THERMAL RADIATION AND MAGNETIC FIELD EFFECTS ON THE FLOW VARIABLES NEAR A STAGNATION POINT

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ABSTRACT

THERMAL RADIATION AND MAGNETIC FIELD EFFECTS ON THE FLOW VARIABLES NEAR A STAGNATION POINT

by Wendelin Schmidt

In this investigation the equations connecting the flow variables with the geometric parameters of the streamlines in three dimensional, inviscid and viscous radiation magnetohydrodynamic gas flow were derived. A simplified mathematical model governing the flow variables distribution near a stagnation point in radiation magnetohydrodynamic flow was developed and used to estimate the combined effects of various physical phenomena on the flow field variables. Specifically we consider the combined effects of thermal radiation, magnetic field, viscosity, heat conductivity and compressibility on the temperature, pressure, electron density, and electric conductivity distribution near a stagnation point.

The first order results obtained from the numerical solutions of the governing equations indicated that the effects of thermal radiation, magnetic field, and

compressibility on the flow field variables are considerable, whereas the effects of viscosity and heat conductivity were found to be very small in the case under consideration here.

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NOMENCLATURE

V	flow velocity, m/sec
ρ	fluid density, kg/m^3
p	fluid static pressure, n/m^2
p_R	radiation pressure, n/m^2
J	electric current density, amp./m^2
H	magnetic field intensity, amp./m
μ	magnetic permeability, webers/amp.-m
u	internal energy of fluid, n-m/kg
Q	energy input, n-m/kg
R	universal gas constant, n-m/kg-^{o}K
T	fluid temperature, ^{o}K
P	$= p + p_R$
$\vec{E}$	electric field intensity, n/coul.
ρ_e	excess electric charge density, coul./m^3
ϵ	electric permittivity of free space, coul.^{2}/n-m^2
∂	electric conductivity, mohs/m
t	time, sec.
η	$= (\partial\mu)^{-1}$ magnetic diffusivity, m^2/sec
p_m	$= \frac{1}{2}\mu H^2$ magnetic pressure, n/m^2
P_t	$= (p + p_R + p_m)$, total scalar pressure, n/m^2
s	arc length along streamline, m
n	arc length along normal to streamline, m
b	arc length along binormal to streamlines, m

k	streamline curvature, 1/m
τ	torsion of streamline, 1/m
x^i	rectangular coordinates, ($i=1,2,3$), x,y,z.
z	cylindrical coordinate
r	cylindrical coordinate
θ	cylindrical coordinate
h^i	component of unit vector, Eq.(3.1.2)
I	$= (u + P/\rho + \frac{1}{2}V^2)$
g_{ij}	metric tensor
M	Mach number
w_k	vorticity vector
e_{ijk}	alternating unit tensor
e	enthalpy of fluid, n-m/kg
B	$=\mu H$ magnetic field webers/m^2
v,w	components of fluid velocity, m/sec
c_v	specific heat at constant volume, n-m/kg$^\circ$K
c_p	specific heat at constant pressure, n-m/kg$^\circ$K
$\vec{Q}_R$	radiative heat flux vector, n-m/m^2-sec
c	velocity of light, m/sec
δ_B	Stefan-Boltzmann constant, n-m/sec.-m^2-$^\circ$K
K_R	Rosseland mean absorption coefficient, 1/m
K_P	Planck mean absorption coefficient, 1/m
ψ	streamfunction

a	$= \mathcal{V}/b$, b = constant, Eq.(6.1.2)
A_M, A_R	constants given by Eq.(6.1.21)
τ^{ij}	viscous stress tensor, Eq.(7.1.2), n/m^2
$\bar{\mu}$	fluid viscosity, kg/m-sec
K_t	thermal conductivity, n-m/m-sec-°K
$\bar{\mu}\phi$	viscous dissipation function, Eq.(7.1.3), $n\text{-}m/sec\text{-}m^3$
n_e	electron number density, $\#/cm^3$

1. INTRODUCTION

The great interest in hypersonic flow around blunt vehicles has been stimulated in the last decade by the intercontinental ballistic missile, the satellite and the deep space programs. Among the phenomena that can be observed during hypersonic atmospheric entry of a vehicle are the thermal radiation emitted by the hot gas flowing around it and the reflection of microwaves by the ionized gas envelope surrounding the vehicle. It is well known that this ionized gas or plasma envelope is the cause of radio, and other communications blackout during atmospheric re-entry. (1, 2, 3, 4)

The influence of the electrons in the ionized gas around the vehicle is felt not only on electromagnetic signal attenuation through radio blackout, but also on aerodynamic quantities such as drag and heat transfer, and on physical quantities such as transport, radiative emission, and absorption properties. From the communications problem point of view the electons are undesirable and should be eliminated. However, the flight magneto-hydrodynamics point of view considers the ionized gas as a phenomenon to be capitalized on by applying a strong magnetic field in such a way so as to provide a re-entry vehicle with braking and other maneuvering capability. (5 to 17).

A solution to the hypersonic blunt-body problem that combines the advantages of minimum computational difficulty with maximum accuracy of results has been sought for more than a decade. The problem under consideration is that of determining the flow field properties (i.e. pressure, temperature, density etc.) around a blunt-nosed configuration traveling through a uniform gas at a flight Mach number greater than unity. In general, the flow field about the body may be divided into two regions based on the magnitude of the local flow Mach number with respect to unity. In the region near the stagnation point of the body the flow Mach number is less then unity and the flow field is therefore subsonic. Much of the effort expended on the blunt-body problem has been confined to the subsonic region since the solution to this region provides the starting data for the well known characteristic method of supersonic flow calculations downstream of the stagnation point. The determination of the fluid properties in the subsonic flow field over the blunt body also provides the necessary data for the subsequent evaluation of the radiant heat transfer as well as for initiating boundary layer calculations to determine wall shear and convective heat transfer to the nose of the body.

From the aerodynamics point of view the main problem associated with the re-entry of a space vehicle is that of convective and radiative heating, and aerodynamic drag.

Since the gas around the vehicle is in a plasma state and, therefore, electrically conducting, the possibility of utilizing an applied magnetic field to reduce surface shear stress and heat transfer has been proposed by a number of authors. (5 to 8). The general approach to the problem consists of dividing the flow field into a viscous boundary layer and an outer inviscid flow. The solution to the boundary layer part of the problem requires a knowledge of the edge of the boundary layer flow conditions which are obtained from the solution to the outer inviscid part of the flow field. Part I of this thesis is directed towards the determination of the combined thermal radiation and magnetic field effects on the inviscid flow field variables near a stagnation point.

The division of the flow field into an inviscid flow region and a boundary layer is only possible for flight altitudes below which the ratio of the vehicle radius to mean free path of the gas molecules is greater than about 75. i.e., $R_b/\lambda > 75$. For altitudes such that the ratio of vehicle radius to mean free path is between about 75 and 1, such a division of the flow field is not possible and either the full or a simplified form of the Navier-Stokes equations must be used as a flow model. (54). In Part II of this thesis we investigate the combined effects of thermal radiation, magnetic field, viscosity and heat conductivity on the flow field variables.

The central problem under consideration consists of the derivation of a mathematical model which allows us to predict the effects of various physical phenomena on the flow field parameters under given conditions. An exact mathematical description of the flow field including thermal radiation, magnetic field, viscosity, and heat conductivity effects, requires a complicated set of non-linear partial differential equations which are very difficult to solve for a given vehicle configuration. Because of this difficulty, many simplified, approximate mathematical models have been proposed which hold under various conditions. One such simplified approach to the calculation of the flow field parameters was proposed by several authors independently and consists of calculating the parameters along flow streamlines under an assumed flow field pressure distribution (49 to 52). The method was used for dissociating, inviscid, non-heatconducting, non-radiating flow without the magnetic field effect, and was found to be quite amenable to parametric study of very complex flow fields.

We shall use the streamline approach in the present investigation with a modification which consists of using an approximate velocity distribution.

PART I: INVISCID RADIATION MAGNETOHYDRODYNAMICS

2. INVISCID GOVERNING SYSTEM OF EQUATIONS

2.1. Fundamental Equations

We consider an inviscid, non-heat-conducting steady flow of an ionized perfect gas in an electro-magnetic field including thermal radiation. The governing hydrodynamical system of equations consists of the mathematical formulation of the physical laws of conservation of mass, momentum, energy, and the equation of state of the gas;

$$\nabla \cdot (\rho \vec{V}) = 0, \tag{2.1.1}$$

$$\vec{V} \cdot \nabla \vec{V} = -\rho^{-1} \nabla P + \rho^{-1} (\vec{J} \times \mu \vec{H}), \tag{2.1.2}$$

$$du + Pd(\rho^{-1}) - dQ = 0, \tag{2.1.3}$$

$$p = \rho RT. \tag{2.1.4}$$

The energy equation (2.1.3) may be used in an integrated form along a streamline to be denoted as the generalized Bernoulli equation.

The equations governing the electromagnetic field are Maxwell's equations and Ohm's law; (55,58)

$$\nabla \cdot \vec{E} = \rho_e / \epsilon \quad ; \qquad \nabla \times \vec{E} = 0, \tag{2.1.5}$$

$$\nabla \cdot \vec{H} = 0 \; ; \qquad \nabla \times \vec{H} = \vec{J}, \tag{2.1.6}$$

$$\vec{J} = \sigma (\vec{E} + \mu \vec{V} \times \vec{H}). \tag{2.1.7}$$

2.2. Reformulation

The above system of equations may be reformulated so as to be more suitable for the present analysis. Substituting for $\vec{J}$ from Eq. (2.1.6) into the equation of motion (2.1.2) we get;

$$\vec{V}\cdot\nabla\vec{V} = -\rho^{-1}\nabla(p + p_R + \tfrac{1}{2}\mu H^2) + \rho^{-1}\mu\vec{H}\cdot\nabla\vec{H}. \qquad (2.2.1)$$

The energy equation suitable for the present analysis may be obtained by starting with Eq.(2.1.3.) as follows;

$$dQ = du + Pd(1/\rho) = d(u + P/\rho) - \rho^{-1}dP,$$

or $$\frac{dQ}{dt} = \vec{V}\cdot\nabla(u + P/\rho) - \rho^{-1}\vec{V}\cdot\nabla P. \qquad (2.2.2)$$

By expanding the left hand side of the equation of motion (2.1.2) and taking the scalar product with $\vec{V}$ we get,

$$\vec{V}\cdot\nabla(\tfrac{1}{2}v^2) - \vec{V}\cdot(\vec{V}\times(\nabla\times\vec{V})) = -\rho^{-1}\vec{V}\cdot\nabla P + \vec{V}\cdot(\vec{J}\times\mu\vec{H}),$$

Using this last result to eliminate $-\rho^{-1}\vec{V}\cdot\nabla P$ in Eq.(2.2.2) we get,

$$\vec{V}\cdot\nabla(u + P/\rho + \tfrac{1}{2}v^2) = \frac{dQ}{dt} + \rho^{-1}\vec{V}\cdot(\vec{J}\times\mu\vec{H}). \qquad (2.2.3)$$

dQ is the heat input from all sources per unit mass, which in our case consists of the Joule heat and the radiation heat flux. Thus we have,

$$\frac{dQ}{dt} = \rho^{-1}(J^2/\sigma) + \rho^{-1}\nabla\cdot\vec{Q}.$$

Using this last result to eliminate $\frac{dQ}{dt}$ in (2.2.3) and substituting (2.1.6) for $\vec{J}$ and expanding $\vec{J} \times \mu\vec{H}$, we get;

$$\vec{V}\cdot\nabla(u + P/\rho + \tfrac{1}{2}V^2) = \rho^{-1}(\nabla \times \vec{H})\cdot(\nabla \times \vec{H})/\sigma + \rho^{-1}\mu\vec{V}\cdot(\vec{H}\cdot\nabla\vec{H}) - \mu\vec{V}\cdot\nabla(\tfrac{1}{2}H^2) + \rho^{-1}\nabla\cdot\vec{Q}, \qquad (2.2.4)$$

which is the energy equation in the required form.

From Maxwell's equations and Ohm's law we develop the following ; solving for $\vec{E}$ from Eq.(2.1.7) and substituting for $\vec{J} = \nabla \times \vec{H}$ we get,

$$\vec{E} = 1/\sigma\, \nabla \times \vec{H} - \mu\vec{V} \times \vec{H} \ ;$$

taking the curl of this last result and accounting for Eq.(2.1.5) we have,

$$(1/\mu\sigma)\nabla \times (\nabla \times \vec{H}) - \nabla \times (\vec{V} \times \vec{H}) = 0.$$

Expanding this last result and using (2.1.6) and replacing $\eta = 1/\mu\sigma$ we get,

$$\eta\,\nabla^2\vec{H} = \vec{V}\cdot\nabla\vec{H} - \vec{H}\cdot\nabla\vec{V} + \vec{H}(\nabla\cdot\vec{V}). \qquad (2.2.5)$$

2.3. Tensor Form

The governing system of equations will be recast in Cartesian tensor form. In this form the equation of continuity becomes,

$$\frac{\partial}{\partial x^i}(\rho V^i) = 0. \qquad (2.3.1)$$

The momentum equation (2.2.1) becomes,

$$\rho V^j\frac{\partial V^i}{\partial x^j} + g^{ij}\frac{\partial P_t}{\partial x^j} - \mu H^j\frac{\partial H^i}{\partial x^j} = 0, \qquad (2.3.2)$$

where $P_t = (p + p_R + \frac{1}{2}\mu H^2)$.

The equation of energy (2.2.4) gives,

$$\rho V^j \frac{\partial I}{\partial x^j} = \sigma^{-1}\left(g^{jr}\frac{\partial H^k}{\partial x^r}\frac{\partial H_j}{\partial x^k} - g^{jr}\frac{\partial H^k}{\partial x^r}\frac{\partial H_k}{\partial x^j}\right) +$$

$$\mu V^j H^i \frac{\partial H_j}{\partial x^i} - V^j \frac{\partial p_m}{\partial x^j} + \frac{\partial Q^i}{\partial x^i}\,, \qquad (2.3.3)$$

where $I = (u + P/\rho + \frac{1}{2}V^2)$; $\quad p_m = \frac{1}{2}\mu H^2$.

The electromagnetic equations (2.2.5) and (2.1.6) become,

$$V^j \frac{\partial H^i}{\partial x^j} - H^j \frac{\partial V^i}{\partial x^j} + H^i \frac{\partial V^j}{\partial x^j} = \eta \frac{\partial}{\partial x^j}\left(g^{jk}\frac{\partial H^i}{\partial x^k}\right), \qquad (2.3.4)$$

$$\frac{\partial H^i}{\partial x^i} = 0. \qquad (2.3.5)$$

To complete the system of equations we add the equation of state of the gas

$$p = \rho RT. \qquad (2.3.6)$$

The unknown quantities consist of three scalars and two vectors; i.e., $p, \rho, T, \vec{V}$, and $\vec{H}$. The three scalar equations required are Eqs. (2.3.1), (2.3.3), and (2.3.6). The two vector equations are Eqs. (2.3.2) and (2.3.4).

3. DYNAMIC AND KINEMATIC RELATIONS

3.1. Basic Decomposition

The equations (2.3.1) to (2.3.5) will be transformed into a coordinate system s^i, n^i, b^i, where the symbols used denote the components of the unit tangent vector (s^i), principal normal (n^i), and binormal (b^i) vectors with respect to a streamline at any point in the flow field. Denoting the magnitudes of the velocity and magnetic field vectors as V, H, respectively we get,

$$\frac{V^i}{V} = s^i, \tag{3.1.1}$$

where s denotes arc length measured along the streamlines in the direction of the flow.
For the magnetic field we have,

$$\frac{H^i}{H} = h^i, \tag{3.1.2}$$

where h^i is a constant unit vector.

A set of relations involving the three unit vectors s^i, n^i, b^i, is given by the well known Frenet-Serret formulas of differential geometry

$$\frac{ds^i}{ds} = kn^i, \qquad \frac{db^i}{ds} = -\tau n^i, \qquad \frac{dn^i}{ds} = \tau b^i - ks^i, \tag{3.1.3}$$

where k is the curvature and τ the torsion of the streamlines.

Substituting Eqs. (3.1.1), (3.1.2) into the appropriate Eqs. (2.3.1) to (2.3.5) the system of equations becomes,

$$\frac{\partial}{\partial x^i}(\rho V s^i) = 0, \tag{3.1.4}$$

$$\rho V s^j \frac{\partial (V s^i)}{\partial x^j} + g^{ij}\frac{\partial P_t}{\partial x^j} - \mu H h^j \frac{\partial (H h^i)}{\partial x^j} = 0, \tag{3.1.5}$$

$$\rho V s^j \frac{\partial I}{\partial x^j} = \delta^{-1}\left(g^{jr}\frac{\partial (H h^k)}{\partial x^r}\,\frac{\partial (H h^j)}{\partial x^k} - g^{jr}\frac{\partial (H h^k)}{\partial x^r}\,\frac{\partial (H h_k)}{\partial x^j}\right) +$$

$$\mu V s^j H h^i \frac{\partial (H h_j)}{\partial x^i} - V s^j \frac{\partial P_m}{\partial x^j} + \frac{\partial Q^i}{\partial x^i}, \tag{3.1.6}$$

$$\frac{\partial (H h^i)}{\partial x^i} = 0, \tag{3.1.7}$$

$$V s^j \frac{\partial (H h^i)}{\partial x^j} - H h^j \frac{\partial (V s^i)}{\partial x^j} + H h^i \frac{\partial (V s^j)}{\partial x^j} = \eta \frac{\partial}{\partial x^j}\left(g^{jk}\frac{\partial (H h^i)}{\partial x^k}\right), \tag{3.1.8}$$

Expanding Eq. (3.1.4) and using $s^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial s}$ we get,

$$\frac{\partial s^i}{\partial x^i} + \frac{\partial}{\partial s}(\ln \rho V) = 0. \tag{3.1.9}$$

Expanding Eq. (3.1.7) we get,

$$\frac{\partial H}{\partial h} = 0. \tag{3.1.10}$$

Expanding Eq. (3.1.5) and using $s^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial s}$,

$h^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial h}$, we get,

$$\rho V \frac{\partial V}{\partial s} s^i + \rho V^2 \frac{\partial s^i}{\partial s} + g^{ij} \frac{\partial P_t}{\partial x^j} - \mu H \frac{\partial H}{\partial h} h^i = 0. \tag{3.1.11}$$

Using Eqs. (3.1.3) and (3.1.10), Eq.(3.1.11) becomes,

$$\rho V \frac{\partial V}{\partial s} s^i + \rho V^2 k n^i + g^{ij} \frac{\partial P_t}{\partial x^j} = 0. \tag{3.1.12}$$

Expanding Eq. (3.1.8) we get,

$$V \frac{\partial H}{\partial s} h^i - H \frac{\partial V}{\partial h} s^i - HV \frac{\partial s^i}{\partial h} + H\left(\frac{\partial V}{\partial s} + V \frac{\partial s^j}{\partial x^j}\right) h^i = \eta \frac{\partial}{\partial x^j}\left(g^{jk} \frac{\partial H}{\partial x^k}\right) h^i, \tag{3.1.13}$$

or

$$\left(\frac{\partial (HV)}{\partial s} + HV \frac{\partial s^j}{\partial x^j} - \eta \frac{\partial}{\partial x^j}\left(g^{jk} \frac{\partial H}{\partial x^k}\right)\right) h^i - H \frac{\partial V^i}{\partial h} = 0. \tag{3.1.14}$$

Expanding Eq.(3.1.6) we get,

$$\rho V \frac{\partial I}{\partial s} = \delta^{-1}\left(g^{jr} h^k \frac{\partial H}{\partial x^r} h_j \frac{\partial H}{\partial x^k} - g^{jr} h^k \frac{\partial H}{\partial x^r} h_k \frac{\partial H}{\partial x^j}\right) +$$

$$\mu V H s^j \frac{\partial}{\partial h}(H h_j) - V \frac{\partial P_m}{\partial s} + \frac{\partial Q^i}{\partial x^i}. \tag{3.1.15}$$

Using Eq.(3.1.10), equation (3.1.15) reduces to,

$$\rho V \frac{\partial I}{\partial s} = -\delta^{-1}\left(g^{jr} \frac{\partial H}{\partial x^r} \frac{\partial H}{\partial x^j}\right) - V \frac{\partial P_m}{\partial s} + \frac{\partial Q^i}{\partial x^i}. \tag{3.1.16}$$

3.2. Variation of Pressure Along and Perpendicular to the Streamlines

To determine the variation of the total pressure P_t along the tangent, principal normal, and binormal directions of the streamlines, we take the scalar product of Eq. (3.1.12) with $g_{ik}s^k$, $g_{ik}n^k$, $g_{ik}b^k$, respectively and get,

$$\rho V\frac{\partial V}{\partial s}s^i g_{ik}s^k + \rho V^2 k n^i g_{ik}s^k + g^{ij}g_{ik}s^k\frac{\partial P_t}{\partial x^j} = 0, \qquad (3.2.1)$$

$$\rho V\frac{\partial V}{\partial s}s^i g_{ik}n^k + \rho V^2 k n^i g_{ik}n^k + g^{ij}g_{ik}n^k\frac{\partial P_t}{\partial x^j} = 0, \qquad (3.2.2)$$

$$\rho V\frac{\partial V}{\partial s}s^i g_{ik}b^k + \rho V^2 k n^i g_{ik}b^k + g^{ij}g_{ik}b^k\frac{\partial P_t}{\partial x^j} = 0. \qquad (3.2.3)$$

Making use of the orthogonal properties of s^i, n^i, b^i, and $g^{ij}g_{ik} = \delta^j_k$, $s^j\frac{\partial}{\partial x^j} = \frac{\partial}{\partial s}$, $n^j\frac{\partial}{\partial x^j} = \frac{\partial}{\partial n}$, etc. we get from Eqs. (3.2.1) to (3.2.3),

$$\rho V\frac{\partial V}{\partial s} + \frac{\partial P_t}{\partial s} = 0, \qquad (3.2.4)$$

$$\rho V^2 k + \frac{\partial P_t}{\partial n} = 0, \qquad (3.2.5)$$

$$\frac{\partial P_t}{\partial b} = 0, \qquad (3.2.6)$$

where $P_t = (p + p_R + p_m)$.

From the result of Eq. (3.2.6) we note that the total pressure remains constant along the binormal direction of the streamlines.

