INVESTMENT RISK CONSIDERING INTRAPERIOD LIQUIDATION: A REEVALUATION OF PRICE VARIANCE

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This is to certify that the

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ABSTRACT

INVESTMENT RISK CONSIDERING INTRAPERIOD LIQUIDATION:

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By

John Steven Zdanowicz

Some of the current theories of investment selection have as their two main variables of analysis, risk and return. There is little controversy over the conceptual formulation of the measure of return, but how risk should be measured is still in dispute. Many conventional models attempt to measure risk by applying probability theory, with some measure of dispersion representing the risk of an investment, i.e. variance, standard deviation, coefficient of variation, etc. However, many conventional models are based on a static analysis, and thereofre, do not consider intraperiod decisions. The conventional model assumes that an investment will be held until and sold in period n, where n is defined as the investment period. Therefore, it is based on the following implicit assumptions: (1) no opportunity for voluntary liquidation prior to period n, (2) no risk of forced liquidation prior to period n, and (3) forced liquidation in period n. Given these assumptions, the conventional model correctly concludes that price variance will always have disutility and thus it should be minimized.

The purpose of this research was to analyze investment risk when the converse of assumption (1) is introduced, i.e., the investor may voluntarily liquidate his investment prior to period n. The remaining two implicit assumptions were maintained. With the introduction of this new assumption set, a dual component measure of risk, based on price variance, was developed. The calculation of the risk parameter is similar to that of the conventional model.

The difference is that the return distribution is not based on the price distribution of the terminal period but on the price distribution over the entire interval. The new risk parameter is defined as the product of the terminal price variance and the probability of exposure to terminal price variance. Both of these components may be expressed as functions of price variance, the former directly related to price variance and the latter inversely related to price variance. Thus it may not always be concluded that price variance will have disutility. An incremental analysis is necessary in order to determine the effect on the total measure of risk due to an incremental change in price variance.

The new model was tested for its internal consistency and a deductive analysis employing the new model along with different assumptions about price probability density functions was also made. The assumptions of the relationship of price probability density functions over time, which were analyzed, included homoscedasticity, heteroscedasticity-direct and heteroscedasticity-inverse. It was shown that in some cases the two components of the new measure of risk vary inversely with respect to each other when there is an incremental change in price variance. Thus proving that an incremental analysis of price variance is required. It was also shown that in some cases an increase in price variance does not increase risk, a conclusion which is contrary to the conclusion of the conventional model. In order to make this analysis, the concepts of elasticity of risk and marginal risk were developed.

Possible extensions of the new model with revised assumptions were also discussed. The revised assumptions include: the introduction of the time value of money, dependent subinterval distributions, and portfolio analysis. The practical applications of the new model were also considered.

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A THESIS

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TO LINDA

WITH ALL MY LOVE

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Of course, I, alone, am responsible for any errors or omissions.

TABLE OF CONTENTS

CHAPTER I INTRODUCTION

"I thank my fortune for it, my ventures are not in one bottom trusted, not to one place; nor is my whole estate upon the fortune of the present year. Therefore, my merchandise makes me not sad."

- Antonio, "The Merchant of Venice"

A. Evolution of the Concept and the Analysis of Risk

Some of the current theories of investment analysis have as their two main variables of analysis, risk and return. There is little controversy over the conceptual formulation of the measure of return, but how risk should be measured is still in dispute. The concept of risk is not new, for it was discussed and analyzed by economists of the classical school as well as those who have followed.

1. Pre-Markowitz

The discipline of Economics has long been concerned with the concept of investment. In both classical and neo-classical economics, the main elements which dominate the theory of investment are interest, profit and risk.

In The Wealth of Nations, Adam Smith analyzed an individuals motivation for saving (thus investment) and states that "consideration of his own private profit is the sole motive which determines the owner of any capital to employ it."[1]

Smith, however, realized that different investments returned different rates of profit. He concluded that profits include both a riskless return, i.e. rate of interest, plus a risk premium, and that "as long as there are

1. Adam Smith, The Wealth of Nations (New York: Random House), 1937, pg. 355.

profits over and above the compensation for risk, to be made by investment, capital accumulation will continue."[2] Smith believed that there was a minimum rate of profit which would be sufficient to compensate for risk.

David Ricardo also believed that profit was the motivational force of investment, for "while profits are high, men will have a motive to accumulate."[3] He also believed that profits from investment should contain some premium for risk taking. "Mens' motives for capital accumulation will diminish with every diminuation of profit, and will cease altogether when profits are so low as not to afford them an adequate compensation for their trouble, and the risk which they must necessarily encounter in employing their capital productively."[4]

Both Smith and Ricardo discussed profit and risk as the basic components of investment theory, and both measured profits as a rate of return. However, it was Ricardo who first attempted to isolate and account for risk in investment selection. In chapter one of Ricardo's Principles of Political Economy and Taxation, he suggested that investment selection should consider, "the time which must elapse before it (the product) can be brought to market, and the rapidity which it (capital) is returned to the employer." Ricardo's analysis of risk was very similar to the payback method which is employed by many firms at the present time.

2. Ibid, pg. 111.

3. David Ricardo, The Principles of Political Economy and Taxation (London : Dent and Son), 1937, pg. 193.

4. Ibid, pg. 73.

The concept of the time value of money was first introduced and developed by Eugen Von Böhm-Bawerk in The Positive Theory of Capital. He acknowledged the existence of an interest rate because future goods are valued less highly than present goods. The profit or interest rate which Böhm-Bawerk developed was an internal average rate of return (Ursprünglicher Zins). It was defined as the average rate of return earned in the production process by the capital tied up in it. The concept of a simple average rate of return was later criticized by Knut Wicksell[5] who introduced the concept of compound interest or rate of return.

Leon Walras introduced a negative cash flow, i.e. insurance premium into his cash flow analysis in order to adjust for risk, and discounted the net cash flows at the market rate of interest to account for the time value of money.

One of Frank Knight's contributions to investment theory was the differentation between risk and uncertainty. Risk, according to Knight, may be analyzed with probabilistic techniques. Uncertainty, which has no a priori probabilities of outcomes, cannot be analyzed in investment theory. Knight's development of the probabilistic aspects of risk was probably the stimulus for the shift to employing the tools of statistics and probability in the theory of risk.

