FORMULATION AND ESTIMATION OF A COMPLETE SYSTEM OF DEMAND EQUATIONS

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ABSTRACT

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SYSTEM OF DEMAND EQUATIONS

By

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This dissertation constitutes an investigation into the principles underlying the formulation and estimation of complete sets of theoretically plausible demand equations. Upon an analysis of these principles, two alternative parametrizations of the convenient double-logarithmic system of demand equations are suggested. These two systems along with two other systems are estimated from five-series data on personal consumption expenditures in the United States. A comparison of the results reveals that the two double-logarithmic systems may indeed be serious competitors to the existing functional forms for demand equations.

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My intellectual debt to Professor A. Goldberger's widely acclaimed review of consumer demand theory would be obvious to the most casual reader. Indeed, the first chapters of this thesis are to a large extent a mere paraphrasing of some of the ideas found in Professor Goldberger's paper. Although I have attempted to record explicit plagiarisms, I may at this point apologize for any passages where I have failed to give his paper due

credit. I can only hope that I have not misinterpreted him in my numerous references to his work.

During the course of preparation of this thesis I have had the opportunity of corresponding with Professors Anton P. Barten, Arthur S. Goldberger, and Henri Theil. They have all been extremely patient in answering to my queries, and have truly served to clarify my thinking on several issues. Professors A. P. Barten and Alan Powell were also kind enough to discuss my thesis with me while they were visiting East Lansing as guests of the Econometrics Workshop here at Michigan State University. To all of them I am sincerely grateful.

Needless to say, I alone remain responsible for any remaining errors.

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CHAPTER 1

THE PURE THEORY OF CONSUMER'S DEMAND

1.1 Introduction

In this chapter we present the main results of the pure theory of consumer's demand under static certainty. This theory, as it is known today, represents a major triumph of axiomatic methods in the analysis of economic behavior. The current formulation, however, has evolved after a long period of extended debate over numerous and assorted issues. In section 1.2 we have attempted to provide a brief survey of the methodological and other issues that were the subject of this historical debate. The emphasis in this section, however, is more toward history. The theoretical issues have been examined in greater detail in sections 1.3 and 1.4. In section 1.3 we make use of hindsight to provide the current axiomatic formulation of consumer demand theory, while in section 1.4 we digress to examine briefly the theory of revealed preference which historically preceded the current formulations. Finally, in sections 1.5 and 1.6, the major results in the pure theory of consumer's demand are derived, and in section 1.7, the conditions under which

aggregation over individuals and commodities may be carried out are formulated.

1.2 Methodology and History

The purpose of demand analysis is to explain variations in consumer expenditures by an analysis of cross-sectional or time-series data on consumption expenditures and prices. For the analysis to be meaningful, a model must be built in conformity with the *a priori* beliefs of the researcher. These *a priori* beliefs may be specified on an *ad hoc* basis, or be deduced from a set of axioms that are specified as the maintained hypotheses. The model should reflect these beliefs in terms of restrictions on the functional form and the parameters of the demand equations. Both the *ad hoc* specification, and the axiomatically derived restrictions approach have been used in the literature on demand theory; and a wide variety of demand models have resulted from combining the two approaches in varying proportions. A broad distinction can, however, be made between two classes of models, which for lack of accepted terminology are labelled as "*ad hoc* models" and "axiomatic models."

The *ad hoc* approach in demand theory is usually credited to the work of Gustav Cassel [1899], [1918], who was the first of the modern economists to revive the approach of Augustin Cournot [1838] and Leon Walras [1874] in viewing demand functions as empirical hypotheses. Unlike Cournot and Walras, however, Cassel expressed a positive distaste for "utility theory" which he considered unrealistic and full of error. Instead, he argued, that a theory of demand could be constructed independent of a utility substructure.[1]

Cassel was not alone in viewing utility theory with some suspicion. Henry L. Moore, credited with being the founder of statistical demand analysis, was yet another major economist who espoused the ad hoc approach in demand theory. With the extensive statistical works of Moore [1914], [1922], [1925-26], the ad hoc tradition in demand analysis was firmly established among empirical demand theorists. Models of demand were specified on an ad hoc basis and restrictions on parameters were imposed in a similar manner. From a methodological point of view it was unclear what the maintained hypotheses were, and

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[1]The references above are from Stigler [1950], and Wold [1943-44]. The extent to which Cassel was successful in construction a demand theory independent of utility is a matter of debate. Wold [1943-44, Part III, pp. 77-90] has shown that Cassel actually imposed sufficient restrictions on his demand functions to make them logically equivalent to the Hicks-Allen [1934] indifference curve approach. Stigler [1950] is of a similar opinion. Houthakker [1961] however, has differed. Taking issue with Wold's [1953] assertion that "revealed preference theory" is a variant of Cassel's approach, Houthakker declares that the basic weakness in Cassel's approach lies in the fact that Cassel did not impose any restrictions on his demand functions.

hence it was difficult to provide an immediate interpretation as to what the estimates of the parameters really signified. Despite these and other weaknesses the ad hoc approach to demand theory went unchallenged until recently.[2] In order to evaluate the merits of the ad hoc approach it will be desirable to look at the more attractive and theoretically desirable axiomatic approach that is also available to the demand theorist. Indeed, an analysis of the axiomatic approach and its historical development provides some insight into the reasons that led to the distaste for utility that had characterized the founders of the ad hoc tradition.

The axiomatic models of demand have evolved from the classical theory of utility. The roots of this theory can be traced to late nineteenth century European writers. The prominent contributors to this theory were Gossen [1854], Jevons [1871], Walras [1874], Edgeworth [1881], Fisher [1892] and Pareto [1895].[3] Unfortunately, the work

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[2]An example of a recent study in this tradition is the work by Houthakker and Taylor [1966] in which an extensive model of consumer demand is built for the United States economy, and the "adding-up" restriction (see section 1.6) is imposed on an ad hoc basis. Since the number of commodities is large (over eighty), this seems to be the only viable approach.

[3]For a reference to these authors, and a discussion of their works, I am relying primarily on Marshall [1890, Bk. III, Ch. III), Hicks and Allen [1934], N. Georgescu-Roegen [1936], Wold [1943-44], Wold in association with Jureen [1953, pp. 81ff. and Notes), and Schumpeter [1954, pp. 1054-1069].

of these writers was seldom unambiguous and was often marred by a pronounced lack of rigor. Moreover, the choice of the word "utility" and the style in which the theory was stated was responsible for a considerable amount of subsequent confusion both with regard to the dependence of utility theory on psychological laws, and also with respect to its philosophical foundations. It was this confusion of economic theory with psychology and philosophy, perhaps, that became the source of discontent with utility theory that many later economists exhibited.[4]

The situation was not long left unremedied. Antonelli [1886] was the first to give a rigorous statement of the theory that had evolved up to that time. Other

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[4]The initial confusion with regard to psychology arose in the context of the importance that the early theorists attached to the ability that goods possessed to fulfill basic biological needs. This hedonistic aspect of the "utility" concept rendered it unacceptable to many. It was quite late in the history of utility theory that scholars realized that this was not a crucial assumption. This has been pointed out by Samuelson [1947a, p. 91]. Yet another aspect of the same issue was the extended debate that arose over the confusion of Gossen's "law of satiable wants" with the Weber-Fechner "fundamental law of psycho-physics" (identical in formal structure to the Bernoulli-Laplace hypothesis about the marginal utility of income, proposed as a solution to the St. Petersburg Paradox). For a discussion see Viner [1925, p. 371ff.], and Schumpeter [1954, p. 1058]. On the philosophical side the ethical and utilitarian convictions of Gossen, Jevons, Bentham, Sidgwick and Edgeworth were the cause of numerous subsequent misunderstandings. On this see Schumpeter [1954, p. 1056]. The fact that utilitarianism had little to do with utility theory was emphatically pointed out by Marshall [1890, Bk. I, Chap. II, Sec. 1, n. 2; and Bk. I, Chap. II, Sec. 1, n. 1].

writers were not far behind. The early twentieth century saw three other writers' successful efforts at a synthesis of the then current theory. Johnson [1913] gave a clear statement to the Jevons-Menger-Walras tradition of the posthumously so-called "cardinal" theory of utility; while both Pareto [1906] and Slutsky [1915] reviewed the existing "ordinal" tradition and made significant contributions of their own.[5] It was with these papers that two distinct traditions arose within utility theory. The orthodox "cardinal" tradition held on to the assumption that the consumer behaved as if he were maximizing a specific utility function, while the "ordinal" theorists were content with the assumption that indifference curves were well-defined, so that the consumer could be maximizing a class of utility functions that were unique only up to monotonic transformations.

The "ordinal" theory, derived independently by Pareto and Slutsky was extensively reviewed and given essentially its current content by Hicks and Allen [1934]. The primary contribution of the "ordinal" theorists was to demonstrate that most of the results derived by the cardinal theorists were accessible to them without

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[5] The references to Antonelli's work are from the sources cited in footnote 2. For Pareto's work I am relying primarily on the extensive discussion in Wold and Jureen [1953]. For the choice of authors to whom the "cardinal utility" tradition can be attributed I am relying on the verdict of Stigler [1950].

assuming the existence of a utility function. They assumed instead the existence of "indifference curves," and using relatively elementary mathematical techniques derived most of the restrictions on demand functions that were known. In an attempt to consider the ramifications of the two existing approaches, Georgescu-Roegen [1936] took up the problem of "integrability" considered by Antonelli [1886], and discussed by Pareto [1906] and Slutsky [1915], and tried to derive its implications for the theory of demand proposed by Hicks and Allen. Georgescu-Roegen, in the same paper, made one of the earliest systematic investigations of the axiomatic structure of the Hicks-Allen theory, which subsequently was a topic of considerable interest. On another front, Wold [1943-44], in an extensive paper used Volterra's [1906] formulation of the integrability conditions[6] to synthesize the three approaches that he found in economic theory: (1) Pareto's theory of indifference maps, (2) The Hicks-Allen theory of marginal substitution, and (3) Cassel's demand function approach.[7] Although both Wold and Georgescu-Roegen provided useful analyses of the approaches found in utility theory, and

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[6] For a discussion of the meaning and relevance of these mathematical conditions, see Wold and Jureen [1953].

[7] The reference to Volterra [1906] is from Wold [1943-44, Part I, p. 86] and Wold and Jureen [1953, p. 90].

raised important issues, the problem of integrability[8] continued to be a blemish on the Hicks-Allen theory. It remained for Samuelson [1950] to demonstrate that the problem of integrability was more of a mathematician's worry than a valid cause for concern among economists. Samuelson demonstrated that the integrability conditions are violated only when we are willing to attribute more than a fair share of inconsistencies to the behavior of the consumer.

The investigation into the axiomatic structure of utility theory was carried further by Samuelson [1938a] through the "revealed preference" theory, which was claimed to have freed the theory of consumer behavior from the "vestigial" motions of utility. In effect, Samuelson laid down a set of axioms with regard to the preference behavior of the consumer, and showed (much like Hicks and Allen) that all of the results of the Hicks-Allen formulation were the consequences of his axioms. Following in his footsteps numerous writers experimented with an array of modified axiom sets that were capable of yielding the same results. Although the "revealed preference" theory was a source of great insight into utility theory, it was

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[8]In simple terms the problem of integrability deals with the conditions under which the existence of "indifference curves" may be shown to imply the existence of a "utility function."

subsequently revealed that Samuelson's theory was not radically different from the Hicks-Allen theory in logical structure.[9] It remained for Uzawa [1960] to provide a synthesis of demand theory by generalizing previous formulations and demonstrating rigorously that the existence of demand functions satisfying certain regularity conditions was implied by, and in turn implied, the existence of a "preference relation" which possessed some specific properties.

The consequence of these developments for the empirical demand theorist is that he "can adopt as his maintained hypothesis a definite" axiomatic structure which yields substantial restrictions on the demand functions. With an explicit statement of the maintained hypotheses, it is very clear that the estimates of the parameters of the "axiomatic model" are conditional on. Further, the empirical worker can put to test the maintained hypotheses themselves and thereby test if the empirical data confirm his *a priori* beliefs regarding the external world.

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[9] Houthakker [1950] showed that the "revealed preference" theory was logically equivalent to the indifference curve approach, while at the same time generalizing Samuelson's "fundamental hypothesis" so that it would imply integrability, and hence the existence of an ordinal utility function. Subsequently, Arrow [1959] showed that if the commodity space contains all its finite subsets, then the equivalence of Samuelson's "weak axiom" to Houthakker's "strong axiom" is assured. Finally, Uzawa [1960] extended Arrow's general theorem and derived conditions for the demand function under which Samuelson's "weak axiom" is equivalent to Houthakker's "strong axiom."

It was stated that the "ordinal" theory yields _most_ of the results of the "cardinal" theory without resorting to the more stringent axioms of the latter. This is not to suggest, however, that the cardinal theory is without merits. In fact, it is a "quasi-cardinal" approach that is found in the empirical analyses of demand.[10] It is somewhat unfortunate that except for very small systems of demand equations (involving very high levels of aggregation), the "ordinal" theory does not provide enough restrictions to enable the empirical demand theorist to estimate all the cross-elasticities of demand, the own-price elasticities and the income elasticities. In these situations the pragmatic approach usually taken is to impose introspective _a priori_ restrictions equating several (often all) cross-elasticities to zero.

As mentioned above, the cardinal approach is better equipped to handle these problems. The pioneer work is of Klein and Rubin [1947], who proposed a specific functional form for a complete set of demand equations that came to be known as the "linear expenditure system." This system was shown by Samuelson [1948] and Geary [1950] to be implied by a specific utility function, which has come to be called the "Stone-Geary Function." Stone [1954a and

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[10] The rest of the discussion in this section anticipates some of the future discussion, and has been included only to provide a very brief outline of current developments in the literature.

others), employed this function to estimate complete sets of demand equations. It was evident that Stone's system yielded substantial economies of parametrization, and resulted in estimates of all cross-elasticities without the imposition of artificial restrictions.

Although it was well-known that if the utility function could be assumed to be of the Stone-Geary type, substantial economies of parametrization could be obtained, the general case of a quasi-cardinal utility function and its consequence for demand theory were not fully exploited. Frisch [1959] and Houthakker [1960a] were the first to analyze a specific type of utility function that resulted in powerful restrictions on the parameters of the demand system. This was the quasi-cardinal property of "additivity" (or Want-Independence in Frisch's terminology) that was in fact possessed also by the Stone-Geary function. In this sense the Frisch-Houthakker case of "separable" utility was a generalization of the Stone-Geary case. The generalization from "additivity" to "almost additivity" was immediate. Barten [1964] gave rigorous content in terms of the Frishc-Houthakker theory to earlier ideas of Strotz [1957]. Based on these ideas, Houthakker [1960a], Barten [1964] and Theil [1965] proposed specific functional forms for the estimation of complete sets of demand equations.

Finally, the work of Pearce [1961], [1964] merits attention. In the tradition of the ordinal theorists

Pearce has made a definite attempt to render the strictly ordinal theory as empirically fruitful as the quasi-cardinal variety. The motivation for such an effort may be derived from a critical look at the restrictive assumptions that cardinal theorists must make. In particular, Pearce objected to the additivity assumption on grounds that it was not invariant with respect to monotonic transformations of the utility function.[11] He proposed, instead, a new form of separability, "neutral want association," which was shown to be invariant under monotonic transformations. From this assumption Pearce claims to have shown that substantial restrictions on the demand functions can be obtained in a manner very similar to the cardinal theorists' models. In fact, Pearce [1964, p. 206] has claimed that all the computational advantages of additivity are available even if we make the much weaker assumption of neutral want association. In the review of empirical models below an analysis of Pearce's claim is made, and there is reason to believe that the case for neutral want association may have been overstated.

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[11] Subsequent discussion reveals that this may not have been a valid criticism of the quasi-cardinal approach. This is because the assumption of "strong separability," which is invariant with respect to an arbitrary non-linear transformation of the utility function, results in much the same restrictions on the demand functions as does the assumption of additivity. For further discussion see Uzawa [1964] and Chapter 2 below.

To summarize, then, the empirical demand theorist has the option of building an "*ad hoc* model" or an "axiomatic" model. It has been pointed out that the consensus is that the former approach is less desirable. Of the latter, there exist two distinct traditions. The "quasi-cardinal" tradition results in the most powerful restrictions on the set of demand equations, although it does so at the cost of restrictive assumptions. The strictly "ordinal" approach is based on more palatable assumptions but does not afford the computational conveniences of the latter.[12]

1.3 The Axiomatic Foundations

As we have noted in the previous section, the foundations of utility theory have only recently been investigated in a satisfactory manner. Prior to this investigation into the aximatic structure of utility, it had been the custom to accept the existence of a well-behaved utility function strictly on faith. Assuming that the consumer maximized his "utility function" which presumably was well-defined and had certain desirable properties, empirical workers conceeded to estimate elasticities

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[12]The advantages of the cardinal models may have been overstated in this section. In subsequent sections it will be pointed out that the three acceptable cardinal models: Stone's Linear Expenditure System, Houthakker's Indirect Addilog Model, and the Rotterdam Model are all non-linear in parameters, so that estimation with restrictions becomes non-trivial.

from available data on expenditures, prices and income. Critics of the approach were at a loss to comprehend the exact implications of such assumptions, and debate over their validity was often confused and universally inconclusive. A need was felt, therefore, to decompose these assumptions into several primitive components in order both to promote understanding of the behavioral implications of this axiom and to aid in a direct statistical verification, should it seem desirable.

The effort is all but concluded with Uzawa's [1960] justly acclaimed synthesis of the investigation into the axiomatic structure of utility theory that began with Georgescu-Roegen [1936], and Wold [1943-44].[13] The results have been reassuring on at least two counts. First, it has been shown that a "well-behaved"[14] utility function does in fact exist under reasonable assumptions with regard to the properties and existence of consumer preferences. Secondly, it has come to emerge that many of the basic results of demand theory can be obtained by using topological methods, directly from the properties of the preference relation, without any mention whatsoever of a

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[13]Although Georgescu-Roegen [1936] did concern himself with the axiomatic bases of utility theory, it seems that Wold [1943-44] was the first writer to be concerned with the conditions under which a real-valued, differentiable, order-preserving (utility) function can be shown to exist.

[14]A definition of "well-behaved" will be given below.

utility function.[15] The theorist thus has his option of using classical calculus methods, or of defining a preference relation which satisfied certain additional conditions, and then use topological methods to derive the same restrictions on the demand equations. Since the economic content of the two approaches is practically the same, the classical calculus method is adopted in this paper. To do so, however, a brief discussion of a set of axioms that imply the existence of a "well-behaved" utility function is given below.[16]

Consider a single consumer who is confronted with a choice between a finite number of commodities which are labelled $i=1,\dots,n$. The quantity consumed of the ith commodity by the consumer is denoted by x_i. A "commodity bundle" will be defined as a n-dimensional vector, with x_j as its jth component, and will be denoted by a non-subscripted lower case English alphabet, for example x or y.

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[15] See, for example, the literature on revealed preference theory: Samuelson [1938a], [1938b], [1947a], Houthakker [1950], etc. In a more general context, these results are derived by Yokoyama [1953]. It might be noted that the term "topological methods" may be a misnomer. The term is used to describe the methods of higher mathematical analysis involving Real Analysis, Topology and Abstract Algebra, as opposed to "calculus methods" by which is meant the elementary calculus approach that is found for example in R. G. D. Allen [1935], and [1956].

[16] The remainder of this section relies heavily on Uzawa [1960], and the extensive discussion of Uzawa's paper, and other formulations by Pearce [1964].

The set of all commodity bundles, C, will be called the "Commodity Space" and will be assumed to be the non-negative orthant of an n-dimensional Euclidean space.

It is assumed that there exists a dichotomous "binary relation" P, defined on C, which is called the consumer's "preference ordering" if it satisfies the following additional requirements:

<u>Property 1</u>: Irreflexivity.

For any x in C, $x\bar{P}x$. (where, $\bar{P}$ denotes the negation of P.)

<u>Property 2</u>: Transitivity.

For all x,y,z in C, If xPy and yPz, Then xPz.

<u>Property 3</u>: Monotonicity.

For all x,y in C, If $x>y$, Then xPy

(Definition: $x>y$ if $x_i \geq y_i$ for all i, and for some j, $x_j > y_j$.)

<u>Property 4</u>: Convexity.

For all x,y in C, If $x \neq y$ and $x\bar{P}y$,

then $(1-\lambda)x+\lambda y$ P x for all $0<\lambda<1$.

are open in C.[17]

An economic interpretation of this assumption may be provided as follows. The relation P can be interpreted as "preference" in the following sense of the word: "xPy"

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[17]For a definition of an "open set" see Rudin [1964, p. 28]. The concept of a "binary relation" is defined in Birkhoff and McLane [1965, p. 29f.]. if, and only if, for all x, y, $x \neq y$, either xPy or yPx.

may be read as "the commodity bundle x is at least as good as the commodity bundle y" in the eyes of a hypothetical consumer. The mathematical assumption of the existence of this binary relation P over the commodity space C is tantamount to assuming that the consumer is able to make pairwise comparisons of commodity bundles and come to the conclusion whether one or the other is preferred, or whether he finds himself indifferent to the two bundles. The assumption of dichotomy implied that the consumer can do this for all possible commodity bundles.

Of the properties that the preference relation is assumed to possess, the first defines the meaning of the relation and states that no commodity is preferred to itself. The transitivity property is of a more substantive nature insofar as it expresses a belief about a certain regularity that the consumer's preference is assumed to have. It states that if the consumer prefers x to y, and y to z, then the consumer must prefer x to z.[18] The assumption of monotonicity states that if a commodity bundle x possesses at least as much of every good as another bundle y, and has more of at least one good, then the bundle x must be preferred to the bundle y. The

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[18] It is this axiom of transitivity that results in the symmetry of the Slutsky substitution term and provides the single most useful restriction on the complete set of demand equation. This is pointed out by Pearce [1964, p. 52], for example.

consequence of this assumption is to rule out the existence of a satiation point.[19] The property of convexity is roughly equivalent to the assumption that if a bundle x is not preferred to a bundle y, then all bundlex z that lie on the straight line connecting x and y and are distinct from y, must not be preferred to the bundle y. The consequence of this property is that on any budget hypersurface there exists a unique point preferred to all others. The property of continuity is the least easily interpretable. Formally it states that both the set of commodity bundles that are preferred to a specific bundle x^o, and the set of bundles over which x^o is preferred, are open in C. In mathematical terms, this implies that there exist limit points for the two set which are not contained in either of the sets. Thus, the consumer is indifferent to a number of points within an E-neighborhood of each point in the commodity space. Roughly speaking, this insures that two commodity bundles that are close together in a spatial sense, must be ranked "close together" in the sense of preference.[20]

[19] Houthakker [1961, p. 713] considers this to be a limitation. For empirical work, however, there would be little doubt that the assumption of non-satiation is not far from reality.

[20] A good discussion of the plausibility of this assumption is found in Pearce [1964, pp. 22ff.]. It is difficult, in general, to find preference orderings that are non-continuous. There is, however, a famous example of the "lexicographic ordering" due to Debreu [1954] that is not continuous and for which a utility function can not

Given the existence of a preference ordering satisfying properties listed above, it has been shown that a utility function can be constructed in the usual sense. More correctly, a class of real-valued, continuous, order-preserving (utility) functions, unique up to monotonic transformations can be shown to exist if there exists an ordering of the type described above. The formal result as stated by Uzawa [1960, p. 135], based on the theorem by Debreu [1954, p. 162, Theorem I] is as follows:

> Theorem: If a dichotomous binary relation, P, defined on an n-dimensional Euclidean (commodity) space, is irreflexive, transitive, monotonic, convex, and continuous, then there exists a real-valued, and continuous function u(x) defined on the Commodity space, such that for any x,y in C:
>
> $$xPy \text{ if and only if } u(x)>u(y).$$

The proof of the theorem is omitted due to the fact that no additional insight of economic relevance may be had from it.

So far, the discussion has been of a general mathematical nature, with no explicit behavioral content. For the empirical analysis of consumer's demand, however, certain behavioral assumptions are needed. Before specifying these, however, we define the concept of an "attainable set of commodities":

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be constructed. Pearce [1964] has discussed why the "lexicographic ordering" does not seem to be of much empirical relevance.

Definition: The "attainable set of commodities" is defined to be a proper subset of the commodity space, C, and is given by $\{x: p'x = y, x \text{ in } C\}$ where p is the (nxl) vector of prices that the consumer must pay, and y is the predetermined amount of total expenditure that the consumer can make, y being a scalar.

The formal model of utility theory, then, rests on the following axioms:

Axiom 1: (Existence of a preference ordering). It is assumed that the consumer possesses a dichotomous irreflexive, transitive, montonic, convex, and continuous "preference ordering" defined over the entire commodity space, C.

Axiom 2: (Axiom of Choice). It is assumed that the consumer chooses that commodity bundle x, which is preferred to all other commodity bundles in the "attainable set" of commodity bundles.

Although these two axioms are sufficient for an analysis of consumer's demand, and the principal results may all be obtained by using methods involving finite differences, it is usually assumed also that:

Assumption: (Differentiability). The set of order-preserving utility functions that result from the preference ordering assumed to exist from Axiom 1, are all at least thrice differentiable.

The need for this assumption arises from the fact that Debreu's Theorem guarantees only the continuity of the utility functions, and makes no claims about differentiability. Having assumed differentiability, classical calculus techniques may be utilized in the analysis of the consumer's problem, and the derivation of demand

functions and restrictions upon them become straightforward.

On the basis of Axioms 1 and 2, and the assumption of differentiability, the consumer's problem may be formulated as:

> Problem: Given a set of prices $\{p_1, p_2, \ldots Pn\}$, and a fixed level of total expenditure or "income" y, the consumer wishes to any utility function that preserves the preference ordering given by Axiom 1; subject to the "budget constraint" $p'x=y$.

Once it is recognized that by virtue of the definition of a "monotonic" transformation, the choice of the specific utility function to be maximized is arbitrary, then the above statement of the problem may easily be seen to be identical in formal content to the classical statement of the consumer's problem. The advantage of the above approach lies, however, in identifying the set of primitive components of this assumption, and hence in increasing the insight into the behavioral implications of the utility assumption.

Before proceeding to an analysis of the problem described above, and the derivation of the classical results of demand theory, we shall digress to discuss very briefly some of the saliant issues in the theory of "revealed preference." In conclusion, we may mention here that two topics of current interest have been completely neglected in our discussion so far. These are the topics of dynamic utility maximization and of stochastic utility

and uncertainty. Our excuse for this omission is that despite their great significance these theories are yet in an embryonic state, and have, therefore, not been a source of many additional empirical tests.

1.4 Revealed Preference: A Digression

No discussion of axiomatic utility theory would be complete without a mention of the revealed preference approach, due to Samuelson [1938a, etc.], which historically preceded the simpler and equally general approach outlined in the preceding section. In fact, the general preference relation approach gained much from the discussions of revealed preference theory. In this section, therefore, we shall briefly survey the principle ideas underlying revealed preference theory, the related debate over transitivity, and the reasons why revealed preference theory may be considered to be a special case of the general theory outline in section 2.

In the tradition of Gustav Cassel, and Henry L. Moore, Paul A. Samuelson attempted to construct a theory of consumer demand independent of the idea of a "utility" function. The resultant revealed preference approach started out with the assumption that given a set of market prices and income, the consumer selects a unique commodity bundle. Thus, a single-valued demand function,

$$x_i = h^i(p_1, \ldots, p_n, y) \qquad (i=1, \ldots, n)$$

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In the tradition of Gustav Cassel, and Henry L. Moore, Paul A. Samuelson attempted to construct a theory of consumer demand independent of the idea of a "utility" function. The resultant revealed preference approach started out with the assumption that given a set of market prices and income, the consumer selects a unique commodity bundle. Thus, a single-valued demand function,

$$x_i = h^i(p_1,\ldots,p_n,y) \qquad (i=1,\ldots,n)$$

was assumed to exist. Also, by nature of definition, the budget constraint,

$$\Sigma_i p_i x_i = y$$

was assumed to hold. It becomes easy to show that from these conditions alone, we may demonstrate the validity of many of the principle results of demand theory to be derived in the next section. However, the two most important results of demand theory: the symmetry and negative definiteness of the Slutsky matrix, are not deduceable from this assumption alone.

To get at these crucial properties of the Slutsky matrix, an attempt was made to work backwards in order to infer the existence of some sort of a quasi-utility function from the axioms of revealed preference.[21] Pursuing this approach, it was found that the symmetry of the Slutsky matrix could not be proved from the assumption given above, alone. With regard to the negative definiteness, however, the revealed preference approach led to some degree of success. To show that the Slutsky matrix was negative definite the revealed preference theorists introduced the so-called "weak axiom" of revealed preference:

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[21]This interpretation is provided by Pearce [1964, p. 67]. This section has greatly benefited from the discussion of revealed preference theory in Pearch [1964, pp. 65-72].

<u>The Weak Axiom</u>: If a specific commodity bundle x^o is purchased at a given set of prices and income, when another commodity bundle x^1 could also have been purchased; then the commodity bundle x^1 will never be purchased when x^o may also be purchased at the prevailing income and prices.

On the basis of this axiom, we may define a revealed preference relation, R, which is antisymmetric, and is given by:

$$x^o R x^1 \text{ if and only if } \Sigma_i \, p_i^o (x_i^o - x_i^1) \geq 0, \; x^o \neq x^1.$$

Assuming only antisymmetry of R, it can be shown that the matrix of Slutsky terms is negative definite. However, the demonstration of symmetry of the Slutsky matrix requires assumptions with regard to the transitivity of R.

Historically, the realization that it was necessary to assume the transitivity of the revealed preference relation, R, came after great difficulty. The first step was provided by Houthakker's [1950] "strong axiom" of revealed preference, which established the relation R*, where $x^o R^* x^s$ if and only if (s-1) commodity bundles can be found such that $x^o R x^1$, $x^1 R x^2, \ldots,$ $x^{s-1} R x^s$. The "strong axiom" required that R* be antisymmetric. The motivation behind the introduction of the "strong axiom" are quite simple. The relation R* is transitive by definition. Combining this with the axiomatized antisymmetry of R*, Houthakker [1950] was able to show that a utility function can, in fact, be demonstrated by appealing to Debreu's Theorem, mentioned in the previous section, and noting

that R* satisfies all of the requirements of a "preference ordering" outlined in section 2. In fact, Uzawa [1960] showed that both the revealed preference relations R, and R* implied transitivity if the choice (demand) functions satisfied certain regularity conditions. With this demonstration, it became clear that the relations R, and R* are in fact equivalent to the relation P introduced in section 2.

To summarize, then, the revealed preference theorists pioneered in an attempt to found a theory of consumer demand upon certain simple axioms derived from the observation of market behavior. It was found, however, that both the weak and the strong hypotheses depended upon the additional assumption of transitivity. Thus, the general "preference relation" approach of section 2, is seen to be equivalent to the revealed preference relations approach. Indeed, the general approach gains considerably in simplicity, without any sacrifice in generality or rigor.

1.5 Theory of Consumer's Demand[22]

In the previous section it has been shown that the problem of the consumer is to

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[22] This section relies heavily on Goldberger [1967].

Maximize: $u = u(x_1, x_2, \ldots x_n)$

Subject to: $\Sigma_i p_i x_i = y$

where $u = u(x_1, \ldots x_u)$ is any utility function which preserves a "preference ordering," x_i is the quantity consumed of the ith commodity, with price, p_i, and y is the predetermined value of total expenditure. In this version, the statement of the consumer's problem corresponds to the "ordinal" model of utility theory.

The analysis of demand is usually conducted, however, in the very closely related "cardinal" model, which may be given as follows:

Maximize: $u = u(x_1, x_2, \ldots, x_u)$

Subject to: $\Sigma_i p_i x_i = y$

where the consumer is assumed to possess a <u>specific</u> utility function u(x), which it is assumed that he maximizes subject to the budget constraint. It is immediately obvious that the latter formulation is far more restrictive than the former. In fact, the validity of the second formulation as a relevant description of the process by which the empirical data under analysis is generated, is highly questionable. Fortunately, the methods of analyzing both formulations are quite parallel. In the following discussion, however, those results that are valid only under the "cardinal" model shall be specifically pointed out. Needless to add, any results that are

true under the "ordinal" model are also true for the "cardinal" case, as the latter is a special case of the former.

