STATE MODEL APPROACH TO THE SYNTHESIS OF LC NETWORKS AND THE CANONIC LC NETWORK TRANSFORMATIONS

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## ABSTRACT


## STATE MODEL APPROACH TO THE SYNTHESIS OF LC NETWORKS AND THE CANONIC LC NETWORK TRANSFORMATIONS

by Radha Krishna Rao Yarlagadda

In this thesis 1) a state model approach to the synthesis of LC networks and 2) a realization procedure for canonic LC networks using the equivalent network transformations are considered.

In Chapter II analysis and formulation of LC networks from the state model point of view is described.

In Chapter III a new definition of canonic LC networks is given. This definition is applicable to general n-port LC networks. A general realization procedure of LC networks from the state model equations is described. A state model approach to the realization of reactance or susceptance matrices with the ideal transformers is given. The state model approach is also utilized in the realization of reactance functions without the use of ideal transformers. This procedure is extended to the reactance matrices of order two having dominant residue matrices; however, it is also applicable to n-port LC networks of the same class.

In Chapter IV a technique for realizing canonic networks is described. This is accomplished by establishing a parametric matrix relation, which relates the parameters of two canonic networks by an orthogonal matrix. Some of the known canonic forms are derived by the application of this technique.

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# CHAPTER I

# INTRODUCTION

In classical network theory, the synthesis problem is stated in terms of the impedance or admittance matrices in the s-domain, and its solution is realized by mathematical operations on these matrices which can be interpreted as the interconnection of certain sub-networks. These mathematical operations, in general, necessitate the use of ideal transformers, even though the given matrix is known to be realizable as a network contains only two terminal R, L and C elements.

Recently it is recognized by many investigators that the topological approach to the synthesis problem might offer new insight. Indeed, the problem of synthesizing n-terminal R networks characterized by the impedance or admittance matrices of order (n-1) is completely solved [PA 1]. Although extension of R-network synthesis to some very special class of multi-terminal RLC networks is also known [CE 4], the solution of the general problem appears at best, to be very difficult.

A natural approach to the RLC network synthesis is by means of state models since, in general, the state model of the network provides more direct information about the network topology than does the network matrices. Recently, some works have been initiated in this direction [RA 1], [DE 2], [KA 1]. Since the properties of the relations between the state models and the topology of the corresponding network are not explored fully, in this thesis a natural decision is made and only the synthesis problem of LC networks is considered.

In recent literature, new canonic networks have been proposed [RE 1, LE 1, YA 1]. The first set of canonic networks were given by Foster in 1924 [FO 1]; later in 1930 Cauer gave another set of canonic networks. In 1955 Reza [RE 1] stated that there are many more canonic forms which cannot be obtained even if a mixture of Foster and Cauer procedures is used. But he gave neither the properties of such new canonic forms nor the realization procedures. In 1963, Lee [LE 1] gave a lattice canonic form and Yarlagadda and Tokad proposed a different lattice canonic form [YA 1]. None of these known canonic forms are obtained as the result of a general theory. Rather, each is derived by a procedure unique to its form.

This thesis presents for the first time two general procedures for deriving one canonic LC network from another. The first procedure is based on the equivalent network transformation introduced by Howitt [HO 1], [HO 2] and extended by Cauer, Guillemin [GU 1], Schoeffler [SC 1] and others. None of these methods, however, consider major changes in the topology of the network, i.e., equivalent networks are always assumed to have the same number of meshes or node-pairs or one equivalent network is generated from the other by rather obvious network modifications.

The second method is based on similarity transformations on the state models of the networks rather than on the mesh or node equations.

In Chapter II analysis and formulation of LC networks from the state model point of view is described along with certain similarity transformations. Such an analysis is the backbone of the synthesis procedures considered in the successive Chapters.

In Chapter III, a new definition of canonic LC network is given which applies to general n-port networks. A general realization procedure for the state model equations as an LC network is described in Section 3.5 and certain conditions for realizability are stated. Although methods exist [KA 1], [GI 1] for derivating the state model from the impedance or admittance matrix, a slightly different approach is given in Section 3.4. The problem of realizing impedance or admittance matrices with the ideal transformers is solved by state models. Although Kalman [KA 3] in a recent talk has suggested that this result is possible, but so far no published material is available. In Section 3.6 a realization procedure for the reactance functions is given in terms of state models and the procedure extended to a class of 2-port LC networks.

In Chapter IV a general procedure for transforming a one-port canonic LC network to another is given. The relation between the element values of one canonic network to another is established and this relation is called parametric matrix relation. For some of the canonic networks given in Table 4.3.1, the nonlinear equations obtained from parametric matrix relation are solved analytically. But, in general, an analytical solution for an arbitrary canonic form may not be possible, although computer solution might conceivably give a solution in a particular application. Such numerical solutions are not considered in this thesis.

Chapter V presents examples, illustrating the procedures described in the thesis.

# CHAPTER II

# ANALYSIS OF LC NETWORKS

## 2.1 General

In this chapter the analysis of n-port LC networks in terms of state model formulation is considered. This analysis establishes the conditions on the short circuit and the open circuit parameters of the n-port LC network which are necessary for its realizability without the ideal transformers.

## 2.2 The State Model

In classical network theory the analysis, in general, is based on the s-domain loop or node equations and natural frequencies are defined as

Definition 2.2.1: The finite zeros of the determinantal equation of the loop or the node matrix of an RLC network are called the natural frequencies, and the number of these natural frequencies is called the "order of complexity" of the network.

The instanteous behavior of the network can also be described by a set of first order differential equations and a set of algebraic equations of the form

$$\begin{aligned} \frac{d}{dt} X &= AX + BY + C\frac{d}{dt} Y \\ \bar{Y} &= PX + QY + R\frac{d}{dt} Y \end{aligned} \qquad (2.2.1)$$

where X is called the state vector and consists of branch capacitor voltages and chord inductor currents, and Y is a vector which contains the specified voltages and currents; $\overline{Y}$ contains the complementary variables of those in Y. The coefficient matrix A is real square and called the operator matrix. All other matrices B, C, P, and Q are real and, in general, they are rectangular. The set of equations in (2.2.1) describes the behavior of the network completely and it is called the state model of the network.

An alternate definition for the natural frequencies of the network as given in terms of the state model is

Definition 2.2.2: Natural frequencies: The natural frequencies of an RLC network is defined as the eigenvalues (not necessarily distinct) of the operator matrix A in the state model (2.2.1).

In the literature, the order of complexity has been discussed by several authors. Some define it in terms of the topological properties of the network [RE 1, GU 1, SE 1 and others], and the others [BR 1, BA 1] define it in terms of state variables. However, the two definitions are equivalent, i.e., the order of complexity is equal to the order of the state vector.

## 2.3 Explicit Expressions for the State Equations of an n-port LC Network

A procedure for determining a state vector of minimum order has been described by several authors [BR 1, BA 1, WI 1, KO 1, and others].

The procedure given originally by Wirth is based on a tree of the system graph defined as follows:

Let G be the graph of such network.

1. Consider the subgraph $G_0$ of G which contains all the elements corresponding to the voltage drivers. Let $T_0$ be a tree (or forest) in $G_0$. For consistent networks it is necessary that $T_0 = G_0$.
2. Let $G_1$ be the subgraph of G which contains $G_0$ and all elements corresponding to capacitors. Select a tree $T_1$ in $G_1$ such that $T_1$ contains $T_0$.
3. Let $G_2$ be the subgraph of G which contains $G_1$ and all elements corresponding to inductors. Select a tree in $G_2$ such that $T_2$ contains $T_1$. Then $T_2$ is a tree in G and all current drivers are included in its co-tree. (For consistent networks current drivers cannot form a cut-set).
4. The voltages of the branch capacitors and the currents of the chord inductors constitute the variables of the state vector.

For an LC network the circuit equations corresponding to a tree selected according to the above rules described has the form

$$\left[\begin{array}{ccc|ccc} B_{11} & B_{12} & 0 & U & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 & U \end{array}\right] \begin{bmatrix} V_0 \\ V_{bc} \\ V_{b\ell} \\ V_{cc} \\ V_{c\ell} \\ V_1 \end{bmatrix} = 0 \qquad (2.3.1)$$

where $B_{ij}$ are the submatrices of the fundamental circuit equations, U, the unit matrix and the variables are classified as follows:

$V_0$ - Voltage drivers

$V_{bc}$ - Branch capacitor voltages

$V_{b\ell}$ - Branch inductor voltages

$V_{cc}$ - Chord capacitor voltages

$V_1$ - Voltages of the current drivers

Similarly, the cut-set equations for the same network corresponding to the same tree are

$$\begin{bmatrix} U & 0 & 0 & -B_{11}^T & -B_{21}^T & -B_{31}^T \\ 0 & U & 0 & -B_{12}^T & -B_{22}^T & -B_{32}^T \\ 0 & 0 & U & 0 & -B_{23}^T & -B_{33}^T \end{bmatrix} \begin{bmatrix} I_0 \\ I_{bc} \\ I_{b\ell} \\ I_{cc} \\ I_{c\ell} \\ I_1 \end{bmatrix} = 0 \qquad (2.3.2)$$

Let the terminal equations of the branch capacitors and chord inductors be written as

$$\begin{bmatrix} C_b & \\ & L_c \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} I_{bc} \\ V_{c\ell} \end{bmatrix} \qquad (2.3.3)$$

and let the terminal equations of the branch inductors and chord capacitors be written as

$$\begin{bmatrix} L_b & \\ & C_c \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_{b\ell} \\ V_{cc} \end{bmatrix} = \begin{bmatrix} V_{b\ell} \\ I_{cc} \end{bmatrix} \qquad (2.3.4)$$

Then the state equations can be obtained as

$$\begin{bmatrix} C_b + B_{12}^T C_c B_{12} & 0 \\ 0 & L_c + B_{23} L_b B_{23}^T \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & B_{22}^T \\ -B_{22} & 0 \end{bmatrix} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix}$$

$$- \begin{bmatrix} B_{12}^T C_c B_{11} & 0 \\ 0 & B_{23} L_b B_{33}^T \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_0 \\ I_1 \end{bmatrix} + \begin{bmatrix} 0 & B_{32}^T \\ -B_{21} & 0 \end{bmatrix} \begin{bmatrix} V_0 \\ I_1 \end{bmatrix}$$

(2.3.5)

From (2.3.1) and (2.3.2) the currents of the voltage sources and the voltages of the current sources are

$$I_0 = + \begin{bmatrix} B_{11}^T & B_{21}^T & B_{31}^T \end{bmatrix} \begin{bmatrix} I_{cc} \\ I_{c\ell} \\ I_1 \end{bmatrix}$$

$$V_1 = - \begin{bmatrix} B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} V_0 \\ V_{bc} \\ V_{b\ell} \end{bmatrix} \qquad (2.3.6)$$

The terminal variables $V_0^*$, $V_1^*$, $I_0^*$, $I_1^*$ are related to the variables associated with the drivers by

$$I_0 = -I_0^* \; , \quad V_1 = V_1^*$$
$$V_0 = V_0^* \; , \quad I_1 = -I_1^* \qquad (2.3.7)$$

also

$$I_{cc} = C_c \frac{d}{dt} V_{cc} = -C_c \frac{d}{dt} (B_{11} V_0 + B_{12} V_{bc})$$

Substituting this expression of $I_{cc}$ and (2.3.7) into (2.3.6) we have

$$I_0^* = -I_0 = B_{11}^T C_c B_{11} \frac{d}{dt} V_0^* + B_{11}^T C_c B_{12} \frac{d}{dt} V_{bc} - B_{21}^T I_{c\ell} + B_{31}^T I_1^*$$

$$V_1^* = - (B_{31} V_0 + B_{32} V_{bc} + B_{33} V_{b\ell})$$

$$= - \left\{ B_{31} V_0^* + B_{32} V_{bc} + \frac{d}{dt} B_{33} L_b B_{23}^T I_{c\ell} - \frac{d}{dt} B_{33} L_b B_{33}^T I_1^* \right\}$$

$$(2.3.8)$$

The state model obtained above reduces to the form given in (2.2.1) upon taking the inversion of the coefficient matrix on the left of (2.3.5) and substituting the resulting expressions for the derivatives of the state variables into (2.3.8). The required inverse always exists as established by the following theorem.

<u>Theorem 2.3.1</u>: Let $C_b$, $C_c$, $L_b$, $L_c$ be diagonal matrices with positive entries and $B_{12}$, $B_{23}$ be arbitrary matrices (in our case they are unimodular matrices), then the matrices $C_b + B_{12}^T C_c B_{12}$ and $L_c + B_{23} L_b B_{23}^T$ are positive definite.

Proof: After Tokad and Kesavan [TO 1]

The final explicit form of the state model for any n-port LC network without ideal transformers is

$$\frac{d}{dt}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & (C_b + B_{12}^T C_c B_{12})^{-1} B_{22}^T \\ -(L_c + B_{23} L_b B_{23}^T)^{-1} B_{22} & 0 \end{bmatrix} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix}$$

$$- \begin{bmatrix} (C_b + B_{12}^T C_c B_{12})^{-1} B_{12}^T C_c B_{11} & 0 \\ 0 & -(L_c + B_{23} L_b B_{23}^T)^{-1} B_{23} L_b B_{33}^T \end{bmatrix} \frac{d}{dt}\begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -(C_b + B_{12}^T C_c B_{12})^{-1} B_{32}^T \\ -(L_c + B_{23} L_b B_{23}^T)^{-1} B_{21} & 0 \end{bmatrix} \begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix}$$

$$I_0^* = \left\{B_{11}^T C_c B_{11} - B_{11}^T C_c B_{12}(C_b + B_{12}^T C_c B_{12})^{-1} B_{12}^T C_c B_{11}\right\} \frac{d}{dt} V_0^*$$

$$+ \left\{B_{11}^T C_c B_{12}(C_b + B_{12}^T C_c B_{12})^{-1} B_{22}^T - B_{21}^T\right\} I_{c\ell}$$

$$- B_{11}^T C_c B_{12}(L_c + B_{23} L_b B_{23}^T)^{-1} B_{21} V_0^* + B_{31}^T I_1^*$$

and

$$V_1^* = - \Bigg[ \left\{ B_{33} L_b B_{33}^T + B_{33} L_b B_{23}^T (L_c + B_{23} L_b B_{23}^T)^{-1} B_{23} L_b B_{33}^T \right\} \frac{d}{dt} I_1^*$$

$$+ \left\{ B_{32} - B_{33} L_b B_{23}^T (L_c + B_{23} L_b B_{23}^T)^{-1} B_{22} \right\} V_{bc}$$

$$- B_{33} L_b B_{23}^T (C_b + B_{12}^T C_c B_{12})^{-1} B_{32}^T I_1^* + B_{31} V_0^* \Bigg] \qquad (2.3.9)$$

Other investigators (e.g., [BR 1, 2]) have given alternate procedures for developing the general form of the state model for LC networks with restricted voltage and current drivers (voltage driver in series with an element and current driver in parallel with an element). For this restricted class of networks the procedure of Bryant and that given here provides a method for investigating the network without drivers. When all voltage drivers are replaced by short circuits and all current drivers are replaced by open circuits then the natural frequencies of the resulting network are invariant.

However, in the procedure given here there is no restriction on the location of the drivers in a network and, in general, replacing voltage drivers by short circuits and the current drivers by open circuits modifies the network and hence the natural frequencies. However, by the following procedure the general LC network with arbitrary drivers, can be transformed into an LC network without the drivers, while holding the number of natural frequencies invariant.

Procedure:

1. Substitute a capacitor for those voltage drivers which form circuits with capacitors in the network and a short circuit for all other voltage drivers.
2. Substitute an inductor for those current drivers which form cut-sets with inductors and open circuit for all others.

In this procedure, the size of the capacitors (inductors) substituted for voltage drivers (current drivers) has no effect on the number of natural frequencies. In the case of an arbitrary n-port network, it is always possible to select the type of excitation at the ports such that the number of natural frequencies of the network will be invariant, i.e., some of the excitations can be taken as voltage drivers, such that they do not form circuits with other capacitors and the remaining excitations can be taken as current drivers, such that they do not form cut-sets with the other inductors.

<u>Theorem 2.3.2:</u> For an LC network the natural frequencies are located on the imaginary axis, i.e., the operator matrix has eigenvalues on the imaginary axis.

Proof: The operator matrix in (2.3.9) can be written as

$$A = \begin{bmatrix} (C_b + B_{12}^T C_c B_{12})^{-1} & 0 \\ 0 & (L_c + B_{23} L_b B_{23}^T)^{-1} \end{bmatrix} \begin{bmatrix} 0 & B_{22}^T \\ -B_{22} & 0 \end{bmatrix} = \begin{bmatrix} X_1^{-1} & \\ & Y_1^{-1} \end{bmatrix} B_1 = PB_1 \tag{2.3.10}$$

where

$$C_b + B_{12}^T C_c B_{12} = X_1$$

$$L_c + B_{23} L_b B_{23}^T = Y_1$$

and

$$\begin{bmatrix} 0 & B_{22}^T \\ -B_{22} & 0 \end{bmatrix} = B_1$$

Since $B_1$ is a skew symmetric matrix eigenvalues of B are all on the imaginary axis (See, e.g., [PE 1], p. 196). To prove that A also has eigenvalues on the imaginary axis, consider the characteristic equation of A:

$$|A - \lambda U| = 0$$
$$|PB_1 - \lambda U| = 0 \qquad (2.3.11)$$

From Theorem 2.3.1, it is known that $\begin{bmatrix} X_1 & \\ & Y_1 \end{bmatrix}$ is a symmetric and positive definite matrix. Therefore, the positive definite matrix,

$$\begin{bmatrix} X^{1/2} & \\ & Y^{1/2} \end{bmatrix}$$

exists, and is unique [PE 1, p. 203]. Hence (2.3.11) can be written as

$$\begin{aligned} \det(PB_1 - \lambda U) &= |P^{1/2}| \, |P^{1/2} B_1 - \lambda P^{-1/2} U| \\ &= |P^{1/2}| \, |P^{1/2} B_1 P^{1/2} - \lambda U| \, |P^{-1/2}| \\ &= |P^{1/2} B_1 P^{1/2} - \lambda U| \end{aligned}$$

where, since $P^{1/2} B_1 P^{1/2}$ is a skew symmetric matrix, all its eigenvalues lie on the imaginary axis. This implies, $A = PB_1$ has eigenvalues on the imaginary axis.

## 2.4 Short-Circuit and Open-Circuit Parameter Matrices from the State Model

To obtain short circuit or open circuit parameters from the state model equations, i.e., from (2.3.9) or from (2.3.5) and (2.3.8), let (2.3.5) and (2.3.8) be repeated here for ready reference.

$$\begin{bmatrix} C_b + B_{12}^T C_c B_{12} & 0 \\ 0 & L_c + B_{23} L_b B_{23}^T \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & B_{22}^T \\ -B_{22} & 0 \end{bmatrix} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix}$$

$$- \begin{bmatrix} B_{12}^T C_c B_{11} & 0 \\ 0 & -B_{23} L_b B_{33}^T \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix} + \begin{bmatrix} 0 & -B_{22}^T \\ -B_{21} & 0 \end{bmatrix} \begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix}$$

(2.4.1)

and

$$I_0^* = B_{11}^T C_c B_{11} \frac{d}{dt} V_0^* + B_{11}^T C_c B_{12} \frac{d}{dt} V_{bc} - B_{21}^T I_{c\ell} + B_{31}^T I_1^*$$

$$V_1^* = - \left\{ B_{31} V_0^* + B_{32} V_{bc} + \frac{d}{dt} B_{33} L_b B_{23}^T I_{c\ell} - B_{33} L_b B_{33}^T \frac{d}{dt} I_1^* \right\}$$

(2.4.2)

Although the derivations of state variables appear explicitly in (2.4.2) they can be eliminated if desired, by direct substitution from the differential equations. Since the coefficient matrix to the left of (2.4.1) is positive definite, its inverse exists. In (2.5.1) introduce the following transformation of variables

$$\begin{bmatrix} (C_b + B_{12}^T C_c B_{12})^{1/2} & 0 \\ 0 & (L_c + B_{23} L_b B_{23}^T)^{1/2} \end{bmatrix} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix}$$

(2.4.3)

where the coefficient matrix represents a positive definite root of the matrix to be inverted. This transformation takes (2.4.1) and (2.4.2) into the form

$$\frac{d}{dt}\begin{bmatrix} V'_{bc} \\ I'_{c1} \end{bmatrix} = \begin{bmatrix} 0 & (C_b+B^T_{12}C_cB_{12})^{-1/2}B^T_{22}(L_c+B_{23}L_bB^T_{23})^{-1/2} \\ -(L_c+B_{23}L_bB^T_{23})^{-1/2}B_{22}(C_b+B^T_{12}C_cB_{12})^{-1/2} & 0 \end{bmatrix}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix}$$

$$- \begin{bmatrix} (C_b + B^T_{12}C_cB_{12})^{-1/2}B^T_{12}C_cB_{11} & 0 \\ 0 & -(L_c + B_{23}L_bB^T_{23})^{-1/2}B_{23}L_bB^T_{33} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} V^*_0 \\ I^*_1 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & (C_b + B^T_{12}C_cB_{12})^{-1/2}B^T_{32} \\ (L_c + B_{23}L_bB^T_{23})^{-1/2} & 0 \end{bmatrix}\begin{bmatrix} V^*_0 \\ I^*_1 \end{bmatrix}$$

and

$$I^*_0 = B^T_{11}C_cB_{11}\frac{d}{dt}V^*_0 + B^T_{11}C_cB_{12}(C_b + B^T_{12}C_cB_{12})^{-1/2}\frac{d}{dt}V'_{bc}$$

$$-(L_c + B_{23}L_bB^T_{23})^{-1/2}B^T_{21}I'_{c\ell} + B^T_{31}I^*_1$$

$$V_1^* = - \Big\{ B_{31} V_0^* + B_{32} (C_b + B_{12}^T C_c B_{12})^{-1/2} V_{bc}'$$

$$+ B_{33} L_b B_{23}^T (L_c + B_{23} L_b B_{23}^T)^{-1/2} \frac{d}{dt} I_{c\ell}' - B_{33} L_b B_{33}^T \frac{d}{dt} I_1^* \Big\} \tag{2.4.4}$$

It is evident that the operator matrix is still skew symmetric, i.e., the skew symmetric property of the operator matrix is invariant under the transformation in (2.4.3).

For convenience define the following variables

$$\begin{aligned}
(C_b + B_{12}^T C_c B_{12})^{1/2} &= X_1^{1/2} \\
(L_c + B_{23} L_b B_{23}^T)^{1/2} &= Y_1^{1/2} \\
B_{12}^T C_c B_{11} &= X_2 \\
B_{23} L_b B_{33}^T &= Y_2 \\
B_{11}^T C_c B_{11} &= X_3 \\
B_{33} L_b B_{33}^T &= Y_3
\end{aligned} \tag{2.4.5}$$

and let (2.4.4) be written as

$$\frac{d}{dt} \begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} = \begin{bmatrix} 0 & X_1^{-1/2} B_{22}^T Y_1^{-1/2} \\ -Y_1^{-1/2} B_{22} X_1^{-1/2} & 0 \end{bmatrix} \begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix}$$

$$
- \begin{bmatrix} X_1^{-1/2} X_2 & 0 \\ 0 & -Y_1^{-1/2} Y_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix} - \begin{bmatrix} 0 & X_1^{-1/2} B_{32}^T \\ -Y_1^{-1/2} B_{21} & 0 \end{bmatrix} \begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix}
$$

$$
I_0^* = X_3 \frac{d}{dt} V_0^* + X_2^T X_1^{-1/2} \frac{d}{dt} V_{bc}' - B_{21}^T Y_1^{-1/2} I_{c\ell}' + B_{31}^T I_1^*
$$

$$
V_1^* = - \left\{ B_{13} V_0^* + B_{32} X_1^{-1/2} V_{bc}' + Y_2^T Y_1^{-1/2} \frac{d}{dt} I_{c\ell}' - Y_3 \frac{d}{dt} I_1^* \right\}
\tag{2.4.6}
$$

### 2.4.1 Short-Circuit Parameters

If an LC network contains only voltage drivers then the s-domain short-circuit parameter matrix is of interest and is obtained from the state model in (2.4.6) by setting $I_1^* = 0$ and replacing $\frac{d}{dt}$ by s. This, of course, implies that all initial conditions are taken equal to zero. The result is

$$
\begin{bmatrix} sU & -X_1^{-1/2} B_{22}^T Y_1^{-1/2} \\ Y_1^{-1/2} B_{22} X_1^{-1/2} & sU \end{bmatrix} \begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} = \begin{bmatrix} sX_1^{-1/2} X_2 \\ Y_1^{-1/2} B_{21} \end{bmatrix} V_0^*
$$

and

$$
I_0^* = sX_3 V_0^* + sX_2^T X_1^{-1/2} V_{bc}' - B_{21}^T Y_1^{-1/2} I_{c\ell}' \tag{2.4.7}
$$

Solving the first expression in (2.4.7) for $V_{bc}'$ and substituting into the second and third gives

$$(sU\, I'_{c\ell} + \frac{1}{s} Y_1^{-1/2} B_{22} X_1^{-1} B_{22}^T Y_1^{-1/2} I'_{c\ell}) =$$

$$Y_1^{-1/2} B_{22} X_1^{-1/2} X_1^{-1/2} X_2 V_0^* - Y_1^{-1/2} B_{21} V_0^*$$

or

$$I'_{c\ell} = \left[ sU + \frac{1}{s} Y_1^{-1/2} B_{22} X_1^{-1} B_{22}^T Y_1^{-1/2} \right]^{-1} \left[ Y_1^{-1/2} B_{22} X_1^{-1} X_2 - Y_1^{-1/2} B_{21} \right] V_0^* \tag{2.4.8}$$

and

$$I_0^* = s(X_3 - X_2^T X_1^{-1} X_2) V_0^* + (X_2^T X_1^{-1} B_{22}^T Y_1^{-1/2} - B_{21}^T Y_1^{-1/2})$$

$$\left[ sU + \frac{1}{s} Y_1^{-1/2} B_{22} X_1^{-1} B_{22}^T Y_1^{-1/2} \right]^{-1} \left[ X_2^T X_1^{-1} B_{22}^T Y_1^{-1/2} - B_{21}^T Y_1^{-1/2} \right]^T V_0^* \tag{2.4.9}$$

using (2.4.5) this last expression can be rewritten as

$$I_0^* = s \left[ B_{11}^T C_c B_{11} - B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1} B_{12}^T C_c B_{11} \right] V_0^*$$

$$+ \left[ \left[ (B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1} B_{22}^T (L_c + B_{23} L_b B_{23}^T)^{-1/2} \right.\right.$$

$$\left. - B_{21}^T (L_c + B_{23} L_b B_{23}^T)^{-1/2} \right] \times$$

$$\left\{ sU+\frac{1}{s}(L_c+B_{23}L_bB_{23}^T)^{-1/2}B_{22}(C_b+B_{12}^TC_cB_{12})^{-1}B_{22}^T(L_c+B_{23}L_bB_{23}^T)^{-1/2}\right\}^{-1}$$

$$\times\left[B_{11}^TC_cB_{12}(C_b+B_{12}^TC_cB_{12})^{-1}B_{22}^T(L_c+B_{23}L_bB_{23}^T)^{-1/2}\right.$$

$$\left.-B_{21}(L_c+B_{23}L_bB_{23}^T)^{-1/2}\right]V_0^* \qquad (2.4.10)$$

### 2.4.2 Open-Circuit Parameters

In the case when the LC network contains only current drivers, the open-circuit parameters can be derived through a procedure similar to that considered in Section 2.4.1. Indeed replacing $\frac{d}{dt}$ by s and setting $V_0^* = 0$ in (2.4.6), we have

$$\begin{bmatrix} sU & -X_1^{-1/2}B_{22}^TY_1^{-1/2} \\ Y_1^{-1/2}B_{22}X_1^{-1/2} & sU \end{bmatrix}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = -\begin{bmatrix} X_1^{-1/2}B_{32}^T \\ -sY_1^{-1/2}Y_2 \end{bmatrix} I_1^*$$

and

$$V_1^* = -\left\{B_{32}X_1^{-1/2}V'_{bc} + sY_2^TY_1^{-1/2}I'_{c\ell} - sY_3I_1^*\right\} \qquad (2.4.11)$$

Solving the second expression in (2.4.11) for $I'_{c\ell}$ and substituting into the first and third gives

$$V_{bc}' = \left\{ sU + \frac{1}{s} X_1^{-1/2} B_{22}^T Y_1^{-1} B_{22} X_1^{-1/2} \right\}^{-1} \left\{ -X_1^{-1/2} B_{32}^T + X_1^{-1/2} B_{22}^T Y_1^{-1} Y_2 \right\} I_1^* \tag{2.4.12}$$

$$\begin{aligned} V_1^* = {} & s(Y_3 - Y_2^T Y_1^{-1} Y_2) I_1^* + (Y_2^T Y_1^{-1} B_{22} X_1^{-1/2} - B_{32} X_1^{-1/2}) \quad \times \\ & \left\{ sU + \frac{1}{s} X_1^{-1/2} B_{22}^T Y_1^{-1} B_{22} X_1^{-1/2} \right\}^{-1} (Y_2^T Y_1^{-1} B_{22}^T X_1^{-1/2} - B_{32} X_1^{-1/2})^T I_1^* \end{aligned} \tag{2.4.13}$$

Using (2.4.5) this last expression can be rewritten as

$$\begin{aligned} V_1^* = {} & s\left[ B_{33} L_b B_{33}^T - B_{33} L_b B_{23}^T (L_c + B_{23} L_b B_{23}^T)^{-1} B_{23} L_b B_{33}^T \right] I_1^* \\ & + \left[ B_{33} L_b B_{23}^T (L_c + B_{23} L_b B_{23}^T)^{-1} B_{22} (C_b + B_{12}^T C_c B_{12})^{-1/2} - B_{32} (C_b + B_{12}^T C_c B_{12})^{-1/2} \right] \\ & \times \left\{ sU + \frac{1}{s} (C_b + B_{12}^T C_c B_{12})^{-1/2} B_{22}^T (L_c + B_{23} L_b B_{23}^T)^{-1} B_{22} (C_b + B_{12}^T C_c B_{12})^{-1/2} \right\}^{-1} \\ & \times \left[ B_{23} L_b B_{23}^T (L_c + B_{23} L_b B_{23}^T)^{-1} B_{22} (C_b + B_{12}^T C_c B_{12})^{-1/2} \right. \\ & \left. - B_{32} (C_b + B_{12}^T C_c B_{12})^{-1/2} \right]^T I_1^* \end{aligned} \tag{2.4.14}$$

Equations (2.4.10) and (2.4.14) are the desired basic equations, and the specifications of the networks are usually given by these forms in network synthesis. In symbolic form (2.4.10) and (2.4.14) can be written as

$$I_0^* = Y V_0^* = (A_1 s + A_2^T (sU + \frac{1}{s} A_3)^{-1} A_2) V_0^* \qquad (2.4.15)$$

and

$$V_1^* = Z I_1^* = (A_4 s + A_5^T (s + \frac{1}{s} A_6)^{-1} A_5) I_1^* \qquad (2.4.16)$$

## 2.5 Certain Necessary Conditions for Transformerless Realization

For the realization of RLC networks without the ideal transformers, Cederbaum has given the following theorem [CE 1].