By the early 1950's there were many techniques of adjusting for risk when employing a discounted cash flow analysis of investment. The first was to adjust the discount rate to include a risk premium, a procedure suggested by Smith and Ricardo.

5. Knut Wicksell, Lectures on Political Economy English Edition - 1934 Translated by E. Classen.

A second method as suggested by Walras was to decrease the expected cash flows to account for risk.[6] After Knight discussed the probabilistic characteristics of risk, Gunar Myrdal developed the certainty equivalent approach, where the expected cash flows were adjusted based on the probability of their occurrence.

The two parameter analysis of risk existed before the writings of Harry Markowitz[7] but it employed only the mean and variance of the individual investments as the variables of analysis. The coefficient of variation for an individual investment is a technique which employs this type of analysis of risk.

2. Harry Markowitz

The main contribution of Harry Markowitz[8] was the introduction of covariance into the risk parameter of the two parameter model, therefore proving the benefits of diversification. Markowitz re-defined portfolio risk to include not only the weighted summation of the risks of the individual investments, but also the correlation of the investments with each other. The subsequent research which evolved from the Markowitz Model added much to the theory of risk.

However, many of the refined two parameter models are still based on the parameters of the Markowitz model, i.e.,

> Return - The expected return of the portfolio which is defined as the weighted sum of the expected returns of the individual investments.

6. Also suggested in, J. R. Hicks, Value and Capital (New York : Oxford University Press, 1939), pg. 126.

7. Frank H. Knight, Risk, Uncertainty and Profit (Boston), Houghton Mifflin Company, 1921.

Variance - The variance of the portfolio which is defined as the variance of a weighted sum which includes the variance of the individual investments and their covariance.

B. The Conventional Two Parameter Model

Many of the refined two parameter models are similar in terms of their required inputs and determination of the parameters, risk and return.[9]

1. Inputs

The inputs for the two parameter model are: (a) the expected returns of the individual investments, (b) the variance of return of the individual investments, and (c) the covariance or correlation of the individual investments with each other. These inputs are calculated as follows:

(a) The expected return of an individual investment is a function of:

8. Harry Markowitz, "Portfolio Selection," Journal of Finance, VII, March, 1952, pp. 77-91. See also, Harry Markowitz, Portfolio Selection : Efficient Diversification of Investments, (New York : John Wiley & Son Inc., 1959), Cowles Foundation Monograph No. 16.

9. James Tobin, "Liquidity Preference as Behavior Towards Risk," The Review of Economic Studies, XXVI, February, 1958, pp. 65-86. Mentions Markowitz analysis for sets of risky investments. Tobins analysis is only of two investments, one of which is riskless. Therefore, covariance is zero for the investment returns are independent of each other.

William F. Sharpe, "Capital Asset Prices : A Theory of Market Equilibrium Under Conditions of Risk," Journal of Finance, XIX September, 1964, pp. 425-42.

John R. Hicks, "Liquidity," The Economic Journal, LXXII, December, 1962, pp. 787-802.

John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," The Review of Economics and Statistics, XLVIII, February 1965, pp. 13-37.

(1)	Time, - the holding period:	$t = 1.....n$
(2)	The initial price at time zero:	Po
(3)	The expected cash flows in period t:	E(At)
(4)	The pure rate of interest:	(i)

The expected return is determined:

$$E = \frac{\left[\sum_{t=1}^{n} \frac{E(At)}{(1+i)^{t}}\right] - Po}{Po} \qquad (1\text{-}1)$$

where the expected cash flows in period t, E(At) are:

(1)	Expected Dividends: E(Dt)	$t = 1.....n$
(2)	Expected Terminal Price: E(Pt)	$t = n$

therefore, (1-2)

$$E = \frac{\left[\frac{E(D1)}{(1+i)^{1}} + \frac{E(D2)}{(1-i)^{2}} + - - - + \frac{E(Dn-1)}{(1+i)^{n-1}} + \frac{E(Dn)}{(1+i)^{n}} + \frac{E(Pn)}{(1+i)^{n}}\right] - Po}{Po}$$

Lintner, for example, states that the investor "makes all purchases and sales of securities and all deposits and loans at discrete points in time, so that selecting his portfolio at any "transaction point," each investor will consider only (i) the cash throwoff (typically interest payments and dividends received) within the period to the next transaction point and (ii) changes in the market prices of the stocks during the same period. The return on any common stock is defined to be the sum of the cash dividends received plus the change in its market price." [10]

(b) The variance of return for an individual investment is a function of the variance of the expected cash flows of the periods $t = 1.....n$.

$$V = \frac{\sum_{t=1}^{n} \frac{V(At)}{(1+i)^{(2)t}}}{Po^{2}} \qquad (1\text{-}3)$$

where V(At) is the variance of an expected cash flow in period t, i.e. variance of dividend, V(Dt) or terminal price, V(Pn).

10. Lintner, p. 16.

Therefore, (1-4)

$$V = \frac{\frac{V(D1)}{(1+i)^2} + \frac{V(D2)}{(1+i)^4} + - - - + \frac{V(Dn-1)}{(1+i)^{2(n-1)}} + \frac{V(Dn)}{(1+i)^{2n}} + \frac{V(Pn)}{(1+i)^{2n}}}{Po^2}$$

(c) Covariance is defined as the expected value of (the deviation of i from its mean) times (the deviation of j from its mean), where i and j are two dependent random variables.

$$COVij = E \ (i - \mu i) \ (j - \mu j) \qquad (1\text{-}5)$$

Covariance measures the degree of dependence between the random variables.

It may also be expressed as a product of the correlation coefficient between two investments and their corresponding variances.

$$COVij = Rij \times Vi^{\frac{1}{2}} \times Vj^{\frac{1}{2}} \qquad (1\text{-}6)$$

where Rij is the correlation coefficient between investment i and investment j.

$$(-1 \leq Rij \leq 1)$$

2. Portfolio Expected Return and Risk

(a) Determination of the parameters

The expected portfolio return is a weighted sum of the expected returns of the individual investments (m).

$$Ep = \sum_{i=1}^{m} (a_i) \cdot (E_i) \qquad i = 1.....m \qquad (1\text{-}7)$$

where a_i is the proportion of the portfolio invested in investment i.

The portfolio variance is the variance of a weighted sum, which includes the variances of the individual investments and their covariances.

$$Vp = \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \cdot a_j \cdot COVij \qquad (1\text{-}8)$$

where i and j are investments in the set of investments m.