From the result of Eq. (3.2.5) we obtain an expression for the curvature k of the streamlines as a function of the fluid density ρ, the velocity V, and the normal pressure gradient as,

$$k = -(\rho V^2)^{-1}\frac{\partial P_t}{\partial n}. \qquad (3.2.7)$$

To obtain a relation for the normal vector of the streamlines n^i as a function of k, ρ, V, and the velocity and total pressure gradients along the streamlines we solve Eq. (3.$.12) for n^i and get,

$$n^i = -(\rho V^2 k)^{-1}(\rho V\frac{\partial V}{\partial s}s^i + g^{ij}\frac{\partial P_t}{\partial x^j}). \qquad (3.2.8)$$

Multiplying the last term of Eq.(3.2.8) by the scalar product of s^i we get,

$$(s^i g_{ik} s^k)g^{ij}\frac{\partial P_t}{\partial x^j} = s^i\frac{\partial P_t}{\partial s}. \qquad (3.2.9)$$

Substituting Eq.(3.2.9) for the last term of (3.2.8) and using Eq.(3.1.1) we get from (3.2.8),

$$n^i = -(\rho V^3 k)^{-1}(\rho V\frac{\partial V}{\partial s} + \frac{\partial P_t}{\partial s})V^i. \qquad (3.2.10)$$

We next obtain a relation for the binormal vector of the streamlines by starting from the definition of b^i,

$$b^i = e^{ijk} g_{jp} g_{kq} s^p n^q. \tag{3.2.11}$$

From Eq.(3.1.12) we have,

$$n^q = -(\rho V^2 k)^{-1} \left(\rho V \frac{\partial V}{\partial s} s^q + g^{qr} \frac{\partial P_t}{\partial x^r} \right). \tag{3.2.12}$$

Substituting Eq.(3.2.12) into (3.2.11) we get,

$$b^i = -(\rho V^2 k)^{-1} \left(\rho V \frac{\partial V}{\partial s} e^{ijk} g_{jp} g_{kq} s^p s^q + e^{ijk} g_{jp} g_{kq} s^p g^{qr} \frac{\partial P_t}{\partial x^r} \right). \tag{3.2.13}$$

Since, $e^{ijk} g_{jp} g_{kq} s^p s^q = 0$, $g_{kq} g^{qr} = \delta_k^r$, $g_{jp} s^p = V_j V^{-1}$,

we get the binormal vector as a function of the flow field parameters,

$$b^i = -(\rho V^3 k)^{-1} \left(e^{ijk} V_j \frac{\partial P_t}{\partial x^k} \right). \tag{3.2.14}$$

<u>Streamline Torsion</u>

To obtain a relation connecting the torsion τ, of the streamlines with the flow field parameters we make use of the Frenet formula Eq.(3.1.3) which is,

$$-\tau n^i = \frac{db^i}{ds}. \tag{3.2.15}$$

Differentiating Eq.(3.2.14) along a streamline we get,

$$\frac{db^i}{ds} = -e^{ijk}\frac{d}{ds}((\rho V^3 k)^{-1} V_j \frac{\partial P_t}{\partial x^k}). \qquad (3.2.16)$$

Expanding Eq.(3.2.16) and substituting the result into Eq.(3.2.15) we get for the torsion of the streamlines,

$$-\tau n^i = e^{ijk} (\rho V^3 k)^{-2} V_j \frac{\partial P_t}{\partial x^k} \frac{\partial}{\partial s}(\rho V^3 k) - (\rho V^3 k)^{-1}(\frac{\partial P_t}{\partial x^k} \frac{\partial V_j}{\partial s} + V_j \frac{\partial}{\partial s}(\frac{\partial P_t}{\partial x^k})) . \qquad (3.2.17)$$

We next determine the static fluid pressure-gradient along a streamline as a function of the Mach number M and the other four variables and their gradients along a streamline; i.e., ρ, V, T, H, and their gradients. From Eq. (3.1.9) we have,

$$\frac{\partial s^i}{\partial x^i} + V^{-1}\frac{\partial V}{\partial s} + \rho^{-1}\frac{\partial \rho}{\partial s} = 0. \qquad (3.2.18)$$

Solving Eq.(3.2.4) for $\frac{\partial V}{\partial s}$ and substituting it into Eq.(3.2.18) we get,

$$-V^2 \rho \frac{\partial s^i}{\partial x^i} - V^2 \frac{\partial \rho}{\partial s} + \frac{\partial p}{\partial s} + \frac{\partial P_R}{\partial s} + \frac{\partial P_m}{\partial s} = 0. \qquad (3.2.19)$$

The velocity of sound is defined by $a^2 = \frac{\partial p}{\partial \rho}$, or

$$a^{-2}\frac{\partial p}{\partial s} = \frac{\partial \rho}{\partial s} . \qquad (3.2.20)$$

Substituting (3.2.20) into (3.2.19) we get,

$$\frac{\partial p}{\partial s} = (M^2-1)^{-1}\left(\frac{\partial p_m}{\partial s} + \frac{\partial p_R}{\partial s} - \rho V^2 \frac{\partial s^1}{\partial x^1}\right). \qquad (3.2.21)$$

Substituting for $-\frac{\partial s^1}{\partial x^1}$ by using (3.2.18), Eq. (3.2.21) becomes,

$$\frac{\partial p}{\partial s} = (M^2-1)^{-1}\left(\mu H \frac{\partial H}{\partial s} + \frac{\partial p_R}{\partial s} + \rho V \frac{\partial V}{\partial s} + V^2 \frac{\partial \rho}{\partial s}\right). \qquad (3.2.22)$$

Equation (3.2.22) indicates the influence that each variable has upon the fluid pressure variation along a streamline.

Expanding Eq.(3.2.5) and solving for the fluid pressure variation in the normal direction to the streamlines we get,

$$\frac{\partial p}{\partial n} = -\left(\mu H \frac{\partial H}{\partial n} + \frac{\partial p_R}{\partial n} + \rho V^2 k\right). \qquad (3.2.23)$$

From equation (3.2.6) we obtain the following expression for the pressure variation in the binormal direction to the streamlines

$$\frac{\partial p}{\partial b} = -\left(\mu H \frac{\partial H}{\partial b} + \frac{\partial p_R}{\partial b}\right). \qquad (3.2.24)$$

3.3. Vorticity Components

The vorticity components are defined as,

$$w_k = e_{kij}g^{jr}\frac{\partial V^i}{\partial x^r} . \tag{3.3.1}$$

Substituting $V^i = Vs^i$ and expanding we get,

$$w_k = Ve_{kij}g^{jr}\frac{\partial s^i}{\partial x^r} + s^i e_{kij}g^{jr}\frac{\partial V}{\partial x^r} . \tag{3.3.2}$$

Taking the scalar product of Eq.(3.3.2) with s^k, n^k, b^k, respectively and recalling the cross-product relations $e_{kij}s^i s^k = 0$, $e_{kij}s^i n^k = b_j$, $e_{kij}s^i b^k = -n_j$, we get,

$$w_k s^k = V(e_{kij}s^k g^{jr}\frac{\partial s^i}{\partial x^r}), \tag{3.3.3}$$

$$w_k n^k = V(e_{kij}n^k g^{jr}\frac{\partial s^i}{\partial x^r}) + \frac{\partial V}{\partial b} , \tag{3.3.4}$$

$$w_k b^k = V(e_{kij}b^k g^{jr}\frac{\partial s^i}{\partial x^r}) - \frac{\partial V}{\partial n} . \tag{3.3.5}$$

In order to obtain the term $\frac{\partial s^i}{\partial x^r}$ as a linear combination of $g_{rp}s^p$, $g_{rp}n^p$, $g_{rp}b^p$, we make use of the following identities,

$$s^r\frac{\partial s^i}{\partial x^r} = \frac{\partial s^i}{\partial s} , \quad n^r\frac{\partial s^i}{\partial x^r} = \frac{\partial s^i}{\partial n}, \quad b^r\frac{\partial s^i}{\partial x^r} = \frac{\partial s^i}{\partial b}, \tag{3.3.6}$$

Each identity of Eq. (3.3.6) is the scalar product of $\frac{\partial s^i}{\partial x^r}$ with s^r, n^r, b^r, respectively and since the later are orthogonal we have,

$$\frac{\partial s^i}{\partial x^r} = \frac{\partial s^i}{\partial s} g_{rp} s^p + \frac{\partial s^i}{\partial n} g_{rp} n^p + \frac{\partial s^i}{\partial b} g_{rp} b^p \ . \tag{3.3.7}$$

Multiplying this last result by g^{jr} and using $g^{jr} g_{rp} = \delta^j_p$ we get,

$$g^{jr} \frac{\partial s^i}{\partial x^r} = \frac{\partial s^i}{\partial s} s^j + \frac{\partial s^i}{\partial n} n^j + \frac{\partial s^i}{\partial b} b^j \ . \tag{3.3.8}$$

Substituting (3.3.8) into the Eqs. (3.3.3) to (3.3.5) we get from the later,

$$w_k s^k = V\left(\frac{\partial s^i}{\partial s} e_{kij} s^k s^j + \frac{\partial s^i}{\partial n} e_{kij} s^k n^j + \frac{\partial s^i}{\partial b} e_{kij} s^k b^j\right), \tag{3.3.9}$$

$$w_k n^k = V\left(\frac{\partial s^i}{\partial s} e_{kij} n^k s^j + \frac{\partial s^i}{\partial n} e_{kij} n^k n^j + \frac{\partial s^i}{\partial b} e_{kij} n^k b^j\right) + \frac{\partial V}{\partial b}, \tag{3.3.10}$$

$$w_k b^k = V\left(\frac{\partial s^i}{\partial s} e_{kij} b^k s^j + \frac{\partial s^i}{\partial n} e_{kij} b^k n^j + \frac{\partial s^i}{\partial b} e_{kij} b^k b^j\right) - \frac{\partial V}{\partial n}. \tag{3.3.11}$$

Taking account of the cross-product relations in equations (3.3.9) to (3.3.11) we get the following results,

$$w_k s^k = V(g_{ir} b^r \frac{\partial s^i}{\partial n} - g_{ir} n^r \frac{\partial s^i}{\partial b}), \tag{3.3.12}$$

$$w_k n^k = V(-g_{ir} b^r n^i k + 0 + g_{ir} s^r \frac{\partial s^i}{\partial b}) + \frac{\partial V}{\partial b} = \frac{\partial V}{\partial b}, \tag{3.3.13}$$

$$w_k b^k = Vk - \frac{\partial V}{\partial n}. \tag{3.3.14}$$

We note again that each equation (3.3.12) to (3.3.14) is the scalar product of w_k with s^k, n^k, b^k, and thus we get the vorticity components as,

$$w_k = V(g_{ir} b^r \frac{\partial s^i}{\partial n} - g_{ir} n^r \frac{\partial s^i}{\partial b}) g_{kp} s^p + (\frac{\partial V}{\partial b}) g_{kp} n^p + (Vk - \frac{\partial V}{\partial n}) g_{kp} b^p. \tag{3.3.15}$$

Multiplying this last result by g^{jk} and using $g^{jk} w_k = w^j$

$g^{jk} g_{kp} = \delta^j_p$, we get

$$w^j = V(g_{ir} b^r \frac{\partial s^i}{\partial n} - g_{ir} n^r \frac{\partial s^i}{\partial b}) s^i + n^j \frac{\partial V}{\partial b} + (Vk - \frac{\partial V}{\partial n}) b^j. \tag{3.3.16}$$

Using the identity developed in Eq.(3.3.8) on V we get,

$$g^{ir} \frac{\partial V}{\partial x^r} = \frac{\partial V}{\partial s} s^i + \frac{\partial V}{\partial n} n^i + \frac{\partial V}{\partial b} b^i. \tag{3.3.17}$$

Multiplying this last result by V and solving for

$V \frac{\partial V}{\partial s} s^i$ we get,

$$\rho V\frac{\partial V}{\partial s}s^i = \rho V g^{ir}\frac{\partial V}{\partial x^r} - \rho V(\frac{\partial V}{\partial n}n^i + \frac{\partial V}{\partial b}b^i). \tag{3.3.18}$$

Substituting Eq.(3.3.18) into Eq.(3.1.12) we get from the later,

$$\tfrac{1}{2}\rho g^{ir}\frac{\partial V^2}{\partial x^r} - \rho V(\frac{\partial V}{\partial n}n^i + \frac{\partial V}{\partial b}b^i) + \rho V^2 kn^i + g^{ir}\frac{\partial P_t}{\partial x^r} = 0. \tag{3.3.19}$$

Adding and subtracting $\tfrac{1}{2}V^2 g^{ir}\frac{\partial \rho}{\partial x^r}$ from Eq.(3.3.19) and rearraging we get,

$$-\rho^{-1}g^{ir}\frac{\partial}{\partial x^r}(p + p_R + \tfrac{1}{2}\rho V^2 + p_m) + \tfrac{1}{2}\rho^{-1}V^2 g^{ir}\frac{\partial \rho}{\partial x^r} = V(Vk - \frac{\partial V}{\partial n})n^i - V\frac{\partial V}{\partial b}b^i . \tag{3.3.20}$$

Introducing a function $\overline{B}$ defined by

$$g^{ir}\frac{\partial \overline{B}}{\partial x^r} = -\rho^{-1}g^{ir}\frac{\partial}{\partial x^r}(p + p_R + p_m + \tfrac{1}{2}\rho V^2) + \tfrac{1}{2}\rho^{-1}g^{ir}\frac{\partial \rho}{\partial x^r} , \tag{3.3.21}$$

With this last result equation (3.3.20) becomes,

$$g^{ir}\frac{\partial \overline{B}}{\partial x^r} = V(Vk - \frac{\partial V}{\partial n})n^i - V\frac{\partial V}{\partial b}b^i, \tag{3.3.22}$$

or

$$\frac{\partial \overline{B}}{\partial x^j} = V(Vk - \frac{\partial V}{\partial n})g_{ji}n^i - V\frac{\partial V}{\partial b}g_{ji}b^i. \tag{3.3.23}$$

Equation (3.3.23) is a vector normal to the surfaces $\bar{B}$ = constant, and if we let its magnitude be $\left|\frac{d\bar{B}}{dN}\right|$ we get from Eq.(3.3.23),

$$\left|\frac{d\bar{B}}{dN}\right| = \left(V^2\left(\frac{\partial V}{\partial b}\right)^2 + V^2\left(Vk - \frac{\partial V}{\partial n}\right)^2\right)^{\frac{1}{2}} . \qquad (3.3.24)$$

Taking the scalar product of equation (3.3.23) with Eq. (3.3.16) we get,

$$w^j\frac{\partial \bar{B}}{\partial x^j} = V\left(Vk - \frac{\partial V}{\partial n}\right)\frac{\partial V}{\partial b}\, g_{ji}n^i n^j - V\left(Vk - \frac{\partial V}{\partial n}\right)\frac{\partial V}{\partial b} g_{ji}b^i b^j = 0. \qquad (3.3.25)$$

Also, taking the scalar product of equation (3.3.23) with s^j we get,

$$s^j\frac{\partial \bar{B}}{\partial x^j} = 0. \qquad (3.3.26)$$

Thus, equations (3.3.25) and (3.3.26) imply that the surfaces $\bar{B}$ = constant contain both the streamlines and the vortex lines.

3.4. Variation of Energy Along the Streamlines

Substituting the identity $\frac{\partial p_m}{\partial s} = \rho(\frac{\partial}{\partial s}(p_m \rho^{-1}) - p_m \frac{\partial \rho^{-1}}{\partial s})$

into equation (3.1.16) we get,

$$\rho V \frac{\partial I}{\partial s} + \rho V \frac{\partial}{\partial s}(p_m \rho^{-1}) = -\delta^{-1}(g^{jr}\frac{\partial H}{\partial x^r}\frac{\partial H}{\partial x^j}) + \rho V p_m \frac{\partial \rho^{-1}}{\partial s} + \frac{\partial Q^i}{\partial x^i}, \tag{3.4.1}$$

Dividing (3.4.1) by (ρV) and combining the two terms on the left we get,

$$\frac{\partial I_t}{\partial s} = -\delta^{-1}(g^{jr}\frac{\partial H}{\partial x^r}\frac{\partial H}{\partial x^j})(\rho V)^{-1} + p_m \frac{\partial \rho^{-1}}{\partial s} + (\rho V)^{-1}\frac{\partial Q^i}{\partial x^i}, \quad (3.4.2.)$$

where $I_t = (u + p\rho^{-1} + p_R\rho^{-1} + p_m\rho^{-1} + \frac{1}{2}V^2)$.

Since $g_{jr}s^r s^j = 1$, we get for the first term on the right of Eq.(3.4.2)

$$\delta^{-1}(g_{jr}s^r s^j g^{jr}\frac{\partial H}{\partial x^r}\frac{\partial H}{\partial x^j}) = \delta^{-1}(\frac{\partial H}{\partial s})^2. \tag{3.4.3}$$

Using the identity developed in Eq.(3.3.8) on Q_i and substituting the result together with Eq. (3.4.3) into Eq.(3.4.2) we get,

$$\frac{\partial I_t}{\partial s} = -\delta^{-1}(\frac{\partial H}{\partial s})^2(\rho V)^{-1} + p_m\frac{\partial \rho^{-1}}{\partial s} +$$

$$(\rho V)^{-1}(\frac{\partial Qi}{\partial s}s^i + \frac{\partial Q_i}{\partial n}n^i + \frac{\partial Q_i}{\partial b}b^i). \tag{3.4.4}$$

Equation (3.4.4) shows that the change of total energy per unit mass, per unit distance along the streamlines depends on: (1) the Joule heat generated, (2) the work of compression done by the magnetic pressure, and (3) the variation of the heat flux vector along and perpendicular to the streamlines.

4. GOVERNING EQUATIONS IN CYLINDRICAL COORDINATES

In this section we formulate the equations of section 2. in cylindrical coordinates for later application to a specific flow problem.

4.1. General Cylindrical Coordinates

Introducing cylindrical coordinates (r,θ,z), we get from Eq.(2.1.1),

$$\frac{\partial(\rho V_r)}{\partial r} + \frac{(\rho V_r)}{r} + \frac{\partial(\rho V_\theta)}{r\,\partial\theta} + \frac{\partial(\rho V_z)}{\partial z} = 0, \qquad (4.1.1)$$

and from equation (2.1.2) we have

$$\rho\left(V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right) = -\frac{\partial(P)}{\partial r} + (J_\theta B_z - J_z B_\theta) \qquad (4.1.2)$$

$$\rho\left(V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right) = -\frac{\partial P}{r\,\partial\theta} + (J_z B_r - J_r B_z) \qquad (4.1.3)$$

$$\rho\left(V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + (J_r B_\theta - J_\theta B_r). \qquad (4.1.4)$$

The energy equation becomes,

$$\rho\left(V_r\frac{\partial e}{\partial r} + \frac{V_\theta}{r}\frac{\partial e}{\partial\theta} + V_z\frac{\partial e}{\partial z}\right) = \left(V_r\frac{\partial P}{\partial r} + \frac{V_\theta}{r}\frac{\partial P}{\partial\theta} + V_z\frac{\partial P}{\partial z}\right) + \frac{J^2}{\sigma} + \left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_\theta}{r\,\partial\theta} + \frac{\partial Q_z}{\partial z}\right), \qquad (4.1.5)$$

where $P = (p + p_R)$, $e = (u + P/\rho)$, $J^2 = J_r^2 + J_\theta^2 + J_z^2$.

From Eq. (2.1.5) we get

$$\left(\frac{\partial E_z}{r\,\partial\theta} - \frac{\partial E_\theta}{\partial z}\right) = 0, \tag{4.1.6}$$

$$\left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right) = 0, \tag{4.1.7}$$

$$\left(\frac{\partial E_\theta}{\partial r} - \frac{\partial E_r}{r\,\partial\theta}\right) = 0, \tag{4.1.8}$$

and from Eq. (2.1.6) we have,

$$\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{\partial B_\theta}{r\,\partial\theta} + \frac{\partial B_z}{\partial z} = 0, \tag{4.1.9}$$

$$\left(\frac{\partial B_z}{r\,\partial\theta} - \frac{\partial B_\theta}{\partial z}\right) = J_r\mu, \tag{4.1.10}$$

$$\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) = J_\theta\mu, \tag{4.1.11}$$

$$\left(\frac{\partial B_\theta}{\partial r} - \frac{\partial B_r}{r\,\partial\theta}\right) = J_z\mu , \tag{4.1.12}$$

and by Eq.(2.1.7)

$$J_r = \sigma(E_r + (V_\theta B_z - V_z B_\theta)), \tag{4.1.13}$$

$$J_\theta = \sigma(E_\theta + (V_z B_r - V_r B_z)), \tag{4.1.14}$$

$$J_z = \sigma(E_z + (V_r B_\theta - V_\theta B_r)), \tag{4.1.15}$$

The above equations (4.1.1) to (4.1.15) are a set of relations for the following unknown quantities,

$p, \rho, T, V_r, V_\theta, V_z, B_r, B_\theta, B_z, E_r, E_\theta, E_z, J_r, J_\theta, J_z$.

4.2. Axially Symmetric Case

In this section we consider Eqs. (4.1.1) to (4.1.15) in axial symmetry for which we have the conditions,

$\frac{\partial}{\partial\theta} = 0$, $V_\theta = 0$, $B_\theta = 0$, and we set $v=V_r$, $w=V_z$.

Introducing the above conditions into the Eqs. (4.1.1) to (4.1.15) we get from (4.1.10) and (4.1.12)

$$J_r = J_z = 0, \tag{4.2.1}$$

from Eqs.(4.1.13) and (4.1.15)

$$J_r = \sigma E_r, \qquad J_z = \sigma E_z, \tag{4.2.2}$$

thus, by (4.2.1) we find that

$$E_r = E_z = 0, \tag{4.2.3}$$

and by Eqs. (4.1.6) to (4.1.8) we find

$$\frac{\partial E_\theta}{\partial r} = 0, \qquad \frac{\partial E_\theta}{\partial z} = 0, \text{ so the } E_\theta = \text{constant}.$$

From Eq.(4.1.14) we find that $J_\theta = 0$, for $v=w=0$, so that $E_\theta = 0$, and therefore it is zero everywhere in the flow field, since it is a constant.

The system of equations (4.1.1) to (4.1.15) now reduce to the following,

$$\frac{\partial(\rho v)}{\partial r} + \frac{(\rho v)}{r} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{4.2.4}$$

$$(v\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z}) = - \frac{\partial P}{\partial r} + \partial(wB_r - vB_z)B_z, \qquad (4.2.5)$$

$$(v\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}) = - \frac{\partial P}{\partial z} - \partial(wB_r - vB_z)B_r, \qquad (4.2.6)$$

$$(v\frac{\partial e}{\partial r} + w\frac{\partial e}{\partial z}) = (v\frac{\partial P}{\partial r} + w\frac{\partial P}{\partial z}) + \partial(wB_r - vB_z)^2 + (\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}), \qquad (4.2.7)$$

where we have used J_Θ from Eq. (4.1.14).

Equations (4.1.9) and (4.1.11) now become by using (4.1.14),

$$\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{\partial B_z}{\partial z} = 0, \qquad (4.2.8)$$

$$(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}) = \mu\partial(wB_r - vB_z). \qquad (4.2.9)$$

The equations (4.2.4) to (4.2.9) together with the equation of state p= RT, are seven equation for the seven unknown quantities; i.e., p, ρ, T, v, w, B_r, B_z. The above quantities are functions of z and r, only.

4.3. Incompressible Case

If the fluid density can be considered as remaining essentially constant in some flow region, the system of equations (4.2.4) to (4.2.9) reduces to the following;

$$\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial w}{\partial z} = 0, \qquad (4.3.1)$$

$$\rho\left(v\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z}\right) + \frac{\partial p}{\partial r} - \sigma B_z(wB_r - vB_z) = 0, \qquad (4.3.2)$$

$$\rho\left(v\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) + \frac{\partial p}{\partial z} + \sigma B_r(wB_r - vB_z) = 0, \qquad (4.3.3)$$

$$\rho c_v\left(v\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) - \sigma(wB_r - vB_z)^2 - \left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}\right) = 0, \qquad (4.3.4)$$

$$\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{\partial B_z}{\partial z} = 0, \qquad (4.3.5)$$

$$\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) - \mu\sigma(wB_r - vB_z) = 0. \qquad (4.3.6)$$

Equations (4.3.1) to (4.3.6) are six equations for the six unknowns; i,e., p, T, v, w, B_r, B_z, ρ = constant.

4.4. Alternate Axially Symmetric Case

In this section we consider equations (4.1.1) to (4.1.15) in axial symmetry for which we have the following conditions;

$$\frac{\partial}{\partial\theta} = 0, \quad V_\theta = 0, \quad J_\theta = 0, \quad \text{and we set} \quad v = V_r, \quad w = V_z.$$

In steady flow the electric field $\vec{E}$ may be taken as constant or in the case of no applied electric field it may be taken as zero, (55).

Introducing the above conditions into the Eqs.(4.1.1) to (4.1.15) we get from (4.1.1),

$$\frac{\partial(\rho v)}{\partial r} + \frac{\rho v}{r} + \frac{\partial(\rho w)}{\partial z} = 0 , \tag{4.4.1}$$

from (4.1.2)

$$\rho(v\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z}) = - \frac{\partial P}{\partial r} - J_z B_\theta , \tag{4.4.2}$$

from (4.1.4)

$$\rho(v\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}) = - \frac{\partial P}{\partial z} + J_r B_\theta, \tag{4.4.3}$$

from (4.1.5)

$$\rho(v\frac{\partial e}{\partial r} + w\frac{\partial e}{\partial z}) = (v\frac{\partial P}{\partial r} + w\frac{\partial P}{\partial z}) + \frac{J^2}{\sigma} + (\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}), \tag{4.4.4}$$

from (4.1.13)

$$J_r = -\sigma w B_\theta , \tag{4.4.5}$$

from (4.1.15)

$$J_z = \partial v B_\Theta , \tag{4.4.6}$$

from (4.1.10) and (4.1.12)

$$\frac{\partial B_\Theta}{\partial z} = - J_r \mu, \qquad \frac{\partial B_\Theta}{\partial r} = J_z \mu. \tag{4.4.7}$$

Eliminating the current density J from Eqs. (4.4.2) to (4.4.4) by using (4.4.5) to (4.4.7) we get,

from (4.4.2)

$$\rho(v\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z}) = - \frac{\partial P}{\partial r} - \mu^{-1}\frac{\partial B_\Theta}{\partial r}B_\Theta , \tag{4.4.8}$$

from (4.4.3)

$$\rho(v\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}) = - \frac{\partial P}{\partial z} - \mu^{-1}\frac{\partial B_\Theta}{\partial z}B_\Theta , \tag{4.4.9}$$

from (4.4.4)

$$\rho(v\frac{\partial e}{\partial r} + w\frac{\partial e}{\partial z}) = (v\frac{\partial P}{\partial r} + w\frac{\partial P}{\partial z}) + B_\phi^2(v^2 + w^2) + (\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}). \tag{4.4.10}$$

Adding the components of (4.4.7) and substituting (4.4.5) and (4.4.6) for the current density we get,

$$\frac{\partial B_\Theta}{\partial r} + \frac{\partial B_\Theta}{\partial z} = \mu \partial B_\Theta(v + w). \tag{4.4.11}$$

Equations (4.4.8) and (4.4.9) may be written as

$$\rho(v\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z}) + \frac{\partial P}{\partial r} + \frac{\partial}{\partial r}(\frac{B_\Theta^2}{2\mu}) = 0, \tag{4.4.12}$$

$$\rho(v\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}) + \frac{\partial P}{\partial z} + \frac{\partial}{\partial z}(\frac{B_\Theta^2}{2\mu}) = 0, \tag{4.4.13}$$

To complete the system of equations we add the equation of state of the gas and the equation of continuity,

$$p = \rho RT, \tag{4.4.14}$$

$$\frac{\partial(\rho v)}{\partial r} + \frac{\rho v}{r} + \frac{\partial(\rho w)}{\partial z} = 0. \tag{4.4.15}$$

Equations (4.4.10) to (4.4.15) are six equations for the six unknown quantities p, ρ, T, v, w, and B_θ .

