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SOME NON-LINEAR ASPECTS OF CRACK-TIP FIELDS IN FINITE ELASTICITY

by

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ABSTRACT

SOME NON-LINEAR ASPECTS OF CRACK-TIP
FIELDS IN FINITE ELASTICITY

by

Jae-Sung Yang

This dissertation has two parts, each concerned with some non-linear aspect of crack-tip fields in finite elasticity. The first part of the current dissertation is concerned with the deformation field near the tip of a crack in an incompressible solid which is loaded in Mode I. The material model used here exhibits strain softening in shear at large deformations, which allows for a loss of ellipticity of the governing differential equations. As a result, one expects zones of localized shear to emanate from each crack-tip and extend into the interior of the body. A local asymptotic analysis near the crack-tip is carried out and both elliptic and non-elliptic solutions to the governing equations are found. Neither of these solutions, alone, satisfy the boundary conditions and so, neither provides a complete solution to the problem. However, a composite deformation field consisting of a "patching together" of these two separate solutions can be constructed. There are certain lines of strain discontinuity emanating from each crack-tip, corresponding to the boundaries between these different solutions, modelling a narrow zone of localized shear.

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In the second part we are concerned with the derivation of simple explicit expressions for J, the energy release rate associated with quasi-static crack growth in nonlinearly elastic solids. The J-integral is the central theoretical quantity behind nonlinear fracture mechanics for rate-independent materials under monotonic loading; can also be regarded as a measure of the intensity of the crack tip singularity fields. The estimation of J for elastic-plastic crack problems is usually achieved by interpolation between the values of J corresponding to linearly-elastic and pure power-law materials. These interpolation schemes involve coefficients determined through a finite element analysis; moreover, different interpolation schemes are required for different constitutive descriptions. It is observed in this dissertation that one can avoid such interpolations by using a certain analytical estimation procedure. A center-cracked strip in Mode III is used to illustrate this. The results for J obtained by this scheme are appropriate for general constitutive relations and are very accurate under certain conditions. Consequently, the need to interpolate between particular materials can be avoided.

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INTRODUCTION

In recent decades, many investigations have been made in fracture mechanics to study the stress and deformation fields near a crack tip, and thereby or otherwise, establish criteria for crack initiation and propagation. Most of these studies have been under linear elastic conditions. The few extensions to nonlinear stress-strain relations were still restricted to small strain behavior, e.g., classical plasticity theory where the strain measure used is still infinitesimal. However, since the analysis of most crack problems leads to the conclusion that the displacement gradients are unbounded at the crack tip, this contradicts the underlying assumptions of any infinitesimal strain theory. Consequently, the results predicted by such a theory cannot possibly be uniformly valid near the crack-tips. It is only recently that fully nonlinear theories, using finite strain measures, have been utilized in the study of fracture mechanics. In this dissertation, we examine two problems using the fully nonlinear theory of elasticity. In the first part we consider a plane strain mode I problem for a crack in an incompressible nonlinearly elastic material and study the crack-tip fields. In the second part, we consider the energy release rate, and discuss an analytical scheme for estimating its value.

PART I

LOCALIZED SHEAR DISCONTINUITIES NEAR THE TIP OF A MODE I CRACK

CHAPTER 1

INTRODUCTION

Throughout the years, many studies have been devoted to the investigation of stress and deformation fields near a crack-tip, most studies being formulated within some small-strain theory. All of these studies, in contradiction to the approximative assumption upon which they rest, give rise to locally unbounded strains at the crack-tips. The predictions of such infinitesimal theories for these problems may therefore be presumed, at best, to be realistic at finite distances from the singular point: They cannot possibly be uniformly valid in the vicinity of these points, no matter how small the applied loads.

A number of recent studies, however, have been concerned with the elastostatic fields under large strains. An asymptotic treatment, consistent with the fully nonlinear equilibrium theory of compressible elastic solids, of the stresses and deformations near the tip of a traction-free crack under plane strain conditions, was given by Knowles and Sternberg, [1]. The analysis considered a particular class of materials. The asymptotic analysis of the foregoing crack problem was reduced to a nonlinear eigenvalue problem, the solution of which was established in closed form, in terms of elementary functions. Later, [2], they reconsidered the previous problem, leading to a clarification and improvement of the results for the lowest-order asymptotic crack tip fields.

In an attempt to determine the deformations and stresses near a crack tip for a class of materials different to that in [1], [2], certain difficulties were encountered. In their efforts to establish an asymptotic solution to a crack problem for the Blatz-Ko material, Knowles and Sternberg, [3], noticed that these difficulties suggested that the problem in this case might not admit a solution of unlimited smoothness. This in turn led them to investigate the possible failure of ellipticity of the appropriate system of displacement equations of equilibrium at solutions involving large deformations. It was shown that ellipticity prevails only if the principal stretches are suitably restricted and breaks down, in particular, at a local state of uni-axial tension or compression of sufficiently severe intensity. They further established necessary and sufficient conditions, in terms of the local principal stretches, for ordinary and strong ellipticity of the equations governing finite plane equilibrium deformations of a general compressible isotropic hyperelastic solid [4].

A *failure of ellipticity* of the elastostatic equations suggests the possible emergence of solution fields that possess finite jump discontinuities in the first displacement gradients across certain curves, while the displacements themselves still remain continuous everywhere. *Equilibrium shocks* of this kind, which resemble in many respects gas-dynamical shocks associated with steady flows, are shown to exist if and only if the governing field equations of equilibrium suffer a loss of ellipticity [5].

Meanwhile, Abeyaratne [6] investigated the corresponding issues for incompressible materials, namely the possibility of the change in type of the differential equations governing finite plane elastostatics and the related issue of the existence of equilibrium fields with discontinuous deformation gradients. He showed that in the particular case of weak elastostatic shocks, ellipticity must necessarily be lost at the preassigned deformation on one side of the shock. Further, in the case of shocks of finite strength, it is shown that a strict failure of ellipticity at a given deformation is sufficient to ensure the existence of such a shock which has associated with it this deformation on one side. Moreover, he shows that a failure of ellipticity at some homogeneous deformation is necessary for the existence of a shock.

Hutchinson and Neale, [7], considered the range of states for which the equations governing incremental responses are elliptic for isotropic, nonlinearly elastic solids obeying a finite strain version of the J_2 deformation theory of plasticity. Rudnicki and Rice examined the hypothesis that localization of deformation into a shear band can be understood as an instability in the macroscopic constitutive description of inelastic deformation of the material, [8]. Specifically, they understood instability in the sense that the constitutive relation may allow a homogeneously material to load to a bifurcation point, at which nonuniform deformation can be incipient in a planar band under conditions of continuing homogeneous deformation outside the band. The condition for the emergence of such deformation fields was shown to be a loss of ellipticity of the governing equations.

Zones of localized deformation in the form of narrow shear bands, are a common feature of certain severely deformed ductile metals. It is thought that such behavior can be explained at the microscopic level by modelling in detail the process of growth and interaction of the many individual fissures that appear (and ultimately join together) in forming the macroscopic surface of rupture. A buckling type instability of these voids at the microscopic-level, into a band-like mode, is a possible explanation for this phenomenon. At the macroscopic level, this manifests itself as a shear band. An investigation of this hypothesis was undertaken by Abeyaratne and Triantafyllidis in [9].

In [10] and [11], Knowles and Sternberg returned to the investigation of the elastostatic field near the tip of a crack, the particular constitutive law now chosen so as to give rise to a loss of ellipticity of the governing displacement equations of equilibrium in the presence of sufficiently severe deformation. In [10] and [11] they consider the small scale nonlinear Mode III problem for an incompressible elastic material. An explicit exact solution, deduced with the aid of the hodograph method, exhibits two symmetrically located lines of displacement-gradient and stress discontinuity issuing from each crack-tip. In [13] Abeyaratne investigated a similar problem and showed the presence of two pairs of equilibrium shocks issuing from points on the crack-faces, different from the crack-tips. In this problem, a class of incompressible, homogeneous, isotropic elastic materials was considered for which the governing displacement equation of equilibrium is elliptic at both sufficiently small and sufficiently large shearing strains but is hyperbolic at an intermediate level of strains. Most recently, Fowler considered the Mode I analog of the aforementioned problem for a particular compressible elastic material, viz. the Blatz-Ko material, using a direct asymptotic method, [14]. Again, the loss of equilibrium ellipticity resulted in the appearance of equilibrium shocks. In the meantime, Hutchinson examined the case of a tensile crack in a perfectly plastic material under conditions of plane stress. The governing differential equations here are non-elliptic (at all non-zero deformations), and he finds two lines issuing from each crack tip across which the stresses are discontinuous [15].

In the present study we will examine the plane strain mode I problem for a crack in an incompressible nonlinearly elastic material. The constitutive law is such that the governing differential equations are elliptic at sufficiently small deformations and non-elliptic at sufficiently large deformations. Specifically, for the present plane strain problem, the material may be completely characterized by its response in simple shear, and this is assumed to be linear for small amounts of shear and of a power-law form at large amounts of shear (see Figure 1).

The analysis of the mode III problems in [11,12,13] is based on a hodograph transformation of the governing second order quasilinear partial differential equation. This method is not available for the present mode I problem since the governing equations are of higher order. Instead, a direct asymptotic analysis of the crack-tip field is carried out.

In Chapter 2 we cite the appropriate equations from finite elastostatics. The crack problem is formulated in Chapter 3 and the governing equations are written in terms of polar coordinates centered at the right crack-tip. In Chapter 4 we find two asymptotic solutions of the differential equations on overlapping subdomains of the crack-tip zone (see Figure 3). One of them is non-elliptic and holds in a zone ahead of the crack; the other is elliptic and is a solution on zones adjacent to the crack-faces. These solutions are pieced-together in

Chapter 5, in order to generate a solution of the complete (asymptotic) boundary-value problem (see Figure 5). The final solution involves two shocks issuing from each crack-tip and emanating into the body. The stress and deformation gradient fields are discontinuous across these curves, though the displacements and tractions remain continuous. The results are summarized in Chapter 6.

CHAPTER 2

PRELIMINARIES PERTAINING TO FINITE PLANE STRAIN

Suppose that R denotes the three-dimensional open region occupied by the interior of an incompressible body in its undeformed configuration. A deformation of the body is described by a sufficiently smooth and invertible transformation

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \quad \text{on } R, \tag{2.1}$$

which maps R onto its image R^* in the deformed configuration, $R^* = \underset{\sim}{y}(R)$. Hence $\underset{\sim}{y}$ is the position vector after deformation of the particle which, in the undeformed configuration, was located at $\underset{\sim}{x}$. We will assume temporarily that the displacement vector field $\underset{\sim}{u}(\underset{\sim}{x})$ is twice continuously differentiable on R.

The deformation gradient tensor $\underset{\sim}{F}$ is defined by

$$\underset{\sim}{F} = \nabla \underset{\sim}{y} \quad \text{on } R, \tag{2.2}$$

and the corresponding Jacobian determinant is given by

$$J(\underset{\sim}{x}) = \det \underset{\sim}{F}(\underset{\sim}{x}) > 0 \quad \text{on } R. \tag{2.3}$$

Since the body is assumed to be incompressible, the deformation must be locally volume-preserving so that

$$J(\underset{\sim}{x}) = 1 \quad \text{on } R. \tag{2.4}$$

The right and left Cauchy-Green tensors $\underset{\sim}{C}$ and $\underset{\sim}{G}$ are defined respectively by

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F}, \quad \underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T. \tag{2.5}$$

Let $\underset{\sim}{\tau}$ be the Cauchy (true) stress tensor field on R^* accompanying the deformation at hand. The equilibrium equations are

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^T \quad \text{on } R^*, \tag{2.6}$$

when body forces are presumed to be absent. The Piola (nominal) stress tensor corresponding to $\boldsymbol{\tau}$ is given, in view of (2.4), by

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\tau}(\hat{\mathbf{y}}(\mathbf{x}))(\mathbf{F}^T(\mathbf{x}))^{-1} \quad \text{on } R. \tag{2.7}$$

Equations (2.2), (2.4), (2.6), and (2.7) lead to the equilibrium equations in the reference configurations

$$\operatorname{div} \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}, \quad \boldsymbol{\sigma}\mathbf{F}^T = \mathbf{F}\boldsymbol{\sigma}^T \quad \text{on } R. \tag{2.8}$$

Furthermore, the nominal and true surface tractions are given by

$$\mathbf{s} = \boldsymbol{\sigma}\mathbf{N} \quad \text{on } S, \ \mathbf{t} = \boldsymbol{\tau}\mathbf{n} \quad \text{on } S^*, \tag{2.9}$$

where S and S^* are surfaces in R and R^*, $S^* = \hat{\mathbf{y}}(S)$, while $\mathbf{N}$ and $\mathbf{n}$ are unit normals to S and S^*, respectively. It follows then that

$$\mathbf{s} = \mathbf{0} \text{ on } S \text{ if and only if } \mathbf{t} = \mathbf{0} \text{ on } S^*, \tag{2.10}$$

which is a useful result since it allows the boundary condition on a traction free surface S^* to be specified on its undeformed image S.