Theorem 2.5.1: A necessary condition for a matrix to be an impedance or admittance matrix of an RLC n-port is that, it is a paramount matrix.

Proof: After Cederbaum [CE 1]

The following corollary is immediate.

Corollary 2.5.1: A necessary condition for a matrix to be impedance or admittance matrix of an LC n-port without ideal transformers is that, the matrix is symmetric, positive real and paramount.

Proof: Follows from Theorem 2.5.1

Theorem 2.5.2: Let Z(s) or Y(s) be a paramount matrix having a pole at the origin or at infinity, then the residue matrices corresponding to these poles are paramount.

Proof: Follows from Theorem 2.5.1

From the state model the following necessary condition for realization of LC network without ideal transformers is developed.

Theorem 2.5.3: A necessary condition for an admittance (or impedance) matrix to have a pole at infinity is that the network contains at least one cut-set consisting of inductors and current drivers only (or at least one circuit consisting of capacitors and voltage drivers only).

Proof: From (2.4.10) and (2.4.15) ( (2.4.14) and (2.4.16) ) the existence of the residue matrix $A_1$ ($A_4$) corresponding to the pole at infinity implies that the submatrices $B_{11}$ and $C_c$ ($B_{33}$ and $L_b$) are non empty and from (2.3.1) the conclusion of the theorem is evident.

## 2.6 An Important Special Case

Consider an LC network in which there are no circuits of capacitors with or without drivers, no cut-sets of capacitors, no circuits of inductors and no cut-sets of inductors with or without current drivers. However, cut-sets of capacitors with the voltage drivers only and circuits of inductors with the current drivers only are allowed. It is shown in Chapter III that this special case corresponds to canonic networks. For this reason the LC network satisfying the above condition will be called a canonic LC network.

The circuit and cut-set equations for this special case take on the form

$$\begin{bmatrix} B_{11} & B_{12} & U & 0 \\ B_{21} & B_{22} & 0 & U \end{bmatrix} \begin{bmatrix} V_0 \\ V_{bc} \\ V_{c\ell} \\ V_1 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} U & 0 & -B_{11}^T & -B_{21}^T \\ 0 & U & -B_{12}^T & -B_{22}^T \end{bmatrix} \begin{bmatrix} I_0 \\ I_{bc} \\ I_{c\ell} \\ I_1 \end{bmatrix} = 0 \qquad (2.6.1)$$

Therefore, the state model is given as

$$\begin{bmatrix} C_b & \\ & L_c \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} 0 & B_{12}^T \\ \hline -B_{12} & 0 \end{array}\right] \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} + \left[\begin{array}{c|c} 0 & B_{22}^T \\ \hline -B_{11} & 0 \end{array}\right] \begin{bmatrix} V_0 \\ I_1 \end{bmatrix}$$

$$I_0 = B_{11}^T I_{c\ell} + B_{21}^T I_1$$

$$V_1 = -B_{21} V_0 - B_{22} V_{bc} \qquad (2.6.2)$$

Again, terminal variables $I_0^*$, $I_1^*$, $V_0^*$ and $V_1^*$ are related to the driving variables by

$$I_0^* = -I_0, \qquad V_0^* = V_0$$

and

$$I_1^* = -I_1, \qquad V_1^* = V_1$$

a) If there are only voltage drivers present then the state model can be simplified further and has the form

$$\frac{d}{dt}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & C_b^{-1}B_{12}^T \\ -L_c^{-1}B_{12} & 0 \end{bmatrix}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} + \begin{bmatrix} 0 \\ -B_{11} \end{bmatrix} V_0^*$$

$$I_0^* = -B_{11}^T\, I_{c\ell} \qquad (2.6.3)$$

By using the transformation of variables (2.6.3) becomes

$$\begin{bmatrix} C_b^{1/2} & \\ & L_c^{1/2} \end{bmatrix}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} \qquad (2.6.4)$$

we have

$$\frac{d}{dt}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & C_b^{-1/2}B_{12}^T L_c^{-1/2} \\ -L_c^{-1/2}B_{12}C_b^{-1/2} & 0 \end{bmatrix}\begin{bmatrix} V'_{bc} \\ I_{c\ell} \end{bmatrix} + \begin{bmatrix} 0 \\ -L_c^{-1/2}B_{11} \end{bmatrix} V_0^*$$

$$I_0 = -B_{11}^T\, L_c^{-1/2}\, I_{c\ell} \tag{2.6.5}$$

The short circuit parameters are obtained from (2.6.5) as

$$I_0^* = B_{11}^T L_c^{-1/2}\left(sU + \frac{1}{s} C_b^{-1/2} B_{12}^T L_c^{-1} B_{12} C_b^{-1/2}\right)^{-1} L_c^{-1/2} B_{11} V_0^* \tag{2.6.6}$$

which has no pole at infinity.

b) If there are only current drivers present in the network then the state model is again simplified and we have

$$\frac{d}{dt}\begin{bmatrix} V_{bc} \\ \\ I_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} 0 & C_b^{-1} B_{12}^T \\ \hline -L_c^{-1} B_{12} & 0 \end{array}\right] \begin{bmatrix} V_{bc} \\ \\ I_{c\ell} \end{bmatrix} + \begin{bmatrix} -C_b^{-1} B_{22}^T \\ \\ 0 \end{bmatrix} I_1^*$$

$$V_1^* = -B_{22}\, V_{bc} \tag{2.6.7}$$

Using the transformation of variables in (2.6.4), the state model transforms into

$$\frac{d}{dt}\begin{bmatrix} V_{bc}' \\ \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & C_b^{-1/2} B_{12}^T L_c^{-1/2} \\ \\ -L_c^{-1/2} B_{12} C_b^{-1/2} & 0 \end{bmatrix} \begin{bmatrix} V_{bc}' \\ \\ I_{c\ell}' \end{bmatrix} + \begin{bmatrix} -C_b^{-1/2} B_{22}^T \\ \\ 0 \end{bmatrix} I_1^*$$

$$V_1^* = -B_{22}\, C_b^{-1/2}\, V_{bc}' \tag{2.6.8}$$

and, the open circuit parameters are given by

$$V_1^* = B_{22}C_b^{-1/2}\left(sU + \frac{1}{s}L_c^{-1/2}B_{12}C_b^{-1}B_{12}^TL_c^{-1/2}\right)^{-1}C_b^{-1/2}B_{22}^TI_1^* \tag{2.6.9}$$

In the above two cases, it is concluded that the short and open circuit parameters cannot have a pole at infinity for the canonic LC networks.

c) The derivation of hybrid parameters for general LC networks from the state model is possible, but the final relation is very complex and are not considered in the thesis. However, for completeness a procedure is given below for canonic LC networks.

Using the transformation of variables in (2.6.4) and the relations $I_0^* = -I_0$, $V_0^* = V_0$, $V_1^* = V_1$, and $I_1^* = -I_1$ (2.6.2) can be written as

$$\frac{d}{dt}\begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} = \begin{bmatrix} 0 & C_b^{-1/2}B_{12}^TL_c^{-1/2} \\ -L_c^{-1/2}B_{12}C_b^{-1/2} & 0 \end{bmatrix}\begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} + \begin{bmatrix} 0 & -C_b^{-1/2}B_{22}^T \\ -L_c^{-1/2}B_{11} & 0 \end{bmatrix}\begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix}$$

$$I_0^* = -B_{11}^TL_c^{-1/2}I_{c\ell}' + B_{21}^T I_1^*$$

$$V_1^* = -B_{21} V_0^* - B_{22} C_b^{-1/2} V_{bc}' \tag{2.6.10}$$

After replacing $\frac{d}{dt}$ by s in (2.6.10), we have

$$\begin{bmatrix} sU & -C_b^{-1/2} B_{12}^T L_c^{-1/2} \\ L_c^{-1/2} B_{12} C_b^{-1/2} & sU \end{bmatrix} \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & -C_b^{-1/2} B_{22}^T \\ -L_c^{-1/2} B_{11} & 0 \end{bmatrix} \begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix} .$$

and

$$I_0^* = -B_{11}^T L_c^{-1/2} I'_{c\ell} + B_{21}^T I_1^*$$

$$V_1^* = -B_{21} V_0^* - B_{22} C_b^{-1/2} V'_{bc} \qquad (2.6.11)$$

From the first equation in (2.6.11)

$$V'_{bc} = \frac{1}{s} C_b^{-1/2} B_{12}^T L_c^{-1/2} I'_{c\ell} - \frac{1}{s} C_b^{-1/2} B_{22}^T I_1^* \qquad (2.6.12)$$

Substituting $V'_{bc}$ into the second equation of (2.6.11) and solving for $I'_{c\ell}$ gives

$$I'_{c\ell} = \frac{1}{s} \left(sU + \frac{1}{s} L_c^{-1/2} B_{12} C_b^{-1} B_{12}^T L_c^{-1/2}\right)^{-1} L_c^{-1/2} B_{12} C_b^{-1} B_{22}^T I_1^*$$

$$- \left(sU + \frac{1}{s} L_c^{-1/2} B_{12} C_b^{-1} B_{12}^T L_c^{-1/2}\right)^{-1} L_c^{-1/2} B_{11} V_0^*$$

$$(2.6.13)$$

By substituting for $I'_{c\ell}$ in (2.6.12) we have

$$V'_{bc} = \frac{1}{s^2} C_b^{-1/2} B_{12}^T L_c^{-1/2} \left(sU + \frac{1}{s} L_c^{-1/2} B_{12} C_b^{-1} B_{12}^T L_c^{-1/2}\right)^{-1} L_c^{-1/2} B_{12} C_b^{-1} B_{22}^T I_1^*$$

$$- \frac{1}{s} C_b^{-1/2} B_{12}^T L_c^{-1/2} \left(sU + \frac{1}{s} L_c^{-1/2} B_{12} C_b^{-1} B_{12}^T L_c^{-1/2}\right)^{-1} L_c^{-1/2} B_{11} V_0^*$$

$$- \frac{1}{s} C_b^{-1/2} B_{22}^T I_1^* \tag{2.6.14}$$

Substituting (2.6.13) and (2.6.14) into the last two equations of (2.6.10) gives the required hybrid parameters as

$$I_0^* = B_{11}^T L_c^{-1/2}(sU + \frac{1}{s} L_c^{-1/2} B_{12} C_b^{-1} B_{12}^T L_c^{-1/2})^{-1} L_c^{-1/2} B_{11} V_0^*$$

$$+ [B_{21}^T - \frac{1}{s} B_{11}^T L_c^{-1/2}(sU + \frac{1}{s} L_c^{-1/2} B_{12} C_b^{-1} B_{12}^T L_c^{-1/2})^{-1} L_c^{-1/2} B_{12} C_b^{-1} B_{22}^T] I_1^*$$

$$V_0^* = (\frac{1}{s} B_{22} C_b^{-1} B_{22}^T - \frac{1}{s^2} B_{22} C_b^{-1} B_{12}^T L_c^{-1/2}(sU + \frac{1}{s} L_c^{-1/2} B_{12} C_b^{-1} B_{12}^T L_c^{-1/2})^{-1} \times$$

$$L_c^{-1/2} B_{12} C_b^{-1} B_{22}^T I_1^*$$

$$- (B_{21} - \frac{1}{s} B_{22} C_b^{-1} B_{12}^T L_c^{-1/2}(sU + \frac{1}{s} L_c^{-1/2} B_{12} C_b^{-1} B_{12}^T L_c^{-1/2})^{-1} L_c^{-1/2} B_{11}) V_0^*$$

$$\tag{2.6.15}$$

Equation (2.6.15) can be written in symbolic form as

$$\begin{bmatrix} I_0^* \\ V_1^* \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix} \tag{2.6.16}$$

Note that in the last relation the entries of all the submatrices A, B, C, and D are rational functions of s. Further, the matrices A and D are symmetric and also $B = -C^T$.

# CHAPTER III

# SYNTHESIS OF LC NETWORKS

## 3.1 General

In this chapter the following problems are considered: 1) canonic forms of LC networks 2) realization of state models with and without ideal transformers 3) realization of state models from the given specifications in s-domain, and 4) realization of s-domain models with and without ideal transformers.

## 3.2 Canonic Forms of LC Networks

Although the properties of canonic one port LC (RL and RC) networks are well defined in network theory a clear definition of canonic LC n-ports has not been given. The following definition is based on the topology of LC n-port networks.

Definition 3.2.1: If all the capacitors of a connected LC network form both a tree and a co-tree in the system graph, then the network is a canonic network.

This definition implies the network N has no drivers. The case where N contains drivers will be considered later.

The number of elements in a canonic network has certain properties which will be used later in the synthesis procedure. Some of these properties are discussed in the form of theorems.

Theorem 3.2.1: Let G be the graph of a connected LC network N which has no drivers. If N has no circuits of capacitors, cut-sets of capacitors, circuits of inductors and cut-sets of inductors, then N is canonic and the number of capacitors is equal to the number of inductors.

Proof: From the hypothesis of the theorem it follows that all capacitors can form a tree, as well as a co-tree. This property implies, by Definition 3.2.1, that the network is canonic. Further, since $e - v + 1 = v - 1$ then the number of capacitors and the number of inductors in N are equal.

Theorem 3.2.2: In an LC network N containing drivers, if the capacitors and the voltage drivers form a tree as well as a co-tree (or equivalently if all the inductors and the current drivers form a tree as well as a co-tree) then in N

$$n_c + n_v = n_\ell + n_i$$

when

$n_c$ = number of capacitors, $n_v$ = number of voltage drivers

$n_\ell$ = number of inductors and $n_i$ = number of current drivers.

Proof: Follows immediately from Theorem 3.2.1.

Theorem 3.2.3: Let G be the graph of a connected LC network N. Also in N let there be

$k_1$ voltage drivers and $k_2$ current drivers,

$r_2$ independent circuits of inductors with the current drivers,

no circuits of capacitors with or without voltage drivers,

no cut-sets of inductors with or without current drivers,

no cut-sets of capacitors only,

and no circuits of inductors only

Then in N

$$n_c + (k_1 - r_1) = n_\ell + (k_2 - r_2)$$

Proof: Replace a voltage driver by a capacitor if the driver does not form a cut-set with the capacitors and by a short circuit if it does. Also replace a current driver by an inductor if the driver does not form a circuit with the inductors and by an open circuit if it does. This procedure reduces N to the network considered in Theorem 3.2.1, and the proof follows.

### 3.2.1 Tests for Canonic LC Networks

Consider the graph G of a connected LC network N. Let n terminal pairs on this network be specified as the ports, i.e., N is considered as an n-port network. Let $G_1$ be the graph of the drivers (the types of drivers yet to be determined) to be connected at the ports of N.

1. Consider the subgraph $G_c$ of $G \cup G_1$ which contains the elements corresponding to the capacitors and the subgraph $G_1$.

In $G_c$ if there is a circuit of capacitors then N cannot be a canonic network. If $G_c$ contains no circuit of capacitors then it is possible to select a tree in $G_c$ such that it contains all the capacitors. Further, this tree may contain elements of $G_1$, in this case the drivers corresponding to these elements are taken as voltage drivers and the remaining drivers in the network are taken as current drivers.

2. Let $G_L$ be the subgraph of $GUG_1$ which contains all inductors, and consider $G_cUG_L$. If there is a cut-set of inductors with or without current drivers then N cannot be a canonic LC. Otherwise consider the next step.

3. Replace a voltage driver by a capacitor if this driver does not form a cut-set with the capacitors and by a short circuit if it does. Replace a current driver by an inductor if this driver does not form a circuit with the inductors and by an open circuit if it does.

If the resulting network satisfies Theorem 3.2.1 then the n-port network under consideration is a canonic LC network.

An n-port canonic LC network can be generated, by the inverse of the above procedure, from a given arbitrary canonic LC network (Definition 3.2.1). If the n-terminal pairs are prescribed as the ports of the network, then the problem is reduced to that of selecting the type of drivers to be connected at the ports. These drivers by necessity must be current drivers, otherwise there would be a circuit of capacitors with the voltage driver. This procedure, in general, introduces a circuit of inductors with the current driver.

If a prescribed port is generated by simply breaking an element and introducing a port, then this port must be excited with a voltage driver. This procedure, in general, introduces a cut-set of capacitors with the voltage driver.

Other canonic LC networks with drivers can be generated from a given canonic LC network by replacing some capacitors by voltage drivers and some inductors with current drivers.

### 3.2.2 Degree of a Rational Matrix

The degree of a rational matrix, defined first by McMillan [MC 1] in 1951, recently has become an important concept in network theory [DU 1, KA 2]. The objective here is to derive a relation between the degree of the Y or Z matrix of an LC network and the minimum number of elements necessary to realize this matrix.

Definition 3.2.2: [DU 1] Let F(s) be an n x n matrix whose elements are rational functions of the complex variables s, and let A be an n x n matrix of complex constants. Then

$$|F(s) + A| = \frac{P_A(s)}{Q_A(s)}$$

where $P_A(s)$ and $Q_A(s)$ are polynomials in s and they are relatively prime. The degree of F(s) is the maximum degree of $P_A(s)$ for all possible choices of the constant matrix A.

McMillan [MC 1] defines the degree of Z or Y matrices as follows: Let $R_0$, $R_1$, ..., $R_n$, $R_\infty$ be the residue matrices of Z or Y with ranks $r_0$, $r_1$, ..., $r_n$, $r_\infty$ respectively. Then the degree of Z or Y is

$$r_0 + 2\Sigma r_i + r_\infty \tag{3.2.1}$$

Kalman [KA 2] has shown that, if Z or Y is regular at $\infty$, the degree, i.e., $(r_0 + 2\Sigma r_i)$ is equal to the number of state variables. When Z or Y has a pole at infinity, he transforms this form to the previous form by considering a Mobius transformation. However, it is easy to show by constructing a state model from Z or Y, as discussed in Section 3.3, that the McMillan's degree is equal to the number of state variables plus the rank of the residue matrix corresponding to the pole at infinity.

In the case of canonic LC networks, the number of elements is equal to the degree of Z or Y matrices. This is true, since the impedance matrix Z (or admittance matrix Y) cannot have a pole at infinity, and also the inductor and capacitor variables are present in the state vector.

## 3.3 Realization of a State Model of an LC n-port without Ideal Transformers

The specification of networks are usually given in the s-domain. However, a state model corresponding to the given s-domain model can always be found and such procedure is presented in Section 3.4. The state model so obtained can be realized by a network containing ideal transformers. This problem is considered in Section 3.5. In this present section, however, certain conditions are derived which are necessary to realize a given state model by an LC network without the ideal transformers. Given the specifications

$$\frac{d}{dt}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & K_1 \\ -K_2 & 0 \end{bmatrix}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} - \begin{bmatrix} K_3 & 0 \\ 0 & K_4 \end{bmatrix}\frac{d}{dt}\begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix} + \begin{bmatrix} 0 & K_5 \\ -K_6 & 0 \end{bmatrix}\begin{bmatrix} V_0^* \\ I_1^* \end{bmatrix}$$

$$I_0^* = K_7 \frac{d}{dt} V_0^* + K_8 I_{c\ell} + K_9 V_0^* + K_{10} I_1$$

$$V_1^* = -[K_{11} \frac{d}{dt} I_1^* + K_{12} V_{bc} + K_{13} I_1^* + K_{14} V_0^*] \tag{3.3.1}$$

a necessary condition for the realization of this model by an LC network is that the operator matrix must have eigenvalues on the imaginary axis.

Comparing (3.3.1) with (2.4.4) and identifying the matrices we have

$$K_1 = (C_b + B_{12}^T C_c B_{12})^{-1} B_{22}$$

$$K_3 = (C_b + B_{12}^T C_c B_{12})^{-1} B_{12}^T C_c B_{11}$$

$$K_5 = (C_b + B_{12}^T C_c B_{12})^{-1} B_{32}^T$$

$$K_2 = (L_c + B_{23} L_b B_{23}^T)^{-1} B_{22}$$

$$K_4 = (L_c + B_{23} L_b B_{23}^T)^{-1} B_{23} L_b B_{33}^T$$

$$K_6 = (L_c + B_{23} L_b B_{23}^T)^{-1} B_{21}$$

$$K_{14} = B_{31} = -K_{10}^T \tag{3.3.2}$$

$$K_7 = B_{11}^T C_c B_{11} - B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1} B_{12}^T C_c B_{11}$$

$$K_8 = B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1} B_{22}^T - B_{21}^T$$

$$K_9 = -B_{11}^T C_c B_{12} (L_c + B_{23} L_b B_{23}^T)^{-1} B_{21}$$

$$K_{11} = B_{33} L_b B_{33}^T + B_{33} L_b B_{23}^T (L_c + B_{23} L_b B_{23}^T)^{-1} B_{23} L_b B_{33}^T$$

$$K_{12} = B_{32} - B_{33} L_b B_{23}^T (L_c + B_{23} L_b B_{23}^T)^{-1} B_{22}$$

$$K_{13} = -B_{33} L_b B_{23}^T (C_b + B_{12}^T C_c B_{12})^{-1} B_{32}^T$$

where $K_i$ is known. The problem is that the matrices $B_{ij}$, $C_b$, $L_b$, $C_c$ and $L_c$ must be determined from the above equations. Let (3.3.2) be written in the form,

$$\begin{aligned}
&(C_b + B_{12}^T C_c B_{12}) K_1 = B_{22}^T \\
&(C_b + B_{12}^T C_c B_{12}) K_3 = B_{12}^T C_c B_{11} \\
&(C_b + B_{12}^T C_c B_{12}) K_5 = B_{32}^T \\
&(L_c + B_{23} L_b B_{23}^T) K_2 = B_{22} \\
&(L_c + B_{23} L_b B_{23}^T) K_4 = B_{23} L_b B_{33}^T \\
&(L_c + B_{23} L_b B_{23}^T) K_6 = B_{21} \\
&K_7 = B_{11}^T C_c B_{11} - B_{11}^T C_c B_{12} K_3 \\
&K_8 = B_{11}^T C_c B_{12} K_1 - B_{21}^T \\
&K_9 = -B_{11}^T C_c B_{12} K_6 \\
&K_{11} = B_{33} L_b B_{33}^T + B_{33} L_b B_{23}^T K_4 \\
&K_{12} = B_{32} - B_{33} L_b B_{23}^T K_2 \\
&K_{13} = -B_{33} L_b B_{23}^T K_5
\end{aligned} \qquad (3.3.3)$$

where each of the matrices

$$(C_b + B_{12}^T C_c B_{12}),\ B_{11}^T C_c B_{12},\ B_{11}^T C_c B_{11},\ B_{33} L_b B_{33}^T$$

$$(L_c + B_{23} L_b B_{23}^T),\ B_{33} L_b B_{23}^T,\ B_{21},\ B_{22},\ B_{32}$$

are considered as unknowns. Once these matrices are determined their constituents, can be found. Since the system of equations in (3.3.3) is linear, a solution exists only if they are consistent. Assuming that the system is consistent, a solution of the unknowns establishes a model of the form given in (2.5.1). In this form the submatrices $C_b + B_{12}^T C_c B_{12}$ and $L_c + B_{23} L_b B_{23}^T$ by necessity are positive definite.