Recall that x_i denotes the quantity consumed of the ith good (i=1,...,n); p_i the price of the ith good (i=1,...,n); and y denotes the predetermined amount of total expenditure, or income. It is convenient to use a vector notation sometimes, and so the (nx1) vectors x and p are defined as:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \text{ and } \quad p = \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix}$$

Given prices p, and income y, the consumer is assumed to select that commodity bundle x, which maximizes the value of an arbitrary (order-preserving) utility function

$$u = u(x_1, x_2, \ldots, x_n) = u(x)$$

subject to his budget constraint:

$$\sum_{i=1}^{n} p_i x_i = y$$

or,

$$p'x = y, \text{ in vector notation.}$$

By virtue of the properties that consumer preferences are assumed to have in Axiom 1, and the assumption of differentiability, the utility function $u=u(x)$ is real-valued, continuous, and thrice differentiable. To develop further notation, $u_i=u_i(x)$ will denote the first partial derivative of $u=u(x)$ with respect to x_i; and $u_{ij}=u_{ij}(x)$ will denote the second partial derivative of $u=u(x)$ with respect to x_i and x_j. Note that the assumption of thrice differentiability implies that $u_{ij}=u_{ji}$, which shall always be assumed to be the case.[23] The parallel matrix notation for these derivatives will be as follows:[24]

$$u_x = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}, \qquad U = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ u_{n1} & \cdots & u_{nn} \end{bmatrix}$$

---

[23]Alternatively, the existence of derivatives to the second order could have been assumed, with the additional assumption that these second order derivatives were continuous. This slightly weaker assumption would, in fact, be sufficient to insure the equality of the cross partials. For a proof of this theorem in advanced calculus, see Cronin-Scanlon [1967, p. 90, Theorem 7]. The stronger assumption is made for simplicity, though the economic consequences of either assumption are practically identical.

[24]The matrix development of utility theory, which has proved of great convenience is due to Barten [1964], [1966], and Theil [1965], and [1967].

Two additional properties of the utility function that are a result of the properties that consumer preferences are assumed to have may be noted here. The monotonicity of preferences implies that all first partials of the utility function u_j, $(j=1,\ldots,n)$ are strictly positive. Also, convexity of preferences imply that the matrix of second order partial derivatives U is negative definite everywhere.[25] This implies, incidentally, that u_{jj} is strictly negative for all $j=1,\ldots,n$; so that each good has diminishing marginal utility.

Using the conventional Lagrange multiplier method for the solution of the constrained maximum problem, the following function is constructed:

$$L(x,\lambda) = u(x) - \lambda(p'x - y)$$

where λ is the Lagrangian multiplier. Maximizing L yields the solution to the constrained maximum problem. Differentiating with respect to x and λ, and setting the derivatives equal to zero, we get the "first order conditions"

$$\text{(FOC)} \qquad \begin{cases} u_x = \lambda p \\ p'x = y \end{cases}$$

---

[25]For a proof of the latter property, which, incidentally, is sufficient to assure a unique solution to the constrained maximization problem, see Lancaster [1968, p. 333].

Or, in the more familiar non-matrix formulation, these are the (n+1) equations:

$$u_i = \lambda p_i \qquad (i=1,2,\ldots,n)$$

(FOC)

$$\Sigma p_i x_i = y \qquad \text{(where the sum is from } i=1, \text{ to } n).$$

These first order conditions are the foundations upon which a vast edifice of demand theory has been built. Borrowing the terminology of macroeconomic models, the set of (n+1) equations (FOC) may be construed as the set of "structural equations" of demand theory, while the set of demand equations and the equation expressing λ as a function of y and p, may be compared to its "reduced form."

To solve for this "reduced form" we first note that the negative difiniteness of the Hessian, U, further guarantees that the set of equations (FOC) may be solved to yield x and λ, as single-valued functions of p and y. These functions are denoted as

$$x = h(p,y), \quad \text{and} \quad \lambda = \lambda(p,y)$$

Or, in customary algebraic notation:

$$x_j = h^j(p_1,\ldots,p_n,y) \qquad (j=1,\ldots,n)$$

$$\lambda = \lambda(p_1,\ldots,p_n,y)$$

The first n equations expressing x_j as a function of all prices and income are referred to as the "<u>complete set of</u>

demand equations." It is readily verified that the demand functions, $h^j (j=1,...,n)$ are invariant under an arbitrary non-linear monotonic transformation of the utility function. Hence, the partial derivatives $\frac{\partial xi}{\partial p_j}$, and $\frac{\partial xi}{\partial y}$ are also "monotonic invariant." These are the equations that are estimated by the empirical demand theorist from an analysis of expenditure and price data.

Using the Barten-Theil notation, a matrix solution of the complete set of demand equations is possible. The matrix solution can then be easily interpreted in terms of the classical development in which Cramer's Rule was the workhorse for deriving various properties of the demand functions. To do so, some additional notation is required. Since the set of demand functions are also differentiable as a consequence of the differentiability of the utility function, we define the (nxl) income-slope vector x_y, and the (nxn) price-slope matrix X_p, as follows:

$$x_y = \begin{bmatrix} \frac{\partial x_1}{\partial y} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial x_n}{\partial y} \end{bmatrix}, \text{ and } \quad X_p = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \cdot\ \cdot\ \cdot & \frac{\partial x_1}{\partial p_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial x_n}{\partial p_1} & \cdot\ \cdot\ \cdot & \frac{\partial x_n}{\partial p_n} \end{bmatrix}$$

Similarly, the λ function is also differentiable, and the scalar λ_y, and the (nxl) price-slope vector λ_p, are defined as follows:

$$\lambda_y = \frac{\partial \lambda}{\partial y} \quad , \quad \text{and} \qquad \lambda_p = \begin{bmatrix} \frac{\partial \lambda}{\partial p_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial \lambda}{\partial p_n} \end{bmatrix}$$

With this notation we may write down what Barten [1966] has called "the fundamental equation of the theory of consumer demand in terms of partial derivatives" as,

$$(1) \ldots \quad \begin{bmatrix} U & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} x_y & X_p \\ -\lambda_y & -\lambda_p \end{bmatrix} = \begin{bmatrix} \emptyset & \lambda I_n \\ 1 & -x' \end{bmatrix}$$

Perhaps the best way to see the validity of equation (1) is to consider the sets of equations that result from a partial differentiation of the first order conditions, (FOC), with respect to income, y, and prices, p. Differentiating the set of equations (FOC) with respect to y, we get:

$$\begin{array}{ccccccccc} u_{11} \frac{\partial x_1}{\partial y} & + & u_{12} \frac{\partial x_2}{\partial y} & + & \cdots & + & u_{1n} \frac{\partial x_n}{\partial y} & - \ p_1 \frac{\partial \lambda}{\partial y} & = 0 \\ \vdots & & \vdots & & & & \vdots & \vdots & \vdots \\ u_{n1} \frac{\partial x_1}{\partial y} & + & u_{n2} \frac{\partial x_2}{\partial y} & + & \cdots & + & u_{nn} \frac{\partial x_n}{\partial y} & - \ p_n \frac{\partial \lambda}{\partial y} & = 0 \\ p_1 \frac{\partial x_1}{\partial y} & + & p_2 \frac{\partial x_2}{\partial y} & + & \cdots & + & p_n \frac{\partial x_n}{\partial y} & & = 1 \end{array}$$

This set of equations is represented by the (n+1) equations that result from equating the elements in the first column to the product of the two matrices on the left hand side of equation (1) to the corresponding elements on the right. In a similar way, the system of (n+1) equations resulting from equating the right hand side and the left hand side elements in the (j+1)th column of equation (1), are seen to be:

$$\begin{array}{ccccccccccc}
u_{11}\dfrac{\partial x_1}{\partial p_j} & + & u_{12}\dfrac{\partial x_2}{\partial p_j} & + & \cdots & + & u_{1n}\dfrac{\partial x_n}{\partial p_j} & - & p_1\dfrac{\partial \lambda}{\partial p_j} & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
u_{j1}\dfrac{\partial x_1}{\partial p_j} & + & u_{j2}\dfrac{\partial x_2}{\partial p_j} & + & \cdots & + & u_{jn}\dfrac{\partial x_n}{\partial p_j} & - & p_j\dfrac{\partial \lambda}{\partial p_j} & = & \lambda \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
u_{n1}\dfrac{\partial x_1}{\partial p_j} & + & u_{n2}\dfrac{\partial x_2}{\partial p_j} & + & \cdots & + & u_{nn}\dfrac{\partial x_n}{\partial p_j} & - & p_n\dfrac{\partial \lambda}{\partial p_j} & = & 0 \\
p_1\dfrac{\partial x_1}{\partial p_j} & + & p_2\dfrac{\partial x_2}{\partial p_j} & + & \cdots & + & p_n\dfrac{\partial x_n}{\partial p_j} & & & = & -x_j
\end{array}$$

The $(n+1)^2$ equations that constitute the matrix equation (1), can be solved for the $(n+1)^2$ partial derivatives. The easiest procedure to adopt is to premultiply both sides of the equation by the inverse of the first matrix on the left hand side of equation (1),[26]

[26] The computation is straightforward. Baretn [1964] first adopted this procedure, and gave the formulae for the inverse. Hadley [1961, pp. 107-109] gives the relevant formulaes for the general case of inverting a partitioned matrix of the form above.

$$\begin{bmatrix} U & p \\ p' & 0 \end{bmatrix}^{-1} = (p'U^{-1}p)^{-1} \begin{bmatrix} (p'U^{-1}p)U^{-1} - (U^{-1}p)(U^{-1}p)' & U^{-1}p \\ (U^{-1}p)' & -1 \end{bmatrix}$$

Carrying out the multiplication, we obtain the solution for the slopes,

$$(2) \begin{bmatrix} x_y & X_p \\ -\lambda_y & -\lambda_p \end{bmatrix} =$$

$$(p'U^{-1}p)^{-1} \begin{bmatrix} U^{-1}p & (p'U^{-1}p)\,\lambda\, U^{-1} - \lambda(U^{-1}p)(U^{-1}p)' - U^{-1}px' \\ -1 & \lambda(U^{-1}p)' + x' \end{bmatrix}$$

Reading off the blocks of equation (2), and writing in simple algebraic form we have

$$(3) \quad . \; . \; . \qquad \lambda_y = (\sum_{ij}\sum u^{ij}p_ip_j)^{-1}$$

where u^{ij} denotes the (i,j)th element of the inverse of the Hessian matrix, U.

$$(4) \quad . \; . \; . \qquad \frac{\partial x_i}{\partial y} = \lambda_y(\Sigma_j\, u^{ij}p_j) \qquad \text{for all } i=1,\ldots,n.$$

$$(5) \quad . \; . \; . \qquad \frac{\partial x_i}{\partial p_j} = \lambda u^{ij} - \lambda\; \lambda_y(\Sigma_s\, u^{is}p_s)(\Sigma_t u^{jt}p_t) -$$

$$\lambda_y(\Sigma_s\, u^{is}p_s)x_j$$

for $i,j=1,2,\ldots,n$.

$$(6) \quad . \; . \; . \qquad \frac{\partial \lambda}{\partial p_i} = -\lambda_y(\lambda\Sigma_s\, u^{is}p_s \; + \; x_i) \qquad (i=1,\ldots,n)$$

The equations (3)-(6) give the slopes of all the demand functions and the λ function, in terms of prices and income. All of the properties that demand functions possess may be obtained from these equations. Before doing this, however, we shall write these equations in a form that is more usual. This is done by substituting equation (4) into (5) and (6). The revised version of these equations are:

(7) . . . $$\lambda_y = (\Sigma_i \Sigma_j\ u^{ij} p_i p_j)^{-1}$$

(8) . . . $$\frac{\partial x_i}{\partial y} = \lambda_y\ (\Sigma_j\ u^{ij} p_j) \qquad (i=1,\ldots,n)$$

(9) . . . $$\frac{\partial x_i}{\partial p_j} = \lambda u^{ij} - \frac{\lambda}{\lambda_y} \left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right) - x_j \left(\frac{\partial x_i}{\partial y}\right) \qquad (i,j=1,\ldots,n)$$

(10) . . . $$\frac{\partial \lambda}{\partial p_j} = -\left[\lambda \left(\frac{\partial x_j}{\partial y}\right) + x_j \left(\frac{\partial \lambda}{\partial y}\right)\right] \qquad (j=1,\ldots,n)$$

We might note that equation (10) is called "Schultz's Relation," after Schultz [1938].

These equations have been used extensively by empirical demand theorists; and specially in the case of "additivy" (discussed below) they have proved to be a source of considerable simplification in computational procedures. In the next section we shall discuss the implications of these equations for demand theory. Before concluding this section, though, it ought to be pointed

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out that equations (8) and (9) are a modified version of the Slutsky equation, or in the terminology of Hicks and Allen [1934], "the fundamental equation of value theory." This can be seen by denoting by K_{ij}, the so-called "Slutsky term"

$$(11) \quad . \; . \; . \quad K_{ij} = \lambda u^{ij} - \frac{\lambda}{\lambda_y} \left(\frac{\partial x_i}{\partial y}\right) \left(\frac{\partial x_j}{\partial y}\right)$$

With this notation, equation (9) may be written as Hicks and Allen wrote it,

$$(12) \quad . \; . \; . \quad \frac{\partial x_i}{\partial p_j} = K_{ij} - x_j \frac{\partial x_i}{\partial y}$$

where K_{ij}, is called the (Hicks-Allen) "substitution effect" and the second term on the right is called the "income effect" of a change in price on the quantity demanded. From equation (12), and from the fact that the demand functions (and their partial derivatives) are "monotonic invariant" it is readily seen that the Slutsky term, K_{ij}, is also invariant with respect to an arbitrary non-linear monotonic transformation of the utility function.

Hicks and Allen attached great importance to equation (9) because they were able to show that K_{ij} represented the change in the quantity purchased by the consumer of the ith good, due to a change in the price of the jth, if the consumer's income was changed so as to compensate for the price change in the sense of keeping utility unchanged. Thus, the consumer's response to a

price change could be decomposed into a (Hicks-Allen) "substitution effect" K_{ij}, and an "income effect" given by the term $x_j\left(\frac{\partial x_j}{\partial y}\right)$

Barten [1964], following Frisch [1959] and Houthakker [1960], has further decomposed the Hicks-Allen "substitution effect" into a "specific substitution effect" and a "general substitution effect." This has been done in response to the realization that there exists a difference between a "specific" substitution of one good for another in terms of the ability of the two goods to fulfill the same needs, and the "general" variety of substitution of one good for another due to the change in real income that is accompanied by a price change. These ideas can be expressed more rigorously by an investigation of the "indirect" utility function.

The "indirect utility function" is obtained from the "direct utility function" $u=u(x)$, by substituting the demand functions $x=x(p,y)$ into the latter, to yield the value of utility that the consumer derives from selecting the optimal good bundle under prices p, and total expenditure y. The indirect utility function is denoted:

$$u = u^*(p,y) = u(x(p,y))$$

Differentiating with respect to y and using the chain rule for differentiation, we have

$$\frac{\partial u^*}{\partial y} \equiv u^*_y = \Sigma_i\ u_i\ \frac{\partial x_i}{\partial y}\ , \quad \text{and} \quad \frac{\partial u^*}{\partial p_j} \equiv u^*_j = \Sigma_i\ u_i\ \frac{\partial x_i}{\partial p_j}$$

where we denote the first partial of u with respect to x_i by u_i, in the usual manner. Substituting for u_i from the first order conditions (FOC) we have

$$u^*_y = \Sigma_i(\lambda p_i)\ \frac{\partial x_i}{\partial y}\ , \quad \text{and} \quad u^*_j = \Sigma_i(\lambda p_i)\ \frac{\partial x_i}{\partial p_j}$$

We now use two results which may be obtained by suitably differentiating the budget constraint in the first order conditions (FOC) and which are formally derived in the next section,

$$\Sigma_i\ p_i\ \frac{\partial x_i}{\partial y} = 1 \qquad \ldots \text{ (Engel aggregation)}$$

and, $$\Sigma_i\ p_i\ \frac{\partial x_i}{\partial p_j} = -x_j \qquad \ldots \text{ (Cournot aggregation)}$$

Substituting these in the equation above, we have

$$(13) \ \ldots \quad u^*_y = \lambda\ , \quad \text{and} \quad u^*_j = -\lambda x_j \qquad (j=1,\ldots,n)$$

It is clear, then, that the Lagrangean multiplier, λ, is the "marginal utility of income" so that the function $\lambda = \lambda(p,y)$ gives the marginal utility of income as a function of prices and income.

These results may now be utilized to analyze an infinitesimal change in utility. Denoting the total differential of the utility function by du, we have

$$du = u^*_y \, dy + \Sigma_j \, u^*_j \, dp_j$$

$$= \lambda \, dy - \lambda \Sigma_j \, x_j dp_j \qquad \text{using (13).}$$

Similarly, the total differential of $\lambda = \lambda(p,y)$, d , is

$$d\lambda = \lambda_y \, dy + \Sigma_j \frac{\partial \lambda}{\partial p_j} dp_j$$

Using Schultze's relation equation (10), we may write this as

$$d\lambda = \lambda_y \, dy - \Sigma_j \left[\lambda \frac{\partial x_j}{\partial y} + x_j \frac{\partial \lambda}{\partial y}\right] dp_j$$

We may now derive two kinds of compensating variations in income. Setting du=0, we get the Hicks-Allen variety of a change in income that is required to offset a change dp in prices, in the sense of leaving <u>utility</u> unchanged,

$$(dy)^* \quad = \quad \Sigma_j \, x_j dp_j$$

On the other hand, in the analysis of Frisch-Houthakker decomposition[27] of the response to price change, we are concerned with that change in income which will compensate the consumer for a price change dp, in the sense of leaving his <u>marginal utility of income</u> unchanged. This is given by equating $d\lambda = 0$.

---

[27]The term is Goldberger's [1967].

$$(dy)^{**} = \Sigma_j \left[\frac{\lambda}{\lambda_y}\left(\frac{\partial x_j}{\partial y}\right) + x_j\right] dp_j$$

$$= \Sigma_j \left[\phi y \left(\frac{\partial x_j}{\partial y}\right) + x_j\right] dp_j$$

where ϕ denotes the inverse of the "income elasticity of the marginal utility of income" denoted E_y, and given by[28]

$$\phi^{-1} \equiv E_\lambda = \left(\frac{\partial \lambda}{\partial y}\right) \frac{y}{\lambda}$$

Consider now the total differential dx_i of the demand function for the ith good,[29]

$$dx_i = \left(\frac{\partial x_i}{\partial y}\right) dy + \Sigma_j \left(\frac{\partial x_i}{\partial p_j}\right) dp_j$$

Substituting from equation (9), we have

$$dx_i = \left(\frac{\partial x_i}{\partial y}\right) dy + \Sigma_j \left[\lambda u^{ij} - \frac{\lambda}{\lambda_y}\left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right) - x_j \left(\frac{\partial x_i}{\partial y}\right)\right] dp_j$$

---

[28] This is the terminology of Barten and Theil, Our is closely related to what Frisch [1932], [1959] called the "money flexibility" which in our terminology would be equal to E . Houthakker [1960] called ϕy the "income flexibility."

[29] The argument could have been carried generally for all goods by considering the (nxl) vector of differentials of the demand functions, dx. For clarity, however, we have chosen a single good. The extension to matrices is obvious and can be found in Goldberger [1967, p. 17].

With this formulation of the differential of the ith demand function and the two concepts of compensation, we may easily analyze the Hicks-Allen and the Frisch-Houthakker decomposition of the response to an infinitesimal price change of the jth commodity dp_j.

In the Hicks-Allen case, consider a change in the demand for the ith good, $(dx_i)^*$ in response to the change in the price of the jth good dp_j, when the consumer is compensated in the utility sense, so that $dy=(dy)^*$. We have then

$$(dx_i)^* = \left(\frac{\partial x_i}{\partial y}\right) dy + \Sigma_j \left(\frac{\partial x_i}{\partial p_j}\right) dp_j$$

and,

$$(dy)^* = \Sigma_j \, x_j dp_j .$$

So that $(dx_i)^* = \Sigma_j \left(\frac{\partial x_i}{\partial p_j}\right)^* dp_j$,

where $\left(\frac{\partial x_i}{\partial p_j}\right)^* = \lambda u^{ij} - \phi y \left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right) = K_{ij}$

and is the Slutsky term introduced before. Thus, K_{ij} measures the response of the quantity demanded of the ith good to the change in the jth price when income is compensated in a manner so as to leave the level of utility unchanged. The total effect of a price change is thus decomposed into a "substitution effect" and an "income effect." Incidentally, goods may be classified in the

Hicks-Allen theory as substitutes, independent, or complements according as K_{ij} is positive, zero, or negative.

In the Frisch-Houthakker decomposition, however, we consider a change in jth price and analyze the effect on the demand for the ith good, when income is compensated so as to leave the marginal utility of income of the consumer unaffected. Thus, we have

$$(dx_i)^{**} = \left(\frac{\partial x_i}{\partial y}\right) dy + \Sigma_j \left(\frac{\partial x_i}{\partial p_j}\right) dp_j$$

and, $$(dy)^{**} = \Sigma_j \left[\phi y \left(\frac{\partial x_j}{\partial y}\right) + x_j \right] dp_j$$

So that on substitution we obtain,

$$(dx_i)^{**} = \Sigma_j \left(\frac{\partial x_i}{\partial p_j}\right)^{**} dp_j$$

where, $$\left(\frac{\partial x_i}{\partial p_j}\right)^{**} = F_{ij}$$

$$= \left(\frac{\partial x_i}{\partial p_j}\right) + \phi y \left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right) + x_j \left(\frac{\partial x_i}{\partial y}\right)$$

$$= \lambda u^{ij}$$

Just as K_{ij} was the Slutsky price slope, F_{ij} is the Frisch[30] price slope, and measures the response in the demand for the ith good due to a change in the jth price, when income

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[30] After Frisch [1959, p. 184] who introduced this type of marginal utility of income compensated price responses in terms of elasticities. The term Frisch slope, is used by Goldberger [1967, p. 18].

is compensated in such a manner so as to leave the marginal utility of income of the consumer unchanged. The total effect of a price change has now been decomposed into a specific substitution effect, a general substitution effect, and an income effect. On a suggestion by Houthakker [1960, p. 248] goods may be classified as substitutes, independent, and complements as the specific substitution effect, F_{ij}, is positive, zero or negative.

A schematic representation due to Goldberger [1967, p. 19] may serve to illustrate the two decompositions of equation (9):

$$\underbrace{\left(\frac{\partial x_i}{\partial p_j}\right)}_{\text{Total Effect}} = \underbrace{\underbrace{\lambda u^{ij}}_{\substack{\text{Specific}\\ \text{Substitution}\\ \text{Effect}}} \quad \underbrace{-\phi y \left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right)}_{\substack{\text{General}\\ \text{Substitution}\\ \text{Effect}}}}_{\text{Hicks-Allen Substitution Effect}} \quad \underbrace{- x_j \left(\frac{\partial x_i}{\partial y}\right)}_{\text{Income Effect}}$$

We might note here, that while the Hicks-Allen decomposition is unaffected by monotonic transformations of the utility function, the further decomposition of the substitution effect in the Frisch-Houthakker fashion is affected by arbitrary non-linear monotonic transformations of the utility function.

1.6 Properties of Demand Functions

The first order conditions and the equations (7)-(10) that were derived from them, have been used in the previous section to decompose the consumer's response to a price change. More fundamentally, however, these results can be used to derive important restrictions on a complete set of demand equations of the consumer for whom Axioms (1) and (2) and the assumption of differentiability applies. Indeed, as Samuelson [1947, p. 97] puts it, "utility analysis is meaningful only to the extent that it places hypothetical restrictions upon these demand functions." Prior to a formal derivation of these results we shall introduce some concepts that have historically been used in the analysis of demand.

Recall that the complete set of demand equations is given by,

$$x_i = x_i(p_1, p_2, \ldots, p_n, y) \qquad (i=1,2,\ldots,n)$$

and the related marginal utility of income function by,

$$\lambda = \lambda(p_1, p_2, \ldots, p_n, y).$$

Although the analysis of demand can easily be conducted with reference to the conventional mathematical concept of "slope" of the demand function, or the partial derivative; it has historically been analyzed with reference to "elasticities" or logarithmic partial derivatives. We define,

then, the (Cournot) "price elasticity of demand" (for good i with respect to the jth price) as,

$$e_{ij} = \frac{\partial(\log x_i)}{\partial(\log p_j)} = \frac{p_j}{x_i}\left(\frac{\partial x_i}{\partial p_j}\right)$$

If i=j, then e_{ii} is referred to as the "own-price elasticity" and otherwise as the "cross-price elasticity." In a similar fashion we may define the "price elasticity of the marginal utility of income" as,

$$e_{\lambda j} = \frac{\partial(\log \lambda)}{\partial(\log p_j)}$$

Where these (Cournot) price elasticities measure the percentage uncompensated change in the demand for ith good with respect to the percentage change in the price of the jth good, the alternative (Slutsky) "price elasticity of demand" is defined as,

$$s_{ij} = \left(\frac{p_i}{x_i}\right) K_{ij}$$

where,

$$K_{ij} = \lambda u^{ij} - \left(\frac{\lambda}{\lambda_y}\right)\cdot\left(\frac{\partial x_i}{\partial y}\right)\cdot\left(\frac{\partial x_j}{\partial y}\right)$$

$$= \left(\frac{\partial x_i}{\partial p_j}\right) + x_j\left(\frac{\partial x_i}{\partial y}\right)$$

as before. From the analysis of the previous section, it is easily seen that s_{ij} measures the utility-maintaining income-compensated percentage variation in the demand for

the ith good with respect to the percentage change in the jth price.

Also, we define the (Engel) "income elasticity of demand" as,

$$E_i = \frac{\partial(\log x_i)}{\partial(\log y)} = \frac{y}{x_i}\left(\frac{\partial x_i}{\partial y}\right)$$

where, E_i measures the percentage change in demand with respect to a percentage change in income. Similarly, we define the "income elasticity of the marginal utility of income" as,

$$E_\lambda = \frac{\partial(\log \lambda)}{\partial(\log y)} = \frac{y}{\lambda}\left(\frac{\partial \lambda}{\partial y}\right).$$

We may note, once again, that E_λ is the celebrated "money flexibility" parameter used by Frisch [1932], [1959], and measures the percentage change in the marginal utility of income associated with a percentage change in income. In addition, we define the "budget share" of the ith good, w_i, as

$$w_i = \frac{p_i x_i}{y}.$$

Note, that purely by definition, the following relation (called "Slutsky's relation") connects the Cournot and Slutsky price elasticities:

Slutsky's Relation: $s_{ij} = e_{ij} + w_j E_i$.

Finally, we define the "elasticity of substitution" between goods i and j, as

$$d_{ij} = \frac{s_{ij}}{w_j} .$$

With these concepts in hand we are ready to derive all of the known restrictions on a complete set of demand equations. The proof of these results may be obtained from the first order conditions, (FOC), directly, or from their solution in terms of the slopes given by equations (7)-(10). The procedure adopted is to give the most simple proof of these results. As a matter of notation, these restrictions are numbered from (R1) to (R7). Recall, the first order conditions, (FOC),

$$\text{(FOC)} \ . \ . \ . \quad \begin{cases} u_i = \lambda p_i & (i=1,\ldots,n) \\ \Sigma_i \, p_i x_i = y \end{cases}$$

The last of these (n+1) equations that must be fulfilled by a complete set of demand equations is

$$\text{(R1)} \ . \ . \ . \quad \Sigma_i \, p_i x_i = y \qquad \text{(Adding-up restriction)}$$

By differentiating partially the (n+1) equation of (FOC), called the "budget equation," and by multiplying and dividing by y/x_i, we get

$$\text{(R2)} \ . \ . \ . \quad \Sigma_i \, w_i E_i = 1 \qquad \text{(Engel aggregation)}$$

Alternatively, we may obtain the same result by multiplying equation (8) by p_i and summing over i, and substituting for λ_y from equation (7). The third restriction, "Cournot aggregation" is similarly obtained by differentiating partially with respect to p_j, the same "budget equation" and multiplying and dividing appropriately to convert slopes into elasticities and collecting terms,

(R3) . . . $\Sigma_i w_i e_{ij} = -w_j \quad (j=1,\ldots,n)$ (Cournot aggregation)

Alternatively, this result can be derived by multiplying (9) by p_i, summing over i, substituting for (R2), multiplying and dividing appropriately, and noting that the inverse of the symmetric matrix U, is itself symmetric.

The fourth restriction is the symmetry of the Slutsky terms, K_{ij}, and can most easily be proved by noting that U^{-1} is a symmetric matrix, so that $u^{ij}=u^{ji}$. This gives,

$$K_{ij} = \lambda u^{ij} - \frac{\lambda}{\lambda y}\left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right)$$

$$= \lambda u^{ji} - \frac{\lambda}{\lambda y}\left(\frac{\partial x_j}{\partial y}\right)\left(\frac{\partial x_i}{\partial y}\right)$$

$$= K_{ji}$$

Recalling the fact that the Slutsky elasticity is given by,

$$s_{ij} = \frac{p_j}{x_i} K_{ij}$$

we may state the symmetry condition in elasticity form:

(R4) . . . $w_i s_{ij} = w_j s_{ji}$ $(i,j=1,...,n)$ (Symmetry Relation)

The restrictions (R2) through (R4) are the main content of the classical theory of consumer demand. They constitute independent restrictions on a complete set of demand equations and are responsible for the considerable economy of parametrization which results when it may be assumed that the consumer behaves in a manner which satisfies Axioms (1), (2). Additional restrictions that are not all independent of these may also be derived from these conditions.

The "homogeneity condition" may be obtained by observing that if prices and income were multiplied by the same arbitrary constant, the first order conditions, (FOC), remain unaffected by proportional changes in income and prices. Stated in a more rigorous way we have

(R5) . . . $\Sigma_j e_{ij} = -E_i$ $(i=1,...,n)$ (Homogeneity)

which states in elasticity form the condition that demand functions are all homogeneous of degree zero in income and prices. This suggests that a proof of the result (R4) may be obtained by applying Euler's theorem after establishing the homogeneity of degree zero of demand functions. More rigorously the result may also be proved by multiplying equation (9) by p_j, summing over j, and substituting

conditions (R2), (R3). This proof points out the fact that the "homogeneity condition" is not independent of (R1), (R2), and (R3), from which it can be derived. In this sense it imposes no additional constraint on the demand functions.

A familiar result, that has been utilized by Pearce [1964] as a restriction on a set of demand functions is

(R6) . . . $\Sigma_i \; p_i K_{ij} \;=\; \Sigma_j \; p_j K_{ij} = 0.$

The validity of this result may most readily be seen by multiplying equation (9) by p_i, summing over i, and using Engel aggregation to clear terms. Repeating the same procedure for p_j, the truth of (R9) can be established. A somewhat longer procedure adopted by Pearce to prove the same result may perhaps be more intuitively appealing. The term $\Sigma_i p_i K_{ij}$ may be proved equal to zero by summing the two equations that may be obtained by (1) partially differentiating the budget equation with respect to p_j, and (2) partially differentiating the budget equation with respect to y and multiplying by x_j. The second term $\Sigma_j \; p_j K_{ij}$ can be shown to be zero by substituting for $\frac{\partial x_i}{\partial p_j}$ from the Hicks-Allen fundamental equation of value theory into the homogeneity condition. It might be noted that a classical consequence of this condition is that not all

goods in the budget of the consumer can be complements in the Hicks-Allen sense.[31]

A final result, which is stated without proof,[32] is that the (nxn) matrix of Slutsky terms, K_{ij}, is negative definite. The reason for this is simply that due to our assumption of convexity of preferences, the Hessian matrix U is negative definite. We state the result formally,

(R7) . . . K is negative definite,

where we have denoted by K the matrix of Slutsky terms. We might point out that from equation (2) we have:

$$K = \lambda U^{-1} - \lambda \lambda_y (U^{-1}p)(U^{-1}p)'$$

Alternatively, (R7) may be stated as,

$$\left|K_{11}\right| < 0, \qquad \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} > 0, \text{ etc.}$$

which is the familiar result that the principal minors of a negative devinite matrix alternate in sign, beginning with the first minor being negative.

[31]The fact that no such condition holds for substitutes points to an asymmetry in the Hicks-Allen theory that Houthakker [1960] has called a "minor blemish." It was in response to this aspect of the Hicks-Allen theory that the further Frisch-Houthakker decomposition of the total substitution effect was proposed. It should be noted that the definition of substitutes and complements on the basis of the sign of the specific substitution effect suffers from no such bias.

[32]For a proof see Pearce [1964, pp. 54-7].