Ep and Vp are the two parameters which are analyzed in the two parameter model (Figure 1).

(b) Efficient Portfolios

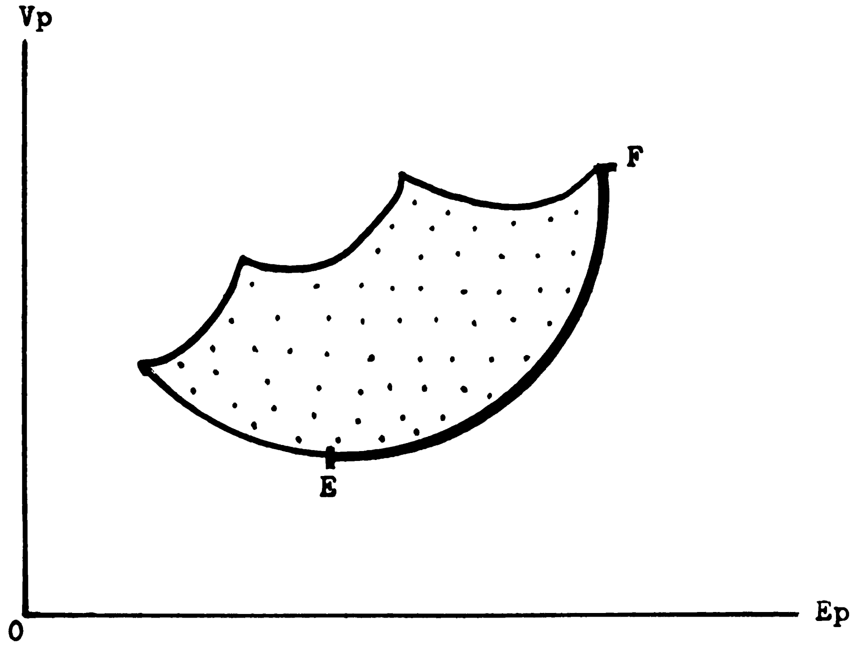

Figure 1

All of the portfolios along the boundary EF, are considered to be efficient portfolios. The line segment EF is considered to be Pareto optimal. Thus, at any point on EF the investor cannot increase return without also increasing risk, and cannot decrease risk without also decreasing return. All other portfolios which are not on EF are considered inefficient. The actual selection of a portfolio from EF will be a function of the investors utility for risk and return, i.e. his indifference curves.

C. The Measure of Investment Risk - Conventional Model

The direction of this analysis is concentrated on the measure of investment risk (V) which is an input into the current portfolio models.

1. Simplifying Assumption - Justification

Before an analysis of the measure of investment risk is made, some assumptions will be introduced in order to simplify the analysis. Other assumptions will also be introduced in order to concentrate the analysis on the specific area of investment risk. Some of these assumptions will be changed later.

(a) Initial price is given: Po

(b) Time horizon or investment period: t = 1.....n

(c) No tax or transaction costs.

(d) One security portfolios: This assumption is introduced in order to concentrate on investment risk only, and to avoid the analysis of covariance which may exist if a portfolio contains more than one security which are not independent of each other. The variance of a one security portfolio is simply the variance of the security. The assumption of independence between all securities would have accomplished the same end. This assumption will be eliminated later.

(e) No dividends: $D_1 \ldots\ldots D_n = 0$

This assumption is introduced because the analysis will concentrate on the relationship between price variance and risk.

(f) Normal price distributions are assumed in order to eliminate the need to consider other moments of the distribution, i.e. skewness and kurtosis.[11]

11. This end could also have been accomplished by assuming the investor had quadratic utility functions, i.e. his utility is a function of only two parameters, portfolio return (Ep) and portfolio risk or variance (Vp).

(g) The time value of money is ignored to simplify the analysis. At this point, the problem of reinvestment will not be considered.

2. Determination of the Parameters

The possible prices in period n and their corresponding probabilities are given as inputs into the model. They may be determined by standard procedures of security analysis. The expected return and variance of return (return density function) for the investment may be determined from these possible prices and their probabilities (price density function) as shown in Figure 2.

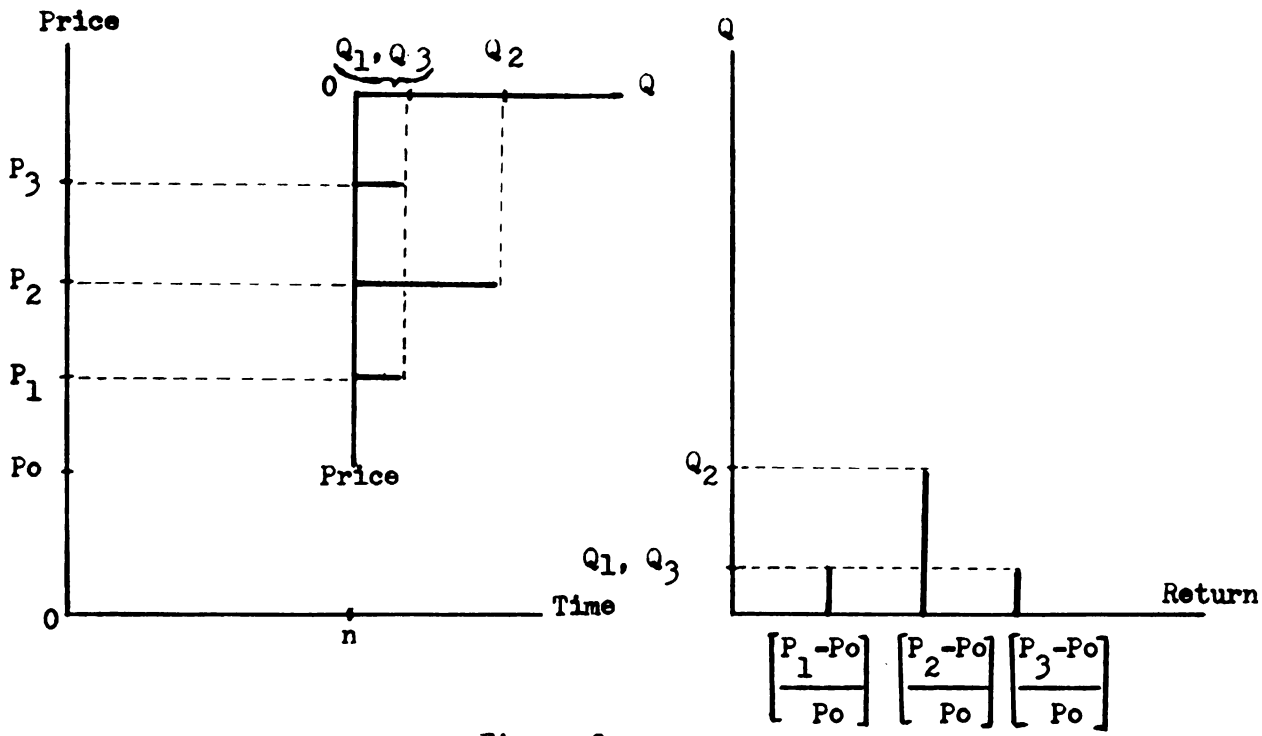

Figure 2

The expected portfolio return and variance of return is the same as the expected investment return and variance of return in the one investment portfolio, and is determined as follows.[12]