From (2.5.1) the submatrices $B_{ij}$, $L_b$, $L_c$, $C_b$ and $C_c$ must be determined. Since from the above solution the submatrices $B_{11}^T C_c B_{11}$, $B_{11}^T C_c B_{12}$ and $C_b + B_{12}^T C_c B_{12}$ are known, construct the matrix

$$Y_c = \begin{bmatrix} B_{11}^T C_c B_{11} & B_{11}^T C_c B_{12} \\ \\ B_{12}^T C_c B_{11} & C_b + B_{12}^T C_c B_{12} \end{bmatrix} \tag{3.3.4}$$

where $Y_c$ corresponds to the admittance matrix of a capacitor network and can be obtained by substituting $V_{b1} = V_{c\ell} = V_1 = 0$ in the node equations of the network.

To determine the matrices $B_{11}$, $B_{12}$, $C_b$ and $C_c$ in (3.3.4) apply the decomposition algorithm of Cederbaum [CE 2] to (3.3.4). Rearranging the rows and columns gives

$$Y_c = \begin{bmatrix} 0 & -B_{11}^T \\ U & -B_{12}^T \end{bmatrix} \begin{bmatrix} C_b & \\ & C_c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -B_{11} & -B_{12} \end{bmatrix} \tag{3.3.5}$$

which is essentially unique due to the nature of this alogarithm. If entries in either $C_b$ and $C_c$ are not positive or if

$$\begin{bmatrix} 0 & -B_{11}^T \\ U & -B_{12}^T \end{bmatrix}$$

is not a cut-set matrix, then, the given state model is not realizable.

The above procedure can be repeated for a new matrix $Z_L$ having the matrices $B_{33} L_b B_{33}^T$, $B_{23} L_b B_{33}^T$ and $L_c + B_{23} L_b B_{23}^T$ as its submatrices, where $Z_L$ corresponds to an impedance matrix of the inductor network. The matrices $B_{23}$, $B_{33}$, $L_c$ and $L_b$ are again determined by the use of the decomposition algorithm. Here the diagonal matrices $L_c$ and $L_b$ must have positive entries and $B_{23}$, $B_{33}$ must be submatrices of an unimodular matrix. From the submatrices determined above and the solution of linear equations in (3.3.2) all the submatrices appearing in (2.5.1) are determined completely, and the circuit matrix is established as

$$B = \left[\begin{array}{ccc|ccc} B_{11} & B_{12} & 0 & U & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 & U \end{array}\right] \tag{3.3.6}$$

The problem is reduced, therefore, to the realization of this circuit matrix. If it is realizable, the topology of the LC network can be determined through any one of the known techniques [TU 1, GO 1, KI 1, BI 1, GU 2, CE 3]. In summary the state model is realizable without ideal transformers if

1. Solution exists for the system (3.3.3)
2. The circuit matrix B in (3.3.6), is realizable, and

3. $C_b$, $C_c$, $L_b$ and $L_c$ are diagonal matrices with positive diagonal entries.

It is interesting to note that if the conditions given in the above are not satisfied, this does not imply that the network is not realizable without the ideal transformers. Indeed, another state model obtained from that given by a similarity transformation might be realized as an LC network without the ideal transformers.

## 3.4 Derivation of a State Model from the s-domain Equations

The derivation of a state model from the s-domain model has been considered by Gilbert and Kalman [GI 1, KA 1, ZA 1] and they have described certain procedures for this derivation. In this thesis, since our main interest is LC networks, a slightly different method for deriving the state model is given. The procedures are the same for the open and short circuit parameters.

Let Y be expanded into partial fractions to obtain

$$Y = R_0 \frac{1}{s} + \Sigma R_i \frac{s}{s^2 + \omega_i^2} + R_\infty s \tag{3.4.1}$$

where $R_0$, $R_i$ and $R_\infty$ are the residue matrices whose properties are well known [CA 1].

The operator matrix in (2.4.6) corresponding to an LC network is skew symmetric and consequently it has pure imaginary eigenvalues $\pm j\lambda_1$, $\pm j\lambda_2$, ... . Note that some of the eigenvalues may be zero.

<u>Theorem 3.4.1</u>: Let A be a skew symmetric matrix of the form

$$A = \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix}$$

Then there exists an orthogonal matrix P such that

$$P^T AP = \Lambda$$

where the matrix $\Lambda$ has one of the two forms

$$\Lambda_1 = \left[\begin{array}{c|c} \bigcirc & \begin{matrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{matrix} \\ \hline \begin{matrix} -\lambda_1 & & & & 0 & \cdots & 0 \\ & -\lambda_1 & & & \vdots & & \vdots \\ & & \ddots & & \vdots & & \vdots \\ & & & -\lambda_k & 0 & \cdots & 0 \end{matrix} & \bigcirc \end{array}\right] = \left[\begin{array}{c|c} \bigcirc & \begin{matrix} D \\ \hline 0 \end{matrix} \\ \hline \begin{array}{c|c} -D & 0 \end{array} & \bigcirc \end{array}\right]$$

$$= \left[\begin{array}{c|c} \bigcirc & D'^T \\ \hline -D' & \bigcirc \end{array}\right] \qquad (3.4.2)$$

or

$$\Lambda_2 = \left[\begin{array}{c|c} \bigcirc & \begin{matrix}\lambda_1 & & & & 0\cdots 0\\ & \lambda_2 & & & \vdots \quad \vdots\\ & & \ddots & & \vdots \quad \vdots \\ & & & \lambda_k & 0\cdots 0\end{matrix} \\ \hline \begin{matrix} -\lambda_1 & & & \\ & -\lambda_2 & & \\ & & \ddots & \\ & & & -\lambda_k \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0\end{matrix} & \bigcirc \end{array}\right] = \left[\begin{array}{c|c|c} \bigcirc & D & 0 \\ \hline -D & \multicolumn{2}{c}{\multirow{2}{*}{$\bigcirc$}} \\ \cline{1-1} 0 & & \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 & D'^{T} \\ \hline -D' & 0 \end{array}\right] \qquad (3.4.3)$$

The expression for $\Lambda_1$ applies when the number of branch capacitors is greater than the number of chord inductors, and $\Lambda_2$ applies when the number of branch capacitors is less than the number of chord inductors. In $\Lambda_1$ and $\Lambda_2$, $\lambda_i$'s are taken as the absolute values of the eigenvalues of A.

Proof: Follows from the application of Theorem 5-11 in Perilis [PE 1]

To apply the above theorem to the operator matrix in the state model of (2.4.6), ($I_1^* = 0$), let P be a transformation matrix such that

$$P^T \begin{bmatrix} V'_{bc} \\ \\ I'_{c\ell} \end{bmatrix} = \begin{bmatrix} V_2 \\ \\ I_3 \end{bmatrix} \tag{3.4.4}$$

The transposed state model is then

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & D'^T \\ \\ -D' & 0 \end{bmatrix}\begin{bmatrix} V_2 \\ \\ I_3 \end{bmatrix} - P^T \begin{bmatrix} X_1^{-1/2} X_2 \\ \\ 0 \end{bmatrix} \frac{d}{dt} V_0^* - P^T \begin{bmatrix} 0 \\ \\ Y_1^{-1/2} B_{21} \end{bmatrix} V_0^*$$

$$I_0^* = X_3 \frac{d}{dt} V_0^* + X_2^T X_1^{-1/2} P \frac{d}{dt} V_2 - B_{21}^T Y_1^{-1/2} P_3^T \tag{3.4.5}$$

and the short circuit equations are

$$I_0^* = s(X_3 - X_2^T X_1^{-1} X_2) V_0^* + (X_2^T X_1^{-1} B_{22}^T Y_1^{-1/2} - B_{21}^T Y_1^{-1/2}) P \times$$

$$[Us + \frac{1}{s} D' D'^T]^{-1} P^T [X_2^T X_1^{-1} B_{22}^T Y_1^{-1/2} - B_{21}^T Y_1^{-1/2}]^T V_0^* \tag{3.4.6}$$

which is of the form

$$I_0^* = [R_\infty s + R_0 \frac{1}{s} + \Sigma \frac{s}{s^2 + \omega_1^2} R_i] V_0^* \tag{3.4.7}$$

The fundamental problem is to derive (3.4.5), or an equivalent state model from (3.4.7).

Since the short circuit and the open circuit matrices are positive real, the residue matrices $R_0$, $R_\infty$ and $R_i$ are all positive definite or semi-definite [CA 1]. Decomposition of each residue matrix $R_j$ into the form $R_j = K_j^T K_j$ is always possible as stated in the following theorem. Even though this theorem is well known, the proof given here presents a procedure for decomposing the residue matrices.

Theorem 3.4.2: The necessary and sufficient condition for the decomposition of any real symmetric matrix R, into the products of the form $K^T K$, is that R is positive definite or positive semi-definite.

Proof: Necessity is evident [HN 1]. To prove the sufficiency, consider an orthogonal matrix V which transforms R into diagonal matrix D, i.e., $V^T R V = D_1$; where $D_1$ is a diagonal matrix consisting of the eigenvalues of R. Since R is positive definite or semidefinite, all the elements in $D_1$ are either positive or zero. Let the positive square root of the matrix of $D_1$ be denoted by $D_1^{1/2}$, i.e., all the elements in $D_1^{1/2}$ are all non-negative. Then R can be written as

$$\begin{aligned} R &= (V D_1^{1/2})(V D_1^{1/2})^T \\ &= K^T K \end{aligned} \tag{3.4.8}$$

which proves the sufficiency.

In (3.4.8) K is a square matrix, consisting of zero rows corresponding to the number of zero eigenvalues of R. These rows in K can be deleted, to give a rectangular matrix $K'$ for which

$$R = K^T K = K'^T K'$$

From the above theorem, (3.4.7) can be written as

$$I_0^* = [K_\infty'^T K_\infty' s + K_0'^T K_0' \frac{1}{s} \Sigma \frac{s}{s^2 + \omega_i^2} K_i'^T K_i] V_0^* \qquad (3.4.9)$$

From the information contained in (3.4.5) and (3.4.6), a state model corresponding to this equation can be constructed as

$$\frac{d}{dt} \begin{bmatrix} V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & D'^T \\ -D' & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 0 \\ K' \end{bmatrix} V_0^*$$

$$I_0^* = K_\infty'^T K_\infty' \frac{d}{dt} V_0^* + K'^T I_3 \qquad (3.4.10)$$

where

$$K' = \begin{bmatrix} K_1' \\ \vdots \\ K_n' \\ K_0' \end{bmatrix} \qquad (3.4.11)$$

and $K_i'$ $i = 1, 2, \ldots n$ are the submatrices obtained from the decomposition of $R_i$, and $D'$ has the form given in (3.4.2) or (3.4.3).

The form of D in (3.4.2) or (3.4.3) is

$$D = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{bmatrix} \tag{3.4.12}$$

where $D_i$ is a scalar matrix associated with $d_i^2 = \omega_i^2$. Note that the order of $D_i$ is equal to the rank of $R_i$, and each $D_i$ is arranged in the same order as of $K_i'$.

An interesting result in the decomposition is that, the procedure gives a spectral decomposition of the residue matrices, indeed

$$R = V D_1 V^T$$

$$= V \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} V^T$$

$$= \sum_{i=1}^{n} V \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & d_i & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} V^T = \sum_{i=1}^{n} V_i d_i V_i^T = \sum_{i=1}^{n} d_i V_i V_i^T$$

where each of the matrices $(V_i V_i^T)$ have the same order as R and each represents a constituent matrix of R.

## 3.5 Realization of Short Circuit and Open Circuit Parameters of an n-port LC Network Using Ideal Transformers

In the literature, the technique for realizing the short circuit and open circuit parameters of an LC network by using the ideal transformers is well established [CA 1]. The procedure described here for the realization of LC n-port networks is slightly different from that of Cauer and utilizes a state model for realizing these parameters. This method also realizes the network with minimum number of reactive elements, this number being equal to the degree of Z or Y. The method is identical for short circuit and open circuit parameters.

Let the given short circuit parameters be

$$I_0^* = Y V_0^* \qquad (3.5.1)$$

where Y is an n x n positive real matrix. Expanding Y into partial fractions gives

$$Y = R_\infty s + R_0 \frac{1}{s} + \Sigma R_i \frac{s}{s^2 + \omega_i^2} \qquad (3.5.2)$$

From Section 3.4, (3.5.1) can be written in terms of the decomposed residue matrices as

$$I_0^* = [K_\infty'^T K_\infty s + K_0'^T K_0' \frac{1}{s} + \Sigma \frac{s}{s^2 + \omega_i^2} K_i'^T K_i'] V_0^* \qquad (3.5.3)$$

and the state model given in (3.4.10) is

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & D'^T \\ -D' & 0 \end{bmatrix}\begin{bmatrix} V_2 \\ I_3 \end{bmatrix}\begin{bmatrix} 0 \\ K' \end{bmatrix} V_0^*$$

$$I_0^* = K_\infty'^T K_\infty' \frac{d}{dt} V_0^* + K'^T I_3 \qquad (3.5.4)$$

In order to realize the state model in (3.5.4), consider an ideal transformer network with the following terminal equations.

$$\begin{array}{c} (n) \\ (r_{bc}) \\ \\ (r_{c\ell}) \end{array} \begin{bmatrix} I_0' \\ I_2' \\ \hline V_3' \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & K'^T \\ 0 & 0 & D'^T \\ \hline -K' & -D' & 0 \end{array}\right] \begin{bmatrix} V_0' \\ V_2' \\ \hline I_3' \end{bmatrix} \qquad (3.5.5)$$

Let the terminal graph of the transformer be selected as shown in Fig. 3.5.1. Further, let two-terminal inductors and capacitors be connected to the external terminals of the transformer network, as shown in Fig. 3.5.2.

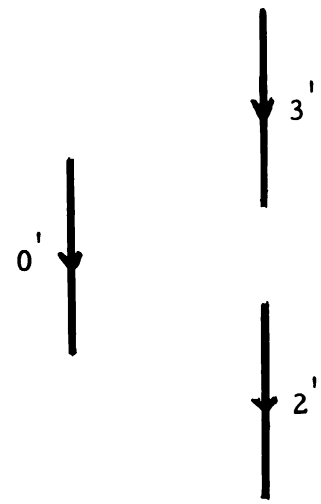



Fig. 3.5.1

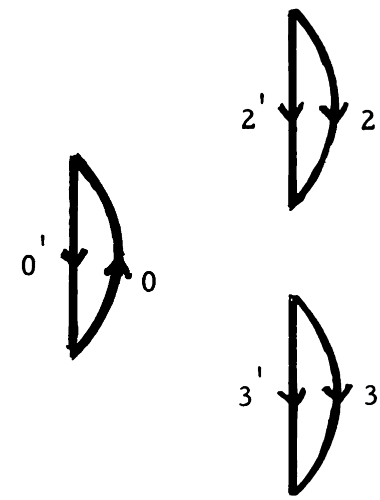



Fig. 3.5.2

Note that the number of ports of the ideal transformer, as indicated in (3.5.5), is equal to the number of state variables in (3.5.4) plus the order of the admittance matrix in (3.5.2). It will be shown that, the ideal transformer network which is loaded by the two-terminal inductors and capacitors has the state model given in (3.5.4) except the first term in the algebraic part of (3.5.4). The transformer network is then modified as required to give a complete solution. From this point the synthesis procedure reduces to the determination of the submatrices in the coefficient matrix of (3.5.5).

If the element values of the inductors and the capacitors are taken equal as unity, then the terminal equations of these components are

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} I_2 \\ V_3 \end{bmatrix} \tag{3.5.6}$$

From Fig. 3.5.2 and (3.5.5) we have

$$\begin{bmatrix} I_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -I_2' \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 & -D'^T \\ -D & 0 \end{bmatrix} \begin{bmatrix} V_2' \\ I_3' \end{bmatrix} + \begin{bmatrix} 0 \\ -K' \end{bmatrix} V_0'$$

$$= \begin{bmatrix} 0 & D'^T \\ -D' & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -K' \end{bmatrix} V_0' \tag{3.5.7}$$

Therefore, the state model is of the form

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & D'^T \\ -D' & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 0 \\ +K' \end{bmatrix} V_0$$

$$I_0 = K'^T I_3' = -K'^T I_3 \tag{3.5.8}$$

In the above model, the residue matrix corresponding to infinity is not included. This matrix is derived separately. Indeed, let the terminal equations of another transformer network, corresponding to the terminal graph in Fig. 3.5.3, be

$$\begin{bmatrix} I_0'' \\ V_3'' \end{bmatrix} = \begin{bmatrix} 0 & K_\infty'^T \\ -K_\infty' & 0 \end{bmatrix} \begin{bmatrix} V_0'' \\ I_3'' \end{bmatrix} \qquad (3.5.9)$$

Connecting capacitors with unit values across the ports indicated by $3''$, the terminal representation of the resulting network with reference to Fig. 3.5.4, can be derived as follows:

The terminal equations of the capacitors are

$$\frac{d}{dt} V_3^{(3)} = I_3^{(3)} \qquad (3.5.10)$$

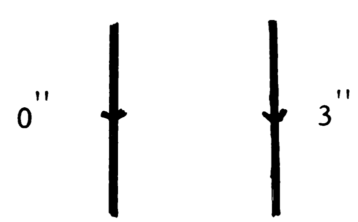



Fig. 3.5.3

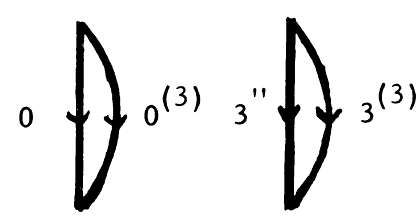



Fig. 3.5.4

From Fig. 3.5.4 and (3.5.9), we have

$$
\begin{aligned}
I_0^{(3)} = -I_0'' &= -K_\infty'^T I_3'' = K_\infty'^T I_3^{(3)} \\
&= K_\infty'^T \frac{d}{dt} V_3^{(3)} = K_\infty'^T \frac{d}{dt} V_3'' \\
&= -K_\infty'^T K_\infty' \frac{d}{dt} V_0'' \\
&= -K_\infty'^T K_\infty' \frac{d}{dt} V_0^{(3)}
\end{aligned}
\tag{3.5.11}
$$

Hence the first term in the algebraic part of (3.5.4) is realized. To combine the transformer networks corresponding to (3.5.5) and (3.5.9) as indicated in Fig. 3.5.5 we have

$$
I_0^* = -(I_0^{(3)} + I_0) \quad \text{and} \quad V_0^* = V_0^{(3)} = V_0 \tag{3.5.12}
$$

Substituting (3.5.8) and (3.5.11) in (3.5.12) gives

$$
\frac{d}{dt}\begin{bmatrix} V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & D'^T \\ -D' & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 0 \\ K' \end{bmatrix} V_0^*
$$

$$
I_0^* = K_\infty'^T K_\infty' \frac{d}{dt} V_0^* + K'^T I_3 \tag{3.5.13}
$$

which is the desired state model. The network corresponding to this state model is shown in Fig. 3.5.6.

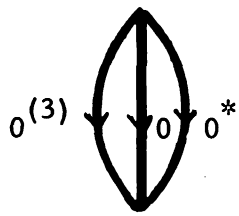



Fig. 3.5.5

The state model given in (3.5.13) is not unique. But any such derived state model can be transformed into the state model (3.5.13) by means of a similarity transformation P. Therefore, the general form for the realization of the given short circuit parameters is given in Fig. 3.5.7.

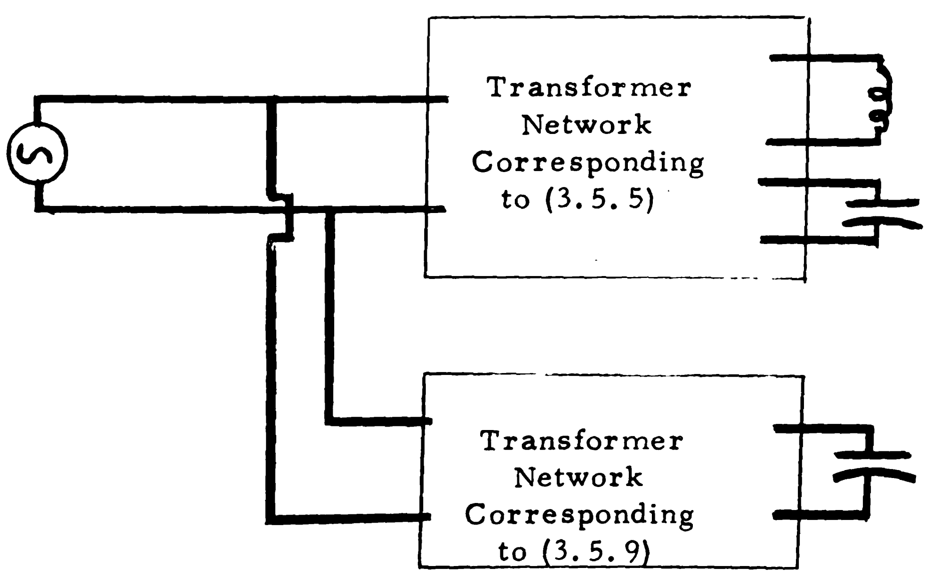



Fig. 3.5.6

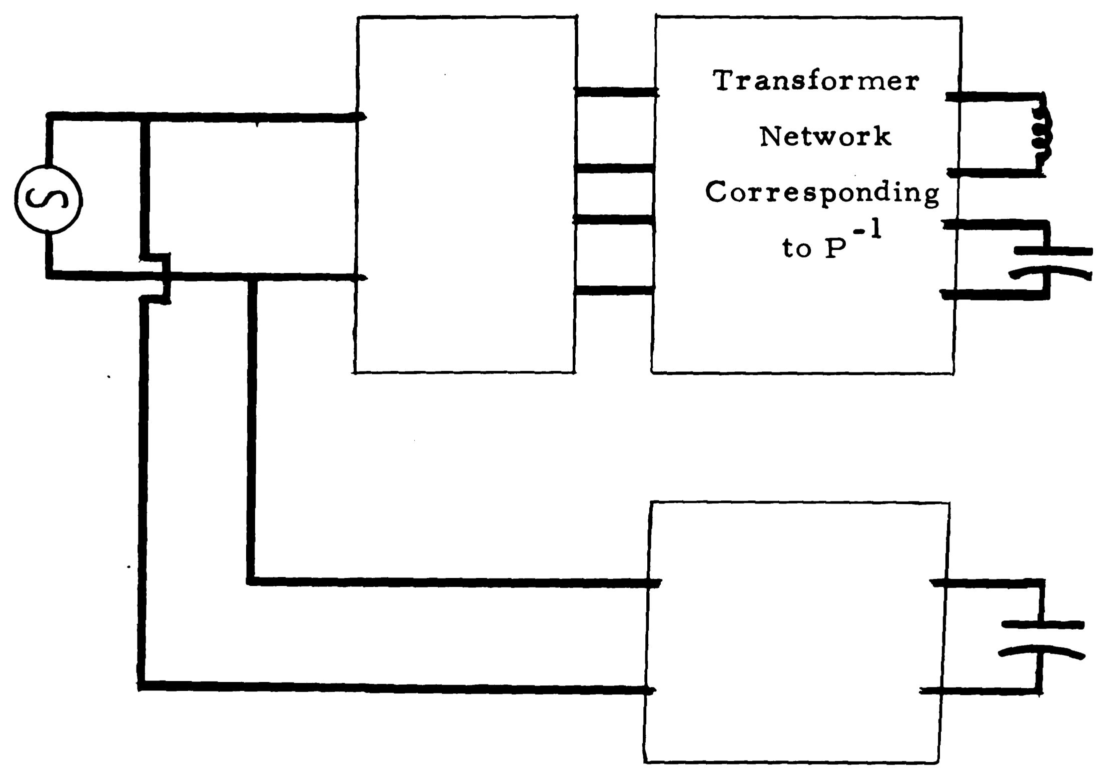



Fig. 3.5.7

The above realization is based on element values of unity magnitude. These values can theoretically be altered to any desired values by including a transformer of the desired ratio. Further, multi-terminal inductance and capacitance networks as well can be used since the realization of one-kind of n-port networks can be reduced to the form discussed above [CA 1].

The realization of the transformer networks corresponding to the terminal equations of the forms given in (3.5.5) and (3.5.9) is classical and can be found elsewhere [BE 1].

## 3.6 Realization of LC Networks without Ideal Transformers

Realization of LC n-port networks without ideal transformers has been an outstanding problem in Network Theory. Only certain sufficient conditions are known to realize a given reactance matrix [SL 1, FO 1, SO 1]. Although general necessary and sufficient conditions are stated by Cederbaum [CE 1], the application of these conditions to the synthesis procedure seem impossible. A new approach to the synthesis problem, not utilizing the reactance matrices directly, is considered here. First attempts in this direction were made by Kalman [KA 3] who gave a procedure for realizing LC driving point admittance function using the state models. The approach given here for one-port LC network synthesis differs from his. Also a procedure for realizing 2-port LC networks is given with the restriction that the residue matrices are dominant.

## I. One-Port Realization Procedure

The procedure developed here is applicable to any driving point admittance or impedance, which does not have a pole at infinity. A consideration of the functions $Y_1(s)$, $Z_2(s)$, $Z_3(s)$ and $Y_4(s)$ is sufficient to cover all possible LC driving point functions (see Section 4.2).