We now turn to the constitutive law and suppose that the body is homogeneous, incompressible, and elastic, and that it possesses an elastic potential $W = \hat{W}(\mathbf{F})$. The nominal stresses are then given by

$$\boldsymbol{\sigma} = \frac{\partial \hat{W}(\mathbf{F})}{\partial \mathbf{F}} - p(\mathbf{F}^T)^{-1}, \tag{2.11}$$

where $p(\mathbf{x})$ is a scalar field arising because of the incompressibility constraint. In the case where the material is isotropic, W depends on $\mathbf{F}$ in a special manner, viz.

$$W = \overset{*}{W}(I_1, I_2), \tag{2.12}$$

where I_1, I_2 are the principal scalar invariants of $\underset{\sim}{G}$:

$$I_1 = \text{tr}\,\underset{\sim}{G},\ I_2 = \tfrac{1}{2}[(\text{tr}\,\underset{\sim}{G})^2 - \text{tr}(\underset{\sim}{G}^2)]. \tag{2.13}$$

Suppose now that the domain R occupied by the undeformed body is a right cylinder with generators parallel to the x_3-axis. Let D be the open region of the (x_1, x_2)-plane occupied by the interior of the middle cross section of this cylinder at $x_3 = 0$. Suppose further that the deformation (2.1) is a plane deformation so that

$$y_\alpha = x_\alpha + u_\alpha(x_1, x_2),\ y_3 = x_3 \quad \text{on } R. \tag{2.14}$$

Throughout this problem, a comma followed by a subscript indicates differentiation with respect to the appropriate coordinate and Latin subscripts take the values 1, 2, 3, while Greek subscripts take the values 1, 2. Repeated subscripts are summed over the proper range. It follows from (2.2) and (2.14) that

$$F_{\alpha\beta} = y_{\alpha,\beta},\ F_{\alpha 3} = F_{3\alpha} = 0,\ F_{33} = 1. \tag{2.15}$$

The nominal stresses are now given by

$$\sigma_{\alpha\beta} = \frac{\partial\hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}} - p\,F^{-1}_{\beta\alpha},\ \sigma_{33} = \frac{\partial\hat{W}(\underset{\sim}{F})}{\partial F_{33}} - p. \tag{2.16}$$

If we assume that the elastic potential W is such that

$$\frac{\partial\hat{W}(\underset{\sim}{F})}{\partial F_{\alpha 3}} = \frac{\partial\hat{W}(\underset{\sim}{F})}{\partial F_{3\alpha}} = 0 \tag{2.17}$$

for every $\underset{\sim}{F}$ such that (2.15) holds, then we further have

$$\sigma_{3\alpha} = \sigma_{\alpha 3} = 0. \tag{2.18}$$

If we now define I by

$$I = F_{\alpha\beta}F_{\alpha\beta} \tag{2.19}$$

we have, because of (2.5), (2.13), and (2.15) that

$$I = I_1 - 1 = I_2 - 1. \tag{2.20}$$

In the case when the material is isotropic, we have from (2.12), (2.20) that, in plane deformations,

$$W = \overset{*}{W}(I+1,\ I+1) \tag{2.21}$$

so that if we define the plane strain elastic potential $W(I)$ By

$$W(I) = \overset{*}{W}(I+1,\ I+1),\ I > 2, \tag{2.22}$$

we have that $\hat{W}(\underset{\sim}{F}) = W(I)$ where $I = F_{\alpha\beta}F_{\alpha\beta}$. It follows from this that

$$\frac{\partial\hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}} = 2F_{\alpha\beta}W'(I). \tag{2.23}$$

From (2.5), (2.16), (2.23), we conclude that

$$\sigma_{\alpha\beta} = 2W'(I)F_{\alpha\beta} - pF^{-1}_{\alpha\beta}. \tag{2.24}$$

The deformation (2.14) is a simple shear in the x_1-direction if it has the form

$$u_1 = kx_2,\ u_2 = 0, \tag{2.25}$$

where k is the amount of shear in the x_1, x_2-plane. Then from (2.19), one gets

$$I = 2 + k^2 \tag{2.26}$$

and the stress of primary interest, τ_{12}, is found from (2.7), (2.15), (2.24), (2.26) to be

$$\tau_{12} = \hat{\tau}(k) = 2kW'(2+k^2),\ 0 < k < \infty; \tag{2.27}$$

$\hat{\tau}(k)$ denotes <u>the response function of the material in simple shear</u>. Its secant, tangent, and infinitesimal shear moduli are given by

$$\mu_s(k) = \frac{\hat{\tau}(k)}{k}, \; \mu_t(k) = \frac{d\hat{\tau}(k)}{dk}, \; \mu_o = \mu_t(0), \tag{2.28}$$

respectively. The secant modulus is assumed to be positive, so that from (2.27), (2.28),

$$\mu_s(k) = 2W'(I) > 0. \tag{2.29}$$

The in-plane behavior of an incompressible material, in plane strain, is essentially governed by its response to simple shear. In particular, <u>any</u> plane deformation of an incompressible body can always be decomposed locally into a rigid-body rotation followed, or preceded, by a simple shear [11]. The amount of this <u>"effective local shear"</u>, k_e, is

$$k_e = (I-2)^{1/2}, \quad I = \operatorname{tr} \underset{\sim}{F}^T \underset{\sim}{F} \; . \tag{2.30}$$

If λ, λ^{-1} are the principal stretches of the deformation, one can equivalently write

$$k_e = |\lambda - \lambda^{-1}|. \tag{2.31}$$

Therefore from (2.24), (2.29), (2.30), the in-plane response of an isotropic, incompressible, elastic material in <u>any</u> plane deformation can be written as

$$\sigma_{\alpha\beta} = \mu_s(k_e)F_{\alpha\beta} - p(\underset{\sim}{x})F^{-1}_{\beta\alpha}, \; k_e = (I-2)^{1/2}. \tag{2.32}$$

From (2.24), (2.8), (2.19), (2.4), (2.3) and (2.1), one may obtain the displacement equations of equilibrium for plane deformations of a homogeneous, isotropic, incompressible hyperelastic material. They are

$$C_{\alpha\beta\gamma\delta}(\underset{\sim}{F})\, u_{\gamma,\beta\delta} - p_{,\beta}F^{-1}_{\beta\alpha} = 0 \quad \text{on } D, \; \det \underset{\sim}{F} = 1 \quad \text{on } D, \tag{2.33}$$

where $F_{\alpha\beta} = u_{\alpha,\beta} + \delta_{\alpha\beta}$ and

$$C_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) = \frac{\partial^2 W}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}} = 2W'(I)\delta_{\alpha\gamma}\delta_{\beta\delta} + 4W''(I)F_{\alpha\beta}F_{\gamma\delta}. \qquad (2.34)$$

It has been shown, (Abeyaratne [11]), that (in the presence of (2.29)) this system of partial differential equation is elliptic at a solution (u_α, p) and at a point (x_1, x_2) if and only if

$$2W''(I)(I-2) + W'(I) > 0. \qquad (2.35)$$

A physical interpretation of this ellipticity condition may be obtained in terms of the concept of the local amount of shear. Recall the definition of the shear stress response function $\hat{\tau}(k)$:

$$\hat{\tau}(k) = 2kW'(2+k^2), \quad |k| < \infty. \qquad (2.36)$$

Differentiating (2.36) with respect to k and observing that the secant modulus is assumed to be positive leads to

$$\hat{\tau}'(k) = 2W'(2+k^2)\{2k^2 \frac{W''(2+k^2)}{W'(2+k^2)} + 1\}. \qquad (2.37)$$

We therefore find that (2.35) is equivalent to

$$\hat{\tau}'(k_e) > 0, \quad k_e = (I-2)^{1/2}, \qquad (2.38)$$

from which we conclude that the system of partial differential equations (2.33) is <u>elliptic</u> at a solution $\underset{\sim}{u}$, p and at a point $\underset{\sim}{x}$ if and only if the tangent modulus evaluated at the effective local shear k_e is positive. Thus, in particular, if the material at hand has a monotonically increasing response curve in shear, it follows that ellipticity prevails throughout.

In this study, we are concerned with the case in which the material behavior allows for ellipticity to be lost at severe levels of deformation. We suppose that the shear response function $\hat{\tau}(k)$ is linear for small amounts of shear and has a power-law form for large amounts of shear. Specifically, we take

$$\hat{\tau}(k) = \begin{cases} \mu_0 k & \text{for } 0 < k < k_0 \\ \mu_0 k_0 \left(\frac{k}{k_0}\right)^{2n-1} & \text{for } k_0 < k < \infty, \end{cases} \tag{2.39}$$

where μ_0, k_0, and n, ($\mu_0 > 0$, $k_0 > 0$, and $0 < n < 1/2$), are material constants. A sketch of $\hat{\tau}(k)$ vs. k is shown in Figure 1. Note that when $0 < n < 1/2$, as is assumed here, ellipticity is lost whenever the effective local shear k_e exceeds k_0.

The loss of ellipticity of the governing partial differential equations leads to the possible occurrence of elastostatic fields which are less smooth than previously assumed. Therefore, we now have to relax the smoothness assumptions made previously in order to account for such "weak solutions." Particular interest lies in the case wherein the field quantities possess the degree of smoothness assumed previously everywhere except on one or more curves on D. Accordingly, it is assumed that, although $\underset{\sim}{u}$ is continuous in D, there is a smooth curve S in D such that (i) p and $\underset{\sim}{u}$ are respectively once and twice continuously differentiable in D-S, (ii) p and $\nabla\underset{\sim}{u}$ suffer finite jumps across S.

Under these new conditions, the field equations discussed previously are to hold in D-S. In addition, it is required that the nominal traction $\underset{\sim}{s}$ be continuous across S; this in turn implies that the true traction $\underset{\sim}{t}$ is continuous across S^*. Therefore, from (2.9), we get

$$[\underset{\sim}{\sigma}]\underset{\sim}{N} = \underset{\sim}{0} \text{ on } S, \quad [\underset{\sim}{\tau}]\underset{\sim}{n} = \underset{\sim}{0} \text{ on } S^*, \tag{2.40}$$

where [.] indicates the jump across the appropriate curve. A curve S (or its image S^*) carrying jump discontinuities in $\underset{\sim}{F}$, p, and $\underset{\sim}{\sigma}$ while preserving continuity of displacement and traction is called an equilibrium shock.

CHAPTER 3

FORMULATION OF THE MODE I CRACK PROBLEM

Let D, the undeformed cross-section of the cylindrical body, be the exterior of the straight-line segment L (Figure 2),

$$L = \{ \underset{\sim}{x} \mid x_2 = 0, \quad -b < x_1 < b \}; \tag{3.1}$$

L represents a traction-free crack of length 2b,

$$\sigma_{\alpha 2}(x_1, 0\pm) = 0, \quad -b < x_1 < b. \tag{3.2}$$[1]

The body is subjected at infinity to a uni-axial stress in a direction normal to the crack

$$\sigma_{22} \to \sigma_\infty, \ \sigma_{12} \to 0, \ \sigma_{11} \to 0, \ \sigma_{21} \to 0 \quad \text{as } x_1^2 + x_2^2 \to \infty. \tag{3.3}$$

The problem to be considered is the following: For the material characterized by (2.32), (2.39), we seek a suitably smooth deformation field $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$, a nominal stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ and a pressure field $p(\underset{\sim}{x})$, all on D, satisfying the field equations (2.2)-(2.4), (2.30), (2.8), (2.32), (2.39) and the traction-free and prescribed-load boundary conditions (3.2), (3.3). As mentioned previously, it is possible for the governing differential equations here to lose ellipticity.[2] This suggests that we seek the solution to the aforementioned boundary-value problem in a class of functions which are less smooth than that in which one would have otherwise sought the solution. Accordingly, we merely require that $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ be continuous and have piecewise continuous first and second partial derivatives on D. Furthermore, $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ is to be bounded near the crack-tip. Note that the preceding smoothness requirements admit

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1. As mentioned previously, Greek subscripts take the values 1,2 and repeated subscripts are summed.

2. The corresponding problem for a material which does not lose ellipticity has been considered by Stephenson [16].

the possibility of finite jump discontinuities in $\underset{\sim}{F}$ and $\underset{\sim}{\sigma}$ across curves in D. In the event that such a curve S exists, equilibrium considerations require that the nominal traction $\underset{\sim}{s}$ be continuous across it. Such a curve S across which the displacements and tractions are continuous but the displacement gradients are discontinuous is referred to as an "equilibrium shock" (or "shock").

Finally, let (r,θ) be local polar coordinates in the undeformed configuration as shown in Figure 2. Then

$$x_1 - b = r\cos\theta, \quad x_2 = r\sin\theta, \quad r > 0, \ -\pi < \theta < \pi. \tag{3.4}$$

On using (2.28) and (2.32), the equilibrium equations $(2.8)_1$ may be shown to imply

$$\begin{aligned} \frac{\partial p}{\partial r} &= \mu_s(k_e)H_r + \frac{\mu_t(k_e)-\mu_s(k_e)}{k_e}\left[g_{rr}\frac{\partial k_e}{\partial r} + \frac{1}{r^2}g_{r\theta}\frac{\partial k_e}{\partial \theta}\right], \\ \frac{\partial p}{\partial \theta} &= \mu_s(k_e)H_\theta + \frac{\mu_t(k_e)-\mu_s(k_e)}{k_e}\left[g_{r\theta}\frac{\partial k_e}{\partial r} + \frac{1}{r^2}g_{\theta\theta}\frac{\partial k_e}{\partial \theta}\right], \end{aligned} \tag{3.5}$$

where we have set

$$\begin{aligned} &H_r = \frac{\partial y_\alpha}{\partial r}\nabla^2 y_\alpha, \quad H_\theta = \frac{\partial y_\alpha}{\partial \theta}\nabla^2 y_\alpha, \quad g_{rr} = \frac{\partial y_\alpha}{\partial r}\frac{\partial y_\alpha}{\partial r}, \\ &g_{r\theta} = \frac{\partial y_\alpha}{\partial r}\frac{\partial y_\alpha}{\partial \theta}, \quad g_{\theta\theta} = \frac{\partial y_\alpha}{\partial \theta}\frac{\partial y_\alpha}{\partial \theta}, \end{aligned} \tag{3.6}$$

and

$$k_e^2 = g_{rr} + \frac{1}{r^2}g_{\theta\theta} - 3. \tag{3.7}$$

Similarly, (2.3), (2.4) yields

$$J = \frac{1}{r}\left(\frac{\partial y_1}{\partial r}\frac{\partial y_2}{\partial \theta} - \frac{\partial y_2}{\partial r}\frac{\partial y_1}{\partial \theta}\right) = 1, \tag{3.8}$$

and the boundary conditions (3.2) on the crack surfaces give

$$\mu_s(k_e)\frac{\partial y_1}{\partial\theta} + rp\frac{\partial y_2}{\partial r} = 0,\quad \mu_s(k_e)\frac{\partial y_2}{\partial\theta} - rp\frac{\partial y_1}{\partial r} = 0 \qquad (3.9)$$

$$\text{on } \theta = \pm\pi,\ 0 < r < 2b.$$

On using (3.6), (3.8), one can show that the boundary conditions (3.9) are equivalent to

$$g_{\theta\theta}g_{rr} = r^2,\ g_{r\theta} = 0,\ p = \mu_s(k_e)/g_{rr} \quad \text{on } \theta = \pm\pi,\ 0 < r < 2b. \qquad (3.10)$$

For the material characterized by (2.32), (2.39) under consideration here, the equilibrium equations (3.5), in view of (2.28), reduce to

$$\partial p/\partial r = \mu_o H_r,\ \partial p/\partial\theta = \mu_o H_\theta \qquad (3.11)$$

at points in D where $0 < k_e < k_o$ and to

$$\frac{\partial p}{\partial r} = \frac{\mu_o k_e^{2n-2}}{k_o^{2n-2}} H_r + (n-1)\frac{\mu_o}{k_o^{2n-2}} k_e^{2n-3}\left[2g_{rr}\frac{\partial k_e}{\partial r} + \frac{2}{r^2} g_{r\theta}\frac{\partial k_e}{\partial\theta}\right],$$

$$\frac{\partial p}{\partial\theta} = \frac{\mu_o k_e^{2n-2}}{k_o^{2n-2}} H_\theta + (n-1)\frac{\mu_o}{k_o^{2n-2}} k_e^{2n-3}\left[2g_{r\theta}\frac{\partial k_e}{\partial r} + \frac{2}{r^2} g_{\theta\theta}\frac{\partial k_e}{\partial\theta}\right],$$

(3.12)

at points where $k_e > k_o$.