The restrictions (R1) to (R7) constitute all of the known restrictions on a complete set of demand equations, and have been utilized with great effect by empirical researchers in demand theory. We might point out that in this section we have stated all of these restrictions in Slutsky price elasticity form instead of the usual Cournot form. The only exceptions being the homogeneity condition (R5) and Cournot aggregation (R3). This has been done because it is the Slutsky elasticities that are usually estimated by researchers. The reason for this is immediately obvious when we consider the Symmetry condition as it is often stated, in Cournot form:

$$E_i + (w_j)^{-1} e_{ij} = E_j + (w_i)^{-1} e_{ji}$$

The validity of this can be seen by direct substitution of the Slutsky relation in (R4).

Due to the importance of the Slutsky elasticity in actual empirical practice, we note two further results that are the direct consequence of Cournot aggregation, the Homogeneity condition, Engel aggregation, and the Slutsky relation:

(R5') . . . $\Sigma_j \, s_{ij} = 0$ $\quad (i=1,\ldots,n)$ (Slutsky homogeneity)

(R3') . . . $\Sigma_i w_i s_{ij} = 0$ $\quad (j=1,\ldots,n)$ (Slutsky aggregation).

1.7 Aggregation Theorems

In the previous sections a considerable amount of theory has been developed with regard to the single consumer's demand for basic goods. It has been shown that demand functions resulting from utility-maximizing behavior must obey certain restrictions. It is unfortunate, however, that typically the data that are available are on a community or nation's expenditure on aggregated "expenditure categories" which subsume within them a large number of basic goods. It is meaningful to ask, then, what additional assumptions are required before we can apply the theory developed so far to the aggregate data that is available. In answer to this question, we give in this section the necessary and sufficient conditions under which, (1) basic-goods may be aggregated into "composite" goods in such a way so as to insure that the aggregate demand functions possess all of the desirable properties of the micro demand functions; (2) single-consumer demand functions may be aggregated into community demand functions with the latter possessing all of the properties of the former; and (3) individual utility functions may be aggregated to community "behavior function" such that the latter gives rise to a community demand function which is both consistent with, and possesses all of the properties of, the individual demand functions.

Of these, the conditions pertaining to the aggregation over commodities is the most straightforward. To restate the problem, we seek a procedure whereby the vast number of basic commodities that enter the individual's utility functions may be condensed into a fewer number of composite commodities that represent sets of the elementary commodities. In doing so, we must obviously sacrifice some of the information that the micro data contain. For aggregation to be "consistent" however, we require that this sacrifice of information should not affect the results obtained. More precisely, aggregation will be said to be "consistent" when a knowledge of the macro relations and the values of the independent macro variables leads to the same values of the dependent macro variables as would be obtained if all the micro relations and the values of all the independent micro variables were known.[33] Specifically, in the context of utility theory, it is useful to think that the consumer maximizes utility in the following manner: he first allocates the total expenditure between composite goods by reference to the price indices of these goods; and then allocates the expenditure on each category among its primitive components, the basic

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[33]These ideas are expressed by H. A. J. Green [1964, pp. 3-5, 35] who provides an excellent survey of the literature on aggregation. As will be obvious, much of this section relies on the initial chapters of Green [1964].

goods, by a reference to their prices.[34] In this framework an aggregation procedure will be said to be "consistent" if there exists a quantity index and a price index for the composite commodity such that, (1) maximizing a utility function whose arguments are the quantity indices gives rise to aggregate demand functions "consistent" in the general sense defined above; and in addition (2) the product of the price and quantity indices for the composite commodities gives rise to the same value of expenditure on the composite commodity as would be obtained by summing the expenditure on each of the elementary commodites contained in the composite commodity.

Before starting the formal result, it would be useful to give the following definition:

Definition: A function $u = u(x_1, x_2, \ldots, x_n)$ is said to be "weakly separable" if and only if the commodities $(x_1, \ldots, x_n)$ can be partitioned into groups G_1, $G_2, \ldots,$ and when $\frac{\partial(u_i/u_j)}{\partial x_k} = 0$, for all $i,j \in G_s$, $k \notin G_s$, where u_i and u_j denote the first partial derivatives of u with respect to x_i and x_j respectively, as before.

We are now in a position to state the basic theorem on the aggregation over commodities:

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[34] This is the celebrated "utility-tree" concept proposed by Strotz [1957]. Strotz [1957, 1959] and Gorman [1959a] analyzed in detail the consequences of this two-stage maximization procedure using the concept of "functional separability" due to Leontieff [1943] and Sono [1961]. In fact, Strotz and Groman anticipated many of the useful results that were later established by Frisch [1959], Houthakker [1960], and Barten [1964]. These latter results are discussed in Cahpter 2.

Theorem: The necessary and sufficient condition for a "consistent" aggregation of goods is that the utility function be weakly-separable, and that each quantity index be homogeneous of degree one in its elementary commodities.[35]

For a proof of the theorem, the reader is referred to Green [1964]. A more complete proof is provided by Gorman [1959a] and Strotz [1959]. To gain an intuitive insight into this theorem we note two facts. The first condition requires weak separability of the utility function, which may be argued for on grounds of the assumption of the two-stage maximizing procedure for the individual consumer. Given weak separability, the only other condition is that there should exist the possibility of constructing a quantity index for the composite commodity such that a proportionate change in all of the quantities of the elementary commodities gives rise to a change in the quantity index of the same proportion. We see, then, that it is

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[35] This is Theorem 4 in Green [1964, p. 25], and is originally due to Gorman [1959a] and Strotz [1959]. Gorman and Strotz offered three alternate conditions that were also necessary and sufficient for consistent aggregation. These were: (i) that there be only two composite commodities; or (ii) that the utility function be weakly-separable with respect to a partition of the goods into two groups, with one group consisting of a single commodity and the other group consisting of composite commodities; where it should be possible to construct quantity indices for the composite commodities which are homogeneous in their arguments; or finally, (iii) that the utility function be strongly separable. It should be noted that the condition we have stated in the theorem is a generalized version of the celebrated "Composite Commodity Theorem" anticipated by Leontieff [1936] and established by Hicks [1939]. The Leontieff-Hicks case of price proportionality is a special case of this theorem. As a parallel degenerate case, with limited economic relevance, Green [1964, p. 25] cites the possibility of quantity proportionality among the elementary commodities.

not unreasonable to consider that the consumer maximizes a utility function whose arguments are quantity indices of composite commodities, subject to the aggregate budget constraint which limits the expenditure on all composite commodities to total expenditure. This procedure gives rise to demand functions for the composite commodities which possess all of the properties possessed by the demand function for the elementary commodities. These demand functions, however, are for the single-consumer. We seek now to find conditions under which these functions may be aggregated over individuals to give rise to community demand functions.

In the case of aggregation over individuals, however, there does not exist a procedure similar to the case of aggregation over goods. The temptation exists, nevertheless, of attempting to find conditions under which utility functions may be aggregated over a community to give rise to a community "behavior function."[36] Indeed,

[36]This approach trespasses on the domain of welfare economics. The "behavior function" has, however, only limited welfare implications; for it cannot be unequivocally asserted that social welfare has increased if the aggregate of utility increases. For a discussion of welfare issues see Pearce [1964, pp. 127-132] and Green [1964, pp. 55-7]. Our interest in the existence of a community behavior function is restricted to the possibility of an additive-type community behavior function which may result from similar individual utility function, thus resulting in the same economies of parametrization for the community demand functions as are available for the individual demand functions. Thus we are interested in the community behavior function only to the extent that it might result in all of the "cardinal" properties of micro demand functions in the macro demand functions.

Samuelson [1956] motivates his discussion of social indifference curves by their bearing on community demand curves. The temptation always is to define a social-welfare function, somewhat like a "composite" utility function, so that by direct appeal to the results derived for the single-consumer theory, we may easily deduce the same results for the community demand theory. The efficacy of this approach has been a source of some controversy, and in this dissertation a position will not be taken on the relevant issues. Instead, we shall take a somewhat neglected path which leads surprisingly to conditions quite similar to the existence conditions for a community behavior functions.

Without recourse to a community utility function, we might ask under what conditions will the aggregate demand functions possess the properties of the single-consumer demand functions. Specifically, we seek necessary and sufficient conditions under which community demand functions satisfy the restrictions (R1) to (R6) that individual demand functions were shown to possess, in section 1.4. To do so we develop some notation[37] as follows:

Let the subscript $i=1,\ldots,n$ refer to goods as usual; and the superscript $k=1,\ldots,K$ refer to individuals. Denote

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[37] This line of attack was adopted by Roy [1952], whose notation we adopt, with slight modification.

the demand function for the ith commodity for the kth individual:

$$x_i^k = x_i^k(p_1,\ldots,p_n, y^k) \qquad \text{for all } i,k$$

where y^k refers to the kth individuals income. Defining community aggregates of demand for the ith good and community income respectively in the natural fashion:

$$x_i = \Sigma_k x_i^k \quad \text{and} \quad y = \Sigma_k y^k$$

we denote community demand functions:

$$x_i = x_i(p_1,\ldots, p_n, y) \qquad \text{for all } i.$$

To facilitate the exposition we define the following parameters:

$$b_i^k = \frac{x_i^k}{x_i}, \quad c^k = \frac{y^k}{y}, \text{ and } d^k = \frac{\partial(\log y^k)}{\partial(\log y)} = \frac{y}{y^k}\left(\frac{\partial y^k}{\partial y}\right)$$

Finally, if we denote as usual the Cournot price elasticity, the Slutsky price elasticity, the Engel income elasticity, and the budget share of the ith commodity in the community demand function respectively as e_{ij}, s_{ij}, E_i, w_i; and their counterparts for the kth individual's demand function by e_{ij}^k, s_{ij}^k, E_i^k, w_i^k; then it is easy to verify that the following relations hold between the macro elasticities and the micro elasticities:

$$e_{ij} = \Sigma_k \; e^k_{ij} \; b^k_i \qquad \text{for all } i,j,k$$

$$E_i = \Sigma_k \; E^k_i \; b^k_i \; d^k \qquad \text{for all } i,k$$

$$w_i = \Sigma_k \; w_i \; w^k_i \; c^k \qquad \text{for all } i,k$$

$$= (w^k_i \; c^k)(b^k_i)^{-1}$$

With these definitions, we state the two main results:

Theorem: The community demand function is homogeneous of degree zero if and only if $d^k = 1$.

Also,

Theorem: The Slutsky term is symmetric in the community demand function if and only if $c^k d^k = b^k_i$ for all i.

$$(c^k d^k = b^k_i \Rightarrow s_{ij} = \Sigma_k s^k_{ij} \; b^k_i)$$

It can also be shown that if the community demand function is homogeneous of degree zero, and possesses symmetric Slutsky terms, then the Adding-up restriction, Engel Aggregation, Cournot Aggregation, and the condition (R6) $\Sigma_i p_i K_{ij} = \Sigma_j p_j K_{ij} = 0$ are fulfilled. As the proofs of these results are trivial we do not state them.[38]

An intuitive interpretation of these results is immediately available. The first theorem states that if the community demand functions are to be homogeneous of degree zero, as they must be, then it is necessary and

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[38] Roy [1952] proves the first Theorem.

sufficient that the "elasticity of income distribution," to use Roy's term, be unity. Alternatively, we must assume that the proportion of national income held by every individual is unaffected by changes in the national income. This is reasonable to expect, for the community demand curve is a function of the prices and community income, while the individual demand functions admit an extremely large number of income variables y^k, $k=1,\ldots,K$. The price that is paid for aggregating all of the income variables is, as we should expect, paid by the assumption that income distribution of the community remains unchanged.

The requirement for the symmetry of Slutsky terms is far more stringent. In combination with the requirement of no change in income distribution, the condition of the second theorem is in fact equivalent to the restriction that $w_i^k = w_i (i=1,\ldots,n;$ and for all $k)$; or, the restriction that every commodity has the same share of every consumer's budget. This is indeed an extremely stringent restriction for it implies that every consumer allocates his expenditure in identical proportions between the commodities in his budget: the actual quantities of each commodity purchased being dependent only on his income. This implies, incidentally, that the Engel curve for each commodity is a straight line through the origin, and is identical for each individual.

Interestingly enough, these are the very conditions on Engel curves that Gorman [1953] has shown to be sufficient for the existence of an aggregate of utility functions, the community "behavior function" mentioned in the beginning of this section. We have come upon these conditions, however, in a more direct manner, and have avoided the relatively difficult proofs that are needed to establish the consistent aggregation of utility functions. For completeness, however, we state the theorems due to Gorman:[39]

Theorem: (Necessity) A necessary condition for consistent aggregation of individual utility functions is that all Engel curves are straight lines, parallel for each individual.

Theorem: (Sufficiency) It is sufficient for consistent aggregation of utility functions that all Engel curves are parallel straight lines through their origin, for each individual.

To summarize, then, aggregate demand functions possess all of the "ordinal" properties of the single-consumer, basic-good demand functions if it may be assumed that the community's income distribution remains unchanged, and also that each individual's Engel curves are straight lines through the origin, and are identical. We have also shown that if the last property of Engel curves may be assumed then a community behavior function may be

[39]These occur as Theorem 9 and Theorem 10 respectively in Green [1964, p. 47, p. 49] where a proof is also given.

constructed by aggregating over the individual's utility functions. Since the aggregation procedure assumes the summing of functions of individual utility functions, we may conclude that if all of the individual utility functions were separable with respect to the <u>same</u> partition, then the community behavior function is also separable with respect to the same partition. Thus, all of the "cardinal" properties resulting from additivity-type assumptions on the individual utility functions carry over to the community behavior function. This has the effect of imposing the same parametric restrictions on the aggregate demand functions. Thus, the community's demand for composite commodities may be treated identically to the single-consumer's demand for elementary commodities if we assume no change in income distribution, and linearity through origin of all Engel curves.

CHAPTER 2

SEPARABLE PREFERENCES

2.1 Introduction

The set of restrictions on a complete set of demand equations that have been derived in section 5 of the previous chapter have been used with great effect in the analysis of data on consumer demand. The major difficulty that arises in the estimation of a complete set of demand equations is the large number of free parameters to be estimated. For n commodities it is required that estimates of $(n^2 + n)$ elasticities be provided. A preliminary reduction in these can result by imposing the independent restrictions of Engel aggregation, (R2), Cournot aggregation, (R3), and Symmetry, (R4), derived in section 1.5. This reduces the number of free parameters from $(n^2 + n)$ to a substantially less, yet fairly large number of $1/2(n^2 + n - 2)$. For example, consider the moderate case of n=10, for which the imposition of these classical restrictions reduces the free parameters from 110 to 54, which is still large in comparison to the number of observations that are usually available.

In the light of these facts a need has been felt to impose additional restrictions on the consumer's preferences in order to further restrict the number of free parameters to be estimated. The next efficacious course, and one that has proved of great importance, has been to incorporate the quasi-cardinal assumption that the consumer's preferences are "separabel" in some manner. In this chapter, then, we present a brief survey of this theory of separable preferences. In the following section (Sec. 2.2) we define the alternative separability assumption and state briefly their implications for the Slutsky terms of the demand equation. Subsequent sections are devoted to a more detailed discussion of the implications for complete sets of demand equations of the assumptions of additive preferences, almost additive preferences and neutral want association, respectively. Finally, we discuss the case of indirectly additive utility in the last section, Sec. 2.6.

2.2 Definitions and Fundamental Results

Before defining "separability" of consumer preferences in a rigorous manner, we might examine intuitively the motivation behind such an assumption. Strotz [1957], [1959], and Gorman [1959a,b] expressed the belief that the total utility that the consumer derives is the sum of the utilities of the "branches" of utility, which may themselves

be split up into further branches. This view of utility as a "tree" reflected the belief that the set of all commodities in the consumer's budget could be partitioned (perhaps repeatedly) into groups of commodities in such a way so as to insure that price changes outside any particular group failed to affect the marginal rates of substitution between goods within the group. Intuitively, this implies that the consumer's allocation of expenditure between several goods in one group of commodities, say food, is unaffected by changes in the price of goods outside the food group. On *a priori* grounds, these assumptions do not appear unreasonable for broad aggregates of goods.

In order to give rigorous content to these ideas, we first define the concept of a "partition" of a set. In doing so, we assume familiarity with the defines of a "set," the operations of "union" and "intersection," the "empty set" and the definition of "mutually exclusive" sets. We have then:

Definition: A *Partition*, P, of a set N, is a set of sets, $P = \{N_1, N_2, \ldots, N_k\}$; where the N_i $(i=1,\ldots,k)$ are mutually exclusive subsets of N, whose "union" is N, i.e., $\bigcup_{i=1}^{k} N_i = N$.

Now we assume that the n goods can be partitioned into groups, $G_1, G_2, \ldots, G_k$, with the number of goods in group G_i being denoted by n_i, for all $i=1,\ldots,k$. Of course, the sum $\Sigma_i\, n_i = n$. We introduce also the notation x^i to denote

an $(n_i \times 1)$ vector of goods in the ith group, G_i. Goods within a group will be identified by a lower subscript, thus x_j^i denotes the jth good in x^i. Denoting the utility functions, once again, by u(x), and the marginal utility of the ith good by u_i, we have:[40]

<u>Definition:</u> u(x) is <u>strongly separable</u> with respect to a partition, P, if and only if

$$\frac{\partial(u_i/u_j)}{\partial x_k} = 0 \qquad \forall,\ i\varepsilon N_s,\ j\varepsilon N_t\ (s\neq t)$$
$$k \not\varepsilon (N_s \cup N_t)$$

<u>Definition:</u> u(x) is <u>weakly separable</u> with respect to a partition, P, if and only if,

$$\frac{\partial(u_i/u_j)}{\partial x_k} = 0 \qquad \forall\ i,\ j\varepsilon N_s;\ k\not\varepsilon N_s$$

It is easy to see that both strong and weak separability are invariant with respect to an arbitrary non-linear monotonic transformation of the utility function.

<u>Definition:</u> u(x) is <u>Pearce-separable</u> with respect to a partition, P, if and only if it is weakly separable with respect to P, and is strongly separable with respect to a pointwise partition of the elements of P. In other words, u(x) is Pearce separable with respect to P, if and only if,

$$\frac{\partial(u_i/u_j)}{\partial x_k} = 0 \qquad \forall i,j\varepsilon N_s;\quad k\neq i,j$$

Once again, it is readily seen that Pearce-separability is a "monotonic invariant" concept.

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[1]This section is but a paraphrasing of Uzawa's [1964] paper which has so succintly related the various concepts of separability, in addition to providing a proof of the necessary and sufficient conditions on the Slutsky terms under "neutral want association."

The consequence of both weak and strong separability, (and hence of Pearce-separability,) both in terms of the functional form of the utility function and of the Slutsky terms have been proved by Uzawa [1964]. As the proof is mathematically involved, and offers little intuitive understanding, we have omitted it, but we state the principal results derived by Uzawa. (In our statement we have actually combined two theorems into one.)

<u>Theorem</u>: $u(x)$ is <u>strongly separable</u> if and only if

either (i) $u(x) = F(u^1(x^1) + \ldots + u^s(x^s))$

or (ii) $K_{ij} = k(x)\left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right)$ $\forall\ i \epsilon N_s,\ j \epsilon N_t (s \neq t)$ & for all x, $k(x)$ being some function of x.

where, for the second condition it is assumed that $u(x)$ is strictly quasi-concave; and the theorem holds in both cases for $s>2$.

Also, a similar theorem for weak separability:

<u>Theorem</u>: $u(x)$ is <u>weakly separable</u> if and only if

either (i) $u(x) = F(u^1(x^1), \ldots, u^s(x^s))$

or (ii) $K_{ij} = k^{st}(x)\left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right)$ $\forall i \epsilon N_s,\ j \epsilon N_t (s \neq t)$ & for all x, $k^{st}(x)$ defined for $s \neq t$.

where, once again, for the latter condition it is assumed that $u(x)$ is strictly quasi-concave.

With the aid of these theorems, and the definition of Pearce-separability, it is readily seen that:

<u>Theorem</u>: u(x) is Pearce-separable if and only if,

either (i) $$\frac{\partial(u_i/u_j)}{\partial x_k} = 0 \qquad i,j\varepsilon N_s; \quad k\neq i,j$$

or (ii) $$u(x) = F(f^1(u^{11}(x_1^1)+\ldots+u^{1n_1}(x_{n_1}^1), \ldots, f^s(u^{s1}(x_1^s)+\ldots+u^{sn_s}(x_{n_s}^s)))$$

or (iii) $$K_{ij} = k^{st}(x)\left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right) \qquad i\varepsilon N_s,\ j\varepsilon N_t;\ \&\ \text{for all } x,$$

$k^{st}(x)$ defined for all s,t.

where for (iii) u(x) is assumed to be strictly quasi-concave.

<u>2.3 Additive Preferences</u>

The pioneering works in the theory of separable preferences are those of Schrotz [1957], [1959], Gorman [1959], Frisch [1959] and Houthakker [1960]. In particular, Houthakker [1960] considered the case of "additive preferences," (or "strongly separable" utility, in the terminology of Sec. 2.2), i.e., the case where the consumer's preferences could be represented by (at least one) utility function which was "directly additive."[41] A

[41] Alternatively, this could be called "complete want-independence" in the terminology of Frisch [1959], who defined "want-independence" between two goods as the case where the marginal utility of one good does not depend on the quantity of the other good.

utility function will be said to be "directly additive" if and only if[42]

$$u = u(x_1,\ldots,l_n) = u(x_1)+\ldots+u(x_n)$$

Now, since the Slutsky equation and the complete set of demand equations are all invariant with respect to an arbitrary non-linear transformation of the utility function, we may proceed to analyze the case of "additive preferences" by considering that specific utility function which is directly additive. We must remember, however, that this additive (canonical) form of the utility function is chosen only for convenience, and all of our results must be checked independtly for monotonic invariance.

To proceed, then, if utility is directly additive then the Hessian matrix of the utility function, U, is diagonal, so that the inverse, U^{-1}, is also diagonal, with elements:

$$(14) \ldots \quad u^{ij} = \begin{cases} 0 & i \neq j \\ (u_{ii})^{-1} & i=j \end{cases}$$

---

[42]Note that "direct additivity" of the utility function is not the same as "strong separability" or its equivalent "additive preferences." The case of strongly separable utility" or "additive preferences" exists if and only if the class of monotonic transforms of any utility function representing the consumer's preferences contain at least one "directly additive" utility function.

The major effect of additivity is that all price elasticities may be derived from income elasticities and the (inverse of) the income elasticity of marginal utility of income, or in Frisch's term the "money flexibility" parameter. To prove this it is helpful to rewrite the solutions to the slopes of the demand equations and the marginal utility of income function that are given as equations (7)-(10), in terms of elasticities. These may be written as:

$$(15) \quad . \; . \; . \quad \phi = \left[\frac{\partial(\log \lambda)}{\partial(\log y)}\right]^{-1} = \frac{\lambda}{y}\,(\Sigma_i \Sigma_j\; u^{ij} p_i p_j)$$

$$(16) \quad . \; . \; . \quad E_i = \frac{\partial(\log x_i)}{\partial(\log y)} = \frac{\lambda\phi^{-1}}{x_i}\,(\Sigma_j\; u^{ij} p_j) \quad (i=1,\ldots,n)$$

$$(17) \quad . \; . \; . \; e_{ij} = \frac{\partial(\log x_i)}{\partial(\log p_j)} = n_{ij} - \phi E_i E_j w_j - E_i w_j \quad (i,j=1,\ldots,n)$$

$$\text{where,} \quad n_{ij} = \frac{\lambda u^{ij} p_j}{x_i} = s_{ij} + \phi E_i E_j w_j$$

$$(18) \quad . \; . \; . \; e_{\lambda j} = \frac{\partial(\log \lambda)}{\partial(\log p_j)} = -\, w_j(E_j + \phi^{-1}) \quad (j=1,\ldots,n)$$

Note now that,

$$\Sigma_j\, n_{ij} = \Sigma_j\, s_{ij} + \phi(\Sigma_j\, w_j E_j) E_i$$

$$= \phi E_i$$

where use has been made of Engel aggregation, (R2), and Slutsky homogeneity, (R5'). Also, in the case of direct additivity,

$$\Sigma_j \, n_{ij} = n_{ii}$$

So that, we have for the price elasticity of demand under directly additive utility:

$$\text{(17DA)} \ldots \quad e_{ij} = \begin{cases} \phi E_i - \phi E_i^2 \, w_i - E_i \, w_i & i=j \\ \\ - \phi E_i E_j w_j - E_i w_j & i \neq j \end{cases}$$

Or, expressing (17DA) in terms of the more conventional expression for the Slutsky price elasticity:

$$s_{ij} = \begin{cases} \phi E_i - \phi E_i^2 \, w_i & i=j \\ - \phi E_i E_j w_j & i \neq j \end{cases}$$

The power of these results is immediately obvious. From the general case of $(n^2 + n)$ unknown free coefficients that are required for the estimation of a complete set of demand equations, the additivity restriction reduces these to the n unknown income elasticities and the "money flexibility" parameter, for a total number of free parameters of $(n+1)$.

Although equation (17DA) expresses concisely the major implication of direct additivity, we record for completeness some of the results stated by Houthakker [1960] in his original derivation. In elasticity form, the following results may be derived by simple manipulation from (17DA):

$$\frac{e_{ik}}{e_{jk}} = \frac{E_i}{E_j} \qquad (i,j\neq k) \tag{19}$$

which states the fact that under direct additivity, the ratio of Cournot price elasticities of two goods with respect to the price of a third good is equal to the ratio of the Engel elasticity of the two goods. This is equation (1) in Houthakker [1960].

Also, Houthakker's equation (11) may be derived from equation (17DA), or more easily from the expression for s_{ij} derived from (17DA), by noting that K_{ij} is, by definition, equal to $\left(\frac{x_i}{p_j}\right)s_{ij}$. This is the result of the fact that the Slutsky term, K_{ij}, is proportional to the product of the income slopes, under direct additivity:

$$K_{ij} = -\frac{\lambda}{\lambda_y}\left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right) \qquad i\neq j \tag{20}$$

Note that this result is the slope form (as opposed to the elasticity form) of our expression for s_{ij}.[43]

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[43] It is sometimes asserted (mistakenly) that the relationship (20) is not "monotonic invariant." That this is not so, is readily seen by defining the "canonical money flexibility" as $(\mu = K_{ij}/(\frac{\partial x_i}{\partial y})(\frac{\partial x_j}{\partial y}))$. Since K_{ij} and the income slopes are monotonic invariant, so is U. Of course, under strong separability, the "cononical money flexibility" is equal to the "money flexibility," λ/λ_y, and for the canonical (or additive) form of the utility function. Thus, while the "canonical money flexibility" is independent of the choice of the utility function, the "money flexibility" is not.

The result (20) is important from another viewpoint. Houthakker's Theorem 1 establishes the condition (20) as being necessary and sufficient for utility to be directly additive. We note the theorem formally without giving a proof:[44]

Theorem: (Houthakker, 1960a) The utility function is directly additive if and only if the Slutsky term, K_{ij}, satisfies equation (20).

To conclude, direct additivity is a forceful restriction on a complete set of demand equations and results in the utmost economies of parametrization. These economies are not costless, however. Houthakker [1960] has pointed out that direct additivity rules out specific substitution: $u^{ij} = 0$ for $i \neq j$. Theil [1967, p. 199] has proved that direct additivity rules out inferior goods. Also, direct additivity rules out complementary goods, as pointed out by Goldberger [1967, p. 31].[45] As a result, direct additivity is a meaningful hypothesis only when

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[44] A more rigorous statement of the theorem and a proof are given by Uzawa [1964, Theorem 4, p. 392]. In fact, Uzawa has shown that the utility function is "strongly separable" if and only if the Slutsky term is proportional to the product of the income slopes. Of course, the factor of proportionality in the latter case is no longer the same as in Houthakker's theorem. Incidentally, Uzawa [1964, n. 6, p. 392] seems to have confused "strong separability" with "direct additivity" and has come to the conclusion that Houthakker's statement of the theorem and his proof suggest more generality than there is. This does not appear to be the case.

[45] These references are given by Goldberger [1967, p. 31].

the data are aggregated to a very high degree.[46] In the estimation of relatively large complete sets of demand equations, where goods are defined in a narrow sense, the assumption of additivity becomes questionable. For in this case, there is reason to suspect the presence of specific substitutability, complementarity and perhaps inferiority.

In view of the severity of these restrictions, the empirical researcher may want to go only half way towards the additivity hypothesis. This can be done by a straightforward generalization of direct additivity that is due to Strotz [1957]. This is the case of "block-independent preferences" (or, "block additivity") where it is assumed that the set of all commodities can be divided into G groups, (with n_g being the number of commodities in the gth group) in such a manner that the marginal utility of any good depends only on the quantity of the goods in the group in which the commodity is placed. This results in the block-diagonality (as opposed to "diagonality") of the Hessian of the utility function, U. Alternatively, we may define a utility function, u, to be block-additive, if it is of the form:

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[46] It might be pointed out that this repeatedly cited effect of "aggregation" is itself only "observed" and there seems to be little theoretical support for such an idea.

$$u = u(x_1, \ldots, x_n)$$

$$= \sum_{g=1}^{n} u^g(x^g),$$

where, x^g denotes the $(n_g x1)$ vector of goods in the gth group. With this formulation, we note that

$$\Sigma_j n_{ij} = \sum_{j}\sum_{g} n_{ij} = E_i$$

Utilizing this as a restriction, along with symmetry and Engel aggregation, we get the number of free parameters to be estimated in the case of block-additive utility as[47]

$$1 + 1/2 \sum_{g=1}^{G} n_g(n_g+1).$$

To see this, note that the price elasticities are given in this case by:

$$e_{ij} = \delta_{gh} n_{ij} - \phi E_i E_j w_j - E_i w_j \qquad i\varepsilon g,\ j\varepsilon h$$

and δ_{gh} is Kronecker's delta.

2.4 Almost Additive Utility

In view of the somewhat restrictive implications of the assumption of direct additivity, it was proposed in the last section that block-additivity be assumed as a half-way measure. Barten [1964] has generalized on these

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[47]This formula is given by Theil [1967, p. 199].

ideas and has proposed the assumption of "almost additivity" which yields the direct additivity case and the block-additivity case as special cases within the almost additivity hypothesis.[48] The motivation behind this assumption lies in the observation that the price elasticities may all be estimated from the income elasticities and the money flexibility parameter, if the off-diagonal elements of the inverse of U are zero. Thus, if the off-diagonal elements could be approximated by some function of the diagonal elements of the Hessian, then there would be reason to believe that a possible relation could be found where cross price elasticities may be estimated from the own-price elasticities and the income elasticities. It is with this in mind that "almost additivity" assumes that the off diagonal elements of the Hessian are not zero, as was the case in additivity, but "small" in comparison to the geometric mean of the corresponding diagonal elements of the Hessian.

Formally, we define a utility function to be "almost additive" if second partial derivatives of the utility function may be written as:

$$u_{ij} = c_{ij}(u_{ii}u_{jj})^{1/2} \qquad i,j=1,\ldots,n$$

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[48] The statement is not quite true for block-additivity, as will be obvious after a definition of "almost additivity" is given.

where the c_{ij} are fixed constants given by:

$$c_{ij} = \begin{cases} 1 & i=j \\ c_{ij} & i\neq j \end{cases}$$

In addition, the c_{ij} are symmetric in the sense that

$$c_{ij} = c_{ji} \qquad i,j=1,\ldots,n$$

and, finally, the c_{ij} (for $i\neq j$) are "small" in the sense that the inverse of the Hessian U, has elements that may be "adequately" approximated by:

$$u^{ij} \approx \begin{cases} (u_{ii})^{-1} & i=j \\ c_{ij}(u_{ii}u_{jj})^{-1/2} & i\neq j \end{cases}$$

We note that the case of direct additivity corresponds to $c_{ij} = 0$, for all i,j. Whereas a specialized block-additivity results if we set blocks of c_{ij} equal to zero. In empirical practice, it is the latter assumption that is frequently used.[49]

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[49] Although almost additivity has been widely acclaimed as an ingeneous and useful hypothesis, as it no doubt is, there remain some implications of the "smallness" of the c_{ij} that have been a source of some discomfort for the present writer. In particular, it seems that there is a need to derive the implications for the functional form of the utility function that must be implied by the second order partial differential equation:

$$u_{ij} = c_{ij}(u_{ii}u_{jj})^{1/2}$$

with the constants being "small." In empirical work, Barten [1964] assumed these constants to be of the order of less than 0.2. It is easily verified that it is

Two consequences of almost additivity are immediately obvious. First, the elements that are zero in U, are zero in U^{-1}. This means that if it is suspected that two commodities have a zero specific substitution effect, then this prior information can be incorporated in the model by specifying the corresponding c_{ij} to be zero. This would result in the u^{ij} being zero, which is necessary and sufficient for the specific substitution effect to be zero; for non-zero prices and finite income. Secondly, the ratio of the non-zero off-diagonal elements to the geometric mean of the corresponding diagonal elements, is the same in U, and U^{-1}.[50]

The consequences of "almost additivity" are seen in a way quite similar to direct additivity. Recall equation (17) which states in general the solution for price elasticities:

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necessary and sufficient for utility to be directly additive, that the c's be zero for $(i \neq j)$. What is unclear is the implications of c's of the order of 0.1, or 0.2. Also, there seems to be a need for examining the order of the error in the approximation used for elements of the inverse.