The partial fraction expansion of

$$Y_1(s) = \frac{(s^2 + a_1^2) \ldots (s^2 + a_{2n-1}^2)}{s(s^2 + a_2^2) \ldots (s^2 + a_{2n}^2)} \tag{3.6.1}$$

gives

$$Y_1(s) = r_0 \frac{1}{s} + \sum_{i=1}^{n} \frac{r_{2i}s}{s^2 + a_{2i}^2} \tag{3.6.2}$$

where $r_i$ $i = 1, \ldots, n$ are the residues of $Y_1$. Therefore, a state model corresponding to (3.6.2) can be constructed, through the procedure given in Section 3.4. This resulting model is of the form

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ \\ I_3 \end{bmatrix} = \left[\begin{array}{cccc|cccc} & & & & 0 & a_2 & & \\ & \bigcirc & & & \vdots & & a_4 & \\ & & & & \vdots & & & \ddots \\ & & & & 0 & & & & a_{2n} \\ \hline 0 & 0 & \ldots & 0 & & & & \\ -a_2 & & & & & & & \\ & -a_4 & & & & \bigcirc & & \\ & & \ddots & & & & & \\ & & & -a_{2n} & & & & \end{array}\right] \begin{bmatrix} V_2 \\ \\ I_3 \end{bmatrix} + \left[\begin{array}{c} \bigcirc \\ \hline \sqrt{r_0} \\ \sqrt{r_2} \\ \sqrt{r_{2n}} \end{array}\right] V \tag{3.6.3}$$

and

$$I = [\sqrt{r_0} \; . \; . \; . \; \sqrt{r_{2n}}] \; I_3$$

Premultiplying both sides of (3.6.3) by the diagonal matrix

$$\left[\begin{array}{c|ccc} U & & & \\ \hline & \frac{1}{\sqrt{r_0}} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{r_{2n}}} \end{array}\right] \qquad (3.6.4)$$

gives a state model of the form

$$\left[\begin{array}{c|cccc} U & & & & \\ \hline & \frac{1}{\sqrt{r_0}} & & & \\ & & \frac{1}{\sqrt{r_2}} & & \\ & & & \ddots & \\ & & & & \frac{1}{\sqrt{r_{2n}}} \end{array}\right] \frac{d}{dt} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix} =$$

$$
\left[\begin{array}{cccc|cccc}
 & & & & 0 & a_2 & & \\
 & & & & \cdot & & \cdot & \\
 & & & & \cdot & & & \cdot \\
 & & & & \cdot & & & \\
 & & & & 0 & & & a_{2n} \\
\hline
0 & \cdot & \cdot & \cdot\ 0 & & & & \\
-\dfrac{a_2}{\sqrt{r_1}} & & & & & & & \\
 & \cdot & & & & & & \\
 & & \cdot & & & & & \\
 & & & \cdot & & & & \\
 & & & -\dfrac{a_{2n}}{\sqrt{r_{2n}}} & & & &
\end{array}\right]
\begin{bmatrix} V_2 \\ \\ I_3 \end{bmatrix}
+
\left[\begin{array}{c} \bigcirc \\ \hline 1 \\ 1 \\ \vdots \\ 1 \end{array}\right] V
\qquad (3.6.5)
$$

and

$$
I = [\sqrt{r_0}, \ldots, \sqrt{r_{2n}}]\ I_3
$$

To retain the skew symmetric property of the operator matrix in (3.6.5), apply the similarity transformation

$$
\begin{bmatrix} V_2 \\ \\ I_3 \end{bmatrix} = \left[\begin{array}{c|cccc} U & & & & \\ \hline & \frac{1}{\sqrt{r_0}} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \frac{1}{\sqrt{r_{2n}}} \end{array}\right] \begin{bmatrix} V_2 \\ \\ I_3' \end{bmatrix} \qquad (3.6.6)
$$

to reduce (3.6.5) to the form

$$
\left[\begin{array}{c|ccccc} U & & & & & \\ \hline & \frac{1}{r_0} & & & & \\ & & \frac{1}{r_2} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \frac{1}{r_{2n}} \end{array}\right] \frac{d}{dt} \begin{bmatrix} V_2 \\ \\ I_3' \end{bmatrix} =
$$

$$
\left[\begin{array}{cccc|cccc}
 & & & & 0 & \frac{a_2}{\sqrt{r_2}} & & \\
 & O & & & \vdots & & \ddots & \\
 & & & & 0 & & & \frac{a_{2n}}{\sqrt{r_{2n}}} \\
\hline
0 & \cdots & & 0 & & & & \\
-\frac{a_2}{\sqrt{r_2}} & & & & & & O & \\
 & \ddots & & & & & & \\
 & & & -\frac{a_{2n}}{\sqrt{r_{2n}}} & & & & 
\end{array}\right]
\begin{bmatrix} V_2 \\ \\ I_3 \end{bmatrix}
+
\left[\begin{array}{c} \\ \hline 1 \\ 1 \\ \vdots \\ 1 \end{array}\right] V
\qquad (3.6.7)
$$

and

$$I = [\,1 \quad 1 \ \ldots \ 1\,]\, I_3'$$

To reduce the operator matrix in (3.6.7) to the desired form, apply the similarity transformation in (3.6.7).

$$
\begin{bmatrix} V_2' \\ \\ I_3' \end{bmatrix}
=
\left[\begin{array}{ccc|c}
\frac{a_2}{\sqrt{r_2}} & & & \\
 & \ddots & & \\
 & & \frac{a_{2n}}{\sqrt{r_{2n}}} & \\
\hline
 & & & U
\end{array}\right]
\begin{bmatrix} V_2 \\ \\ I_3' \end{bmatrix}
\qquad (3.6.8)
$$

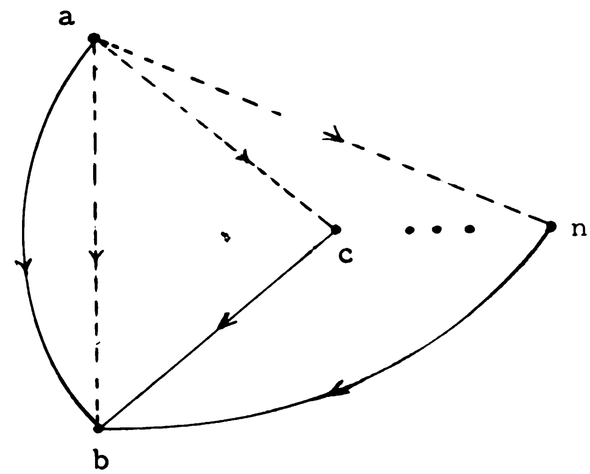
Then

$$
\left[\begin{array}{cccc|cccc}
\frac{r_2}{a_2^2} & & & & & & & \\
 & \ddots & & & & & & \\
 & & \frac{r_{2n}}{a_{2n}^2} & & & & & \\
\hline
 & & & & \frac{1}{r_0} & & & \\
 & & & & & \frac{1}{r_2} & & \\
 & & & & & & \ddots & \\
 & & & & & & & \frac{1}{r_{2n}}
\end{array}\right]
\frac{d}{dt}
\begin{bmatrix} V_2' \\ \\ I_3' \end{bmatrix}
=
$$

$$
\left[\begin{array}{cccc|cccc}
 & & & & 0 & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & \ddots \\
 & & & & 0 & & & 1 \\
\hline
0 & 0 & & & & & & \\
-1 & & & & & & & \\
 & -1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & -1 & & & &
\end{array}\right]
\begin{bmatrix} V_2' \\ \\ I_3' \end{bmatrix}
+
\left[\begin{array}{c}
\bigcirc \\
\hline
1 \\
1 \\
\vdots \\
1
\end{array}\right] V
\qquad (3.6.9)
$$

and

$$
I = [\,1 \quad 1 \;\ldots\; 1\,]\; I_3'
$$


a
c
n
b

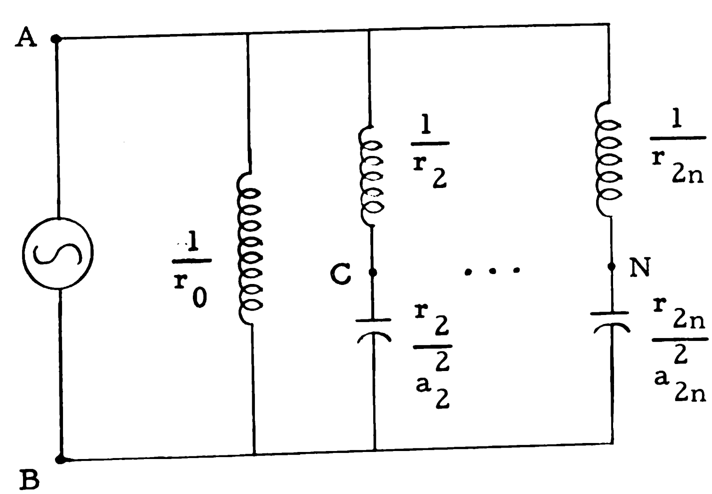

A
B
$\frac{1}{r_0}$
$\frac{1}{r_2}$
C
$\frac{r_2}{a_2^2}$
$\frac{1}{r_{2n}}$
N
$\frac{r_{2n}}{a_{2n}^2}$

Fig. 3.6.1

The circuit matrix as obtained is (see Chapter II),

$$\left[\begin{array}{c|ccc|c} -1 & 0 & & & \\ -1 & 1 & & & U \\ \vdots & & \ddots & & \\ -1 & & & 1 & \end{array}\right] \begin{bmatrix} V \\ V_2' \\ I_3' \end{bmatrix} = 0 \qquad (3.6.10)$$

The circuit matrix in (3.6.10) corresponds to second Foster form indicated in Fig. 3.6.1. The element values obtained from the coefficient matrix in (3.6.9) are indicated in Fig. 3.6.1.

## II. Two-Port LC Network Realization Procedure

This part is devoted to the realization of a special class of 2 x 2 reactance or susceptance matrices having dominant residue matrices. Slepian and Weinberg [SL 1] have described a general procedure of realizing n-port Z(Y) matrices having dominant residue matrices. Using state model approach, we shall show that 2 x 2 Z or Y matrices can be realized yielding an identical solution to that obtained by Slepian and Weinberg. Since the realization procedure is identical to both Z and Y matrices, only Y matrices are considered here. Consider first the susceptance matrix Y, which does not have a pole at infinity.

The general form of the network is shown in Fig. 3.6.2.

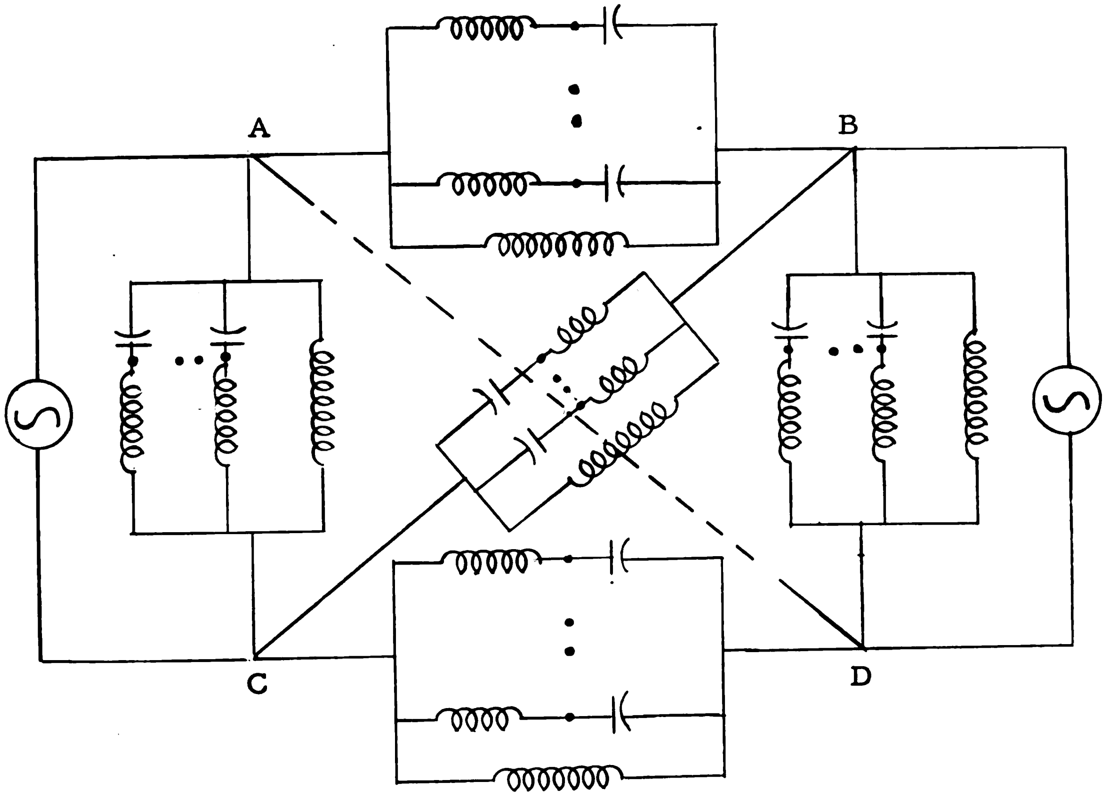
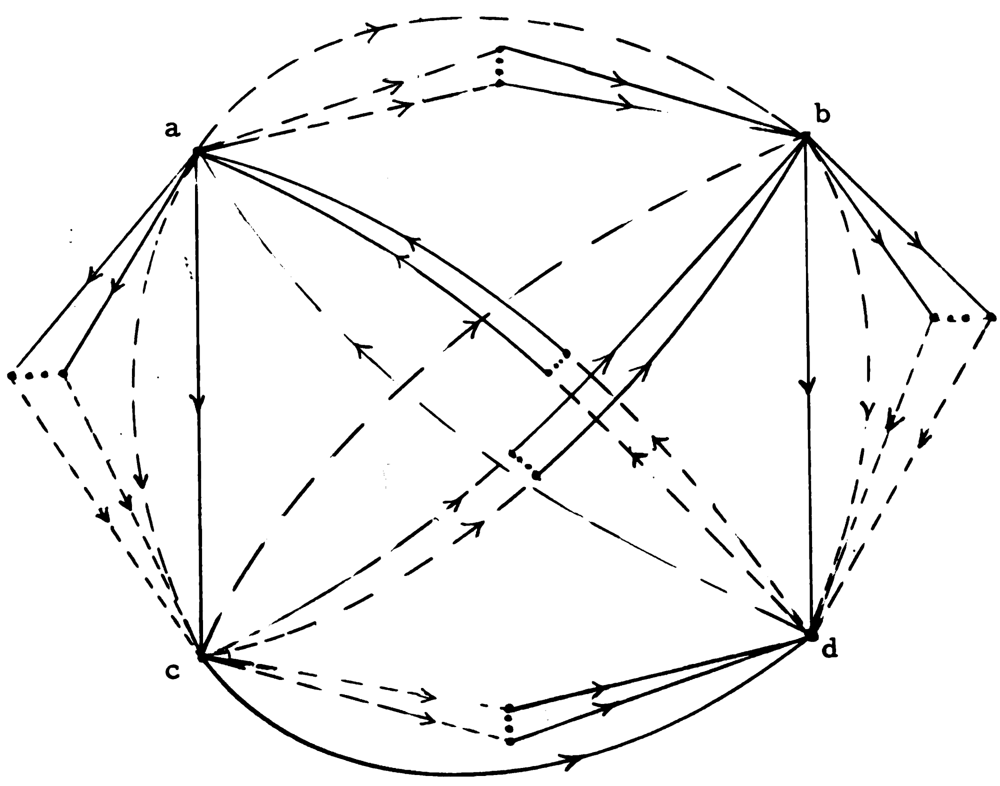



Fig. 3.6.2



In this network, it can be observed that all the capacitors and the voltage drivers can be included in a tree T, which also contains an inductor. This inductor has to be selected from a series or a cross arm of the lattice. Select an inductor between any one of the terminal pairs (A, B), (A, D), (B, C) or (C, D). If such an inductor does not exist, select an inductor from the resonators between any one of these terminal pairs. Let the graph of this network be shown in Fig. 3.6.3 with the arbitrary orientations of edges. Because of the above selections of the inductors we have two possible cases.

1. An inductor exists between any of the terminal pairs. Then the circuit equations can be written as

$$\begin{bmatrix} B_{11} & U & \vline & B_{13} & \vline & U & \\ B_{21} & 0 & \vline & B_{14} & \vline & & U \end{bmatrix} \begin{bmatrix} V \\ V_{bc} \\ V_{b1} \\ V_{c\ell} \\ V_{\ell 2} \end{bmatrix} = 0 \qquad (3.6.11)$$

where

$V$ = voltage drivers

$V_{bc}$ = capacitor voltages

$V_{b\ell}$ = voltage of the branch inductor

$V_{c\ell}$ = voltages of the inductors which are in the resonators

$V_{\ell 2}$ = voltages of the inductors which are not mentioned above.

Then the state model can be constructed as follows.

$$\begin{bmatrix} C_b & 0 & 0 \\ 0 & L_{c\ell}+\ell_{b\ell}B_{13}B_{13}^T & \ell_{b\ell}B_{13}B_{14}^T \\ 0 & \ell_{b\ell}B_{14}B_{13}^T & L_2+\ell_{b\ell}B_{14}B_{14}^T \end{bmatrix} \frac{d}{dt}\begin{bmatrix} V_{bc} \\ I_{c\ell} \\ I_{\ell 2} \end{bmatrix}$$

$$= \left[\begin{array}{c:c:c} \bigcirc & U & \bigcirc \\ \hdashline -U & \bigcirc & \bigcirc \\ \hdashline \bigcirc & \bigcirc & \bigcirc \end{array}\right] \begin{bmatrix} V_{bc} \\ I_{c\ell} \\ I_{\ell 2} \end{bmatrix} + \left[\begin{array}{c} \bigcirc \\ \hdashline -B_{11} \\ -B_{21} \end{array}\right] V^* \qquad (3.6.12)$$

and

$$I^* = -[B_{11}^T \quad B_{21}^T] \begin{bmatrix} I_{c\ell} \\ I_{\ell 2} \end{bmatrix}$$

where $I^*$ and $V^*$ represents the terminal variables which are related to the driving function as

$$I^* = -I$$

and

$$V^* = V$$

2. There is no inductor between any of the terminal pairs (A, B), (A, D), (B, C), or (C, D) in Fig. 3.6.2. In this case the synthesis procedure can be reduced to the previous case. Indeed, in this case, since Y matrix does not have a pole at the origin, then we introduce such a pole by simply adding a term corresponding to the pole at the origin. After the realization for this augmented matrix the inductances corresponding to the pole at the origin are removed to yield the network for Y.

In the following we shall discuss the procedure for realizing 2 x 2 susceptance matrices which have dominant residue matrices.

Consider first the susceptance matrix $Y_1$ which does not have a pole at infinity. The partial fraction expansion gives

$$Y_1 = R_0 \frac{1}{s} + \sum \frac{s}{s^2 + \omega_i^2} R_i \qquad (3.6.13)$$

where $R_0$ and $R_i$ are the residue matrices which are assumed to be dominant. The procedure for decomposing the residue matrices differ from the procedure described in Section 3.4. Consider the following two possibilities:

a) Let

$$R_j = \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \qquad (3.6.14)$$

represent any one of the residue matrices in (3.6.13), such that $a_1 \geq b_1$, $c_1 \geq b_1$ and $b_1 \geq 0$, i.e., $R_j$ is dominant. Then $R_j$ in (3.6.14) can be decomposed into the form

$$R_j = \begin{bmatrix} b_1 & b_1 \\ b_1 & b_1 \end{bmatrix} + \begin{bmatrix} a_1 - b_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_1 - b_1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{b_1} \\ \sqrt{b_1} \end{bmatrix} [\sqrt{b_1} \quad \sqrt{b_1}] + \begin{bmatrix} \sqrt{a_1 - b_1} \\ 0 \end{bmatrix} [\sqrt{a_1 - b_1} \quad 0] + \begin{bmatrix} 0 \\ \sqrt{c_1 - b_1} \end{bmatrix} [0 \quad \sqrt{c_1 - b_1}]$$

$$= \begin{bmatrix} \sqrt{b_1} & \sqrt{a_1 - b_1} & 0 \\ \sqrt{b_1} & 0 & \sqrt{c_1 - b_1} \end{bmatrix} \begin{bmatrix} \sqrt{b_1} & \sqrt{b_1} \\ \sqrt{a_1 - b_1} & 0 \\ 0 & \sqrt{c_1 - b_1} \end{bmatrix} = K_j^T K_j \tag{3.6.15}$$

b) If the residue matrix is of the form

$$R_k = \begin{bmatrix} a_2 & -b_2 \\ -b_2 & c_2 \end{bmatrix} \tag{3.6.16}$$

where $a_2 \geq b_2$, $c_2 \geq b_2$ and $b_2 \geq 0$. Then $R_k$ in (3.6.16) can be decomposed as follows

$$R_k = \begin{bmatrix} b_2 & -b_2 \\ -b_2 & b_2 \end{bmatrix} + \begin{bmatrix} a_2 - b_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_2 - b_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{b_2} \\ -\sqrt{b_2} \end{bmatrix} [\sqrt{b_2} \ -\sqrt{b_2}] + \begin{bmatrix} \sqrt{a_2-b_2} \\ 0 \end{bmatrix} [\sqrt{a_2-b_2} \quad 0] + \begin{bmatrix} 0 \\ \sqrt{c_2-b_2} \end{bmatrix} [0 \quad \sqrt{c_2-b_2}]$$

$$= \begin{bmatrix} \sqrt{b_2} & \sqrt{a_2-b_2} & 0 \\ -\sqrt{b_2} & 0 & \sqrt{c_2-b_2} \end{bmatrix} \begin{bmatrix} \sqrt{b_2} & -\sqrt{b_2} \\ \sqrt{a_2-b_2} & 0 \\ 0 & \sqrt{c_2-b_2} \end{bmatrix}$$

$$= K_k^T K_k \tag{3.6.17}$$

The corresponding state model can be constructed by the procedure already described in Section 3.4. The result is

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I_3 \end{bmatrix} = \left[\begin{array}{c|c} 0 & D^T \\ \hline -D & 0 \end{array}\right] \begin{bmatrix} V_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 0 \\ K' \end{bmatrix} V$$

$$I = K'^T I_3 \tag{3.6.18}$$

where

$$K' = \begin{bmatrix} K_1 \\ K_2 \\ \cdot \\ \cdot \\ \cdot \\ K_n \\ K_0 \end{bmatrix}$$

and D has the same form as in (3.4.2) or (3.4.3). Applying a procedure identical to that applied for one ports, i.e., premultiplying both sides of (3.6.18) by a diagonal matrix and applying the similarity transformation, the equivalent state model is

$$\begin{bmatrix} C_2 & \\ & L_3 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_2' \\ I_3' \end{bmatrix} = \left[\begin{array}{cccc|cccc} & & & & 1 & & & 0 \\ & \bigcirc & & & & 1 & & 0 \\ & & & & & & \ddots & \vdots \\ & & & & & & 1 & 0 \\ \hline -1 & & & & & & & \\ & -1 & & & & & & \\ & & \ddots & & & \bigcirc & & \\ & & & -1 & & & & \\ 0 & 0 & \cdots & 0 & & & & \end{array}\right] \begin{bmatrix} V_2' \\ I_3' \end{bmatrix} + \begin{bmatrix} 0 \\ B' \end{bmatrix} V$$

and

$$I = B'^T I_3 \tag{3.6.19}$$

where $B'$ contains $\pm 1$, 0 as its entries and $C_2$ and $L_3$ are diagonal matrices.

To determine the network corresponding to (3.6.19) the only requirement is that $B'$ must be a submatrix of a circuit matrix. If it is, then the realization follows immediately. If $B'$ is not a submatrix of a circuit matrix, then the state model obtained above must be modified by using a proper similarity transformation on the state vector so that the new model is realizable. This is always possible and is shown next. The first column of $B'$ contains only the entries +1, or 0 but not -1 and the second column of each $B'_i$ (the submatrix of $B'$) has the following possible forms

$$\text{(a)} \quad \begin{bmatrix} \pm 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{(b)} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (3.6.20)$$

$$\text{(c)} \quad [\pm 1]$$

In (3.6.20a), the plus sign is taken if the off-diagonal entry in the residue matrix is positive, and the negative sign is taken if otherwise. Equation (3.6.20b) appears if the residue matrix is diagonal and (3.6.20c) appears if the entries in the residue matrix are equal in absolute value. Again the plus sign is taken if the off-diagonal entry of the residue matrix is positive and the negative sign is taken if otherwise. In the following, three cases are considered separately. In Case 1 it is assumed that the off-diagonal entries in the residue matrices are all positive, in Case 2 all are negative and in Case 3 they are arbitrary.

<u>Case 1:</u> If each submatrix $B_i$ of $B'$ contains one of the following matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.6.21}$$

then $B'$ is a submatrix of a circuit matrix. Then the circuit equations as obtained are

$$\left[\begin{array}{c|cccc|c} & 1 & & & & \\ & & 1 & & & \\ -B' & & & 1 & & U \\ & & & & \ddots & \\ & & & & 0 & \end{array}\right] \begin{bmatrix} V \\ V_2 \\ V_3 \end{bmatrix} = 0 \tag{3.6.22}$$

These equations correspond to the network shown in Fig. 3.6.4.

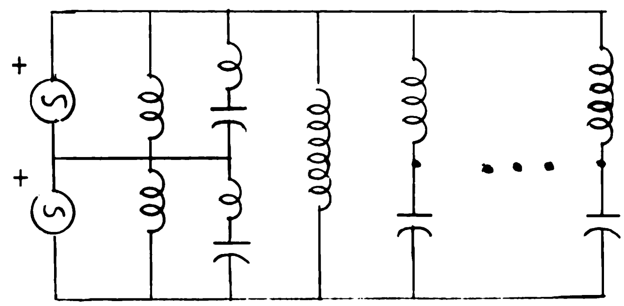


Fig. 3.6.4

The element values in this network are dictated by the diagonal matrices $C_2$ and $L_3$ in the state model.

<u>Case 2:</u> If each $B_i$ represents any one of the following matrices

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.6.23}$$

Then $B'$ is a submatrix of a circuit matrix of the form

$$\left[\begin{array}{c|cccc|c} & 1 & & & & \\ & & 1 & & & \\ -B' & & & 1 & & U \\ & & & & \ddots & \\ & & & & 0 & \end{array}\right] \begin{bmatrix} V \\ V_2 \\ V_3 \end{bmatrix} = 0$$

and the corresponding network for the circuit matrix is very similar to Fig. 3.6.4 and is shown in Fig. 3.6.5.

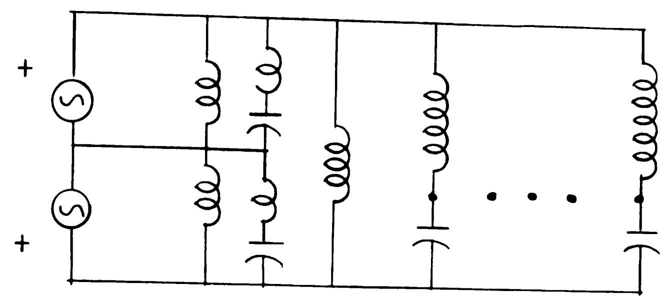


Fig. 3.6.5

Case 3: If some $B_i^{'}$ have the form in (3.6.21) and some $B_i^{'}$ have the form in (3.6.23), then the resultant $B^{'}$ may not be a submatrix of circuit matrix. A typical form of $B^{'}$ in this case is

$$B^{'} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \qquad (3.6.24)$$

In this case, although $B^{'}$ may not be a realizable submatrix, the state model can be transformed to a realizable state model by first augmenting the state model and then using a similarity transformation on the state vector.

The augmentation of the state model can be achieved by decomposing the residue matrices as follows:

$$R_j = \begin{bmatrix} \sqrt{b_1/2} & \sqrt{b_1/2} & \sqrt{a_1-b_1} & 0 \\ \sqrt{b_1/2} & \sqrt{b_1/2} & 0 & \sqrt{c_1-b_1} \end{bmatrix} \begin{bmatrix} \sqrt{b_1/2} & \sqrt{b_1/2} \\ \sqrt{b_1/2} & \sqrt{b_1/2} \\ \sqrt{a_1-b_1} & 0 \\ 0 & \sqrt{c_1-b_1} \end{bmatrix} \qquad (3.6.28)$$

$$R_k = \begin{bmatrix} \sqrt{b_2/2} & \sqrt{b_2/2} & \sqrt{a_2-b_2} & 0 \\ -\sqrt{b_2/2} & -\sqrt{b_2/2} & 0 & \sqrt{c_2-b_2} \end{bmatrix} \begin{bmatrix} \sqrt{b_2/2} & -\sqrt{b_2/2} \\ \sqrt{b_2/2} & -\sqrt{b_2/2} \\ \sqrt{a_2-b_2} & 0 \\ 0 & \sqrt{c_2-b_2} \end{bmatrix} \qquad (3.6.9)$$

In order to illustrate the procedure,the following example is considered.

Example: Let

$$Y(s) = \begin{bmatrix} \dfrac{s(2s^2+5)}{(s^2+1)(s^2+4)} & -\dfrac{3s}{(s^2+1)(s^2+4)} \\ -\dfrac{3s}{(s^2+1)(s^2+4)} & \dfrac{s(2s^2+5)}{(s^2+1)(s^2+4)} \end{bmatrix} \qquad (3.6.30)$$

be the given susceptance matrix which has no pole at the origin.