CHAPTER 4

ASYMPTOTIC SOLUTIONS TO THE FIELD EQUATIONS

In this chapter we proceed to calculate asymptotic expressions for the deformation field near the right crack-tip. We will determine two solutions of the governing differential equations, each valid on some sub-domain of the crack-tip zone. In the next chapter, these solutions will be combined in a suitable manner in order to generate a solution of the complete (asymptotic) boundary-value problem. One of the deformation fields determined is a non-elliptic one, and satisfies (3.8), (3.12) in a zone H ahead of the crack-tip, (see Figure 3). The other is elliptic, and is determined from (3.8), (3.11) in zones E^+, E^- adjacent to the crack-faces. The regions E^+, E^-, and H are described by:

$$
\begin{aligned}
E^+ &= \{ (r,\theta) \,|\, 0 < \theta < \pi,\ 0 < r < r_0\}, \\
H &= \{ (r,\theta) \,|\, -\theta_0 < \theta < \theta_0,\ 0 < r < r_0\}, \qquad (4.1) \\
E^- &= \{ (r,\theta) \,|\, -\pi < \theta < 0,\ 0 < r < r_0\},
\end{aligned}
$$

where θ_0 $(0,\pi)$ is an angle to be determined.

4.1 Non-elliptic solution on H

We assume that the deformation field in H admits an asymptotic representation of the form

$$\hat{y}_\alpha(x_1,x_2) \sim r^m u_\alpha(\theta) \qquad \text{as } r \to 0, \qquad (4.2)$$

where $u_\alpha(\theta)$, $(u_\alpha u_\alpha \not\equiv 0)$, are smooth functions defined on some range $-\theta_0<\theta<\theta_0$, and m is a constant. We suppose also that the associated pressure field is of the form

$$p(x_1,x_2) \sim r^q p(\theta) \text{ as } r \to 0. \tag{4.3}$$

It is further assumed that the asymptotic identities (4.2), (4.3) may be formally differentiated as many times as necessary. Since we are interested in the asymptotic structure of the solution near the crack-tip, we seek the smallest values of m and q (i.e. the most singular terms)for which (4.2), (4.3) are asymptotically consistent with the governing boundary-value problem. Accordingly we first restrict attention to the case

$$m < 0. \qquad (4.4)^3$$

If a solution consistent with (4.4) can be found, we need not consider the range $m > 0$.

Equations (4.2), (4.4) and the incompressibility condition (3.8) yield

$$u_1 \dot{u}_2 - \dot{u}_1 u_2 = 0 \quad \text{on } -\theta_0 < \theta < \theta_0, \tag{4.5}$$

which may be integrated to give

$$u_\alpha = a_\alpha u(\theta) \quad \text{on } -\theta_0 < \theta < \theta_0. \tag{4.6}$$

Here a_1, a_2 are constants, $(a_\alpha a_\alpha \neq 0)$, and u $(\not\equiv 0)$ is a smooth function on $-\theta_0 < \theta < \theta_0$. In view of the nature of the prescribed loading (3.3), and the geometric symmetry present in the current problem, one expects the deformation field to possess the following symmetries:

3. As mentioned previously, the deformation field must remain bounded at the crack-tip. Despite (4.4), the deformation (4.2) is bounded when the crack-tip is approached from within the (eventual) hyperbolic domain (see Chapter 6).

$$y_1(r,\theta) = y_1(r,-\theta),\ y_2(r,\theta) = -y_2(r,-\theta) \text{ for } 0<r<r_0, -\pi<\theta<\pi. \quad (4.7)$$

Therefore, in view of (4.2), (4.6), (4.7), we find that one of the following must hold:

$$\begin{aligned} &\text{(i)} \quad u(\theta) \equiv 0 \text{ for } -\theta_0 < \theta < \theta_0 \text{ if } a_1 a_2 \neq 0, \\ &\text{(ii)} \quad u(\theta) = -u(-\theta) \text{ for } -\theta_0 < \theta < \theta_0 \text{ if } a_1 = 0,\ a_2 \neq 0, \qquad (4.8) \\ &\text{(iii)} \quad u(\theta) = u(-\theta) \text{ for } -\theta_0 < \theta < \theta_0 \text{ if } a_1 \neq 0,\ a_2 = 0. \end{aligned}$$

The first, of course, is inadmissible, and so we note that either a_1 or a_2 must vanish. From a physical point of view, one expects the deformation in the x_2-direction to be at least as "severe" (singular) as the deformation in the x_1-direction, which would imply that it is a_1, rather than a_2, that vanishes. Consequently in what follows we assume this to be the case, and so,

$$y_1 = o(r^m),\ y_2 \sim r^m u(\theta) \text{ as } r \to 0,\ u(\theta) = -u(-\theta) \text{ on } -\theta_0 < \theta < \theta_0. \quad (4.9)$$

With no loss of generality, the constant a_2 has been absorbed into $u(\theta)$. Note that

$$u(0) = 0. \quad (4.10)$$

Next, from (3.6), (3.7), (4.9), we find

$$k_e^2 \sim r^{2m-2} G(\theta),\ G(\theta) = m^2 \dot{u}^2(\theta) + u^2(\theta). \quad (4.11)$$

We assume that $G(\theta)$ does not vanish on $[-\theta_0, \theta_0]$. On using (4.9), (4.11) and (3.6), the equilibrium equations (3.12) in H, give

$$\begin{aligned} \partial p/\partial r &\sim \mu_0 k_0^{2-2n} m r^{2(m-1)n-1} u(\theta) Z(\theta), \\ \partial p/\partial \theta &\sim \mu_0 k_0^{2-2n} r^{2(m-1)n} u(\theta) Z(\theta), \end{aligned} \quad (4.12)$$

on $-\theta_0 < \theta < \theta_0$, in which we have set

$$Z(\theta) = G^{n-2}[G\{m^2 u + \ddot{u}\} + (n-1)\{2m(m-1)Gu + \dot{G}\dot{u}\}]. \quad (4.13)$$

The equations of equilibrium (4.12), after the elimination of the pressure p, yield the single equation

$$u(\theta)\dot{Z}(\theta) + \{1+ 2(1-m)m^{-1}n\}\dot{u}(\theta)Z(\theta)=0 \quad \text{on } -\theta_0<\theta<\theta_0. \tag{4.14}$$

Since $u(\theta)=-u(-\theta)$ on $(-\theta_0,\theta_0)$, it follows that $u(0)=\ddot{u}(0)=\dot{G}(0)=0$. Consequently, (4.13) implies that

$$Z(0) = 0. \tag{4.15}$$

Equations (4.14), (4.15) necessitate

$$Z(\theta) = 0 \quad \text{for } -\theta_0 < \theta < \theta_0, \tag{4.16}$$

which, by (4.13), is equivalent to

$$G\{m^2u+\ddot{u}\} + (n-1)\{2m(m-1)Gu+\dot{G}\dot{u}\}=0 \quad \text{for } -\theta_0<\theta<\theta_0. \tag{4.17}$$

Differential equations of the form (4.17) have been previously encountered and successfully analyzed by the phase-plane method (see, for example, Knowles and Sternberg [1]). We follow their procedure and introduce functions $\rho(\theta)$ (> 0) and $\psi(\theta)$ through

$$mu(\theta)=\rho(\theta)\cos\psi(\theta),\ \dot{u}(\theta)=\rho(\theta)\sin\psi(\theta) \quad \text{on } -\theta_0 < \theta < \theta_0. \tag{4.18}$$

Eliminating $u(\theta)$ from (4.18) gives

$$\dot{\rho}(\theta)/\rho(\theta) = \{m+\dot{\psi}(\theta)\} \tan \psi(\theta). \tag{4.19}$$

On the other hand (4.11), (4.17) and (4.18) provide

$$(2n-1)(\dot{\rho}(\theta)/\rho(\theta))\tan\psi(\theta) + \{\dot{\psi}(\theta) + m + 2(m-1)(n-1)\} = 0. \tag{4.20}$$

Equation (4.19) can now be used to eliminate $\dot{\rho}/\rho$ in (4.20) which leads to

$$\dot{\psi}\ \{n/(1-n)+\cos 2\psi\} + \{w_0+\cos2\psi\} = 0 \quad \text{on } -\theta_0 < \theta < \theta_0, \tag{4.21}$$

where we have set

$$w_0 = 1 - m(1-2n)/(1-n) > 1. \tag{4.22}$$

The symmetry condition (4.10), in view of (4.18) requires that

$$\psi(0) = \pi/2. \tag{4.23}$$

Integrating (4.21) and using (4.23) gives $\psi(\theta)$ implicitly as

$$\theta = -(\psi - \frac{\pi}{2}) + \frac{(1-m)(1-2n)}{1+(1-m)(1-2n)} w_1 \tan^{-1}[w_1 \tan(\psi - \frac{\pi}{2})], \tag{4.24}$$

where

$$w_1 = ((w_0+1)/(w_0-1))^{1/2} > 0, \tag{4.25}$$

and w_0 is given by (4.22). Equation (4.24) cannot be inverted explicitly to furnish $\psi = \psi(\theta)$. A typical graph of θ vs. ψ as described by (4.24) is shown in Figure 4. It indicates that $\psi(\theta)$ is multi-valued in general but that a single-valued branch, which satisfies $\psi(0) = \pi/2$, may be chosen. This corresponds to the solid portion of the curve in Figure 4. From here on, it is this single-valued branch that we consider. The graph indicates that $\psi(\theta)$ increases monotonically on $(-\theta_0, \theta_0)$ from the value $\psi_0 = 1/2 \cos^{-1}(-n/1-n)$ to $\pi - \psi_0$ and is antisymmetric about $\theta = 0$, $\psi = \pi/2$. Furthermore, we find from (4.24) that

$$\theta_0 = -\tan^{-1}(\sqrt{1-2n}) + \frac{(1-m)(1-2n)}{1+(1-m)(1-2n)} w_1 \tan^{-1}\{w_1\sqrt{1-2n}\}, \; 0 < \theta_0 < \pi. \tag{4.26}$$

We now proceed to find $u(\theta)$ in terms of $\psi(\theta)$. Eliminating $\rho(\theta)$ from (4.18) gives

$$\dot{\psi}\,\dot{u} - mu\dot{\psi}\tan\psi = 0 \quad \text{on } -\theta_0 < \theta < \theta_0, \tag{4.27}$$

so that (4.21) can be written in the form

$$\{w_0 + \cos 2\psi\}\dot{u} + mu\dot{\psi}\{n/(1-n) + \cos 2\psi\}\tan\psi = 0. \tag{4.28}$$

Integrating (4.28) yields the following expression for $u(\theta)$:

$$u(\theta) = a \cos\psi(\theta)\{\cos 2\psi(\theta) + w_0\}^{-(1-m)/2}, \tag{4.29}$$

with $\psi(\theta)$ being the single-valued inverse of (4.24) on $-\theta_0 < \theta < \theta_0$. In equation (4.29), a is a positive constant and w_0 is given by (4.22).

For later purposes it will turn out that we need an explicit expression for $u(\theta)$ at small values of θ. This can be readily obtained by using a Taylor expansion of $u(\theta)$ about $\theta = 0$. From (4.10), (4.18), (4.19), (4.23), we have

$$u(0) = 0,\ \dot{u}(0) = \rho(0), \tag{4.30}$$

$$\ddot{u}(0) = \dot{\rho}(0), \tag{4.31}$$

$$\dddot{u}(0) = \ddot{\rho}(0) - \rho(0)\ (\dot{\psi}(0))^2, \tag{4.32}$$

while (4.21), (4.23) provide

$$\dot{\psi}(0) = -m. \tag{4.33}$$

On the other hand $(4.18)_1$, (4.29) give

$$\rho(\theta) = am\{\cos 2\psi(\theta) + w_o\}^{-(1-m)/2} \text{ on } -\theta_o < \theta < \theta_o, \tag{4.34}$$

so that

$$\rho(0) = am\{\frac{m(2n-1)}{(1-n)}\}^{-(1-m)/2}, \tag{4.35}$$

$$\dot{\rho}(0) = 0,\ \ddot{\rho}(0) = \frac{2(1-m)(1-n)m}{(1-2n)}\,\rho(0). \tag{4.36}$$

Therefore, (4.30)-(4.36) indicate that

$$u(\theta) = \rho(0)[\theta - \frac{m\{2(1-n)+m(4n-3)\}}{2n-1}\frac{\theta^3}{6}] + O(\theta^5) \text{ as } \theta \to 0. \tag{4.37}$$

At this stage the displacement field in H has been found to be

$$y_1 = o(r^m), \quad y_2 = r^m u(\theta) + o(r^m) \quad \text{as } r \to 0, \tag{4.38}$$

where $u(\theta)$ is given by (4.24) and (4.29). We now proceed to obtain a better estimate on y_1. Accordingly, we suppose that

$$y_1 = r^{\ell} v(\theta) + o(r^{\ell}),\ y_2 = r^m u(\theta) + o(r^m) \quad \text{as } r \to 0, \tag{4.39}$$

n which ℓ (>m) is an unknown exponent and $v(\theta)$ is an unknown smooth function. Symmetry (4.7), requires that $v(\theta) = v(-\theta)$. The incompressibility condition (3.8), on using (4.39), leads to

$$r^{\ell+m-2}\,(\ell\dot{v}u - m v\dot{u}) + o\,(r^{\ell+m-2}) = 1, \tag{4.40}$$

so that necessarily,

$$\ell \leq 2-m. \tag{4.41}$$

One must consider the two cases $\ell < 2-m$ and $\ell = 2-m$ separately. First, if $\ell < 2-m$, (4.40) implies that

$$\ell v\dot{u} - m\dot{v}u = 0 \quad \text{on } -\theta_0 < \theta < \theta_0, \tag{4.42}$$

which may be integrated to give $v(\theta)$ as

$$v(\theta) = c|u(\theta)|^{\ell/m}, \tag{4.43}$$

where c is a constant. Since $\ell/m < 1$ and $u(0) = 0$, $\dot{u}(0) \neq 0$, it follows that $\dot{v}(0)$ is bounded only if $\ell = 0$. Thus one finds that

$$v(\theta) = \text{constant}, \ \ell = 0. \tag{4.44}$$

This corresponds to a rigid-body translation in the x_1-direction.