[50]These results are stated by Barten [1964, p. 4]. He states the second result with the qualification, "apart from sign" due to the way he formulated his definition of "almost additivity." The formulation in this paper avoids this by neglecting to append a minus sign before the u_{ii}, as Barten does. Barten's reason for doing so are to insure that diagonal elements "represent decreasing marginal utilities." Barten [1964, p. 4]. His $\theta_{ij} = -c_{ij}$.

$$(17) \ldots \quad e_{ij} = n_{ij} - \phi E_i E_j w_j - E_i w_j \qquad (i,j=1,\ldots,n)$$

In the case of almost additivity we have,

$$n_{ij} = \begin{cases} \dfrac{\lambda p_i}{x_i \; u_{ii}} & i=j \\[2ex] \dfrac{\lambda p_j}{x_i (u_{ii} u_{jj})^{-1/2}} & i \neq j \end{cases}$$

It is easy to see, therefore, that the following relation holds between the $n_{ij}(i \neq j)$, and the n_{ii}, n_{jj}:[51]

$$n_{ij} = c_{ij} \; (n_{ii} n_{jj} \; w_{j/w_i})^{1/2}$$

Also, the constraint that the specific substitution effects sum to money flexibility times the income elasticity holds for each good. Thus, if m_i non-zero c_{ij} are specified for each row of U, then we have to estimate $1+2n+\Sigma m_i$ elasticities, and ϕ. There are $1+n+1/2\Sigma m_i$ constraints; Engel aggregation, $\Sigma_j \; n_{ij} = \phi E_i$, and symmetry of specific substitution effects. This leaves $n+1/2\Sigma m_i$ free parameters to be estimated.[52] This is a substantial reduction in the parameters, and shows the effectiveness of almost additivity as a more sophisticated additivity assumption.

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[51] In Barten's [1964] actual work, a different formula was used as an approximation to this formula. This is discussed in the section on almost additivity in Chapter 3, below.

[52] Barten [1964, pp. 607].

2.5 Neutral Want Association

A hypothesis, quite similar to Barten's almost additivity hypothesis, is that of "neutral want association" proposed by Pearce [1961], [1964]. Once again the restriction is on the off-diagonal elements of the Hessian of the utility function, and it is assumed, as before, that commodities can be divided into groups, but this time the off-diagonal elements of the Hessian are assumed to be proportional to the product of the marginal utilities of the row and column goods, the factor of proportionality being the same for goods in the same group. Unlike almost additivity, though, neutral want association does not place any restriction on the size of these factors of proportionality. Also, neutral want association introduces all of the first derivatives of the utility function into the specification of the Hessian. Both these factors have the effect of making it a non-trivial problem to solve for the inverse of the Hessian, U, and to analyze the effect on the Frisch-Houthakker specific substitution effect as we have done so far. We resort, therefore, to an alternative mode of analysis used by Uzawa [1964] in which the implication of neutral want association is seen upon the separability of the utility function, and results are derived for the Slutsky term under these assumptions.

To proceed, we need a definition of neutral want association.

<u>Definition</u>: (Pearce, 1961) Two goods i and j are defined <u>to be neutrally want associated</u> to a third good, k, if and only if,

$$\frac{\partial(u_i/u_j)}{\partial x_k} = 0$$

where, u_i, etc. denote the first partial derivatives of the utility function, as usual. To get an intuitive idea of the implication of neutral want association, note that what the definition amounts to is simply the assumption that the marginal rate of substitution between goods i and j is not affected by the amount consumed of the kth good. This is a straightforward generalization of the additivity assumption that the marginal utility of one good was unaffected by the quantity consumed of another good. Of course, the additivity relation was binary and hence symmetric. This is obviously not true of the neutral want association relation.

In the previous sections we were able to demonstrate the effect of the hypotheses of additivity and almost additivity directly on the inverse of the Hessian matrix, and hence on the Slutsky elasticities. Although we shall not be able to carry out this procedure in the case of neutral want association, we note, however, the effect on the Hessian matrix of the assumption of neutral want association. The elements of the Hessian, U, are esaily seen to be:

$$u_{ij} = \begin{cases} u_{ii} & i=j \\ \lambda^2 a_{st} p_i p_j & i \varepsilon N_s,\ j \varepsilon N_t;\ \text{for all } s,t. \end{cases}$$

where the a_{st} are constants, N_s and N_t are the sth and t-th commodity groups, λ is the marginal utility of income, and use has been made of the first order conditions. Thus, very much like Barten's almost additivity hypothesis, the neutral want association hypothesis also assumes that the off-diagonal elements of the Hessian matrix are functions of the known prices, the marginal utility of income, and a constant which remains the same for each commodity group. The comparison between almost additivity and neutral want association, however, cannot be carried on much further. The reason is that Barten formulates his hypothesis with the sole purpose of getting manageable and well-behaved off-diagonal elements in the inverse of U, while Pearce makes no such attempt. Thus, the methods of analysis, utilizing the Frisch-Houthakker decomposition, are no longer available to us for the purpose of analyzing the effects of neutral want association on economies of parametrization.

The alternative approach in analyzing the consequences of neutral want association is due to Uzawa [1964]. Consider the entire set of commodites grouped into k groups: G_1, G_2, ..., G_k, with x^i denoting the $(n_i x\ 1)$ vector of

goods in the i-th group, and n_i being the number of goods in the i-th group. We have, then,

<u>Theorem</u>: (Uzawa, 1964)

$$\frac{\partial(u_i/u_j)}{\partial x_k} = 0, \qquad i,j \varepsilon G_s,\ k \neq i,j.$$

if and only if:

$$K_{ij} = k^{st}(x) \left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right) \qquad i \varepsilon G_s,\ j \varepsilon G_t;\ \text{and for all } x,\ k^{st}(x) \text{ defined for all } s,t.$$

In other words, if goods may be partitioned in such a way that all pairs of goods in any group are neutrally want associated with any third good, then the income compensated (Slutsky) price elasticity between two goods is proportional to their income elasticities. The constant of proportionality, however, is the same for each pair of groups to which the respective commodities belong. Under neutral want association, then, the number of unrestricted parameters to be estimated becomes $1/2g(g+1)+n$, where g is the number of groups, and n the number of commodities, [Pearce, 1964, p. 214]. This is indeed a substantial reduction in the number of parameters.

Unfortunately, no empirical results seem to be available on the application of neutral want association to the estimation of demand functions, in the form outlined above. Instead, it seems, Pearce [1964] used additional assumptions in the implementation of neutral want

association to British data. In his estimation procedure, Pearce [1964, pp. 213ff.] adopted the following assumptions, which appear to be quite restrictive:

Assumption 1: The expenditure on any group of commodities bears a fixed proportion to total expenditure.

Assumption 2: The income slope of the demand functions is assumed constant with respect to variations in prices and expenditures.

The first assumption is equivalent to assuming that each commodity group may be treated independently, in the sense that price changes outside the group do not affect the allocation of expenditures within any group. If enforced exactly, rather than as an approximation, this would be equivalent to arbitrarily equating all the cross-elasticities outside each group to zero. Such an assumption would cast severe doubts on the results. The second assumption seems equally restrictive in its implications. Thus, a valid test of the plausibility of the neutral want association hypothesis remains to be carried out.

2.6 Indirectly Additive Utility

A final theoretical restriction on the form of the utility function which results in economies of parametrization is the restriction of "indirect additivity," proposed by Houthakker [1960] along with its counterpart, direct additivity, mentioned above. Unfortunately, the consequences of indirect additivity are difficult to

reconcile with ordinarily held beliefs about the behavior of the consumer. Also, preliminary empirical results on a specific indirectly additive utility model of demand have offered evidence of serious statistical limitations of the model. For completeness, however, we discuss the consequences of indirect additivity briefly.

It has been noted in Chapter 1 that by substituting for the demand functions into the direct utility function, we can get the indirect utility function:

$$u = u^*(p_1, \ldots, p_n, y)$$

$$= u(x_1(p,y), \ldots, x_n(p,y))$$

where, for notational convenience we denote by p, as usual, the (nxl) price vector. Now, each demand function is homogeneous of degree zero, so that proportional increases in prices and income leaves the quantities demanded of each good unchanged. This means, then, that the indirect utility function is also homogeneous of degree zero in prices and income. Thus, the indirect utility function can be written as:

$$u = u^*(^y/p_1, \ldots, ^y/p_n).$$

This is the canonical form of the indirect utility function, and all discussions of the functional form refer to this function rather than the one above.

Rigorously, then, a utility function is said to be "indirectly additive" if the corresponding indirect utility function can be written as:

$$u = u^*(p,y) = \Sigma_i \; u^{*i}(y/p_i)$$

The consequences of indirectly additive utility are best analyzed by the use of an identity due to Rene Roy [1943]. To derive this identity recall equation (13) of Chapter 1:

$$(13) \;.\;.\;.\quad u^*_y = \lambda, \quad \text{and} \quad u^*_j = -\lambda x_j \qquad (j=1,\ldots,n)$$

where u^*_y, and u^*_j denote the partial derivatives of the indirect utility function u^* with respect to income y, and the jth price p_j, respectively. As a direct consequence of (13) we have Roy's Identity:

$$x_i = -\frac{u^*_i}{u^*_y} \qquad (i=1,\ldots,n)$$

Using Roy's identity and equation (13) it is relatively easy to derive all of the expressions for price and income slopes of the demand and the marginal utility of income functions. Note that from (13), we have

$$\lambda_y = u^*_{yy}$$

$$\text{and} \qquad \frac{\partial\lambda}{\partial p_j} = u^*_{yj} = u^*_{jy} \qquad (j=1,\ldots,n)$$

where u^*_{ab} denote second partials of u^* with respect to the variables a, b. The income slope is obtained by

differentiating Roy's identity; where for all i=1,...,n we have

$$\frac{\partial x_i}{\partial y} = -\frac{(u^*_y\, u^*_{iy} - u^*_i\, u^*_{yy})}{u^{*2}_y} \qquad (i=1,\ldots,n)$$

$$= - (u^*_y)^{-1}\ (u^*_{yi} + x_i\ u^*_{yy}) \qquad \text{by (13)}$$

$$= - \lambda^{-1}(\frac{\partial \lambda}{\partial p_i} + x_i\ \frac{\partial \lambda}{\partial y}) \qquad \text{by (13)}$$

Similarly, by differentiating with respect to the jth price, we derive the price slope of the demand functions: For all i,j=1,...,n

$$\frac{\partial x_i}{\partial p_j} = -\frac{(u^*_y\, u^*_{ij} - u^*_i\, u^*_{yj})}{u^{*2}_y}$$

$$= - (u^*_y)^{-1}(u^*_{ij} + x_i\ \frac{\partial \lambda}{\partial p_j}) \qquad \text{by (13)}$$

$$= - (u^*_y)^{-1}\{u^*_{ij} + x_i\,(-\lambda\frac{\partial x_j}{\partial y} - x_j\frac{\partial \lambda}{\partial y})\} \qquad \text{by (10)}$$

$$= -\lambda^{-1}u^*_{ij} + \lambda^{-1}\lambda_y x_i x_j + x_i\frac{\partial x_j}{\partial y} \qquad \text{by (13)}$$

Collecting results, we may write in concise form all of the solutions to the slopes in terms of the indirect utility function quite generally as:

(71) $\frac{\partial \lambda}{\partial y}$ $\qquad \lambda_y = u^*_{yy}$

$$(81) \qquad \frac{\partial x_i}{\partial y} = \lambda^{-1}\left(\frac{\partial \lambda}{\partial p_i} + x_i \frac{\partial \lambda}{\partial y}\right) \qquad (i=1,\ldots,n)$$

$$(91) \qquad \frac{\partial x_i}{\partial p_j} = -\lambda^{-1} u^*_{ij} + \lambda^{-1}\lambda_y \, x_i \, x_j + x_i \left(\frac{\partial x_j}{\partial y}\right) \qquad (i,j=1,\ldots,n)$$

$$(101) \qquad \frac{\partial \lambda}{\partial p_j} = u^*_{yj} = u^*_{jy} \qquad (j=1,\ldots,n)$$

These relations in fact can be viewed as a sort of "dual" to the equations (7)-(10), of Chapter 1.[53]

The consequences of indirect additivity are now obvious. If the utility function is indirectly additive then $u^*_{ij} = 0$ for $i \neq j$. This means that the price elasticities of the demand equations are given by:

$$e_{ij} = n^*_{ij} + \phi^{-1} w_j + E_j \, w_j \qquad (i,j=1,\ldots,n)$$

where,

$$n^*_{ij} = -\frac{p_j}{x_i}\frac{u^*_{ij}}{\lambda}$$

and ϕ is the money flexibility parameter. Note that $n^*_{ij} = 0$, $i \neq j$ is the consequence of indirect additivity, so that the Cournot price elasticity of demand depends

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[53] Several comments are in order. The duality relations linking direct and indirect additivity are explored by Samuelson [1965]. Further discussion of this aspect is at the end of this section where the case of "simultaneous additivity" is briefly touched upon. These relations are derived elegantly by Goldberger [1967, pp. 85-86] using matrix notation, similar to the Barten-Theil notation for the directly additive case.

only on the good whose price is changing and not on the good whose quantity is affected. This is the classical consequence of indirect additivity and was expressed by Houthakker [1960a] as:

$$\frac{\partial x_i/\partial p_k}{\partial x_j/\partial p_k} = \frac{x_i}{x_j} \qquad (i,j\neq k)$$

or, in our elasticity notation, quite simple as,

$$e_{ik} = e_{jk} \qquad (i,j\neq k)$$

Houthakker [1960] has given formulas for the price elasticities of demand that are related to our expression for e_{ij} given above.[54] We derive these formulas by noting, as Goldberger [1967] does, the effect of the Cournot aggregation condition on our results. Multiplying through by w_i and summing over i; we have for j=1,...,n:

$$\begin{aligned} -w_j &= \Sigma_i\ w_i e_{ij} \\ &= \Sigma_i\ w_i n^*_{ij} + (\phi^{-1} + E_j)\ w_j\ \Sigma_i\ w_i \\ &= \Sigma_i \left(\frac{p_i p_j}{y}\right)\frac{u^*_{ij}}{\lambda} + (\phi^{-1} + E_j)\ w_j \end{aligned}$$

[54]Goldberger [1967, pp. 87-88] has derived the correct version of the original Houthakker formula, which is in error. Our derivation is somewhat different from either Houthakker or Goldberger, but the substantive results are the same.

So that, in general, we have,

$$(\phi^{-1} + E_j)w_j = \Sigma_i \frac{p_i p_j}{y} \frac{u^*_{ij}}{\lambda} - w_j \qquad (j=1,\ldots,n)$$

Now in the case of indirect additivity this reduces to:

$$(\phi^{-1} + E_j)\; w_j = (p^2{}_j/y)(u^*_{jj}/\lambda) - w_j$$

This in turn implies that

$$(\phi^{-1} + E_j) = - n^*_{jj} - 1$$

or,

$$n^*_{jj} = - (\phi^{-1} + E_j) - 1$$

We may write, therefore, the following equations that give the price elasticities of demand under indirect additivity:

$$e_{ij} = \begin{cases} (p^2_j/y)(u^*_{jj}/\lambda) - w_j & \ldots\ldots \quad (i\neq j) \\ (p^2_i/y)(u^*_{ii}/\lambda) - w_i - (p_i/x_i)(u^*_{ii}/\lambda) & (i=j) \end{cases}$$

where, the first of these quations is cited by Houthakker [1960] as equation (22). The own price elasticity is derived, however, in a somewhat different form. For symmetry, we state the cross elasticity formulas corresponding to Houthakker's own-price formulas, also.[55] This can be done by substituting for n^*_{ii} in the expression for e_{ij}, to

[55] This corresponds to Houthakker [1960a] equation (23), which was corrected by Goldberger [1967] and appears as equation (4.36) in the latter paper.

give an alternative expression for the price elasticities under indirect additivity:

$$e_{ij} = \begin{cases} (\phi^{-1} + E_j)\, w_j & (i \neq j) \\ (\phi^{-1} + E_j)(w_j - 1) - 1 & (i = j) \end{cases}$$

where, the latter formula is equation (4.36) of Goldberger [1967].

We see, then, that the case of indirect additivity imposes substantial restrictions on the number of free parameters to be estimated, for all price elasticities are estimable from income elasticities and the money flexibility parameter. Unfortunately, there is no direct motivation for assuming indirect additivity comparable to the direct additivity case. The result is that the indirect additivity hypothesis has to be introduced solely on the grounds of resulting computational convenience. The empirical consequence of equal Cournot price elasticities of two goods with respect to a third price, seems dubious for obvious reasons. Thus, the indirect additivity hypothesis has not much to recommend itself to the empirical worker.[56]

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[56] In practice, indirect additivity has been imposed only within the context of the "indirect addilog" utility function proposed by Houthakker [1960b]. As Goldberger [1967, p. 92] has noted, Houthakker arrived at this function by attempting to force the "constant elasticity of demand" system to satisfy the budget constraint. In doing so, however, the indirect addilog function became nonlinear in parameters, so that the most attractive feature of the

In conclusion, we might mention the case of "simultaneous additivity" mentioned by Houthakker [1960a] and explored by Samuelson [1965]. A utility function is defined to be "simultaneously" additive if both the direct and the indirect utility functions are additive. It was proved (incorrectly, as it turned out) by Houthakker [1960a] that simultaneous additivity implied unitary income elasticities. Also, Samuelson [1965] proved (once again, incorrectly) that simultaneous additivity implied unitary price elasticities. The exception to these results was given by Hicks [1969], who showed that these results were not true for the following utility function:

Hicks exception: $u = u(x_1, \ldots, x_r) + \sum_{r+1}^{n} a_j (\log x_j)$ [57]

Samuelson [1969] has corrected the two theorems, and has shown that Houthakkers theorem [1960, Th. 3] holds for n=3, with the Hicks exception being the only exception. Also, Samuelson's theorem holds, except that at most one good may not have unitary price elasticity.

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constant elasticity of demand system - ease of estimation - was lost.

[57] We give the generalized Hicks' exception, given by Samuelson [1969].

CHAPTER 3

EMPIRICAL MODELS OF CONSUMER DEMAND: STOCHASTIC SPECIFICATION, AND ESTIMATION

3.1 Introduction

In Chapter 1 it was shown that a "utility-maximizing" consumer possesses a complete set of demand equations which obey a set of restrictions on their partial derivatives. In addition, if it may be assumed that the consumers in a given community or nation possess identical preference patterns, with linear Engel curves possessing zero intercepts, then it was shown that the community's demand for aggregates of goods is also a function of all prices and national income. Further, this complete set of demand functions for the community possesses all of the properties possessed by the individual's demand functions. Finally, it was pointed out that under the same assumptions about Engel curves, a community "behavior function" can be constructed. The community's demand functions may then be derived alternatively by viewing the community as a single consumer attempting to maximize the community behavior function subject to the community budget constraint. Thus it was established that the

community's complete set of demand curves possess all of the properties that the individual demand curves were shown to possess.

Under the assumptions above, it has been shown that the community's demand for aggregates of goods is a function of all prices and national income, with constraints on the partial derivatives of the demand functions. Unfortunately, this is not sufficient to determine a unique functional form for the demand equations. The choice for functional form must be guided, however, by the condition that the demand equations must obey the restrictions derived above. Historically, this point was often ignored by empirical researchers, and it is only in the past decade that a great deal of attention has been directed at the search for a "complete system of theoretically plausible demand functions."

In this chapter we discuss the three leading "theoretically plausible" functional forms that have been proposed in the literature. In addition, we discuss the "Constant Elasticity of Demand System" which is of historical importance, but is not theoretically adequate.

3.2 The Constant Elasticity of Demand System

The "Constant Elasticity of Demand System" is based upon the double-logarithmic specification of the functional form of the demand equations, and has been the

most widely used specification in the estimation of demand functions.[58] Despite the fact that it was well-known that the function was inconsistent with utility theory, the ease of estimation and direct interpretation of the parameters led empirical researchers to utilize this form and justify their actions by arguing that the specification was a good first-order local approximation.

The "Constant Elasticity of Demand System" (CEDS) may be specified in its non-stochastic form as follows:

$$\log x_i = a_i + E_i (\log y) + \Sigma_j e_{ij} (\log p_j) \quad (i,j=1,\ldots,n)$$

where, as before, X_i is the quantity of the i^{th} good, p_i is its price, y is the total expenditure or "income," E_i is the income (Engel) elasticity, and e_{ij} is the price (Cournot) elasticity of demand. In addition, it is assumed that E_i and e_{ij} are constant with respect to variations in prices and income. The a_i are constants which may be interpreted as trend terms. This specification will be referred to as the "Cournot specification" of the CEDS, as distinguished from the "Slutsky specification" which may be derived from the above by using the Slutsky Relation:

$$\log x_i = a_i + E_i (\log y - \Sigma_j w_j \log p_j) + \Sigma_j s_{ij} (\log p_j) \quad (i,j=1,\ldots,n)$$

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[58] Schultz (1938, p. 83, n. 46) cites a July, 1929 paper by Leontieff in which this form was used.

where w_j is the j-th commodity's budget share, $w_j \equiv p_j x_j / y$, and s_{ij} is the price (Slutsky) elasticity.[59] The simplicity of the system is apparent. All coefficients are directly interpretable as elasticities. Finally, if an additive disturbace term is appended to the form, estimation becomes a trivial problem.

The CEDS is, unfortunately, inconsistent with utility theory. This was demonstrated by Wold and Jureen [1953, pp. 105-7], for example, in the context of integrability conditions. They showed, to be precise, that the double-logarithmic demand function does not satisfy the integrability conditions, and hence is inconsistent with any utility-maximizing process, unless it is assumed that the indifference curves are of an empirically implausible form. A simpler and intuitively appealing argument can be made in fact against the plausibility of any system of demand equations in which elasticities are assumed constant. The argument rests upon the fact that if the income elasticity is constant then unless it is equal to unity, there exists a finite income for which the commodity will either not be bought at all, or be bought solely. Thus, if income elasticities are constant then they must all be

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[59]The distinction between "Cournot" and "Slutsky" specifications will be made throughout, and is dependent upon which price elasticity appears in the functional form. The latter specification was used by Stone (1954a).

unity. This may be proved rigorously as follows: We have for all $i=1,\ldots,n$. By definition:

$$\log w_i \equiv \log p_i + \log x_i - \log y$$

Taking total differentials, and assuming prices constant,

$$d(\log w_i) \equiv d(\log x_i) - d(\log y)$$

$$\equiv (E_i - 1)\, d(\log y)$$

where E_i denotes the income elasticity as before, and use has been made of the fact that the ratio of total logarithmic differentials is equal to the partial logarithmic derivative if prices are assumed unchanged. Rewriting, we have:

$$\frac{dw_i}{w_i} \equiv (E_i - 1)\ \frac{dy}{y}$$

Now assume, in all generality, that at income y^o and prices p^o the budget share of the ith commodity was w_i^o. Using the above relation we can calculate the incomes for which the budget share of the ith commodity becomes unity and zero. To do so we note that:

$$dw_i < (1-w_i^o) \quad \text{iff} \quad (E_i-1)dy < \frac{(1-w_i^o)}{w_i^o}\ y^o$$

and $\quad dw_i > -w_i^o \quad \text{iff} \quad (E_i-1)dy > -y^o$

We have, therefore, the condition that: $0 < (w_i^o + dw_i) < 1$ if and only if:

$$\left[(E_i-1) - 1\right] y^o < (E_i-1)(y^0 + dy) < \left[(E_i-1) + \frac{(1-w_i^o)}{w_i^o}\right] y^o$$

Depending upon the sign of (E_i-1), the above equation gives us the limits (finite) within which the income, $(y^o - dy)$, must remain if the ith commodity's budget share is to lie between zero and unity. We see immediately that only under the case $E_i=1$ do we permit income to lie between plus or minus infinity.[60] Thus, it has been shown that if income elasticities are constant, and are all not equal to unity, then the budget constraint is violated at a finite income.[61]

The estimation of the CEDS is not discussed in any detail due to its trivial nature. The stochastic specification consists of assuming that a normally distributed disturbance term with zero mean and finite variance

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[60] Actually, we are only concerned with removing the upper bound on income, for the natural lower bound of zero is always assumed to apply.

[61] The need for such a proof was suggested by a reading of a paper by Yoshihara [1969] in which he proved this for the specific case of the double-logarithmic function (as opposed to our case of any constant income elasticity of demand function). Unfortunately, Yoshihara's proof implicitly assumes no negative income elasticities, and is, therefore, not completely general.

appears additively in each of the demand equations. Single equation OLS estimates are best linear unbiased, and with the assumption of normality are also maximum likelihood.[62] Alternatively, the disturbances may be assumed to be non-independent. Ignoring parametric restrictions, the assumption of non-independence of the disturbances across equations does not affect the properties of the OLS estimates. However, if some of the price elasticities are assumed to be zero and if the disturbances are assumed non-independent, OLS estimates are no longer efficient. An appropriate procedure is Zellner's (asymptotically) efficient procedure, ZEF.

Finally, unless cross-elasticities are assumed zero, the CEDS does not provide a useful model for even a medium sized model. This is partly due to the severe problem of multicollinearity which is almost always present in the data on prices. On the other hand, if the cross-elasticites are arbitrarily equated to zero, then the validity of the CEDS specification can be seriously questioned. As is well-known, an incorrect specification leads to biased parameter estimates.

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[62] This is not strictly correct because there are restrictions on the parameters which are ignored primarily due to the fact that imposing them is non-trivial.

3.3 Rotterdam School of Demand Models

A very elegant rectification of the CEDS results from considering differentials of logarithms in the specification of the functional form. This differential double-logarithmic form has given rise to the "Rotterdam School of Demand Models" which retain some of the flavor of the CEDS, but are entirely consistent with the theory developed in the previous chapter.[63] Indeed, the approach adopted by Barten and Theil admits of considerable generality, though the specific formulations are admittedly of an approximate nature.

To arrive at the Barten-Theil specification, consider the total differential of the demand functions:

$$dx_i = \left(\frac{\partial x_i}{\partial y}\right) dy + \Sigma_j \left(\frac{\partial x_i}{\partial p_j}\right) dp_j \qquad (i,j=1,\ldots,n)$$

If we denote by $\bar{D}x$ the logarithmic differential:

$$\bar{D}x \equiv d(\log x) \equiv \frac{dx}{x}$$

then we have the following equation, derived from the above equation by dividing by x_i and rearranging:

$$\bar{D}x_i = E_i \bar{D}y + \Sigma_j e_{ij} \bar{D}p_j \qquad (i,j=1,\ldots,n)$$

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[63] These models are the work of Barten [1964], [1967a], and Theil [1965], [1967]. The name "Rotterdam School" is due to Goldberger [1967].

where E_i and e_{ij} are as usual the income (Engel) elasticities, and the price (Cournot) elasticities, respectively. Note that if E_i and e_{ij} are considered as functions of income and prices (and not as parameters) then the above equation is completely general and is true for any specification of functional form of the demand equation. In fact, the subsequent discussion maintains this spirit of generality, which is abandoned only when we spcify which terms in the specification are assumed to be constant and hence assumed to be parameters instead of variables. It has been noted that the symmetry condition of Cournot elasticities is rather clumsy to handle. As before, then, we consider the "Slutsky form" of the above equation, which is obtained by substituting for e_{ij} from the Slutsky Relation, to give:

$$\bar{D}x_i = E_i(\bar{D}y - \Sigma_j\, w_j\bar{D}p_j) + \Sigma_j\, s_{ij}\bar{D}p_j \qquad (i,j=1,\ldots,n)$$

where the s_{ij} are the Slutsky price elasticities, and the w_j are the budget shares, as usual. Multiplying both sides of the equation by w_i we get the form that is fundamental to the Rotterdam School models:

$$(21) \qquad w_i\bar{D}x_i = w_iE_i(\bar{D}y - \Sigma_j\, w_j\bar{D}p_j) + \Sigma_j w_i s_{ij}\bar{D}p_j \qquad (i,j=1,\ldots,n)$$

We note, once again, that equation (21) is completely general and holds for any specification of the functional form. Several substantive assumptions are now made, which reflect in fact, the maintained hypotheses underlying the Rotterdam models. First, it is assumed that $\bar{D}s_i$, $\bar{D}y$, $\bar{D}p_j$ can be suitably approximated by the finite differences in their logarithms, Dx_i, Dy, Dp_j; where the operator D is defined by:

$$Dx_t \equiv (\log x_t) - (\log x_{t-1})$$

where x_t, and x_{t-1} are the observations on the variable x in the periods t, and t-1, respectively. More importantly, it is assumed that the income elasticity, E_i, depends in the following manner upon prices and income:

$$w_i\ E_i = b_i, \text{ where } b_i \text{ is a constant;}$$

which is to say that the income elasticity of demand for all goods in a rectangular hyperbola when plotted against the budget share of the commodity. This is, in fact, the assumption which renders the Rotterdam models theoretically plausible, in contrast to the CEDS.

It may further be assumed that $(w_i s_{ij}) = c_{ij}$, another constant. In this version, the Rotterdam model may be written as:

$$(22)\quad w_i Dx_i = b_i(Dy - \Sigma_j\ w_j Dp_j) = \Sigma_j\ c_{ij}\ Dp_j \qquad (i,j=1,\ldots,n)$$

It is easy to see that the following restrictions on the parameters apply:

Engel Aggregation: $\Sigma_i \; b_i = 1$

Symmetry: $c_{ij} = c_{ji}$ (for all $i \neq j$)

Homogeneity (Slutsky): $\Sigma_j \; c_{ij} = 0$ (for all $i=1,\ldots,n$)

In practice, the first two conditions are imposed in estimation; while homogeneity is imposed by deflating all prices by the nth (or any other) price, and estimating the remaining (n-1) equations. The parameters of the remaining equations can then be estimated by using the restrictions given above.[64] In this specification there has been no appeal to the cardinal properties resulting from additivity type hypotheses. Substantial economies of parametrization result from decomposing the Slutsky price elasticity into the Frisch-Houthakker "specific" and "general" substitution effects discussed in Cahpter 1.

Before doing this, however, we note that in practice it is desirable to alter equation (22) in two ways. First, the budget shares, w_i, are usually replaced by the arithmetic means of the budget shares, w_i^*, given by:

[64] With slight modification, this is the procedure used by Parks [1969]. The modifications, which are almost always made, consist of using the (backward) arithmetic means of the budget shares instead of the ordinary budget shares; and using an average of the logarithmic difference of the quantities, weighted by the arithmetic means of budget shares, for the real income term given by $(Dy-\Sigma_j w_j Dp_j)$. These modifications are discussed below.

$$w^*_{it} = 1/2(w_{it} + w_{i,t-1}).$$

Also, in order that the equations add up, the real income term, $(Dy-\Sigma_j w_j Dp_j)$ is replaced by Dx_t, where

$$Dx_t = \Sigma_j w^*_{jt} Dx_{jt}$$

The motivation for the latter operation is as follows. The term, $(Dy_t - \Sigma_j w_{jt} Dp_{jt})$, is a local quadratic approximation to the log-change in the "true index" of real income. This is also true of Dx_t. Now the difference between these two approximations is of the third-order in the logarithmic differences of prices.[65] Thus, it is not unreasonable to replace the former term by the latter, when in doing so the equations can be made to satisfy the adding-up criterion. With these changes, equation (22) can be written in the form in which Parks [1969], for example, has estimated them:

$$(23) \quad w^*_{it} Dx_{it} = b_i Dx_t + \Sigma_j c_{ij} Dp_{jt} \qquad (i,j=1,\ldots,n)$$

where the t subscript refers to the observations in the t-th period.