We will now integrate the equations of motion and energy along a streamline. Thus, multiplying (4.4.12) by dr, (4.4.13) by dz, and (4.4.10) by dz and using (4.3.13) for the streamlines we get,

$$\rho\left(v\frac{\partial v}{\partial r}dr + v\frac{\partial v}{\partial z}dz\right) + \frac{\partial P}{\partial r}dr + \frac{\partial}{\partial r}\left(\frac{B^2}{2\mu}\right)dr = 0, \tag{4.4.16}$$

$$\rho\left(w\frac{\partial w}{\partial r}dr + w\frac{\partial w}{\partial z}dz\right) + \frac{\partial P}{\partial z}dz + \frac{\partial}{\partial z}\left(\frac{B^2}{2\mu}\right)dz = 0, \tag{4.4.17}$$

$$\rho\left(w\frac{\partial e}{\partial r}dr + w\frac{\partial e}{\partial z}dz\right) = \left(w\frac{\partial P}{\partial r}dr + w\frac{\partial P}{\partial z}dz\right) + \partial B^2(v^2 + w^2)dz + \left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}\right)dz, \tag{4.4.18}$$

Factoring v, and w, and noting that $dv = \frac{\partial v}{\partial r}dr + \frac{\partial v}{\partial z}dz$ etc. we get,

$$\rho v dv + \frac{\partial P}{\partial r}dr + \frac{\partial}{\partial r}\left(\frac{B^2}{2\mu}\right)dr = 0, \tag{4.4.19}$$

$$\rho w dw + \frac{\partial P}{\partial z}dz + \frac{\partial}{\partial z}\left(\frac{B^2}{2\mu}\right)dz = 0, \tag{4.4.20}$$

$$\rho w de = w dP + \partial B^2(v^2 + w^2)dz + (\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z})dz, \quad (4.4.21)$$

Adding equations (4.4.19) and (4.4.20) we get,

$$d(\tfrac{1}{2}v^2 + \tfrac{1}{2}w^2) + dP + d(\frac{B^2}{2\mu}) = 0. \quad (4.4.22)$$

Dividing Eq. (4.4.22) by the fluid density and integrating we get,

$$\tfrac{1}{2}v^2 + \tfrac{1}{2}w^2 + \int\frac{dP}{\rho} + \int\frac{d(\frac{B^2}{2\mu})}{\rho} = \text{constant}. \quad (4.4.23)$$

For constant fluid density we get from (4.4.23) by integration from some reference point,

$$(\tfrac{1}{2}\rho v^2 + \tfrac{1}{2}\rho w^2) - (\tfrac{1}{2}\rho v_o^2 + \tfrac{1}{2}\rho w_o^2) + (P-P_o) + (\frac{B^2}{2\mu} - \frac{B_o^2}{2\mu}) = 0. \quad (4.4.24)$$

Solving equation (4.4.24) for the fluid static pressure p, we get

$$p = P_o + \frac{B_o^2}{2\mu} + \tfrac{1}{2}\rho(v_o^2 + w_o^2) - \tfrac{1}{2}\rho(v^2 + w^2) - p_R - \frac{B^2}{2\mu}. \quad (4.4.25)$$

We note that equation (4.4.25) reduces to the classical Bernoulli equation for the non-radiating, non-magnetic case.

From equation (4.4.21) we get with $e = c_pT$,

$$\left.\frac{dT}{dz}\right|_{\gamma=\text{const.}} = (\rho c)^{-1}\frac{dP}{dr} + \frac{\partial B^2}{\rho c_p w}(v^2 + w^2) + (\rho c_p w)^{-1}(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}). \quad (4.4.26)$$

For the incompressible case ρ = constant we get from equation (4.4.21)

$$\rho w d(c_v T + P/\rho) = \rho w d(c_v T) + w dP$$
$$= w dP + \partial B^2(v^2 + w^2)dz + \left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}\right)dz. \tag{4.4.27}$$

Upon cancelling wdP and dividing by $\rho w c_v$ we get from equation (4.4.27)

$$\left.\frac{dT}{dz}\right|_{\psi=\text{const.}} = \frac{\partial B^2}{\rho c_v w}(v^2 + w^2) + (\rho c_v w)^{-1}\left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}\right). \tag{4.4.28}$$

We will return to the above equations in section 6., after we establish the general fluid flow conditions and the fluid properties.

5. THERMAL RADIATION AND IONIZATION

In this section we give a brief outline of the governing equations of radiative transfer, and develop the equations for calculating ionization and electric conductivity of the gas.

5.1. The Equation of Transfer

A high temperature gas emits radiation energy as a result of rotational, vibrational, and electronic transitions from exited energy levels to lower energy levels. The emitted radiant energy corresponding to these transitions is distributed over a wide wave length region. The total radiant intensity emitted from a volume of gas is obtained by summing the radiant intensities from the individual energy transitions. For gas dynamic calculations the simplest approach to the determination of the radiative intensity of gases is to determine overall emissivities as a function of pressure and temperature of the gas.

The fundamental quantity sought in radiative transfer of energy through an absorbing, emitting, and scattering medium is the specific intensity I_ν defined by,

$$\frac{dE_\nu}{\cos\theta \, dA \, dw \, d\nu \, dt} = I_\nu, \qquad (5.1.1)$$

where dE_ν is the amount of energy transmitted in the frequency interval $(\nu, \nu + d\nu)$, through dA in time dt, in a direction making an angle θ with the normal to dA, and lying within the solid angle dw.

The distribution of the intensity I_ν in the radiation field is governed by a conservation equation called the radiative transfer equation. This equation, as given by Chandrasekhar and Kourganoff, is (36,37)

$$-\frac{dI_\nu}{ds} = \rho k_\nu I_\nu - \rho j_\nu \, , \qquad (5.1.2)$$

where, ρ = fluid density

k_ν = absorption coefficient

j_ν = emission coefficient

The emission coefficient j_ν for the case in which both scattering and absorption and emission are present, is given by Kourganoff as (36),

$$j_\nu = k_\nu \bar{w}_o \bar{I} + (1 - \bar{w}_o) k_\nu B_\nu(T), \qquad (5.1.3)$$

where $\bar{w}_o$ represents the fraction of energy loss due to scattering and is called the albedo for single scattering, and $B_\nu(T)$ is the Planck function given by,

$$B_\nu(T) = 2h\nu^3 c^{-2} (\exp(\frac{h}{kT}) - 1)^{-1} \qquad (5.1.4)$$

where k and h are the Boltzmann and Planck constants respectively.

The two special cases of local thermodynamic equilibrium and perfect isotropic scattering are obtained from Eq. (5.1.3) by letting $\bar{w}_o = 0$, and $\bar{w}_o = 1$, respectively.

Substituting Eq. (5.1.3) into (5.1.2) and dividing by ρk_ν we get,

$$-\frac{dI_\nu}{\rho k_\nu ds} = I_\nu - (\bar{w}_0 \bar{I}_\nu + (1 - \bar{w}_0)B_\nu(T)), \qquad (5.1.5)$$

where $\bar{I}_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu d\mu_0 , \qquad (\mu_0 = \cos\theta).$

For local thermodynamic equilibrium $\bar{w}_0 = 0$, and Eq. (5.1.5) becomes,

$$-\frac{dI_\nu}{\rho k_\nu ds} = I_\nu - B_\nu(T). \qquad (5.1.6)$$

For isotropic scattering $\bar{w}_0 = 1$, and Eq. (5.1.5) gives,

$$-\frac{dI_\nu}{\rho k_\nu ds} = I_\nu - \bar{I}_\nu \qquad (5.1.7)$$

The optical thickness of the medium between the points s' and s is defined by,

$$\tau_0(s',s) = \int_{s'}^{s} \rho k_\nu \; ds, \qquad (5.1.8)$$

so that $\qquad d\tau_0 = k_\nu \rho \, ds. \qquad (5.1.9)$

5.2. Radiation Flux and Pressure

In the general case equation (5.1.5) must be solved for the specific intensity I_ν. The heat flux vector $\vec{Q}_R$ is then obtained by integration as,

$$\vec{Q}_R = \int_0^\infty \int_0^{2\pi} \int_0^\pi I_\nu \sin\theta \cos\theta d\theta d\phi d\nu , \qquad (5.2.1)$$

and the radiation pressure is given by,

$$p_R = 2\pi c^{-1} \int_{-1}^{+1} I_\nu \mu_o^2 \, d\mu_o \quad . \qquad (5.2.2)$$

Since the fluid dynamic equations of motion and energy in which the above two terms appear are a set of differential equations, it is desireable to obtain the expressions for $\vec{Q}_R$ and p_R as a function of the fluid properties or their derivatives. this is possible if local thermodynamic equilibrium may be assumed such that a local fluid temperature T may be defined at each point in the flow field. In such a case the governing equation for the intensity I_ν is Eq. (5.1.6), and for the optically thick case a solution may be obtained by a Taylor series expansion of I_ν about $B_\nu(T)$. The expressions for $\vec{Q}_R$ and p_R as obtained by Zhigulev (18), Goulard (20), Scala and Sampson (31), and Pai (38), are

$$p_R = 4/3(\frac{\sigma_b}{c})T^4, \qquad (5.2.3)$$

$$\nabla \cdot \vec{Q}_R = - 4\sigma_b K_p T^4, \quad \text{optically thin gas}, \qquad (5.2.4)$$

$$\vec{Q}_R = 16/3(\frac{\sigma_s T^3}{K_R})\nabla T, \text{ optically thick gas}, \qquad (5.2.5)$$

where K_p is the Planck mean absorption coefficient defined by,

$$K_p = B(T)^{-1}\int_0^\infty k_\nu B_\nu(T) d \;, \quad B(T) = \int_0^\infty B_\nu(T) d\nu = \frac{\sigma_s T^4}{\pi}, \quad (5.2.6)$$

and K_R is the Rosseland mean absorption coefficient defined by,

$$1/K_R = \frac{\int_0^\infty \frac{dB_\nu}{k_\nu\, dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu} . \tag{5.2.7}$$

The Rosseland mean absorption coefficient K_R as given by Scala and Sampson (31) for air as a function of temperature and pressure is,

$$K_R = (4.52 \times 10^{-7}) p^{1.31} \exp(5.18 \times 10^{-4} T - 7.13 \times 10^{-9} T^2), \tag{5.2.8}$$

where K_R is expressed in cm^{-1}, the pressure p in atmospheres, and the temperature T in ^oK.

The Planck mean absorption coefficient K_P for air was also given as,

$$K_P = 8.3 K_R. \tag{5.2.9}$$

5.3. Ionization and Electric Conductivity

One of the most important transport properties in magnetogasdynamics is the electric conductivity of the gas which in part depends on the number of free electrons present or the degree of ionization of the gas. The ionization occuring in high temperature gases, such as that surrounding the space vehicle, is referred to as thermal ionization which is a general term applied to the ionizing action of molecular collisions, radiation, and electron collisions.

To determine the degree of ionization we consider a gas mixture of neutral particles, positive ions, and electrons which produce partial pressures and are related to the total gas pressure by,

$$p = p_n + p_i + p_e \ . \qquad (5.3.1)$$

The pressure p is related to the temperature T by,

$$p = nk_oT, \qquad N/m^2 \qquad (5.3.2)$$

where n is the number of molecules per unit volume and k_o is the gas constant per molecule or the Boltzmann constant. If we define the degree of ionization as

$$x = \frac{n_e}{n} = \frac{n_i}{n}, \qquad (5.3.3)$$

where $n_e = n_i$ are the number of electrons and ions per unit volume, and $n = n_n + n_e$, then the relation developed by Saha is (48)

$$\frac{x^2}{1-x^2} = (3.158 \times 10^{-7})\frac{T^{5/2}}{p_a}\exp(-\frac{q}{k_oT}), \qquad (5.3.4)$$

where, p_a = total pressure in atmospheres,

q = ionization energy in joules,

T = temperature in ^oK,

k_o = Boltzmann constant in joule/ ^oK.

Substituting $p_a = p/(1.013 \times 10^5)$ into Eq. (5.3.4)

$$\frac{x^2}{1-x^2} = (.032T^{5/2}p^{-1})\exp(-\frac{q}{K_oT}) = K(T,p), \qquad (5.3.5)$$

where p is in Newtons per m^2.

Solving Eq.(5.3.5) for the degree of ionization x we get,

$$x = \frac{n_e}{n} = (\frac{K(T,p)}{1 + K(T,p)})^{\frac{1}{2}} . \qquad (5.3.6)$$

Substituting Eq.(5.3.2) for n into (5.3.6) we get the electron number density as a function of temperature and pressure of the gas,

$$n_e = \frac{p}{K_oT} (\frac{K(T,p)}{1 + K(T,p)})^{\frac{1}{2}}. \qquad (5.3.7)$$

The number of neutral particles may be obtained from

$$n_n = n - n_e . \qquad (5.3.8)$$

Using Eqs. (5.3.2) and (5.3.7) we get the neutral particles as a function of temperature and pressure of the gas,

$$n_n = \frac{p}{K_oT}(1 - (\frac{K(T,p)}{1 + K(T,p)})^{\frac{1}{2}}. \qquad (5.3.9)$$

We note that in the limit as the temperature T becomes large the quantity containing $K(T,p)$ in Eq.(5.3.9) approaches unity so that $n_n \rightarrow 0$, and we have a fully ionized gas, and as T becomes small the quantity approaches zero and we have a neutral gas.

An equation for the electrical conductivity of a partially ionized gas which was found to agree very well with experiment is (41)

$$\partial = \frac{n_e(e^2)}{m_e\bar{v}(n_n\bar{Q}_{en} + n_i\bar{Q}_{ei})}, \quad \frac{\text{mohs}}{\text{m}} \tag{5.3.10}$$

Where, m_e = electron rest mass, kg,

e = electron charge, coulomb,

$\bar{v}$ = mean thermal velocity of an electron, m/sec,

$\bar{Q}_{en}$ = electron-atom mean collision cross section, m^2,

$\bar{Q}_{ei}$ = electron-ion mean collision cross section, m^2,

The mean electron thermal velocity $\bar{v}$ is given as a function of temperature by,

$$\bar{v} = \left(\frac{8k_oT}{m_e}\right)^{\frac{1}{2}} . \tag{5.3.11}$$

Substituting this last result into Eq.(5.3.10) we get,

$$\partial = \left(\frac{e^4}{8k_oTm_e}\right)^{\frac{1}{2}}\left(\frac{n_e}{n_n\bar{Q}_{en} + n_i\bar{Q}_{ei}}\right) . \tag{5.3.12}$$

From equation (5.3.8) we get by using (5.3.6),

$$\frac{n_n}{n_e} = \frac{n}{n_e} - 1 = (1 + K^{-1})^{\frac{1}{2}} - 1. \tag{5.3.13}$$

From Eq.(5.3.5) we have,

$$K^{-1} = \frac{p\exp(q/k_oT)}{.032\ T^{5/2}} . \tag{5.3.14}$$

Substituting Eq.(5.3.14) into (5.3.13) we get,

$$\frac{n_n}{n_e} = (1 + \frac{p\exp(q/k_oT)}{.032T^{5/2}}) - 1. \qquad (5.3.15)$$

Since in our case $n_e = n_i$ we get from Eq.(5.3.12) by dividing top and bottom of the last term by n_e and using Eq.(5.3.13)

$$\delta = (\frac{e^4}{8k_o m_e T})^{\frac{1}{2}}(((1 + K^{-1}) - 1)\overline{Q}_{en} + \overline{Q}_{ei})^{-1}. \qquad (5.3.16)$$

The mean electron-ion collision cross section $\overline{Q}_{ei}$ is (41)

$$\overline{Q}_{ei} = \frac{1.714(10^{-10})}{.5816\ T^2} \ln(\frac{1.241(10^4)T^2}{(n_eT)^{\frac{1}{2}}(2)^{\frac{1}{2}}}). \quad m^2 \qquad (5.3.17)$$

from Eqs.(5.3.7) and (5.3.14) we have (n_e in $\#/cm^3$)

$$(n_eT)^{\frac{1}{2}} = 10^{-3}(\frac{(p/k_o)^2}{1 + K^{-1}})^{\frac{1}{4}} . \qquad (5.3.18)$$

Substituting (5.3.18) into (5.3.17) we get the collision cross section as a function of temperature and pressure,

$$\overline{Q}_{ei} = \frac{2.95(10^{-10})}{T^2} \ln(\frac{8.8(10^6)T^2}{(\frac{(p/k_o)^2}{1+ K^{-1}})^{\frac{1}{4}}}) . \qquad (5.3.19)$$

Equation (5.3.16) together with (5.3.19) gives the electrical conductivity of a partially ionized gas as a function of temperature and pressure of the gas. ($\overline{Q}_{en}$= constant).

6. SPECIAL CASE OF v(r), w(z), ONLY

In this section we consider a solution to the equations along streamlines as obtained in section 4.4., by choosing the form of the streamlines so that v(r) only and w(z) only.

6.1. Equation of the Streamlines

Introducing a streamfunction such that,

$$v = \frac{\partial \psi}{r\,\partial z}, \qquad -w = \frac{\partial \psi}{r\,\partial r}, \tag{6.1.1}$$

and the equation of continuity (4.3.7) is automatically satisfied. If we let

$$v = br, \qquad w = -2bz, \tag{6.1.2}$$

where b is a costant, we have by Eq.(6.1.1),

$$\frac{\partial \psi}{r\,\partial z} = br, \qquad \frac{\partial \psi}{r\,\partial r} = 2bz. \tag{6.1.3}$$

From this last result we find that

$$\psi = br^2z, \tag{6.1.4}$$

which is the required streamfunction.
It is readily verified that Eq.(6.1.4) satisfies the Laplace equation

$$\frac{\partial^2\psi}{\partial r^2} - \frac{\partial \psi}{r\,\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0. \tag{6.1.5}$$

From equation (6.1.4) we find that for $\psi = 0$, either $r = 0$, or $z = 0$, so that the z-axis is the stagnation streamline and at $z=0$, we have the r-plane through that point. For $\psi = \psi_i =$ constant we get from Eq.(6.1.4)

$$z = \frac{\psi_1}{b} r^{-2} , \qquad (6.1.6)$$

which is the equation of the streamlines and represents flow against a disk. To obtain a particular set of streamlines it is necessary to evaluate the constant "b" in Eq. (6.1.6). For this purpose we use the definition of the Stokes streamfunction; i.e., $2\pi\psi$ is equal to the volume flow rate between any two streamlines for constant density flow. Thus, at any point z upstream of the wall the volume flow rate between the stagnation streamline and any other streamline r distance away from it is given by

$$2\pi\psi = (\pi r^2)V, \qquad (6.1.7)$$

where V is the fluid velocity of the oncoming stream. Thus, by Eq.(6.1.7) and (6.1.4) we have

$$\psi = \tfrac{1}{2}Vr^2 = br^2z, \qquad (6.1.8)$$

and

$$b = \tfrac{1}{2}V/z. \qquad (6.1.9)$$

Now if the velocity V is known at some point $z = z_1$ upstream from the wall; i.e., at $z = z_1$, $V = V_1$, and we have by Eq.(6.1.9)

$$b = \tfrac{1}{2}V_1/z_1 . \qquad (6.1.10)$$

Substituting the result of Eq.(6.1.10) into (6.1.4) we get,

$$\psi = (\tfrac{1}{2}V_1/z_1)r^2z \ . \qquad (6.1.11)$$

Solving (6.1.11) for r we get,

$$r = \left(\frac{\psi}{(\frac{1}{2}V_1/z_1)z}\right)^{\frac{1}{2}}, \qquad \psi > 0. \qquad (6.1.12)$$

We may now obtain explicit expressions for the pressure and temperature distribution along the streamlines given by Eq.(6.1.12).

From Eq.(6.1.2) we have

$$v^2 + w^2 = b^2(r^2 + 4z^2). \qquad (6.1.13)$$

Substituting Eqs.(6.1.13) and (5.2.3) into (4.4.25) and noting that $V^2 = v^2 + w^2$ we get,

$$p = P_1 + \frac{B_1^2}{2\mu} + \tfrac{1}{2}\rho V_1^2 - \tfrac{1}{2}\rho b^2\left(\frac{\psi}{bz} + 4z^2\right) - \frac{4\partial_s T^4}{3c} - \frac{B^2}{2\mu}, \quad (6.1.14)$$

where we have also used (6.1.12) to eliminate r.

We next obtain the expression for the temperature distribution along the streamlines from Eq.(4.4.28) by using Eqs.(5.2.4), (6.1.2), (6.1.12) and (6.1.13),

$$\left.\frac{dT}{dz}\right|_{\psi = \text{const.}} = \frac{-\partial B^2 b^2}{c_v \rho 2bz}\left(\frac{\psi}{bz} + 4z^2\right) + \frac{4\partial_s K_p T^4}{\rho c_v 2bz} \ . \qquad (6.1.15)$$

Equations (6.1.14) and (6.1.15) are two equation for the two unknowns p and T along the streamlines given by (6.1.12).

We now consider the magnetic field of the following form

$$B = \frac{B_1 r_1^2}{r^2} . \qquad (6.1.16)$$

Introducing

$$a = (\psi/b) \qquad (6.1.17)$$

and using Eq.(6.1.12) to eliminate r in (6.1.16) we get

$$B = \frac{B_1 r_1^2 z}{a} . \qquad (6.1.18)$$

Substituting (6.1.18) into (6.1.14) and (6.1.15) we get,

$$p = P_1 + p_m + \tfrac{1}{2}\rho V_1^2 - \tfrac{1}{2}\rho b^2(a/z + 4z^2) - p_R - \frac{B_1^2 r_1^4 z^2}{2\mu a^2}, \qquad (6.1.19)$$

$$\left.\frac{dT}{dz}\right|_{\psi = \text{const.}} = A_R T^4/z - A_M(1/a + \frac{4z^3}{a^2}), \qquad (6.1.20)$$

where
$$A_R = \frac{2\partial_B K_p}{\rho c_v b} , \quad \text{and} \quad A_M = \frac{\partial b(B_1^2 r_1^4)}{\rho 2\ c_v} . \qquad (6.1.21)$$

6.2. Temperature and Pressure Distribution Along Streamlines

The pressure distribution is given by Eq.(6.1.19) which can be evaluated once the temperature distribution is known. The temperature distribution is given by (6.1.20) which is a first order non-linear ordinary differential equation of the following general form,

$$\frac{dT}{dz} = f(z,T), \tag{6.2.1}$$

with the condition of $T = T_1$, at $z = z_1$.

We propose a solution of Eq.(6.1.20) by a method of successive approximation. A proper development of this method is given by Coddington (57). The successive approximate solutions to Eq.(6.2.1) are defined to be the functions T_1, T_2, T_3, ·····, given recursively by the formulas,

$$\begin{aligned} T_1(z_1) &= T_1 \qquad \text{(initial condition)} \\ T_2(z) &= T_1 + \int_{z_1}^{z} f(z,T_1)dz, \\ T_3(z) &= T_1 + \int_{z_1}^{z} f(z,\ T_1(z))dz, \\ T_{n+1}(z) &= T_1 + \int_{z_1}^{z} f(z,\ T_n(z))dz, \end{aligned} \tag{6.2.2}$$

where $n = 1,2,3,\cdots\cdots\cdots$.

It may be noted that the more nearly correct a particular approximation $T_n(z)$ is, the better will be its successor $T_{n+1}(z)$. In our case we will obtain a good first approximation by integrating Eq.(6.1.20) with the magnetic term neglected. Thus, by neglecting the magnetic term in Eq. (6.1.20) and integrating by separation of variables we get for our first approximation,

$$T_2 = (c_1 - 3A_R \ln z)^{-1/3}, \qquad (6.2.3)$$

where c_1 is obtained from the initial condition $T = T_1$, at $z = z_1$, as

$$c_1 = T_1^{-3} + 3A_R \ln z_1. \qquad (6.2.4)$$

To obtain the second approximation we substitute Eq.(6.2.3) into (6.1.20) and (6.2.2), which gives

$$T_3 = T_1 + \int_{z_1}^{z} \frac{A_R}{z}(c_1 - 3A_R \ln z)^{-4/3} dz - \frac{A_M}{a}\int_{z_1}^{z}(1 + 4z^3/a)dz. \qquad (6.2.5)$$

Integrating and using (6.2.4) for c_1 we get from (6.2.5)

$$T_3 = (T_1^{-3} - 3A_R \ln(z/z_1))^{-1/3} + \frac{A_M}{a}(z_1 - z) + \frac{A_M}{a^2}(z_1^4 + z^4). \qquad (6.2.6)$$

We note that for $z = z_1$, $T_3 = T_1$ as required by the initial condition, and as $z \to 0$, $T_3 \to (A_M z_1/a + A_M z_1^4/a^2)$.