Consequently, it is the second possibility,

$$\ell = 2-m, \tag{4.45}$$

that is relevant. In this case, (4.40) implies that $(2-m)v\dot{u}-m\dot{v}u = 1$ on $(-\theta_0, \theta_0)$. The solution $v(\theta)$ of this equation, which is bounded at $\theta = 0$ is

$$v(\theta) = -\frac{1}{m}[u(\theta)]^{(2-m)/m}\int_0^{\theta}[u(\phi)]^{-2/m}d\phi \quad \text{for } 0 < \theta < \theta_0, \tag{4.46}$$

with symmetry, $v(\theta) = v(-\theta)$, defining $v(\theta)$ on $(-\theta_0, 0)$. From (4.46), (4.35)-(4.37), we find that for small angles θ, $v(\theta)$ can be approximated to be

$$v(\theta) = \frac{1}{(2-m)\rho(0)}\left[1+\frac{m(m-2)\{2(1-n)+m(4n-3)\}}{2(2n-1)(3m-2)}\theta^2\right] + O(\theta^4). \qquad (4.47)$$

In <u>summary</u>, we have found that the deformation field near the crack-tip in region H is given, asymptotically, by (4.39). Here the exponent m is yet to be determined. The function $u(\theta)$ is known (except for the amplitude a) and is given by (4.24), (4.29). The exponent ℓ = 2-m and the function $v(\theta)$ is given by (4.46). The region H is defined by (4.1), (4.26).

4.2. <u>Elliptic solution on $E^{\pm}$</u>

Next, in E^+, we take $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ to be of the form

$$y_\alpha \sim b_\alpha + r^\nu u_\alpha(\theta) \quad \text{as } r \to 0. \qquad (4.48)$$

Since we have ellipticity here $k_e < k_0$, and this requires $\nu > 1$. Furthermore, not both u_1, u_2 can vanish identically. Incompressibility (3.8) together with (4.48) provides

$$\nu r^{2\nu-2}\,(u_1\dot{u}_2-u_2\dot{u}_1) + o(r^{2\nu-2}) = 1 \quad \text{as } r \to 0. \qquad (4.49)$$

Since $2\nu-2 > 0$, (4.49) requires that $\nu = 1$. Thus

$$u_1\dot{u}_2 - u_2\dot{u}_1 = 1 \quad \text{on } 0 < \theta < \pi. \qquad (4.50)$$

On the other hand the equilibrium equations (3.11), on using (3.6), (4.48) and ν=1, lead to

$$\partial p/\partial r \sim \mu_0 u_\alpha(\ddot{u}_\alpha+u_\alpha)/r, \quad \partial p/\partial\theta \sim \mu_0\, \dot{u}_\alpha(\ddot{u}_\alpha+u_\alpha). \qquad (4.51)$$

Eliminating the pressure p from (4.51) yields

$$\frac{d}{d\theta}\,(u_\alpha\ddot{u}_\alpha) + 2\dot{u}_\alpha u_\alpha = 0,$$

which may be integrated[4] to give

4. The constant of integration can be shown to be zero by using (4.51), (4.53) and the boundary condition $(3.10)_3$.

$$u_1(\ddot{u}_1+u_1) + u_2(\ddot{u}_2+u_2) = 0 \quad \text{on } 0 < \theta < \pi. \tag{4.52}$$

On the other hand, differentiating (4.50) with respect to θ gives

$$u_1(\ddot{u}_2+u_2) - u_2(\ddot{u}_1+u_1) = 0 \quad \text{on } 0 < \theta < \pi. \tag{4.53}$$

Since $u_1^2 + u_2^2 \neq 0$, the pair of equations (4.52), (4.53) require that

$$\ddot{u}_1+u_1 = 0 \ , \ \ddot{u}_2+u_2 = 0 \quad \text{on } 0 < \theta < \pi. \tag{4.54}$$

Next, on using (4.48) with ν=1 and (3.6), the boundary conditions $(3.10)_1$, $(3.10)_2$, on the crack-faces give

$$u_\beta u_\beta \dot{u}_\alpha \dot{u}_\alpha = 1, \quad u_\alpha \dot{u}_\alpha = 0 \quad \text{on } \theta = \pi. \tag{4.55}$$

Thus on solving (4.54), subject to (4.50), (4.55), one finds

$$\begin{aligned} u_1(\theta) &= a_o \cos\theta - (b_o/(a_o^2+ b_o^2)) \sin\theta, \\ u_2(\theta) &= b_o \cos\theta + (a_o/(a_o^2+b_o^2)) \sin\theta, \end{aligned} \tag{4.56}$$

for $0 < \theta < \pi$, where a_o, b_o are unknown constants, $a_o^2 + b_o^2 \neq 0$.

In order to calculate the leading terms for the stress components in E^+, it turns out that we need an expression for the deformation to an order higher than in the foregoing. Therefore, we now assume

$$y_\alpha \sim b_\alpha + ru_\alpha(\theta) + r^k f_\alpha(\theta) \quad \text{as } r \to 0, \tag{4.57}$$

where $k > 1$ is a constant and $f_\alpha(\theta)$ are smooth functions on $[0,\pi]$. The higher order consideration (4.57) yields, through the incompressibility condition (3.8),

$$u_1\dot{f}_2 - u_2\dot{f}_1 + k(\dot{u}_2 f_1 - \dot{u}_1 f_2) = 0 \quad \text{on } 0 < \theta < \pi, \tag{4.58}$$

while the equilibrium equations (3.11), together with (3.6), give

$$\partial p/\partial r \sim \mu_o r^{k-2}\, u_\alpha(\ddot{f}_\alpha+k^2 f_\alpha), \ \partial p/\partial\theta \sim \mu_o r^{k-1}\, \dot{u}_\alpha(\ddot{f}_\alpha+k^2 f_\alpha). \tag{4.59}$$

This indicates that the pressure field is of the form $p(x_1, x_2) \sim p_0 + r^q f_3(\theta)$ as $r \to 0$, where $q = k - 1 > 0$ and

$$(k-1)f_3 = \mu_0 u_\alpha(\ddot{f}_\alpha + k^2 f_\alpha), \quad \dot{f}_3 = \mu_0 \dot{u}_\alpha(\ddot{f}_\alpha + k^2 f_\alpha) \quad \text{on } 0 < \theta < \pi. \tag{4.60}$$

The boundary conditions $(3.10)_1$ and $(3.10)_2$, in view of (3.6), (4.55), (4.57), yield

$$\begin{aligned} &u_\alpha \dot{u}_\beta (u_\alpha \dot{f}_\beta + k\dot{u}_\beta f_\alpha) = 0 \quad \text{on } \theta = \pi, \\ &u_\alpha \dot{f}_\alpha + k\dot{u}_\alpha f_\alpha = 0 \quad \text{on } \theta = \pi, \end{aligned} \tag{4.61}$$

while the third of (3.10), together with $(4.59)_1$, requires that

$$u_\gamma u_\gamma u_\alpha u_\alpha u_\beta (\ddot{f}_\beta + k^2 f_\beta) + 2k(k-1) f_\alpha u_\alpha = 0 \quad \text{on } \theta = \pi. \tag{4.62}$$

In summary, we have found that the deformation field near the crack-tip in region E^+ is given, asymptotically, by (4.57). Here the exponent $k > 1$ is yet to be determined. The functions $u_1(\theta)$, $u_2(\theta)$ are given by (4.56) while $f_1(\theta)$, $f_2(\theta)$ are to be determined from (4.58), (4.60)-(4.62). Symmetry (4.7), gives the deformation in E^-.

CHAPTER 5

MATCHING

In the previous section we found two solutions to the governing differential equations, each valid on some subdomain of the crack-tip zone. Figure 3 shows the domain of validity of each solution. In this chapter, we will combine these deformation fields in a suitable manner in order to obtain a solution of the complete (asymptotic) boundary-value problem.

It is sufficient to restrict attention to the upper half-plane. One can show from (4.24), (4.26), (4.29), (4.39), (4.45), (4.46), (4.56), (4.57) that[5] y_α^E does not match continuously onto y_α^H across the line $\theta = \theta_0$. Thus we are led to seek two curves S^+, S^- lying in H across which the deformation is continuous, (see Figure 5). These curves are defined in a neighborhood of the right crack-tip by

$$S^\pm: \quad \theta = \pm \hat{\theta}(r), \quad 0 < r < r_0, \tag{5.1}$$

where we suppose that

$$\hat{\theta}(r) = Ar^s + o(r^s) \quad \text{as } r \to 0. \tag{5.2}$$

Here $A > 0$ and $s \geq 0$ are constants. Note that if $s = 0$ the curves $S^\pm$ make angles $\pm A$ with the x_1-axis at the crack-tip. In this case one must have $0 < A < \theta_0$. If $s > 0$, the curves $S^\pm$ are tangent to the x_1-axis at the crack-tip.

The crack-tip zone may now be divided into three zones $\mathcal{E}^+$, $\mathcal{E}^-$ and $\mathcal{H}$ as follows (see Figure 5):

5. The superscripts E and H will be used in this section to denote quantities associated with the elliptic and non-elliptic solutions, respectively.

$$\mathcal{E}^+ = \{(r,\theta)\,|\,\hat{\theta}(r) < \theta < \pi,\ \ 0 < r < r_0\},$$
$$\mathcal{H} = \{(r,\theta)\,|\,-\hat{\theta}(r) < \theta < \hat{\theta}(r),\ \ 0 < r < r_0\}, \qquad (5.3)$$
$$\mathcal{E}^- = \{(r,\theta)\,|\,-\pi < \theta < -\hat{\theta}(r),\ \ 0 < r < r_0\}.$$

The deformation field near the crack-tip is now taken to be

$$y_\alpha = \begin{cases} \overset{E}{y_\alpha} \text{ on } \mathcal{E}^+, \mathcal{E}^-, \\ \overset{H}{y_\alpha} \text{ on } \mathcal{H}, \end{cases} \qquad (5.4)$$

where y_α^H and y_α^E were found in Sections 3.1 and 3.2 respectively. It is clear from the preceding analysis that (5.4) does in fact satisfy all requirements, provided that this deformation field and the corresponding traction field are continuous across S^+ and S^-, i.e.

$$y_\alpha^E = y_\alpha^H \text{ on } S^+, \qquad (5.5)$$

$$s_\alpha^E = s_\alpha^H \text{ on } S^+. \qquad (5.6)$$

First we match y_1 across the shock and find through (5.2), (4.39), (4.45), (4.47), (4.57), (4.56), (5.5), that

$$b_1 + a_o r \sim \frac{1}{(2-m)\rho(0)}\, r^{2-m} \quad \text{as } r \to 0, \qquad (5.7)$$

and so, recalling that $m < 0$,

$$b_1 = 0, \quad a_o = 0. \qquad (5.8)$$

Similarly, matching y_2 across the shock leads to

$$b_2 + b_0 r + r^k f_2(0) \sim A\rho(0) r^{s+m} \quad \text{as } r \to 0. \qquad (5.9)$$

Observe from (4.57) that b_2 denotes the crack-opening displacement. Thus we expect $b_2 > 0$ and so (5.9) gives

$$b_2 = A\rho(0), \ s = -m > 0. \qquad (5.10)$$

We may also match the deformation y_1 to a higher order to find, through (5.2), (4.39), (4.45), (4.47), (4.57), (4.56), (5.5) and (5.8), that

$$-b_o^{-1} Ar^{1-m} + r^k f_1(0) \sim \frac{1}{(2-m)\rho(0)} r^{2-m} \quad \text{as } r \to 0. \quad (5.11)$$

Assuming $f_1(0) \neq 0$, this gives

$$k = 1-m, \; f_1(0) = Ab_o^{-1}. \quad (5.12)$$

Next we compute expressions for the Piola stress components in E^+ and H. First, we calculate the pressure field using (3.12), (3.11), (3.6), (4.39), (4.45), (4.56), (4.57), (5.8) to get, in H and E^+, respectively $p^H(r,\theta) = O(r^{2(m-1)(n-2)})$,

$$p^E(r,\theta) \sim p_o + \mu_o \frac{r^{k-1}}{k-1} \{-b_o^{-1} \sin\theta(\ddot{f}_1 + k^2 f_1) + b_o \cos\theta(\ddot{f}_2 + k^2 f_2)\}, \quad (5.13)$$

where p_o is a constant. We now calculate the stress fields through (2.2), (2.30), (2.28), (2.32), (2.39) to get, in H,

$$\begin{aligned}
\overset{H}{\sigma}_{11} &= O(r^{(m-1)(2n-3)}), \\
\overset{H}{\sigma}_{12} &= O(r^{(m-1)(2n-3)}), \\
\overset{H}{\sigma}_{21} &= \mu_o k_o^{2-2n} G(\theta)^{n-1} [mu\cos\theta - \dot{u}\sin\theta] r^{(m-1)(2n-1)} \\
&\quad + o(r^{(m-1)(2n-1)}), \\
\overset{H}{\sigma}_{22} &= \mu_o k_o^{2-2n} G(\theta)^{n-1} [mu\sin\theta + \dot{u}\cos\theta] r^{(m-1)(2n-1)} \\
&\quad + o(r^{(m-1)(2n-1)}),
\end{aligned} \quad (5.14)$$

and <u>in</u> E^+,

$$\begin{aligned}
\overset{E}{\sigma}_{11} &= \{\mu_o(kf_1\cos\theta - \dot{f}_1\sin\theta) - p_o(kf_2\sin\theta + \dot{f}_2\cos\theta)\} r^{k-1} + o(r^{k-1}), \\
\overset{E}{\sigma}_{12} &= (p_o b_o - \mu_o b_o^{-1}) + [\mu_o(kf_1\sin\theta + \dot{f}_1\cos\theta) + p_o(kf_2\cos\theta - \dot{f}_2\sin\theta) + \\
&\quad (b_o\mu_o/(k-1))\{-b_o^{-1}\sin\theta(\ddot{f}_1 + k^2 f_1) + b_o\cos\theta(\ddot{f}_2 + k^2 f_2)\}] r^{k-1} + o(r^{k-1}), \\
\overset{E}{\sigma}_{21} &= \mu_o b_o - p_o b_o^{-1} + O(r^{k-1}),
\end{aligned} \quad (5.15)$$

$$\overset{E}{\sigma}_{22} = \{\mu_o(kf_2\sin\theta + \dot{f}_2\cos\theta) - p_o(kf_1\cos\theta - \dot{f}_1\sin\theta)\}r^{k-1} + o(r^{k-1}).$$

The traction-free condition $\sigma_{12} = \sigma_{22} = 0$ on the crack requires that

$$p_o = \mu_o b_o^{-2} \tag{5.16}$$

as well as that (4.61), (4.62) hold.