The Barten-Theil approach, however, does not stop here. The Slutsky elasticity, s_{ij}, may be decomposed as follows:

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[65] This is proved by Theil [1967, pp. 223-226], where a detailed discussion of these issues is presented.

$$s_{ij} = n_{ij} - \phi E_i E_j w_j$$

as can be seen from equation (9), and the definitions of $s_{ij} \equiv \frac{p_j}{x_i} K_{ij}$. Thus, we have:

$$\begin{aligned} c_{ij} &= w_i \, s_{ij} \\ &= w_i n_{ij} - \phi w_i E_i E_j w_j \\ &= m_{ij} - b_i b_j \phi \end{aligned}$$

where by m_{ij} we denote the product of w_i and n_{ij} which is assumed constant, and the b_i, b_j, are as above, and are called the "marginal budget shares" of the commodities i and j, respectively. Note that the constancy of the m_{ij} implies that the specific substitution effects are assumed to vary inversely (as a rectangular hyperbola) with the budget shares of the commodity whose quantity is affected. Substituting for c_{ij} into (23) we get:

$$w^*_{it} Dx_{it} = b_i \, Dx_t + \Sigma_j \, (m_{ij} - \phi b_i b_j) Dp_{jt} \quad (i,j=1,\ldots,n)$$

Before proceeding to simplify this equation, we note the parametric restrictions implied by the theory developed in the previous chapter. First, Engel aggregation requires, as before, that $\Sigma b_i = 1$. Secondly, recalling that by definition $n_{ij} \equiv \lambda u^{ij} p_j / x_i$, where λ is the marginal utility of income and u^{ij} is the (i,j) the element of the inverse of Hessian, U, of the utility function, we see that:

$$m_{ij} = \frac{\lambda p_i p_j u^{ij}}{y}$$

so that, $m_{ij}=m_{ji}$ for all i,j. Thirdly, the following condition is obvious from equation (8):

$$\Sigma_j m_{ij} = \frac{p_i}{y} \Sigma_j \lambda u^{ij} p_j$$

$$= \frac{p_i}{y} \frac{\lambda}{\lambda_y} \frac{\partial x_i}{\partial y}$$

$$= \phi b_i$$

Which, in turn, implies the final condition:

$$\Sigma_i \Sigma_j m_{ij} = \phi < 0,$$

where, the inequality is based upon the negative difiniteness of the U matrix.

The fact that $\Sigma_j m_{ij} = b_i$, can be used to simplify the above equation further:

(24) $w^*_{it} Dx_{it} = b_i Dx_t + \Sigma_j m_{ij} (Dp_{jt} - \Sigma_k b_k Dp_{kt})$

Equation (24) gives the functional form that is fundamental to the Rotterdam models. An interpretation of the various terms is readily available. The parameters, b_i, and m_{ij}, are the "marginal budget share" of the ith commodity, and the "specific substitution effect" on the ith commodity of a change in the price of the jth commodity, respectively. The term, Dx_t, is an approximation to the log-change

in real income, $(Dy_t - \Sigma_j w^*_{jt} Dp_{jt})$, where the latter is the difference between the log-changes in absolute income and the "cost of living price index." The price term, $(Dp_{jt} - \Sigma_k b_k Dp_{kt})$, may also be interpreted readily. This is, in fact, the log-change in the relative price of the jth commodity; and is obtained by subtracting the log-change in the "marginal price index" from the log-change in the price of the jth commodity.[66]

Denoting by Dp'_{jt}, the log-change in the relative price of the jth commodity, i.e., $(Dp_{jt} - \Sigma_k b_k Dp_{kt})$, we may summarize all of the above as follows:

The Non-stochastic Rotterdam Model:

$$(25) \quad w^*_{it} Dx_{it} = b_i Dx_t + \Sigma_j m_{ij} Dp'_{jt}$$

where, $$w^*_{it} = 1/2(w_{it} + w_{i,t-1})$$

$$Dx_t = \Sigma_i w^*_{it} Dx_{it}$$

and, $$Dp'_{jt} = (Dp_{jt} - \Sigma_k b_k Dp_{kt}).$$

In addition, the parameters are restricted as follows:

$$\Sigma_i b_i = 1$$

$$m_{ij} = m_{ji} \qquad \text{(for all i,j)}$$

$$\Sigma_j m_{ij} = \phi b_i \qquad \text{(for all i=1,...,n)}$$

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[66] These interpretations are given, for example, by Theil [1967, pp. 197, 200-201].

and $$\Sigma_i \Sigma_j m_{ij} = \phi < 0.$$

If the number of commodities is n, the total number of free parameters to be estimated, including ϕ, is $1/2n(n+1)$. Note that in the case of direct additivity, $m_{ij} = 0$ for $i \neq j$. Thus, in this case, the number of free parameters reduces to n. In the case of the existence of a partition into groups, with respect to which the utility function is strongly separable, i.e., in the case of "block-independent" preferences, it is easy to show that the number of free parameters to be estimated is given by $1+1/2 \sum_{g=1}^{G} n_g(n_g+1)$, where G is the total number of groups, and n_g is the number of commodities in group g.

In the highly pragmatic spirit that characterizes the Rotterdam approach, Barten and Theil have explored a concise stochastic specification of this model. This has been termed the "Marginal Utility Shock Model" and consists of assuming the existence of a stochastic component in the first order conditions of utility-maximization. The implications of this specification have been explored in detail by Theil [1967], and Barten [1968],[67] in the context

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[67]One of the first attempts to specify in some detail the stochastic structure of demand relationships is that of Theil and Neudecker [1957], and predates the marginal utility shock model. However, the models presented in that paper are more complicated, and admittedly less elegant. See Theil [1967, p. 228, n. 2].

of a quadratic utility function. Since the derivation of the actual estimator proposed by Theil [1964] is somewhat complicated, we present a brief outline of the procedure adopted.

The stochastic specification of the functional form for the demand equations is given by assuming that an additive disturbance term appears in equation (25) given above. Thus, we have:

$$(26) \qquad w^*_{it}\ Dx_{it} = b_i\ Dx_t + \Sigma_j\ m_{ij}\ Dp'_{jt} + v_{it}$$

Note that by our definition of the variable, Dx_t, we have

$$\Sigma_i\ v_{it} \ = \ 0.$$

This means that the disturbances are not independent, so that the covariance matrix of the disturbance term is singular. To cope with this problem, a more specific formulation of the nature of the disturbance term is needed. The marginal utility shock model consists of the specification that the utility function is of the following quadratic form:

$$u = u(x_1,\ldots,x_n) = 1/2\ \Sigma_i\Sigma_j\ x_i x_j u_{ij} + \Sigma_i\ a_i x_i$$

where the a_i are assumed to be random variables.[68] This results in the following first order conditions:

$$u_i(x) = \Sigma_j x_j u_{ij} + a_i = \lambda p_i \qquad (i,j=1,\ldots,n)$$

where the marginal utilities are stochastic, with x_j, a_i, and stochastic; u_{ij} fixed. In addition, the budget constraint is assumed to hold in a non-stochastic manner.

To avoid random coefficients in the demand equations, we replace λ, by $E(\lambda)$, in the definition of the parameters m_{ij}; and redefine m_{ij} as follows:

$$m_{ij} = \frac{E(\lambda)\ p_i p_j u^{ij}}{y}$$

From this equation it is easy to solve for u_{ij} in terms of the m_{ij}, p_i, p_j, y, and $E(\lambda)$. To do so, let M denote the (nxn) matrix with m_{ij} as its (i,j)th element; and let P denote the (nxn) diagonal matrix with the elements of the (nx1) price vector P appearing on its diagonal. The above equation may be written in the following matrix form:

$$M = \frac{E(\lambda)}{y} PU^{-1}P$$

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[68] Theil[1967, p. 228] points out that the quadratic approximation suffices when considering "first-order effects in the neighborhood of the consumer's optimal point." A generalization of the above approach has been pointed out incidentally in a footnote by Pollak and Wales [1969, p. 617, n. 13], and has not been explored at all. Pollak and Wales suggest that the Theil-Barten approach is tantamount to assuming a stochastic utility function of the form $u = u(x)+\Sigma_i a_i x_i$.

where, once again, U refers to the Hessian matrix. Since it is assumed that prices are positive, P^{-1} exists; and hence, M^{-1} exists. We have therefore:

$$U = \frac{E(\lambda)}{y} PM^{-1}P$$

or in simple algebraic form,

$$u_{ij} = \frac{E(\lambda)}{y} (p_i p_j m^{ij})$$

where, m^{ij}, denotes the (i,j)th element of the matrix M^{-1}. By substituting this in the first order conditions derived above, we get:

$$\frac{E(\lambda)}{y} (\Sigma_j p_i p_j m^{ij} x_j) + a_i = \lambda p_i \qquad (i=1,\ldots,n)$$

or, alternatively, in terms of budget shares:

$$(\Sigma_j w_j m^{ij}) + \frac{a_i}{p_i E(\lambda)} = \frac{\lambda}{E(\lambda)} \qquad (i=1,\ldots,n)$$

Now consider random shocks (da_i) in the stochastic term of the expression for the marginal utility of good i. This will cause random shocks in the budget share w_i and the marginal utility of income, λ, which will be related to each other as follows:

$$(\Sigma_j m^{ij} dw_j) + s_i = \frac{d\lambda}{E(\lambda)} \qquad (i=1,\ldots,n)$$

where, s_i, is the ratio of the initial shift in the marginal utility to the expected value of the marginal utility

of good i, and is given by $\frac{da_i}{E(\lambda)}$. Now, dw_i is subject to the restriction: $\Sigma_i\ dw_i = 0$, by definition. Recalling that M is a symmetric matrix, with the sum of the ith row equal to $\Sigma_j\ m_{ij} = \phi b_i$; and the sum of all elements $\Sigma_i \Sigma_j\ m_{ij} = \phi$; we may solve the above (n+1) equations for dw_i, and $\frac{d\lambda}{E(\lambda)}$, in terms of m_{ij}, b_i, s_i, and ϕ, to give the following result:[69]

$$dw_i = -\Sigma_j\ (m_{ij} - \phi b_i b_j) s_j$$

Note that in this analysis prices and income have been assumed constant, so that the budget discrepancy, dw_i, is identically equal to the disturbance term in the ith demand equation. If we additionally assume that the random marginal utility shocks are uncorrelated with prices and income, then the result carries over to the case where income and prices do vary, as is the case in actuality. We therefore have:

$$v_{it} = -\Sigma_j\ (m_{ij} - \phi b_i b_j) s_{jt}$$

where the time subscripts have been attached, and in particular, s_{jt} will reflect the random shifts in the marginal utility of the jth good, over time.

Note the similarity between the expression for v_{it} and the price term in the demand equations (25):

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[69] For a complete derivation see Barten [1968, pp. 220-221] or Theil [1967, pp. 228-230].

$$\Sigma_j \, m_{ij} \, Dp'_{jt} \;=\; \Sigma_j \, (m_{ij} - \phi b_i b_j) \, Dp_{jt}$$

An interpretation of this similarity provides insight into the assumptions underlying the marginal utility shock model. In effect, the assumptions with regards to the stochastic disturbance terms in the demand equations can be summed up as follows. It is assumed that marginal utilities of goods are subject to a random shock, and that this random shock (apart from its sign) has the same effect on the dependent variable, as the effect of log-changes in relative prices. In fact, it is possible to decompose the effect of the random shock into a specific and a general component much like the effect of a price change. It should also be noted that with this definition the singularity of the covariance matrix of the disturbance terms has been built in within the stochastic structure. For we have:

$$\begin{aligned} - \; \Sigma_i \, v_{it} &= \Sigma_i \Sigma_j (m_{ij} - \phi b_i b_j) s_{jt} \\ &= \Sigma_j (\Sigma_i m_{ij} - \phi b_j) s_{jt} \\ &= \Sigma_j (\phi b_j - \phi b_j) s_{jt} \\ &= 0 \end{aligned}$$

To proceed with estimation, the specification of the first and second moments of the disturbance are required. This is done by a direct assumption of these

moments for a_i, which define the moments of s_i, which in turn define the moments of v_i. In particular the following assumptions are made about the moments of the disturbance terms:

$$E\ (v_{it}) = 0 \qquad \text{(for all i,t)}$$

$$\text{Cov}\ (v_{it}, v_{jt'}) = 0 \qquad \text{(for all i,j; and } t \neq t'\text{)}$$

$$\text{Var}\ (s_{it}) = \frac{\sigma^2}{\phi} m^{ii} \qquad \text{(for all i,t)}$$

$$\text{and, Cov}\ (s_{it}, s_{jt}) = \frac{\sigma^2}{\phi} m^{ij} \qquad \text{(for all i,t)}$$

This completes the stochastic specification. The implication for the covariance matrix of the disturbances, v_{it}, is as follows:

$$E\ (V_t V_t') = \frac{\sigma^2}{\phi}\ (M - \phi bb')$$

where, V_t is the (nxl) disturbance vector with elements (v_{it}); M is the matrix described above; and b is the (nxl) vector of the marginal budget shares, with elements (b_i). We note without proof that the covariance matrix defined above is positive semidefinite, with rank (n-l). The latter fact is in fact consistent with our requirement that $\Sigma_i v_{it} = 0$.

With these stochastic specifications, Theil [1964], has suggested an estimator for the covariance matrix of the disturbance term, which can be constructed with *a priori* specification of the M matrix of specific

substitution effects. This estimator is best linear unbiased under certain very restrictive conditions. Barten [1968, pp. 230ff.] discusses the properties, and the restrictiveness of the estimator proposed by Theil. He then applies this model to a four commodity breakdown of Dutch data on consumption covering the periods 1923-1939 and 1950-1961, which have also been used by Theil and Mnookin [1966] and Theil [1967]. Barten's [1964] *a priori* information is derived from the studies by Frisch [1959] and Houthakker [1965]. Utilizing Frisch's estimate of the marginal utility of income, and deriving estimates of marginal budget shares from Houthakker's estimates of income elasticities, the standard errors of these estimates are derived from an estimate of the covariance matrix. A knowledge of the covariance matrix of the disturbance is used to calculate the covariance matrix of the parameters. Theoretical plausibility of the model is then tested primarily by checking for the negative definiteness of the matrix of Slutsky terms, which was the only restriction which was not incorporated in estimation.

In conclusion, we note that the Rotterdam model does effect considerable economies of parametrization, but the nonlinearity in parameters results in making estimation extremely cumbersome.

3.4 Stone's Linear Expenditure System

A serious competitor to the double-logarithmic type of demand models discussed in the previous two sections, is the so-called "Stone's Linear Expenditure System," (SLES), proposed by Klein and Rubin [1948], and implemented extensively by Stone [1954a], [1954b], and elsewhere, and by Stone, Brown and Rowe [1964]. The functional specification of the demand functions is given in its non-stochastic form as:

$$x_i = c_i + \frac{b_i}{p_i}(y - \Sigma_k p_k c_k) \qquad (i=1,\ldots,n)$$

The "expenditure functions" corresponding to the demand functions above are given by:

$$p_i x_i = p_i c_i + b_i(y - \Sigma_k p_k c_k) \qquad (i=1,\ldots,n)$$

In this form, there is an immediate economic interpretation available for the behavioral implications of the SLES [Stone, 1954, p. 512]. The consumer may be thought to allocate his expenditures in a two-step fashion. He first buys the minimum required quantities of each good, c_i. The cost of this "subsistence bundle" may be termed "subsistence income," and is given by $\Sigma_k p_k c_k$. In the second step, the consumer is seen as allocating a fixed proportion, b_i, of his "supernumerary income," $(y-\Sigma_k p_k c_k)$, to his supernumerary consumption. On intuitive grounds,

this interpretation provides a most convincing motivation for the introduction of SLES in the estimation of demand relationships.

Additional desirable properties of SLES result from the fact that a utility interpretation is also available. Samuelson [1947b] and Geary [1949] have demonstrated that the SLES may be derived from the constrained maximation of the so-called "Stone-Geary utility function," which is a generalization of a specific form of the "Bergson family" of utility functions.[70] This utility basis of the SLES assures that the resulting demand functions are consistent with the classical utility-maximizing assumptions. Additionally, the utility basis may be identified as the source of the comparatively few parameters that occur in the SLES. These utility aspects of SLES are considered in the next section.

Purely from the functional form, however, it can readily be seen that the income (Engel) elasticities, and the (Slutsky) price elasticities are given by:

$$w_i E_i = b_i \qquad (i=1,\ldots,n)$$

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[70]The term "Bergson family" is due to Pollak [1967] and Samuelson [1965], and is used in reference to utility functions considered in Bergson [1936]. The indifference maps of these functions are identical to the isoquant maps of the "constant elasticity of substitution" production functions. This is noted by Pollak [1967, p. 3, n. 6].

and, $$w_i s_{ij} = \begin{cases} b_i - \phi\, b_i^2 & (i=j) \\ - \phi\, b_i b_j & (i \neq j) \end{cases}$$

where, $-\phi$ denotes the ratio of supernumerary income to the total income, and is given by:

$$-\phi = \frac{(y - \Sigma_k p_k c_k)}{y}$$

(This notation anticipates future results, where it will be shown that with a "Stone-Geary" utility function, the Frisch "money flexibility" parameter is, in fact, as defined above.) It is important to note that SLES assumes the b_i, and c_i to be constant, and hence the elasticities E_i, and s_{ij} appearing in the above equations are not constant with respect to variations in expenditures and prices. Additionally, SLES, like the Rotterdam models, assumes the constancy of the marginal budget shares, as opposed to that of income elasticities, and is, therefore, free of the inconsistencies resulting in CEDS.

The expression for the Slutsky price elasticities given above may be derived as follows. By direct differentiation of the demand equations, we can get the Cournot price elasticities:

$$e_{ij} = -(1 - \frac{c_i}{x_i})\ \delta_{ij} - b_i (\frac{p_j c_j}{p_i x_i}) \qquad (i,j=1,\ldots,n)$$

where, δ_{ij}, is Kronecker's delta. Utilizing the Slutsky relation, of section 1.5, it is easily seen that:

$$s_{ij} = -(1 - \frac{c_i}{x_i})\ \delta_{ij} + b_i(1 - \frac{c_j}{x_j})(\frac{w_j}{w_i}) \quad \text{(for all i,j)}$$

where, the w_i are budget shares as usual. The expression for s_{ij} is then derived from the above equation by noting that

$$(1 - \frac{c_i}{x_i}) = -\phi E_i$$

a result easily verified by manipulation of terms in the functional form for the demand equations under SLES.

With the expressions above we may examine the effect of the parametric restrictions implied by demand theory as derived in the first chapter. It is readily seen that Engel aggregation implies that the marginal budget shares sum to unity:

$$\Sigma_i\ b_i = 1.$$

This, in fact, is the only restriction on the parameters. For, as may be readily verified, the SLES satisfies the homogeneity condition, the adding-up condition, and the symmetry of the Slutsky terms implicitly. This is, indeed, the most attractive feature of SLES. For estimation, then, there is only one restriction. The number of free parameters to be estimated is, therefore, given by (2n-1), where n is the number of commodities. This is a considerably small amount in comparison with most other models.

Having noted how the parameters of the SLES relate to the conventional elasticities of demand, we proceed to examine the stochastic formulations available. Usually, it has been the custom to assume that an additive random variable appears in the non-stochastic formulation of the "expenditure" equation. Estimation is then carried out by an iterative method due to Stone [1962-66], and Stone, Brown and Rowe [1964], and reported in Malinvaud [1966, pp. 310-14]. The estimation procedure suggested by Stone ignored, however, the singularity of the covariance matrix implied by the adding-up criterion. A maximum likelihood procedure has been suggested by Parks [1968] which takes this problem into account. An alternative stochastic specification results from appending an additive disturbance term to the demand equation as opposed to the expenditure equation. This results in the non-constancy of the covariance matrix over time, and renders the estimation procedure suggested by Parks, ineffective. Pollak and Wales [1969] have explored this problem, and have considerably refined the specification of the error structure for the SLES, and have proposed another maximum likelihood procedure for estimation. We conclude this section with a brief discussion of the several models proposed, and briefly examine the motivation behind the estimation procedures suggested.

In the original formulations, the SLES, was given the following stochastic formulation:

$$p_i x_i = p_i c_i + b_i(y - \Sigma_k p_k c_k) + u_i \quad (i=1,\ldots,n)$$

where the u_i are independently and identically distributed random variables with zero mean and constant variance. The estimation of SLES is always accomplished by exploiting the fact that given the $b_i (i=1,\ldots,n)$, the SLES is linear in the c_i, and vice versa. Thus, using initial estimates of either of the n parameters, an iterative procedure is used to estimate the other, and so on, until some measure of convergence is reached. Of course, the estimation procedure must take into account the restriction that the b_i sum to unity. Fortunately, the definition of the variable y as $\Sigma_i p_i x_i$ ensures that the least squares coefficients b_i obtained by regressing $p_i x_i$ on y, will sum to unity. This is a well-known result in least squares estimation, and is examined in the context of SLES, for example, by Goldberger and Gamalestos [1967, pp. 28ff.] where the estimation procedure outlined above is discussed. We might note, that the estimation procedure given here is due to Stone [1954b], and has been the most extensively used procedure in the estimation of SLES.

Parks [1968] criticized the estimation procedure outlined above on the grounds that the constraint on the b_i's and the definition of the variable y, implies that

$\Sigma_i u_i = 0$. Thus, Stone's procedure does not take into account the singularity of the convariance matrix, which is implicit in the model. Parks has suggested that one equation of the system may be deleted (the choice of the equation is arbitrary, if the disturbance covariance matrix is constant over time, as is assumed) and the resulting system of (n-1) equations be estimated, either by a straightforward application of the Gauss-Newton procedure for nonlinear estimation, or by Stone's procedure applied to the reduced system of the (n-1) equations. Parks points out, however, that the Gauss-Newton procedure has the added advantage that it provides an estimate of the variance-covariance matrix of the estimates, while Stone's procedure results in estimates of the variance-covariance matrix of the n parameters, b_i, conditional on the previous estimates of the c_i, or vice versa. In addition, Parks' experience with Swedish data from 1861-1955 suggests that convergence is faster with the Gauss-Newton procedure.

Finally, it might be noted that if there is reason to believe that the disturbances may be serially correlated, then both the modified Stone's procedure, and the Parks' (Gauss-Newton) procedure may be readily extended along the lines suggested by Parks [1967], and Kmenta and Gilbert [1970]. Essentially, this is done by estimating the above equations after correcting for autocorrelation. The

correction consists of a weighted first-differencing of the variables, the weights being estimates of the first order autocorrelation parameter obtained from either the single-equation OLS residuals (Parks), or the ZEF residuals (Kmenta and Gilbert). Utilizing the former procedure, Parks [1969] has provided estimates of SLES fitted to Swedish data.

Pollak and Wales [1969], however, looked at a somewhat different specification of SLES. They specified that the random disturbance term, u_{it}, enters the demand functions additively, i.e., for $i=1,\ldots,n$; $t=1,\ldots,T$, the expenditure equations are:

$$p_{it}x_{it} = p_{it}c_i + b_i(y_t - \Sigma_k\, p_{kt}c_k) + p_{it}u_{it}$$

where the u_{it} are i.i.d. random variables with zero mean and finite variance. Thus, the composite error term of the expenditure equation, $v_{it} = p_{it}u_{it}$, is in general not constant over time. In addition, v_{it} does not retain the property of homoskedasticity. Thus, in the Pollak-Wales specification of SLES, Parks' procedure fails to produce maximum likelihood estimates of the parameters, due to the presence of heteroskedasticity in the disturbances.[71]

[71] It might be noted that there seems to be some confusion in the literature with regard to the uniqueness of the parameter estimates of SLES that result from utilizing the modified Stone's procedure, as suggested by Parks [1968]. Parks [1968, Appendix, p. A-1] states that the choice of the equation to be deleted "does not matter." Pollak and Wales [1969, pp. 618-619] claim, however, that

Pollak and Wales have developed an alternative estimation procedure in the context of "dynamic" models proposed by them. These dynamic models proposed by them are of the "distributed lags" variety, and are considerably more sophisticated than the initial dynamic specification that was suggested originally by Stone. The simplest dynamic specification, due to Stone, may be specified as follows:

$$b_{it} = b_i^* + b_i^{**}\, t$$

and, $$c_{it} = c_i^* + c_i^{**}\, t$$

where, b_{it}, c_{it}, replace the coefficients b_i, c_i in the original specification of SLES, to incorporate the belief that the parameters are subject to a linear trend, instead of the static assumption that they are constant over time. In this version, SLES has been estimated by Parks [1969].

Pollak and Wales object to the above specification on the grounds that a secular increase (assuming $a_i^{**}>0$) in

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the estimates depend on which equation is omitted." (This statement by Pollak and Wales is also cited in an unpublished paper by Murray Brown and Dale Heien, on page 18). Additional evidence for the uniqueness of estimates, i.e., the irrelevance of the choice of equation to be deleted, are provided by Powell [1969] in the context of the general (not Stone's) linear expenditure system. Powell has conclusively demonstrated that in any <u>linear</u> expenditure system of demand equations, the choice of equation to be deleted is arbitrary.

the subsistence bundles, c_i, over time, irrespective of prices and income is an implausible specification. In addition, it might be pointed out that this kind of an assumption for the marginal budget shares, b_i, is even more questionable, since the b_i cannot exceed unity. Instead, Pollak and Wales propose "habit formation" models with regard to c_{it}, while assuming b_{it} to be constant over time. In addition, the chief contribution of Pollak and Wales has been to develop a dynamic version of SLES, in which considerable attention is given to the specification of the structure of the disturbance terms. With their stochastic specifications, they develop a more satisfactory procedure for dealing with the implied singularity of the covariance matrix of the disturbance terms.

The Pollak-Wales linear expenditure system (PWLES) is derived from the non-stochastic specification of SLES as follows. Consider the non-stochastic SLES:

$$p_{it}x_{it} = p_{it}c_{it} + b_i(y_t - \Sigma_k\, p_{kt}c_{kt})$$

where the t-subscript on the c_i is appended to incorporate the belief that the subsistence bundles, c_i, change over time. The crucial assumption of the PWLES is that the subsistence bundle is a random variable, satisfying certain stochastic assumptions. Thus, PWLES specifies the exact nature of the disturbance terms appearing in each demand equation. More rigorously, PWLES assumes that:

$$e_{it} = e^*_{it} + u_{it} \qquad (i=1,\ldots,n;\ t=1,\ldots,T)$$

where the u_{it} are random variables, assumed to possess the following properties:

(i) $E(u_{it}) = 0$ (For all i,t)

(ii) $E(u^2_{it}) = \sigma^2_i$ (For all i,t)

(iii) $E(u_{it}u_{jt}) = 0$ (for all $i \neq j$, t)

(iv) $E(u_{it}u_{jt'}) = 0$ (for all i,j, $t \neq t'$)

Substituting the equation for e_{it} into the non-stochastic equation for SLES, given above, we get:

$$p_{it}x_{it} = p_{it}c_{it}^* + b_i(y_t - \Sigma_k\, p_{kt}c_{kt}^*) + p_{it}v_{it}$$

where, $p_{it}v_{it} = p_{it}u_{it} - b_i(\Sigma_k\, p_{kt}u_{kt})$.

The implications for the distribution of the v_{it} are easily derived from the specification of the moments of u_{it}. If the u_{it} are assumed to be multivariate normal, in addition, the v_{it} are also distributed as multivariate normal, with mean zero. The covariance between v_{it} from two different time periods is zero. However, under assumption (iv) the variance of the disturbances v_{it} are independent of income and quantities but are inversely related to prices. Since prices fluctuate, the covariance matrix of the disturbances may not be assumed constant over time. Pollak and Wales

consider (ii) to be implausible, and suggest an alternative specification:

$$\text{(ii)}' \qquad E(u_{it}^2) = \sigma_i^2 \, E(x_{it}^2) = \sigma_i^2 \, \hat{x}_{it}$$

where, (ii)' expresses the belief that the variance of the disturbance terms in the demand equations are larger for higher levels of consumption.

Finally, to complete the specification of PWLES, the nature of variations in a_{it} over time need to be specified. Pollak and Wales suggest the following alternative hypotheses:

$$\text{Hypothesis 1:} \qquad c_{it}^* = k_i + k_i^* \, t \qquad (i=1,\ldots,n)$$

$$\text{Hypothesis 2:} \qquad c_{it}^* = k_i + k_i^* \, x_{it-1}.$$

where the first hypothesis is of the Stone variety, while the second is the "linear habit formation" hypothesis which subsumes under itself the cases of constant c_{it} (over time), and the "proportional" habit formation hypothesis, which results when $k_i=0$ in Hypothesis 2.

Using these specifications, Pollak and Wales derived maximum likelihood estimators for the coefficients, and estimated demand for four commodities for the United States, using data from 1948-1965. They also estimated these models for prewar (1930-1941) data, and found significant differences in the underlying utility functions for the prewar and the postwar periods. Unfortunately, their estimates suggest that for postwar data the stochastic specification of variances of disturbances proportional to quantities is valid only for the "proportional" habit formation model under Hypothesis 2. However, the dynamic specification and the estimation technique were shown to affect the results of estimation significantly. We discuss these results in some detail here because the model to be proposed in Chapter 3 is also estimated using the same data.

3.5 Utility Aspects of SLES

Samuelson [1948] and Geary [1949] demonstrated that Stone's linear expenditure system could be derived from a constrained maximization of the so-called "Stone-Geary" utility function, which is given as:

$$u = u(x_1,\ldots,x_n) = \Sigma_i\ b_i\ \log(x_i - c_i)$$

where the function is defined for $(x_i - c_i)>0$, and for all $i=1,\ldots,n$, $0 \leq b_i \leq 1$, and $c_i \geq 0$. In addition, $\Sigma_i b_i=1$. Maximizing the Stone-Geary utility function subject to the budget constraint, gives rise to SLES:

$$p_i x_i = p_i c_i + b_i(y - \Sigma_k p_k c_k).$$

Thus, the *a priori* specification of SLES has been shown to be equivalent to the assumption that the consumers possess a definite pattern of preferences, which they attempt to maximize subject to their budget constraint. In addition, the utility basis of SLES is a source of additional insight into the behavioral implications of SLES.

The first point to be noted is that the Stone-Geary utility function is directly additive. Thus, without further analysis, the results of section 1.6 ensure that goods represented by SLES must be Hicks-Allen substitutes. Also, these goods cannot be inferior, nor can they be complements, nor specific substitutes in the Frisch-Houthakker sense. The absence of specific substitutability, inferiority, and complementarity suggest that SLES is a plausible specification only for broad aggregates of expenditure categories.

In addition, it is easy to show that for SLES the implied Engel curves are linear, and the own-price elasticities are less than unity in absolute value. The fact that Engel curves are linear is obvious from the constancy of the marginal budget shares b_i, which are defined as

$$b_i = p_i \left(\frac{\partial x_i}{\partial y}\right) \qquad (i=1,\ldots,n)$$

Thus, given a set of prices, the constancy of b_i ensures that the slope of the Engel curves, $\left(\frac{\partial x_i}{\partial y}\right)$, is also constant. The restriction on own-price elasticities is apparent

from the formulas derived in the last section. Recall that the Cournot own-price elasticities are given by:

$$e_{ii} = -(1 - \frac{c_i}{x_i}) - b_i(\frac{c_i}{x_i}) \qquad (i=1,\ldots,n)$$

$$= -1 + (1 - b_i)\frac{c_i}{x_i} \qquad (i=1,\ldots,n)$$

Now, since $0 \leq b_i \leq 1$ $(i=1,\ldots,n)$, the own-price elasticities, e_{ii}, lie between zero and minus unity. An examination of the similar expression for the Slutsky own-price elasticities will reveal that they are subject to the same upper and lower bounds. These results imply, therefore, that SLES is a valid specification only for goods whose demand is "price-inelastic" in the usual sense. Further, the linearity of the Engel curves implies that the model should be applied only to those samples for which the sample variance of income is relatively small.[72]

The linearity of the Engel curves also result in an interesting analysis of the behavior of the budget shares as income varies. These results due to Goldberger [1967, pp. 53 ff.] may be derived as follows. Dividing the expenditure equation under SLES, by income, y, we get

$$w_i = (p_i c_i)/_y + b_i(y - \Sigma_k\, p_k c_k)/_y \qquad (i=1,\ldots,n)$$

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[72]These points are made by Stone [1965, p. 275], and Goldberger [1967, pp. 61ff.].

where w_i denotes the (average) budget share of the ith good, as before. This may be written as

$$w_i = (1 - \alpha)\, w_i^* + \alpha\, b_i$$

where, $w_i^* = \dfrac{p_i c_i}{\Sigma_k\, p_k c_k}$, the "subsistence budget shares"

and, $\alpha = (y - \Sigma_k p_k c_k)/_y$, is the ratio of supernumerary income to income. It is easy to see, then, that in SLES the budget shares are weighted averages of the "subsistence" budget shares and the "marginal" budget shares for each commodity.