A higher approximation may be obtained by re-substituting Eq.(6.2.6) into (6.2.2) which gives,

$$T_4 = T_1 + \frac{A_M}{a}(z_1 - z) + \frac{A_M}{a^2}(z_1^4 - z^4) +$$

$$\int_{z_1}^{z} \frac{A_R}{z}\left[(c_1 - 3A_R \ln z)^{-1/3} + \frac{A_M}{a}(z_1 - z) + \frac{A_M}{a^2}(z_1^4 - z^4)\right]^4 dz. \quad (6.2.7)$$

From Eq.(6.2.7) it is apparent that the formal integration process becomes more and more complicated for higher approximations so that a numerical process would have to be used sooner or later in order to obtain the nth order of approximation. Therefore, we propose a piecewise application of our second order approximation (6.2.6) over a number of smaller intervals by dividing the range of integration into a finite number of smaller intervals. Thus, dropping the subscript 3 in Eq.(6.2.6) which denoted the 2nd approximation, we may use Eq. (6.2.6) to compute the temperature in the range $z_2 \leq z \leq z_1$ where z_2 may be taken as close to z_1 as desired to obtain the necessary accuracy. After computing the temperature at z_2 we may consider this point as our initial condition and apply Eq.(6.2.6) over the next interval $z_3 \leq z \leq z_2$ with z_2 playing the role of z_1. We may continue in this manner until the entire range of interest is covered.

In general we may write Eq.(6.2.6) in the following form,

$$T_{i+1} = (T_i^{-3} - 3A_R \ln(z/z_i))^{-1/3} + \frac{A_M}{a}(z_i - z) + \frac{A_M}{a^2}(z_i^4 - z^4), \quad (6.2.8)$$

where T_{i+1} is the temperature at any point in the interval $z_{i+1} \leq z \leq z_i$ and T_i is the temperature at the point z_i; $i = 1,2,3,\cdots\cdots$, represents the number of intervals under consideration. Thus, we consider equation (6.2.8) as the solution to the temperature distribution over the entire range of interest. i,e., $0 < z \leq z_1$.

PART II: VISCOUS RADIATION MAGNETOHYDRODYNAMICS

7. GOVERNING EQUATIONS OF VISCOUS FLOW

7.1. Fundamental Equations

We consider a viscous, heat-conducting, steady flow of an ionized gas in an electro-magnetic field with thermal radiation. The governing equations for the present case may be obtained by modifying the system of equations derived in section 2.3. The modification consists of adding the viscous stress terms to the equations of momentum (2.3.2), and the viscous dissipation term to the equation of energy (2.3.3). The heat flux vector $\vec{Q}$ is also modified to account for the heat conductivity of the gas.

The viscous stress term is given by, (56)

$$\frac{\partial \tau^{ij}}{\partial x^j}, \tag{7.1.1}$$

where τ^{ij} are the components of the stress tensor given by

$$\tau^{ij} = \bar{\mu}\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right) - \frac{2}{3}\bar{\mu}\frac{\partial v^k}{\partial x^k}\delta^{ij} \; . \tag{7.1.2}$$

The viscous dissipation function is obtained as,

$$\bar{\mu}\phi = g_{ki}\tau^{kj}\frac{\partial v^i}{\partial x^j}. \tag{7.1.3}$$

The heat flux vectors now become from (5.2.4) and (5.2.5)

$$\vec{Q} = (K_t + \frac{16\partial_B T^3}{3K_R})\nabla T, \qquad \text{optically thick gas,} \qquad (7.1.4)$$

$$\nabla\cdot\vec{Q} = \nabla\cdot(K_t\nabla T) - 4K_p\partial_B T^4, \text{ optically thin gas,} \qquad (7.1.5)$$

The system of equations now become,

continuity,

$$\frac{\partial}{\partial x^i}(\rho V^i) = 0, \qquad (7.1.6)$$

momentum,

$$\rho V^j\frac{\partial V^i}{\partial x^j} + g^{ij}\frac{\partial P_t}{\partial x^j} - \mu H^j\frac{\partial H^i}{\partial x^j} - \frac{\partial\tau^{ij}}{\partial x^j} = 0, \qquad (7.1.7)$$

energy,

$$\rho V^j\frac{\partial I}{\partial x^j} = \partial^{-1}(g^{jr}\frac{\partial H^k}{\partial x^r}\frac{\partial H_j}{\partial x^k} - g^{jr}\frac{\partial H^k}{\partial x^r}\frac{\partial H_k}{\partial x^j}) +$$

$$\mu V^j H^i\frac{\partial H_j}{\partial x^i} - V^j\frac{\partial P_m}{\partial x^j} + \frac{\partial Q^i}{\partial x^i} + \bar{\mu}\phi. \qquad (7.1.8)$$

magnetic field Eq.

$$V^j\frac{\partial H^i}{\partial x^j} - H^j\frac{\partial V^i}{\partial x^j} + H^i\frac{\partial V^j}{\partial x^j} = \eta\frac{\partial}{\partial x^j}(g^{jk}\frac{\partial H^i}{\partial x^k}), \qquad (7.1.9)$$

and the equation of state, $\quad p = \rho RT. \qquad (7.1.10)$

We note that the above system of equations are considerably more complex than the classical Navier-Stokes equations of classical fluid dynamics.

7.2. Transformation to Streamline Coordinates

The transformation of equations (7.1.6) and (7.1.9) was given in section 3. and will not be repeated here. By introducing the velocity and magnetic field components from (3.1.1), (3.1.2), into the equations of momentum (7.1.7) and the equation of energy (7.1.8) we get,

$$\rho V s^j \frac{\partial (V s^i)}{\partial x^j} + g^{ij}\frac{\partial P_t}{\partial x^j} - \mu H h^j \frac{\partial (H h^i)}{\partial x^j} - \frac{\partial \tau^{ij}}{\partial x^j} = 0, \qquad (7.2.1)$$

$$\rho V s^j \frac{\partial I}{\partial x^j} = \delta^{-1}\left(g^{jr}\frac{\partial (Hh^k)}{\partial x^r}\frac{\partial (Hh_j)}{\partial x^k} - g^{jr}\frac{\partial (Hh^k)}{\partial x^r}\frac{\partial (Hh_k)}{\partial x^j}\right) +$$

$$\mu V s^j H h^i \frac{\partial (Hh_j)}{\partial x^i} - V s^j \frac{\partial P_m}{\partial x^j} + \frac{\partial Q^i}{\partial x^i} + g_{ki}\tau^{kj}\frac{\partial (V s^i)}{\partial x^j}. \qquad (7.2.2)$$

Expanding (7.2.1) and using $s^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial s}$ etc., and also (3.1.3) and (3.1.11) we get

$$\rho V \frac{\partial V}{\partial s} s^i + \rho V^2 k n^i + g^{ij}\frac{\partial P_t}{\partial x^j} - \frac{\partial \tau^{ij}}{\partial x^j} = 0. \qquad (7.2.3)$$

Expanding equation (7.2.2) and using (3.1.11) we get,

$$\rho V \frac{\partial I}{\partial s} = -\delta^{-1}\left(g^{jr}\frac{\partial H}{\partial x^r}\frac{\partial H}{\partial x^j}\right) - V\frac{\partial P_m}{\partial s} + \frac{\partial Q^i}{\partial x^i} + g_{ki}\tau^{kj}\frac{\partial (V s^i)}{\partial x^j}.$$

$$(7.2.4)$$

7.3. Streamline-Pressure Variation, Curvature and Torsion

The variation of the total pressure P_t along the tangent, principal normal, and binormal directions of the streamlines may be obtained by taking the scalar product of (7.2.3) with $g_{ik}s^k$, $g_{ik}n^k$, $g_{ik}b^k$, respectively,

$$\rho V\frac{\partial V}{\partial s}s^i g_{ik}s^k + \rho V^2 kn^i g_{ik}s^k + g^{ij}g_{ik}s^k\frac{\partial P_t}{\partial x^j} - g_{ik}s^k\frac{\partial \tau^{ij}}{\partial x^j} = 0, \tag{7.3.1}$$

$$\rho V\frac{\partial V}{\partial s}s^i g_{ik}n^k + \rho V^2 kn^i g_{ik}n^k + g^{ij}g_{ik}n^k\frac{\partial P_t}{\partial x^j} - g_{ik}n^k\frac{\partial \tau^{ij}}{\partial x^j} = 0, \tag{7.3.2}$$

$$\rho V\frac{\partial V}{\partial s}s^i g_{ik}b^k + \rho V^2 kn^i g_{ik}b^k + g^{ij}g_{ik}b^k\frac{\partial P_t}{\partial x^j} - g_{ik}b^k\frac{\partial \tau^{ij}}{\partial x^j} = 0. \tag{7.3.3}$$

Making use of the orthogonal properties of s^i, n^i, b^i, and $g^{ij}g_{ik} = \delta^j_k$, $s^j\frac{\partial}{\partial x^j} = \frac{\partial}{\partial s}$, $n^j\frac{\partial}{\partial x^j} = \frac{\partial}{\partial n}$, etc., we get from (7.3.1) to (7.3.3),

$$\rho V\frac{\partial V}{\partial s} + \frac{\partial P_t}{\partial s} - g_{ik}s^k\frac{\partial \tau^{ij}}{\partial x^j} = 0, \tag{7.3.4}$$

$$\rho V^2 k + \frac{\partial P_t}{\partial n} - g_{ik}n^k\frac{\partial \tau^{ij}}{\partial x^j} = 0, \tag{7.3.5}$$

$$\frac{\partial P_t}{\partial b} - g_{ik}b^k\frac{\partial \tau^{ij}}{\partial x^j} = 0, \tag{7.3.6}$$

where the total pressure $P_t = (p + p_m + p_R)$.

From Eq. (7.3.4) we find that the pressure variation along the streamlines depends on the momentum change as well as on the viscous stresses. The same holds true for the pressure variation in the normal direction of the streamlines. From Eq. (7.3.6) we see that the pressure is no longer constant in the binormal direction of the streamlines for the viscous case under consideration here.

Streamline Curvature

An expression for the curvature k of the streamlines may be obtained in terms of the fluid density ρ, the velocity V, the normal pressure gradient $\frac{\partial P_t}{\partial n}$, and the viscous stress term by solving Eq.(7.3.5) for k,

$$k = (\rho V^2)^{-1}\left(g_{ik}n^k \frac{\partial \tau^{ij}}{\partial x^j} - \frac{\partial P_t}{\partial n}\right). \tag{7.3.7}$$

Torsion

To develop an expression for the torsion of the streamlines as a function of the flow field parameters we begin with the Frenet formula

$$-\tau n^i = \frac{db^i}{ds}. \tag{7.3.8}$$

The unit binormal vector in Eq.(7.3.8) is by definition

$$b^i = e^{ijk} g_{jp} g_{kq} s^p n^q. \tag{7.3.9}$$

An expression for the normal vector n^q may be obtained by solving Eq.(7.2.3) as,

$$n^q = (\rho V^2 k)^{-1}(-\rho V \frac{\partial V}{\partial s} s^q - g^{qr}\frac{\partial P_t}{\partial x^r} + \frac{\partial \tau^{qr}}{\partial x^r}). \qquad (7.3.10)$$

Substituting Eq.(7.3.10) into (7.3.8) and making use of the following identities,

$$e^{ijk} g_{jp} g_{kq} s^p s^q = 0, \quad g_{kq} g^{qr} = \delta^r_k, \qquad g_{jp} s^p = V_j V^{-1},$$

we get the binormal vector as a function of the flow field parameters,

$$b^i = (\rho V^3 k)^{-1} e^{ijk}(V_j g_{kq}\frac{\partial \tau^{qr}}{\partial x^r} - V_j \frac{\partial P_t}{\partial x^k}). \qquad (7.3.11)$$

differentiating (7.3.11) along a streamline and substituting into (7.3.8) we get the following expression for the torsion of the streamlines,

$$-\tau n^i = e^{ijk}\frac{\partial}{\partial s}\left\{(\rho V^3 k)^{-1} V_j (g_{kq}\frac{\partial \tau^{qr}}{\partial x^r} - \frac{\partial P_t}{\partial x^k})\right\}. \qquad (7.3.12)$$

7.4. Vorticity

Substituting Eq.(3.3.18) into (7.2.3) we get

$$-\rho V\left(\frac{\partial V}{\partial n}n^i + \frac{\partial V}{\partial b}b^i\right) + \rho V^2 k n^i + V g^{ir}\frac{\partial V}{\partial x^r} + g^{ir}\frac{\partial P_t}{\partial x^r} - \frac{\partial \tau^{ij}}{\partial x^j} = 0. \tag{7.4.1}$$

Adding and subtracting $\frac{1}{2}V^2 g^{ir}\frac{\partial \rho}{\partial x^r}$ from Eq. (7.4.1) we get,

$$\left(\rho V^2 k - \rho V\frac{\partial V}{\partial n}\right)n^i - \rho V\frac{\partial V}{\partial b}b^i + g^{ir}\frac{\partial(\frac{1}{2}\rho V^2)}{\partial x^r} + g^{ir}\frac{\partial P_t}{\partial x^r} - \frac{\partial \tau^{ir}}{\partial x^r} - \frac{1}{2}V^2 g^{ir}\frac{\partial \rho}{\partial x^r} = 0. \tag{7.4.2}$$

Dividing (7.4.2) by ρ and transposing some terms we get,

$$-\rho^{-1} g^{ir}\frac{\partial}{\partial x^r}\left(P_t + \tfrac{1}{2}\rho V^2\right) + \tfrac{1}{2}\rho^{-1}V^2 g^{ir}\frac{\partial \rho}{\partial x^r} + \frac{\partial \tau^{ir}}{\partial x^r}\rho^{-1} = \left(V^2 k - V\frac{\partial V}{\partial n}\right)n^i - V\frac{\partial V}{\partial b}b^i. \tag{7.4.3}$$

Introducing a function $\bar{B}$ defined by,

$$g^{ir}\frac{\partial \bar{B}}{\partial x^r} = -\rho^{-1} g^{ir}\frac{\partial}{\partial x^r}\left(P_t + \tfrac{1}{2}\rho V^2\right) + \tfrac{1}{2}\rho^{-1}V^2 g^{ir}\frac{\partial \rho}{\partial x^r} + \rho^{-1}\frac{\partial \tau^{ir}}{\partial x^r}, \tag{7.4.4}$$

equation (7.4.3) becomes,

$$g^{ir}\frac{\partial \bar{B}}{\partial x^r} = (V^2 k - V\frac{\partial V}{\partial n})n^i - V\frac{\partial V}{\partial b}b^i. \tag{7.4.5}$$

Multiplying equation (7.4.5) by g_{ji} we get,

$$\frac{\partial \bar{B}}{\partial x^j} = (V^2 k - V\frac{\partial V}{\partial n})g_{ji}n^i - V\frac{\partial V}{\partial b}g_{ji}b^i . \tag{7.4.6}$$

Taking the scalar product of Eq.(7.4.6) with (3.3.16) we get,

$$w^j\frac{\partial \bar{B}}{\partial x^j} = 0. \tag{7.4.7}$$

Taking the scalar product of Eq.(7.4.6) with s^j,we get

$$s^j\frac{\partial \bar{B}}{\partial x^j} = 0. \tag{7.4.8}$$

Thus, equations (7.4.7) and (7.4.8) imply that the surfaces $\bar{B}$ = constant contain both the streamlines and the vortex lines.

7.5. General Cylindrical Coordinates

The governing equations of viscous, radiation magneto-hydrodynamics in cylindrical coordinates are as follows,

$$\frac{\partial}{\partial r}(\rho V_r) + \frac{\rho V_r}{r} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0, \qquad (7.5.1)$$

$$\rho\left(V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right) =$$

$$-\frac{\partial P}{\partial r} + (J_\theta B_z - J_z B_\theta) - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right), \qquad (7.5.2)$$

$$\rho\left(V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right) =$$

$$-\frac{1}{r}\frac{\partial P}{\partial \theta} + (J_z B_r - J_r B_z) - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right), \qquad (7.5.3)$$

$$\rho\left(V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right) =$$

$$-\frac{\partial P}{\partial z} + (J_r B_\theta - J_\theta B_r) - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right), \qquad (7.5.4)$$

$$\rho\left(V_r\frac{\partial e}{\partial r} + \frac{V_\theta}{r}\frac{\partial e}{\partial \theta} + V_z\frac{\partial e}{\partial z}\right) = \left(V_r\frac{\partial P}{\partial r} + \frac{V_\theta}{r}\frac{\partial P}{\partial \theta} + V_z\frac{\partial P}{\partial z}\right) +$$

$$\frac{J^2}{\sigma} + \bar{\mu}\phi + \left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{\partial Q_z}{\partial z}\right), \qquad (7.5.5)$$

where $P = p + p_R$, $e = (u + P/\rho)$, $J^2 = J_\theta^2 + J_r^2 + J_z^2$,

and
$$\bar{\mu}\phi = \tau_{rr}\left(\frac{\partial V_r}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right) + \tau_{zz}\left(\frac{\partial V_z}{\partial z}\right) +$$
$$\tau_{r\theta}\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right) + \tau_{\theta z}\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z}\right) +$$
$$\tau_{rz}\left(\frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z}\right). \qquad (7.5.6)$$

7.6. Axially Symmetric Case

The governing equations (7.5.1) to (7.5.6) may be considerably simplified for the axially symmetric case for which we have the following conditions;

$$\frac{\partial}{\partial \theta} = 0, \quad V_\theta = 0, \quad J_\theta = 0, \quad \tau_{i\theta} = \tau_{\theta i} = 0. \qquad (7.6.1)$$

We also let $V_r = v$, and $V_z = w$, and get by introducing these conditions into (7.5.1) to (7.5.6),

$$\frac{\partial}{\partial r}(\rho v) + \frac{\rho v}{r} + \frac{\partial}{\partial z}(\rho w) = 0, \qquad (7.6.2)$$

$$\rho\left(v\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial r} - J_z B_\theta - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z}\right), \qquad (7.6.3)$$

$$\rho\left(v\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + J_r B_\theta - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}\right), \qquad (7.6.4)$$

$$\rho\left(v\frac{\partial e}{\partial r} + w\frac{\partial e}{\partial z}\right) = \left(v\frac{\partial P}{\partial r} + w\frac{\partial P}{\partial z}\right) + \frac{J^2}{\sigma} + \bar{\mu}\phi +$$
$$\left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}\right),, \qquad (7.6.5)$$

where,

$$\bar{\mu}\phi = \tau_{rr}(\frac{\partial v}{\partial r}) + \tau_{zz}(\frac{\partial w}{\partial z}) + \tau_{rz}(\frac{\partial w}{\partial r} + \frac{\partial v}{\partial z}). \tag{7.6.6}$$

From Eqs. (4.4.5), (4.4.6) we have,

$$J_r = -\sigma w B_\theta, \qquad J_z = -\sigma v B_\theta. \tag{7.6.7}$$

We now have $J^2 = J_r^2 + J_z^2 = \sigma^2 B_\theta^2 (v^2 + w^2)$. (7.6.8)

The current density in (7.6.3) to (7.6.5) may now be eliminated by using (7.6.7) and (7.6.8) as follows,

$$\rho(v\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z}) = -\frac{\partial P}{\partial r} - \sigma v B_\theta^2 - (\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z}), \tag{7.6.9}$$

$$\rho(v\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}) = -\frac{\partial P}{\partial z} - \sigma w B_\theta^2 - (\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}), \tag{7.6.10}$$

$$\rho(v\frac{\partial e}{\partial r} + w\frac{\partial e}{\partial z}) = (v\frac{\partial P}{\partial r} + w\frac{\partial P}{\partial z}) + \sigma B_\theta^2(v^2 + w^2) +$$

$$\bar{\mu}\phi + (\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}). \tag{7.6.11}$$

The above equations (7.6.9) to (7.6.11) may be integrated along a streamline by multiplying Eq. (7.6.9) by dr and (7.6.10), (7.6.11) by dz and using the equation of the streamline $w\,dr = v\,dz$ and noting that

$dv = \frac{\partial v}{\partial r}dr + \frac{\partial v}{\partial z}dz$ etc. we get,

$$\rho v dv + \frac{\partial P}{\partial r}dr + \partial vB_\theta^2 dr + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z}\right)dr = 0, \quad (7.6.12)$$

$$\rho w dw + \frac{\partial P}{\partial z}dz + \partial wB_\theta^2 dz + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}\right)dz = 0, \quad (7.6.13)$$

$$\rho w de = wdP + \partial B_\theta^2(v^2 + w^2)dz + \bar{\mu}\phi dz + \left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}\right)dz. \quad (7.6.14)$$

Adding Eqs. (7.6.12) and (7.6.13) we get,

$$d(\tfrac{1}{2}v^2 + \tfrac{1}{2}w^2) + dP + \partial B_\theta^2(vdr + wdz) +$$
$$\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z}\right)dr + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}\right)dz = 0. \quad (7.6.15)$$

We now let V be the fluid velocity along the streamlines and noting that $vdr + wdz = \frac{(dr)^2}{dt} + \frac{(dz)^2}{dt} = \frac{(ds)^2}{dt} = Vds$

we get from (7.6.15)

$$\rho d(\tfrac{1}{2} V^2) + dP + \partial VB_\theta^2 ds + \tau_s = 0, \quad (7.6.16)$$

where

$$\tau_s = \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z}\right)dr + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}\right)dz. \quad (7.6.17)$$

Noting that $w = dz/dt$ we get from Eq.(7.6.14)

$$(dz/dt)de = (dz/dt)dP + \partial B_\theta^2(V^2)dz + \bar{\mu}\phi dz + Q_s dz, \quad (7.6.18)$$

where, $$Q_s = \left(\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{\partial Q_z}{\partial z}\right). \quad (7.6.19)$$

Dividing Eq.(7.6.18) by dz and multiplying by ds and noting that $ds/dt = V$ we get,

$$\rho V de = V dP + \partial B_0^2 V^2 ds + \bar{\mu}\phi ds + Q_s ds, \tag{7.6.20}$$

or $$\rho V\frac{de}{ds} = V\frac{dP}{ds} + \partial B_0^2 V^2 + \bar{\mu}\phi + Q_s \ . \tag{7.6.21}$$

To complete the system of equations we add the equations of continuity and fluid state along a streamline

$$\frac{d}{ds}(\rho V) = 0, \tag{7.6.22}$$

and

$$\frac{dp}{ds} = R(\rho\frac{dT}{ds} + T\frac{d\rho}{ds}). \tag{7.6.23}$$

Eqs. (7.6.16), (7.6.21), (7.6.22), and (7.6.23) are four differential equations along streamlines for the four quantities P, ρ, T, and V.

The four equations (7.6.16), (7.6.21) to (7.6.23) may be solved for the fluid variable gradients along a streamline as follows; from (7.6.16) we have,

$$\rho V\frac{dV}{ds} + \frac{dP}{ds} + \partial VB^2 + \frac{\tau_s}{ds} = 0, \tag{7.6.24}$$

and from (7.6.22) $$\frac{d\rho}{ds} = -\frac{\rho}{V}\frac{dV}{ds} \ . \tag{7.6.25}$$

Multiplying (7.6.24) by V and adding the result to (7.6.21) we get from the later,

$$\rho V\frac{de}{ds} + \rho V^2\frac{dV}{ds} - \bar{\mu}\phi - Q_s + \frac{V\tau_s}{ds} = 0. \tag{7.6.26}$$

The velocity gradient may be eliminated from (7.6.26) and (7.6.24) by using (7.6.25); thus,

$$\rho V c_p\frac{dT}{ds} - V^3\frac{d\rho}{ds} - \bar{\mu}\phi - Q_s + \frac{V\tau_s}{ds} = 0, \tag{7.6.27}$$

$$-V^2\frac{d\rho}{ds} + \frac{dp}{ds} + \frac{dp_R}{ds} + \partial VB^2 + \frac{\tau_s}{ds} = 0. \tag{7.6.28}$$

The fluid pressure gradient may be eliminated from (7.6.28) by using (7.6.23) and using (5.2.3) for p_R we get,

$$(R\rho + \frac{16\partial_B T^3}{3c})\frac{dT}{ds} + (RT - V^2)\frac{d\rho}{ds} + \partial VB^2 + \frac{\tau_s}{ds} = 0, \tag{7.6.29}$$

Solving (7.6.27) for $d\rho/ds$ we get,

$$\frac{d\rho}{ds} = V^{-2}(\rho c_p\frac{dT}{ds} - (\frac{\bar{\mu}\phi + Q_s}{V}) + \frac{\tau_s}{ds}). \tag{7.6.30}$$

Eliminating $d\rho/ds$ from (7.6.29) by using (7.6.30) and solving the result for the temperature gradient we get,

$$\frac{dT}{ds} = \frac{(\frac{RT}{V^2} - 1)(\frac{\bar{\mu}\phi + Q_s}{V}) - (\frac{RT}{V^2})(\frac{\tau_s}{ds}) - \partial VB^2}{\frac{16\partial_B T^3}{3c} + R\rho + c_p\rho(\frac{RT}{V^2} - 1)}. \tag{7.6.31}$$

Introducing $c_p = \frac{R\gamma}{\gamma-1}$ and $RT = p/\rho$ equation (7.6.31) becomes

$$\frac{dT}{ds} = \frac{(\frac{RT}{V^2} - 1)(\frac{\bar{\mu}\phi + Q_s}{V}) - (\frac{RT}{V^2})(\frac{\tau_s}{ds}) - \sigma V B^2}{\frac{16\sigma_B T^3}{3c} + \frac{p}{T}(1 + \frac{\gamma}{\gamma-1}(\frac{RT}{V^2} - 1))} = f_1 \ . \qquad (7.6.32)$$

The temperature gradient may now be eliminated from Eq. (7.6.30) by using (7.6.32) which gives.

$$\frac{d\rho}{ds} = V^{-2}(\frac{p}{T}(\frac{\gamma}{\gamma-1})f_1 - (\frac{\bar{\mu}\phi + Q_s}{V}) + \frac{\tau_s}{ds}) = V^{-2}f_2 \ . \qquad (7.6.33)$$

The velocity gradient may now be obtained by substituting (7.6.33) into (7.6.25) which gives

$$\frac{dV}{ds} = -\frac{1}{V\rho}(\frac{p}{T}(\frac{\gamma}{\gamma-1})f_1 - (\frac{\bar{\mu}\phi + Q_s}{V}) + \frac{\tau_s}{ds}) = -\frac{f_2}{\rho V} \ . \qquad (7.6.34)$$

The pressure gradient may now be obtained by substituting (7.6.32) and (7.6.33) into (7.6.23) which gives,

$$\frac{dp}{ds} = (\frac{p}{T})f_1 + (\frac{RT}{V^2})f_2. \qquad (7.6.35)$$

The fluid variables p, ρ , T, and V may be obtained by simultaneous solution of equations (7.6.32) to (7.6.35).

