,60.,WV4)
           WV1D=DELAY(1,.020,WV11)
           WV2D=DELAY(1,.020,WV22)
           WV3D=DELAY(1,.020,WV33)
           WV4D=DELAY(1,.020,WV44)
           ETA1=1.-WV11/U1
           ETA2=1.-WV22/U2
           ETA3=1.-WV33/U3
           ETA4=1.-WV44/U4
           T1=1.66*P11
           T2=1.66*P22
```

```
          T3=1.66*P33
          T4=1.66*P44
          B1=MU1*W1*COS(7.85*ALPH1)
          B2=MU2*W2*COS(7.85*ALPH2)
          B3=MU3*W3*COS(7.85*ALPH3)
          B4=MU4*W4*COS(7.85*ALPH4)
          L1=MU1*W1*SIN(7.85*ALPH1)
          L2=MU2*W2*SIN(7.85*ALPH2)
          L3=MU3*W3*SIN(7.85*ALPH3)
          L4=MU4*W4*SIN(7.85*ALPH4)
          ETA1R=0.5*U
          ETA1A=U-.25*(WV1D+WV2D+WV3D+WV4D)
          Y1=DERIV(0.,ETA1A)
          ERRIN1=(ETA1A+.15*Y1)-ETA1R
          ERROT1=INSW(ERRIN1,1.,-3.)
          E1=15000.*ERROT1
          P1=INTGRL(0.,X1)
          P2=P1
          P3=P1
          P4=P1
          X1=X11*E1
          X11=IOR(Y111,Y112)
          Y111=AND(Y11,P1)
          Y112=AND(E1,Y12)
          Y11=NOT(E1)
          Y12=NOT(P1-1200.)
          P11=DELAY(1,.020,P1)
          P22=P11
          P33=P11
          P44=P11
          HEAD=INTGRL(0.,R)
          YDOT=U*COS(HEAD)-V*SIN(HEAD)
          XDOT=U*SIN(HEAD)+V*COS(HEAD)
          Y=INTGRL(0.,YDOT)
          X=INTGRL(0.,XDOT)
          TIMER   FINTIM=4.,DELT=.0002,DELMIN=.000000001,PRDEL
FINISH     U=1.
PRINT    MU1,WV1,WV2,WV3,WV4,U,V,R,P1,Y,X,HEAD
PRTPLT   MU1,WV1,WV2,WV3,WV4,U,V,R,P1,MU2,MU3,MU4
END
PARAM     A1=1.,A2=.7,A3=.7,A4=1.
RESET LABEL
LABEL     70 PERCENT MU
END
PARAM     A=1.,A2=.5,A3=.5,A4=1.
RESET LABEL
LABEL     50 PERCENT MU ON ONE SIDE
END
STOP
```

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