Adding the term

$$\frac{1}{s}R_0 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{s}$$

to Y(s) we have

$$Y(s) + R_0\frac{1}{s} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{s} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\frac{s}{s^2+1} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{s}{s^2+4} \tag{3.6.31}$$

$$= R_0\frac{1}{s} + R_1\frac{s}{s^2+1} + R_2\frac{s}{s^2+4}$$

Note that $R_0$ is arbitrary but it is dominant. Decomposing the residue matrices as

$$R_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$R_0 = \begin{bmatrix} 1 \\ \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

A state model can be constructed:

$$\frac{d}{dt}\begin{bmatrix} v_3 \\ v_4 \\ v_5 \\ v_6 \\ i_7 \\ i_8 \\ i_9 \\ i_{10} \\ i_{11} \end{bmatrix} = \left[\begin{array}{cccc|ccccc} & & & & 1 & & & & 0 \\ & \bigcirc & & & & 1 & & & 0 \\ & & & & & & 2 & & 0 \\ & & & & & & & 2 & 0 \\ \hline -1 & & & & & & & & \\ & -1 & & & & & & & \\ & & -2 & & & & \bigcirc & & \\ & & & -2 & & & & & \\ 0 & 0 & 0 & 0 & & & & & \end{array}\right] \begin{bmatrix} v_3 \\ v_4 \\ v_5 \\ v_6 \\ i_7 \\ i_8 \\ i_9 \\ i_{10} \\ i_{11} \end{bmatrix} + \left[\begin{array}{cc} & \\ \multicolumn{2}{c}{\bigcirc} \\ & \\ \hline \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & -1 \end{array}\right] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

and

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} i_7 \\ i_8 \\ i_9 \\ i_{10} \\ i_{11} \end{bmatrix} \qquad (3.6.32)$$

Applying the transformation of variables we have

$$
\begin{bmatrix}
\frac{1}{2} & & & & & & & & \\
& \frac{1}{2} & & & & & & & \\
& & \frac{1}{8} & & & & & & \\
& & & \frac{1}{8} & & & & & \\
& & & & 2 & & & & \\
& & & & & 2 & & & \\
& & & & & & 2 & & \\
& & & & & & & 2 & \\
& & & & & & & & 1
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix} v_3' \\ v_4' \\ v_5' \\ v_6' \\ i_7' \\ i_8' \\ i_9' \\ i_{10}' \\ i_{11}' \end{bmatrix}
=
$$

$$
\left[\begin{array}{cccc:ccccc}
 & & & & 1 & & & & 0 \\
 & \bigcirc & & & & 1 & & & 0 \\
 & & & & & & 1 & & 0 \\
 & & & & & & & 1 & 0 \\
\hdashline
-1 & & & & & & & & \\
 & -1 & & & & & & & \\
 & & -1 & & & & \bigcirc & & \\
 & & & -1 & & & & & \\
0 & 0 & 0 & 0 & & & & &
\end{array}\right]
\begin{bmatrix} v_3' \\ v_4' \\ v_5' \\ v_6' \\ i_7' \\ i_8' \\ i_9' \\ i_{10}' \\ i_{11}' \end{bmatrix}
+
\left[\begin{array}{cc}
 & \\
\bigcirc & \\
 & \\
 & \\
\hdashline
1 & 1 \\
1 & 1 \\
1 & -1 \\
1 & -1 \\
1 & -1
\end{array}\right]
\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}
\qquad (3.6.33)
$$

The submatrix

$$B' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

is not a submatrix of a circuit matrix, therefore, a similarity transformation T must be used to reduce $B'$, to the desired form. General description of the transformation matrix is given later.

Let the transformation matrix be taken as of the form

$$T_1 = \left[\begin{array}{cccc|c} 1 & & & & -1 \\ & 1 & & & 1 \\ & & 1 & & 0 \\ & & & 1 & 0 \\ \hline & & & & \\ & & & & 1 \end{array}\right] \qquad (3.6.34)$$

Applying the transformation to the inductor currents in (3.6.33) and then using another set of transformation of variables similar to that used in one-port case, the final state model is

$$
\left[\begin{array}{cccc|c}
2 & & & & \\
 & 2 & & & \\
 & & \frac{1}{8} & & \\
 & & & \frac{1}{8} & \\
\hline
 & & & & T_1' \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 1 \end{pmatrix} T_1'^{T}
\end{array}\right]
\frac{d}{dt}
\begin{bmatrix} v_3^* \\ v_4^* \\ v_5^* \\ v_6^* \\ i_7^* \\ i_8^* \\ i_9^* \\ i_{10}^* \\ i_{11}^* \end{bmatrix}
=
$$

$$
\left[\begin{array}{cccc|ccccc}
 & & & & 1 & & & & 0 \\
 & & \bigcirc & & & 1 & & & 0 \\
 & & & & & & 1 & & 0 \\
 & & & & & & & 1 & 0 \\
\hline
-1 & & & & & & & & \\
 & -1 & & & & & & & \\
 & & -1 & & & & \bigcirc & & \\
 & & & -1 & & & & & \\
0 & 0 & 0 & 0 & & & & &
\end{array}\right]
\begin{bmatrix} v_3^* \\ v_4^* \\ v_5^* \\ v_6^* \\ i_7^* \\ i_8^* \\ i_9^* \\ i_{10}^* \\ i_{11}^* \end{bmatrix}
+
\left[\begin{array}{cc}
 & \\
\bigcirc & \\
 & \\
\hline
0 & -1 \\
1 & 0 \\
1 & -1 \\
1 & -1 \\
1 & -1
\end{array}\right]
\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}
$$

and

$$\begin{bmatrix} I_1 \\ \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ \\ -1 & 0 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_7^* \\ i_8^* \\ i_9^* \\ i_{10}^* \\ i_{11}^* \end{bmatrix} \tag{3.6.35}$$

where

$$T_1' = \begin{bmatrix} \frac{1}{2} & & & & \\ & \frac{1}{2} & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} T_1 \tag{3.6.36}$$

Hence

$$T_1' \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 1 \end{bmatrix} T_1'^{T}$$

$$= \begin{bmatrix} \frac{1}{2} & & & & \\ & \frac{1}{2} & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & \frac{1}{2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} [-1 \quad 1 \quad 0 \quad 0 \quad 1] \tag{3.6.37}$$

Then the circuit matrix can be constructed from (3.6.35) and (3.6.37) as

$$B = \left[\begin{array}{cc|cccc|c|ccccc} 0 & +1 & 1 & & & & -1 & 1 & & & & \\ -1 & 0 & & 1 & & & 1 & & 1 & & & \\ -1 & 1 & & & 1 & & 0 & & & 1 & & \\ -1 & 1 & & & & 1 & 0 & & & & 1 & \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & & & & & 1 \end{array}\right] \tag{3.6.38}$$

The network can be realized from (3.6.38) and the element values are determined from (3.6.36) and (3.6.37), and the result is shown in Fig. 3.6.6.

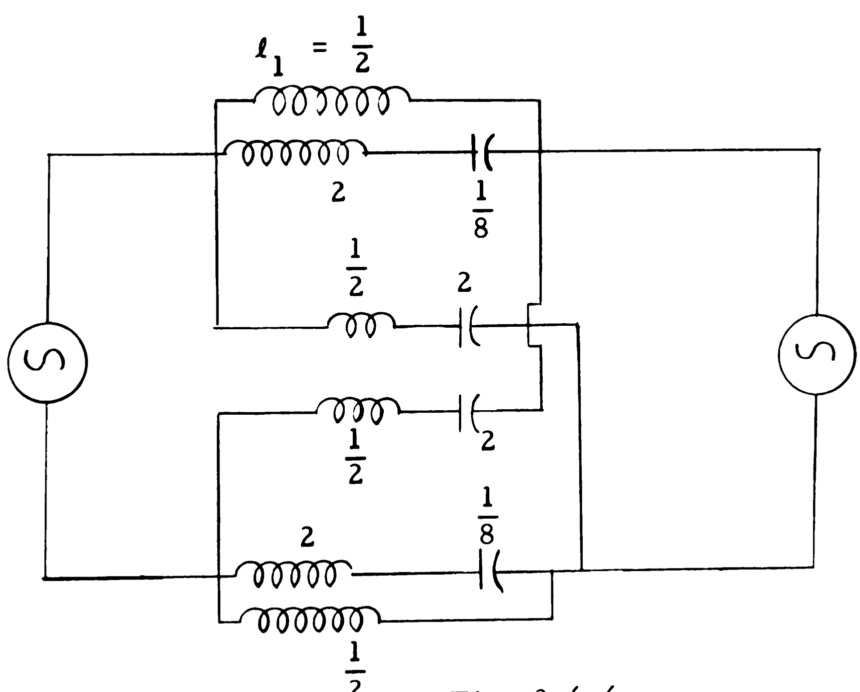



Fig. 3.6.6

The network corresponding to the given matrix Y can be obtained by removing the two inductors $\ell_1$ and $\ell_2$ in this figure.

Selection of the Transformation Matrix T

Consider

$$B' = \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_k \\ B_0 \end{bmatrix}$$

where $B_i$ is obtained from the residue matrices as discussed earlier. The leading two rows of $B_i$ $(i \neq 0)$ are one of the forms: [1 1], [1 -1], [1 0], [0 1]. The last row of $B_0$ can be taken as [1 1], or [1 -1]. Then T must have the form

$$T = \left[\begin{array}{c|c} U & T' \\ \hline 0 & 1 \end{array}\right] \qquad (3.6.39)$$

where

$$T' = \begin{bmatrix} T_1 \\ \cdot \\ \cdot \\ \cdot \\ T_k \\ T_0 \end{bmatrix}$$

contains $\pm 1$, 0 as entries. If the last row of $B_0$ is $[1 \quad 1]$, then considering all the submatrices, $B_j$, whose first two rows contain $[1 \quad -1]$, we see that it is necessary to take

$$\begin{aligned} T_{j1} &= -1 \\ T_{j2} &= 2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} T_{j1} &= 1 \\ T_{j2} &= -1 \end{aligned}$$

and for all the other entries $T_i = 0$. If the last row of $B_0$ is $[1 \quad -1]$, consider all the submatrices $B_j$ whose first two rows contain $[1 \quad 1]$, then we shall select either

$$\left.\begin{aligned} T_{j1} &= -1 \\ T_{j2} &= 1 \end{aligned}\right\} \qquad \text{or} \qquad \left.\begin{aligned} T_{j1} &= 1 \\ T_{j2} &= -1 \end{aligned}\right\}$$

In the above discussion we assumed that Y has no pole at infinity. If it has a pole at infinity, the residue matrix can be realized without considering the state model approach and the corresponding network can be connected to that obtained from the realization of the other residue matrices.

The above procedure can be extended to n-port LC networks for which the residue matrices are dominant [SL 1]. If the dominancy condition is not imposed on the residue matrices, the realization by the procedure given here, in general, may not be possible.

## CHAPTER IV

## EQUIVALENT NETWORKS

### 4.1 General

This chapter is concerned with the parameter transformations on one port canonic networks.

The theory of equivalent networks goes back to 1930's. In the classical theory one equivalent network is obtained from another by applying a non-singular transformation to the mesh impedance or node admittance matrices such that certain parameters of the network are invariant. In the following sections, these classical transformation techniques are summarized along with the development of certain new contributions to equivalent networks.

In the last section state model equations are used to generate one equivalent network from the other.

### 4.2 Equivalent Networks - Transformation Matrix

Two n-port networks are said to be equivalent if they have identical terminal equations corresponding to a given terminal graph.

The important point is that the two n-port networks have the same terminal characteristics. Their internal construction need not be identical.

In general, the transformation of a network into another equivalent network is possible only if the internal constructions of these two networks are known. If only the terminal equations are given such transformation is possible if the network is of one element-kind. This type of transformation, called the congruent transformation, is discussed by Cauer [CA 1] and has very limited applications.

Howitt in his doctoral thesis, later in his papers [HO 1, HO 2] discussed the equivalence transformation of RLC networks, and showed that the equivalent electrical networks form a group. He also considered the necessary minimum number of elements to realize the given driving point impedance function.

The equivalent networks are also considered by Guillemin [GU 1], [GU 3], who attempted to simplify the problem by using the normal coordinate transformations. Recently Schoeffler [SC 1], [SC 2] considered a transformation of equivalent networks, keeping the same network configuration but with different element values. Guillemin applies the transformation on the mesh impedance or node admittance matrices, while Schoeffler applies such transformations to the parameter matrices, thereby introducing additional constraints on the transformation matrix. In all of these existing techniques the topology of the equivalent networks is held rigid, i.e., in general, equivalent networks derived have identical topologies (networks having either the same number of meshes or the same number of nodes). In the following, the principles of Howitt transformation is considered briefly with some additional remarks.

Let the mesh equations of an n-port RLC network in the s-domain be written as

$$V = (R + L\,s + \frac{1}{s}\,K)\,I \tag{4.2.1}$$

where

$$V = [v_1, v_2, \ldots, v_n]^T \qquad \text{(mesh voltages)}$$

$$I = [i_1, i_2, \ldots, i_n]^T \qquad \text{(mesh currents)}$$

and R, L, K are square real matrices, i.e., mesh resistance, mesh inductance and mesh elastance matrices. Let the vector I and V be transformed into vectors $I'$ and $V'$ by

$$\begin{aligned} I &= C\,I' \\ V' &= C^T\,V \end{aligned} \tag{4.2.2}$$

where C is a non-singular matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \tag{4.2.3}$$

This transformation carries (4.2.1) into a new form for which the mesh resistance, mesh inductance and mesh elastance matrices are of the form

$$
\begin{aligned}
R' &= C^T R C \\
L' &= C^T L C \\
K' &= C^T K C
\end{aligned}
\qquad (4.2.4)
$$

To hold the driving point impedance invariant through the transformation, it is necessary that $i_1 = i_1'$ and $v_1 = v_1'$. Consequently, if C is taken as

$$
\begin{bmatrix}
1 & 0 & \dots & 0 \\
c_{21} & c_{22} & & c_{2n} \\
\cdot & & & \\
\cdot & & & \\
\cdot & & & \\
c_{n1} & & & c_{nn}
\end{bmatrix}
\qquad (4.2.5)
$$

In general, it is easy to show that for the invariance of first k mesh voltages and currents, C must have the form

$$
C = \left[\begin{array}{c|c}
U_k & 0 \\
\hline
C_2 & C_3
\end{array}\right]
\qquad (4.2.6)
$$

where $U_k$ is a unit matrix of order k.

## 4.3 Canonic Transformation for One-Port Canonic Networks

The oldest known one-port LC canonic networks were obtained by Foster [FO 1] through the partial fraction expansion of the reactance functions. Later Cauer [CA 1] gave other forms, which are canonic by considering the continued fraction expansion of the reactance functions. Recently Lee [LE 1] gave a lattice canonic form for the one-port LC networks which are derived by applying a combination of the Foster and Cauer procedures and modifying the final network by an ideal transformer. Yarlagadda and Tokad [YA 1] considered a rather restricted lattice canonic form which differes from that given by Lee and is derived from two cascaded Brune sections. The known canonic networks introduced by Foster, Cauer and Lee are shown in Table 4.3.1.

In Section 3.2 procedures have already been given for deriving one-port canonic networks from a general canonic network. There are four possible classes of one-port canonic networks obtainable by this process. These four classes correspond to the four reactance functions.

<u>Class 1</u>:

$$Z_1(s) = \frac{s(s^2 + a_2^2) \ldots (s^2 + a_{2n}^2)}{(s^2 + a_1^2) \ldots (s^2 + a_{2n-1}^2)}$$

or

$$Y_1(s) = \frac{(s^2 + a_1^2) \ldots (s^2 + a_{2n-1}^2)}{s(s^2 + a_2^2) \ldots (s^2 + a_{2n}^2)}$$

<u>Class 2</u>:

$$Z_2(s) = \frac{(s^2 + a_1^2) \ldots (s^2 + a_{2n-1}^2)}{s(s^2 + a_2^2) \ldots (s^2 + a_{2n}^2)}$$

Class 3:

$$Z_3(s) = \frac{s(s^2 + a_2^2) \ . \ . \ . \ (s^2 + a_{2n-1}^2)}{(s^2 + a_1^2) \ . \ . \ . \ (s^2 + a_{2n}^2)}$$

Class 4:

$$Z_4(s) = \frac{(s^2 + a_1^2) \ . \ . \ . \ (s^2 + a_{2n}^2)}{s(s^2 + a_2^2) \ . \ . \ . \ (s^2 + a_{2n-1}^2)} \qquad (4.3.1)$$

or

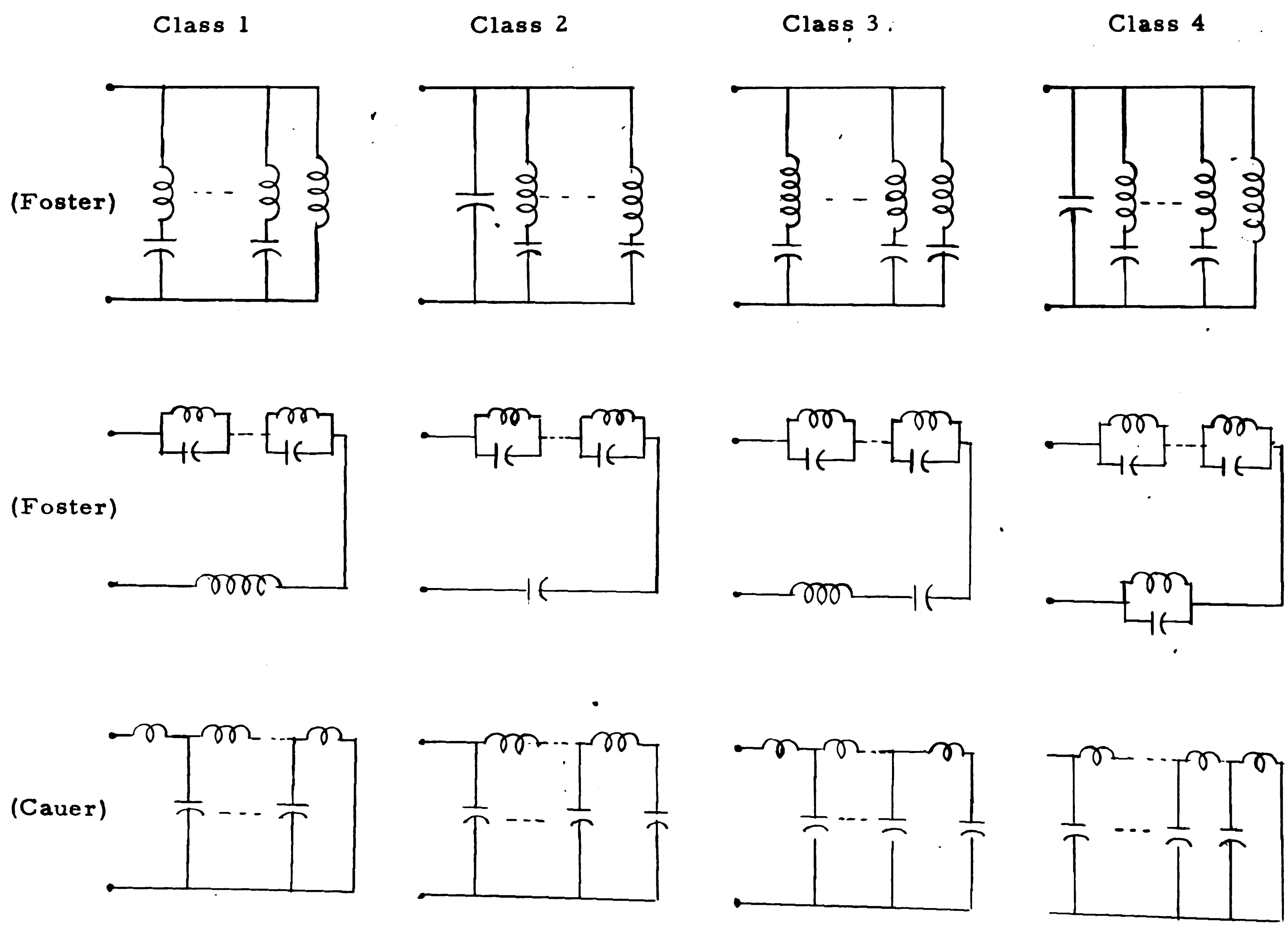
$$Y_4(s) = \frac{s(s^2 + a_2^2) \ . \ . \ . \ (s^2 + a_{2n-1}^2)}{(s^2 + a_1^2) \ . \ . \ . \ (s^2 + a_{2n}^2)}$$

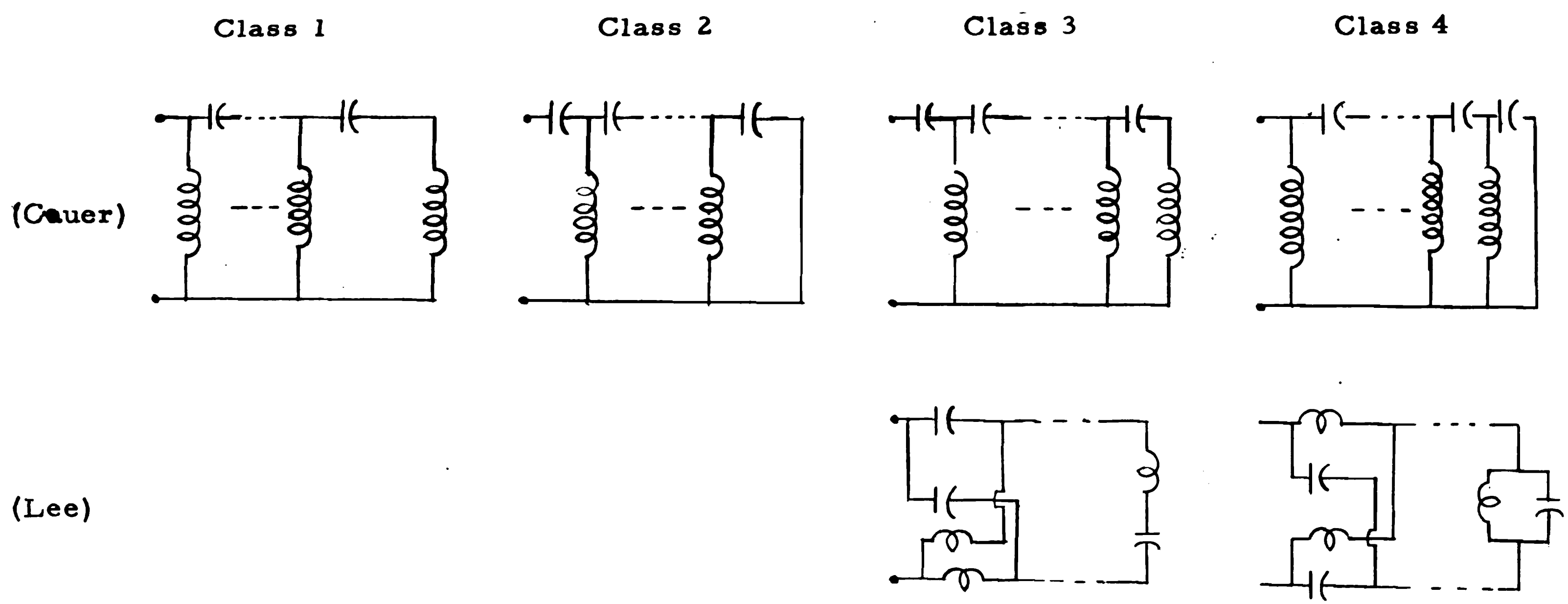
Each class corresponds to a class of canonic networks as follows.

Class 1: Class 1 is obtained by replacing a capacitor by a voltage driver in a canonic network. Therefore, the number of inductros is one more than the number of capacitors. For $Y_1(s)$ in (4.3.1), one-port canonic network contains (n + 1) inductors, n capacitors and a voltage driver.

Class 2: This class of canonic networks are obtained by replacing an inductor by a current driver in a canonic network. Therefore, the number of capacitors is one more than the number of inductors. For $Z_2(s)$ in (4.3.1), one-port canonic network of Class 2 contains n + 1 capacitors, n inductors and a current driver.

Table 4.3.1




Class 1
Class 2
Class 3
Class 4
(Cauer)
(Lee)

Class 3: This class of canonic networks are obtained from a canonic network by inserting a current driver between arbitrary nodes. Therefore, the number of capacitors is equal to the number of inductors. For $Z_3(s)$ in (4.3.1), one port canonic network of Class 3 contains n inductors, n capacitors and a current driver. Note that the current driver forms a circuit with some of the inductors in the network.

Class 4: If in a canonic network, a branch or a chord is opened, and a voltage driver is inserted in series with this branch or chord, the fourth class of canonic networks results. In this class of networks the number of capacitors is equal to the number of inductors. For $Y_4(s)$ in (4.3.1), one port canonic network of Class 4 contains n inductors, n capacitors and a voltage driver. Note that the voltage driver forms a cut-set with some of the capacitors.

The admittance functions are considered for Classes 1 and 4, and impedance functions are for Classes 2 and 3 for the obvious reasons that $Y_1(s)$, $Z_2(s)$, $Z_3(s)$ and $Y_4(s)$ have no poles at infinity.

## Transformation Matrix C Corresponding to Canonic Networks

In considering the equivalent networks, it is generally assumed that one canonic network is given, and the problem is to determine the element values of another canonic network of known topology. Since the topology of both networks are known, certain restrictions are imposed on the transformation matrix C.

Class 1: Let $N_1$ be a one-port canonic network having the cut-set equations

$$[U\ A_1] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = 0 \tag{4.3.2}$$

and let $N_2$ be second one-port canonic network of the class having the cut-set equations

$$[U\ A_2] \begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = 0 \tag{4.3.3}$$

where $N_1$ is assumed to be known completely (i.e., topology and element values are known) and for $N_2$ only the topology is known. The problem is solved if the element values of $N_2$ are determined. Because of the nature of this class of canonic forms, the known submatrices $A_1$ and $A_2$ in (4.3.2) and (4.3.3) are nonsingular. The branch equations for network $N_1$, are

$$I_1^* = \left[\begin{array}{c|c} 0 & \\ \hline U & \end{array}\ A_1\right] \begin{bmatrix} D_1 s & \\ & \frac{1}{s} L_1 \end{bmatrix} \left[\begin{array}{c|c} 0 & U \\ \hline \multicolumn{2}{c}{A_1^T} \end{array}\right] V_1^* \tag{4.3.4}$$

and for network $N_2$

$$I_1^{*'} = \left[\begin{array}{c|c} 0 & \\ \hline & A_2 \\ U & \end{array}\right] \left[\begin{array}{c|c} D_2 s & \\ \hline & \frac{1}{s} L_2 \end{array}\right] \left[\begin{array}{c|c} 0 & U \\ \hline \multicolumn{2}{c}{A_2^T} \end{array}\right] V_1^{*'} \qquad (4.3.5)$$

where the entries in the diagonal matrices $D_1$ and $D_2$ are the element values of the capacitors, the entries in the diagonal matrices $L_1$ and $L_2$ are the inverses of the element values of the inductors in these canonic networks, and $I_1^* = -I_1$, $V_1^* = V_1$, $I_1^{*'} = -I_1'$ and $V_1^{*'} = V_1'$ are the terminal variables.