8. GEOMETRIC PARAMETERS OF STREAMLINES IN AXIAL SYMMETRY

8.1. Streamlines of the Form $z = f(r)$

In this section we derive a set of relations for the geometric parameters $\hat{s}$, $\hat{n}$, $\hat{b}$, k, and τ of the streamlines which are expressed by an equation of the form $z = f(r)$, where z is the axis of symmetry.

The position vector $\vec{R}$ of any point on the streamline is given by,

$$\vec{R} = ri + zj, \qquad (8.1.1)$$

where i and j are unit vectors in the r and z directions respectively.

A vector tangent to the streamline is given by,

$$\frac{d\vec{R}}{dr} = i + \frac{dz}{dr}j , \qquad (8.1.2)$$

and the unit tangent vector $\hat{s}$ is obtained as

$$\hat{s} = \frac{d\vec{R}}{dr} \Big/ \left|\frac{d\vec{R}}{dr}\right| = (i + \frac{dz}{dr}j)(1 + (\frac{dz}{dr})^2)^{-\frac{1}{2}} . \qquad (8.1.3)$$

Since $\hat{s}\cdot\hat{s} = 1$, we have $\hat{s}\cdot\frac{d\hat{s}}{ds} = 0$, so that $\frac{d\hat{s}}{ds}$ is a vector in the direction normal to $\hat{s}$, and the magnitude of this vector is the curvature k of the curve. Differentiating Eq.(8.1.3) with respect to r we have,

$$\frac{d\hat{s}}{dr} = -(i+\frac{dz}{dr}j)(1+(\frac{dz}{dr})^2)^{-3/2}(\frac{dz}{dr})\frac{d^2z}{dr^2} + (1+(\frac{dz}{dr})^2)^{-\frac{1}{2}}\frac{d^2z}{dr^2}j. \quad (8.1.4)$$

We also have

$$\frac{d\hat{s}}{ds} = \frac{d\hat{s}}{dr}\frac{dr}{ds} = \frac{d\hat{s}}{dr}\Big/\frac{ds}{dr} = \frac{d\hat{s}}{dr}\Big/\left|\frac{d\vec{R}}{dr}\right|. \quad (8.1.5)$$

Using (8.1.2) and (8.1.4) in (8.1.5) we get

$$\frac{d\hat{s}}{ds} = \frac{d^2z}{dr^2}(j - \frac{dz}{dr}i)(1 + (\frac{dz}{dr})^2)^{-2}. \quad (8.1.6)$$

The curvature k of the streamline is now obtaine as,

$$k = \left|\frac{d\hat{s}}{ds}\right| = \frac{d^2z}{dr^2}(1 + (\frac{dz}{dr})^2)^{-3/2} . \quad (8.1.7)$$

The unit vector normal to the streamline may be obtained from the Frenet formula as,

$$\vec{n} = k^{-1}\frac{d\hat{s}}{ds} = (j - \frac{dz}{dr}i)(1 + (\frac{dz}{dr})^2)^{-\frac{1}{2}}. \quad (8.1.8)$$

The unit binormal vector of the streamline is given by,

$$\vec{b} = \vec{s} \times \vec{n}. \quad (8.1.9)$$

Substituting Eqs.(8.1.3) and (8.1.8) into (8.1.9) we get,

$$\vec{b} = (\vec{\theta} + (\frac{dz}{dr})^2\vec{\theta})(1 + (\frac{dz}{dr})^2)^{-1} = \vec{\theta} ; \quad (8.1.10)$$

thus, $\hat{b}$ is a constant unit vector in the direction of θ and perpendicular to both $\hat{s}$ and $\hat{n}$.

From the Frenet formula for the torsion of the streamline we get by taking the dot product,

$$-\tau = \hat{n}\cdot\frac{d\hat{b}}{ds} . \qquad (8.1.11)$$

Since $\hat{b}$ is a constant unit vector we have $\frac{d\hat{b}}{ds} = 0$, and by Eq.(8.1.11) we find that the torsion of the streamlines is zero. Hence, the streamlines are plain curves.

8.2. Streamline Curvature k for $\Psi = f(z,r)$

For the special case of the streamfunction of (6.1.4) we have

$$\Psi = br^2z, \qquad (8.2.1)$$

or

$$z = (\Psi/b)r^{-2}. \qquad (8.2.2)$$

Differentiating (8.2.2) with respect to r we get,

$$\frac{dz}{dr} = -(\Psi/b)r^{-3} , \qquad \frac{d^2z}{dr^2} = 6(\Psi/b)r^{-4} . \qquad (8.2.3)$$

Substituting (8.2.3) into (8.1.7) we get,

$$k = 6(\Psi/b)r^{-4}(1 + 4(\Psi/b)^2r^{-6})^{-3/2} . \qquad (8.2.4)$$

Using Eq.(8.2.2) in (8.2.4) we get

$$k = 6z^2(\Psi/b)^{-1}(1 + 4(\Psi/b)^{-1}z^3)^{-3/2}, \quad \text{for } \Psi \neq 0, \qquad (8.2.5)$$

We also have for arc length, $(ds)^2 = (dz)^2 + (dr)^2$, (8.2.6) so that

$$\frac{ds}{dz} = (1 + \frac{(\Psi/b)}{4}z^{-3})^{\frac{1}{2}}. \qquad (8.2.7)$$

PART III: NUMERICAL SOLUTIONS AND RESULTS

9. INVISCID INCOMPRESSIBLE FLOW RESULTS

The general purpose of Part III is to investigate numerically the effects of various physical phenomena on the flow field variables. Specifically, we are interested in evaluating the effects of the geometric streamline parameters, such as the curvature, the effects of the magnetic field, and the combined effects of thermal radiation and Joule heating on the temperature, electron density, and electric conductivity distribution along streamlines. The procedure consists of numerically evaluating the governing system of equations which were developed in Parts I and II.

9.1. Physical Streamlines and Parameters

The streamlines to be considered in this investigation are those developed in section (6.1) and are given by Eq. (6.1.12) for various values of $\psi = \psi_i =$ constant > 0. By using Eq. (6.1.8) for various values of ψ we obtain the system of streamlines shown in Figure III-1. Since the streamlines are symmetric about the z-axis, only those on the positive r side are shown. We note that the streamlines approach both axes as we move in an increasing direction of z and r. The streamlines may be considered to be those of a fluid flowing in the negative z - direction against a disk of radius r located at z = 0. Our main concern here will be to calculate the flow field variables along these streamlines and determine the combined effects of the streamline curvature k, magnetic field strength B, thermal radiation and Joule heating.

We begin by computing the streamline curvature variation along the streamlines by using Eq. (8.2.5). The result is shown in Figure III-2, as a function of distance z from the surface of the disk. From the figure we note that the streamline curvature for all the streamlines approaches zero very rapidly with distance from the wall. As we approach the wall along some streamline the curvature increases rapidly and then drops to zero again near the wall. It is also noted that the curvature of the streamlines increases more rapidly for those that are closest to the z - axis, so that the streamline ψ_1 has the largest curvature increase.

9.2. Pressure Distribution Along and Normal to Streamlines

In this section we evaluate the effect of the streamline curvature k, the magnetic field B, and the thermal radiation pressure p_R , on the fluid pressure gradient normal to the streamlines and on the pressure variation along the streamlines.

By solving Eq.(3.2.5) for the normal pressure gradient we get,

$$\frac{dp}{dn} = -\rho V^2 k - \frac{dp_m}{dn} - \frac{dp_R}{dn} , \qquad (9.2.1)$$

where k is geven by Eq. (8.2.5), p_R by Eq. (5.2.3), and

$$\frac{dp_m}{dn} = \frac{dp_m}{dr}\left(\frac{dr}{dn}\right) = -\frac{2(B_1 r_1^2)^2}{\mu(\Psi/bz)^{5/2}(1 + 4z^3(\Psi/b)^{-2})^{\frac{1}{2}}}, \qquad (9.2.2)$$

where we have used Eqs. (6.1.12), (6.1.16).

The normal pressure gradient given by (9.2.1) evaluated along streamlines is shown in Figure III-3 as a function of distance z from the wall. From the analysis the following was found:

1. The effect of the radiation pressure gradient is negligible.
2. The effect of the magnetic pressure gradient is small.
3. The effect of the streamline curvature is largest in the region of higher fluid velocity.

The fluid pressure distribution along the streamlines was obtained by evaluating Eq. (6.1.19) and the result is shown in Figure III - 4 as a function of distance from the wall. We briefly note the following results:

1. The effect of the radiation pressure p_R is negligible.
2. The effect of the magnetic field B is considerable as indicated for the pressure distribution along the streamline .
3. The pressure increases and decreases with the curvatures of the streamlines.

9.3. Temperature, Electron Density and Electric Conductivity Distribution

In this section we consider the effect of thermal radiation and magnetic field on the temperature, electron density, and electric conductivity distribution along the streamlines. The temperature distribution is obtained by using Eq. (6.2.8), and the electron density and electric conductivity distribution is computed by using Eqs. (5.3.7) and (5.3.16) respectively.

The results for various values of the magnetic field strength B are shown in Figures III -5 to III - 7 for the streamline ψ_1 as a function of distance from the wall z . We note from Figure III - 5 that the temperature decreases continuously due to radiation cooling for the case of zero magnetic field B. The effect of an increase in magnetic field strength B is to increase the temperature for a short distance after which it drops more rapidly due to radiation cooling. Thus, we see that the two phenomena create opposing effects.

By using the Planck mean absorption coefficient K_p as a parameter we see from Figure III - 5 that a greater decrease in temperature occures for small increases in values of K_p. The same remarks apply to the electron density and electric conductivity distribution shown in Figures III - 6 and III - 7 respectively.

In order to determine the effect of different streamlines on the flow variables distribution, two solutions are shown for streamlines ψ_1 and ψ_2 in Figures III - 8 to III - 10. We note that the magnetic field effect is not nearly as great for streamline ψ_2 as it is for ψ_1. This behavior may be explained through the fact that the magnetic field B was taken to be inversely proportional to r^2 as given by Eq. (6.1.16), so that the magnetic field decreases as we move away from the z - axis.

Finally, it may be noted that the temperature, electron density, and electric conductivity variation along the streamlines is considerable and therefore, is an indication of the importance of considering thermal radiation and magnetic field effects in high temperature MHD flow field calculations.

9.4. Velocity Distribution

The velocity distribution along the streamlines is obtained by solving Eqs. (6.1.13), (6.1.12). The result is shown in Figure III - 11 as a function of distance z from the wall. We note the following results;

1. The velocity decreases almost linearly with z for all streamlines at a distance greater than about 20 cm.
2. The velocity at each point along the streamlines increases as the distance of the streamlines from the z - axis increases.

3. The minimum velocity along each streamline occurs very closely at the point of maximum curvature of the streamline as may be seen by inspection of Figure III -2 , and III - 11.

It was assumed that at z = 40 cm the magnitude of the velocity V for all streamlines was the same.

10. VISCOUS, COMPRESSIBLE FLOW RESULTS

In section 9 we investigated the variation of the flow field parameters along the streamlines under the simplifying assumption of constant density flow; i.e. $\rho =$ constant. The principal purpose of the present section is to determine the effect of variable density on the flow field parameters distribution along the streamlines including viscous, heat conduction, radiation and magnetic field effects.

10.1. Temperature, Pressure, and Density Distribution

The temperature T, pressure p, and density ρ, variation along the streamlines is obtained by simultaneous solution of Eqs. (6.1.19), (7.6.23), and (7.6.32) respectively. The velocity distribution contained in these equations is taken as a first approximation as that given by Eq. (6.1.13) for the incompressible case. We also require an expression for $\bar{\mu}\phi$, τ_s/ds, and Q_s, which we evaluate as follows:

The viscous stress tensor components given by Eq. (7.1.2) in cylindrical coordinates for the axially symmetric case are,

$$\tau_{rr} = \bar{\mu}(2\frac{\partial v}{\partial r} - 2/3(\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial w}{\partial z})$$

$$\tau_{zz} = \bar{\mu}(2\frac{\partial w}{\partial z} - 2/3(\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial w}{\partial z})$$

$$\tau_{rz} = \tau_{zr} = \bar{\mu}(\frac{\partial w}{\partial r} + \frac{\partial v}{\partial z}) .$$

By using Eq.(6.1.2) the stress components become,

$$\tau_{rr} = \bar{\mu}(2b - 2/3(b + b - 2b)) = 2\bar{\mu}b,$$

$$\tau_{zz} = \bar{\mu}(-4b - 2/3(b + b - 2b)) = -4\bar{\mu}b \qquad (10.1.1)$$

$$\tau_{rz} = \tau_{zr} = 0.$$

By using Eqs.(10.1.1) in (7.6.17) we get

$$\tau_s = \left(\frac{2\bar{\mu}b}{r}\right)dr, \qquad \text{or} \qquad \frac{\tau_s}{ds} = \left(\frac{2\bar{\mu}b}{r}\right)\left(\frac{dr}{ds}\right). \qquad (10.1.2)$$

By using Eqs.(8.2.2), (8.2.3), and (8.2.6) in (10.1.2) we get

$$\frac{\tau_s}{ds} = 2\bar{\mu}b\left(\frac{\gamma/b}{z} + 4z^2\right)^{-\frac{1}{2}} . \qquad (10.1.3)$$

By using (10.1.1) in (7.6.6) we get,

$$\bar{\mu}\phi = \tau_{rr}\left(\frac{\partial v}{\partial r}\right) + \tau_{zz}\left(\frac{\partial w}{\partial z}\right) = (2\bar{\mu}b)b + (4\bar{\mu}b)(2b) = 10\bar{\mu}b^2. \qquad (10.1.4)$$

From Eq.(7.1.5) we have

$$Q_s = K_t\frac{d^2T}{ds^2} - 4K_p\delta_B T^4. \qquad (10.1.5)$$

The viscosity $\bar{\mu}$ and thermal conductivity K_t are given by, (53)

$$\bar{\mu} = 1.462(10^{-6})T^{3/2}(1 + \frac{112}{T})^{-1} \quad \frac{kg}{m\text{-}sec},$$

$$K_t = 1.994(10^{-3})T^{3/2}(1 + \frac{112}{T})^{-1} \quad \frac{n\text{-}m}{m\text{-}sec\,^{o}K}. \qquad (10.1.6)$$

The solution to the system of equations for ρ, p, and T was obtained by using a modified Runge-Kutta step-by-step integration process. The modification to the standard fourth-order process was made to allow integration in the decreasing direction of z. The modified process was tested by comparing the solutions obtained with the standard and modified process when applied to a differential equation. The same solution was obtained with both processes.

The well known standard fourth-order Runge-Kutta integration process is as follows;

$$y_{n+1} = y_n + 1/6(b_1 + 2b_2 + 2b_3 + b_4),$$

$$b_1 = hf(z_n, y_n),$$

$$b_2 = hf(z_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}b_1),$$

$$b_3 = hf(z_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}b_2),$$

$$b_4 = hf(z_n + h,\ y_n + b_3), \qquad (10.1.7)$$

where h is the step-size and $\frac{dy}{dz} = f(z,y)$.

To allow integration in the decreasing direction of the independent variable z the standard process given in (10.1.7) was modified to the following form;

$$y_{n+1} = y_n - 1/6(b_1 + 2b_2 + 2b_3 + b_4),$$

$$b_1 = hf(z_n, y_n),$$

$$b_2 = hf(z_n - \tfrac{1}{2}h,\ y_n - \tfrac{1}{2}b_1),$$

$$b_3 = hf(z_n - \tfrac{1}{2}h,\ y_n - \tfrac{1}{2}b_2),$$

$$b_4 = hf(z_n - h,\ y_n - b_3), \qquad (10.1.8)$$

where h is the step-size and $\frac{dy}{dz} = f(z,y)$ as before.

The numerical computations were performed on an IBM 1620 digital computer at the General Motors Institute computing laboratory with the following initial values;

$$V_1 = 2000 \quad m/sec,$$

$$z_1 = 40 \quad cm,$$

$$T_1 = 20{,}000 \quad {}^oK,$$

$$\rho_1 = .04429 \quad kg/m^3,$$

$$p_1 = \rho_1 RT_1 = 2.542246(10^5) \quad n/m^2,$$

$$R = 287 \quad n\text{-}m/kg\text{-}{}^oK, \qquad (10.1.9)$$

$$\gamma = 1.4$$

$$B_1 = 0,\ .2,\ \text{Or}\ .4 \quad webers/m^2.$$

The step-size used in the computations was $h = .01$.

Inasmuch as each step in the calculations is the same, we will briefly describe and show the results for a single step away from the initial conditions along streamline ψ_2.

We note from Eq. (10.1.8) that each step in the solution for the dependent variable y (y = T in our case) requires four evaluations of the differential equation to be integrated. In our case the temperature distribution is given by Eq. (7.6.32) where the right hand side is evaluated by using the initial conditions given in (10.1.9) together with Eqs. (10.1.3) to (10.1.6) and the electrical conductivity is evaluated from Eq. (5.3.20). Thus, the temperature T may now be calculated one step away from the initial conditions. i.e., at z = .39 meters. By using the just calculated temperature we may obtain the pressure at this point by evaluating Eq. (6.1.19) and the density is now obtained from the equation of state of the gas $\rho = p/RT$. The process may now be repeated for the next step.

The results for the first step are shown below:

z = .39000000E+00   T = .19864611E+05   p = .25422460E+06

ρ = .45366864E-01 n_e = .74155286E+18 δ = .12683507E+05

The density distribution $\rho(z,\psi)$ is shown in Fig.(III-12). We note the following results:

1. The density increases and then decreases as we approach the wall for all streamlines ψ, shown.

2. The increase in density is less for streamlines located farther away from the z - axis.

The pressure distribution p (z, γ) is shown in Fig.(III-13) and indicates the following results:

1. The pressure increases and then decreases as we approach the wall just as in the incompressible case.
2. A comparison of the compressible, Fig.(III-13), and the incompressible pressure distribution, Fig. (III-4), shows that the results are qualitatively the same, but differ considerably on a quantitative bases.
3. The compressible pressures are higher than those given by the incompressible flow model for all streamlines shown.

A direct comparison of the compressible and incompressible temperature distribution $T(z, \gamma)$ for two neighboring streamlines is shown in Fig. (III-14) from which we note the following results:

1. The temperature distribution is qualitatively the same for the compressible and incompressible flow model.
2. From a quantitative point of view the compressible flow model yields a higher temperature at all points along the streamlines.
3. The effect of radiation cooling is to decrease the temperature considerably.

10.2. Electron Density and Electric Conductivity Distribution

Having obtained the compressible pressure and temperature distribution it is now possible to calculate the electron number density and the electric conductivity distribution by using Eqs.(5.3.7) and (5.3.16) respectively.

A direct comparison of the results obtained from the compressible and incompressible flow models is shown in Figs.(III-15) and (III-16).

From the two figures we note the following results:

1. Both the electron density $n_e(z,\psi)$, and electric conductivity $\sigma(z,\psi)$ distribution show a qualitative similarity to the incompressible case.
2. From a quantitative point of view we note higher values for both n_e and σ in the compressible case.
3. The effect of radiation cooling is to decrease both the electron number density as well as the electric conductivity.

10.3. Summery of Results and Conclusions

The primary objective of this investigation was to determine the combined effects of thermal radiation and magnetic fields on the flow variables distribution near a stagnation point of a blunt vehicle moving through a gas at hypersonic velocity. Inasmuch as a streamline approach was chosen to carry out this investigation it seemed appropriate to consider the general three dimensional dynamic and kinematic relations connecting the flow variables with the geometric parameters of the streamlines as a secondary objective.

The general relations for the tangent, principal normal, and binormal vectors and the curvature and torsion of the streamlines were derived in terms of the flow field variables for both the inviscid and viscous radiation magneto-hydrodynamic case in Parts I and II respectively. We also found that for the inviscid case the total pressure P_t, remains constant in the binormal direction of the streamlines, but not in the viscous case.

From the numerical results obtained in Part III, which are plotted in Figures III-1 to III-16, we find that the physical phenomena of thermal radiation, magnetic field, and compressibility have a considerable effect on the flow field variables, whereas the viscosity and heat conductivity effects were found to be very small in the case under consideration here.

A typical example of the calculations made in this investigation is shown in the Appendix, page 98, together with the Fortran program for the IBM 1620 computer.