Further, the weights appearing in the expression for w_i may be identified as the Frisch "money flexibility" function. To see this, substitute the SLES demand functions into the Stone-Geary utility function to get the indirect utility function associated with the Stone-Geary function:

$$u = u^*(p_1, \ldots, p_n, y)$$

$$= \Sigma_i\, b_i \log b_i \;+\; \log(y - \Sigma_i\, p_i c_i) - \Sigma_i\, b_i \log p_i$$

Thus, the marginal utility of income is given by

$$\lambda = (y - \Sigma_k\, p_k c_k)^{-1}$$

The inverse of the income elasticity of the marginal utility of income, which by definition is the "money flexibility," ϕ, is easily seen to be

$$\phi = - \frac{(y - \Sigma_k p_k c_k)}{y} = \frac{\partial(\log \lambda)}{\partial(\log y)}^{-1}$$

Thus, the ratio of supernumerary income to income, apart from its sign, is identical to the α above. In view of this, we may write the final expression for the (average) budget shares under SLES

$$w_i = (1 - |\phi|) w_i^* + |\phi| b_i \qquad (i=1,\ldots,n)$$

Noting first that w_i and w_i^* sum to unity, and secondly that w_i, w_i^*, $|\phi|$, and b_i all lie between zero and unity; an interesting implication of the response of budget shares to changes in real income, becomes available. We see that the (average) budget shares under SLES are bounded from below by the "subsistence" budget shares, and from above by the marginal budget shares. Their proximity to the upper and lower bounds is determined by the value of $|\phi|$ for the particular levels of prices and income. As income rises, with prices constant (or vice versa), $|\phi|$ tends to its upper limit of unity, and the (average) budget shares, w_i, tend towards the marginal budget shares, b_i. Similarly, a fall in real income due to changes in income or prices, results in $|\phi|$ to fall towards its lower limit of zero, and the (average) budget shares approach the "subsistence" budget shares.[73]

[73]The fact that the Frisch "money flexibility" function, lies between zero and unity in absolute value, and is the ratio of supernumerary income to total income,

In addition to the above properties, SLES has also the desirable property of aggregating perfectly over both individuals and commodities. This should be obvious from the section of aggregation in the previous chapter where we stated the conditions on Engel curves which were necessary and sufficient for aggregation. Thus, the linearity of Engel curves plays a crucial role in the existence of desirable properties in the linear expenditure system. Curiosly enough, it is this linearity of the Engel curves which constitutes one of the serious restrictions of SLES too.

Finally, we point out that a utility basis may also be provided for the Pollak-Wales modification of SLES, by a direct extension of the Stone-Geary function. Pollak and Wales [1969] have shown that their stochastic specifications for their dynamic demand functions are obtainable from a constrained maximization of the following stochastic utility function:

$$
\begin{aligned}
u &= u(x_1,\dots,x_n,v_1,\dots,v_n) \\
&= \Sigma_i\ b_i \log(x_i - c_i - v_i)
\end{aligned}
$$

where the v_i are random variables with a specified distribution. Maximizing this utility function subject to

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lends particular credance to Frisch's [1932], [1959], proposition that "money flexibility" be treated as an index of welfare.

the budget constraint, yields the PWLES, given in the previous section:

$$p_i x_i = p_i c_i + b_i (y - \Sigma_k p_k c_k) + v_i^*$$

where, $\quad v_i^* = p_i v_i - b_i (\Sigma_k p_k v_k)$.

Thus, much like SLES, the PWLES may also be justified on grounds of a specific utility function. Indeed, the Pollak-Wales approach towards a stochastic formulation of the consumer's utility function admits of somewhat more generality than the Rotterdam approach, which is the only other model whose stochastic underpinnings have been explored in detail. As Pollak-Wales [1969] point out, the "marginal utility shcok model" utilized by Theil and Barten (discussed in section 3 above) assumes that the stochastic terms enter the utility function in a specific manner, as follows:

$$u = u(x_1, \ldots, x_n) + \sum_{k=1}^{n} x_k \, v_k$$

where v_k are random variables.

3.6 General Linear Expenditure Systems (GLES)

Although Stone's linear expenditure system has attracted the most attention, other linear expenditure systems have also been investigated and estimated. These "general" linear expenditure systems were proposed by

Stone [1954b], and were in fact specialized by Stone to yield SLES. The procedure by which this was achieved was simply to impose the classical restrictions on the general linear expenditure system. In doing this, Stone demonstrated that GLES satisfied the classical restrictions implied by demand theory only if it had the form of SLES. An alternative approach has been to consider the GLES and impose (approximately) the classical restrictions on the demand function in estimation. This approach was adopted by Leser [1958], [1960], [1961], and subsequently by Powell [1965], [1966], and Powell, Hoa, and Wilson [1967]. Estimation of the linear expenditure system was analyzed definitively by Powell [1969]. As an introduction to these models it might be instructive to consider the method by which Stone [1954b] arrived at SLES from GLES.[74]

Stone [1954] considered the GLES,

$$\text{(GLES)} \ldots \quad p_i x_i = a_i + b_i y + \Sigma_k c_{ik} p_k \qquad (i=1,\ldots,n)$$

where the a_i, b_i, and the c_k are assumed to be constant. By direct differentiation and manipulation of terms it is easily verified that the income (Engel) elasticities, and the price (Cournot) elasticities are given by,

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[74]Our exposition follows Frisch [1954] due to his simplified presentation. Stone [1954], and Goldberger [1967, pp. 50-52] also present the same material.

$$E_i = \frac{b_i}{w_i} \qquad (i=1,\ldots,n)$$

and, $$e_{ij} = \frac{c_{ij}p_j}{x_i p_i} - \frac{p_j}{p_i}\,\delta_{ij} \qquad (i,j=1,\ldots,n)$$

where, w_i is the budget share as before, and δ_{ij} is Kronecker delta. It is easy to see, therefore, that the Slutsky price elasticities, s_{ij}, are given by

$$s_{ij} = \frac{w_j b_i}{w_i} + \frac{c_{ij}p_j}{x_i p_i} - \frac{p_j}{p_i}\,\delta_{ij} \qquad (i,j-1,\ldots,n)$$

The effect of the classical restrictions may now be examined. Homogeneity requires that

$$\Sigma_j\, e_{ij} + E_i = 0 \qquad (i=1,\ldots,n)$$

Substituting from above, we see that for the GLES, this implies that

$$\Sigma_j\, e_{ij} + E_i = -\left(\frac{a_i}{p_i x_i}\right) = 0 \qquad (i=1,\ldots,n)$$

Thus, GLES satisfies the "homogeneity" condition if and only if all the intercepts, $a_i = 0$. Similarly, the "adding-up" criterion, $\Sigma_i p_i x_i = y$, if fulfilled if and only if,

$$\Sigma_i a_i = 0, \quad \Sigma_i b_i = 1, \text{ and } \Sigma_j c_{ij} = 0 \ (i=1,\ldots,n)$$

The symmetry condition, $w_i s_{ij} = w_j s_{ji}$, requires,

$$w_j b_i + c_{ij}\frac{p_j}{y} - w_j\frac{x_i}{x_j}\,\delta_{ij} = w_i b_j + c_{ji}\frac{p_i}{y} - w_i\frac{x_j}{x_i}\,\delta_{ij}$$

Multiplying through by y, cancelling the last terms on each side, and substituting for $p_i x_i$ and $p_j x_j$ from GLES, we have after transfering all terms to the left hand side,

$$b_i(a_j + b_j y + \Sigma_k c_{jk} p_k) + p_j c_{ij} - b_j(a_i + b_i y + \Sigma_k c_{ik} p_k)$$
$$+ p_i c_{ji} = 0$$

or, $$(b_i a_j - b_j a_i) + \Sigma_k \; p_k(b_i c_{jk} - b_j c_{ik})$$
$$+ (p_j c_{ij} - p_i c_{ji}) = 0$$

Utilizing Kronecker deltas, we may write down equation (21) of Frisch [1954, p. 509]: (in our notation)

$$(b_i a_j - b_j a_i) + \Sigma_k \; p_k(b_i c_{jk} - b_j c_{ik}$$
$$+ c_{ik}\delta_{kj} - c_{jk}\delta_{ki}) = 0$$

where this equation holds for all i,j=1,...,n. Since this equation holds for all prices, both the first term, and the coefficient of the price term are identically zero:

$$(b_i a_j - b_j a_i) = 0, \qquad (i,j-1,\ldots,n)$$

and, $$\frac{c_{ik}}{\delta_{ki} - b_i} = \frac{c_{jk}}{\delta_{kj} - b_j} \qquad (i,j=1,\ldots,n)$$

The first condition implies that

$$a_i = b_i \overline{y} \qquad (i=1,\ldots,n)$$

while the second is true only if,

$$c_{ij} = (\delta_{ji} - b_i)\, \bar{x}_j \qquad (i,j-1,\ldots,n)$$

where $\bar{y}$, $\bar{x}_j (j=1,\ldots,n)$, are $(n+1)$ constants that characterize the demand equations.

Introducing these into the original expenditure equations under the GLES, we get

$$p_i x_i = b_i y + \Sigma_k (\delta_{ki} - b_i) p_k \bar{x}_k \quad (i=1,\ldots,n) \quad (\Sigma b_i = 1)$$

$$= p_i \bar{x}_i + b_i (y - \Sigma_k p_k \bar{x}_k) \quad (i-1,\ldots,n) \quad (\Sigma b_i = 1)$$

which is Stone's linear expenditure system, SLES, with c_k[75] replacing $\bar{x}_k$ in our previous notation. Thus, it has been shown that Stone's LES is the only form of the GLES which satisfies the homogeneity condition, the adding-up criterion, and the symmetry conditions. Indeed, the derivation above points quite forcefully towards the power of the classical restrictions.

The impact of this, however, is to make the model nonlinear in parameters, and hence rob the GLES of its single most attractive feature: linearity. The estimation of SLES is quite cumbersone, as we have seen, and has been satisfactorily analyzed only recently. Leser [1958],

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[75]A single-subscripted c_k refers to the subsistence consumption level of the k-th commodity, in conformity with our previous notation; and should not be confused with the double subscripted c_{ik} which were arbitrary coefficients in the GLES.

therefore, examined the GLES with the view of maintaining the convenient parametric linearity of the system at the expense of only an approximate enforcement of the classical restrictions. In Leser's linear expenditure system, (LLES), the expenditure functions are taken to be of the GLES form, initially:

$$p_i x_i = b_i y + \Sigma_j a_{ij} p_j \qquad (i=1,\ldots,n)$$

where, it is understood that only the "adding-up" criterion is to hold globally, while the homogeneity and symmetry conditions are to hold only at the sample means of expenditures and prices.

It is apparent that the adding-up criterion will be met for all values of the independent variables if and only if

$$\Sigma_i a_{ij} = 0 \qquad (j=1,\ldots,n)$$

and $\quad \Sigma_i b_i = 1.$

(The latter condition, incidentally, insures that Engel aggregation conditions are satisfied globally). To achieve further economy of parametrization, Lese introduces the Hicks-Allen "elasticity of substitution" between goods i and j, a^*_{ij}, defined as

$$w_j a^*_{ij} = s_{ij} \qquad (i,j=1,\ldots,n)$$

where s_{ij}, and w_j are the Slutsky price elasticity and the budget share of the jth commodity, respectively. Using the Slutsky relation, it is easily verified that the Slutsky price elasticity for the GLES is given by

$$s_{ij} = p_j a_{ij}/v_i - \delta_{ij} + w_j b_i/w_i \qquad (i,j=1,\ldots,n)$$

where, v_i denotes the expenditure, $p_i x_i$, on the ith good. By a suitable rearrangement of terms, and substitution from the expression for a^*_{ij}, we have

$$a_{ij} = \bar{w}_i \bar{x}_j a^*_{ij} - b_i \bar{x}_j + \bar{x}_i \delta_{ij} \qquad (i,j=1,\ldots,n)$$

which is derived by Leser [1960, p. 105] and is cited by Powell [1969, p. 921, equation (A.3)]. The equation is understood to hold at sample means of budget shares and quantities, $\bar{w}_i$, and $\bar{x}_i$, respectively. Substituting this in the GLES, we have

$$p_i x_i = p_i \bar{x}_i + b_i (y - \Sigma_j p_j \bar{x}_j) + \Sigma_j a^*_{ij} (\bar{w}_i p_j \bar{x}_j)$$

$$(i=1,\ldots,n)$$

Leser's linear expenditure system (LLES) is derived from this equation by noting the fact that the demand functions will be homogeneous (at the sample means) if and only if

$$\Sigma_j \; w_j a^*_{ij} \;=\; 0 \qquad (i=1,\ldots,n)$$

Substituting for a^*_{ii} from this condition we have, for LLES

$$(\text{LLES}) \ldots \quad p_i x_i = p_i \bar{x}_i + b_i (y - \Sigma_j p_j \bar{x}_j)$$
$$+ \sum_{j \neq i} a^*_{ij} (\bar{w}_i p_j \bar{x}_j - \bar{w}_j p_i \bar{x}_i)$$

which is given by Leser [1960, p. 105, equation 2]. Unfortunately, LLES is still quite rich in parameters, so that for purposes of estimation, Leser arbitrarily equated all cross-elasticities of substitution between goods, $a^*_{ij} (i \neq j)$. With this assumption, it is readily seen that LLES has (2n+1) free parameters. Also, it has been shown that LLES satisfies the adding-up criterion globally, and the homogeneity and symmetry conditions at the sample means. Estimation of the model is discussed by Leser, but the definitive solution is provided by Powell [1969]. Powell's procedure takes account of the implied singularity of the covariance matrix of the disturbances. A generalized Aitken [1935] type of estimator is derived by Powell with the use of the Moore-Penrose generalized inverse for a matrix of less than full rank. We discuss briefly both Leser's procedure and the one suggested by Powell.

Having equated the cross elasticities of substitution, Leser was confronted by a linear model in which one parameter, $a^*_{ij} = a^*$ (identical for all $i \neq j$), occurred in all of the equations. Leser [1960, pp. 107ff.] adopted the procedure of accepting those estimates of a* which

minimized certain arbitrary linear combinations of residual sums of squares from the n fitted equations. Thus, denoting $v_i = p_i x_i$, Leser obtained estimates of the parameters in LLES, which minimized:

$$S = \Sigma_i A_i \Sigma_t (v_{it} - \hat{v}_{it})^2$$

where, A_i are arbitrary constants, and $\hat{v}_i$ represent the estimated values of v_i. To derive actual estimates, Leser proposed the following two assumptions:

(i) $A_i = 1$ (for all i)

or, (ii) $A_i = 1/\Sigma_t (v_{it} - \hat{v}_{it})^2$

which correspond respectively to the minimization of the total sum of squares, and to the maximization of the sum of the R^2s. Powell [1966, p. 665, n. 3] subsequently credits a referee for pointing out that only under criterion (i) are the estimates of the parameters linear. Utilizing this procedure, Leser obtained the estimates of a*. Estimates of the other parameters were obtained in a similar fashion in a second round. It was noted that the least squares procedure ensured that $\Sigma_i b_i = 1$; and $\Sigma_i t_i = 0$, where t_i were coefficients of the time trend variable.

Powell [1965] has proposed an alternative estimation procedure for LLES which takes also into account the fact that the classical restrictions imply that the covariance matrix of the disturbance terms is singular. Although

Powell considers the general LLES, (a^*_{ij} unequal for $i \neq j$), the "restricted" LLES under consideration may be estimated by deleting one equation (the choice is arbitrary) and using a restricted ($a^*_{ij} = a^*$, all $i \neq j$) Zellner's [1962] two-step procedure. Recovering the coefficients of the deleted equation by parametric restrictions, this estimation procedure leads to efficient and unbiased estimates under appropriate assumptions. The details of the procedure parallel the discussion in Chapter 4 for the model proposed in this paper.

On another front, Powell [1965], and elsewhere, proposed a modification of LLES. The motivation behind Powell's model lay in the dissatisfaction with Leser's procedure of arbitrarily equating the cross elasticities of substitution. Indeed, from a strictly theoretical point of view, Frisch [1959] has questioned the use of the elasticity of substitution as a parameter in demand models, due to its peculiar disadvantage of tending towards infinity as budget shares become small. (Thus, Leser's [1960] estimate of 0.5 as the value of the elasticity of substitution seems questionable when the mean budget shares for the nine commodities under question are reported between 0.020 and 0.259 [Leser, 1960, p. 108]). Of course, the advantage of using the elasticity of substitution as a parameter lies in its symmetry.

To achieve symmetry, however, Powell took the approach of utilizing results from the theory of additive preferences. Thus, if it may be assumed that the class of utility functions which give rise to the GLES have as a member an additive utility function, then the results of section 1.6 of the previous chapter may be used in achieving a more parsimonious parametrization of the GLES. This, in fact, was Powell's approach. To derive his "model of additive preferences" (PMAP) consider the GLES:

$$p_i x_i = b_i y + \Sigma_j \, a_{ij} p_j$$

where, the intercept (not used by Powell), and an additive time trend term (used by Powell) are omitted for simplicity. With the above specification, it is easy to see that

$$p_i \frac{\partial x_i}{\partial y} = b_i \, , \text{ and } \; p_i \frac{\partial x_i}{\partial p_j} = a_{ij} - x_i \delta_{ij} \qquad \text{(all } i,j\text{)}$$

An expression for a_{ij} may be derived as follows:

$$a_{ij} = p_i \frac{\partial x_i}{\partial p_j} + x_i \delta_{ij} \qquad (i=1,\ldots,n)$$

$$= p_i (K_{ij} - x_j b_i / p_i) \qquad \text{(for } i \neq j\text{)} \text{ (where, } K_{ij} \text{ is the Slutsky term).}$$

$$= k \, b_i b_j / \bar{p}_j - \bar{x}_j b_i \qquad \text{(for } i \neq j\text{)}$$

where, $k = -\frac{\lambda}{\lambda_y}$,

and use has been made of the fact that under additivity

$$K_{ij} = -\frac{\lambda}{\lambda_y}\left(\frac{\partial x_i}{\partial y}\right)\left(\frac{\partial x_j}{\partial y}\right)$$

Note, however, that the consequences of additivity are not imposed globally, but are instead assumed to hold only at the sample means of the prices and quantities.

Also, the adding-up criterion is met only if

$$\Sigma_i \Sigma_j \; a_{ij} p_j = 0.$$

Which implies, since prices are arbitrary, that

$$\Sigma_i \; a_{ij} = 0.$$

Using this relation, we deduce:

$$a_{ii} = (b_i - 1)(kb_i/\overline{p}_i - \overline{x}_i),$$

where, we have used the expression for a_{ij} above. Finally, we may substitute these expressions for a_{ij} into the original formulation of the expenditure equation under GLES, to get Powell's "model of Additive preferences" (PMAP):

$$\text{(PMAP)} \ldots \; p_i x_i = p_i \overline{x}_i + b_i (y - \Sigma_j p_j \overline{x}_j)$$

$$+ kb_i \Sigma_j b_j (p_j/\overline{p}_j - p_i/\overline{p}_i)$$

In this version, an interpretation of PMAP becomes available. Given a set of income and prices the consumer is assumed to pruchase "typical" quantities of goods, $\bar{x}_i$ (i=1,...,n). The consumer then allocates his "supernumerary income," $(y - \Sigma_j p_j \bar{x}_j)$, proportionally among goods in accordance with their respective marginal budget shares, b_i. This is parallel to the behavioral interpretation given for SLES with the "subsistence bundles," c_i, of SLES playing the role of the "typical bundles," $\bar{x}_i$, of PMAP. Unlike SLES, however, PMAP assumes that the consumer adjusts his allocation of his supernumerary income to account for substitution effects resulting from price changes. This price response is given by the third term in PMAP, which, incidentally, sums to zero across equations, and thus ensures that the adding-up criterion is met for all prices and incomes.[76] Powell [1966, p. 663], further notes that if the two terms, $(y - \Sigma_j p_j \bar{x}_j)$, and $(p_i x_i - p_i \bar{x}_i - kb_i \Sigma_j b_j (p_j/\bar{p}_j - p_i/\bar{p}_i))$, are deflated by the ith price, p_i, then (in terms of the purchasing power of the ith good) we would obtain, respectively, an index of "real" supernumerary income, and a quantity index for the ith good.

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[76]This interpretation, and much of the subsequent discussion of PMAP, relies to a great extent on the exposition of PMAP given by Goldberger [1967, pp. 95-101], alongwith Powell [1966, pp. 663ff.].

From a statistical viewpoint, PMAP possesses considerably more attractive properties than SLES. Firstly, the introduction of sample mean values (observable) instead of the "minimum" or "subsistence" values, c_i, reduces by n the number of parameters to be estimated for PMAP, as compared to SLES. Thus, the number of total free parameters to be estimated under PMAP is seen to be just n. In other respects, however, PMAP retains the same assumptions as SLES. In particular, the behavior of price and income elasticities with respect to variations in expenditures and prices are assumed to be identical under the two specifications. It might be noted, however, that PMAP assumes the parameter "k" to be constant. This implies that Frisch's "money flexibility" (or the inverse of the income elasticity of the marginal utility of income) when plotted against income yields a rectangular hyperbola, at mean values of prices and income. This is due to the equality of k with $(-\phi y)$ at sample means. SLES, on the other hand, defines $(-\phi y)$ to be identical to "supernumerary income," a crucial variable. Finally, it has been noted that PMAP imposes the classical restrictions only approximately. Thus, PMAP is not strictly consistent with utility-maximizing behavior.[77]

[77] It ought to be pointed out, however, that Goldberger [1967, pp. 99ff.] has shown that PMAP may be derived from a constrained maximization of the Stone-Geary utility function subject to the additional restrictions that $(\overline{x}_i - kb_i/\overline{p}_i) = c_i (i=1,\ldots,n)$; where c_i are parameters of

Evaluating PMAP with respect to ease of estimation, we see that PMAP is non-linear in parameters like SLES. Thus, the estimation of PMAP is carried out by iterative procedures described under SLES. The comments of that section apply to the estimation of PMAP.

3.7 Other Models of Consumer Demand

In the previous sections we have explored the most widely utilized empirical models in demand theory. There remain, however, an infinite variety of models that may be derived from any utility function which satisfies the several properties necessary for qualifying as a utility function. In this section we consider three specific models of consumer demand that are a result of three famous utility functions proposed in demand theory. These are the Quadratic utility demand models, the "direct addilog," and the "indirect addilog" utility models of demand. Although all three are consistent with utility maximizing behavior, and hence satisfy the classical constraints, yet on intuitive and empirical grounds these models leave much to be desired. Their main contribution lies in providing additional insight into theoretically plausible models of consumer demand.[78]

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the Stone-Geary function identified under SLES as subsistence bundles.

[78] Our discussion relies on Goldberger [1967] and Houthakker [1960].

Quadratic Utility:

The "Quadratic utility function" is given by

$$\text{(QUF)} \ldots \; u = u(x_1,\ldots,x_n) = \Sigma_i \; a_i x_i \; - \; 1/2\Sigma_i\Sigma_j \; b_{ij}x_i x_j$$

where the a_i and the b_{ij} are constants. Although the Quadratic utility function was considered by Allen and Bowley [1935], it received little attention subsequently in the analysis of consumer demand, with the notable exception of Rotterdam models where it was discussed by Theil [1967, pp. 183-188, 228-229], who used it as an illustration.

To facilitate exposition, we adopt the following notation:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

where a is an (nxl) vector, and B an (nxn) pos. definite matrix of coefficients. The Quadratic utility function may then be given in its matrix form by

$$\text{(QUF)} \ldots \quad u = u(x) = a'x - 1/2 \; x'Bx$$

where, x, is the (nxl) vector of quantities defined in

Chapter 1. By the usual methods adopted before, it is easy to see that the demand functions are given by:

$$x_i = \Sigma_j a_j b^{ij} - (\Sigma_i \Sigma_j b^{ij} p_i p_j)^{-1} (\Sigma_i \Sigma_j a_j b^{ij} p_i - y) \Sigma_j b^{ij} p_j$$

where, b^{ij} refers to the (i,j)th element of B^{-1}. (For details of the derivation, Goldberger [1967, p. 73ff.] may be consulted.) From this we see that the expenditure functions under the Quadratic utility specification are given by:

$$p_i x_i = p_i \Sigma_j a_j b^{ij} - (\Sigma_i \Sigma_j b^{ij} p_i p_j)^{-1} (\Sigma_i \Sigma_j a_j b^{ij} p_i - y) p_i \Sigma_j b^{ij} p_j$$

or in matrix notation:

$$\hat{p}x = \hat{p}B^{-1}a - (p'B^{-1}p)^{-1}(p'B^{-1}a - y)\hat{p}B^{-1}p$$

where, $\hat{p}$, denotes the (nxn) matrix whose off-diagonal elements are zeros, and the ith diagonal element is p_i.

An ingenious, but unreasonable interpretation may now be offered with regard to the behavior of the consumer reflected by the expenditure functions given above. To do so we note that the marginal utility of income function may be derived to be:

$$\lambda = (p'B^{-1}p)^{-1}(p'B^{-1}a - y)$$

So that the positivity of the marginal utility vector u_x ensures that:

$$y < p'B^{-1}a.$$

This means that $p'B^{-1}a$ may be thought of as maximum attainable income, or "bliss income." If actual income, y, were equal to the "bliss income" the consumer would buy the "bliss bundle" of goods, given by $B^{-1}a$. Thus, an interpretation for the consumer's behavior may be offered as follows. The consumer is thought to "buy" the "bliss bundle" conceptually, but since his income is below the "bliss income" he "sells back" the goods, receiving a fixed proportion, $p_i(\Sigma_j b^{ij}p_j/\Sigma_i\Sigma_j b^{ij}p_ip_j)$, of his "infra-bliss deficit" of, $(\Sigma_i\Sigma_j a_j b^{ij}p_i - y) > 0$, in return [Goldberger, 1967, pp. 74-75]. This interpretation is (admittedly) at best tenuous. A further discussion of the Quadratic utility model is omitted due to its limited relevance for empirical work. We note, however, the principle results.

It is relatively easy to show that under the Quadratic utility function, the demand functions may give rise to negative quantities for some income-prices. Also, it is possible that some goods possess negative marginal utilities at low income levels while the opposite holds true for other goods. From the demand functions, we see that the marginal budget shares are constant for given prices. Thus, the Engel curves are linear. However, under the Quadratic utility model, the marginal budget shares are not bounded between zero and unity, (although they sum to unity as usual). This is because the elements of $B^{-1}p$ may be negative. The possibilities of negative marginal budget shares also implies that inferior goods are permitted under the Quadratic utility function. An additional property of this function concerns the perverse behavior exhibited by the Frisch money flexibility function. In the case of Quadratic utility, for a given set of prices, the money flexibility falls from zero to minus infinity as income rises from zero to bliss income. Thus, its inverse (and not itself) may serve as a welfare indicator in this case. Finally, we note that if it is specified that $b_{ij} = 0$, for $i \neq j$, then utility becomes additive, and the additive quadratic utility function gives rise to the so-called Gossen map (see Samuelson [1947a, p. 93] and Allen and Bowley [1935, p. 139]).

From the point of view of estimation, the Quadratic utility model possesses little attraction. In fact, the demand functions that result are quite complex, and have not been estimated by anyone.[79]

Direct Addilog Utility:

The direct addilog utility function (DAUF) was proposed by Houthakker [1960], alongwith its operational counterpart, the indirect addilog utility function, (IAUF). The DAUF is given by:

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[79] We have but paraphrased Goldberger [1967, pp. 73-80] in this section.

(DAUF) ... $u = u(x_1,\ldots,x_n) = \Sigma_i \; a_i \; x_i^{b_i}$

where it is assumed that $a_i > 0$, and $0 < b_i < 1$ for all $i=1,\ldots,n$. It has been shown by Houthakker [1960, p. 253] that only a partial solution to the demand functions is available. This is due to the fact that the first order conditions are difficult to solve explicitly for the x's in terms of the prices and income. To get a partial solution, the ratio of marginal utilities may be equated to price ratios to give:

$$(1 - b_i)\log x_i - (1 - b_j)\log x_j = \log(a_i b_i / a_j b_j) - \log(p_i/p_j) \qquad \text{(for all } i,j)$$

Since the direct addilog model is non-operational, (unless extremely tedious methods of estimation are adopted), we omit further discussion, but note briefly the few properties that are known about this model. Houthakker's [1960] results that the ratio of income elasticities under DAUF are constant, may be easily derived by taking a total differential of the equation above, assuming prices constant:

$$(1 - b_i)d \log x_i - (1 - b_j)d \log x_j = 0 \qquad \text{(all } i,j)$$

which implies, after clearing terms and dividing by dlog y,

$$E_i/E_j = (1 - b_j)/(1 - b_i) \qquad \text{(all i,j)}$$

A special case of the DAUF results when all the b_i are equal, say to b. It is easy to show, then, that the resulting demand functions are given by

$$x_i = \frac{(p_i/a_i b)^{(b-1)^{-1}}}{\Sigma_j p_j (p_j/a_j b)^{(b-1)^{-1}}} \; y \qquad (i=1,\ldots,n)$$

Hence, all Engel curves are straight lines through the origin, so that income elasticities are all unity. Pollak [1967, p. 3] shows that this special case of the direct addilog utility function is a monotonic trasformation of the constant elasticity of substitution utility function. This is easy to see if we set $a_i = \rho_i$, $b = -\rho$. Then

$$v(x_1,\ldots,x_n) = u^{1/b} = (\Sigma_i \rho_i x_i^{-\rho})^{-1/\rho}$$

which is the CES utility function. Finally, for the specialized indirect addilog function, it can be shown that the money flexibility is given by:

$$\phi = (b - 1),$$

which is negative but independent of income.

<u>Indirect Addilog Utility</u>:

Finally, we have the operational case of "indirect addilog" utility function due also to Houthakker [1960]. Estimation of demand functions under this utility function

was carried out by Houthakker [1960], although the demand functions which result had already been explored by Somermeijer and Witt [1956]. Somermeijer [1961], Russel [1965], Parks [1969], and Yoshiahara [1969], have applied the model to data from Netherlands, U.S.A., Sweden, and Japan, respectively. The extreme difficulty in estimation, the lack of good empirical fits, and the admitted lack of intuitive justification render the indirect addilog case as somewhat of a curiosity among demand models.

The indirect addilog utility function proposed by Houthakker [1960] is given as

$$\text{(LAUF)} \ldots\ u = u^*(y, p_1, \ldots, p_n) = \Sigma_i\ a_i(y/p_i)^{b_i} (a_i<o;\ -1<b_i<o)$$

where, u* is understood to be the "indirect utility function" defined in Chapter 1. This is, in fact, the only explicit function that has been considered in the literature, which exhibits the property of "indirect additivity" examined in Chapter 1, Section   .

Obtaining demand functions under indirect addilog utility is relatively easier, as noted earlier. Differentiating with respect to y gives the marginal utility of income function:

$$\lambda = \frac{\partial u^*}{\partial y} = (\Sigma_k\ a_k b_k (y/p_k)^{b_k})/y$$

Differentiating with respect to the ith price gives:

$$\frac{\partial u^*}{\partial p_i} = -(1/p_i)\ a_i b_i\ (y/p_i)^{b_i} \qquad (i=1,\ldots,n)$$

Now, using Roy's identity (see p.), we get the demand functions:

$$x_i = -\frac{1}{\lambda}\ \frac{\partial u^*}{\partial p_i} = \frac{a_i b_i (y/p_i)^{1+b_i}}{\Sigma_k\ a_k b_k (y/p_k)^{b_k}} \qquad (i=1,\ldots,n)$$

which is Houthakker's equation (29), [1960].

It is readily verified that for this system, the income (Engel) elasticities are given by:

$$E_i = (1+b_i) - \Sigma_k b_k w_k \qquad (i=1,\ldots,n)$$

where, the w_k are the usual budget shares. Similarly, for the (Cournot) price elasticities, we have

$$e_{ij} = b_j w_j - (1+b_i)\ \delta_{ij} \qquad (i,j=1,\ldots,n)$$

Thus, as is true in any indirectly additive model, the cross-price elasticities depend only on the good whose price is changing. Note, also, that for this particular model the differences between income elasticities are constant.

By direct differentiation of the λ function, we obtain the income elasticity of the marginal utility of income, or the inverse of money flexibility, after some rearrangement:

$$\frac{\partial \lambda}{\partial y} = (1/y)\lambda \Sigma_k (b_k - 1) w_k$$

So that the income flexibility, ϕ, is given by:

$$\phi = -1 + \Sigma_k b_k w_k$$

In the case of indirect addilog utility, then, the money flexibility follows Frisch's conjecture. As income rises, high income elasticity goods occupy larger shares of the budget. Thus, both b_k and w_k rise together, with the result that $\Sigma_k b_k w_k$ rises with income; and hence, ϕ rises with income [Goldberger, 1967, p. 91].