Expressing (4.3.4) and (4.3.5) in symbolic form, we have

$$I_1^* = Y_1 V_1^* \qquad (4.3.6)$$

and

$$I_1^{*'} = Y_1 V_1^{*'} \qquad (4.3.7)$$

To derive the transformation relating (4.3.4) and (4.3.5), let

$$\begin{aligned} V_1^* &= C V_1^{*'} \\ I_1^{*'} &= C^T I_1^* \end{aligned} \qquad (4.3.8)$$

from which it follows that

$$I_1^{*'} = C^T Y_1 C V_1^{*'} \qquad (4.3.9)$$

Comparing (4.3.9), (4.3.7), (4.3.5) and (4.3.4) gives

$$C^T \left[\begin{array}{c|c} 0 & \\ \hline U & A_1 \end{array}\right] \begin{bmatrix} D_1 s & \\ & \frac{1}{s} L_1 \end{bmatrix} \left[\begin{array}{c|c} 0 & U \\ \hline \multicolumn{2}{c}{A_1^T} \end{array}\right] C \equiv \left[\begin{array}{c|c} 0 & \\ \hline U & A_2 \end{array}\right] \begin{bmatrix} D_2 s & \\ & \frac{1}{s} L_2 \end{bmatrix} \left[\begin{array}{c|c} 0 & U \\ \hline \multicolumn{2}{c}{A_2^T} \end{array}\right]$$

(4.3.10)

which is an identity for all values of s. For large values of $s = j\omega$, (4.3.10) gives

$$C^T \begin{bmatrix} 0 \\ U \end{bmatrix} D_1 [0 \quad U] C = \begin{bmatrix} 0 \\ U \end{bmatrix} D_2 [0 \quad U] \qquad (4.3.11)$$

Equation (4.3.11) can be rewritten as

$$C^T \begin{bmatrix} 0 \\ U \end{bmatrix} D_1 [0 \quad U] C = \begin{bmatrix} 0 \\ U \end{bmatrix} D_2^{1/2} V \, V^T D_2^{1/2} [0 \quad U] \qquad (4.3.12)$$

where V is an arbitrary orthogonal matrix. From (4.3.12)

$$C^T \begin{bmatrix} 0 \\ U \end{bmatrix} D_1^{1/2} = \begin{bmatrix} 0 \\ U \end{bmatrix} D_2^{1/2} V \qquad (4.3.13)$$

From (4.2.5) the driving point impedance remains invariant if

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \tag{4.3.14}$$

and from (4.3.13) it follows that

$$C_{21} = \cdots = C_{n1} = 0$$

The final form of C is, therefore,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix} \tag{4.3.15}$$

and (4.3.13) can be rewritten as

$$C_1^T D_1^{1/2} = D_2^{1/2} V \tag{4.3.16}$$

$$C_1^T = D_2^{1/2} V D_1^{-1/2} \tag{4.3.17}$$

where it is evident that $C_1$ is a non-singular matrix.

For small values of $s = j\omega$, (4.3.10) gives

$$C^T A_1 L_1 A_1^T C = A_2 L_2 A_2^T \tag{4.3.18}$$

or

$$\begin{bmatrix} 1 & \\ & C_1^T \end{bmatrix} A_1 L_1 A_1^T \begin{bmatrix} 1 & \\ & C_1 \end{bmatrix} = A_2 L_2 A_2^T \qquad (4.3.19)$$

Substituting the expression of $C_1$ from (4.3.17) into the above equation gives

$$\begin{bmatrix} 1 & \\ & D_2^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \\ & V \end{bmatrix} \begin{bmatrix} 1 & \\ & D_1^{-1/2} \end{bmatrix} A_1 L_1 A_1^T \begin{bmatrix} 1 & \\ & D_1^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} \begin{bmatrix} 1 & \\ & D_2^{1/2} \end{bmatrix}$$

$$= A_2 L_2 A_2^T \qquad (4.3.20)$$

or

$$\begin{bmatrix} 1 & \\ & V \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \\ & D_1^{-1/2} \end{bmatrix} A_1 L_1 A_1^T \begin{bmatrix} 1 & \\ & D_1^{-1/2} \end{bmatrix}}_{\text{parametric matrix of } N_1} \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} A_2 L_2 A_2^T \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix}}_{\text{parametric matrix of } N_2}$$

$$(4.3.21)$$

This result indicates that the parametric equations of network $N_1$ can be transformed to the parametric equations of network $N_2$ by an orthogonal matrix. The unknowns in this transformation are the matrices $V$, $D_2$ and $L_2$.

Class 2: The procedure for the derivation of the transformation for this class is similar to that given above except for the fact that a current driver replaces voltage drivers and circuit equations rather than cut-set equations are considered. The result then has the same general form as that in (4.3.21) with cut-set matrices replaced by the circuit matrices.

Class 3: Let the canonic network $N_1$ have the circuit equations

$$[B_1 \quad U] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \tag{4.3.22}$$

and let the canonic network $N_2$ have the circuit equations

$$[B_2 \quad U] \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = 0 \tag{4.3.23}$$

where again $N_1$ is known completely but only the topology of canonic network $N_2$ is known. Note that unlike in Class 1 and 2, $B_1$ and $B_2$ are rectangular matrices. Deriving the chord equations for network $N_1$, we have

$$V_2^* = \left[\begin{array}{c|c} B_1 & 0 \\ \hline & U \end{array}\right] \begin{bmatrix} D_1 \frac{1}{s} & \\ & L_1 s \end{bmatrix} \left[\begin{array}{c|c} \multicolumn{2}{c}{B_1^T} \\ \hline 0 & U \end{array}\right] I_2^* \tag{4.3.24}$$

and for network $N_2$

$$V_2^{*'} = \left[\begin{array}{c|c} & 0 \\ B_2 & -- \\ & U \end{array}\right] \begin{bmatrix} D_2 \frac{1}{s} & \\ & L_2 s \end{bmatrix} \left[\begin{array}{c|c} \multicolumn{2}{c}{B_2^T} \\ \hline 0 & U \end{array}\right] I_2^{*'} \qquad (4.3.25)$$

where the entries in the diagonal matrices $D_1$ and $D_2$ are the inverses of the capacitor element values, similarly the entries in $L_1$, $L_2$ are the inductor element values, and $V_2^* = V_2$, $I_2^* = -I_2$, $V_2^{*'} = V_2'$, and $I_2^{*'} = -I_2'$ are the terminal variables.

Consider the following transformation

$$\begin{aligned} I_2 &= C\, I_2' \\ V_2' &= C^T\, V_2 \end{aligned} \qquad (4.3.26)$$

Equations (4.3.24) and (4.3.25) gives

$$C^T \left[\begin{array}{c|c} & 0 \\ B_1 & -- \\ & U \end{array}\right] \begin{bmatrix} D_1 \frac{1}{s} & \\ & L_1 s \end{bmatrix} \left[\begin{array}{c|c} \multicolumn{2}{c}{B_1^T} \\ \hline 0 & U \end{array}\right] C = \left[\begin{array}{c|c} & 0 \\ B_2 & -- \\ & U \end{array}\right] \begin{bmatrix} D_2 \frac{1}{s} & \\ & L_2 s \end{bmatrix} \left[\begin{array}{c|c} \multicolumn{2}{c}{B_2^T} \\ \hline 0 & U \end{array}\right]$$

(4.3.27)

For large values of $s = j\omega$, from (4.3.27) we have

$$C^T \begin{bmatrix} 0 \\ U \end{bmatrix} L_1 [0 \quad U] C = \begin{bmatrix} 0 \\ U \end{bmatrix} L_2 [0 \quad U] \qquad (4.3.28)$$

which implies

$$C^T \begin{bmatrix} 0 \\ U \end{bmatrix} L_1^{1/2} = \begin{bmatrix} 0 \\ U \end{bmatrix} L_2^{1/2} V \tag{4.3.29}$$

where V is an arbitrary orthogonal matrix. The transformation matrix C has the form

$$C = \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix} \tag{4.3.30}$$

For small values of $s = j\omega$, (4.3.27) gives

$$\begin{bmatrix} 1 & \\ & V \end{bmatrix} \begin{bmatrix} 1 & \\ & L_1^{-1/2} \end{bmatrix} B_1 D_1 B_1^T \begin{bmatrix} 1 & \\ & L_1^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & \\ & V^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \\ & L_2^{-1/2} \end{bmatrix} B_2 D_2 B_2^T \begin{bmatrix} 1 & \\ & L_2^{-1/2} \end{bmatrix} \tag{4.3.31}$$

where the relation in (4.3.29) has been used in the final form. The unknowns in the transformation are V, $L_2$ and $D_2$.

Class 4: The derivation of the transformation for Class 4 canonic forms is identical to that in Class 3 except for obvious interchange of voltage and current drivers and circuit and cut-set matrices.

## 4.4 Realization of One-Port Canonic Networks by the Canonic Transformation

Consider first the realization of some of the known one-port canonic networks by using the canonic transformation developed in the last two sections. The parameters for equivalent LC networks are related by (4.3.21), where the matrices V, $D_2$ and $L_2$ are yet to be determined.

For a canonic network of Class 1, containing n inductors and n-1 capacitors, the number of unknowns in (4.3.21) is

$$n + (n-1) + \frac{(n-1)(n-2)}{2} = \frac{n(n+1)}{2} \qquad (4.4.1)$$

where $\frac{(n-1)(n-2)}{2}$ corresponds to the number of unknown entries in the orthogonal matrix V. In (4.3.21) the matrix

$$\begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} A_2 L_2 A_2^T \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix}$$

is symmetric. Therefore, the number of equations is equal to the number of unknowns. However, the equations in (4.3.21) can be simplified by rewriting

$$\begin{bmatrix} 1 & \\ & V \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} = \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} \begin{bmatrix} B'_{11} & B'_{12} \\ B'^T_{12} & B'_{22} \end{bmatrix} \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix}$$

(4.4.2)

Combining $D_2^{1/2}$ with the matrix V, we have

$$\begin{bmatrix} 1 & \\ & D_2^{1/2} V \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} 1 & \\ & V^T D_2^{1/2} \end{bmatrix} = \begin{bmatrix} B'_{11} & B'_{12} \\ B'^T_{12} & B'_{22} \end{bmatrix} \tag{4.4.3}$$

from which it follows that

$$\begin{aligned} A_{11} &= B'_{11} \\ A_{12} V^T D_2^{1/2} &= B'_{12} \\ D_2^{1/2} V A_{22} V^T D_2^{1/2} &= B'_{22} \end{aligned} \tag{4.4.4}$$

If we let

$$V^T D_2^{1/2} = X^T \tag{4.4.5}$$

the (4.4.5) can be written as

$$\begin{aligned} A_{11} &= B'_{11} \\ A_{12} X^T &= B'_{12} \\ X A_{22} X^T &= B'_{22} \end{aligned} \tag{4.4.6}$$

These equations are quadratic in the unknowns. Steepest descent, Gauss-Scidel or Newton's method can perhaps be used to obtain a numerical solution [HH 1, ZU 1]. The difficulty, however, is that the approximate solution must be given. This approximate solution can perhaps be obtained by using the analog simulation methods discussed by Rybashov [RY 1]. For some of the known canonic forms an analytical solution is possible.

In the following we shall discuss the realization of certain known canonic forms of Class 1. In obtaining these forms, parametric equations in (4.3.21) are considered. Canonic network $N_1$ is assumed to be known.

I. Cauer's First Canonic Form

The topology of the first Cauer form is given in Table 4.3.1. In (4.3.21), $A_2$ corresponds to the submatrix of the cut-set matrix for the Cauer network.

$$A_2 = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & \ddots & & \\ & & & & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

It is evident that $A_2 L_2 A_2^T$ is in tridiagonal form, for any diagonal matrix $L_2$, and this property is not altered after pre- and post-multiplication by the diagonal matrix

$$\begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix}$$

Let (4.3.21) be written in the form

$$\begin{bmatrix} 1 & \\ & V \end{bmatrix} \underbrace{\begin{bmatrix} b_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}}_{B} \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}}_{A} \qquad (4.4.7)$$

where A is a tridiagonal matrix and all other matrices are defined by comparison with (4.3.21). The known matrix B can be reduced to A by an orthogonal matrix transformation described by Frame [FR 1]. An alternate procedure, and one which has advantage over the tridiagonal method in that it also provides a proof that element values so obtained are all positive, is presented here.

Since B is positive definite, it has real positive eigenvalues

$$B = P \Lambda P^T \qquad (4.4.8)$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \cdots & P_{1n} \\ P_{21} & P_{22} \cdots & P_{2n} \\ \vdots & \vdots & \vdots \\ P_{n1} & P_{n2} & P_{nn} \end{bmatrix} \qquad (4.4.9)$$

and $\Lambda$ is a diagonal matrix containing the eigenvalues of A. Substituting (4.4.8) in (4.4.7) we have

$$\begin{bmatrix} 1 & \\ & V \end{bmatrix} P \Lambda P^T \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} = A \tag{4.4.10}$$

or

$$= P^T \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} A \begin{bmatrix} 1 & \\ & V \end{bmatrix} P = Q^T A Q \tag{4.4.11}$$

where

$$Q^T = \begin{bmatrix} q_{11} \cdots & q_{n1} \\ \vdots & \vdots \\ q_{1n} \cdots & q_{nn} \end{bmatrix} = P^T \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} = \begin{bmatrix} p_{11} & p_{21} \cdots & p_{n1} \\ \vdots & & \\ p_{1n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \\ 0 & V \end{bmatrix} \tag{4.4.12}$$

Comparing the terms in the above matrix equation, we obtain

$$q_{11} = p_{11}, \quad q_{12} = p_{12}, \quad \ldots, \quad q_{1n} = p_{1n} \tag{4.4.13}$$

Since Q in (4.4.11) is an orthogonal matrix, it transforms A into a diagonal form. The matrices Q and A are obtained, simultaneously, from the above information as demonstrated in the following.

Since $q_i$ is an eigenvector

$$(A - \lambda_i U)\, q_i = 0 \quad (i = 1, 2, \ldots n) \tag{4.4.14}$$

or

$$\begin{bmatrix} a_{11}-\lambda_i & -a_{12} & 0 & 0 & 0\dots \\ -a_{12} & a_{22}-\lambda_i & -a_{23} & 0 & 0\dots \\ 0 & -a_{23} & a_{33}-\lambda_i & -a_{34} & 0\dots \\ 0 & 0 & -a_{34} & a_{44}-\lambda_i & 0\dots \\ \vdots & \vdots & 0 & \vdots & \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ni} \end{bmatrix} = 0 \tag{4.4.15}$$

From the first equation of (4.4.15)

$$\begin{aligned} (a_{11} - \lambda_1)\, q_{11} &= a_{12}\, q_{21} \\ (a_{11} - \lambda_2)\, q_{12} &= a_{12}\, q_{22} \\ \vdots \qquad & \qquad \vdots \\ (a_{11} - \lambda_n)\, q_{1n} &= a_{12}\, q_{2n} \end{aligned} \tag{4.4.16}$$

where $b_{11} = a_{11}$ and $q_{1i}$ are known. After squaring and adding the equations in (4.4.16) we have

$$a_{12}^2(q_{21}^2 + \dots + q_{2n}^2) = a_{12}^2 = (a_{11} - \lambda_1)^2\, q_{11}^2 + \dots + (a_{11} - \lambda_n)^2\, q_{1n}^2 \tag{4.4.17}$$

Taking the positive value of $a_{12}$ as determined from (4.4.17), the values of $q_{2i}$ can be obtained from (4.4.16). Then

$$a_{22} = q_{21}^2\, \lambda_1 + \dots + q_{2n}^2\, \lambda_n$$

The remaining coefficients $a_{ij}$ in (4.4.15) are obtained by an identical procedure. After obtaining the matrix A, the network parameters are determined as follows. Let A be written as

$$A = \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & & \ddots \end{bmatrix} L_2 \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} \tag{4.4.18}$$

with

$$D_2^{-1/2} = \begin{bmatrix} d_2 & & & \\ & d_3 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

and

$$L_2 = \begin{bmatrix} \ell_1 & & & \\ & \ell_2 & & \\ & & \ddots & \\ & & & \ell_n \end{bmatrix}$$

Forming the matrix products in (4.4.18)

$$A = \begin{bmatrix} \ell_1 & -\ell_1 d_2 & 0 & \cdot & \cdot & \cdot & 0 \\ -\ell_1 d_2 & (\ell_1+\ell_2)d_2^2 & -\ell_2 d_2 d_3 & 0 & \cdot & \cdot & 0 \\ 0 & -\ell_2 d_2 d_3 & (\ell_2+\ell_3)d_3^2 & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & & & & \\ \cdot & 0 & \cdot & & & & \\ \cdot & \vdots & & & & & \\ 0 & 0 & \cdot & & & & \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} & 0 & \cdot & \cdot & \cdot & 0 \\ -a_{12} & a_{22} & -a_{23} & 0 & \cdot & \cdot & \cdot \\ 0 & -a_{23} & a_{33} & -a_{34} & \cdots & & \\ \vdots & & \cdot & & & & \\ \cdot & 0 & \cdot & & & & \\ & \vdots & & & & & \\ 0 & 0 & \cdot & & & & \end{bmatrix} \tag{4.4.19}$$

Therefore, network parameters are

$$a_{11} = \ell_1$$

$$d_2 = \frac{a_{12}}{a_{11}}$$

$$\ell_2 = \frac{a_{11}^2 a_{22}}{a_{12}^2} - a_{11} = \frac{a_{11}}{a_{12}^2} (a_{11} a_{22} - a_{12}^2) \tag{4.4.20}$$

$$d_3 = \frac{a_{23} a_{12}}{a_{11} a_{22} - a_{12}^2}$$

$$\ell_3 = \frac{a_{33}}{d_3^2} - \frac{a_{23}}{d_2 d_3} = \frac{1}{a_{12} d_3} \left[ \frac{a_{33} (a_{11} a_{22} - a_{12}^2) - a_{11} a_{23}^2}{a_{12} a_{23}} \right]$$

$$\vdots$$

Note that the numerical values of the capacitors is $\frac{1}{d_i^2}$ and inductors $\frac{1}{\ell_i}$. The positiveness of the parameters is established by the following theorem.

Theorem 4.4.1: Let B be a symmetric and positive definite matrix of order n and let N be a real non-singular matrix, then $NBN^T$ is positive definite.

Proof: After Hohn [HN 1]

Since the matrix $A_1 L_1 A_1^T$ is positive definite, the matrix A is positive definite and the submatrices corresponding to the principal minors of A are positive definite. In general, the expression for $\ell_i$ contains the first principal minor of order i x i, which is positive. Therefore, there exists a first Cauer form for a given canonic network with positive element values.

II. Second Foster Form

The topology of the second Foster form is given in Table 4.3.1. In this case the matrix $A_2$ corresponds to the submatrix of the cut-set matrix and has the form

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \dots 1 \\ & 1 & \bigcirc \\ & & 1 \\ \bigcirc & & & \ddots \\ & & & & 1 \end{bmatrix} \qquad (4.4.21)$$

Therefore, for any diagonal $L_2$ and $D_2$ matrices we have

$$\begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} A_2 L_2 A_2^T \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} & \\ a_{12} & a_{22} & \bigcirc \\ \vdots & & \ddots \\ a_{1n} & \bigcirc & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \tag{4.4.22}$$

where $A_{22}$ is also a diagonal matrix. Writing the above parametric equation in the form

$$\begin{bmatrix} 1 & \\ & V \end{bmatrix} \begin{bmatrix} b_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} = \begin{bmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \tag{4.4.23}$$

and equating the submatrices gives

$$\begin{aligned} a_{11} &= b_{11} \\ V\, B_{12}^T &= A_{12}^T \\ V\, B_{22}\, V^T &= A_{22} \end{aligned} \tag{4.4.24}$$

Since V is an orthogonal matrix which transforms $B_{22}$ into its diagonal form, the eigenvalues of $B_{22}$ are the diagonal entries in $A_{22}$. The matrix V can be obtained by any one of several methods, and the matrix A is obtained as a solution to (4.4.24).

The parameter values can be obtained by writing (4.4.22) explicitly in terms of $L_2$ and $D_2$ as

$$\begin{bmatrix} 1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} \sum \ell_i & \ell_2 & \cdots & \ell_n \\ \ell_2 & \ell_2 & & \\ \vdots & & \ddots & \\ \ell_n & & & \ell_n \end{bmatrix} \begin{bmatrix} 1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & & \\ \vdots & & \ddots & \\ a_{1n} & & & a_{nn} \end{bmatrix}$$

(4.4.25)

which gives

$$\underset{i \neq 1}{d_i^2} = \left.\frac{a_{ii}}{a_{1i}}\right.^{2}$$

$$\underset{i \neq 1}{\ell_i} = \frac{a_{1i}^2}{a_{ii}} \qquad (4.4.26)$$

and

$$\ell_1 = a_{11} - \sum_{i=2}^{n} \frac{a_{1i}^2}{a_{ii}}$$

where $\frac{1}{d_i^2}$ represents the capacitor element values and $\frac{1}{\ell_i}$ are the inductor element values. Since $B_{22}$ is positive definite, the eigenvalues of $B_{22}$ are all positive, implying $a_{ii}$ ($i \neq 1$) is positive ($a_{11}$ is also positive since $b_{11} > 0$). Since A is positive definite, $\ell_1$ is positive. By the above reasoning all $\ell_i$'s and $d_i^2$'s are positive. Note that $a_{ii}$, $i \neq 1$ corresponds to the resonant frequencies of the resonators in the second Foster form and that they are distinct.

## III. Second Cauer Form

The topology of the second Cauer form is given in Table 4.3.1. The matrix $A_2$ corresponds to the submatrix of the cut-set matrix and has the form

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \dots 1 \\ & 1 & 1 & 1 & 1 \dots 1 \\ & & 1 & 1 & 1 \dots 1 \\ & & & 1 & 1 \dots 1 \\ & & & & \ddots \quad \vdots \\ & & & & \quad 1 \end{bmatrix} \tag{4.4.27}$$

For canonic networks of Class 1, $A_2^{-1}$ exists and it is given as

$$A_2^{-1} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & \ddots & \ddots \\ & & & & \ddots & -1 \\ & & & & & 1 \end{bmatrix} \tag{4.4.28}$$

In this case, taking the inverse of the corresponding parametric equation, i.e., taking the inverse of each matrix in (4.3.21) we have

$$\begin{bmatrix} 1 & \\ & V^T \end{bmatrix} \begin{bmatrix} 1 & \\ & D_1^{1/2} \end{bmatrix} (A_1^T)^{-1} L_1^{-1} A_1^{-1} \begin{bmatrix} 1 & \\ & D_1^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \\ & V \end{bmatrix}$$

$$\begin{bmatrix} 1 & \\ & D_2^{1/2} \end{bmatrix} (A_2^T)^{-1} L_2^{-1} A_2 \begin{bmatrix} 1 & \\ & D_2^{1/2} \end{bmatrix} \tag{4.4.29}$$

Since the right hand side of (4.4.29) has the tridiagonal form, the techniques already presented can be used to find the matrix A, and the network parameters. It can be shown that all the element values are positive.

## IV. First Foster Form

The topology of the first Foster form is given in Table 4.3.1. In this case the matrix $A_2$ corresponding to the submatrix of the cut-set matrix has the form

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & & \bigcirc \\ 1 & & 1 & \\ 1 & \bigcirc & & \ddots\; 1 \end{bmatrix} \tag{4.4.30a}$$

The parametric equation in symbolic form is

$$\begin{bmatrix} 1 & \\ & V \end{bmatrix} \begin{bmatrix} b_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} = \begin{bmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} b_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} = \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} \begin{bmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} 1 & \\ & V \end{bmatrix} \qquad (4.4.30)$$

Since B is a symmetric and positive definite matrix, there exists an orthogonal matrix P which transforms B into its diagonal form, i.e.,

$$B = P\Lambda P^T = P\Lambda^{1/2}\Lambda^{1/2} P^T \qquad (4.4.31)$$

where P and $\Lambda$ are calculatable. If Q is an arbitrary orthogonal matrix, then (4.4.30) can be written as

$$\begin{bmatrix} 1 & \\ & V^T \end{bmatrix} \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} A_2 L_2^{1/2} Q Q^T L_2^{1/2} A_2^T \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & \\ & V \end{bmatrix}$$

$$= (P\Lambda^{1/2}) (\Lambda^{1/2} P^T) \qquad (4.4.32)$$

It follows from (4.4.32) that

$$\begin{bmatrix} 1 & \\ & V^T \end{bmatrix} \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} A_2 L_2^{1/2} Q = P\Lambda^{1/2} \qquad (4.4.33)$$

or

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & V_{22} & V_{32} & \dots & V_{n2} \\ \cdot & V_{23} & V_{33} & \dots & V_{n3} \\ \cdot & & & & \\ 0 & V_{2n} & \cdot & \cdot & V_{nn} \end{bmatrix} \begin{bmatrix} \ell_1^{1/2} & & & \\ d_2\ell_1^{1/2} & d_2\ell_2^{1/2} & & \\ \vdots & & \ddots & \\ d_n\ell_1^{1/2} & & \cdot & d_n\ell_n^{1/2} \end{bmatrix} \begin{bmatrix} q_{11} & \cdot & \cdot & \cdot & q_{1n} \\ q_{21} & & q_{22} & & q_{2n} \\ \vdots & & & & \vdots \\ q_{n1} & \cdot & \cdot & \cdot & q_{nn} \end{bmatrix}$$

$$= P\Lambda^{1/2} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & \cdot & \cdot & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1^{1/2} & & & \\ & \lambda_2^{1/2} & & \\ & & \ddots & \\ & & & \lambda_n^{1/2} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11}\lambda_1^{1/2} & p_{12}\lambda_2^{1/2} & \cdots & p_{1n}\lambda_n^{1/2} \\ p_{21}\lambda_1^{1/2} & p_{22}\lambda_2^{1/2} & \cdots & p_{2n}\lambda_n^{1/2} \\ p_{n1}\lambda_1^{1/2} & \cdot & \cdot & \cdot \; p_{nn}\lambda_n^{1/2} \end{bmatrix}$$

$$= \begin{bmatrix} \ell_1^{1/2} & 0 & \dots & 0 \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & & \vdots \\ s_{n1} & \cdot & \cdot & s_{nn} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & & \vdots \\ q_{n1} & \cdot & \cdot & q_{nn} \end{bmatrix} \qquad (4.4.34)$$

where $s_{ij}$ is a function of $v_{ij}$, $\ell_i$ and $d_i$.