$$Ep = E = \frac{E(Pn) - Po}{Po} = \frac{\left[\sum_{x=1}^{z} (Pnx) \cdot (Qnx) \right] - Po}{Po} \quad (1\text{-}9)$$

$$Vp = V = \frac{V(Pn)}{Po^2} = \frac{\sum_{x=1}^{z} (Pnx - E(Pn))^2 \cdot (Qnx)}{Po^2} \quad (1\text{-}10)$$

where Pnx, x = 1.....z are the possible prices in period n

Qnx, x = 1.....z is the probability that Px will occur in period n

$$\sum_{x=1}^{z} Qnx = 1$$

3. Criteria for Selection

The selection of the optimal portfolio is based on the law of dominance.

(a) Law of Dominance

Because of the many assumptions and problems encountered when indifference curves are applied as a technique of selection criteria, an attempt will be made to limit their use. The analysis will be structured so that the selection of an optimal portfolio may be made by employing only the law of dominance and, therefore, determine conclusions without employing indifference curves.

12. The following example is stated in terms of discrete probability distributions. If price was expressed as a continuous price density function the equations would be expressed as follows:

$$Ep = \frac{\left[\int_0^\infty Pnx\, f(Pnx)\, dPnx \right] - Po}{Po}$$

$$Vp = \frac{\int_0^\infty (Pnx - E(Pn))^2 f(Pnx)\, dPnx}{Po^2}$$

The law or Axiom of Dominance[13] is derived from the discipline of price theory. Assume there are two goods X and Y, and that an individuals wealth is comprised of a bundle of these goods. Assume the individual initally holds bundle A which is comprised of (X_A, Y_A). (Figure 3)

Employing the law of dominance, it may be determined which bundles in the (X, Y) commodity space dominate (are preferred to) bundle (A) or which bundles (A) dominates.

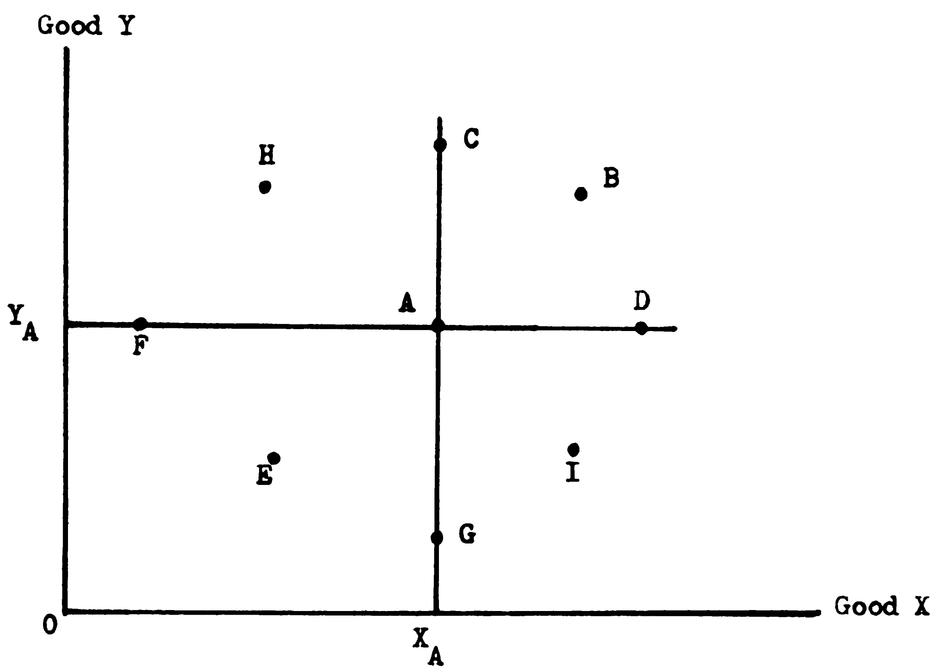

Figure 3

Strict Dominance - B strictly dominates A because

$$X_B > X_A \quad \text{and} \quad Y_B > Y_A$$

13. See Appendix A

Weak Dominance - C weakly dominates A because

$$X_C = X_A \quad \text{and} \quad Y_C > Y_A$$

D weakly dominates A because

$$X_D > X_A \quad \text{and} \quad Y_D = Y_A$$

Strict Dominance - A strictly dominates E because

$$X_A > X_E \quad \text{and} \quad Y_A > Y_E$$

Weak Dominance - A weakly dominates F because

$$X_A > X_F \quad \text{and} \quad Y_A = Y_F$$

A weakly dominates G because

$$X_A = X_G \quad \text{and} \quad Y_A > Y_G$$

It cannot be determined a priori whether bundles H and I are better or worse than A. Such information can come only from a more detailed knowledge of the individuals preference structure. Information on quantities alone is of no help.

The law of dominance assumes non-saturation of goods X and Y, i.e.

$$\frac{\partial U}{\partial X} > 0 \qquad (1\text{-}11)$$

$$\frac{\partial U}{\partial Y} > 0 \qquad (1\text{-}12)$$

where U is the individuals utility.

The two parameter model may also employ this type of analysis for it also is analyzing and comparing bundles (portfolios) of two commodities, risk and return. The basic difference in employing the law of dominance in this situation is that one of the goods (risk) is a nusiance good. Therefore, assuming nonsaturation in this two commodity space implies risk aversion, i.e.

$$\frac{\partial U}{\partial \text{return}} > 0 \tag{1-13}$$

$$\frac{\partial U}{\partial \text{risk}} < 0 \tag{1-14}$$

where U is the investors utility.