In conclusion, we might mention again that unlike its counterpart, direct additivity, indirect additivity is hard to justify on inutitive grounds. It is, in fact, unclear what behavioral implications of indirect additivity are. Thus, the relevance of the indirect additivity hypothesis as a source of meaningful models of demand is restricted. Further, the estimation of the set of demand equations under indirect addilog utility function is quite cumbersome. Houthakker [1960] took ratios of quantities to suppress the nonlinearity in parameters that is inherent in the model. This leads, of course, to several estimates of the parameters. The estimation of the indirect addilog model was solved by Parks [1969], who proposed that the system of (n-1) equations obtained by taking ratios of quantities be estimated jointly, with restrictions across equations being imposed in order to yield unique estimates of the parameters. Preliminary empirical results indicate that the model does not compare favorably with models like SLES, or the Rotterdam models. It might be noted that Houthakker arrived at the indirect addilog specification in attempting to adjust the CEDS to satisfy the adding-up criterion. In doing so, unfortunately, the model became nonlinear in parameters, and hence lost its only desirable property: linearity.

To conclude this chapter, we recall that of all the empirical models suggested, only the linear expenditure systems, and the Rotterdam models are both theoretically plausible, and give rise to cogent empirical explanations. Among these, the only model which is linear in parameters is the linear expenditure system proposed by Leser. Unfortunately, LLES has the disadvantage that it has a rather large number of parameters to be estimated. To correct for this the elasticities of substitution may be equated for each pair of distinct goods. However, this solution is far from attractive. Thus, the empirical worker in demand theory is almost inevitably confronted with parametrically nonlinear models, if his choice is to be plausible on theoretical grounds.

To partially remedy this situation, we propose in the next chapter a modification of the CEDS which renders it consistent with the classical utility maximizing theory

developed in Chapter 1. In addition, this model is linear in parameters and is relatively easy to estimate for a small number of commodities. The model also has the advantage that the results of additivity theory may be introduced (at the expense of nonlinearities) for the reduction in the number of parameters. The model is applied to U.S. data, and results presented.

CHAPTER 4

A MODIFICATION OF THE DOUBLE-LOGARITHMIC SYSTEM OF DEMAND EQUATIONS

4.1 Introduction

In the previous chapter we have examined all of the theoretically plausible functional forms for a complete set of demand equations that have been utilized in the empirical work on demand analysis. This class of admissible demand equations is further restricted by the criterion of empirical plausibility. Thus, in a direct comparison of three of the alternative models, Parks [1969] reports that estimates based on a fairly long time series on Swedish data reveal that the Rotterdam model has a slight advantage over SLES, while the Indirect Addilog model gives extremely poor fits. The evidence against the Indirect Addilog model is confirmed by Yoshihara's [1969] estimates based on Japanese data. These results were, in fact, anticipated by Houthakker [1960], who admitted that the lack of an intuitive interpretation of the Indirect Addilog model and the restrictive features inherent in it, led him to believe that the model may not be of great empirical relevance. A consensus has, therefore,

seemed to emerge that the "Linear Expenditure Systems" and the Rotterdam models are the only theoretically plausible models which are known to be consistent both with the theory and with the data. Even among these two models, only the Rotterdam model admits of a parametrically linear version if the number of commodities is small.[80] For a large number of goods in the consumer's budget, both these models become nonlinear in parameters, and are not easy to estimate.

Clearly, a need exists for the development of additional theoretically plausible models, which also meet the test of data. In this chapter we propose two such models, based upon the CEDS and bearing a great degree of formal resemblance to the Rotterdam models. The models proposed in this chapter share the linearity property of the Rotterdam model for a small number of commodities, and can be extended in the fashion of the Rotterdam models to nonlinear versions. Using a four-commodity breakdown of U.S. expenditure data from 1929-1968 utilized by Pollak and Wales [1969] in the estimation of SLES, we estimate

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[80] The parametrically linear version of the Rotterdam model is estimated by Parks [1970]. This version does not involve any spearability or additivity assumptions, as was done in the original formulations. For this reason each demand equation contains as many explanatory variables as the number of commodities under analysis. Multicollineauity in price variable and the lack of extensive data therefore restrict its use to a "small" number of commodities. Roughly speaking, n may be considered small whenever $1/2(n^2+n-2)$, (the number of free parameters to be estimated), is less than the number of independent observations.

both the models proposed here, the Rotterdam model, and Leser's LES. The models are then compared with respect to the negative definiteness of the matrix of the Slutsky substitution terms, the plausibility of the values of marginal budget shares, and two measures of goodness of fit. On the basis of these comparisons, conclusions are drawn with regard to the potential empirical usefulness of the two models proposed.

4.2 Two Theoretically Plausible Demand Models

Before proceeding to derive the models proposed in this section, we examine briefly the principles underlying the construction of demand models. From Chapter 1 we know that under appropriate assumptions we can be certain of the existence of a continuous and differentiable complete set of demand functions:

$$x_i = h^i(y, p_1, \ldots, p_n) \qquad (i=1,\ldots,n)$$

which are determined entirely by the preference ordering of the consumers, in the sense that they are invariant with respect to an arbitrary monotonic transformation of any specific utility function representing the preference ordering. In addition, we have shown that these functions satisfy certain partial differential equations in quantities, prices and income. In other words, we know that a complete set of demand equations constitute a specific

solution to a set of partial differential equations, which are usually formulated in terms of logarithmic partial derivatives.[81] The search for theoretically plausible functional forms is, therefore, a search for specific solutions to a set of partial differential equations. With restrictions expressed in logarithmic partial derivatives, and using our earlier notation with x_i referring to quantity, p_i to price, and y to income, we seek functions $h^i(i=1,\ldots,n)$,

$$x_i = h^i(y,p_1,\ldots,p_n)$$

which satisfy the following logarithmic partial differential equations:

$$\Sigma_i\ w_iE_i = 1$$

$$\Sigma_j\ e_{ij} = E_i \qquad (i=1,\ldots,n)$$

$$s_{ij} = s_{ji} \qquad (i,j=1,\ldots,n)$$

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[81]It has sometimes been argued that nothing would be lost if demand analysis were to rid itself of the classical notions of "elasticity" and logarithmic derivatives and adopt the more mathematically conventional concept of partial derivatives. It must be pointed out, however, that the logarithmic partial derivatives are well-defined mathematical functions that have the added advantage of being without dimension. (For a discussion of the concept of dimension, which has been used to great advantage in the natural sciences, see de Jong [1967]. de Jong also points out [pp. 139-142] that Samuelson's statement [1947a, p. 126n.] that the elasticity concept is not dimensionless, may be in error.)

and the linear equation $\Sigma_i p_i x_i = y$, where w_i is the budget share of the i-th good, E_i the income (Engel) elasticity, e_{ij} the price (Cournot) elasticity, and s_{ij} the Slutsky price elasticity of demand. Strictly speaking, the functions h^i must also satisfy the condition that the matrix of Slutsky terms K_{ij} be negative definite. However, in practice, this condition is difficult to satisfy in this general a form. The practice has been to estimate the sample values (evaluated at a specific value of expenditures and prices), of the Slutsky terms, after estimation, and then use the negative definiteness criterion as a check on the empirical validity of the models.

With this formulation, it is easy to identify two lines of attack. First, we can generate a wide variety of demand models by specifying a specific utility function and then deriving the demand functions. From the results in Chapter 1, it is clear that this procedure would yield demand functions which meet the criteria outlined above. Indeed, examples of this approach are the quadratic utility demand model, the direct and indirect addilog demand specifications, and to some extent, the linear expenditure system due to Stone. There are two weaknesses in this approach. First, it often happens that the resulting demand functions possess a functional form which is not trivial to estimate. Secondly, it is unclear what behavioral implications result from the assumption that the

consumer is maximizing the class of utility functions which admit the specific utility function as a monotonic transform.

For these reasons, we prefer the second approach in which functional forms of demand equations are specified on a priori grounds first, and are then subjected to the classical restrictions. Indeed, the two leading functional forms are derived on these principles. The SLES model is the result of restricting the GLES in order to satisfy the classical restrictions, while the Rotterdam model is based on a similar approach with respect to a differential expansion of the demand functions. In developing our models, we use as our starting point the CED system discussed in Chapter 2. The CED system is particularly easy to estimate, and its parameters being elasticities are dimensionless, and easy to interpret.[82] Of course, the model is not theoretically plausible. Our purpose, in brief, is to remedy this situation.

To motivate the formulation of CED system, we note that the concept of "elasticities" of demand possesses the

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[82]For these reasons Houthakker [1965] was willing to overlook the theoretical deficiencies of the CEDS, and claim that the CEDS was "without serious rivals in respect of goodness of fit, ease of estimation, and immediacy of interpretation." [1965, p. 278]. Goldberger and Gamalestos [1967] took issue with this claim and offered that SLES, "while not without its share of weaknesses, retains some attraction as a rival to constant elasticity of demand models. . ." [1967, p. 73].

desirable feature of being without dimension, and have been used extensively in the literature on demand analysis. Thus, we may define, without loss of generality, the elasticity functions:

$$E_i = f^i(y, p_1, \ldots, p_n) = \frac{y}{h_i}\frac{\partial h_i}{\partial y} \qquad (i=1,\ldots,n)$$

and, $$e_{ij} = g^{ij}(y, p_1, \ldots, p_n) = \frac{p_j}{h_i}\frac{\partial h_i}{\partial p_j} \qquad (i,j=1,\ldots,n)$$

and, $$s_{ij} = e_{ij} + w_j E_i$$

where, the E_i, e_{ij}, s_{ij} are respectively the Engel, Cournot, and Slutsky elasticities, and w_i are the budget shares. Now, since these equations are definitional, they are satisfied by all functions h^i which may be considered as demand functions. Hence, we may look at a particularly simple functional form for h^i in which the elasticities, E_i, e_{ij}, s_{ij}, (if assumed constant) appear as coefficients of the independent variables. This is the familiar CED system of demand equations: (in Slutsky form)

$$\log x_i = E_i[(\log y) - \Sigma_j w_j(\log p_j)] + \Sigma_j s_{ij}(\log p_j)$$

Notice, however, that if the E_i and s_{ij} are assumed to be functions, and not parameters, then the demand function given above becomes merely a particular solution to the n(n+1) definitional partial differential equations given above. Thus, the CED system violates the classical

restrictions only when it is assumed that these elasticities are constant. There is, however, no need to parametrize at this stage.

We consider, therefore, the function given above, where it is understood that the E_i and s_{ij} are some general functions of prices and income, defined by the equations given above. In all generality, we multiply through by w_i, the budget share, to get:

$$w_i (\log x_i) = w_i E_i (\log \bar{y}) + \Sigma_j w_i s_{ij} (\log p_j)$$

where

$$(\log \bar{y}) = (\log y) - \Sigma_j w_j (\log p_j)$$

With this formulation, several parametrizations are possible. First, it may be assumed that the income elasticity, E_i, varies as a rectangular hyperbola with respect to variations in the budget shares, w_i, i.e., let $w_i E_i = b_i$, where b_i $(i=1,2,\ldots,n)$ is a constant. This is the assumption that marginal budget shares, which are the slopes of the Engel curves, are constant with respect to variations in expenditures and prices. This assumption of linearity of each Engel curve is crucial for aggregation to go through, as we have seen in section 1.6, and is, in fact, utilized both by Stone, and by Theil and Barten in the formulation of their models. Denoting by b_i, the respective marginal budget shares, we obtain a partial

parametrization of the model. If, in addition, we assume that the quantity $(w_i s_{ij})$ is constant with respect to variations in expenditures and prices, as the Rotterdam models do, we obtain the following specification, which may be called the "double-logarithmic Rotterdam elasticity specification" model

$$\text{(DOLRES)} \ldots \quad w_i(\log x_i) = b_i(\log \bar{y}) + \Sigma_j\, c_{ij}(\log p_j)$$

where, c_{ij}, denotes the constant $(w_i s_{ij})$, by assumption. Notice, that the above specification is, in fact, identical to the Rotterdam model, if the logarithms of the variables are replaced by changes in the logarithms of these same variables.

Before proceeding to discuss further similarities between the DOLRES and the Rotterdam models, an alternative specification may also be derived. Note that the Hicks-Allen "elasticity of substitution" between goods, d_{ij}, is related to the Slutsky price elasticity, s_{ij}, by the following relation:

$$d_{ij} = s_{ij}/w_j.$$

Thus, the Rotterdam assumption that the c_{ij} are constant with respect to variations in expenditures and prices, reflects the belief that the elasticity of substitution between goods is determined solely by the budget shares

of the two goods, and is, in fact, given by the ratio of a constant and a product of the two budget shares. Formally:

$$d_{ij} = \frac{c_{ij}}{w_i \; w_j}$$

where, c_{ij} is constant. Although, with the exception of Leser's LES, demand models have not been formulated in terms of the elasticity of substitution, we see that this can easily be accomplished in this model. The constancy of the elasticity of substitution between factors has been hypothesized in the theory of production and has been used to great advantage. Incorporating the same assumption for consumer goods, we propose the following double-logarithmic (constant elasticity of substitution) model, (DOLCES):

$$\text{(DOLCES)} \; \ldots \; w_i(\log x_i) = b_i(\log \bar{y}) + \Sigma_j \; d_{ij} \; w_i w_j(\log p_j)$$

where, d_{ij} is the (constant) elasticity of substitution between goods i and j.[83]

Having given two parametrizations of the traditional double-logarithmic model, we consider the implications of the classical restrictions on the parameters. These are easy to derive. Engel aggregation implies that

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[83] It might be mentioned that a similar version of the Rotterdam model may also be constructed if it is so desired. In fact, the theory of production is rich in hypotheses with regard to the elasticity of substitution, and it may, perhaps, be a useful source of alternative parametrization for demand models.

the marginal budget shares, b_i, sum to unity. The symmetry condition implies that the coefficients $c_{ij} = c_{ji}$ for the DOLRES model, and the $d_{ij} = d_{ji}$ for the DOLCES model, for all i, j. For the DOLRES model, the homogeneity condition implies that the c_{ij} sum to zero for each equation: $\Sigma_j c_{ij} = 0$; while, for the DOLCES model the condition is that $\Sigma_j w_j d_{ij} = 0$, for all i. For both models, the homogeneity condition may be incorporated into the functional form, by deflating all prices by one of the prices. Without loss of generality, we may deflate all prices by the nth price to obtain the following specifications for the two models:

$$\text{(DOLRES)} \ldots w_i(\log x_i) = b_i(\log \bar{y}) + \Sigma_j c_{ij}(\log p_j - \log p_n)$$

$$\text{(DOLCES)} \ldots w_i(\log x_i) = b_i(\log \bar{y})$$

$$+ \Sigma_j d_{ij} w_i w_j(\log p_j - \log p_n)$$

where the summation over the index j is understood to be from 1 to (n-1).

In this version, with the b_i's restricted by $\Sigma_i b_i = 1$, the DOLRES and DOLCES models satisfy all the classical restrictions other than the adding-up criterion, and, of course, the negative definiteness condition on the matrix of Slutsky terms. The DOLRES model, in particular, is identical to the preliminary formulations of the

Rotterdam models if the logarithms are replaced by changes in logarithms. Thus, the parameters of the DOLRES models have exactly the same interpretations as the parameters of the Rotterdam models, and provide a useful basis of comparison. This similarity between DOLRES and the Rotterdam model is not surprising. The Rotterdam model uses an exact expansion of the logarithmic differential of the demand function and approximates the infinitesimal changes by finite changes, while the DOLRES model is derived by considering a particular solution to the set of partial differential equations defining the logarithmic partial derivatives which appear as coefficients in the Rotterdam model. Thus, in a sense, we have merely formulated the Rotterdam model in an absolute, rather than differential, version. It is this similarity between the two models which we exploit to the advantage of the DOLRES and the DOLCES models, in attempting to solve the problem of the non-additivity of the demand equations.

Recall that a similar problem arose with respect to a preliminary version of the Rotterdam model, in which the adding-up criterion was also not met. In the Rotterdam model the solution was found by replacing the log-change in real income by an index of the log-change in the volume of consumption. This procedure was justified [Theil, 1967, p. 224ff.] by noting that both the log-change in real income and the log-change in the volume of

consumption were local quadratic approximations to the true cost of living index, involving errors of third order in logarithms of changes in prices and income. Thus, the two approximations were considered interchangeable for empirical purposes. This would suggest, admittedly on intuitive grounds, that an index of the absolute level of the volume of consumption--similar to the log-change volume index--may provide an adequate approximation to the level of real income. Replacing the logarithmic real income term ($\log \bar{y}$), by a log-volume index of the level of consumption, $x^*_t = {}_k w_{kt} \log x_{kt}$, we may rewrite the DOLRES and and DOLCES models in their final non-stochastic form:

$$\text{(DOLRES)} \; \ldots \quad w_{it}(\log x_{it}) = b_i \; x^*_t + \sum_{j=1}^{n-1} c_{ij}(\log p_{jt} - \log p_{nt})$$

$$\text{(DOLCES)} \; \ldots \quad w_{it}(\log x_{it}) = b_i \; x^*_t + \sum_{j=1}^{n-1} d_{ij} \; w_{it} w_{jt}(\log p_{jt} - \log p_{nt})$$

where the index i ranges over commodities, and the index t refers to time. Note that in doing this the b_i may no longer be interpreted as marginal budget shares. We may, however, still consider them as approximations to the marginal budget shares.

To gain additional insight into the nature of approximation involved, we might consider the relationship between the absolute log-volume index utilized here, and the log-change volume index used in the formulation of the Rotterdam models. The Rotterdam volume index is given by:

$$Dx_t \quad = \quad \Sigma_k \; w^*_{kt}(\log x_{kt} - \log x_{k,t-1})$$

where, $w^*_{kt} \quad = \quad 1/2(w_{kt} + w_{k,t-1})$.

Note that the use of arithmetic means of the budget shares as weights in the construction of the Rotterdam index insures that the index meets the time-reversal test. This is in fact, the reason why the Rotterdam model, in its current formulation, utilizes the arithmetic means of the budget shares as the common multiple corresponding to the role of the current period budget shares used in this section in the derivation of DOLRES and DOLCES models. An additional feature of the Rotterdam index is its purely "statistical" property of being zero for the case when $x_{kt} = x_{k,t-1}$ for all k.

In comparison, the index we propose is an "economic" index, and is given by:

$$x^*_t \quad = \quad \Sigma_k \; w_{kt}(\log x_{kt}).$$

Even if all quantities were to remain unchanged, on purely statistical grounds it would be possible for the budget shares w_{kt} to change due to changes in prices, thus making the volume index proposed by us to fluctuate due to variations in prices even though the volume of consumption does not change. Clearly, this would be an undesirable property for a volume index to possess. To insure that the log-volume index proposed above does not change if all quantities remain unchanged, we must invoke the homogeneity condition of economic theory, which insures that if <u>all</u> quantities remain unchanged, then the budget shares of the respective commodities could not have changed either.

The log-volume index proposed here bears an explicit relation to the log-change in volume index utilized in the Rotterdam models.[84] Using the notation developed above, it may be easily verified that the following relationship exists between the two indices:

$$Dx_t = x_t^* - x_{t-1}^* - \Sigma_i \quad 1/2(\log (x_{it}x_{i,t-1}))(w_{it}-w_{i,t-1})$$

Thus, we see that the change in the log-volume index proposed in this section diverges from the log-change in the Rotterdam volume index by a weighted average of the geometric means of the volume of consumption in the current and the preceding period; the weights being the differences

[84] I am indebted to Professor Henri Theil for pointing out this relation in correspondence.

in the budget shares of the respective commodities over the two periods. This weighted average term reflects the difference in the two indices due to the use of absolute budget shares in one, and the arithmetic means of the budget shares in the other index, and, in fact, may be considered to be the degree of error involved in approximating the Rotterdam log-change index by changes in the log-volume index. Although the relative magnitude of the term,

$$\Sigma_i 1/2[\log (x_{it} x_{i,t-1})] (w_{it} - w_{i,t-1})$$

is not immediately obvious, we may still justify the use of the log-volume index on two grounds. First, since the Rotterdam index is itself an approximation, the degree of divergence of the log-volume index from the Rotterdam index is not directly relevant. Secondly, this difference arises due to the use of arithmetic means of budget shares instead of the budget shares themselves. Thus, the volume index proposed here does not seem an unreasonable approximation to the true real income.

4.3 Stochastic Specification and Estimation

The stochastic specification of complete sets of demand equations has not been given the degree of attention that it merits. The notable exceptions to this statement are the Rotterdam models, for which the "marginal

utility shock model" (discussed in section 3.2) is developed with the aim of incorporating a stochastic element in the utility maximizing behavior itself; and the Pollak-Wales versions of the LES (discussed in section 3.3) for which a similar attempt has been made. Although the formal similarity between the Rotterdam model and the DOLRES and the DOLCES models could, perhaps, have been exploited with the intent of developing a stochastic model aprallel to the marginal utility shock model, we have chosen not to do so. Instead, we adopt the most convenient stochastic specification available, and assume that the disturbance term enters additively into the demand equations. With this specification, the disturbance terms in each of the demand equations may be interpreted as the allocation discrepancy due to random factors.

To facilitate exposition, we shall write the demand equations for both the DOLRES and the DOLCES models as:

$$(27) \qquad y_{it} = b_i\, y_t + \Sigma_j\, k_{ij}\, z_{jt} + u_{it} \qquad \begin{matrix}(i=1,\ldots,n)\\(j=1,\ldots,n-1)\\(t=1,\ldots,T)\end{matrix}$$

where the b_i and k_{ij} are constants, y_{it} is the t-th observation on the i-th dependent variable, and y_t $(=\Sigma_i y_{it})$, and x_{it} are the t-th observation on the non-stochastic independent variables, y, and x_i, respectively. The u_{it} are interpreted to be the t-th value of the i-th disturbance terms, and are assumed to possess a specified

variance-covariance structure. It is readily seen that with a suitable interpretation of the parameters and variables, (27) gives rise to both the DOLRES and the DOLCES models.

If the u_{it} are to be interpreted as allocation discrepancies, as we have suggested, then the sum of these allocation discrepancies in any time period must be zero. This can also be seen by recalling that the b_i sum of unity and the k_{ij} are symmetric, and in the case of DOLRES, the k_{ij} sum to zero for each equation. Then adding over the i, (27) yields:

$$\Sigma_i u_{it} = \Sigma_i y_{it} - y_t - \Sigma_i \Sigma_j k_{ij} z_{jt}$$

$$= \Sigma_j (z_{jt} \Sigma_i k_{ij})$$

$$= \Sigma_j (z_{jt} \Sigma_i k_{ji}) = 0$$

A similar proof for the DOLCES is readily constructed. Thus, it has been shown that the disturbances in the demand equations sum to zero, and hence are not mutually independent. This is the cause of the singularity of the covariance matrix of the disturbance terms which must be present in any model of consumer demand which seeks to allocate total expenditure into its various components.

Adopting the simplest assumptions with regard to the moments of the distribution of the disturbances,

u_{it}, the complete model may be formulated as follows:

(28a) $$y_{it} = b_i \, y_t + \Sigma_j \, k_{ij} \, z_{jt} + u_{it} \qquad \begin{matrix}(i=1,\ldots,n)\\(j=1,\ldots,n-1)\\(t=1,\ldots,T)\end{matrix}$$

(28b) $\Sigma_i b_i = 1; \;\; k_{ij} = k_{ji}$ (for all i,j);

(28c) $\Sigma_i u_{it} = 0$ (for all t)

(28d) $E(u_{it}) = 0$ (for all i,t)

(28e) $E(u_{is}u_{jt}) = 0$ (for $i \neq j$, $s \neq t$)

(28f) $E(u_{it}u_{jt}) = \sigma_{ij}$ (for all i,j,t)

(28g) $E(u_{is}u_{it}) = 0$ (for all $s \neq t$).

In this formulation the set of equations (28) have a convenient matrix representation due to Zellner [1962]:

(20) $$\begin{bmatrix} y_1 \\ y_n \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & X_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}$$

where, for all i=1,...,n:

$$y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{bmatrix}, \quad X_i = \begin{bmatrix} y_1 & z_{11} & z_{21} & \cdots & z_{n1} \\ y_2 & z_{12} & z_{22} & \cdots & z_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ y_T & z_{1T} & z_{2T} & \cdots & z_{nT} \end{bmatrix}, \quad u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iT} \end{bmatrix}$$

with y_{it}, y_t, z_{ij} defined as in (28).[85] Even more compactly, (29) may be written as:

(30) $$y = X\beta + u$$

where, y is the (nTx1) vector of observations on all of the dependent variables, and X is the (nT x n^2) matrix of observations on the non-stochastic explanatory variables, β is an (n^2 x 1) vector of parameters, and u is an (nTx1) vector of disturbances with mean zero, and covariance matrix,

$$E(uu') = \Sigma.$$

With this notation, the model represented by the equations (28a-g) may be completely represented as:

(31) $$y = X\beta + u$$
$$R\beta = r$$
$$E(uu') = \Sigma, \text{ and } \Sigma_i u_{it} = 0, \text{ for all } t.$$

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[85]Note that the ells of x_i do not depend on i, so that $x_i = x_j$.

where, R is an appropriate restrictions matrix which together with the vector r expresses the restrictions (3.3.2b), and Σ is the covariance matrix whose elements are defined by the equation (28e,f,g).[86]

The derivation of a best linear unbiased (restricted) estimator, which is a maximum-likelihood estimator under the assumption of normality of the disturbances, is trivial in the case that Σ^{-1} exists, and is a direct extension of the simple restricted least squares estimator outlined for example in Goldberger [1964, pp. 256-258]. Unfortunately, the restriction that the budget discrepancies, u_{it}, sum to zero for each period imply the singularity of the covariance matrix Σ. Thus, neither the likelihood function exists, nor is the "generalized sum of squared deviations" [Goldberger, 1964, p. 233]

$$S = (y - X\beta)' \, \Sigma^{-1} \, (y - X\beta)$$

defined. In a pathbreaking paper, Powell [1969] considered the set of all restricted estimators of $\beta, \overset{n}{\{\beta\}}$, and showed that a specific estimator, say $\hat{\beta}^{o}$, of any independent subset of the parameters, possesses the property that the generalized sum of the squared deviations associated with this estimator (which is defined by construction) is equal

[86]The y and x of (31) should not be confused with the earlier notation in which y was income and x referred to quantities of goods!

to the following generalized sum of squared deviations associated with $\tilde{\beta} \in \{\tilde{\beta}\}$:

$$S^* = (y - X\tilde{\beta})\ \Sigma^+\ (y - X\tilde{\beta}),$$

where Σ^+ is the unique Moore-Penrose generalized inverse of the covariance matrix Σ. The Moore-Penrose inverse of a matrix of less than full rank is defined as

$$\Sigma^+ = Q'(QQ')^{-1}(T'T)^{-1}T'$$

where, T is any column basis for Σ, and Q is a matrix which satisfies the condition:

$$\Sigma = TQ$$

The Moore-Penrose generalized inverse, is a g-inverse in the sense that:

$$\Sigma\ \Sigma^+\Sigma = \Sigma.$$

The procedure suggested by Powell [1969] is computationally equivalent to deleting as many equations as are necessary to render the covariance matrix to be of full rank, then estimating the remaining parameters using the conventional restricted least squares estimator, and finally recovering the left out parameters by the use of the restrictions matrix R. The force of Powell's result, which derives from the uniqueness of the Moore-Penrose inverse, is that the estimates of the parameters so obtained are numerically invariant with respect to the choice of the equation to be deleted. We omit the details of Powell's proof, but heuristically derive an alternative formulation of the same result.[87]

Instead of looking at the generalized sums of squared deviations, we may alternatively consider the following generalized "likelihood" function which may be defined when the disturbances are assumed to be multivariate singular normal, with the covariance matrix, Σ, of less than full rank:

$$L^* = -1/2n(\log 2\pi) - 1/2(y-X\beta)' \Sigma^+ (y-X\beta)$$

where, Σ^+, as before is the Moore-Penrose generalized inverse. We might seek a restricted maximum "likelihood" estimator which maximizes L* subject to an arbitrary linear constraint on the parameters:

$$R\beta = r.$$

Note that the "likelihood" function L* is well-defined in the sense of being single valued. Thus, the solution of the constrained maximization of the "likelihood" function will give rise to an estimator which is also unique and

[87]It ought to be mentioned that Powell [1969] did not present his results for the case of arbitrary linear restrictions, but considered instead, some specific restrictions on the parameters. However, his results seem to hold for the general case which we have presented above.

well-defined. The derivation is strightforward. Construct the Lagrangean function:

$$L = -\ 1/2n(\log 2\pi) - 1/2(y-X\beta)\ \Sigma^{+}\ (y-X\beta) + 2\ \lambda'(R\beta - r)$$

where, λ, is the vector of Lagrange multipliers of dimension (Jxl), where J is the rank of the restriction matrix R. To maximize L, take partial derivatives with respect to β,

$$\frac{\partial L}{\partial \beta} = X'\Sigma^{+}y - (X'\Sigma^{+}X)\beta + R'\lambda$$

Equating to zero, we have:

$$\tilde{\beta} = (X'\Sigma^{+}X)^{-1}(X'\Sigma^{+}y) + (X'\Sigma^{+}X)^{-1}R'\tilde{\lambda}$$

Utilizing the fact that $R\beta = r$, we have from this equation by multiplying through by R, and rearranging,

$$\tilde{\lambda} = (R(X'\Sigma^{+}X)^{-1}R')^{-1}(r - Rb),$$

where $b \equiv (X'\Sigma^{+}X)^{-1}(X'\Sigma^{+}y)$

Substituting for $\tilde{\lambda}$ in the original equation we have the restricted maximum "likelihood" estimator which is equivalent to Powell's estimator:

$$\tilde{\beta} = (X'\Sigma^{+}X)^{-1}(X'\Sigma^{+}y)$$
$$+ (X'\Sigma^{+}X)^{-1}R'(R(X'\Sigma^{+}X)^{-1}R')^{-1}(r-Rb)$$

Since Σ^{+} is numerically invariant with respect to the choice of any column basis, we may for computational convenience delete arbitrarily the last equation, and estimate the following reduced system of demand equations:

$$(32) \qquad y_{it} = b_i\, y_t + \Sigma_j\, k_{ij}\, z_{jt} + u_{it} \qquad \begin{matrix}(i,j=1,\ldots,n-1;\\ t=1,\ldots,T)\end{matrix}$$

with the restriction that $k_{ij} = k_{ji}$ for all $i,j=1,\ldots,n-1$. The parameters of the n-th demand equation may be recovered by the left out restrictions of symmetry, and the Engel aggregation conditions. Powell's theorem insures that this procedure gives rise to the same numerical estimates of the parameters.

To estimate the reduced system of equations (32) we note first that the disturbances are correlated across equations. This would suggest using Zellner's (asymptotically) efficient two-stage Aitken procedure. However, as is well known, in the case where the explanatory variables in each equation are the same, Zellenr's procedure (ZEF) reduces to the ordinary least squares procedure (OLS). To impose the symmetry restrictions, however, we must once again resort to the Zellner-Aitken procedure. The formulation (28g) restricts the disturbances to be non-autoregressive. Since the data utilized in the present study are time series, some sort of autocorrelation may be present. If it is assumed that the disturbances

follow a first-order autoregressive (Markov) scheme, then a modification of the Zellner procedure due to Kmenta and Gilbert [1970] may be utilized. This is a four-stage procedure in which the Zellner-Aitken two-stage residuals are utilized to obtain a weighted first-differencing of the relevant variables before estimating the parameters by the two-stage ZEF again.

Formally, we consider the case where (28g) does not hold, but instead we have:

$$E(u_{it}u_{i,t-s}) = \rho_i^s \sigma_{ii} \tag{28g'}$$

which is the specification which results if the disturbances u_{it} follow a first-order autoregressive scheme:

$$u_{it} = \rho_i u_{i,t-1} + v_{it}$$

with the v_{it} distributed independently and with zero mean, and constant variance: Var $(v_{it}) = (1 - \rho_i^2)\sigma_{ii}$. The demand model may then be written as:

$$\begin{aligned} y_{it} &= b_i \, y_t + \Sigma_j \, k_{ij} \, z_{jt} + u_{it} \\ u_{it} &= \rho_i \, u_{i,t-1} + v_{it} \end{aligned} \tag{33}$$

with v_{it} satisfying the assumptions of the classical linear regression model. To estimate the DOLRES and DOLCES models, we use the direct extension of the Kmenta-Gilbert four-stage procedure, ZEF-ZEF.