From the first row of (4.4.34)

$$\begin{aligned}
\ell_1^{1/2} q_{11} &= p_{11} \lambda_1^{1/2} \\
\ell_1^{1/2} q_{12} &= p_{12} \lambda_2^{1/2} \\
&\vdots \\
\ell_1^{1/2} q_{1n} &= p_{1n} \lambda_n^{1/2}
\end{aligned} \tag{4.4.35}$$

where $\ell_1^{1/2} = b_{11}^{1/2}$, $p_{ij}$ and $\lambda_i$'s are known from (4.4.31). Therefore, from (4.4.35) $q_{1i}$ can be calculated. To obtain the other $q_{ij}$, pre-multiply the matrix equation (4.4.33) by its transpose, which can be written as

$$Q^T L_2^{1/2} A_2^T \begin{bmatrix} 1 & \\ & D_2^{-1} \end{bmatrix} A_2 L_2^{1/2} Q = \Lambda = Q^T C Q \tag{4.4.36}$$

Furthermore, because of the special form of $A_2$ in (4.4.30a), C has the form

$$C = \begin{bmatrix} c_{11} & c_{12} & & c_{1n} \\ c_{12} & c_{22} & & \bigcirc \\ & & \ddots & \\ c_{1n} & \bigcirc & & c_{nn} \end{bmatrix} \tag{4.4.37}$$

From (4.4.36), for $q_i$ an eigenvector of C,

$$(C - \lambda_i U)\, q_i = 0$$

or

$$\begin{bmatrix} c_{11}-\lambda_i & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{12} & c_{22}-\lambda_i & 0 & \cdots & & 0 \\ \cdot & & & & & \\ \cdot & 0 & c_{33}-\lambda_i & \cdots & & 0 \\ \cdot & \vdots & \vdots & & & \vdots \\ c_{1n} & 0 & 0 & \cdots & & c_{nn}-\lambda_i \end{bmatrix} \begin{bmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ni} \end{bmatrix} = 0 \qquad (4.4.38)$$

where $c_{11}$, $q_{1i}$ are known quantities. From the second row of the matrix equation in (4.4.37) we have

$$\begin{aligned} (c_{22}-\lambda_1)\, q_{21} &= -q_{11}\, c_{12} \\ (c_{22}-\lambda_2)\, q_{22} &= -q_{12}\, c_{12} \\ &\vdots \\ (c_{22}-\lambda_n)\, q_{2n} &= -q_{12}\, c_{12} \end{aligned} \qquad (4.4.39)$$

for $q_{1i} \neq 0$, and

$$\begin{aligned} \frac{q_{21}}{q_{11}} &= \frac{c_{12}}{\lambda_1 - c_{22}} \\ \frac{q_{22}}{q_{12}} &= \frac{c_{12}}{\lambda_2 - c_{22}} \\ &\vdots \\ \frac{q_{2n}}{q_{1n}} &= \frac{c_{12}}{\lambda_n - c_{22}} \end{aligned} \qquad (4.4.40)$$

Multiplying each one of the equations in (4.4.40) by $q_{11}^2$, $q_{12}^2$, ..., $q_{1n}^2$ respectively, gives

$$
\begin{aligned}
q_{11}\, q_{21} &= \frac{c_{12} q_{11}^2}{\lambda_1 - c_{22}} \\
q_{12}\, q_{22} &= \frac{c_{12} q_{12}^2}{\lambda_2 - c_{22}} \\
&\vdots \\
q_{1n}\, q_{2n} &= \frac{c_{12} q_{1n}^2}{\lambda_r - c_{22}}
\end{aligned}
\tag{4.4.41}
$$

In (4.4.37) $c_{ii}$ $(i \neq 1)$ corresponds to the resonant frequencies of the resonators in the first Foster form and they are all distinct. Furthermore, $c_{1i}$ is unequal to zero in the matrix C. Therefore, the following theorem can be stated.

Theorem 4.4.2: Let

$$
C = \begin{bmatrix}
c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\
c_{12} & c_{22} & & & \bigcirc & \\
\cdot & & c_{33} & & & \\
\cdot & & & \ddots & & \\
\cdot & \bigcirc & & & \ddots & \\
c_{1n} & & & & & c_{nn}
\end{bmatrix}
\tag{4.4.42}
$$

be a non-singular symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ $(\lambda_i > 0)$ such that the entries $c_{1i}$ and $c_{ii}$ are all non-zero. Let $c_{ii} \neq c_{jj}$ for $i \neq 1$, $i \neq j$. Then $c_{ii} \neq \lambda_j$ for all $i \neq 1$.

Proof: The proof of this theorem is established by contradiction. Let Q be an orthogonal matrix which transforms C into the diagonal form. Assume that $c_{22} = \lambda_k$. Then for $\lambda_k$

$$\begin{bmatrix} c_{11}-\lambda_k & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{12} & 0 & & & \bigcirc & \\ \cdot & & & & & \\ \cdot & & c_{33}-\lambda_k & & & \\ \cdot & \bigcirc & & \ddots & & \\ c_{1n} & & & & & c_{nn}-\lambda_k \end{bmatrix} \begin{bmatrix} q_{1k} \\ q_{2k} \\ \vdots \\ q_{nk} \end{bmatrix} = 0 \qquad (4.4.43)$$

From the second row of the matrix equation (4.4.43), $c_{12}\, q_{2k} = 0$, which implies $q_{2k} = 0$, since $c_{1i} \neq 0$ by hypothesis. Then from the third, fourth, ..., nth row of (4.4.43)

$$q_{3k} = q_{4k} = \ldots = q_{nk} = 0$$

Also from the first row

$$c_{12}\, q_{2k} = 0$$

which implies

$$q_{2k} = 0$$

Therefore, the eigenvector $q_k$ corresponding to the eigenvalue $\lambda_k$ is identically zero and a contradiction is established --- by definition an eigenvector is a non-zero vector. An identical proof can be repeated for all $c_{ii}$ $(i \neq 1)$ to establish the theorem.

Now return to the solution of (4.4.41), since Q is an orthogonal matrix

$$q_{11}\,q_{21} + q_{12}\,q_{22} + \dots + q_{1n}\,q_{2n} = 0 \qquad (4.4.44)$$

Substituting for $q_{1j}\,q_{2j}$ from (4.4.41) into (4.4.51) gives

$$\frac{c_{12}q_{11}^2}{\lambda_1 - c_{22}} + \frac{c_{12}q_{12}^2}{\lambda_2 - c_{22}} + \dots + \frac{c_{12}q_{1n}^2}{\lambda_n - c_{22}} = 0 \qquad (4.4.45)$$

But since $c_{12} \neq 0$, it follows that

$$\frac{q_{11}^2}{\lambda_1 - c_{22}} + \frac{q_{12}^2}{\lambda_2 - c_{22}} + \dots + \frac{q_{1n}^2}{\lambda_n - c_{22}} = 0 \qquad (4.4.46)$$

Repeating the procedure given above establishes all $c_{ii}$ $(i \neq 1)$ as a solution to the polynomial obtained from

$$\frac{q_{11}^2}{\lambda_1 - c_{ii}} + \frac{q_{12}^2}{\lambda_2 - c_{ii}} + \dots + \frac{q_{1n}^2}{\lambda_n - c_{ii}} = 0 \qquad (4.4.47)$$

The degree of this polynomial in $c_{ii}$ is $(n - 1)$. By using the positive pair property, it can be shown that all $c_{ii}$ $(i \neq 1)$ are positive real and distinct (from the unpublished notes of Dr. Tokad). Therefore, the roots of (4.4.47) gives

$$\begin{aligned} q_{i1} &= q_{11}\,\frac{c_{1i}}{\lambda_1 - c_{ii}} \\ q_{i2} &= q_{12}\,\frac{c_{1i}}{\lambda_2 - c_{ii}} \\ &\vdots \\ q_{in} &= q_{1n}\,\frac{c_{1i}}{\lambda_n - c_{22}} \end{aligned} \qquad (4.4.48)$$

After squaring and adding, we have

$$q_{i1}^2 + \ldots + q_{in}^2 = 1 = c_{1i}^2 \left\{ \left( \frac{q_{11}}{\lambda_1 - c_{ii}} \right)^2 + \ldots + \left( \frac{q_{1n}}{\lambda_n - c_{ii}} \right)^2 \right\} \tag{4.4.49}$$

where $c_{1i}$ is the only unknown. The network parameters are found by applying a procedure similar as described before for the previous cases.

The procedures for Class 2 canonic networks are similar to those of Class 1 and are not consider further.

The procedures for Class 3 and 4 are similar to 1, except when it needs an inverse of the submatrix $B_2$. But, in general, Class 3 and 4 can be reduced to Class 2 and 1, respectively, by the following procedure; a canonic network of Class 3 is reduced to Class 2 by introducing a capacitor $C_0$ in series with the current driver and of Class 4 is reduced to Class 1 by connecting an inductor parallel to the voltage driver. In considering the solution, $N_1$ and $N_2$ of Class 3 (4) are augmented by the above procedure with an identical capacitor (an identical inductor).

In conclusion of this section, realization of canonic networks by using transformation matrix needs a solution of nonlinear algebraic equations. In general, an analytical solution may not be possible. However, in certain special cases, these equations can be linearized.

## 4.5 A Realization of Canonic Networks by Tridiagonalization

In this procedure, the transformation from one canonic network, $N_1$, to the other, $N_2$, is realized in two steps. First, transform the given network $N_1$ into its equivalent first Cauer form $N_c$; then transform $N_c$ into $N_2$. The reason for the use of Cauer network as an intermediate step is obvious --- the parameter matrix for $N_c$ is tridiagonal.

The transformation matrix T from $N_1$ to $N_2$ is then obtained as the product of two transformation matrices $T_1$ and $T_2$ where $T_1$ is used to transform $N_1$ into $N_c$ and $T_2$ transforms $N_c$ into $N_2$. The construction of $T_1$ is already given in Section 4.4. To construct $T_2$ let the parametric equation be written as

$$T_2 A T_2^T = \begin{bmatrix} 1 & \\ & V \end{bmatrix} A \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} = B \qquad (4.5.1)$$

where A is tridiagonal and corresponds to the first Cauer form $N_c$, B corresponds to the unknown canonic form $N_2$ and V is an orthogonal matrix. The topology of $N_2$ is assumed to be known.

The transformation in (4.5.1) can also be written as

$$A = \begin{bmatrix} 1 & \\ & V^T \end{bmatrix} B \begin{bmatrix} 1 & \\ & V \end{bmatrix} \qquad (4.5.2)$$

The matrices B and A have the following form

$$B = \begin{bmatrix} b_{11} & b_{12} \cdots & b_{1n} \\ b_{12} & b_{22} \cdots & b_{2n} \\ \vdots & \vdots & \vdots \\ b_{1n} & b_{2n} \cdots & b_{nn} \end{bmatrix} \tag{4.5.3}$$

and

$$A = \begin{bmatrix} a_{11} & -a_{12} & & & & \\ -a_{12} & a_{22} & -a_{23} & & & \\ & -a_{23} & a_{33} & & & \\ & & & \ddots & & \\ & & & & a_{n-1,n-1} & -a_{n,n-1} \\ & & & & -a_{n,n-1} & a_{nn} \end{bmatrix} \tag{4.5.4}$$

The first row in B is reduced to tridiagonal form by the symmetric orthogonal matrix [FR 1]

$$V_1 = U - \frac{2}{X_1^T X_1} X_1 X_1^T$$

where

$$X_1 = \begin{bmatrix} b_{12} + a_{12} \\ b_{13} \\ \vdots \\ b_{1n} \end{bmatrix}$$

and

$$\sum_{i \neq 1}^{n} b_{1i}^2 = a_{12}^2$$

Applying this transformation to B gives

$$\begin{bmatrix} 1 & \\ & V_1 \end{bmatrix} B \begin{bmatrix} 1 & \\ & V_1 \end{bmatrix} = B_1$$

$$= \begin{bmatrix} b_{11} & -b_{12}^{(1)} & 0 & \cdots & 0 \\ -b_{12}^{(1)} & b_{22}^{(1)} & b_{23}^{(1)} & \cdots & b_{2n}^{(1)} \\ 0 & b_{23}^{(1)} & & & \vdots \\ \vdots & \vdots & & & \\ 0 & b_{2n}^{(1)} & \cdots & & b_{nn}^{(1)} \end{bmatrix} \tag{4.5.5}$$

where $b_{ij}^{(1)}$ are functions of $b_{ij}$, and $b_{11} = a_{11}$, $b_{12} = a_{12}$, $b_{22} = a_{22}$. The second row of $B_1$ is transformed to tridiagonal form by an orthogonal matrix

$$V_2 = U - \frac{2}{X_2^T X_2} X_2 X_2^T$$

where

$$X_2 = \begin{bmatrix} b_{23}^{(1)} + a_{23} \\ b_{24}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{2n}^{(1)} \end{bmatrix}$$

and

$$a_{23}^2 = \sum_{i \neq 1, 2}^{n} b_{2i}^2$$

The matrix $B_2$ has the form

$$B_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & V_2 \end{bmatrix} B_1 \begin{bmatrix} 1 & & \\ & 1 & \\ & & V_2 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} & 0 & \cdot & \cdot & \cdot & 0 \\ -a_{12} & a_{22} & -b_{23}^{(2)} & 0 & \cdots & & 0 \\ 0 & -b_{23} & b_{33}^{(2)} & b_{34}^{(2)} & \cdots & & b_{3n}^{(2)} \\ \cdot & & \cdot & & & & \cdot \\ \cdot & 0 & \cdot & & & & \cdot \\ \cdot & \vdots & \cdot & & & & \cdot \\ 0 & 0 & b_{3n}^{(2)} & \cdot & \cdot & \cdot & b_{nn}^{(2)} \end{bmatrix}$$

(4.5.6)

where $b_{23}^{(2)} = a_{23}$, $b_{33}^{(2)} = a_{33}$ and the entries $b_{ij}^{(2)}$ $(i, j \neq 2, 3)$ are obtained in terms of $b_{ij}^{(1)}$. This procedure is continued until B is tridiagonalized. The number of orthogonal transformations involved in this procedure is n - 2 and in the last stage, we have

$$B_{n-2} = \begin{bmatrix} a_{n-1,n-1} & \pm a_{n-1,n} \\ \pm a_{n-1,n} & a_{n,n} \end{bmatrix} = V_{n-2} \begin{bmatrix} b^{(n-3)}_{n-1,n-1} & b^{(n-3)}_{n-1,n} \\ b^{(n-3)}_{n-1,n} & b^{(n-3)}_{nn} \end{bmatrix} V_{n-2}$$

(4.5.7)

The equations of equality involved in the successive transformation are

$$a_{11} = b_{11}$$

$$a_{12} = \sum_{i=1}^{n} b^{2}_{1i}$$

$$a_{22} = f\,(b_{ij})$$

$$a_{23} = \sum_{i \neq 1,2}^{n} b^{(1)^{2}}_{2i}$$

$$a_{33} = f\,(b^{(1)}_{ij}) = f\,(b_{i,j}) \qquad (4.5.8)$$

$\cdot$
$\cdot$
$\cdot$

$$a_{n-1,n-1} = f\,(b^{(n-3)}_{ij}) = f\,(b_{i,j})$$

$$a_{n-1,n_3} = f\,(b^{(n-3)}_{ij}) = f\,(b_{i,j})$$

$$a_{n,n} = f\,(b^{(n-3)}_{ij}) = f\,(b_{i,j})$$

In the above set there are

$$n + n-1 = 2n-1$$

equations, corresponding to the 2n-1 unknowns. However, these equations are nonlinear, and have no special advantage over the quadratic equations of Section 4.4 insofar as solution is concerned, except the number of equations are minimized and are expressed in terms of the network parameters.

Synthesis equations similar to those in (4.5.8) can be obtained by a different procedure, which is as follows. Since the topology of the canonic network $N_2$ is known, the entries of the admittance matrix $Y_2$ can be obtained as an explicit function of the network parameters. The driving point admittance is the ratio of the principal minor and the determinant of $Y_2$. Due to the topological properties of canonic networks, the principal minor and the determinant of $Y_2$ do not have any factors in common.

In a similar manner the driving point admittance of $N_1$ can be obtained. The desired synthesis equations are obtained by equating the two admittance functions. The resulting equations are also nonlinear in network parameters.

## 4.6 Equivalent Networks - A State Model Approach

In Section 4.2 the properties of equivalent networks are studied from their mesh or nodal equations. Equivalent networks can also be generated from each other by considering their state models with the restriction, of course, that the operator matrices are related by a similarity transformation.

The state model for canonic networks considered in Chapter 2 is of the form

$$\begin{bmatrix} C_b & \\ & L_c \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & B_{12}^T \\ -B_{12} & 0 \end{bmatrix} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} + \begin{bmatrix} 0 \\ -B_{11} \end{bmatrix} V_0^*$$

(4.6.1)

$$I_0^* = -B_{11}^T I_{c\ell}$$

and its transformed state model is

$$\frac{d}{dt} \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \begin{bmatrix} 0 & C_b^{-1/2} B_{12}^T L_c^{-1/2} \\ L_c^{-1/2} B_{12} C_b^{-1/2} & 0 \end{bmatrix} \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \begin{bmatrix} 0 \\ -L_c^{-1/2} B_{11} \end{bmatrix} V_0^*$$

and (4.6.2)

$$I_0 = -B_{11}^T L_c^{-1/2} I'_{c\ell}$$

or

$$\frac{d}{dt} \begin{bmatrix} V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix} \begin{bmatrix} V_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -B_1 \end{bmatrix} V_0^*$$

(4.6.3)

$$I_0^* = -B_1^T I_3$$

Since a similarity transformation does not effect the terminal characteristics of the corresponding network, it is always possible to transform the state model of canonic network $N_1$ into the state model of canonic network $N_2$. Let the state model of $N_2$ be written in symbolic form as

$$\frac{d}{dt}\begin{bmatrix} V_2' \\ I_3' \end{bmatrix} = \begin{bmatrix} 0 & A_2^T \\ -A_2 & 0 \end{bmatrix} \begin{bmatrix} V_2' \\ I_3' \end{bmatrix} + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} V_0^*$$

$$I_0^* = -B_2^T I_3^* \qquad (4.6.4)$$

Note that $V_0^*$ and $I_0^*$ in (4.6.3) and (4.6.4) are identical. A procedure for constructing the transformation matrix is given by the following theorems.

Theorem 4.6.1: Let the operator matrix corresponding to (4.6.2) be

$$A = \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix} \qquad (4.6.5)$$

Then the non-zero eigenvalues of $A_1 A_1^T$ and $A_1^T A_1$ are identical.

Proof: Consider

$$|sU - A| = \det \left\{ [sU] - \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix} \right\} = \begin{vmatrix} S_1 & -A_1^T \\ A_1 & S_2 \end{vmatrix} \tag{4.6.6}$$

where $S_1$ and $S_2$ are the scalar matrices of the forms $sU_1$ and $sU_2$ where subscripts 1 and 2 are used to distinguish the order of these matrices.

Expanding the determinant given in (4.6.6) [GA 1], we have

$$\begin{aligned} |sU - \overset{(1)}{A}| &= |S_2|\ |S_1 + A_1^T (S_2)^{-1} A_1| \\ &= |S_2|\ |S_1 + \tfrac{1}{s} A_1^T A_1| \\ &= |S_2|\ |s^2 U_1 + A_1^T A_1|\ |S_1^{-1}| \end{aligned} \tag{4.6.7}$$

or expanding by the second row

$$|sU - \overset{(1)}{A}| = |S_1|\ |s^2 U_2 + A_1 A_1^T|\ |S_2^{-1}| \tag{4.6.8}$$

Therefore, from (4.6.7) and (4.6.8) we have

$$|sU - \overset{(1)}{A}| = |S_2|\ |s^2 U_1 + A_1^T A_1|\ |S_1^{-1}| = |S_1|\ |s^2 U_2 + A_1 A_1^T|\ |S_2^{-1}| \tag{4.6.9}$$

It follows from (4.6.9) that the non-zero eigenvalues of $A_1^T A_1$ and $A_1 A_1^T$ are identical.

Theorem 4.6.2: Let

$$A^{(1)} = \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix} \tag{4.6.10}$$

and

$$A^{(2)} = \begin{bmatrix} 0 & A_2^T \\ -A_2 & 0 \end{bmatrix} \tag{4.6.11}$$

be two operator matrices having the same eigenvalues. Then there exists an orthogonal matrix of the form

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{4.6.12}$$

which transforms $A^{(1)}$ into $A^{(2)}$.

Proof: Consider first the lemma

Lemma 4.6.1: Let

$$A^{(1)} = \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} 0 & D^T \\ -D & 0 \end{bmatrix}$$

where D has one of the forms shown in (3.4.2) or (3.4.3). Let $A^{(1)}$ and $A_3$ have the same eigenvalues. Then there exists an orthogonal matrix of the form

$$P^{(1)} = \begin{bmatrix} P_1' & \\ & P_2' \end{bmatrix} \tag{4.6.13}$$

which transforms $A_1$ to $A_3$.

Proof: Since $(A_1 A_1^T)$ is a symmetric matrix, there exists an orthogonal matrix $P_1$ [PE 1] such that

$$P_1^T (A_1 A_1^T) P_1 = D D^T \tag{4.6.14}$$

To show that this is true we follow a technique similar to that used in the proof of Theorem 4.6.1. Indeed, since $A^{(1)}$ and $A_3$ have the same eigenvalues we can write

$$P A^{(1)} P^{-1} = A_3$$

where P is orthogonal, hence

$$| sU - P A^{(1)} P^{-1} | = |sU - A_3|$$

$$|P| \; |sU - A^{(1)}| \; |P^{-1}| = |sU - A_3|$$

$$|sU - A^{(1)}| = |sU - A_3|$$

From (4.6.9) we have

$$|S_1| \; |s^2U + A_1^T A_1| \; |S_2^{-1}| = |S_1| \; |s^2U + D^T D| \; |S_2^{-1}|$$

$$= |S_2| \; |s^2U + A_1 A_1^T| \; |S_1^{-1}| = |S_2| \; |s^2U + D D^T| \; |S_1^{-1}| \tag{4.6.15}$$

which shows that $A_1^T A_1$ and $D^T D$ have identical eigenvalues and (4.6.14) is established. Also from (4.6.15) it follows that $A_1 A_1^T$ and $D D^T$ have identical eigenvalues. Therefore, there exists an orthogonal matrix $P_2$ such that

$$P_1^T (A_1 (P_2 P_2^T) A_1^T) P_1 = D D^T$$

or

$$P_1^T A P_2 = D \tag{4.6.16}$$

Continuing with the proof of the main theorem; since $A^{(2)}$ has the same eigenvalues as $A^{(1)}$, it follows from the Lemmas that $A^{(2)}$ can be transformed into $A_3$ using an orthogonal matrix $P''$ of the form given in (4.6.13). Thus

$$(P')^T A^{(1)} P' = A_3$$

and

$$(P'')^T A^{(2)} P'' = A_3$$

Therefore

$$(P'')(P')^T A_1 P' (P'')^T = A_2$$

where

$$P = P' (P'')^T$$

Consider now the similarity transformation from $N_1$ to $N_2$

$$\begin{bmatrix} V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix} \begin{bmatrix} V_2' \\ I_3' \end{bmatrix} \tag{4.6.17}$$

From (4.6.3) we have

$$\frac{d}{dt}\begin{bmatrix} V_2' \\ I_3' \end{bmatrix} = \begin{bmatrix} P_1^T & \\ & P_2^T \end{bmatrix} \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix} \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix} \begin{bmatrix} V_2' \\ I_3' \end{bmatrix} + \begin{bmatrix} P_1^T & \\ & P_2^T \end{bmatrix} \begin{bmatrix} 0 \\ -B_1 \end{bmatrix} V_0^*$$

$$I_0^* = -B_1^T P_2 I_3' \tag{4.6.18}$$

which is to be identical to (4.6.4), i.e., we require that

$$\begin{bmatrix} P_1^T & \\ & P_2^T \end{bmatrix} \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix} \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix} = \begin{bmatrix} 0 & A_2^T \\ -A_2 & 0 \end{bmatrix}$$

(4.6.19)

or

$$P_2^T B_1 = B_2$$

$$P_1^T A_1^T P_2 = A_2^T$$

and (4.6.20)

$$P_2^T B_1 = B_2$$

Since $N_1$ is given and the topology of $N_2$ is known, the forms of $A_2$ and $B_2$ are known. Hence the problem is to find orthogonal matrices $P_1$ and $P_2$ such that (4.6.20) is satisfied. The network parameters can be obtained by compariing (4.6.2) with (4.6.4).

The realization of any canonic form by this procedure may not be possible, in general, because of the nonlinear algebraic equations involved in (4.6.20). However, some of the known canonic froms can be realized by this procedure. An example is illustrated in Chapter V.

The development for n-port canonic networks is identical to the above, except $V_0^*$ is now taken as a vector corresponding to the drivers.

CHAPTER V

EXAMPLES

## 5.1 General

This chapter is devoted to the examples which illustrate the various techniques presented in the previous chapters.

## 5.2 Example 1

Realize the LC network whose specifications are given by the following state model.

$$\frac{d}{dt}\begin{bmatrix} v_2 \\ v_3 \\ i_7 \\ i_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \\ i_7 \\ i_8 \end{bmatrix} - \begin{bmatrix} -\frac{1}{4} \\ 0 \\ 0 \\ 0 \end{bmatrix}\frac{d}{dt}V_0^* + \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} v_0^* \\ i_1^* \end{bmatrix} \tag{5.2.1}$$

and

$$i_0^* = \frac{23}{4}\frac{d}{dt}v_0^* + \left[\ \frac{1}{4} \quad \frac{3}{4}\ \right]\begin{bmatrix} i_7 \\ i_8 \end{bmatrix} - i_1^*$$

$$v_1^* = -\left\{3\frac{d}{dt}i_1^* + \left[\ -1 \quad 0\ \right]\begin{bmatrix} v_2 \\ v_3 \end{bmatrix} + v_0^*\right\}$$

Solution: Since the eigenvalues of the operator matrix are pure imaginary, one of the necessary conditions is satisfied.