In calculating the Piola tractions, note that the unit normal vector $\underset{\sim}{N}$ to S^+ has components

$$N_1 = -A(1+s)r^s + o(r^s),\ N_2 = 1 + O(r^{2s}). \tag{5.17}$$

We may now match the nominal tractions across S^+. On using (2.9), (5.14)-(5.17) we find that the continuity of $s_1 = \sigma_{1\beta}N_\beta$ requires

$$\{\mu_o\dot{f}_1(0) + p_o kf_2(0) + \frac{b_o^2\mu_o}{k-1}(\ddot{f}_2(0) + k^2f_2(0))\}r^{k-1} + o(r^{k-1}) = 0. \tag{5.18}$$

This implies that

$$\mu_o\dot{f}_1(0) + p_o kf_2(0) + \frac{b_o^2\mu_o}{k-1}(\ddot{f}_2(0) + k^2f_2(0)) = 0. \tag{5.19}$$

Similarly, on matching the traction component $s_2 = \sigma_{2\beta}N_\beta$, we get

$$\mu_o k_o^{2-2n}\ \rho(0)^{2n-1}\ r^{(m-1)(2n-1)}$$
$$\sim -(\mu_o b_o - p_o b_o^{-1})A(1-m)r^{-m} + (\mu_o f_2(0) - p_o kf_1(0))r^{-m}, \tag{5.20}$$

from (2.9), (5.14), (5.15), (5.17), (5.10), which provides

$$m = 1 - 1/2n, \tag{5.21}$$

$$\mu_o\dot{f}_2(0) - p_o kf_1(0) = A(1-m)(\mu_o b_o - p_o b_o^{-1}) + \mu_o k_o^{2-2n}\rho(0)^{2n-1}. \tag{5.22}$$

CHAPTER 6

SUMMARY OF RESULTS

In summary, we have investigated the near-tip displacement and stress fields under plane strain mode I conditions for the piecewise power-law material characterized by (2.39), (2.32). We found two shock lines S^+, S^- emanating from the crack-tip (Figure 5) and described asymptotically by

$$\theta \sim b_0 f_1(0) r^{-1+1/2n}. \tag{6.1}$$

The curves S^+, S^- separate the crack-tip zone into regions $\mathcal{E}^+$, $\mathcal{E}^-$ (adjacent to the crack-faces) and $\mathcal{H}$ (ahead of the crack), defined by (5.3), (5.1), (6.1). Note that S^+ and S^- are tangential to the x_1-axis at the crack-tip. The deformation and stress fields vary smoothly within $\mathcal{E}^+$, $\mathcal{E}^-$ and $\mathcal{H}$; the deformation and the tractions are continuous across S^+ and S^-. The deformation gradient and stress are discontinuous across these curves.

The <u>deformation field</u> in the elliptic zone $\mathcal{E}^+$ is given by

$$\left.\begin{aligned} y_1 &\sim -rb_0^{-1}\sin\theta + r^{1/2n} f_1(\theta), \\ y_2 &\sim b_2 + rb_0\cos\theta + r^{1/2n} f_2(\theta), \end{aligned}\right\} \text{ on } \mathcal{E}^+. \tag{6.2}$$

Here $b_2 = b_0 f_1(0)\rho(0) > 0$ and $b_0 > 0$ are constants with $\rho(0)$ given by (4.35), (5.21). Further, the smooth functions f_1, f_2 (and f_3) are given by

$$f_i(\theta) = k_0^{2-2n}\ \rho(0)^{2n-1}\ g_i(\theta), \quad i = 1,\ 2 \text{ and } 3, \tag{6.3}$$

where according to (4.58), (4.60)-(4.62), (5.19), (5.22), (4.56), (5.8), (5.12), (5.16) and (5.21), g_1, g_2 (and g_3) are to be found from the linear boundary-value problem

$$
\begin{aligned}
&b_0\cos\theta\, \dot g_1 + b_0^{-1}\sin\theta\, \dot g_2 + k(b_0\sin\theta\, g_1 - b_0^{-1}\cos\theta\, g_2) = 0 \quad \text{on } (0,\pi),\\
&\mu_0\{-b_0^{-1}\sin\theta(\ddot g_1+k^2 g_1) + b_0\cos\theta(\ddot g_2+k^2 g_2)\} = (k-1)g_3 \text{ on } (0,\pi),\\
&\mu_0\{b_0^{-1}\cos\theta(\ddot g_1+k^2 g_1) + b_0\sin\theta(\ddot g_2+k^2 g_2)\} = -\dot g_3 \quad \text{on } (0,\pi),\\
&\ddot g_2 + k\{k+2(k-1)b_0^{-4}\}g_2 = 0 \quad \text{on } \theta = 0 \text{ and } \theta = \pi, \qquad\qquad (6.4)\\
&\dot g_2 - kb_0^{2}g_1 = 1 \quad \text{on } \theta = 0,\ \dot g_2 - kb_0^{-2}g_1 = 0 \quad \text{on } \theta = \pi,
\end{aligned}
$$

where $k = 1/2n$. We have been unable to determine a closed form solution of this linear boundary-value problem. On the other hand, for a given value of b_0, it may be solved numerically for each value of n. The leading order homogeneous deformation characterized by (6.2) describes a state of uni-axial tension in the y_2-direction. The principal stretches of this deformation are b_0, b_0^{-1}, and in view of ellipticity, are restricted by $|b_0-b_0^{-1}| < k_0$. The Piola stress components in $\mathcal{E}^+$ are given by (5.15) with $p_0=\mu_0 b_0^{-2}$, $k = 1/2n$.

In the hyperbolic region $\mathcal{H}$, the deformation field is given by

$$
\left.\begin{aligned}
y_1 &\sim r^{1+1/2n}v(\theta),\\
y_2 &\sim r^{1-1/2n}u(\theta),
\end{aligned}\right\} \text{ on } \mathcal{H}, \qquad (6.5)
$$

with $u(\theta)$ given by

$$
\begin{aligned}
&u(\theta) = a\cos\psi(\theta)\{\cos 2\psi(\theta) + (1-2n+2n^2)/(2n(1-n))\}^{-1/4n},\\
&\theta = -(\psi-\pi/2) + \tan^{-1}\{(1-2n)^{-1}\tan(\psi-\pi/2)\},
\end{aligned} \qquad (6.6)
$$

and $v(\theta)$ given by (4.46) with $m = 1-1/2n$. On using the fact that the shock-lines $S^{\pm}$ are tangential to the x_1-axis at the origin, and $u(0)=0$, it may be easily verified that the deformation (6.5) is bounded in $\mathcal{H}$. The Piola stress components in $\mathcal{H}$ are given by (5.14) with $m = 1-1/2n$.

Finally we note that the asymptotic expressions for the various field quantities determined here are known completely in terms of two unknown constants a and b_0. Necessarily, an asymptotic solution procedure cannot provide the value of both of these constants since they must depend on the global problem. It may however be possible to determine the value of one of them through a higher order calculation.

PART II

ESTIMATION OF ENERGY RELEASE RATES: AN ALTERNATIVE TO INTERPOLATION

CHAPTER 7

INTRODUCTION

Fracture mechanics is focused in two principal directions, the development of phenomenological explanations of crack extension, and the description of the micromechanical process of material separation on the microscale. In the first, emphasis is placed on predicting crack extension behavior, usually in terms of a single parameter which characterizes the near-tip stress field. Linear elastic fracture mechanics is a case in point, where predictions are made in terms of the elastic stress-intensity factor K, which serves to characterize the influence of applied loads and geometry on the near-tip field under elastic (or even small-scale yielding) conditions. Hence, the analytical problem in elastic fracture mechanics is to determine the stress intensity-factor. Rice showed that in the elastic range, the value of the so-called J -integral is proportional to the square of the stress intensity-factor [17].

The J - integral continues to be path independent even for nonlinearly elastic materials. It is the central theoretical quantity behind nonlinear fracture mechanics for rate - independent materials under monotonic loading, since it can be regarded, from a physical point of view, as a measure of the intensity of the crack-tip singularity fields. Also, it represents the energy release rate (see, for example, the recent survey article, Hutchinson [18]).

If a crack in a deformed solid is to propagate, it requires energy and the "energy release rate" represents the rate at which energy is made available to the crack for this purpose. Experiments carried out by Begley and Landes [19] have demonstrated the potential that this parameter has for use as a fracture initiation criterion. The importance of the energy release rate as a fracture initiation criterion has prompted many studies to calculate its value under nonlinear conditions. These have primarily consisted of numerical and experimental investigations, e.g. [20-23], as well as analytical approximations based on interpolation, e.g. [24,25]. The only exact solution appears to be that due to Amazigo [26] for a pure power - law material under conditions of anti - plane shear.

In his paper [24], Shih proposed some approximate (but accurate) formulas for estimating the relation between the path - independent integral, J, the applied stress, the load point displacement, and the constitutive parameters for cracked bodies of strain hardening elastic -plastic materials. The formula makes use of results from the elastic and fully plastic solutions so as to interpolate from the small - scale yielding range to the fully plastic range. However, this interpolation scheme involves coefficients that are determined through a finite element analysis. Moreover, different constitutive descriptions (e.g. a Ramberg - Osgood material as against power - law material) require different schemes.

Recently, Abeyaratne [27] suggested an analytical estimation procedure for calculating the value of the energy release rate associated with Mode I and Mode III crack problems for an infinite body composed of a nonlinearly elastic material. The results are accurate even up to moderately large loads. This scheme does not suffer from the drawbacks mentioned above and adopts an approximation that is a natural extension of the small-scale yielding concept. The small-scale yielding approximation for the energy release rate uses the argument that when the applied loads are small, the nonlinearities in the problem are confined to a small region near the crack tips and that the linear elastic solution provides a good approximation elsewhere. The present scheme, essentially, replaces the linear elastic solution in the preceding argument by a better approximation. Consider, for example, a Mode III crack problem under remotely applied uniform shear. In the absence of the crack, the body is in a homogeneous state of (finite) anti-plane shear. If one were to assume that the presence of the crack causes only a small perturbation of this homogeneous state, one can then carry out an analysis based on the theory of small deformations superposed on a large deformation. The results based on such an approximation would clearly be invalid in the vicinity of the crack-tips, but one would expect it to provide a reasonable approximation at points distant from the crack. Since our interest here lies solely in estimating the energy release rate, and since this can be written in the form of a path independent line integral J taken over a contour that is far from the crack-tips over most of its length, one expects such an approximation to be suitable for our purposes, at least when confined to moderate levels of loading.

The analytical scheme in [27] is appropriate for general constitutive relations and is not restricted to a particular model. Consequently, one does not need to go through an elaborate calculation each time the material changes. This important observation was not made in [27] and it is this point that we wish to make in this study. We use a center - cracked strip in Mode III to illustrate this.

CHAPTER 8

PRELIMINARIES PERTAINING TO FINITE ANTI-PLANE STRAIN

Suppose that an isotropic, homogeneous, incompressible elastic body occupies the region R in the undeformed state, and consider the deformation

$$\underset{\sim}{y} = \underset{\sim}{y}\,(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}\,(\underset{\sim}{x}) \quad \text{on } R, \tag{8.1}$$

where $\underset{\sim}{x}$ is the position vector in R, $\underset{\sim}{y}$ the position vector in the deformation image R^* of R, and $\underset{\sim}{u}$ is the displacement vector. This mapping is one-to one. The deformation gradient tensor $\underset{\sim}{F}$ is defined by

$$\underset{\sim}{F}\,(\underset{\sim}{x}) = \nabla\,\underset{\sim}{y}\,(\underset{\sim}{x}); \tag{8.2}$$

the incompressibility of the material requires that

$$J = \det \underset{\sim}{F}\,(\underset{\sim}{x}) = 1 \quad \text{on } R, \tag{8.3}$$

for every admissible deformation, so that the deformation is locally volume-preserving. let

$$\underset{\sim}{G} = \underset{\sim}{F}\,\underset{\sim}{F}^T \tag{8.4}$$

be the left Cauchy - Green tensor for the deformation, and set

$$I_1 = \operatorname{Tr} \underset{\sim}{G},\ I_2 = 1/2\,[(\operatorname{Tr} \underset{\sim}{G})^2 - \operatorname{Tr}\,(\underset{\sim}{G}^2)], \tag{8.5}$$

so that I_1, I_2 are two of the three principal invariants of $\underset{\sim}{G}$. Here a superscript T stands for transpose and the third invariant is $I_3 = \det \underset{\sim}{G} = J^2$, and is equal to 1 by the incompressibility condition (8.3). At the undeformed state $I_1 = I_2 = 3$, and $I_1 > 3$, $I_2 > 3$ for all deformations.

The mechanical response of an isotropic, incompressible, hyperelastic material is governed by its strain energy function per unit undeformed volume W, which depends only on the strain invariants I_1, I_2: $W = W(I_1, I_2)$. If $\underset{\sim}{\tau}$ is the Cauchy stress tensor (force per unit deformed surface area), the stress - deformation relation is

$$\underset{\sim}{\tau} = 2[\frac{\partial W}{\partial I_1}\underset{\sim}{G} + \frac{\partial W}{\partial I_2}(I_1\underset{\sim}{1} - \underset{\sim}{G})\underset{\sim}{G}] - p\underset{\sim}{1} \tag{8.6}$$

where $\underset{\sim}{1}$ is the unit tensor and the scalar p is an arbitrary pressure whose presence is necessary to accomodate the incompressibility constraint (8.3). The Piola stress $\underset{\sim}{\sigma}$ (force per unit undeformed area) is then related to $\underset{\sim}{\tau}$ according to

$$\underset{\sim}{\sigma} = \underset{\sim}{\tau}(\underset{\sim}{F}^T)^{-1}. \tag{8.7}$$

From (8.6), (8.7), and (8.4) it follows that

$$\underset{\sim}{\sigma} = 2[\frac{\partial W}{\partial I_1}\underset{\sim}{F} + \frac{\partial W}{\partial I_2}(I_1\underset{\sim}{1} - \underset{\sim}{G})\underset{\sim}{F}] - p(\underset{\sim}{F}^T)^{-1}. \tag{8.8}$$

The differential equations of equilibrium may be expressed either in terms of $\underset{\sim}{\tau}$, regarded as a function on R^*, or in terms of $\underset{\sim}{\sigma}$, regarded as a function on R. In the absence of body forces, these equations are

$$\begin{aligned} \operatorname{div}\underset{\sim}{\tau} &= \underset{\sim}{0} \quad \text{on } R^*, \\ \operatorname{div}\underset{\sim}{\sigma} &= \underset{\sim}{0} \quad \text{on } R. \end{aligned} \tag{8.9}$$

The tensor $\underset{\sim}{\tau}$ is symmetric ($\underset{\sim}{\tau} = \underset{\sim}{\tau}^T$); $\underset{\sim}{\sigma}$ in general is not. If S^* is a traction free portion of the boundary of the region R^* occupied by the deformed body, $\underset{\sim}{\tau}$ satisfies

$$\underset{\sim}{\tau}\,\underset{\sim}{n} = \underset{\sim}{0} \quad \text{on } S^* \tag{8.10}$$

where $\underset{\sim}{n}$ is a unit normal vector on S^*. If S is that portion of the boundary of R which is carried onto S^* by the deformation (8.1), as a consequence of (8.7), (8.10), $\underset{\sim}{\sigma}$ satisfies

$$\underset{\sim}{\sigma}\,\underset{\sim}{N} = \underset{\sim}{0} \quad \text{on } S \tag{8.11}$$

where $\underset{\sim}{N}$ is a unit normal vector on S.