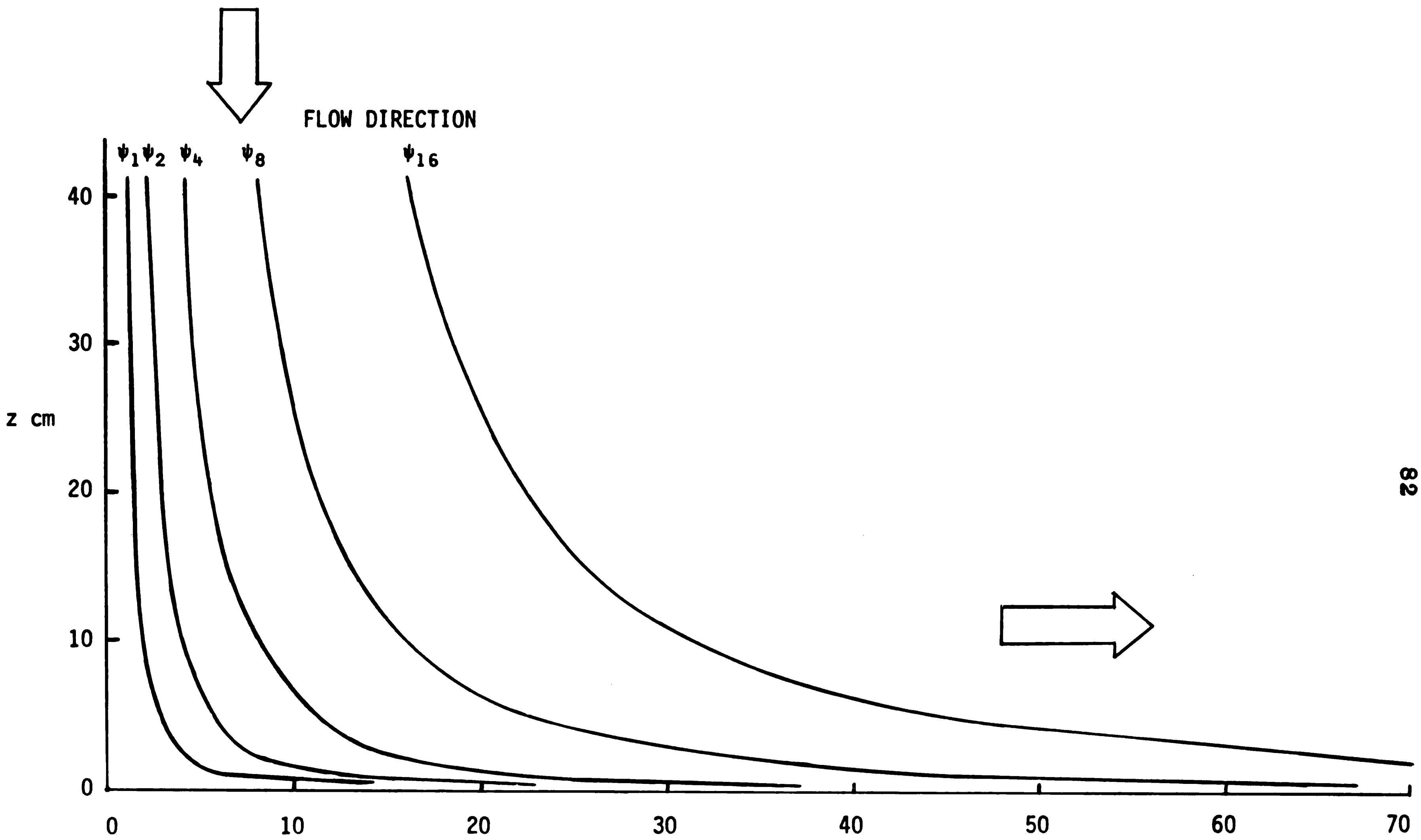

Figure III-1. Streamline Pattern for Flow Against a Disk

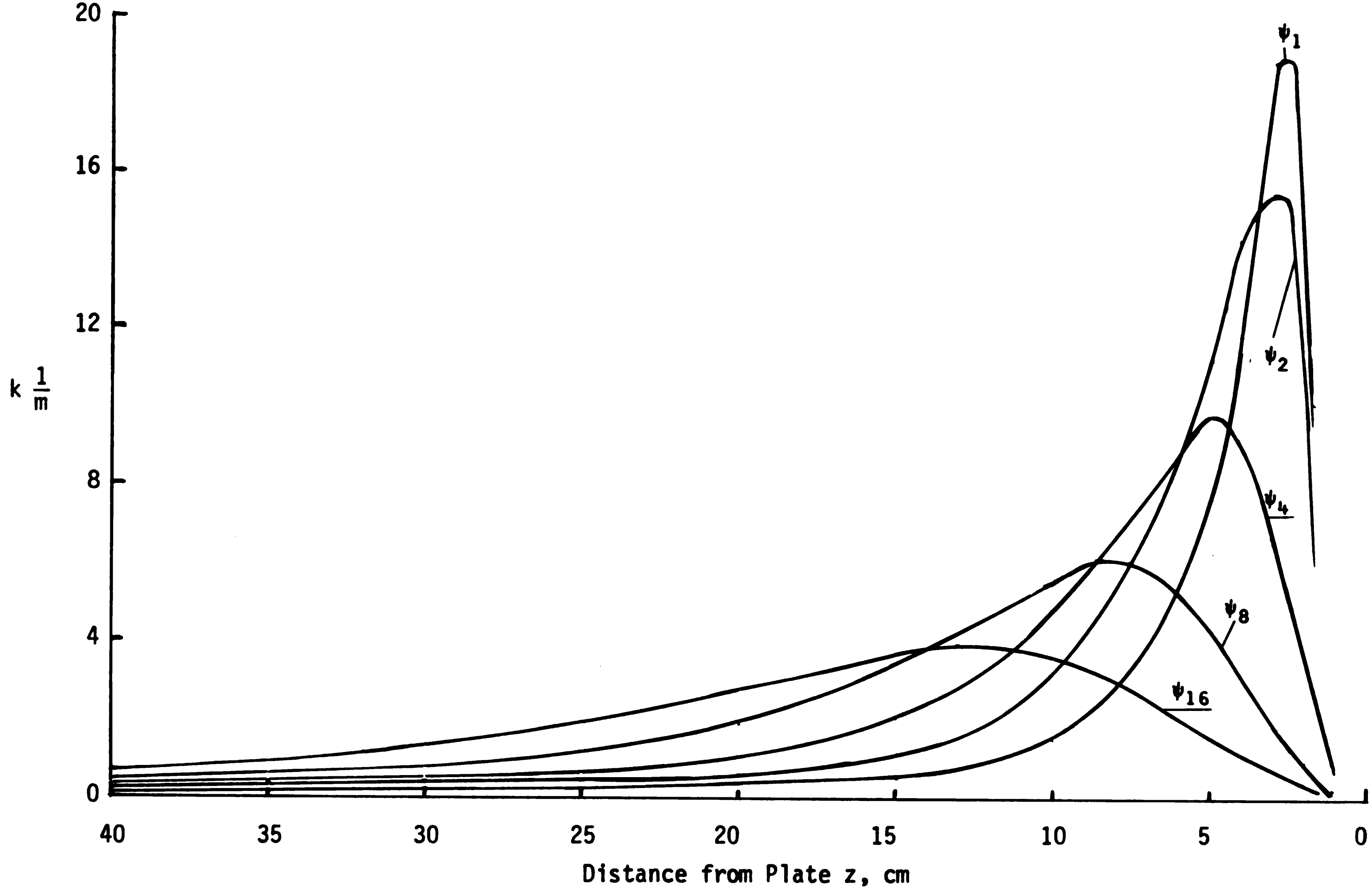

Figure III-2. Streamline Curvature k along Streamlines ψ

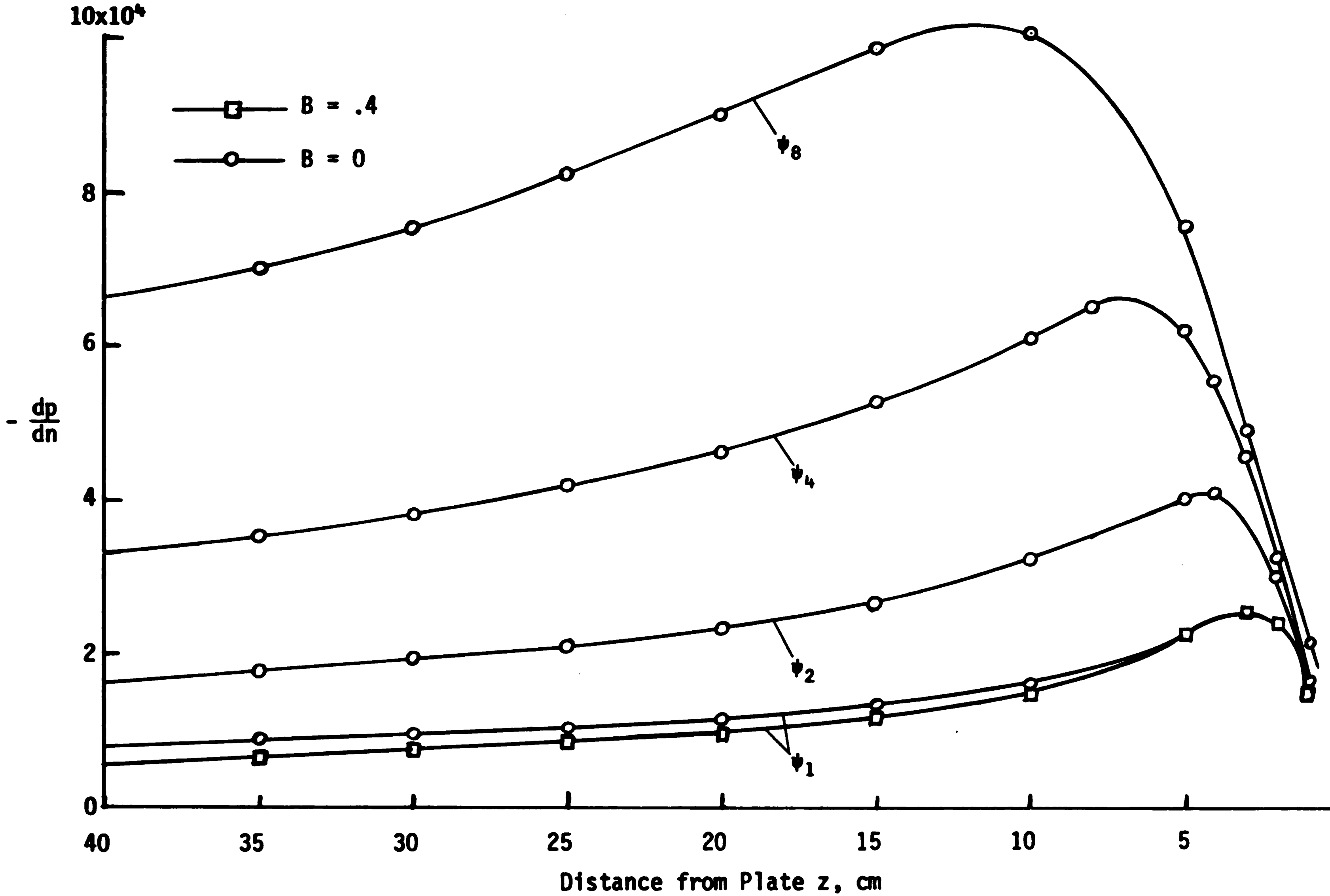

Figure III-3. Pressure Gradiant Variation Normal to Streamlines ψ

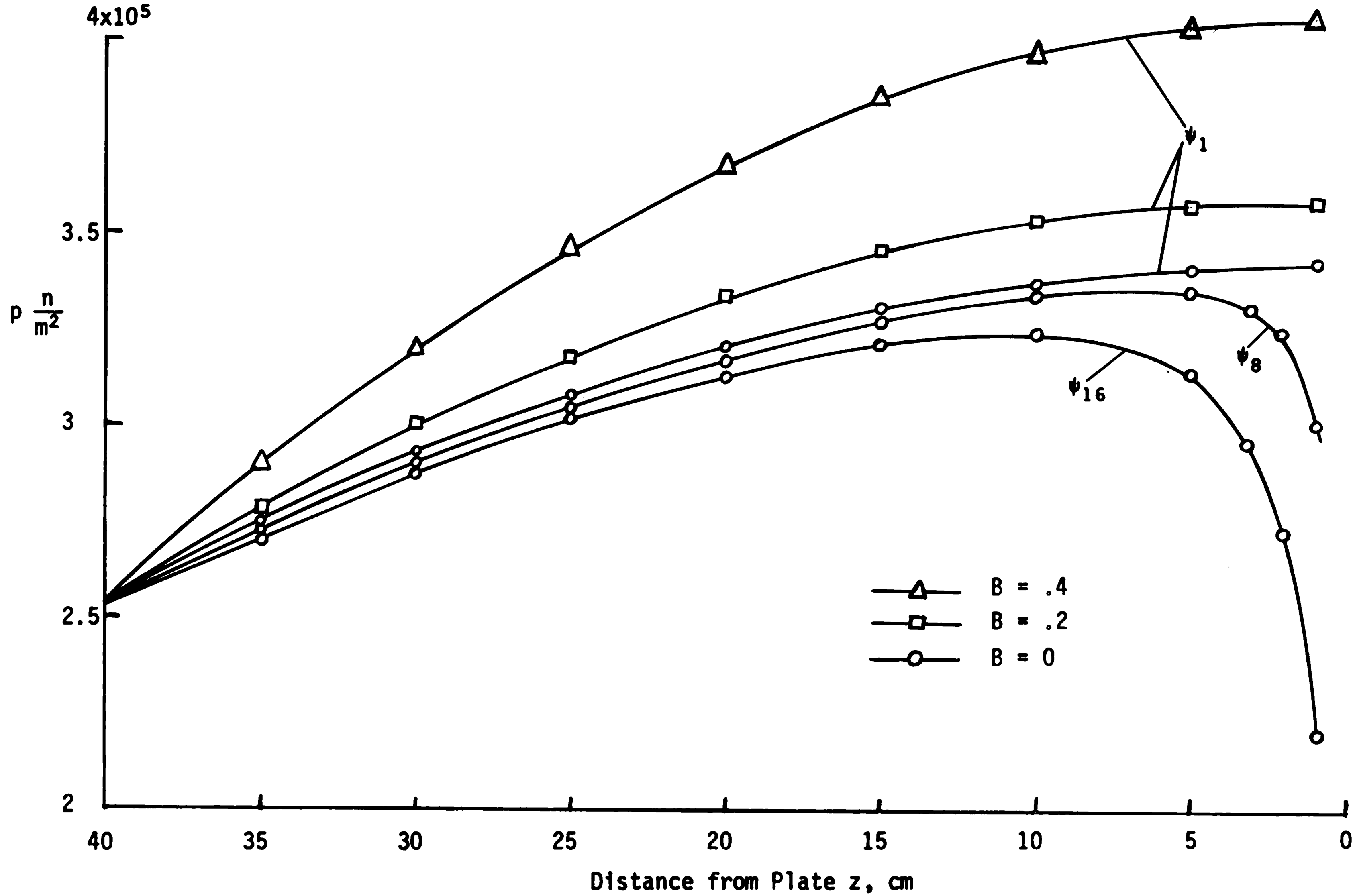

Figure III-4. Effect of Magnetic Field on Pressure Distribution along Streamlines. (Incompressible)

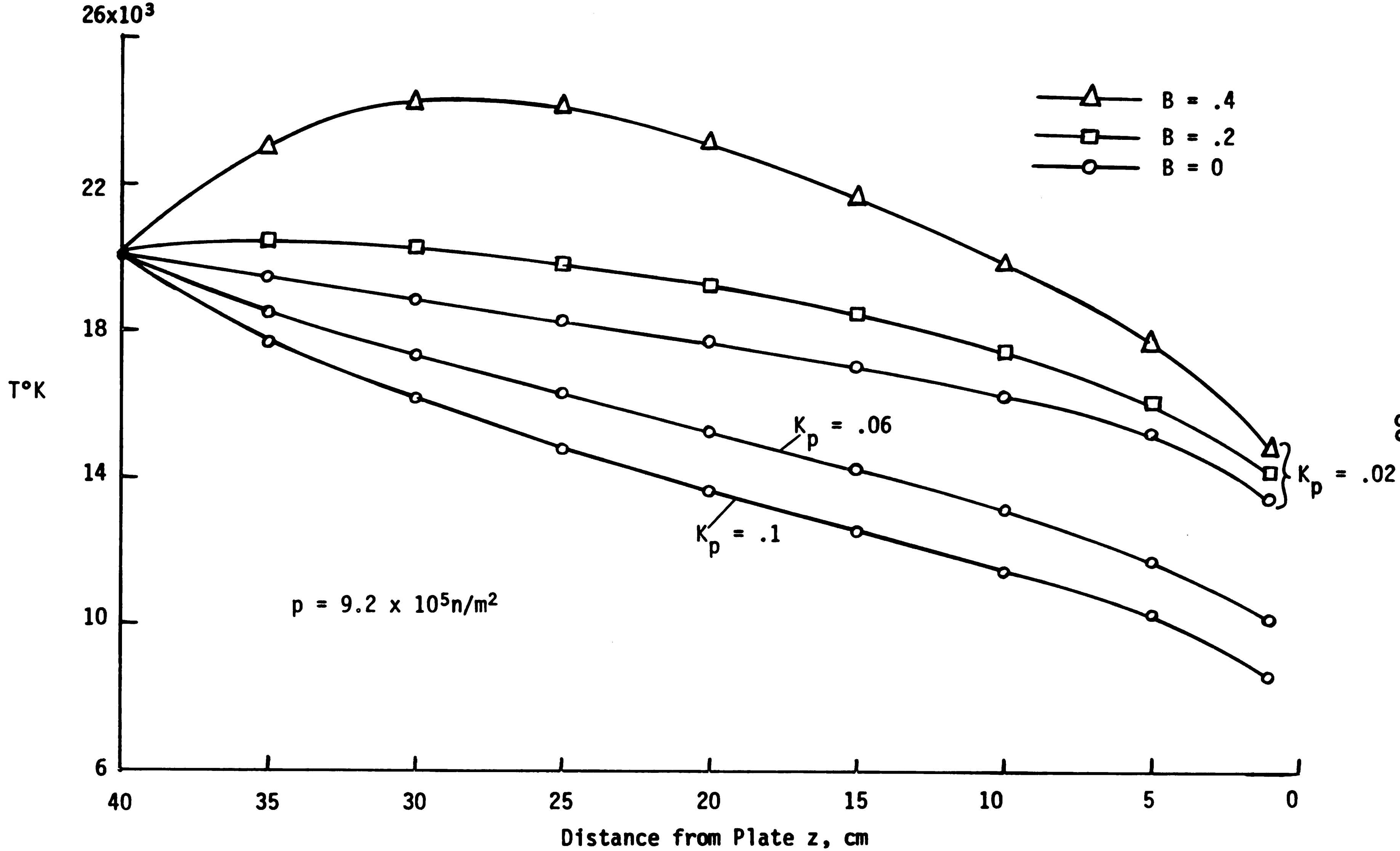

Figure III-5. Temperature Distribution along Streamline ψ_1. (Incompressible)

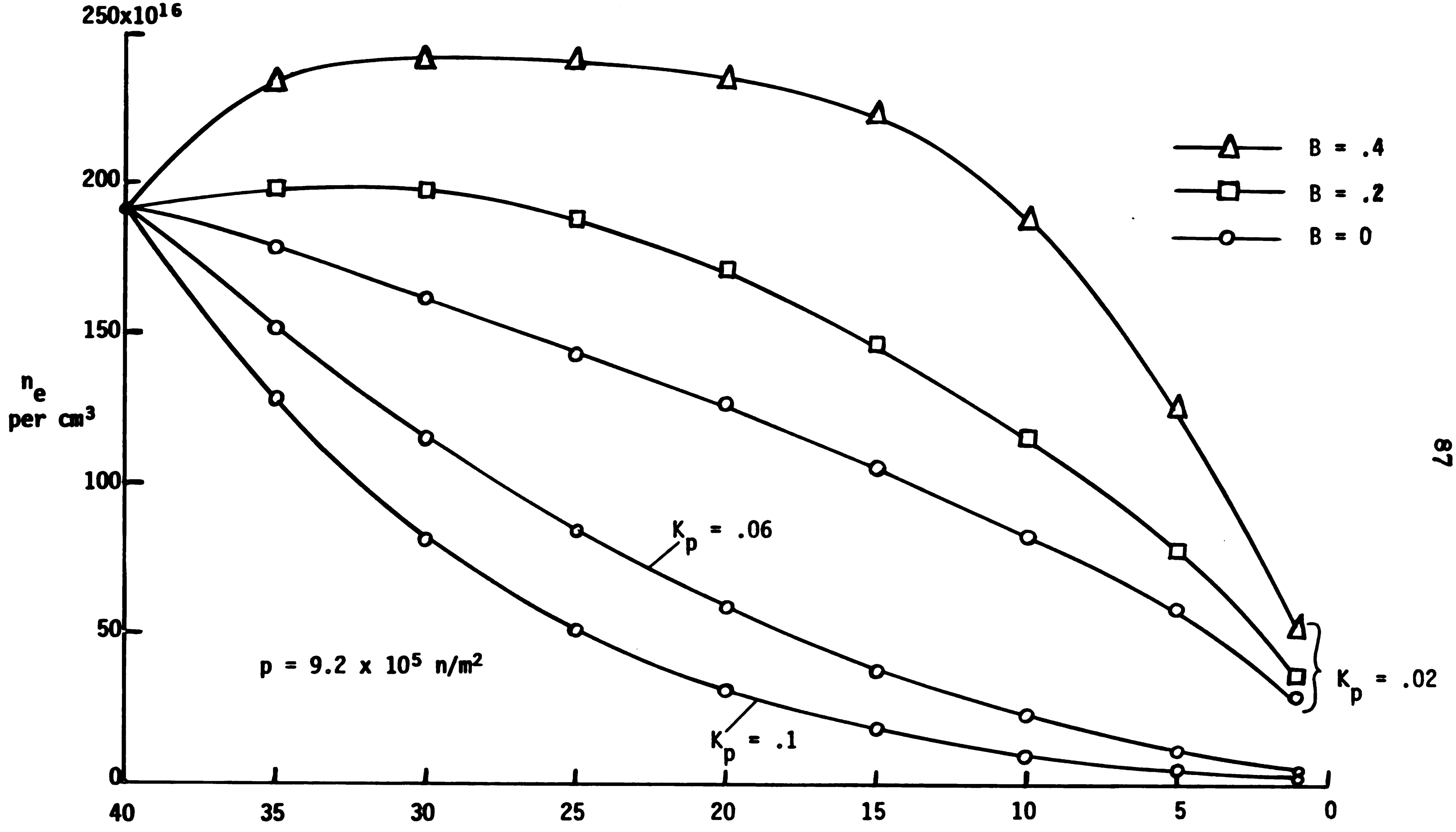

Figure III-6. Electron Density Distribution Along Streamline ψ_1. (Incompressible)

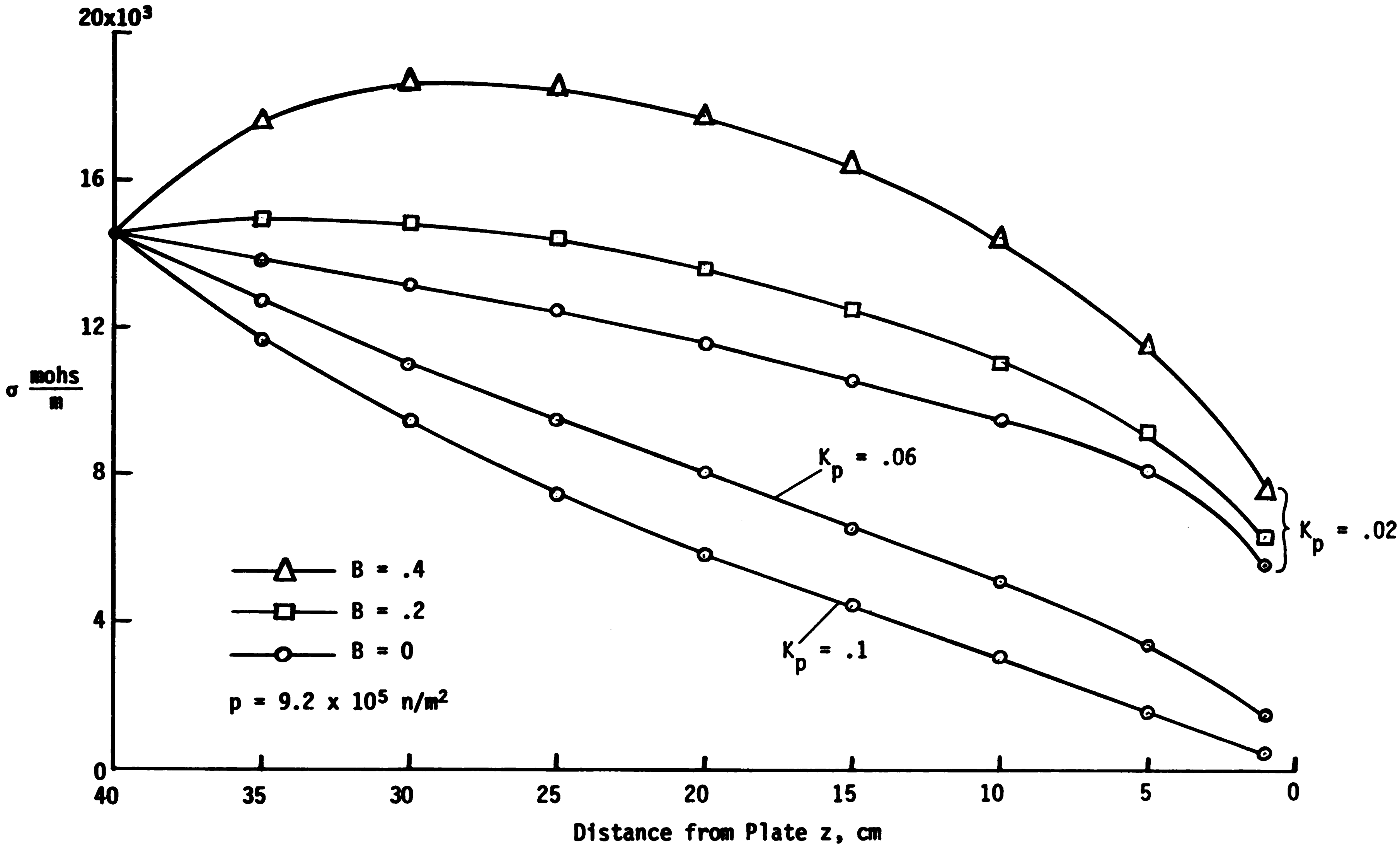

Figure III-7. Electric Conductivity Distribution along Streamline ψ_1. (Incompressible)