A graphical explanation of the application of the law of dominance is shown in Figure 4.

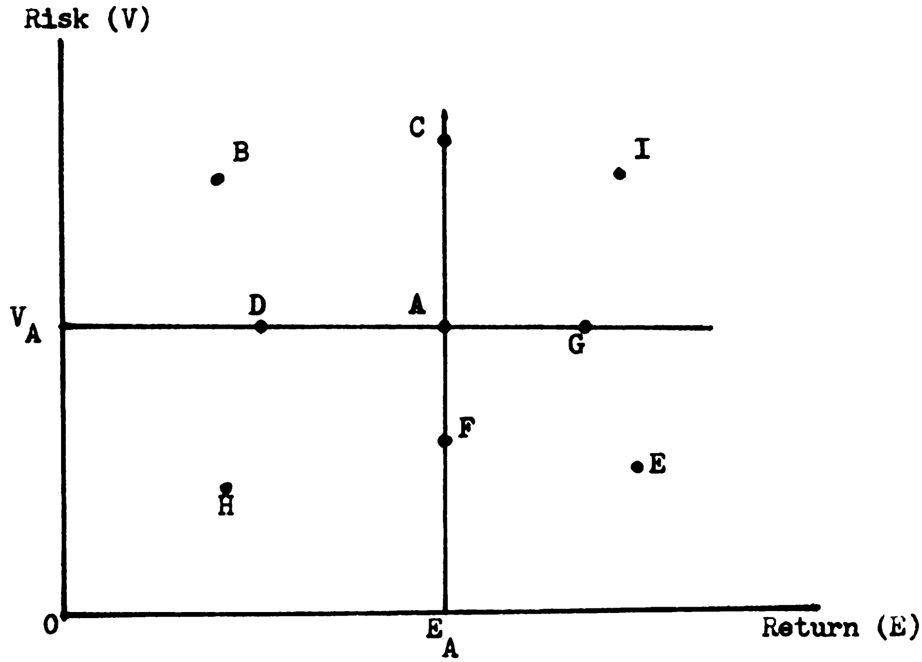

Figure 4

Assume an individual initially holds portfolio (A) which has an expected return of E_A and risk of V_A. Employing the law of dominance, it may be determined which portfolios in the (E, V) commodity space dominate (are preferred to) portfolio A or which portfolios A dominate.

Strict Dominance - Portfolio A strictly dominates portfolio B because

$$E_A > E_B \quad \text{and} \quad V_A < V_B$$

Weak Dominance - Portfolio A weakly dominates portfolio C because

$E_A = E_C$ and $V_A < V_C$

Portfolio A weakly dominates portfolio D because

$E_A > E_D$ and $V_A = V_D$

Strict Dominance - Portfolio E strictly dominates Portfolio A because

$E_E > E_A$ and $V_E < V_A$

Weak Dominance - Portfolio F weakly dominates portfolio A because

$E_F = E_A$ and $V_F < V_A$

Portfolio G weakly dominates portfolio A because

$E_G > E_A$ and $V_G = V_A$

As before, it cannot be determined a priori whether portfolios H and I are better or worse than portfolio A. The comparison of the quantities of expected return and risk is of no help; a more detailed knowledge of the investors preference structure, i.e. his utility for return and disutility for risk is necessary to make a comparison. Therefore, some assumptions will be made later which will allow operation in the quadrants where the law of dominance may be applied, and the detailed analysis of an investors utility is unnecessary.

(b) Example of the Application of the Law of Dominance

Assume there are two (one security) portfolios, A and B which have the same beginning prices, same expected price in period n, but have different price variances. (Figure 5)

$Po_A = Po_B$

$E_A(Pn) = E_B(Pn)$

$V_A < V_B$

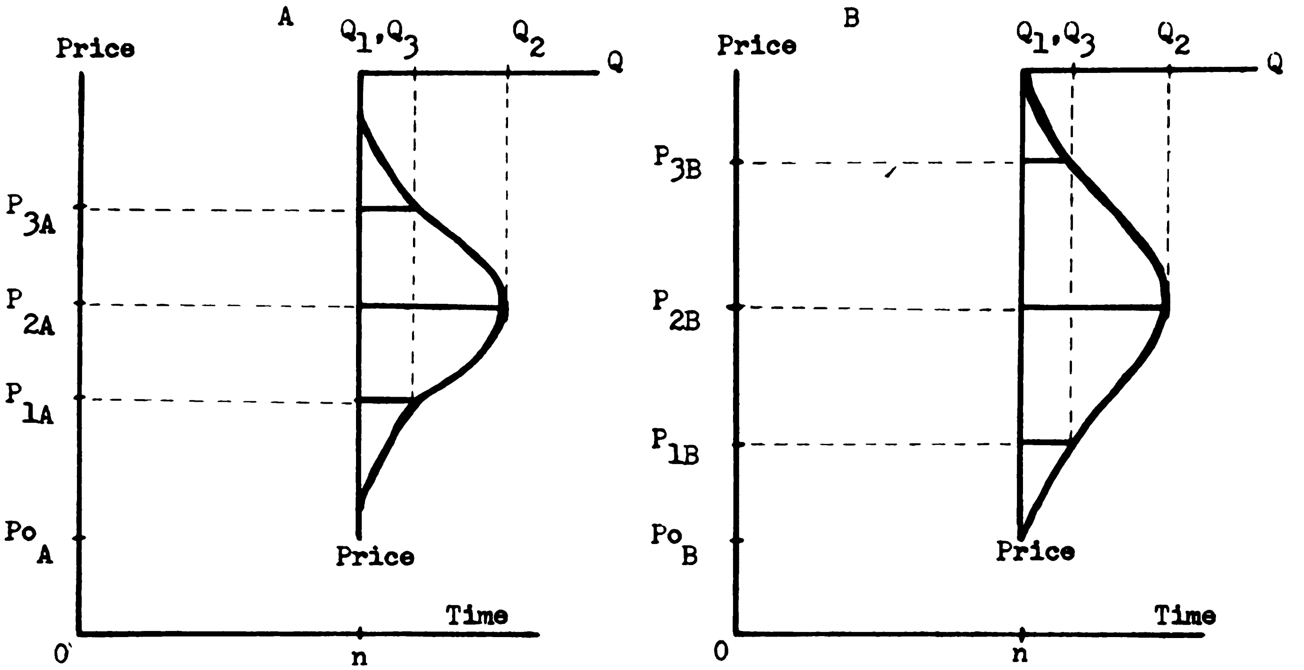

Figure 5

The price distributions can be converted into return distributions as shown earlier. Both A and B would have the same expected return, but B would have a larger variance of return. (Figure 6)