Suppose now that R is a cylindrical region, and choose rectangular cartesian coordinates x_1, x_2, x_3 with the x_3 - axis parallel to the generator of R. The deformation (8.1) is an <u>anti-plane shear</u> if it is of the form

$$y_1 = x_1,\; y_2 = x_2,\; y_3 = x_3 + u(x_1,x_2). \tag{8.12}$$

It is convenient to think of the "out-of-plane" displacement u as a function on the plane cross-section D of R for which $x_3 = 0$. Here x_i, y_i are the components of $\underset{\sim}{x}$, $\underset{\sim}{y}$, respectively, in the given frame.

In the present problem we consider a class of incompressible materials for which W is independent of the second invariant

$$W = W(I_1), \tag{8.13}$$

so as to ensure that the governing field equations have a solution of anti-plane shear type (Knowles [13]). We assume that $W(I_1)$ is twice continuously differentiable for $I_1 > 3$ and that it vanishes in the undeformed state, $W(3) = 0$. If (8.13) holds, the two forms (8.6), (8.8) of the constitutive law take the simpler forms

$$\left.\begin{aligned} \underset{\sim}{\tau} &= 2W'(I_1)\underset{\sim}{G} - p\,\underset{\sim}{1}\,, \\ \underset{\sim}{\sigma} &= 2W'(I_1)\underset{\sim}{F} - p(\underset{\sim}{F}^T)^{-1}, \end{aligned}\right. \tag{8.14}$$

where the prime indicates differentiation with respect to the argument of W. For the deformation (8.12), the matrices of components of the tensors $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are readily found from (8.2), (8.4) to be

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_{,1} & u_{,2} & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & u_{,1} \\ 0 & 1 & u_{,2} \\ u_{,1} & u_{,2} & 1 + |\nabla u|^2 \end{pmatrix}, \tag{8.15}$$

and

$$|\nabla u|^2 = u_{,\alpha} u_{,\alpha}. \tag{8.16}$$

Here a subscript preceded by a comma indicates partial differentiation with respect to the corresponding x - coordinate; and Greek subscripts have the range 1, 2, while Latin subscripts take the values 1, 2, 3; repeated subscripts are summed over the appropriate range. From the first of (8.15), one confirms that the incompressibility condition (8.3) is automatically satisfied, while the first of (8.5) and the second of (8.15) furnish

$$I_1 = I_2 = 3 + |\nabla u|^2. \tag{8.17}$$

From (8.14), (8.15) we find the Cauchy and Piola stress components in terms of u, p. Through the equilibrium equations for the Piola stresses, one can determine p which when substituted back into the expressions for the stresses gives (e.g. Knowles [14])

$$\begin{aligned} &\tau_{3\alpha} = \tau_{\alpha 3} = \sigma_{3\alpha} = 2W'(I_1)u_{,\alpha}, \\ &\sigma_{\alpha 3} = [2W'(I_1) + d_0(x_3+u) + d_1]u_{,\alpha}, \\ &\tau_{\alpha\beta} = \sigma_{\alpha\beta} = -[d_0(x_3+u) + d_1]\delta_{\alpha\beta}, \\ &\tau_{33} = 2W'(I_1)\ |\nabla u|^2 - d_0(x_3+u) - d_1, \\ &\sigma_{33} = -[d_0(x_3+u) + d_1]. \end{aligned} \tag{8.18}$$

Here $\delta_{\alpha\beta}$ is the Kronecker delta, and d_0, d_1 are constants. The differential equations of equilibrium require that

$$[2W'(3+|\nabla u|^2)u_{,\alpha}]_{,\alpha} = d_0 \quad \text{on } D \tag{8.19}$$

The constants d_0 and d_1 must be determined from boundary conditions in a particular anti-plane shear problem.

The deformation (8.12) is a <u>simple</u> <u>shear</u> in the x_3-direction if u has the form

$$u(x_1,x_2) = k\ x_2, \tag{8.20}$$

where k is the amount of shear in the x_3, x_1 - plane. Then from (8.17), one gets

$$I_1 = I_2 = 3 + k^2 \tag{8.21}$$

and the stress of primary interest, τ_{32}, is found from the first of (8.18), (8.20), (8.21), to be

$$\tau_{32} = \hat{\tau}(k) = 2kW'(3+k^2). \tag{8.22}$$

The graph of $\hat{\tau}(k)$ vs. k is henceforth to be referred to as the <u>response curve in simple shear</u>. Then the corresponding secant, tangent, and infinitesimal shear moduli (all assumed to be positive) are defined by

$$\mu_s(k) = \frac{\hat{\tau}(k)}{k},\ \mu_t(k) = \frac{d\hat{\tau}(k)}{dk},\ \mu_o = \mu_t(0). \tag{8.23}$$

Finally, returning to a general anti-plane deformation (8.12) and comparing (8.17) with (8.21), leads one to define the "effective shear" as

$$k_e(\underset{\sim}{x}) = |\nabla u(\underset{\sim}{x})| \quad \text{on } D. \tag{8.24}$$

CHAPTER 9

FORMULATION

9.1 The crack problem.

Consider now an infinite strip of width 2b, whose cross section D contains a crack of length 2a. Let (x_1,x_2,x_3) be rectangular Cartesian coordinates chosen such that the x_3-axis is parallel to the edges of the crack with the origin lying on the crack, midway between its edges (see Figure 6). Suppose that the strip is subjected at infinity to a simple shear parallel to the edges of the crack, and that both the crack and the long sides at $x_1 = \pm b$, remain traction-free in the deformed configuration. Thus by (8.11), on the plane surface of the crack, the Piola stresses must satisfy

$$\sigma_{12} = \sigma_{22} = \sigma_{23} = 0 \quad \text{for } x_2 = 0\pm,\ -a < x_1 < a. \tag{9.1}$$

Reference to (8.17) shows that the first conditon in (9.1) is automatically satisfied, while the second conditon requires that

$$d_0 = d_1 = 0. \tag{9.2}$$

Because of (9.2), one can reduce, with the aid of (8.17), the Cauchy and Piola stress components given in (8.18) to the following forms:

$$\begin{aligned} &\tau_{3\alpha} = \tau_{\alpha 3} = \sigma_{3\alpha} = \sigma_{\alpha 3} = 2W'(3+|\nabla u|^2)u_{,\alpha} \quad \text{on } D,\\ &\tau_{\alpha\beta} = \sigma_{\alpha\beta} = 0 \quad \text{on } D,\\ &\tau_{33} = 2W'(3+|\nabla u|^2)|\nabla u|^2, \quad \sigma_{33} = 0 \qquad \text{on } D. \end{aligned} \tag{9.3}$$

Recalling that $\tau = \hat{\tau}(k)$ represents the response function in simple shear of the current material, in view of (8.23) one can now write

$$\mu_s(k_e) = 2W'(3+k_e^2), \tag{9.4}$$

$$\mu_t(k_e) = 2W'(3+k_e^2) + 4\, k_e^2\, W''(3+k_e^2)$$

where the first invariant I_1 is now given, from (8.17), (8.24), by

$$I_1 = 3 + k_e^2 \ . \tag{9.5}$$

Here k_e designates the effective amount of the shear $|\nabla u|$. The constitutive law of the current incompressible material $(8.14)_1$ yields the components of the Cauchy (true) stresses accompanying an anti-plane deformation, through (9.3), (9.4), in terms of the out-of-plane displacement as

$$\tau_{3\alpha} = \tau_{\alpha 3} = \mu_s(k_e)u_{,\alpha},$$
$$\tau_{\alpha\beta} = 0, \quad \tau_{33} = \mu_s(k_e)k_e^2 \ . \tag{9.6}$$

Consequently, the boundary value problem governing the out-of-plane displacement $u(x_1,x_2)$ consists of the equilibrium equation $(8.9)_1$ which, in view of (9.6), is

$$[\mu_s(k_e)u_{,1}]_{,1} + [\mu_s(k_e)u_{,2}]_{,2} = 0 \quad \text{on } D \tag{9.7}$$

with the traction free boundary conditions on crack and side surfaces

$$\tau_{32} = \mu_s(k_e)u_{,2} = 0 \quad \text{at } x_2 = 0\pm, \ -a < x_1 < a,$$
$$\tau_{31} = \mu_s(k_e)u_{,1} = 0 \quad \text{at } x_1 = \pm b, \ -\infty < x_2 < \infty, \tag{9.8}$$

In addition to the free-surface conditions (9.8), one imposes the requirement that, at infinity, the displacement field should approach that of a simple shear parallel to the crack surface and perpendicular to the cross-section D,

$$u \sim k_\infty x_2 \quad \text{as } x_1^2 + x_2^2 \to \infty, \ -b < x_1 < b, \tag{9.9}$$

where k_∞ is the remotely applied shear. The differential equation (9.7), the boundary conditions (9.8), (9.9) and the further requirement that u be suitably smooth on D and bounded near the crack tips, comprise a nonlinear boundary value problem for the out-of-plane displacement $u(x_1,x_2)$.

9.2. The energy release rate.

There is associated with the anti-plane shear problem formulated above, a path independent integral of the type first discovered by Eshelby [28] and later exploited by Rice [17] in connection with crack problems. The path independent integral associated with finite anti-plane shear is

$$J = \int_\Gamma [W(3+k_e^2)\, n_1 - \tau_{3\alpha} n_\alpha u_{,1}]ds. \tag{9.11}$$

Here Γ is any simple closed curve which encloses the right crack tip in its interior but does not include the left one and n_α are the components of the unit outward normal to Γ while s is arc length along Γ.

Now let P(a) denote the total potential energy of the body and loading system currently under consideration. Let $P(a+\Delta a)$ denote the total potential energy of a system which is identical to the previous one, except that the crack in this body has length $2(a+\Delta a)$. Then the energy release rate G associated with the first body is defined by

$$\begin{aligned} G &= -\lim_{\Delta a\to 0} \frac{P(a+\Delta a) - P(a)}{\Delta a} \\ &= -\frac{\partial P}{\partial a}. \end{aligned} \tag{9.12}$$

Rice [17] has shown that J = G, provided the path of integration Γ is as described above. Therefore, the value of the path-independent integral J, (9.11), taken around the crack tip can be viewed as representing the energy release rate. Our purpose in this problem is to calculate the value of G (or equivalently J) associated with the particular boundary value problem (9.7)-(9.9).

CHAPTER 10

SOLUTION

In the absence of the crack, the deformation of the body is one of simple shear and the associated displacement field is $u = k_\infty x_2$ on D. If we assume that the presence of the crack causes only a small disturbance of this crack-free equilibrium state, we would have

$$u(x_1,x_2) = k_\infty x_2 + \hat{u}(x_1,x_2), \text{ where } |\nabla\hat{u}| << 1 \quad \text{on D.} \tag{10.1}$$

While such a hypothesis is undoubtedly invalid in the vicinity of the crack, it seems reasonable to assume that it would be true at points distant from the crack tips. Since our interest here lies <u>solely</u> in calculating the value of the path-independent integral J, and since for this purpose we may consider a path that is essentially far removed from the crack tip (provided the crack length 2a is not too close to the strip width 2b), the error introduced by such an assumption is probably small. Thus motivated, we now assume that the displacement field is of the form (10.1). Then from (10.1), we get

$$u_{,1} = \hat{u}_{,1}, \quad u_{,2} = k_\infty + \hat{u}_{,2} \tag{10.2}$$

so that

$$k_e^2 = |\nabla u|^2 = k_\infty^2 + 2\, k_\infty \hat{u}_{,2} + |\nabla\hat{u}|^2, \; |\nabla\hat{u}|^2 = u_{,\alpha} u_{,\alpha}. \tag{10.3}$$

Next, we calculate the Cauchy stresses to leading order with the aid of (9.4), (9.6), (10.3) as

$$\begin{aligned}
\tau_{31} &= \mu_s(k_e)u_{,1} \\
&= 2\hat{W}'(3+k_\infty^2+2k_\infty\hat{u}_{,2})\hat{u}_{,1} \\
&= 2[\hat{W}'(3+k_\infty^2) + \hat{W}''(3+k_\infty^2)\ (2K_\infty\hat{u}_{,2})]\hat{u}_{,1} \\
&= 2\hat{W}'(3+k_\infty^2)\hat{u}_{,1} \\
&= \mu_s(k_\infty)\hat{u}_{,1}
\end{aligned} \tag{10.4}$$

where $\mu_s(k_\infty) = 2\hat{W}'(3+k_\infty^2)$ is the secant modulus at infinity. Similarly, we get

$$\begin{aligned}
\tau_{32} &= \mu_s(k_e)u_{,2} \\
&= 2\hat{W}'(3+k_\infty^2)\ k_\infty + 4W''(3+k_\infty^2)\ k_\infty^2\hat{u}_{,2} + 2\hat{W}'(3+k_\infty^2)\hat{u}_{,2} \\
&= \tau_\infty + \mu_t(k_\infty)u_{,2}
\end{aligned} \tag{10.5}$$

where $\mu_t(k_\infty) = 2\hat{W}'(3+k_\infty^2) + 4\hat{W}''(3+k_\infty^2)k_\infty^2$ is the tangent modulus at infinity. We have set

$$\tau_\infty = 2\hat{W}'(3+k_\infty^2)k_\infty,$$

so that τ_∞ is the <u>remotely applied shear stress</u>.