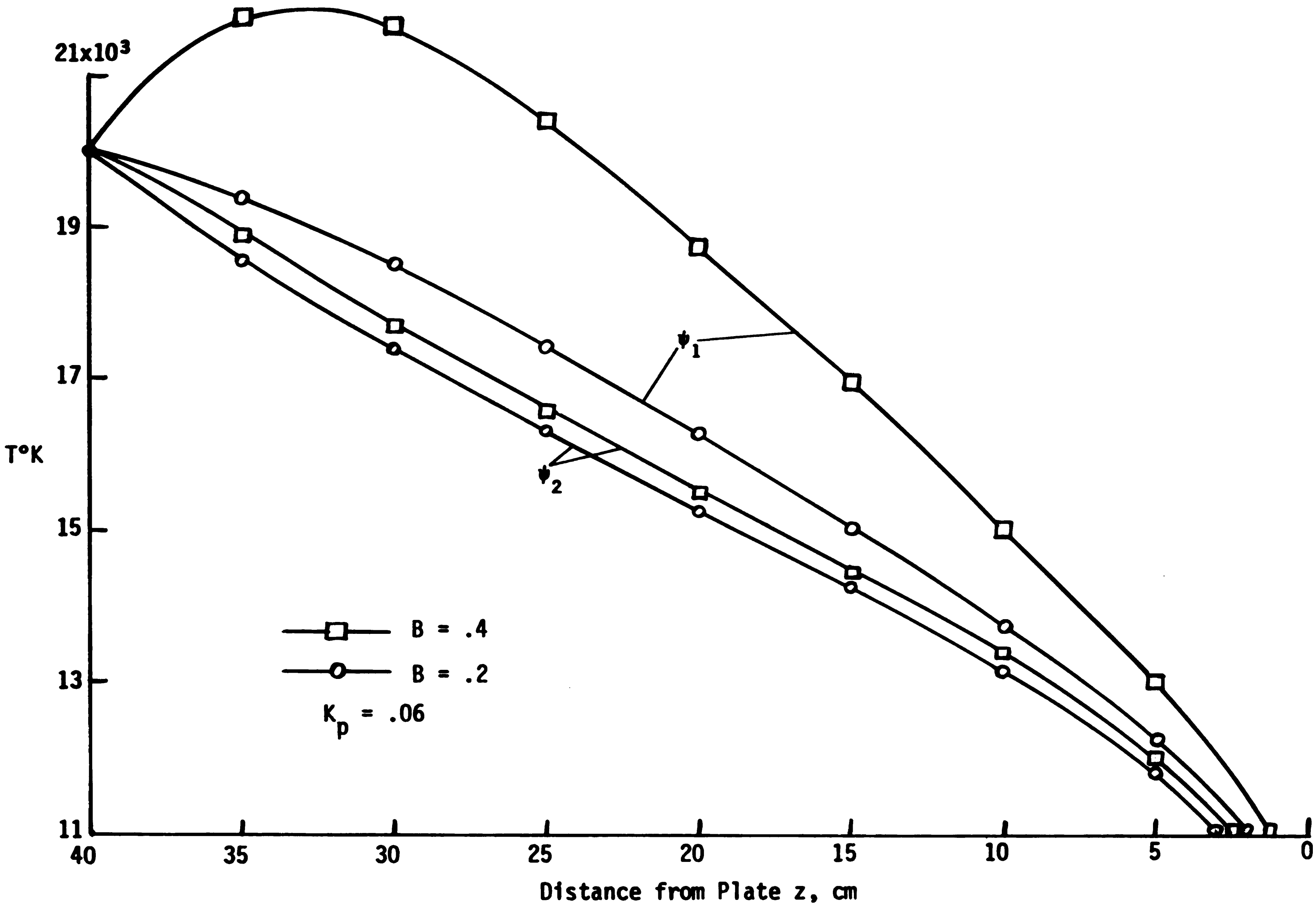

Figure III-8. Comparison of Temperature Distribution along ψ_1 and ψ_2. (Incompressible)

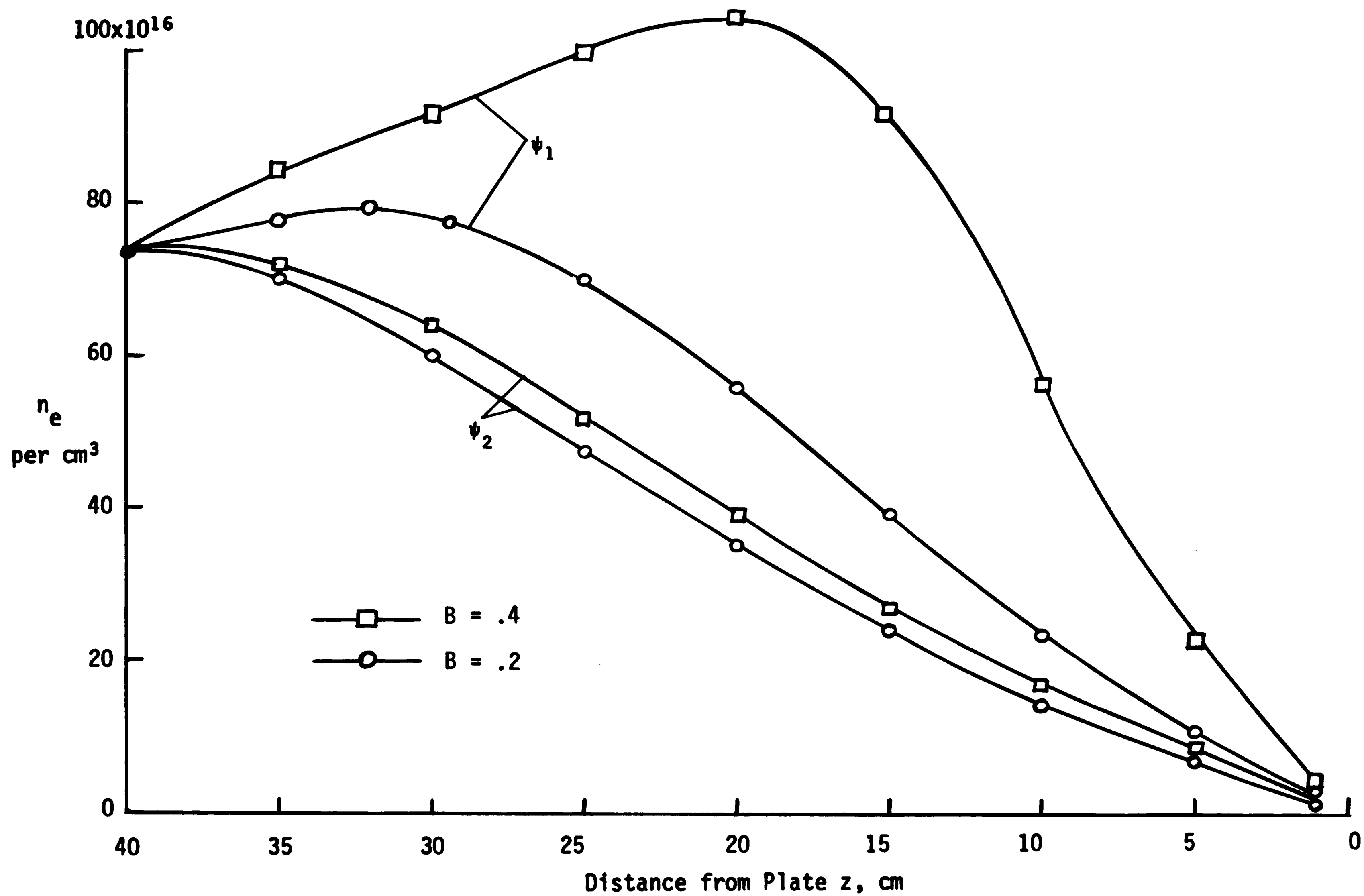

Figure III-9. Comparison of Electron Density Distribution along ψ_1 and ψ_2. (Incompressible)

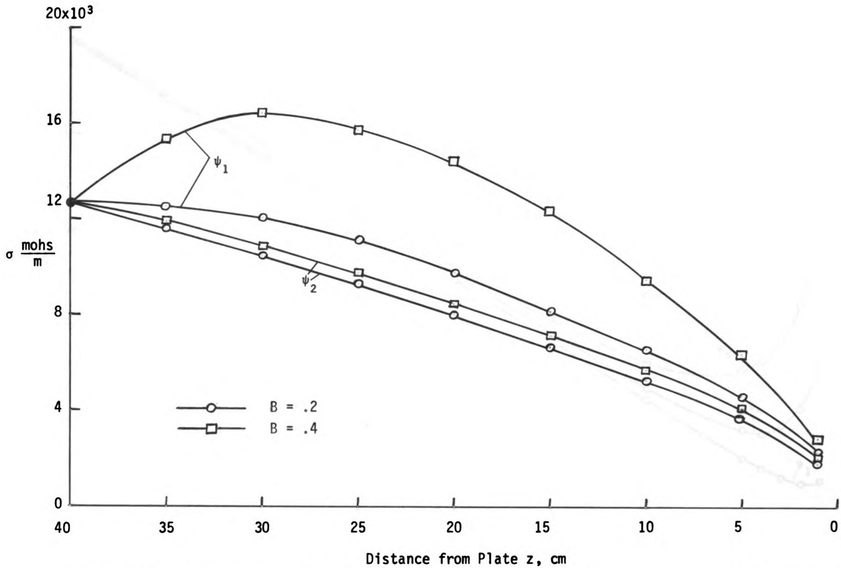

Figure III-10. Comparison of Electric Conductivity Distribution along ψ_1 and ψ_2. (Incompressible)

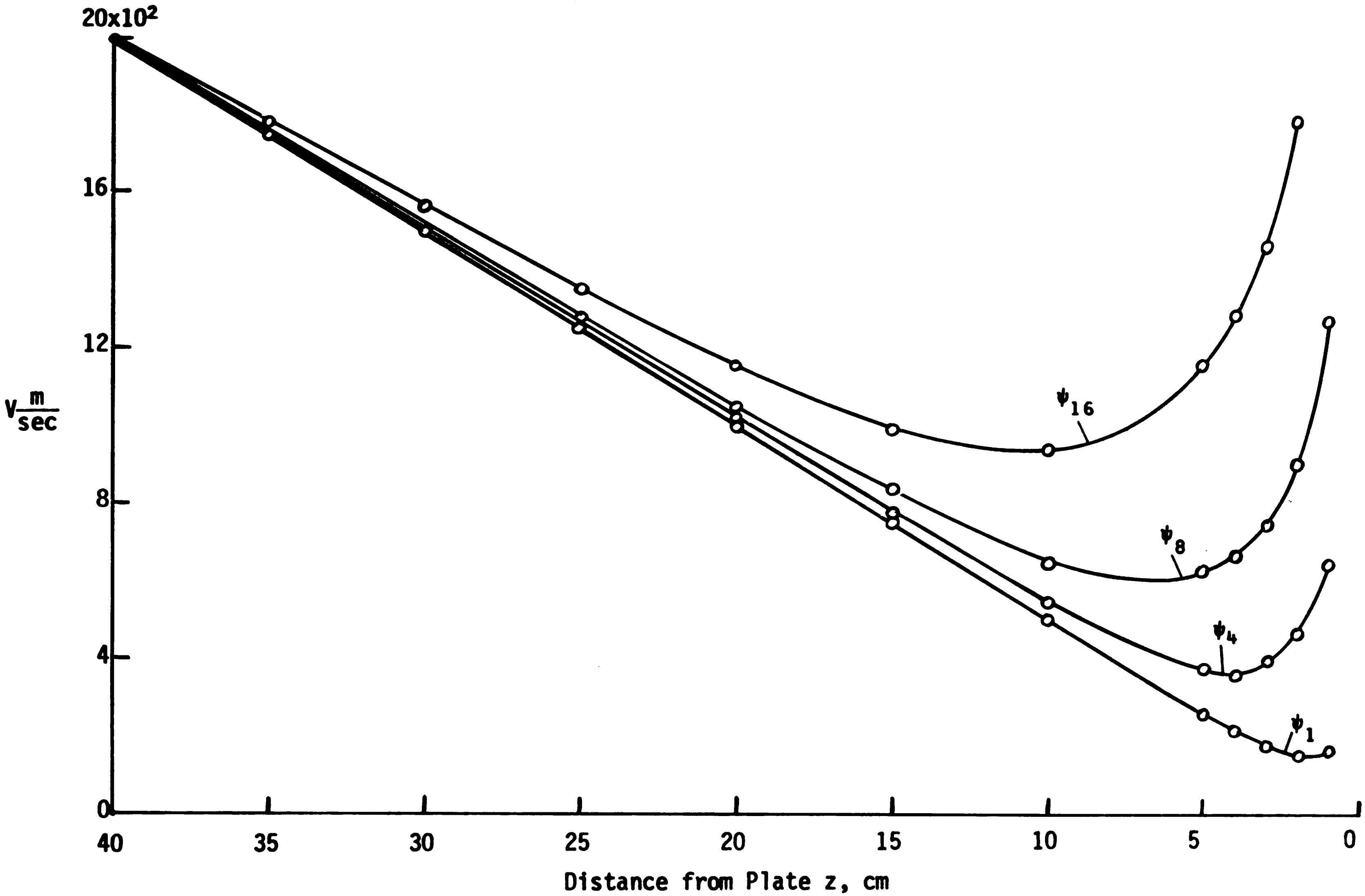

Figure III-11. Velocity Distribution along Streamlines ψ

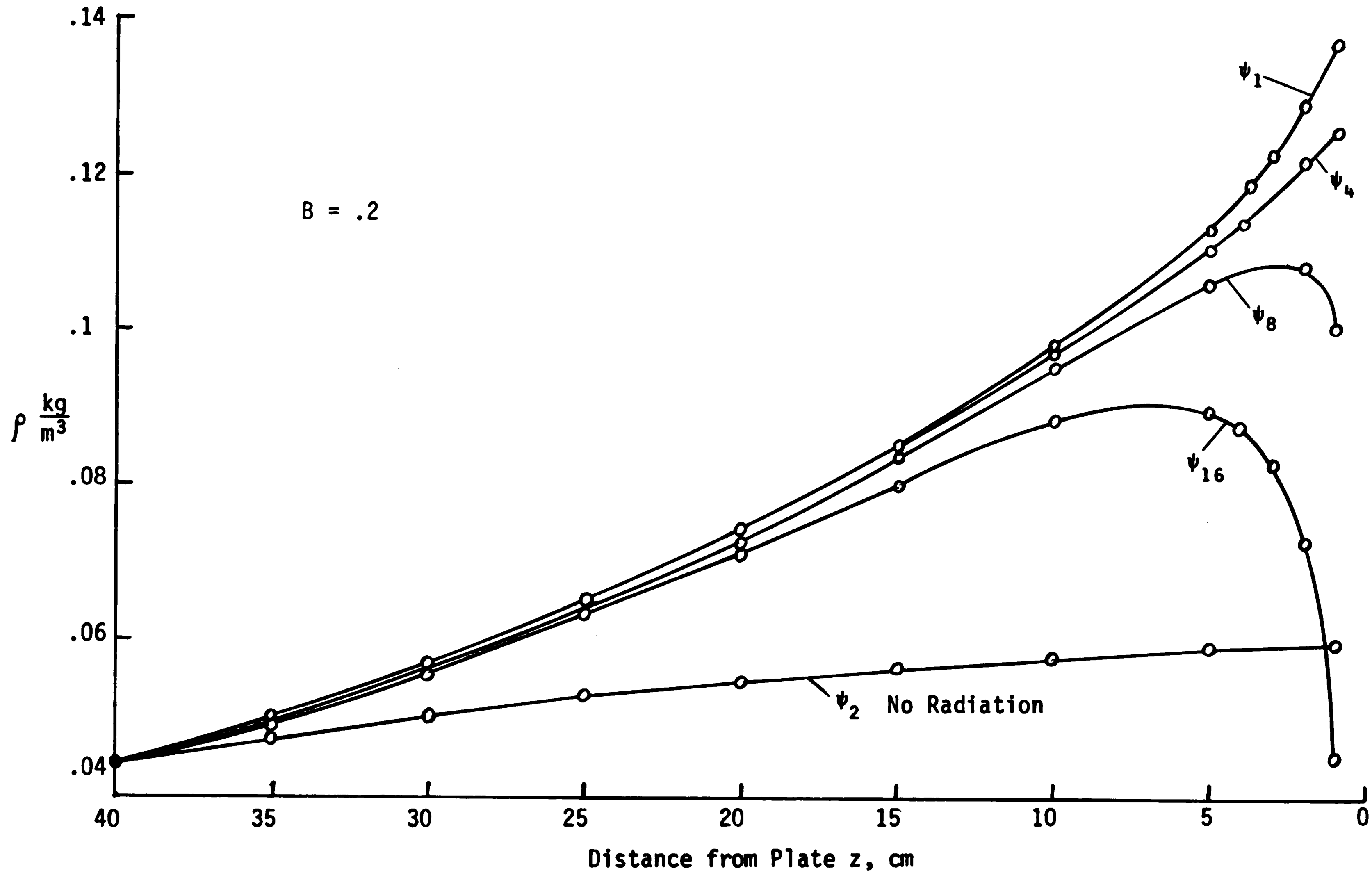

Figure III-12. Density Distribution along Streamlines ψ

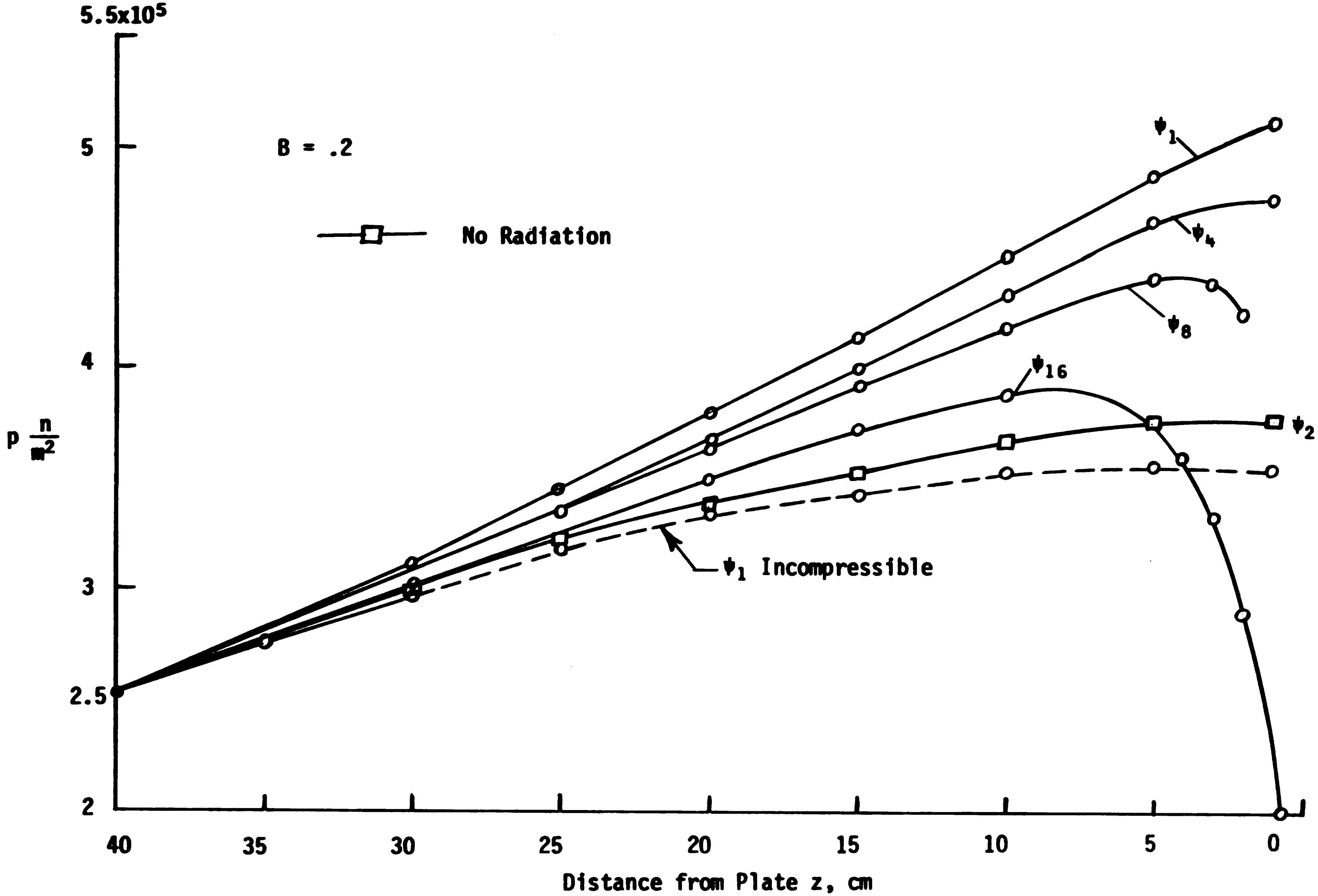

Figure III-13. Pressure Distribution along Streamlines ψ. (Compressible)

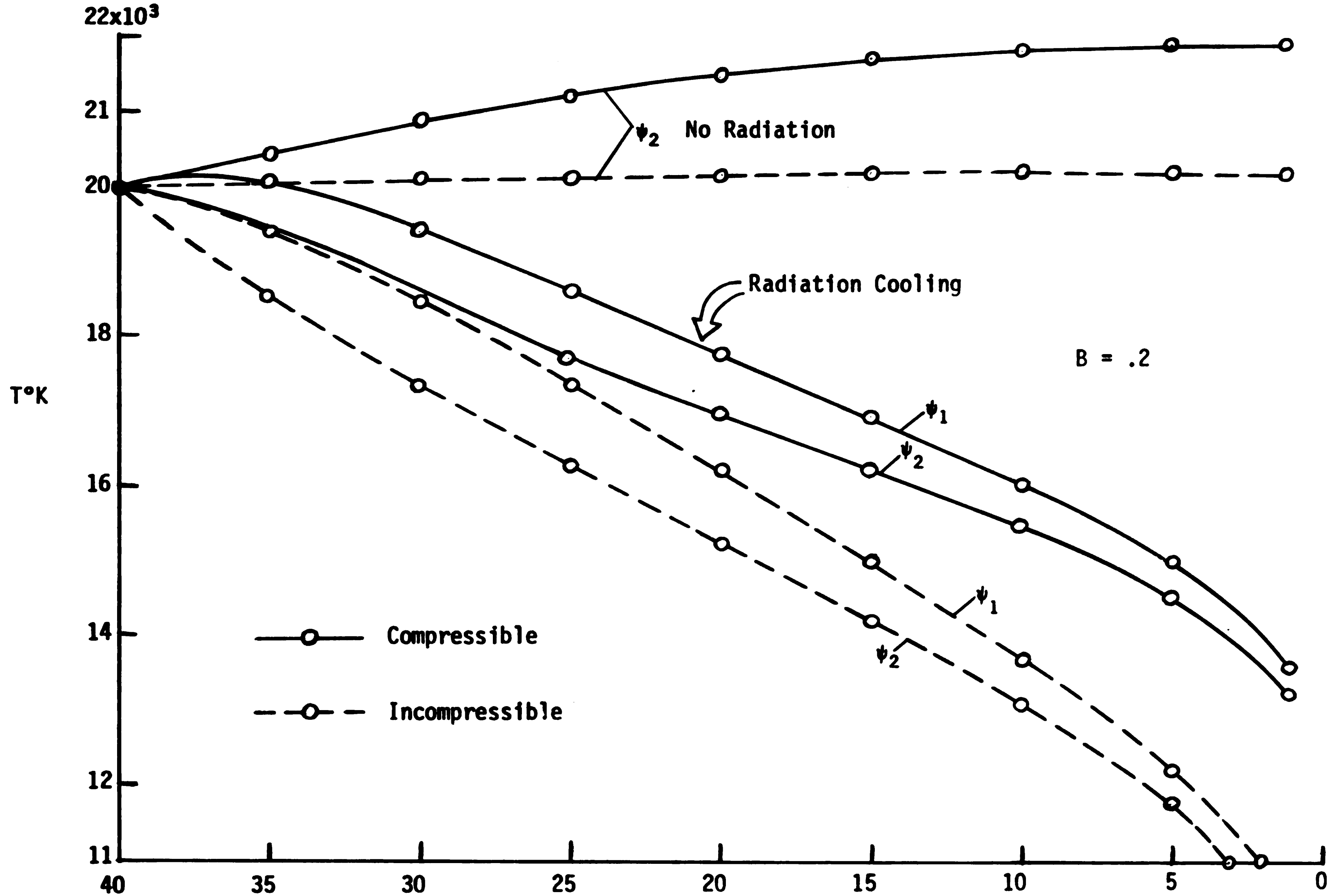

Figure III-14. Effect of Thermal Radiation Cooling on Temperature Distribution

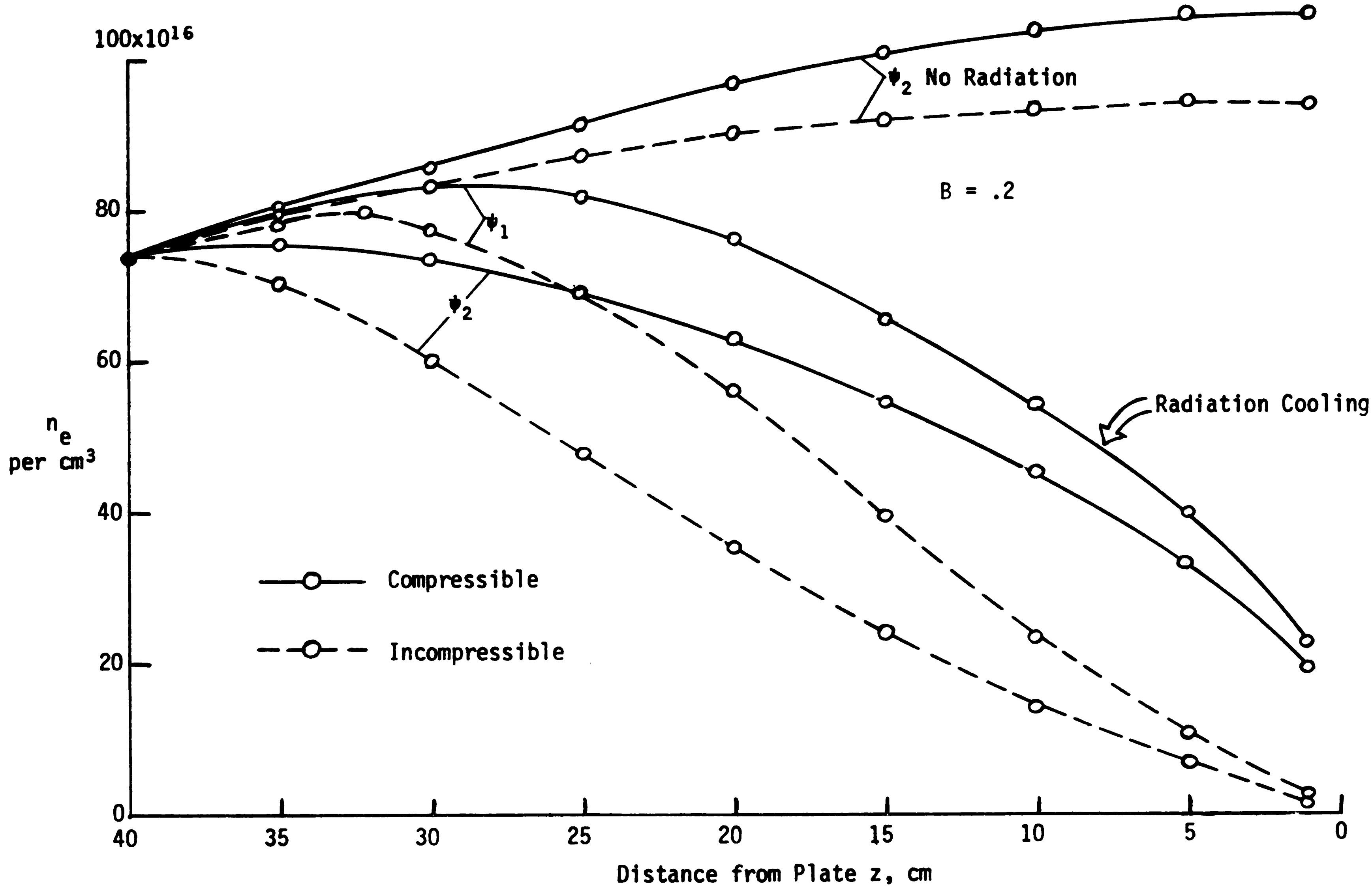

Figure III-15. Effect of Thermal Radiation Cooling on Electron Density Distribution