Comparing (3.3.2) and the state model in (5.2.1), gives

$$
\begin{aligned}
K_1 &= \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{bmatrix} \\
-K_2 &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\
K_3 &= \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix} \\
K_4 &= 0 \\
K_5 &= \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} \\
-K_6 &= \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \\
K_7 &= \frac{23}{4} \\
K_8 &= [\frac{1}{4} \quad \frac{3}{4}]
\end{aligned}
\qquad (5.2.2)
$$

$$K_9 = 0$$

$$K_{10} = -1$$

$$K_{11} = 3$$

$$K_{12} = [\,-1 \quad 0\,]$$

$$K_{13} = 0$$

$$K_{14} = 1$$

Let the following matrices be identified as

$$C_b + B_{12}^T C_c B_{12} = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}$$

$$B_{22}^T = \begin{bmatrix} b_{32} & b_{34} \\ b_{42} & b_{43} \end{bmatrix}$$

$$B_{12}^T C_c B_{11} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{5.2.3}$$

$$B_{32}^T = \begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

$$B_{23} L_b B_{33}^T = \begin{bmatrix} b_9 \\ b_{10} \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} b_5 \\ b_6 \end{bmatrix}$$

$$B_{11}^T C_c B_{11} = b_7$$

and

$$L_c + B_{23} L_b B_{23}^T = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{12} & \ell_{22} \end{bmatrix}$$

Therefore, (3.3.3) can be written in matrix form:

$$
\begin{array}{r}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24
\end{array}
\left[\begin{array}{cccccccccccccccccccc}
-\frac{1}{4} & & & -1 & & & & & & & & & & & & & & & & \\
\frac{1}{4} & \frac{1}{4} & & & -1 & & & & & & & & & & & & & & & \\
 & -\frac{1}{4} & & & & -1 & & & & & & & & & & & & & & \\
 & \frac{1}{4} & \frac{1}{4} & & & & -1 & & & & & & & & & & & & & \\
\frac{1}{4} & & & & & & & & & & 1 & & & & & & & & & \\
 & \frac{1}{4} & & & & & & & & & & 1 & & & & & & & & \\
\frac{1}{4} & & & & & & & & & & & & -1 & & & & & & & \\
 & \frac{1}{4} & & & & & & & & & & & & -1 & & & & & & \\
 & & & -1 & & & & -1 & -\frac{1}{2} & & & & & & & & & & & \\
 & & & & & -1 & & & \frac{1}{2} & & & & & & & & & & & \\
 & & & & -1 & & & & -1 & \frac{1}{2} & & & & & & & & & & \\
 & & & & & & -1 & & & \frac{1}{2} & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & \frac{1}{2} & & & & & & 1 & & & & & \\
 & & & & & & & & & \frac{1}{2} & & & & & & 1 & & & & \\
 & & & & & & & & & & \frac{1}{4} & & & & & & 1 & & & \\
 & & & & & & & & & & -\frac{1}{4} & & & & -1 & & & & & \\
 & & & & & & & & & & \frac{1}{4} & \frac{1}{4} & & & & -1 & & & & \\
 & & & & & & & & & & & \frac{1}{2} & & & & & & & & \\
 & & & & & & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & 1 & & & & & & 1 & -\frac{1}{2} \\
 & & & & & & & & & & & & & 1 & & & & & & -\frac{1}{2} \\
 & & & & & & & & & & & & & & & & & & -\frac{1}{4} &
\end{array}\right]
\left[\begin{array}{c}
c_{11}\\c_{12}\\c_{22}\\b_{32}\\b_{34}\\b_{42}\\b_{43}\\\ell_{11}\\\ell_{12}\\\ell_{22}\\b_1\\b_2\\b_3\\b_4\\b_5\\b_6\\b_7\\b_8\\b_9\\b_{10}
\end{array}\right]
=
\left[\begin{array}{c}
0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\\frac{23}{4}\\\frac{1}{4}\\\frac{3}{4}\\0\\3\\-1\\0\\0
\end{array}\right]
\tag{5.2.4}
$$

In (5.1.4) there are 24 linear equations, in 20 unknowns. However, equations 6, 7, 19 and 24 are redundant, and they can be omitted. By writing the remaining 20 equations in matrix form, it is verified that the coefficient matrix is nonsingular and the solution is given by

$$c_{11} = 4, \; c_{12} = 0, \; c_{22} = 4, \; b_{32} = -1, \; b_{34} = 1, \; b_{42} = 0,$$

$$b_{43} = 1, \; \ell_{11} = 1, \; \ell_{12} = 0, \; \ell_{22} = 2, \; b_1 = -1, \; b_2 = 0,$$

$$b_3 = -1, \; b_4 = 0, \; b_5 = 0, \; b_6 = -1, \; b_7 = 6, \; b_8 = 3, \; b_9 = 0,$$

$$b_{10} = 0 \tag{5.2.5}$$

From (5.2.3) and (5.2.5)

$$Y_c = \begin{bmatrix} B_{11}^T C_c B_{11} & B_{11}^T C_c B_{12} \\ B_{12}^T C_c B_{11} & C_b + B_{12}^T C_c B_{12} \end{bmatrix} = \left[\begin{array}{c|cc} 6 & -1 & 0 \\ \hline -1 & 4 & 0 \\ 0 & 0 & 4 \end{array}\right]$$

Applying Cederbaum's algorithm to $Y_c$ and rearranging the rows and columns, we have

$$Y_c = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} 3 & & & \\ & 4 & & \\ & & 1 & \\ & & & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{5.2.6}$$

Similarly

$$Z_L = \begin{bmatrix} L_c + B_{23} L_b B_{23}^T & B_{23} L_b B_{33}^T \\ B_{33} L_b B_{23}^T & B_{33} L_b B_{33}^T \end{bmatrix} = \left[\begin{array}{cc|c} 1 & & \\ & 2 & \\ \hline & & 3 \end{array}\right] \qquad (5.2.7)$$

and after the application of algorithm

$$Z_L = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] \begin{bmatrix} 3 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (5.2.8)$$

Also from (5.2.3) and (5.2.5) it is found that

$$B_{22} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$B_{32} = [\,-1 \quad 0\,]$$

$$B_{21} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$B_{31} = 1$$

Therefore, the circuit matrix is

$$
\begin{bmatrix}
-1 & 1 & 0 & 0 & 1 & & & & \\
-1 & 0 & 0 & 0 & & 1 & & & \\
0 & -1 & 0 & 0 & & & 1 & & \\
-1 & 1 & 1 & 0 & & & & 1 & \\
1 & -1 & 0 & 1 & & & & & 1
\end{bmatrix}
\begin{bmatrix}
v_0 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_1
\end{bmatrix}
= 0
$$

which is realizable and the corresponding network is given in Fig. 5.2.1, with the element values taken from (5.2.6) and (5.2.8).

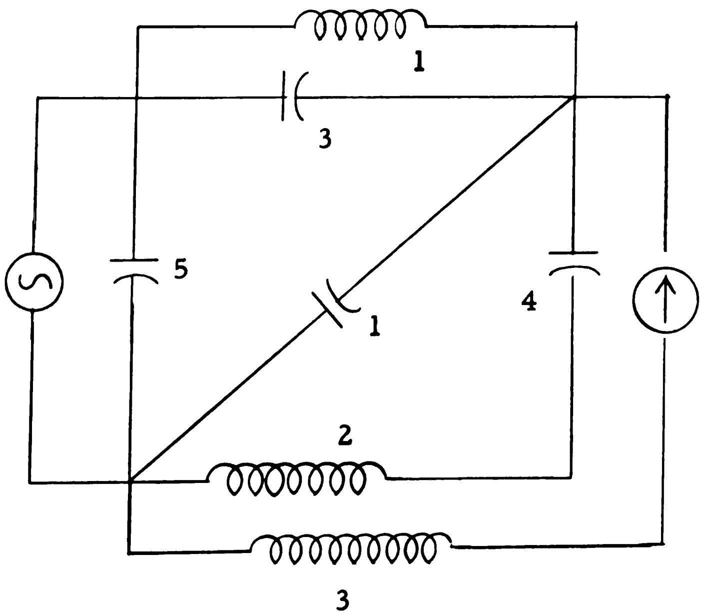



Fig. 5.2.1

5.3 The following example is given to illustrate the procedure described in Chapter IV, Section 5.

Example 2

Determine the element values of the lattice canonic network in Fig. 5.3.2 which has identical driving point impedance with the Cauer network given in Fig. 5.3.1.

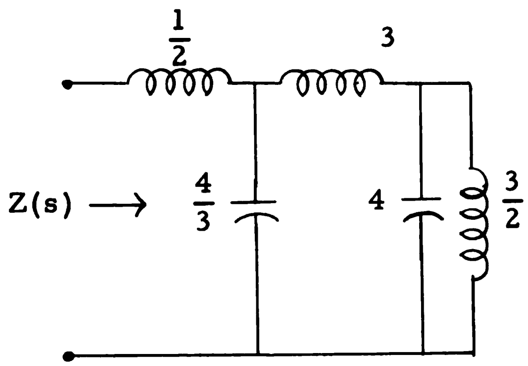



Fig. 5.3.1

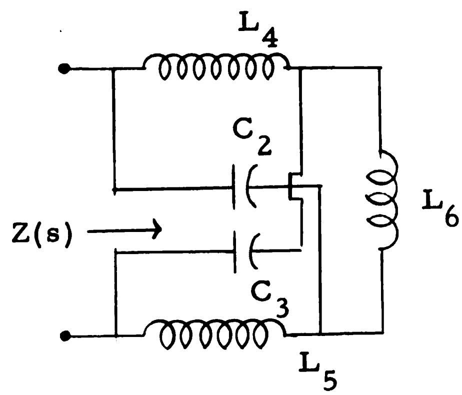



Fig. 5.3.2

Solution: Let the linear graphs of these equivalent LC networks be drawn as in Fig. 5.3.3.

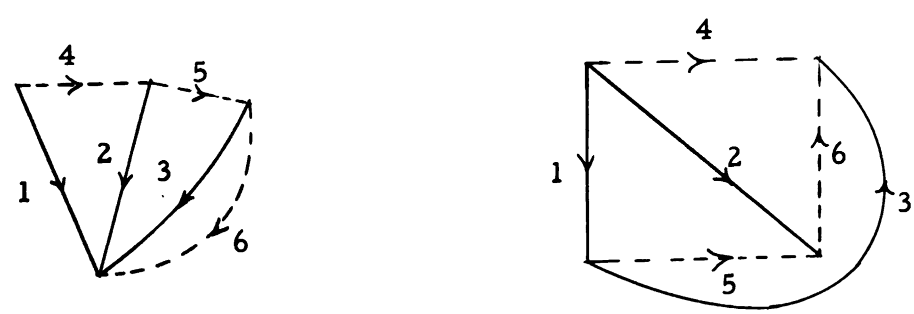



Fig. 5.3.3

The parameter matrix for the Cauer form is

$$A = \begin{bmatrix} 2 & -\sqrt{3} & 0 \\ -\sqrt{3} & \frac{7}{4} & -\frac{\sqrt{3}}{12} \\ 0 & -\frac{\sqrt{3}}{12} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} & 0 \\ -a_{12} & a_{22} & -a_{23} \\ 0 & -a_{23} & a_{33} \end{bmatrix}$$

(5.3.1)

The parameter matrix for the lattice section will be of the form

$$B = \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix} A_2 L_2 A_2^T \begin{bmatrix} 1 & \\ & D_2^{-1/2} \end{bmatrix}$$

$$= \begin{bmatrix} \ell_4 + \ell_5 + \ell_6 & -(\ell_5 + \ell_6)\, d_2 & (\ell_4 + \ell_5)\, d_3 \\ -(\ell_5 + \ell_6)\, d_2 & (\ell_5 + \ell_6)\, d_2^2 & -\ell_5\, d_2\, d_3 \\ (\ell_4 + \ell_5)\, d_3 & -\ell_5\, d_2\, d_3 & (\ell_4 + \ell_5)\, d_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & -b_{12} & b_{13} \\ -b_{12} & b_{22} & -b_{23} \\ b_{13} & -b_{23} & b_{33} \end{bmatrix}$$

(5.3.2)

Since the first row of B is not in tridiagonal form, an orthogonal matrix defined by

$$V_1 = U - \frac{2}{X^T X} X X^T$$

is used to transform B into a tridiagonal form where

$$X = \begin{bmatrix} -b_{12} + a_{12} \\ + b_{13} \end{bmatrix} , \quad b_{12}^2 + b_{13}^2 = a_{12}$$

Therefore, $V_1$ can be expressed in terms of $b_{ij}$ as

$$V_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} b_{12} & -b_{13} \\ -b_{13} & -b_{12} \end{bmatrix} \tag{5.3.3}$$

Since

$$A = \begin{bmatrix} 1 & \\ & V_1 \end{bmatrix} \left[\begin{array}{c|cc} b_{11} & -b_{12} & b_{13} \\ \hline -b_{12} & b_{22} & -b_{23} \\ b_{13} & -b_{23} & b_{33} \end{array}\right] \begin{bmatrix} 1 & \\ & V_1 \end{bmatrix} = \begin{bmatrix} b_{11} & -b_{12}^{(1)} & 0 \\ -b_{12}^{(1)} & b_{22}^{(1)} & b_{23}^{(1)} \\ 0 & b_{23}^{(1)} & b_{33}^{(1)} \end{bmatrix}$$

(5.3.4)

and

$$b_{12}^{(1)} = a_{12}$$

then

$$\begin{bmatrix} b_{22}^{(1)} & \pm b_{23}^{(1)} \\ \pm b_{23}^{(1)} & b_{33}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{bmatrix} \tag{5.3.5}$$

Although in (5.3.5), there are two possibilities for the sign of $b_{23}^{(1)}$, we shall take the positive sign. In this case

$$V_1 = \begin{bmatrix} b_{22} & -b_{23} \\ -b_{23} & b_{33} \end{bmatrix} V_1 = \begin{bmatrix} \frac{7}{4} & \frac{\sqrt{3}}{12} \\ \frac{\sqrt{3}}{12} & \frac{1}{4} \end{bmatrix} \tag{5.3.6}$$

or

$$\begin{bmatrix} b_{22} & -b_{23} \\ -b_{23} & b_{33} \end{bmatrix} = \left(\frac{1}{\sqrt{3}}\right)^2 \begin{bmatrix} b_{12} & -b_{13} \\ -b_{13} & -b_{12} \end{bmatrix} \begin{bmatrix} \frac{7}{4} & \frac{\sqrt{3}}{12} \\ \frac{\sqrt{3}}{12} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} b_{12} & -b_{13} \\ -b_{13} & -b_{12} \end{bmatrix} \tag{5.3.7}$$

From (5.3.2) and the tridiagonalization procedure we also have the relations

$$3 = b_{12}^2 + b_{13}^2 \qquad \text{or} \qquad b_{12} = + \sqrt{3 - b_{13}^2} \tag{5.3.8}$$

and

$$b_{12}^2 b_{33} + b_{13}^2 b_{22} - b_{12} b_{13} b_{23} = 2 b_{22} b_{33} \tag{5.3.9}$$

Substituting the expressions of $b_{12}$, $b_{22}$, $b_{23}$, $b_{33}$ given in (5.3.7) and

(5.3.8) into (5.3.9), we obtain

$$\frac{1}{54} b_{13}^4 - \frac{1}{18} b_{13}^2 + \frac{1}{8} + \frac{\sqrt{3}}{36} (3 - b_{13}^2)^{3/2} b_{13} - \frac{\sqrt{3}}{36} (3 - b_{13}^2)^{1/2} b_{13}^3 = 0 \tag{5.3.10}$$

which is equivalent to a fourth degree equation in the unknown $b_{13}^2$.

One of the solutions of (5.3.10) is

$$b_{13} = \frac{3}{2}$$

Corresponding to this solution from (5.3.8) and (5.3.7), we have

$$b_{12} = \frac{\sqrt{3}}{2} , \quad b_{22} = \frac{1}{2} , \quad b_{23} = \frac{1}{\sqrt{3}} , \quad b_{33} = \frac{3}{2} \tag{5.3.11}$$

Hence from (5.3.2), the network parameters are determined as follows:

$$d_2^2 = \frac{1}{3} , \quad d_3^2 = 1 , \quad \ell_5 = 1 , \quad \ell_4 = \frac{1}{2} , \quad \ell_6 = \frac{1}{2}$$

and the element values are

$$c_2 = 3 , \quad c_3 = 1 , \quad L_4 = \frac{1}{\ell_4} = 2, \; L_5 = \frac{1}{\ell_5} = 2 , \quad L_6 = \frac{1}{\ell_6} = 2 \tag{5.3.12}$$

Other possible solutions can be investigated similarly.

5.4 The procedure for the transformation of the state model described in Chapter IV, Section 6 is illustrated by the following example.

<u>Example 3</u>

Determine the element values of the Cauer network, $N_2$, shown in Fig. 5.4.1, which is equivalent to the Foster network, $N_1$, given in Fig. 5.4.2.

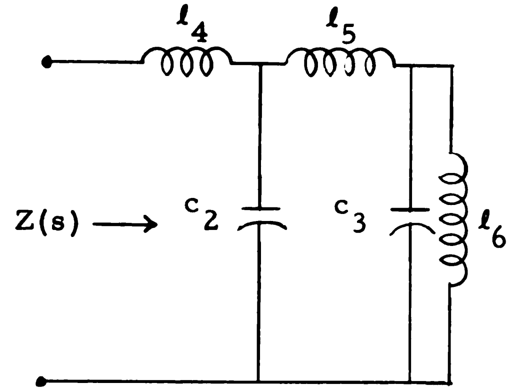



Fig. 5.4.1

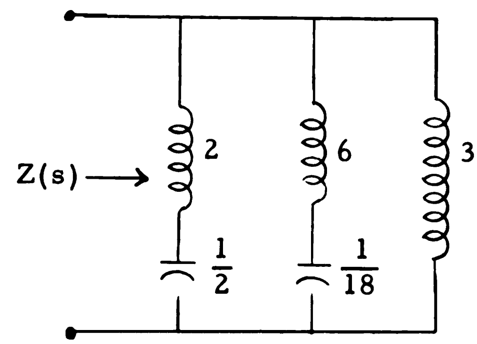



Fig. 5.4.2

Solution: The linear graphs for $N_1$ and $N_2$ are given in Fig. 5.4.3.

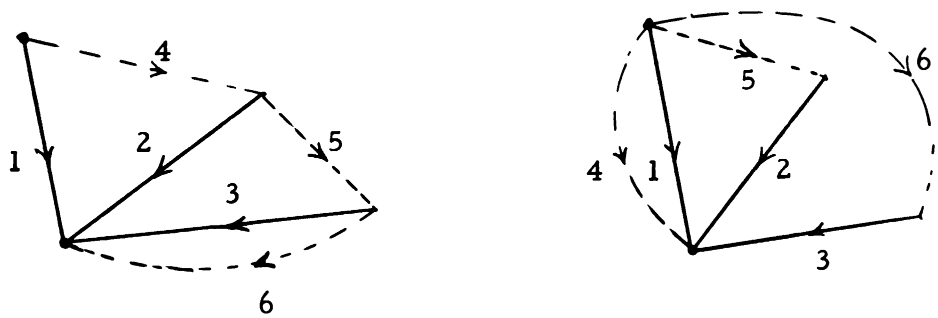



Fig. 5.4.3

The state model of $N_2$ is

$$
\begin{bmatrix} c_2 & & & & \\ & c_3 & & & \\ & & \ell_4 & & \\ & & & \ell_5 & \\ & & & & \ell_6 \end{bmatrix}
\frac{d}{dt}\begin{bmatrix} v_2 \\ v_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}
=
\left[\begin{array}{cc|ccc} & & 1 & -1 & 0 \\ & & 0 & 1 & -1 \\ \hline -1 & 0 & & & \\ 1 & -1 & & & \\ 0 & 1 & & & \end{array}\right]
\begin{bmatrix} v_2 \\ v_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} V_1
\qquad (5.4.1)
$$

and

$$
I_1 = -\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_4 \\ i_5 \\ i_6 \end{bmatrix}
$$

Applying the transformation described in Chapter III, the state model in (5.4.1) can be brought to the form

$$
\frac{d}{dt}\begin{bmatrix} v_2' \\ v_3' \\ i_4' \\ i_5' \\ i_6' \end{bmatrix}
=
\left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{\multirow{2}{*}{$\bigcirc$}} & (c_2\ell_4)^{-1/2} & -(c_2\ell_5)^{-1/2} & \\ & & 0 & (c_3\ell_5)^{-1/2} & -(c_3\ell_6)^{-1/2} \\ \hline -(c_2\ell_4)^{-1/2} & 0 & & & \\ (c_2\ell_5)^{-1/2} & -(c_3\ell_5)^{-1/2} & & \bigcirc & \\ 0 & (c_3\ell_6)^{-1/2} & & & \end{array}\right]
$$

$$
\times \begin{bmatrix} v_2' \\ v_3' \\ i_4' \\ i_5' \\ i_6' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ell_4^{-1/2} \\ 0 \\ 0 \end{bmatrix} V_1
$$

and (5.4.2)

$$
i_1 = -[\ \ell_4^{-1/2} \quad 0 \quad 0\ ] \begin{bmatrix} i_4' \\ i_5' \\ i_6' \end{bmatrix}
$$

and making use of the terminal variables

$$
v_1^* = v_1
$$

and

$$
i_1^* = -i_1
$$

we have

$$
\frac{d}{dt} \begin{bmatrix} v_2' \\ v_3' \\ i_4' \\ i_5' \\ i_6' \end{bmatrix} = \left[\begin{array}{cc|ccc} & & a_1 & -a_2 & 0 \\ \multicolumn{2}{c|}{\bigcirc} & 0 & a_3 & -a_4 \\ \hline -a_1 & 0 & & & \\ a_2 & -a_3 & \multicolumn{3}{c}{\bigcirc} \\ 0 & a_4 & & & \end{array}\right] \begin{bmatrix} v_2' \\ v_3' \\ i_4' \\ i_5' \\ i_6' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix} V_1^*
$$

$$i_1^* = [\, b_1 \quad 0 \quad 0 \,] \begin{bmatrix} i_4' \\ i_5' \\ i_6' \end{bmatrix}$$

or in symbolic form

$$\frac{d}{dt}\begin{bmatrix} V_{bc}' \\ V_{c\ell}' \end{bmatrix} = \begin{bmatrix} 0 & A_2^T \\ -A_2 & 0 \end{bmatrix} \begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} V_1 \qquad (5.4.4)$$

$$I_1 = -B_2^T I_{c\ell}'$$

Repeating the same transformation for $N_1$ its state model takes on the form

$$\frac{d}{dt}\begin{bmatrix} v_2^* \\ v_3^* \\ i_4^* \\ i_5^* \\ i_6^* \end{bmatrix} = \left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{\multirow{2}{*}{$\bigcirc$}} & 0 & \sqrt{3} & 0 \\ & & 0 & 0 & 1 \\ \hline 0 & 0 & & & \\ -\sqrt{3} & 0 & \multicolumn{3}{c}{\bigcirc} \\ 0 & -1 & & & \end{array}\right] \begin{bmatrix} v_2^* \\ v_3^* \\ i_4^* \\ i_5^* \\ i_6^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} V_1^* \qquad (5.4.5)$$

and

$$I_1^* = \left[ \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{2}} \right] \begin{bmatrix} i_4^* \\ i_5^* \\ i_6^* \end{bmatrix}$$

or symbolically

$$\frac{d}{dt} \begin{bmatrix} V_{bc}^* \\ I_{c\ell}^* \end{bmatrix} = \begin{bmatrix} 0 & A_1^T \\ -A_1 & 0 \end{bmatrix} \begin{bmatrix} V_{bc}^* \\ I_{c\ell}^* \end{bmatrix} + \begin{bmatrix} 0 \\ -B_1 \end{bmatrix} V_1 \tag{5.4.6}$$

and

$$I_1 = -B_1^T I_{c\ell}$$

Now consider the transformation described in Section 4.6. To transform the state model of $N_1$ into that of $N_2$, from (4.6.20), we write

$$P_2^T B_1 = B_2 \qquad \text{or} \qquad B_1 = P_2 B_2$$

and (5.4.7)

$$P_2^T A_1 P_1 = A_2$$

where $P_1$ and $P_2$ are orthogonal matrices. The first equation gives

$$P_2 B_2 = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{5.4.8}$$

or

$$p_{11}\,b_1 = \frac{1}{\sqrt{3}}$$

$$p_{21}\,b_1 = \frac{1}{\sqrt{6}} \qquad (5.4.9)$$

$$p_{31}\,b_1 = \frac{1}{\sqrt{2}}$$

squaring and adding these equations, we have

$$b_1^2\,(p_{11}^2 + p_{21}^2 + p_{31}^2) = 1 = b_1^2$$

or

$$b_1 = 1$$

where the sign of $b_1$ is decided from the topology of $N_2$. Therefore, (5.4.9) gives

$$p_{11} = \frac{1}{\sqrt{3}}\,, \quad p_{21} = \frac{1}{\sqrt{6}}\,, \quad p_{31} = \frac{1}{\sqrt{2}} \qquad (5.4.10)$$

Also from the second equation in (5.4.7), it follows that

$$P_2\,A_2\,A_2^T\,P_2^T = A_1\,A_1^T = \begin{bmatrix} 0 & & \\ & 3 & \\ & & 1 \end{bmatrix} \qquad (5.4.11)$$

where

$$A_2 A_2^T = \begin{bmatrix} a_1^2 & -a_1 a_2 & 0 \\ -a_1 a_2 & a_2^2 + a_3^2 & -a_3 a_4 \\ 0 & -a_3 a_4 & a_4^2 \end{bmatrix} = A \qquad (5.4.12)$$

Equation (5.4.11) indicates the eigenvalues of A are 0, 3 and 1. From (5.4.10) the first eigenvector is known. Then

$$a_1^2 = p_{11}^2 \lambda_1 + p_{21}^2 \lambda_2 + p_{31}^2 \lambda_3$$

$$= 0 + \frac{1}{6} \cdot 3 + \frac{1}{2} \cdot 1 = 1$$

or

$$a_1 = 1$$

Further from the relation

$$(A - \lambda_i U) p_i = 0$$

or

$$\begin{bmatrix} a_1^2 - \lambda_i & -a_1 a_2 & 0 \\ -a_1 a_2 & a_2^2 + a_3^2 - \lambda_i & -a_3 a_4 \\ 0 & -a_3 a_4 & a_4^2 - \lambda_i \end{bmatrix} \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{bmatrix} = 0 \qquad (5.4.13)$$

we have

$$a_1^2 p_{11} = a_1 a_2 p_{12}$$

$$(a_1^2 - 3) p_{21} = a_1 a_2 p_{22} \qquad (5.4.14)$$

$$(a_1^2 - 1) p_{31} = a_1 a_2 p_{32}$$

Substituting the values of $p_{i1}$ and $a_1$, (5.4.14) gives

$$\frac{1}{\sqrt{3}} = a_1 a_2 p_{12}$$

$$-\frac{2}{\sqrt{6}} = a_1 a_2 p_{22} \qquad (5.4.15)$$

$$0 = a_1 a_2 p_{32}$$

from which

$$\frac{1}{3} + \frac{4}{6} = a_1^2 a_2^2$$

or

$$a_2 = 1$$

Therefore, from (5.4.14), we have

$$p_{12} = \frac{1}{\sqrt{3}}, \quad p_{22} = -\frac{2}{\sqrt{6}}, \quad p_{32} = 0$$

Since

$$a_2^2 + a_3^2 = p_{12}^2 \lambda_1 + p_{22}^2 \lambda_2 + p_{32}^2 \lambda_3 = 2$$

we also have

$$a_3 = 1$$

The second row of (5.4.13), after substituting the known variables, gives

$$-\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = a_3\, a_4\, p_{13}$$

$$-\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} = a_3\, a_4\, p_{23}$$

$$-\frac{1}{\sqrt{2}} = a_3\, a_4\, p_{33}$$

from which we have

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1 = a_3^2\, a_4^2 = 1$$

or

$$a_4 = 1$$

Therefore,

$$p_{13} = \frac{1}{\sqrt{3}}, \quad p_{23} = \frac{1}{\sqrt{6}}, \quad p_{33} = -\frac{1}{\sqrt{2}}$$

Since all the entries are found, (5.4.3) is of the form

$$\frac{d}{dt}\begin{bmatrix} v_2' \\ v_3' \\ i_4' \\ i_5' \\ i_6' \end{bmatrix} = \left[\begin{array}{cc|ccc} & & 1 & -1 & 0 \\ \multicolumn{2}{c|}{\bigcirc} & & & \\ & & 0 & 1 & -1 \\ \hline -1 & 0 & & & \\ 1 & -1 & \multicolumn{3}{c}{\bigcirc} \\ 0 & 1 & & & \end{array}\right] \begin{bmatrix} v_2' \\ v_3' \\ i_4' \\ i_5' \\ i_6' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} V_1^*$$

and

$$I_1^* = [\,1 \quad 0 \quad 0\,] \begin{bmatrix} i_4' \\ i_5' \\ i_6' \end{bmatrix}$$

Note that, with this procedure, although the transformation matrix P is necessary to be evaluated, it is not needed for the final part of synthesis. Comparing (5.4.2) and (5.4.16), we have the element values

$$c_2 = 1 \ , \quad c_3 = 1 \ , \quad \ell_4 = 1 \ , \quad \ell_5 = 1 \ , \quad \ell_6 = 1$$

# CHAPTER VI

# CONCLUSION

The use of state models is shown to be effective in realizing reactance functions and they have served as a basis for extending the realization of 2-port Z and Y matrices having dominant residue matrices. The same procedure can be applied to n-port matrices, having dominant residue matrices. One of the significant features of the state model approach is that the realization procedures do not differ from one- to n-port networks.

The similarity transformations given in Chapter IV provide simple and practical ways of relating the parameters of one canonic network to another. Although the solution to the resulting equations is given explicitly for certain classes of canonic networks, their solution for certain other classes may be very difficult either analytically or numerically. One of the difficulties with numerical techniques is an initial guess. As an aid in establishing the initial estimation, a procedure is given for reducing the parametric matrix to tridiagonal form. This form is sometimes effective in obtaining an analytic solution.

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