Similarly we can linearize the nonlinear boundary value problem (9.7)-(9.9) to get

$$\begin{aligned}
&\mu_s(k_\infty)\hat{u}_{,11} + \mu_t(k_\infty)\hat{u}_{,22} = 0 && \text{on } D, \\
&\hat{u}_{,2} = \frac{-\tau_\infty}{\mu_t(k_\infty)} && \text{on } x_2 = 0\pm,\ |x_1| < a, \\
&\hat{u}_{,1} = 0 && \text{on } x_1 = \pm b,\ |x_2| < \infty, \\
&\hat{u} \to 0 && \text{as } |x_2| \to \infty,\ |x_1| < b.
\end{aligned} \tag{10.6}$$

Correspondingly, we can calculate the linearized version of J as follows. First, $W(3+k_e^2)$ in the integrand of (9.11) is linearized by

$$W(3+k_e^2) = \hat{W}(3+k_\infty^2+2k_\infty\hat{u}_{,2}+k_e^2),$$

$$= \hat{W}(3+k_\infty^2) + \hat{W}'(3+k_\infty^2)(2k\ \hat{u}_{,2}+k_e^2) + \frac{1}{2}\hat{W}''(3+k_\infty^2)($$

$$2k_\infty + \hat{u}_{,2} + k_e^2)^2, \tag{10.7}$$

$$= \hat{W}(3+k_\infty^2) + \tau_\infty\hat{u}_{,2} + \hat{W}'(3+k_\infty^2)\hat{u}_{,1}^2 + \frac{1}{2}\mu_t(k_\infty)\hat{u}_{,2} ,$$

and $\tau_{3\alpha}n_\alpha u_{,1}$ by

$$\tau_{3\alpha}n_\alpha u_{,1} = \{2\hat{W}'(3+k_\infty^2)\hat{u}_{,1}n_1 + (\tau_\infty+\mu_t(k_\infty)\hat{u}_{,2})n_2\}\hat{u}_{,1} , \tag{10.8}$$

so that the J integral (9.11) is now linearized to yield

$$J = \int_\Gamma 1/2\{\mu_t(k_\infty)\hat{u}_{,2}^2n_1-2\mu_t(k_\infty)\hat{u}_{,2}\hat{u}_{,1}n_2-\mu_s(k_\infty)\hat{u}_{,1}^2n_1\}ds. \tag{10.9}$$

Our present task therefore is reduced to solving the <u>linear</u> boundary value problem (10.6) for $\tilde{u}(x_1,x_2)$ and to then evaluate the integral (10.9).

The linear boundary-value problem (10.6) may be further simplified by the scaling $\xi_1 = x_1$, $\xi_2 = mx_2$, $m^2 = \mu_s(k_\infty)/\mu_t(k_\infty)$ and by setting

$$U(\xi_1,\xi_2) = \tilde{u}(x_1,x_2) + k_\infty m^2x_2. \tag{10.10}$$

This leads to

$$\tilde{u}_{,1} = U_{,1},\ \tilde{u}_{,11} = U_{,11},$$

$$\tilde{u}_{,2} = mU_{,2} - k_\infty m^2,\ \tilde{u}_{,22} = m^2U_{,22}, \tag{10.11}$$

so that the boundary-value problem (10.6) yields

$$
\begin{array}{lll}
\nabla^2 U(\xi_1,\xi_2) = 0 & \text{on } D, & \\
U_{,2} = 0 & \text{on } \xi_2 = 0\pm,\ |\xi_1| < a, & \\
U_{,1} = 0 & \text{on } \xi_1 = \pm b,\ |\xi_2| < \infty, & (10.12) \\
U \to (k_\infty m)\xi_2 & \text{on } \xi_2 \to \infty,\ |\xi_1| < b. &
\end{array}
$$

The mathematical problem (10.12) is identical to the linear elasticity problem governing the Mode III center-cracked strip problem. The solution to this problem is well-known and we find ([31])

$$U \sim 2\,\frac{K_{III}}{\mu_s(k_\infty)/m}\sqrt{\frac{r}{2\pi}}\,\sin\frac{\theta}{2} \quad \text{as } r \to 0,\ |\theta| < \pi. \tag{10.13}$$

where (r,θ) are polar coordinates in the (ξ_1,ξ_2) plane such that $r\cos\theta = \xi_1 - a$, $r\sin\theta = \xi_2$ and

$$K_{III} = \tau_\infty \sqrt{\pi a}\left[\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]^{1/2}. \tag{10.14}$$

On using (10.11), (10.13) we may calculate

$$
\begin{aligned}
\hat{u}_{,1} &= -\frac{K_{III}}{\mu_s(k_\infty)/m}\,\frac{\sin\frac{\theta}{2}}{\sqrt{2\pi r}} \\
\hat{u}_{,2} &= m\,\frac{K_{III}}{\mu_s(k_\infty)/m}\,\frac{\cos\frac{\theta}{2}}{\sqrt{2\pi r}} - k_\infty m^2
\end{aligned}
\tag{10.15}
$$

Thus we may now evaluate the J integral (10.9) (by choosing the contour to be an ellipse of major axis a and minor axis ma in the (x_1,x_2)-plane):

$$J = \int_{-\pi}^{\pi} 1/2[-\mu_s(k_\infty)(\frac{-K_{III}}{\mu_s(k_\infty)/m}\ \frac{\sin\frac{\theta}{2}}{\sqrt{2\pi r}})^2\ \frac{r\cos\theta}{m} +$$

$$\mu_t(k_\infty)(m\frac{K_{III}}{\mu_s(k_\infty)/m}\ \frac{\cos\frac{\theta}{2}}{\sqrt{2\pi r}} - k_\infty m^2) r\ \frac{\cos\theta}{m} -$$

$$2\mu_t(k_\infty)(-\frac{K_{III}}{\mu_s(k_\infty)/m}\ \frac{\sin\frac{\theta}{2}}{\sqrt{2\pi r}})(m\frac{K_{III}}{\mu_s(k_\infty)/m}\ \frac{\cos\frac{\theta}{2}}{\sqrt{2\pi r}}$$

$$- k_\infty m^2)\ r\sin\theta\}d\theta,$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\frac{\sqrt{\mu_s(k_\infty)\mu_t(k_\infty)}}{2\pi}\ (\frac{K_{III}}{\mu_s(k_\infty)/m})^2\{(\cos^2\frac{\theta}{2} -$$

$$\sin^2\frac{\theta}{2})\cos\theta + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\theta\}d\theta,$$

$$= \frac{1}{2}\sqrt{\mu_s(k_\infty)\mu_t(k_\infty)}(\frac{K_{III}}{\mu_s(k_\infty)/m})^2, \tag{10.16}$$

which in view of (10.14), gives the desired result

$$J = \frac{1}{2}\frac{\tau_\infty^2}{\sqrt{\mu_s(k_\infty)\mu_t(k_\infty)}}\ \pi a\ [\frac{2b}{\pi a}\tan\frac{\pi a}{2b}]. \tag{10.17}$$

Here $\mu_s(k_\infty)$ and $\mu_t(k_\infty)$ are the values of the secant and tangent moduli far from the crack and τ_∞ is the remotely applied shear stress.

CHAPTER 11

DISCUSSION

Equation (10.17) is a simple formula for the value of J in terms of constitutive, geometric and load parameters. We now specialize it to certain special material. In the case of a linear material, $k/k_0 = \tau/\tau_0$, we have from (8.23) that

$$\mu_s(k_\infty) = \mu_t(k_\infty) = \tau_0/k_0 \tag{11.1}$$

and so (10.17) yields (the "small-scale yielding" result)

$$J/\tau_0 k_0 a = \frac{1}{2}\left(\frac{\tau_\infty}{\tau_0}\right)^2 \pi\left[\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]. \tag{11.2}$$

On the other hand, in the case of a pure power-law material described by $k/k_0 = \alpha(\tau/\tau_0)^n$, we have

$$\mu_s(k_\infty) = \frac{\tau_0}{(\alpha k_0)^{1/n}} k_\infty^{(1-n)/n},$$
$$\mu_t(k_\infty) = \frac{\tau_0}{n(\alpha k_0)^{1/n}} k_\infty^{(1-n)/n}, \tag{11.3}$$

so that (10.17) yields (the "fully plastic solution")

$$J/\tau_0 k_0 a = \frac{\alpha}{2}\left(\frac{\tau_\infty}{\tau_0}\right)^{1+n}\sqrt{n}\ \pi\left[\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]. \tag{11.4}$$

If one were now interested in a, say, Ramberg-Osgood material,

$$k/k_0 = \tau/\tau_0 + \alpha(\tau/\tau_0)^n \qquad (11.5)$$

the traditional procedure for determining J has been to interpolate between (11.2) and (11.4). [The need for such a procedure stems from the fact that (11.2) and (11.4) are usually found directly, by exploiting certain special features of those material, rather than from a more general result such as (10.17): no such specialized methods have been available to study more general materials such as (11.5)]. The interpolation formula proposed by Shih [24] for the Ramberg-Osgood material is

$$J/\tau_0 k_0 a = \left(\frac{\tau_\infty}{\tau_0}\right)^2 \frac{f_1(a/b,1)}{(1-a/b)} + \alpha\left(\frac{\tau_\infty}{\tau_0}\right)^{n+1} \frac{f_1(a/b,n)}{(1-a/b)^n}, \qquad (11.6)$$

where the function $f_1(a/b, n)$ is determined numerically through a finite element analysis (see Table 3 in [24]). Improvements to (11.6) based on the notion of an effective crack length, have also been proposed in [24].

The advantage of the present procedure is that the result (10.17) is not restricted to a particular shear response function. For a Ramberg-Osgood material one obtains, from (10.17) and (11.5), the expression (in the "elastic-plastic range")

$$J/\tau_0 k_0 a = \frac{1}{2}\left(\frac{\tau_\infty}{\tau_0}\right)^2 \left[1+\alpha(n+1)\left(\frac{\tau_\infty}{\tau_0}\right)^{n-1} + \alpha^2 n\left(\left(\frac{\tau_\infty}{\tau_0}\right)^{2n-2}\right.\right.$$
$$\left.\right]^{1/2} \frac{2b}{a} \tan\left(\frac{\pi a}{2b}\right). \qquad (11.7)$$

Figure 7 shows the variation of the energy release rate $J/\tau_0 k_0 a$ with the applied load τ_∞/τ_0 (in the case $n = 3$, $\alpha = 3/7$ and for $a/b = 1/8$ and

1/4). The solid curve corresponds to the formula (11.7) derived here while the dashed curve describes the result according to the interpolation formula (11.6). The agreement is seen to be good.

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FIGURES

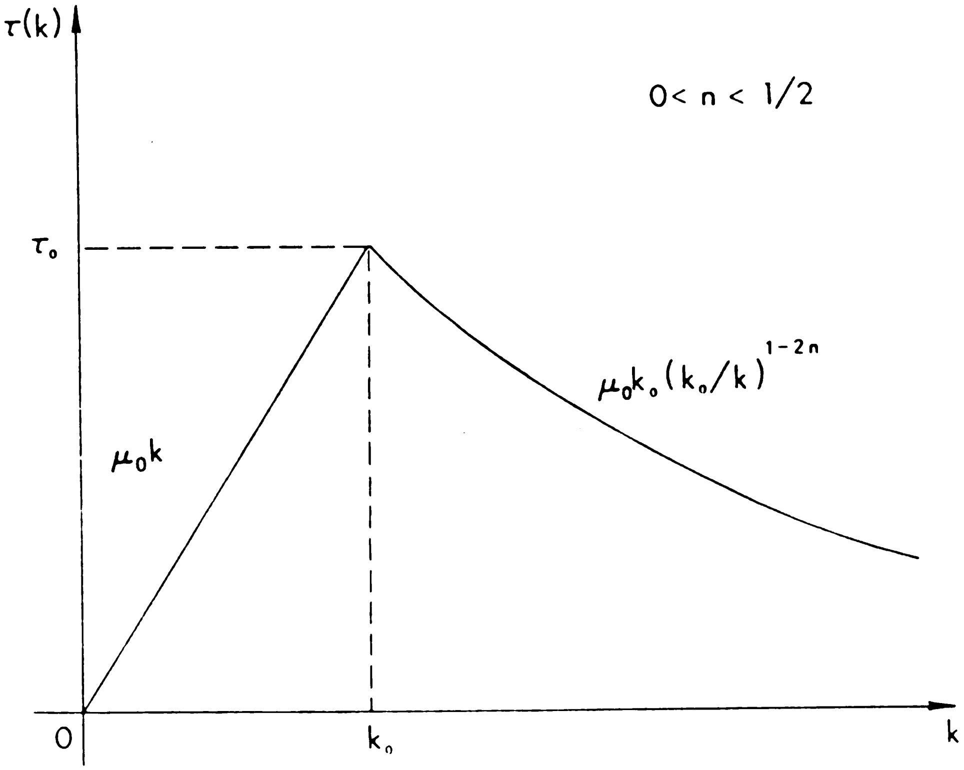

Figure 1. Response curve in simple shear for the piecewise power-law material.