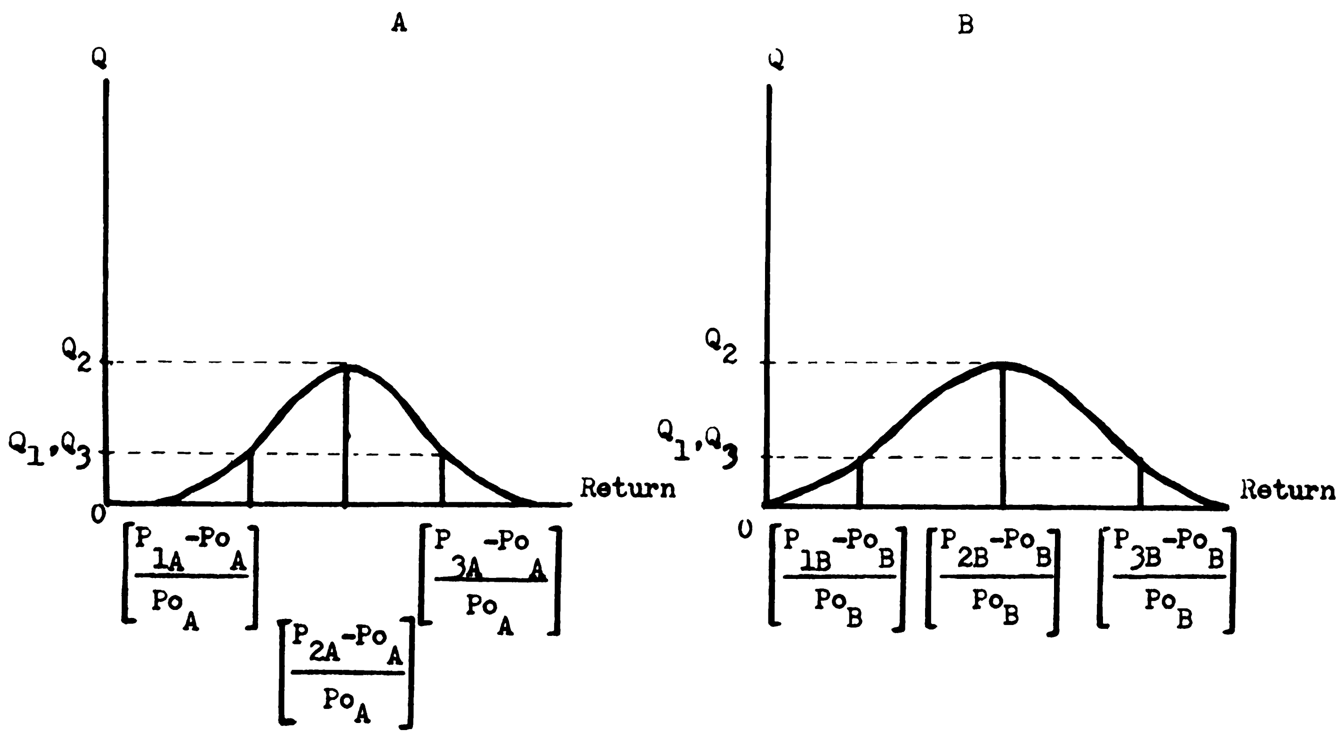

Figure 6

When plotted in the two parameter (E, V) space, portfolio A would be preferred to portfolio B, because portfolio A weakly dominates portfolio B. i.e. (Figure 7)

Portfolio A weakly dominates portfolio B because:

EpA = EpB

VpA < VpB

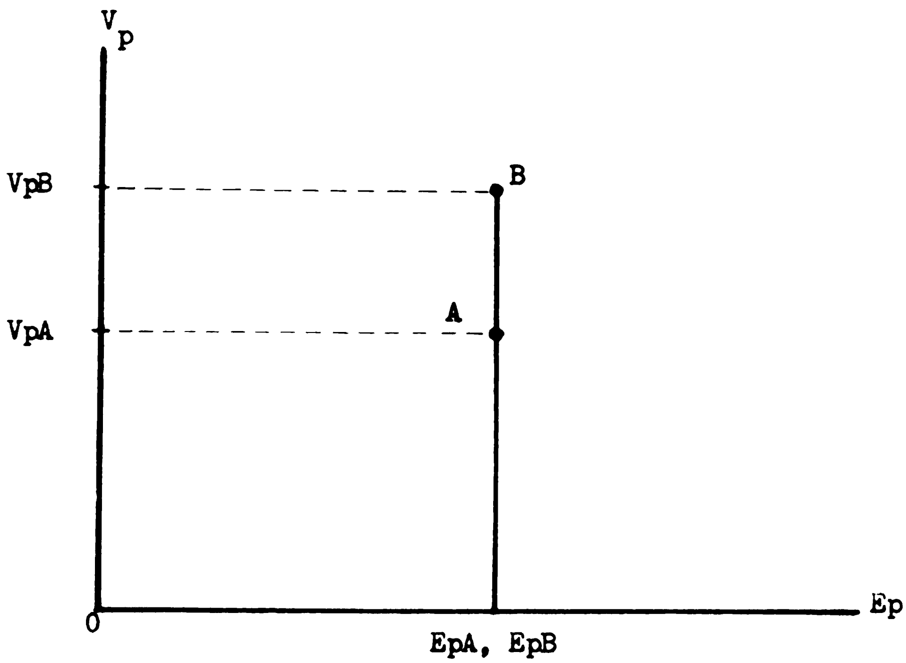

Figure 7

D. Utility and Price Variance

As stated earlier, we are assuming risk aversion or a disutility for risk, (assume U is a measure of investors utility). Therefore, the conventional two parameter model would conclude by the law of dominance, that for every expected rate of return, the investor would prefer the portfolio with the minimum amount of risk. In this model, risk is measured in terms of the portfolio variance of return. Therefore, it follows that the conventional model would also conclude that for every portfolio expected rate of return the investor should minimize portfolio variance of return. i.e.