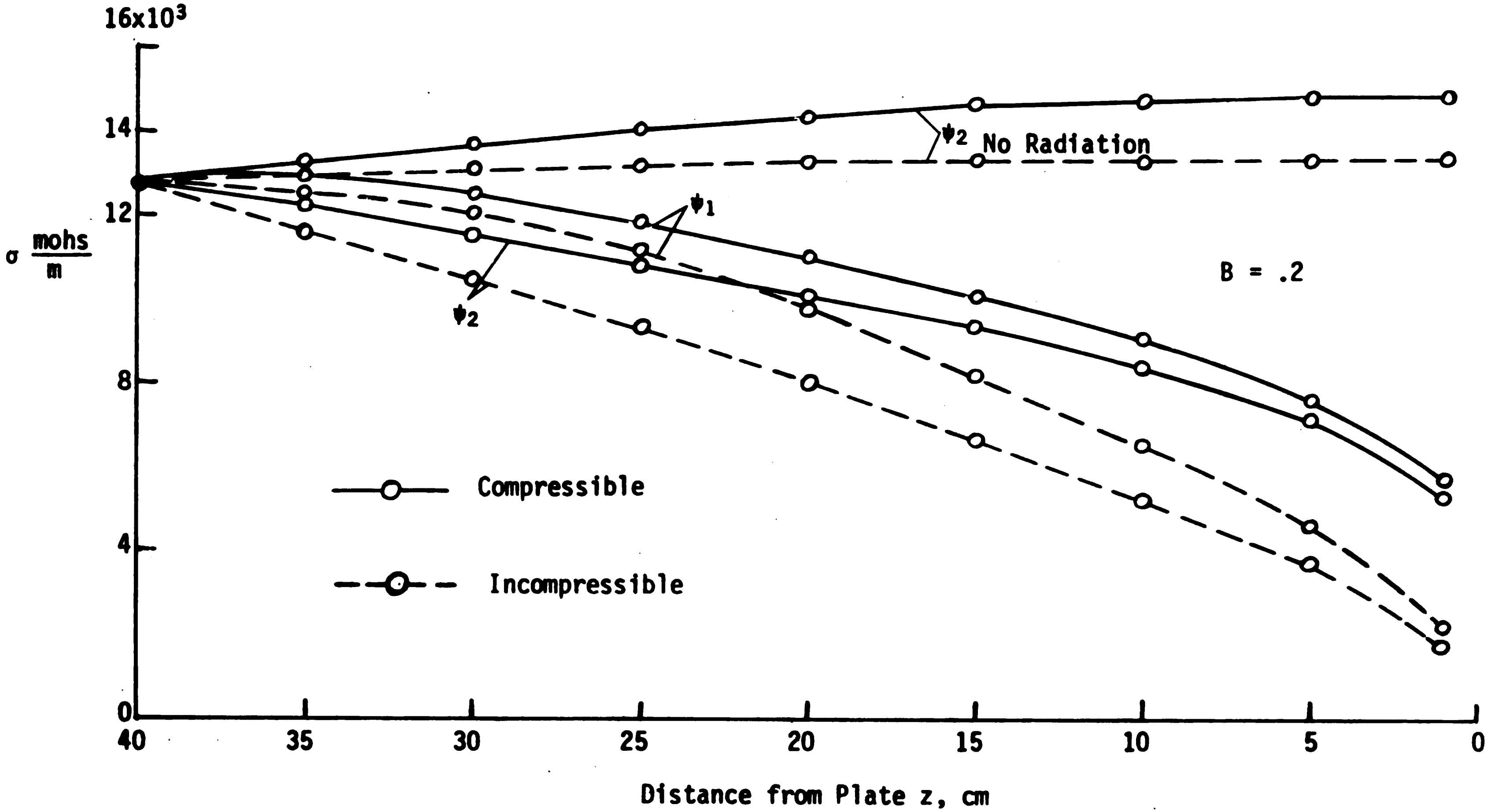

Figure III-16. Effect of Thermal Radiation Cooling on Electric Conductivity Distribution

APPENDIX

Typical Fortran Program and Results

This program integrates the differential equation for the temperature distribution by the modified Runge-Kutta method taking account of variable viscosity, heat conductivity, electron-ion collision cross-sections, and electric conductivity. At the same time the program calculates the pressure, density, velocity, electron density, and electric conductivity distribution along the streamlines. Inasmuch as all flow variables are calculated as a function of z and ψ along the streamlines we also obtain the coordinates r as a function of z and ψ so that the final results may be interpreted as having the flow variables distribution given as a function of the two coordinates r and z. Thus, by choosing as many streamlines as desired it is possible to obtain the flow variables distribution throughout the flow field under consideration as a function of the field coordinates, r and z.

FORTRAN PROGRAM FOR IBM 1620

```
C       VISCOUS COMPRESSIBLE FLOW PROGRAM FOR THE
C       CALCULATION OF THE FLOW VARIABLES DISTRIBUTION
C       NEAR A STAGNATION POINT
C       RS=STREAMLINE ENTRANCE RADIUS AT Z =40 CM
C       Z=COORDINATE NORMAL TO DISK
C       RZ=COORDINATE PARALLEL TO DISK
C       T=FLUID TEMPERATURE
C       P=FLUID PRESSURE
C       RO=FLUID DENSITY
C       ECC=ELECTRON NUMBER DENSITY
C       C=ELECTRIC CONDUCTIVITY OF THE FLUID
C       V=FLUID VELOCITY
C       QN=ELECTRON-ATOM COLLISION CROSS SECTION
C       QI=ELECTRON-ION COLLISION CROSS SECTION
        RS=.01
        DO 200 I=1,5
        PUNCH 2,RS
      2 FORMAT(40HSTREAMLINE ENTRANCE RADIUS IN METERS RS=E14.8)
        T=20000.
        RO=.000086*515.
        R=287.
        P=RO*R*T
        Z=.4
```

```
      PUNCH 13,Z,T,P,RO
      H=.01
      ZL=.01
      C=10000.
      QTK=0.
    5 K=0
      GO TO 100
   10 AK1=FZ*H
      FZO=FZ
      Z1=Z
      T1=T
      Z=Z1-H/2.
      T=T1-AK1/2.
      GO TO 100
   15 AK2=FZ*H
      T=T1-AK2/2.
      GO TO 100
   20 AK3=FZ*H
      Z=Z1-H
      T=T1-AK3
      GO TO 100
   25 AK4=FZ*H
      FZ1=FZ
      T=T1-(AK1+2.*AK2+2.*AK3+AK4)/6.
      Z=Z1-H
      RZ=(RS*RS*Z2/Z)**.5
```

```
   30 TK=.00199*T**.5/(1.+112./T)
   31 QTK=TK*(FZ1-FZ0)/H
      PUNCH 13,Z,T,P,RO
   13 FORMAT (5H  Z=E14.8,5H  T=E14.8,5H  P=E14.8,5H  RO=E14.8)
C     ELECTRIC CONDUCTIVITY PROGRAM NEXT
      T2=T
      P2=P/47.88
      QN=1./(10.**19)
      UI=166000.
      X=(.032*T2**2.5)/(47.88*P2*EXPF(UI/T2))
      Y=(47.88*P2*10.**23)/(1.38*T2)
      E=Y*SQRTF(X/(1.+X))
      EN=Y-E
      ECC=E/(10.**6)
      QI=(2.95/(T2**2*10.**10))*LOGF(8780.*T2**1.5/SQRTF(ECC))
      G=(20.5/T2)**.5/(10.**12)
      C=(G*E)/(EN*QN + E*QI)
      PUNCH 65,ECC,C,V,RZ
   65 FORMAT  (5H ECC=E14.8,5H  C=E14.8,5H  V=E14.8,5H  RZ=E14.8)
      IF(ZL-Z)5,110,110
  100 R=287.
      GA=1.4
      STB=5.67/10.**8
      CL=16./(9.*10.**8)
      CR=STB*CL*T**3
      W=2000.
```

```
      Z2=.4
      SI1=W*RS*RS/2.
      B1=.2
      R1=.01
      BS=B1*R1*R1/RS**2
      B2=BS*Z/Z2
      PBS=(BS*BS*10.**7)/(8.*3.14)
      PVS=.5*RO*W*W
      PV=.5*RO*(W/(2.*Z2))**2*(RS*RS*Z2/Z + 4.*Z*Z)
      PB=B2*B2*10.**7/(8.*3.14)
      P1=254224.6
      P=P1+PBS+PVS-PV-PB
      RO=P/(R*T)
      V=(W*W*Z*Z/(Z2*Z2) + SI1*W/(2.*Z2*Z))**.5
      CV=R*T/(V*V)
      BO=.2
      B=(BO/10.**4)*Z/(RS*RS*Z2)
      PK=.06
      QR=-4.*PK*STB*T**4
      ZS=(1. + RS*RS*Z2/(4.*Z**3))**.5
C     VISCOUS PROGRAM NEXT
  101 WZ=W/Z2
      U=.000001462*T**.5/(1.+112./T)
      UO=2.5*U*WZ*WZ
      STRES=U*WZ/(RS*RS*Z2/Z + 4.*Z*Z)**.5
      QV=(QR+UO+QTK)/V
```

```
102 DTN =ZS*(CV-1.)*QV + ZS*C*V*B*B-ZS*CV*STRES
    DTD=CR + (P/T)*(1.+(GA/(GA-1.))*(CV-1.))
    FZ=-DTN/DTD
    K=K+1
    GO TO (10,15,20,25),K
110 CONTINUE
200 RS=RS*2.
    END
```

Typical program printout for calculations along streamline ψ_2; only intermittent results are shown.

```
STREAMLINE ENTRANCE RADIUS IN METERS RS= .20000000E-01

  Z=.40000000E+00   T=.20000000E+05   P=.25422460E+06   RO=.44290000E-01

  Z=.39000000E+00   T=.19864611E+05   P=.25864267E+06   RO=.45366864E-01
ECC=.74155286E+18   C=.12683507E+05   V=.19506569E+04   RZ=.20254787E-01
................    ..............    ..............    ...............
  Z=.35000000E+00   T=.19287799E+05   P=.27761739E+06   RO=.50151164E-01
ECC=.75311657E+18   C=.12279440E+05   V=.17508155E+04   RZ=.21380899E-01
................    ..............    ..............    ...............
  Z=.30000000E+00   T=.18529958E+05   P=.30401610E+06   RO=.57165924E-01
ECC=.74016460E+18   C=.11665416E+05   V=.15011105E+04   RZ=.23094012E-01
................    ..............    ..............    ...............
```

```
   Z=.25000000E+00  T=.17785594E+05  P=.33345811E+06  RO=.65326054E-01
 ECC=.69753902E+18  C=.10972201E+05  V=.12515987E+04  RZ=.25298223E-01
 ................   ..............   ..............   ...............
   Z=.20000000E+00  T=.17064096E+05  P=.36563279E+06  RO=.74657600E-01
 ECC=.63172579E+18  C=.10219858E+05  V=.10024968E+04  RZ=.28284271E-01
 ................   ..............   ..............   ...............
   Z=.15000000E+00  T=.16347266E+05  P=.39987440E+06  RO=.85229311E-01
 ECC=.54957608E+18  C=.94057760E+04  V=.75443113E+03  RZ=.32659862E-01
 ................   ..............   ..............   ...............
   Z=.10000000E+00  T=.15589313E+05  P=.43542470E+06  RO=.97317954E-01
 ECC=.45384093E+18  C=.84905995E+04  V=.50990171E+03  RZ=.40000001E-01
 ................   ..............   ..............   ...............
   Z=.50000000E-01  T=.14667850E+05  P=.47285149E+06  RO=.11231728E+00
 ECC=.33847680E+18  C=.73309736E+04  V=.28722804E+03  RZ=.56568544E-01
 ................   ..............   ..............   ...............
   Z=.10000000E-01  T=.13332793E+05  P=.50944799E+06  RO=.13294087E+00
 ECC=.19624103E+18  C=.56150074E+04  V=.32015619E+03  RZ=.12649111E+00
```

BIBLIOGRAPHY

1. Huber, P.W.; and Sims, T.E.: The Entry-Communications Problem. Astronaut. Aero., vol.2, no.10, Oct.1964, 30-40.

2. Lehnert, R.; and Rosenbaum, B.: Plasma Effects on Apollo Re-entry Communication. NASA TN D-2732, 1965.

3. Gold, R.R.: Analysis of Electromagnetic Wave Propagation Through Magnetoactive Plasmas. Phys. Fluids 7, 1964, 446.

4. Gold, R.R.; and Cashen, J.F.: Communication Through Plasmas Bounding Slender Entry Vehicles. AIAA J. vol.5, no.5, May 1967, 917-922.

5. Resler, E.L., Jr.; and Sears, W.R.: The Prospects for Magneto-Aerodynamics. J. Aero. Sci., April 1958, 235-245.

6. Sears, W.R.: Magnetohydrodynamic Effects in Aerodynamic Flows. ARS J. June 1959, 397-405.

7. Neuringer, J.L.; and McIlroy, W.: Incompressible Two-Dimensional Stagnation Point Flow of an Electrically conducting Viscous Fluid in the Presence of a Magnetic Field. J. Aero. Sci. March 1958, 194-198.

8. Rossow, V.J.: Magnetohydrodynamic Analysis of Heat Transfer Near a Stagnation Point. J. Aero. Sci. May 1958, 334-335.

9. Shercliff, J.A.: Magnetogasdynamics and Its Possible Aeronautical Applications. J. Royal Aero. Society, Sept. 1959, 518-521.

10. Kemp, N.H.; and Petschek, H.E.: Two-Dimensional Incompressible Magnetohydrodynamic Flow Across an Elliptical Solenoid. J. Fluid Mech. 4, 1958, 553-584.

11. Phillips, R.L.: Effects of Magnetic Drag on Re-Entry Body Heating. ARS J. 1961, 672-674.

12. Levy, R.H.; and Petschek, H.E.: Magnetohydrodynamically Supported Hypersonic Shock Layer. Phys. Fluids, 6, 1963, 946-961.

13. Levy, R.H.; Gierasch, P.J.; and Henderson, D.B.: Hypersonic Magnetohydrodynamics With or Without a Blunt Body. AIAA J. Dec. 1964, 2091-2099.

14. Hooks, L.E.: Simplified Magnetoaerodynamic Flow Relations for Axisymetric Blunt Bodies, AIAA J. April 1967, 644-650.

15. Porter, R.W.; and Cambel, A.B.: Magnetic Coupling in Flight Magnetoaerodynamics. AIAA J. April 1967, 803-805.

16. Porter, R.W.; and Cambel, A.B.: Hall Effect in Flight Magnetogasdynamics. AIAA J. Dec. 1967, 2208-2213.

17. Seemann, G.R.; and Cambel, A.B.: Observations Concerning Magnetoaerodynamic Drag and Shock Standoff Distance. Proc. Nat. Aca. Sci. vol.55. no.3, March 1966. 457-465.

18. Zhigulev, V.N.; Romishevskii, Ye.A.; and Vertushkin, V.K.: Role of Radiation in Modern Gasdynamics. AIAA J. June 1963, 1473-1485.

19. Kivel, B.: Radiation from Hot Air and Its Effects on Stagnation-Point Heating. J. Aero. Scie. February 1961, 96-101.

20. Goulard, R.; and M.: One -Dimensional Energy Transfer in Radiant Media. Int. J. Heat and Mass Transfer, vol.1. 1960, 81-91.

21. Gille, J.; and Goody, R.: Convection in a Radiating Gas. J. Fluid Mech. vol. 20, part 1, 1964, 47-79.

22. Gruszczynski, J.S.; and Warren, W.R.: Study of Equilibrium Air Total Radiation. AIAA J. March 1967, 517-525.

23. Yoshikawa, K.K.; and Wick, B.H.: Radiative Heat Transfer During Atmosphere Entry at Parabolic Velocity. NASA TN D-1074,1961.

24. Fay, J.A.; Moffatt, W.C.; and Probstein, R.F.: An Analytical Study of Meteor Entry. AIAA J. May 1964, 845-853.

25. Goulard, R.: The Coupling of Radiation and Convection in Detached Shock Layers. J. Quant. Spectroscopy and Rad. Trans. vol.1, Dec. 1961, 249-257.

26. Yoshikawa, K.Y.; and Chapman, D.R.: Radiative Heat Transfer and Absorption Behind a Hypersonic Normal Shock Wave. NASA TN D-1424, 1962.

27. Viskanta, R.; and Grosh, R.J.: Boundary Layer in Thermal Radiation Absorbing and Emitting Media. Int. J. Heat and Mass Transfer, vol.5, Sept. 1962, 795-806.

28. Koh, J.C.Y.; and DeSilva, C.N,: Interaction Between Radiation and convection in the Hypersonic Boundary Layer on a Flat Plate. ARS J. May 1962, 739-743.

29. Koh, J.C.Y.: Radiation From Nonisothermal Gases to the Stagnation Point of a Hypersonic Blunt Body. ARS J. Sept. 1962, 1374-1377.

30. Goulard, R.: Radiation Transfer Regimes in Hypersonic Flight. Supersonic Flow, Chem. Processes and Radiative Transfer, edited by D.B.Olfe and V. Zakkay, Pergamon Press, N.Y., 1964, 307-317.

31. Scala, S.M.; and Sampson, D.H.: Heat Transfer in Hypersonic Flow with Radiation and Chemical Reaction. Supersonic Flow, Chem. Processes and Radiative Transfer, ed. by D.B. Olfe and V. Zakkay, Perg. Press, N.Y., 1964.

32. Thomas, P.D.: Transparency Assumption in Hypersonic Radiative Gas Dynamics. AIAA J. August 1965, 1401-1407.

33. Cohen, I.M.: Couette Flow of a Radiating Fluid: Optically Thick Medium. AIAA J. Jan. 1967, 78-83.

34. Cohen, I.M.: Strongly Radiating Hypersonic Flow over a Sphere: An Asymptotic Sketch. AIAA J. March, 1967, 501-506.

35. Hoshizaki, H.; and Wilson, K.H.: Convective and Radiative Heat Transfer During Superorbital Entry. AIAA J. Jan. 1967, 25-35.

36. Kourganoff, V.: Basic Methods in Transfer Problems. Dover Pub. Inc. New York, 1963.

37. Chandrasekhar, S.: Radiative Transfer. Dover Pub. Inc. New York, 1960.

38. Pai, S.I.: Radiation Gas Dynamics. Springer-Verlag, New York Inc. 1966.

39. Cohen, R.S.; Spitzer, L.,Jr.; and Routley, P.McR.: The Electrical conductivity of an Ionized Gas. Phys. Rev., vol. 80, Oct. 1950, 230-238.

40. Spitzer, L., Jr.; and Härm, R.: Transport Phenomena in a Completely Ionized Gas. Phys. Rev., vol.89, March 1953, 977-981.

41. Viegas, J.R.; and Peng, T.C.: Electrical Conductivity of Ionized Air in Thermodynamic Equilibrium. ARS J. May 1961, 654-657.

42. Kerrebrock, J.L.: Nonequilibrium Ionization Due to Electron Heating: I. Theory. AIAA J. June 1964, 1072-1080.

43. Kerrebrock, J.L.; and Hoffman, M.A.: Nonequilibrium Ionization Due to Electron Heating: II. Experiment. AIAA J. June 1964, 1080-1087.

44. Zukoski, E.E.; Cool, T.A.; and Gibson, E.G.: Experiments Concerning Nonequilibrium Conductivity in a Seeded Plasma. AIAA J. Aug. 1964, 1410-1417.

45. Reilly, J.P.: Stagnation Point Heating in Ionized Monatomic Gases. Phys. Fluids, vol. 7, Dec. 1964, 1905-1912.

46. Schweitzer, S.; and Mitchner, M.: Electrical Conductivity of Partially Ionized Gases. AIAA J. June 1966, 1012-1019.

47. Maecker, H.; and Peters, Th.: Einheitliche Dynamik und Thermodynamik des Thermischen Plasmas. Z.fur Physik, Bd.144, 1956, 586-611.

48. Samaras, D.G.: Theory of Ion Flow Dynamics. Prentice-Hall, Englewood Cliffs, N.J., 1962.

49. Bloom, M.H.; Steiger, M.H.: Inviscid Flow with Nonequilibrium Molecular Dissociation for Pressure Distributions Encountered in Hypersonic Flight. J. Aerospace Sci. Nov. 1960, 821-835.

50. Lin, S.C.; and Teare, J.D.: A Streamtube Approximation for Calculation of Reaction Rates in the Inviscid Flow Field of Hypersonic Objects. Sixth Symp. Ballistic Missile and Aerospace Tech., 1961.

51. Maslen, S.H.: Inviscid Hypersonic Flow Past Smooth Symmetric Bodies. AIAA J. June 1964, 1055-1061.

52. Perini, L.L.; and Melnik, W.L.: Nonequilibrium Hypersonic Flow Past Smooth Symmetric Bodies. J. Spacecraft, March 1968, 309-313.

53. Hansen, C.F.: Approximations for the Thermodynamic and Transport Properties of High-Temperature Air. NASA TR. R-50. 1959.

54. Adams, M.C.; and Probstein, R.F.: On the Validity of Continuum Theory for Satellite and Hypersonic Flight Problems at High Altitudes. Jet Propulsion, Feb. 1958. 86-89.

55. Sutton, G.W.; and Sherman, A.: Engineering Magnetohydrodynamics. McGraw-Hill Book Co. N.Y. 1965.

56. Schlichting, H.: Boundary Layer Theory, McGraw-Hill Book Co. N.Y. 1960.

57. Coddington, E.E.; and Levinson, N. : Theory of Ordinary Differential Equations. McGraw-Hill Book Co. N.Y. 1955.

58. Carson, D.R.; and Lorrain, P.: Introduction to Electromagnetic Fields and Waves. W.H. Freeman and Company, San Francisco, 1962.

59. Belotserkovskii, O.M.; and Chushkin, P.I.: The Numerical Solution of Problems in Gas Dynamics. "Basic Developments in Fluid Dynamics ", ed. by M. Holt, Academic Press, N.Y. 1965, 1-126.