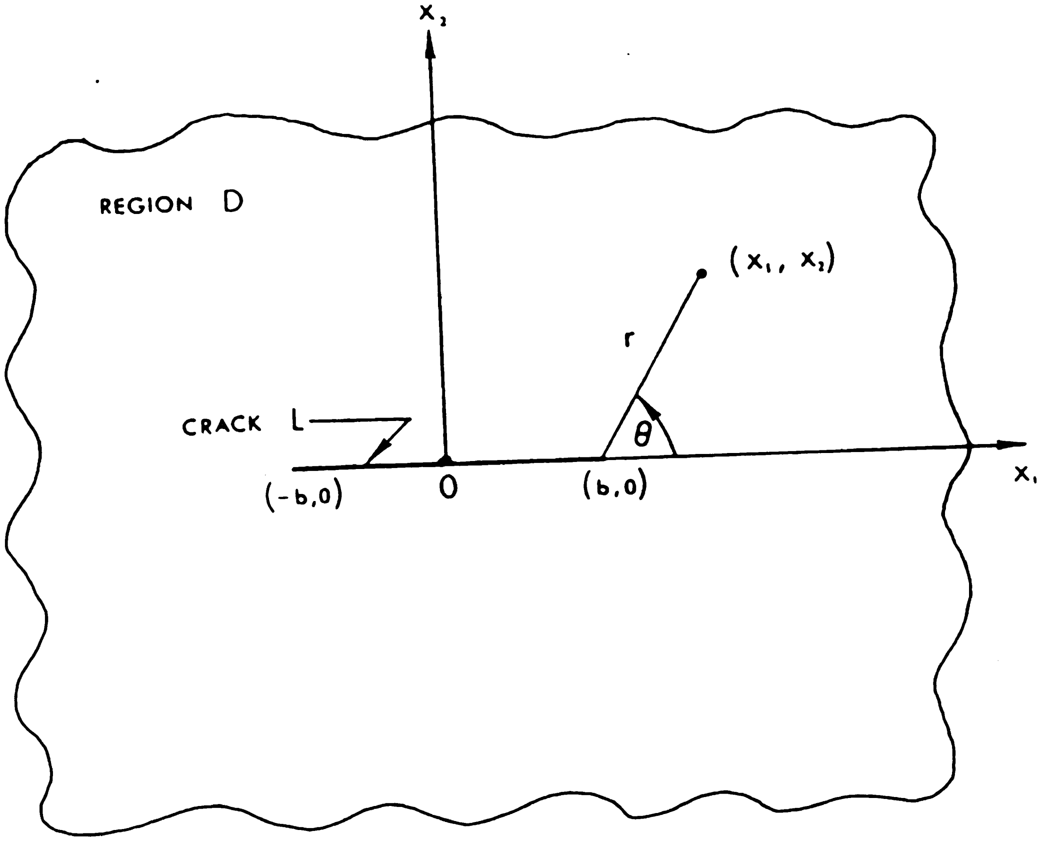

Figure 2. Geometry of the global crack problem.

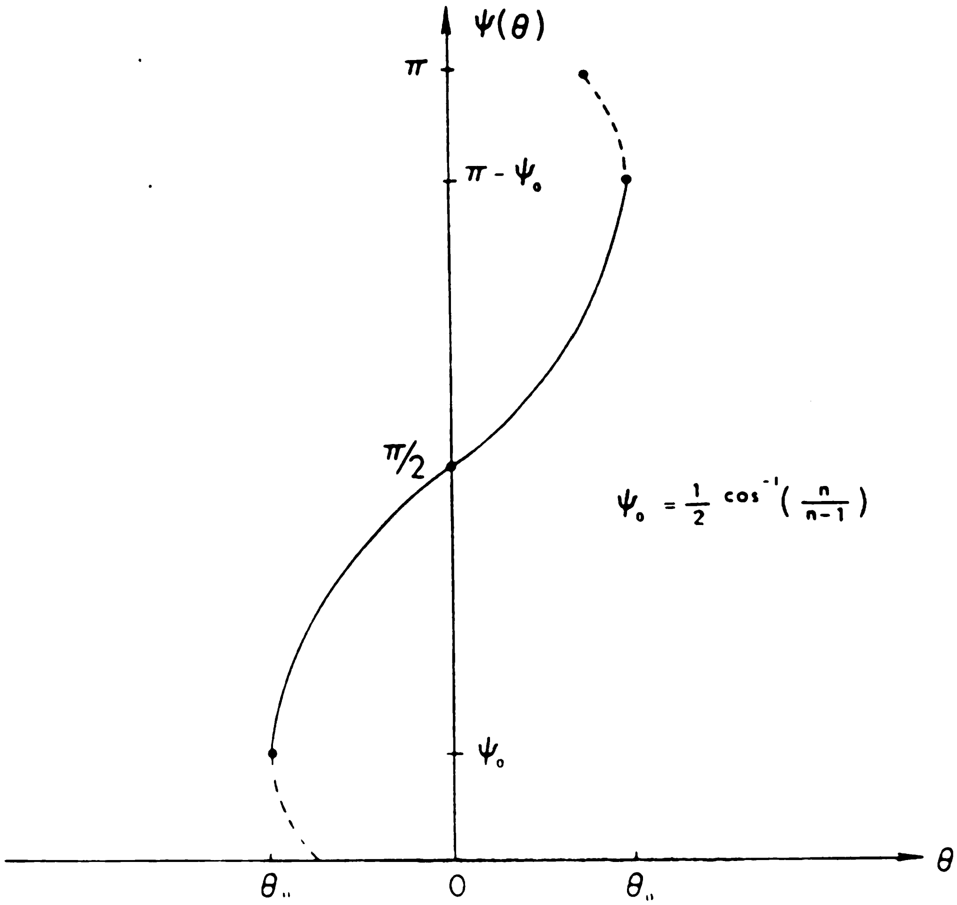

Figure 4. Sketch of $\psi(\theta)$ vs. θ as defined by (4.24) with $0 < n < 1/2$, $m < 0$.

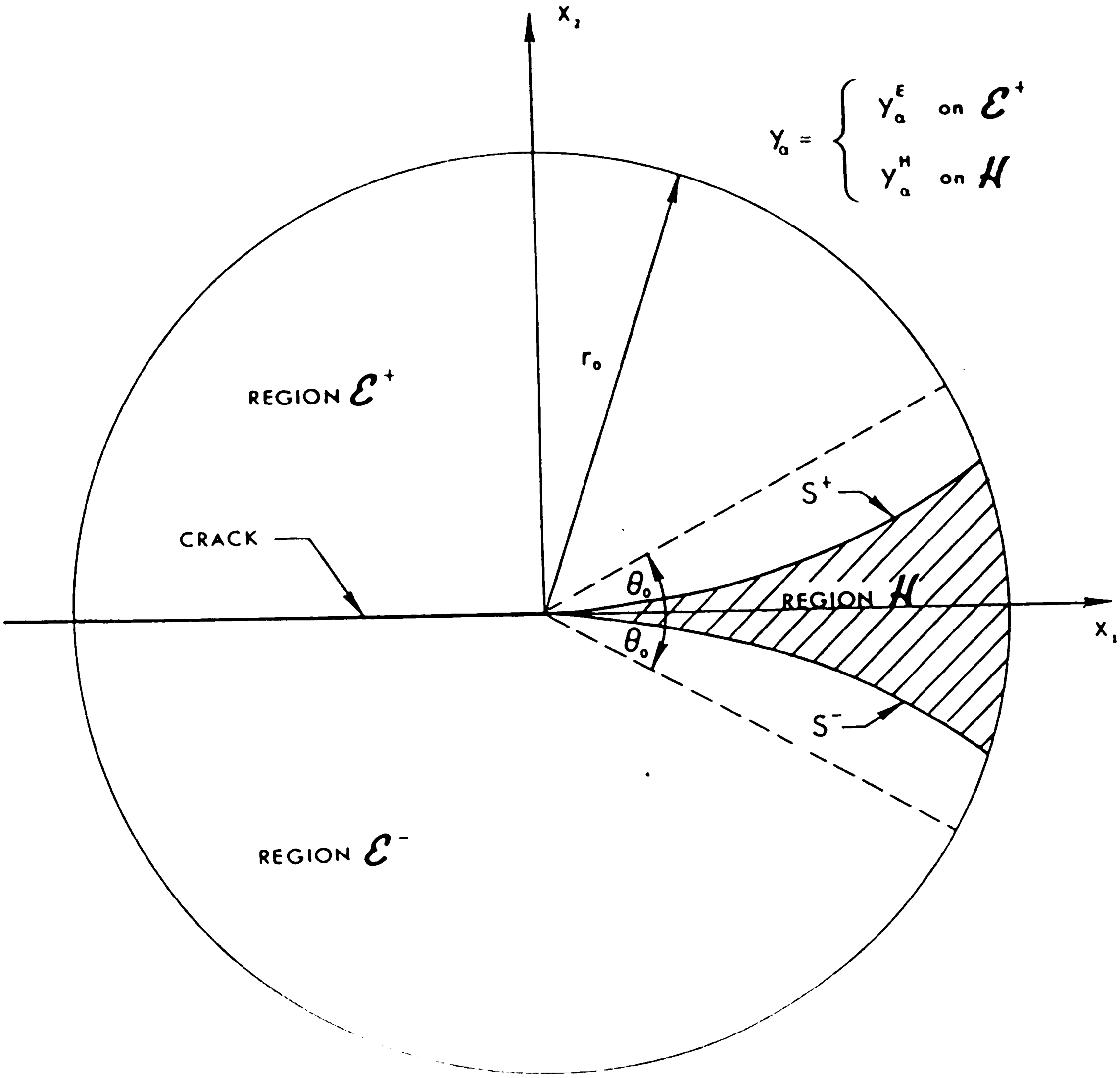

Figure 5. Asymptotic solution to the problem: schematic description.

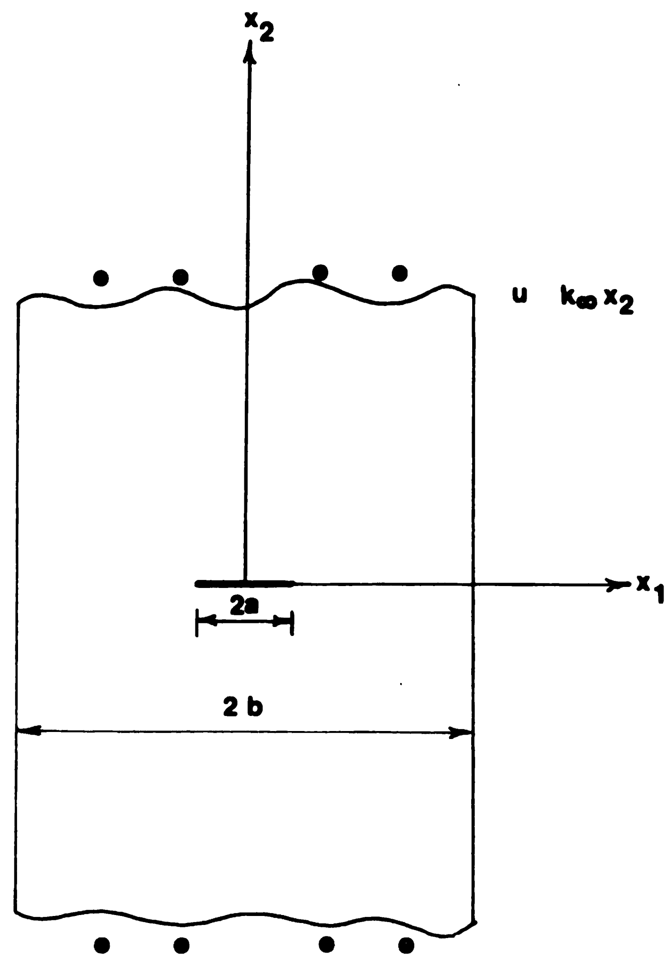

Figure 6. Geometry of Center-cracked strip.

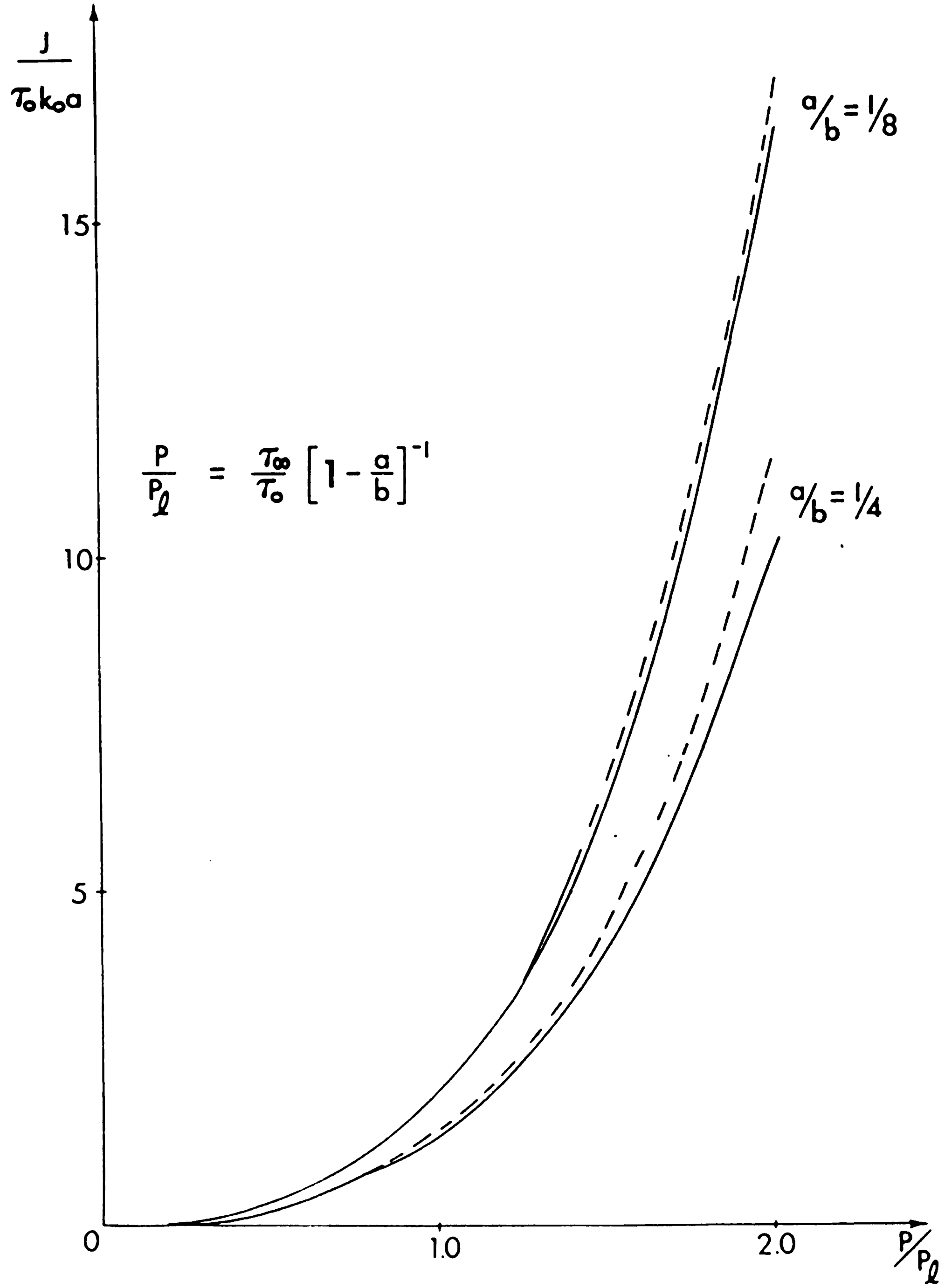

Figure 7. Variation of J with load for a Ramberg-Osgood material with n=3, α=3/7. Solid curve corresponds to the formula (11.7) derived here; dashed curve corresponds to interpolation formula (11.6).