Assume: $\frac{\partial U}{\partial \text{risk}} < 0$ Equation (1-14)

Risk = Variance of Return = Vp

Therefore, $\frac{\partial U}{\partial Vp} < 0$ (1-15)

However, variance of return is positively related to variance of price. As shown earlier, portfolio return variance is calculated by dividing price variance by the initial price squared, i.e.

$$Vp = \frac{V(Pn)}{Po^2}$$ Equation (1-10)

Assume: Po is given: (Assumption a)

Po $>$ 0

Therefore, $\frac{\partial Vp}{\partial V(Pn)} = \frac{1}{Po^2} > 0$ (1-16)

Therefore, if for every expected rate of return, the investor is to minimize risk, i.e. return variance, then it follows (employing the chain rule of derivatives) that for all securities with the same expected rate of return, price variance should be minimized, for price variance has disutility to the investor. i.e.

$$\frac{\partial U}{\partial \text{risk}} = \frac{\partial U}{\partial Vp} < 0$$ Equation (1-14), (1-15)

$$\frac{\partial Vp}{\partial V(Pn)} > 0$$ Equation (1-16)

Therefore: $\frac{\partial U}{\partial V(Pn)} = \frac{\partial U}{\partial Vp} \cdot \frac{\partial Vp}{\partial V(Pn)}$

(-) = (-) $\cdot$ (+)

$$\frac{\partial U}{\partial V(Pn)} < 0$$ (1-17)

Price variance is not considered in absolute dollar terms, but relative to the initial purchase price (Po). For simplicity it was and will continue to be assumed that the initial prices of the securities are equal and certain, therefore, the investor should minimize absolute price variance. However, if this assumption were not made, the measure of risk (return variance) would still be defined as an increasing linear function of price variance with a slope of $(1/Po^2)$ and the conventional model would conclude that the investor should minimize relative price variance for every expected rate of return. (Figure 8)

Given the assumption of this model, the conclusion that there is a disutility for price variance is correct (equation 1-17).

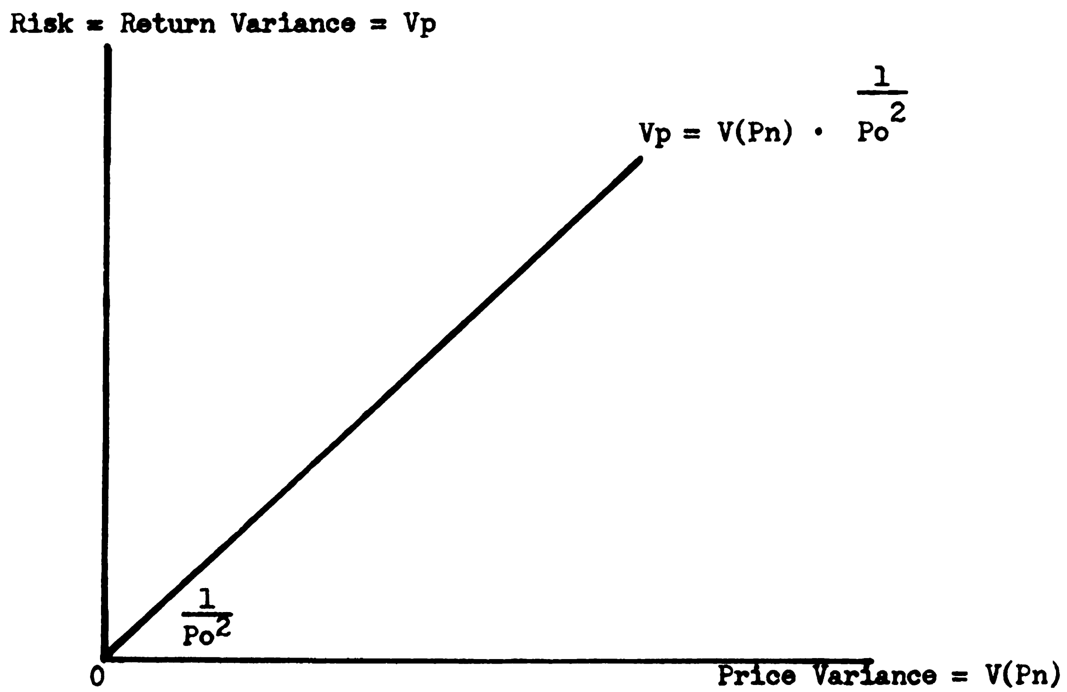

Figure 8

E. Implicit Assumptions and Conclusions of the Conventional Model

The basic charateristic of the conventional model is that it is a static model. It analyzes two discrete points in time and except for dividends, ignores any interim analysis of possible events. It is assumed that dividends are zero in order to concentrate on the implications of this static model, and especially its analysis of price variance. Because of this static analysis, the model has some implicit assumptions which warrant further investigation.

1. Implicit Assumptions About Investment Period

The conventional model assumes that the securities in the portfolio will be held until and sold in period n, where n is defined as the investors time horizon or investment period. Because of this general assumption, the model makes the following implicit assumptions.[14]

(a) The model implicitly assumes that the investor will not have the opportunity or the desire to voluntarily liquidate part or all of his portfolio before period n.

(b) It also implicitly assumes that the investor will not be forced to liquidate part or all of his portfolio before period n.

(c) An additional implicit assumption is that the investor will be forced to liquidate all of his portfolio in period n.

2. Conclusions of the Conventional Model

The measure of risk and the conclusion of disutility for price variance are sound, given the assumption, both explicit and implicit, and the structure of the model, i.e. the model is internally consistent or valid.

14. Throughout the remainder of this analysis, the "conventional model" will be defined as a model which makes the above assumptions.

CHAPTER II EXTENSION AND RE-EVALUATION OF THE CONVENTIONAL TWO PARAMETER MODEL

The purpose of this research is to analyze investment selection criteria, when different assumptions are introduced into the conventional model.

A. Development of a Model

A new model will be developed which will analyze investments and portfolios throughout the entire time interval, specified by the investors time horizon or holding period. The current model only analyzes the difference between the beginning and end of the interval.

The new model will introduce the possibility of making intraperiod decisions and actions. The conventional model assumes that action will be taken at the end of the period (n), and completely ignores the interim periods, except for dividends.