First, OLS residuals are used to get consistent estimates of the covariance matrix of the system of (n-1) equations obtained by deleting the last equation. Secondly, using these estimates we obtain the restricted least squares estimates of the parameters, where the restrictions imposed are those of symmetry. Thirdly, assuming that the scheme of autocorrelation is a first-order Markov scheme, we used the residuals of the restricted ZEF estimates to estimate the autocorrelation parameters, ρ_i. Finally, we obtain once again the two-stage restricted ZEF estimates of the parameters, after correcting for autoregression by lagging the variables in the usual manner.[88]

4.4 Data and Variables

The data utilized in the empirical section of this study are from The National Income and Product Accounts of the United States, 1929-1965: Statistical Tables. The basic sources are Tables 2.6 and 8.6. The former give constant dollar expenditure on some forty-six expenditure

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[88]Some small sample results by Kmenta and Gilbert [1968] reveal that the use of ZEF residuals, instead of OLS residuals for the estimation of the ρ_i lead to considerably more efficient results in small samples. Indeed, the most efficient procedures found by Kmenta and Gilbert [1968] is to estimate the ρ_i jointly on the basis of ZEF residuals, (called JOINTEST procedure by them). This procedure was, in fact, considered by Parks [1967, p. 503, n.], but was rejected as unnecessary.

categories, and the latter lists implicit price deflators (1958=100) for the same categories. From this data we constructed four aggregates: Food, Clothing, Shelter, and Miscellaneous. These aggregates were used by Pollak and Wales [1969], who estimated several variants of SLES from the same series that we have used. Pollak and Wales, however, used only the data from 1948-1965 for the major part of their study, although they made comparisons between pre-war and post-war data, and found significant differences in the two samples. In this study, we have used the entire series from 1929-1964, but have constructed our aggregates exactly as Pollak and Wales [1969] did. We have also used the additions to the time series that became available recently. Thus, we added to our 1929-1964 data, additional data covering the period 1965-1968, which are provided in the July, 1969 issue of the Survey of Current Business.

In the construction of the four commodity aggregates, Pollak and Wales excluded all durable goods, transportation services, and gasoline and oil. The classifications of expenditures that were aggregated into the four commodities are listed below: (the numbers in the parentheses refer to the two tables cited above).

I. Food:

1. Food and Beverages (15)

II. Clothing:

1. Clothing and shoes (21)
2. Shoe cleaning and repair (54)
3. Cleaning, dyeing, pressing, etc. (55)

III. Shelter:

1. Housing (35)
2. Household operation services (39)
3. Semidurable housefurnishings (29)
4. Cleaning and polishing preparations, etc. (30)
5. Other fuel and ice (31)

IV. Miscellaneous:

1. Tobacco products (27)
2. Toilet articles and preparations (28)
3. Nondurable toys and sport supplies (33)
4. Barbershops, beauty parlors, and baths (56)
5. Medical care services (57)
6. Admission to specified spectator amusements (61)
7. Drug preparations and sundries (32).

The price index for the aggregates were constructed by taking a weighted average of the implicit price deflators for each primary expenditure item. The weights used were the ratios of expenditures on the primitive items to the total expenditure on the aggregate.

In the definition of variables, we differ somewhat from the procedure used by Pollak and Wales. First, unlike Pollak and Wales, we estimate community demand functions in the aggregate, instead of estimating per capita expenditure functions. Secondly, we define the quantity of the i-th commodity, (x_i), as the constant dollar expenditure divided by the implicit price deflator for the commodity. Pollak and Wales, on the other hand, chose to consider the constant dollar expenditures as "quantities"

(x_i). Both procedures seem perfectly acceptable to us, thought, it appears that our definition might be a more natural one.

4.5 Empirical Results

In this section we present the estimates, based on the four commodity data for the United States, for the parameters of the DOLCES and the DOLRES models proposed in sections 4.2 and 4.3. For the purposes of comaprison, we have also estimated from the same data, the Rotterdam model and Leser's linear expenditure system. Before considering each model in turn, we make a few remarks with regard to some statistical peculiarities of the data under consideration. In the estimation of SLES models from the same data, Pollak and Wales [1969] assumed the data to be free of serial correlation. It seemed to us that this was not a plausible specification for any time series. However, our results lead us to believe that if autocorrelation is present, it does not seem to follow a first-order Markov scheme. This is because our corrections for first-order autocorrelation were not very effective for most of the models considered. Specifically, even after we transformed the data, the estimates obtained showed the Durbin-Watson statistic to be significantly indicative of the presence of first order autocorrelation. A cursory visual examination of residuals plotted against time failed to

reveal any quadratic or higher order pattern. This would suggest then that the Durbin-Watson statistic is inappropriate for the data under consideration. In the presentation of our results, however, we continue to give the Durbin-Watson statistic, although it should be realized that its use as a test statistic may be limited.[89]

As it was indicated in the previous sections, the DOLRES model is estimated from the following specification:

$$w_{it}(\log x_{it}) = b_i(\Sigma_k w_{kt}\log x_{kt})$$
$$+ \Sigma_j c_{ij}(\log p_{jt} - \log p_{nt}) + u_{it}$$

where, $i=1,\ldots,n-1$; $t=1,\ldots,T$; $k=1,\ldots,n$.
The parameters b_i are approximations to the marginal budget shares, due to the fact that the real income term has been replaced by a volume index, although we shall continue to use the term "marginal budget shares" to describe them. The c_{ij} are the income compensated (Slutsky) price elasticities, weighted by the budget share of the i-th commodity. Alternatively, the c_{ij} may be interpreted as the Hicks-Allen elasticity of substitution between goods i and j, weighted by the inverse of the product of the

[89] It would have been extremely interesting to examine the applicability of some specification error tests devised by Ramsey [ ] for single equation estimates. Further research in this direction will be followed in another study.

respective budget shares. These parameters with the same interpretations, also appear in the Rotterdam model, which, it may be recalled, is specified as follows:

$$w^*_{it}\ Dx_{it} = b_i\ Dx_t + \Sigma_j\ c_{ij}\ Dp'_{jt} + u_{it}$$

where, $w^*_{it} = 1/2(w_{it}+w_{i,t-1})$, $Dx_{it} = (\log x_{it} - \log x_{i,t-1})$,

and $Dp'_{jt} = (\log p_{jt} - \log p_{nt}) - (\log p_{j,t-1} - \log p_{n,t-1})$.

The parameters b_i are once again approximations to the marginal budget shares, though they are slightly different from the b_i in DOLRES, by virtue of the fact that we have used an absolute index instead of a differential index (see pp. 3-10ff.). The two models may, therefore, be compared directly with respect to the values of the parameter estimates of c_{ij}, and to some degree with respects to the estimates of b_i. The estimates of DOLRES obtained by using a restricted ZEF-ZEF (Kmenta-Gilbert) procedure, are reported in Tables 1.1, 1.2, and 1.3.

It should be noted that at each step of the estimation procedure, the marginal budget shares are all positive and lie between zero and one, as we should expect. In addition, the income compensated own-price (Slutsky) elasticities are all negative, as theory would lead us to believe, and the matrix (estimated) of the coefficients c_{ij} is negative definite. The DOLRES model, therefore,

Table 1.1.--OLS/ZEF Estimates of Unconstrained DOLRES Model.

	"Income"	Food	Clothes	Shelter	
Food	0.1532 (0.0138)	-0.4643 (0.1479)	0.1643* (0.1501)	0.1014* (0.0699)	R^2=0.7328 DW=0.1125
Clothes	0.3143 (0.0048)	0.1373 (0.0508)	-0.0894* (0.0516)	0.1544 (0.0240)	R^2=0.7796 DW=0.1809
Shelter	0.2176 (0.0056)	0.2101 (0.0594)	-0.1711 (0.0603)	-0.4563 (0.0281)	R^2=0.9487 DW=0.2064

*Indicates that coefficients are less than twice their standard errors, and hence are not significantly different from zero. Note: Although parameter estimates are identical for OLS and ZEF, the estimates of standard errors differ. This table records OLS estimates of the standard errors.

Table 1.2.--ZEF Estimates of Unconstrained DOLRES Model Correcting for First Order Autocorrelation.

	"Income"	Food	Clothes	Shelter	
Food	0.2978 (0.0269)	-0.3575 (0.0413)	0.0965* (0.0526)	-0.0315* (0.0790)	R^2=0.9350 DW=0.6058
Clothes	0.2707 (0.0112)	0.1283 (0.0221)	-0.1170 (0.0285)	0.2022 (0.0446)	R^2=0.8231 DW=0.9145
Shelter	0.2214 (0.0114)	0.1411 (0.0230)	-0.0367* (0.0296)	-0.3463 (0.0488)	R^2=0.8421 DW=0.6100

*indicates coefficients not significantly different from zero.

Table 1.3.--ZEF Estimates of Constrained DOLRES model (With Correction for Autocorrelation).[a,b]

	"Income"	Food	Clothes	Shelter	
Food	0.2300	-0.3737	0.1432	0.1123*	R^2=0.9266 DW=0.4953
Clothes	0.2659		-0.1693	0.0304*	R^2=0.7603 DW=0.5295
Shelter	0.2266			-0.2328	R^2=0.7916 DW=0.2564

[a]The missing entries may be recovered by symmetry.

[b]Asymptotic F statistic for restrictions: $F_{2,99}$ = 12.052 (sig. Prob. < 0.005).

*indicates coefficients not larger than twice the standard errors of the unconstrained estimates. See text for a discussion.

fulfills the negative definiteness criterion (for this sample) which it should be recalled, was the only <u>a priori</u> knowledge that we held but did not impose in estimation.

Note, also, that the demand equation for the fourth commodity group "Miscellaneous" may easily be obtained from the Tables above, by virtue of the fact that we restrict the marginal budget shares to sum to unity, and the c_{ij} to be symmetric and sum to zero for each equation. In Table 1.3, it should be pointed out, that the standare errors of the parameters are not reported due to the unavailability of this feature in the statistical computer programs available at M.S.U. As an alternative, we used the well-known result (see Goldberger, 1964, p. 257) that the standard errors of the unrestricted estimators is an upper bound to the standard errors of the restricted estimators. With the use of this result, we see that with the exception of two parameters (about which we may not say anything), the rest of the parameters in Table 1.3 are significantly different from zero.

In Tables 3.1 and 3.2 we report, respectively, the unconstrained and constrained estimates of the Rotterdam model. No correction for autocorrelation was attempted for the Rotterdam model, which uses a differenced version of the data. We assumed that the differencing procedure would reduce the presence of any autocorrelation in the original time series. (This was, in fact true of Swedish data employed by Parks [1969], who reported that although the data indicated the presence of high serial correlation, the Rotterdam model indicated values of the Durbin-Watson statistic which were not indicative of the presence of serial correlation in its residuals.) In the case of our data, however, two of the three equations yield values of the Durbin-Watson statistic which would indicate the presence of first-order autocorrelation, if such a scheme was assumed to exist. We attribute this result, once again, to the peculiar nature of our data, for which the hypothesis of <u>first-order</u> autocorrelation may not be tenable.

For the Rotterdam model also, the estimates lie within the theoretically expected range. The estimates indicate, once again, the negative definiteness of the estimated matrix of Slutsky terms. More interesting, however, is the considerable similarity in numerical magnitude of the estimated c_{ij} for the Rotterdam model and the DOLRES model. This is, indeed, what we had expected. Although we have not tested the hypothesis of equivalence of the parameter estimates under the two models, a cursory comparison seems to suggest that this might, in fact, be the case. Note also the differences in the estimates of the b_i, which we had also anticipated.

Table 3.1.--Unconstrained OLS/ZEF Estimates of the Rotterdam Model.

	"Income"	Food	Clothes	Shelter	
Food	0.4626 (0.0376)	-0.3358 (0.0344)	0.1978 (0.0444)	0.1064 (0.0688)	R^2=0.9618 DW=0.8002
Clothes	0.2174 (0.0267)	0.1299 (0.0243)	-0.1638 (0.0314)	0.1739 (0.0486)	R^2=0.8596 DW=1.8211
Shelter	0.1856 (0.0372)	0.1340 (0.0339)	-0.0619 (0.0437)	-0.3120 (0.0676)	R^2=0.4732 DW=0.9556

Table 3.2.--Constrained ZEF Estimates of the Rotterdam Model.[a]

	"Income"	Food	Clothes	Shelter	
Food	0.4491	-0.3075	0.1420	0.0979	R^2=0.9599 DW=0.7306
Clothes	0.1816		-0.2217	0.0435*	R^2=0.8230 DW=1.4268
Shelter	0.2328			-0.1750	R^2=0.3472 DW=0.7595

[a]Asymptotic F statistic for restrictions: $F_{3,99}=3.764$. Corresponds to Sig. Probability of 0.013.

In Table 2.1, 2.2, and 2.3 we present the estimates of the parameters of the DOLCES model, which, it might be recalled, is specified by:

$$w_{it}(\log x_{it}) = b_i(\Sigma_k w_{kt}\log x_{kt}) + \Sigma_j d_{ij}(\log p_{jt} - \log p_{nt}) + u_{it}$$

where the b_i are to be interpreted as identical to their counterparts in the DOLRES MODEL, and hence are approximations to the marginal budget shares; and the d_{ij} represent the (constant) Hicks-Allen elasticities of substitution in the DOLCES model. It should also be noted that the assumption of constancy of the elasticities of substitution, will affect the parameter estimates of the b_i, so that the estimates of the b_i should be expected to

Table 2.1.--OLS/ZEF Estimates of Unconstrained DOLCES Model.[a]

	"Income"	Food	Clothes	Shelter	
Food	0.1545 (0.0121)	-3.4778 (0.7854)	3.8905 (1.9128)	0.7654* (0.5759)	R^2=0.7683 DW=0.1254
Clothes	0.3143 (0.0040)	2.3113 (0.6270)	-3.9855 (1.5654)	3.0470 (0.4151)	R^2=0.8175 DW=0.2109
Shelter	0.2266 (0.0055)	1.8475 (0.5632)	-3.0030 (1.4068)	-5.1947 (0.3288)	R^2=0.9431 DW=0.1586

[a]Although OLS and ZEF yield identical estimates of the coefficients, they differ on estimates of the standard errors. In this Table we report OLS estimates of the standard errors.

*indicates coefficient not significantly different from zero.

Table 2.2.--ZEF Estimates of Unconstrained DOLCES Model Correcting for First Order Autocorrelation.

	"Income"	Food	Clothes	Shelter	
Food	0.2855 (0.0330)	-2.1652 (0.2573)	1.3156* (0.7564)	-0.6238* (0.8084)	R^2=0.9296 DW=0.7031
Clothes	0.1729 (0.0039)	1.1809 (0.1100)	-5.0622 (0.3343)	1.8112 (0.3418)	R^2=0.9783 DW=0.7436
Shelter	0.1736 (0.0132)	0.9274 (0.1818)	-0.5776* (0.5452)	-3.6817 (0.4470)	R^2=0.8511 DW=1.2317

*Indicates coefficients not significantly different from zero.

Table 2.3.--Constrained ZEF Estimates of DOLCES Model (with Correction for Autocorrelation).[a]

	"Income"	Food	Clothes	Shelter	
Food	0.2777	-2.1017	1.1731*	0.4770*	R^2=0.9254 DW=0.6552
Clothes	0.1738		-5.0854	1.1051	R^2=0.9768 DW=0.5832
Shelter	0.1806			-3.6738	R^2=0.8080 DW=1.0774

[a]Asymptotic F statistic for restrictions: $F_{3,99}$ = 4.740 (Sig. P. = 0.004).

differ for the DOLRES and DOLCES models, although the economic meaning of the parameter is identical for the two models.

In evaluating the parameter estimates, we might note that for the DOLCES model, as before, the marginal budget shares lie between zero an unity as should be expected. Also, the Hicks-Allen own-elasticities-of-substitution, which are equivalent to the income compensated own (Slutsky) price elasticities, are all negative as should be expected. The magnitude of the estimates of the d_{ij} parameters may be compared to the estimates of c_{ij} in the Rotterdam model and the DOLRES model, only for specific values of the budget shares, by using the relationship

$$c_{ij} = w_i w_j d_{ij}.$$

Before making this comparison, we shall examine the estimates of Leser's LES, for which direct estimates of the cross-elasticities of substitution are obtained. We present these results in Table 4.1.

Leser's linear expenditure system, LLES, is estimated by the following specification:

$$p_{it}(x_{it} - \bar{x}_i) = b_i(\Sigma_k\, p_{kt}(x_{kt} - \bar{x}_k)) + \sum_{j\neq i} d_{ij}\, z_{ijt} + c_i{}^t + u_{it}$$

where, $\bar{x}_i = (1/T)\Sigma_t\, x_{it}$, and if we denote

$$\bar{w}_i = (1/T)\Sigma_t\, w_{it}\ ,$$

$$z_{ijt} = (\bar{w}_i p_{jt}\bar{x}_j - \bar{w}_j p_{it}\bar{x}_i).$$

Here, the d_{ij} are directly comparable to the d_{ij} of the DOLCES model. In the DOLCES model, however, we imposed the homogeneity condition by deflating one of the prices by the last price. This is not the case for LLES. In the LLES, the homogeneity condition is imposed only at the sample means of the expenditures, prices, and budget shares. Thus, the own-elasticity of substitution can be obtained in LLES by the following relation:

$$d_{ii} = -(1/\bar{w}_i) \sum_{j\neq i} \bar{w}_j d_{ij}$$

Table 4.1.--Unconstrained OLS/ZEF Estimates of Leser's LES.[a,b]

	Income	Food	Clothes	Shelter	Misc.	Trend	
Food	0.5411 (0.0883)		7.4281 (1.7940)	3.1830 (1.4324)	-7.4620* (3.9860)	-0.1776* (0.1485)	R^2=0.8605 DW=0.2004
Clothes	0.2671 (0.0236)	-0.1098 (0.0140)		3.3310 (0.9857)	-9.2532 (2.6657)	-0.1943 (0.0409)	R^2=0.9663 DW=1.2656
Shelter	0.2791 (0.0318)	2.6450 (0.6626)	-6.2158 (1.6376)		3.2341 (2.4796)	0.5139 (0.1013)	R^2=0.9710 DW=0.2984
Misc.	0.0975 (0.0093)	1.2518 (0.4709)	-1.6995 (1.0511)	-0.6571 (0.2512)		0.0858 (0.0192)	R^2=0.9476 DW=0.3843

[a]The standard errors of the estimated coefficients presented in this table are based on OLS estimates.

[b]The entries on the diagonal, $d_{ii} = -(1/\bar{w}_i) \sum_{j \neq i} \bar{w}_j d_{ij}$, where $\bar{w}_k$ denotes the sample mean of the k-th budget share.

*Indicates coefficients which are less than twice their standard errors.

Table 4.2.--Constrained ZEF Estimates of Leser's LES.[a,b]

	Income	Food	Clothes	Shelter	Misc.	Trend	
Food	0.3401		-0.1260*	-0.0091*	1.6258*	-0.4620	R^2=0.7817 DW=0.2243
Clothes	0.2016			0.2421*	-0.9747*	-0.3071	R^2=0.9560 DW=0.8194
Shelter	0.3153				0.5779*	0.6450	R^2=0.9525 DW=0.2974

[a]The off-diagonal entries may be recovered by invoking the symmetry of the elasticity of substitution. The elements of the diagonal are obtained as in Table 4.3.

[b]The asymptotic F statistic for testing the (linear) restrictions of symmetry, is in this case: $F_{3,99} = 12.480$, and has a significance probability of less than 0.0005.

*Indicates estimates of the restricted parameters which are less than twice the value of the geometric mean of the standard errors of the corresponding unrestricted estimates.

This is somewhat inconvenient for the purposes of direct comparison. Also, we might note that the DOLCES model imposes all of the classical restrictions globally, while LLES imposes only the adding-up restriction globally. Thus, the similarity between the DOLCES and LLES is not as great as for example between the DOLRES and the Rotterdam model.

We estimated LLES by a direct parallel to the estimation procedure used for DOLRES and DOLCES models. The ZEF/ZEF procedure was extended to the case of restricted estimation, where the restrictions were those of symmetry of d_{ij}. Once again, we did not correct for the presence of autocorrelation, due to the peculiar nature of the time series under consideration, despite the fact that the Durbin-Watson statistics indicated the presence of first-order autocorrelation, if such a specification was assumed to exist.

Although the estimates of LLES may not be directly compared to the DOLCES model because of the fact that own-elasticities of substitution are absent in LLES, we may still observe that the estimates of the cross-elasticities of substitution show a marked difference for the two models. Within LLES itself, the elasticities of substitution between two goods show considerable variations in their unconstrained estimates. Thus, a cursory examination of LLES in its unconstrained form reveals, for our data, that the elasticities of substitution do not appear to be symmetric. We may test the linear hypothesis of symmetry by observing the value of the (asymptotic) F-statistic, which is, in our case, associated with a probability value of less than 0.0005. Using the asymptotic F-statistic as a proxy for our small sample test, we would reject the hypothesis of symmetry conditional on Leser's specification of the functional form. Although we impose the restriction, despite its implausibility, we note that this result raises some doubts with regard to the empirical plausibility of LLES. Another consequence of imposing restrictions, which are implausible, may be seen in the values of the restricted estimates of the elasticities of substitution which are all considerably less than twice the geometric means of the standard errors of the corresponding unconstrained parameter estimates. Of course, this latter conclusion must be tempered by the fact that the geometric means of the standard errors are at best an approximation to the upper bound on the actual standard errors.

4.6 A Comparison of the Models

In the previous section we have reported the results of the estimation of the Rotterdam, DOLRES, and DOLCES models, and Leser's LES, on the basis of U.S. data. In this section we compare these models, both with regard

to differentiating between plausible and unplausible specifications, and with respect to comparing the values of the income and price elasticities implied by these models.

Several criteria may be used in the comparison of consumer demand models. The most obvious criterion is the comparison of the R^2 values for each equation, under the alternative specifications. These values give the proportion of variation in the dependent variable about its mean, explained by the explanatory variables. Alternatively, Parks [1969] and others have looked upon $(1-R^2)$, as a measure of "badness of fit" in opposition to the conventional "goodness of fit" criterion. The problem with R^2 comparisons lies in the upward bias of R^2 as an estimator of the population squared multiple correlation coefficient. This has been pointed out by Barten [1962].

An alternative criterion, proposed by Theil [1965], and Theil and Mnookin [1966], is the "average information inaccuracy of prediction" criterion, used with considerable success by Theil himself, Goldberger and Gamalesos [1967], Parks [1969], and others. This is based upon looking at budget shares, w_{it}, as probabilities that a given dollar in the consumer's budget will be spent on the ith commodity at time t. Thus, the w_{it} can be treated as prior probabilities, and their predicted values as

posterior probabilities, $\hat{w}_{it}$. With this interpretation, the "information inaccuracy of prediction" at time t is defined to be

$$I_t = w_{it} \log (w_{it}/\hat{w}_{it})$$

Thus, a measure of how well a model predicts over the sample period is the arithmetic mean of the information inaccuracy of prediction, or the "average information inaccuracy of prediction"

$$\bar{I} = (1/T) \sum_{t=1}^{T} w_{it} \log(w_{it}/\hat{w}_{it})$$

Unfortunately, the estimation procedure does not ensure that the predicted values of budget shares will not be negative. This has, in fact, been the case for the models estimated with our data. The first reason for this is that we are not, in fact, predicting budget shares directly, but the logarithms of quantities weighted by budget shares. Thus, the estimates of the budget shares are second round estimates, which turn out to be negative in several cases.

Yet another index of the plausibility of the model suggests itself from the paper by Parks [1969]. Parks' procedure has been to test the symmetry hypothesis for each model before imposing the symmetry restriction. Actually, the test of the symmetry hypothesis is conditional

on the validity of the functional specification. However, since our belief in the symmetry of the Slutsky terms may be stronger than our belief in the validity of the specific functional form, we may consider the significance probability of the asymptotic F statistic for the test of the (linear) symmetry hypothesis itself as an indicator of the plausibility of the model. Roughly speaking, this view of the F statistic expresses the belief that if it were known that either the functional form or the symmetry of the Slutsky terms (or both) were not valid, then we would be willing to reject the validity of the functional form before we reject the symmetry hypothesis.

In Table 5 we present the $(1-R^2)$, and the significance probability of the F statistic under the alternative specifications. On the least "badness of fit" criterion, the DOLCES model clearly dominates all the other models, with the exception of LLES on Shelter. The Rotterdam model seems to do better than the DOLRES model, with the exception of the estimates for the Shelter equation, for which the Rotterdam model has an extremely low R^2. Leser's LES seems to outperform the Rotterdam model on all but the Food equation. However, this must be tempered by the realization that the estimates for LLES violate the negative definiteness of the Slutsky matrix, implied by theory. This can readily be seen from Table 4.2, where the diagonal elements are not all negative. For this

reason we shall omit further discussion of LLES. It seems, then, that on the basis of the "badness of fit" criterion, DOLCES clearly dominates all models, while the Rotterdam model seems to outperform the DOLRES model. On the "concordance with the symmetry hypothesis" criterion, reflected by significance probabilities of the asymptotic F statistic for symmetry, the Rotterdam model ranks highest. The DOLCES model is also plausible, but the DOLRES model does not appear to be a serious competitor.

Table 5.--A Comparison of the Models with Regard to Badness of Fit and Significance Probability of the Asymptotic F statistic for the Symmetry Hypothesis.

	$(1-R^2)$			(Sig. P.)[a]
	Food	Clothing	Shelter	
Rotterdam	0.0401	0.1770	0.6528	0.013
DOLRES	0.0734	0.2397	0.2084	<0.005
DOLCES	0.0745	0.0232	0.1920	0.004
LLES	0.2183	0.0440	0.0475	<0.0005

[a]A high Sig. P. indicates that the Symmetry hypothesis is not rejected, and hence indicates a better model.

The three models may also be compared with respect to the implied values of the price and income elasticities, and the elasticities of substitution. In Tables 6.1, 6.2, 6.3 and 6.4 we present respectively, the income (Engel) ealsticities, the income compensated (Slusky) price elasticities, the price (Cournot) elasticities, and the Hicks-Uzawa elasticities of substitution, evaluated at the sample mean values of the budget shares, for the alternative models.

In comparing the income elasticities, it should be recalled that for the Rotterdam model, the DOLRES and the DOLCES models, the "income" elasticity is, in fact, approximated by a volume elasticity. This is not true for the Leser's LES. Apart from LLES, then, the estimates of the income elasticities show considerable uniformity in the DOLCES and DOLRES models, and even in the Rotterdam model. With the usual definitions, Food and Shelter come out as "necessities" under all three models, while Clothing is a "luxury" under all three models. The DOLRES and DOLCES models would lead us to classify the Miscellaneous items as "luxury" while the Rotterdam model would not. Clearly, then, while the numerical values of the estimates are comparable, differences do exist across models.

Tables 6.2, 6.3 and 6.4 provide estimates of the price elasticities and the elasticities of substitution. Leser's LES is easily seen to violate the condition that

Table 6.1.--Income (Engel) Elasticities Evaluated at Sample Means of the Budget Shares.

	Food	Clothing	Shelter	Miscel-laneous
Rotterdam	0.1242	1.1202	0.7792	0.9774
DOLRES	0.5757	1.6403	0.7585	1.9871
DOLCES	0.6952	1.0721	0.6045	2.6344
LLES	0.8514	1.2436	1.0554	1.0240

Table 6.2.--The Slutsky Matrix Evaluated at Sample Means of Budget Shares.

		Food	Clothing	Shelter	Miscel-laneous
Food	Rotterdam	-0.7697	0.3555	0.2451	0.1692
	DOLRES	-0.9355	0.3585	0.2811	0.2959
	DOLCES	-0.8396	0.1902	0.1425	-0.5069
	LLES	-0.0503	-0.0015	0.4857	0.4339
Clothes	Rotterdam	0.8760	-1.3676	0.2683	0.2233
	DOLRES	0.8834	-1.0444	0.1875	-0.0265
	DOLCES	0.4686	-0.8244	0.3302	-0.0256
	LLES	-0.0036	0.0392	-0.2912	-0.2556
Shelter	Rotterdam	0.3277	0.1456	-0.5858	0.1125
	DOLRES	0.3759	0.1018	-0.7792	0.3016
	DOLCES	0.1906	0.1791	-1.0976	-0.7279
	LLES	0.6495	-0.1580	0.1727	0.6641
Misc.	Rotterdam	0.4841	0.2592	0.2406	-0.9839
	DOLRES	0.8464	-0.0308	0.6452	-1.4608
	DOLCES	-1.4501	-0.0297	-1.5572	-3.0370
	LLES	1.2412	-0.2967	1.4208	2.3653

Table 6.3.--The Cournot Price Elasticity Matrix Evaluated at Sample Mean Values of Budget Shares.

		Food	Clothing	Shelter	Miscel-laneous
Food	Rotterdam	-1.2188	0.1732	-0.0908	0.0122
	DOLRES	-1.1655	0.2651	0.1091	0.2155
	DOLCES	-1.1173	0.0775	-0.0652	-0.6040
	LLES	-0.3904	-0.1395	0.2314	0.3150
Clothes	Rotterdam	0.4284	-1.5492	-0.0663	0.0669
	DOLRES	0.2281	-1.3103	-0.3025	-0.2556
	DOLCES	0.0403	-0.9982	0.0099	-0.1753
	LLES	-0.5004	-0.1624	-0.6627	-0.4292
Shelter	Rotterdam	0.0164	0.0193	-0.8186	0.0036
	DOLRES	0.0729	-0.0212	-1.0058	0.1957
	DOLCES	-0.0509	0.0811	-1.2782	-0.8123
	LLES	0.2279	-0.3291	-0.1426	0.5167
Misc.	Rotterdam	0.0936	0.1008	-0.0514	-1.1204
	DOLRES	0.0526	-0.3529	0.0515	-1.7383
	DOLCES	-2.5025	-0.4568	-2.3442	-3.4049
	LLES	0.8322	-0.4627	1.1149	2.2223

the matrix of Slutsky terms be negative definite. This would cast serious doubt on the plausibility of the LLES specification. The values reported for the double-logarithmic models show considerable concordance as well as a few marked divergences. For all of the three double-logarithmic models, the signs of the income compensated own-price elasticities are negative. In fact, all three models seem plausible on theoretical grounds. For only three price (Slutsky) elasticities (in the Miscellaneous column) the three models differ in sign from each other.

Table 6.4.--The Hicks-Uzawa Elasticity of Substitution Matrix Evaluated at Sample Mean Values of Budget Shares.

		Food	Clothing	Shelter	Miscel-laneous
Food	Rotterdam		2.1927	0.8203	1.2177
	DOLRES		2.2113	0.9409	2.1187
	DOLCES		1.1731	0.4770	-3.6299
	LLES		-0.0091	1.6258	3.1071
Clothes	Rotterdam			0.8982	1.2117
	DOLRES			0.6277	-0.1899
	DOLCES			1.1051	-0.1833
	LLES			-0.9747	-1.8302
Shelter	Rotterdam				0.8053
	DOLRES				2.1595
	DOLCES				-5.2121
	LLES				4.7556
Misc.	Rotterdam				
	DOLRES				
	DOLCES				
	LLES				

Thus, there seems to be a fiar degree of concordance among the three models in the classification of the commodity groups as "substitutes" and "complements" on the Hicks-Allen definition. The Cournot elasticities show wider variations between estimates, from one model to the other. Remarkably enough, however, all Cournot own-price elasticities are negative, lending credance to the belief that "demand curves are downward sloping." The values of the Hicks-Uzawa elasticity of substitution may be looked as the curvature of the indifference curves. These values vary considerably across models, and is positive in all but four cases.

4.7 Summary and Conclusions

To summarize, then, of the four models estimated, the Rotterdam model, the DOLRES and DOLCES models seem all to provide theoretically plausible estimates of the parameters; while Leser's LES seems to be deficient on this count. The Rotterdam model and the DOLCES model seem to be comparable in terms of desirable properties, while the DOLRES model seems to be in some conflict with the symmetry hypothesis with regard to the Slutsky terms. It would appear, then, that the constant elasticity of substitution hypothesis, which proved to be of so much value in the theory of production may hold some answers to the choice of an efficacious functional form for a complete set of demand equations. It would indeed by interesting to compare the performance of the Rotterdam model under its original specification, to the CES specification to which it can easily be transformed. For the non-differential double logarithmic case, the CES specification seems to do clearly better, and the question with regard to the differential double-logarithmic (Rotterdam) model seems quite open.

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