B. Assumptions of the Model

For simplicity, the proposed model will retain the simplifying assumptions of the model discussed earlier. Some additional assumptions, both simplifying and critical to the new model will also be made.

1. Previous Simplifying Assumptions

These are discussed and justified in Chapter I, Section C-1

(a) Initial price is given: Po

(b) Time horizon or investment period: $t = 1.....n$

(c) No tax or transaction costs.

(d) One security portfolios.

(e) No dividends: $D_1.....D_n = 0$

(f) Normal price distributions.

(g) No time value of money.

2. Additional Assumptions

Some additional assumptions will be introduced into the new model. The first, (a) will be a change of one of the implicit assumptions of the conventional model. The second, (b) will be a corollary assumption based on the first. The third and fourth assumptions, (c) and (d) are the remaining implicit assumptions of the conventional model. The fifth and sixth, (e) and (f) are simplifying assumptions and their change will be discussed later.[1]

(a) The investor has the freedom to trade before period n. He may voluntarily liquidate part or all of his portfolio before his expected holding period n has expired. This assumption is the converse of the first implicit assumption of the conventional model, and is a critical assumption of the new model.

(b) The investor may make a trade (sale) in any of the n-1 subintervals. The time horizon or investment period may be divided into n-1 discrete subintervals. The investor may liquidate part or all of his portfolio in any of these n-1 subintervals.

(c) The new model will explicitly continue to assume that the investor will not be forced to liquidate part or all of his portfolio before period n.

(d) The new model will also continue to assume that the investor must liquidate all of his portfolio in period n, which is an implicit assumption of the current model.

(e) The n-1 price density functions are normally distributed about a trend line (Po, E(Pn)). The assumption of homoscedasticity of the price density functions is made in order to simplify the analysis. The assumption of heteroscedasticity will be introduced later in the analysis.

1. Throughout the remainder of this analysis, the "new model" will be defined as a model which is based on assumptions, (a), (b), (c) & (d).

(f) The n price density functions are independent of each other.[2] This simplifying assumption is introduced in order that we may continue to employ the law of dominance and avoid employing the use of indifference curves. Independence of subinterval price density functions will allow the new model to maintain the same expected rate of return for a security as the conventional model. All changes will be reflected by changes in variance of return or risk. Therefore, conclusions may be made about the preference of securities by the law of weak dominance. The introduction of dependent subinterval price density functions will be discussed later.

C. The New Measure of Investment Risk

The new model will employ the same two parameters that were introduced by Markowitz, but it will introduce a new method of calculating risk, or variance of return. The proposed risk measure will also be a function of price variance, but it will have two components.

1. Terminal Risk

The terminal risk is exactly the same risk that is analyzed in the conventional model. It is the risk of period n, when it is assumed that the investor must liquidate his portfolio, and is calculated by the same method as in Chapter I, Section C-2 (Equation 1-10). Terminal risk is defined as the terminal return variance, and is a positive function of price variance (equation 1-16). As terminal price variance is increased, terminal return variance is also increased, ceteris paribus.

2. The assumptions of homoscedasticity and independent price density functions are not contradictory. Independent price density functions implies that price movements over time are independent. Homoscedasticity implies that the parameters of the density functions are not independent.

2. Risk of Exposure to Terminal Risk

The second component to the risk parameter is the risk of ever reaching period n, and facing the terminal risk defined in the previous section. This risk is an inverse function of the subinterval price variance. As variance around the price trend line increases, the risk of reaching period n decreases because there is some probability of reaching and selling the security at the price, $\beta = (E(Pn))$ before period n.[3] The new model assumes that if the price in a subinterval (1.....n-1) reaches the price, $\beta = (E(Pn))$, the investor will voluntarily liquidate, but if it remains below the expected price, he will hold the security. This risk will be measured in terms of the probability of reaching period n and being exposed to the terminal risk.

3. Total Risk of the Interval

The calculation of the risk parameter is similar to that of the conventional model. The basic difference is that the return distribution is not based on the price distribution of the terminal period, but on the price distribution over the entire interval. The possible prices of the distribution will not change but the probability of their occurrence will. If there is some variance through the interval, there may be some probability of not reaching period n. Thus, the probabilities of the terminal possible prices will be the product of their original probabilities and the probability of reaching period n. The probability of not reaching period n is the probability of selling at an earlier time at $\beta = E(Pn)$. Therefore, the actual price distribution over the time interval will be more leptikurtic than the terminal

3. The subinterval liquidation value is defined as β. In order to maintain the same expected price of the conventional model, the new model will assume that $\beta = E(Pn)$.

price distribution. The return distribution may be determined from this interval price distribution as shown earlier. This new expected return and variance of return will be the two parameters the new model will employ in selecting investments.

D. Analysis of the Disutility of Price Variance

With this new measure of risk, it may not be concluded that price variance will always have disutility, because one component of total risk, i.e. the risk of exposure to terminal risk decreases as price variance is increased. Therefore, a marginal analysis of price variance must be made. The new model will account for the marginal increase in terminal risk and the marginal decrease in the risk of exposure to terminal risk due to an incremental increase in price variance. It may be found that risk is not a positive linear function of price variance and possibly that total risk as the new model has defined it may actually decrease over a range, as price variance is increased. Given the assumption of the new model, it may reveal that some price variance will have utility for the investor, which is contradictory to the conclusions of the conventional model (equation 1-17).

E. Problem of Employing the Conventional Model for This Analysis

One might argue that the conventional model may be employed by analyzing each of the subintervals (1.....n-1). However, this approach would be improper because there is no risk of forced liquidation during the subintervals. The conventional model would analyze each subinterval distribution as if it were a terminal distribution. This would be an improper analysis, because if there is no risk of forced liquidation in a subinterval, then there is no downside risk in that subinterval. The investor may voluntarily hold the security until the next subinterval. However, there may be upside potential if there is some probability of the security reaching the expected price.

The conventional model would conclude that there was a larger amount of risk than actually exists in the investment. It would also always imply a disutility for price variance. The conventional model would be employed if, in fact, every subinterval had risk of forced liquidation.

F. The New Model - Determination of the Parameters

The following section will explain how an investor would determine the two parameters (risk and return), that will be employed in the new model. It is assumed that the investor has employed some method for determining his expectations about future prices.

1. Discrete Case

The investor will specify some holding period (n) which has n-1 subintervals in which he may make one trade. As inputs into the model, the investor will determine possible prices and their probabilities for periods t = 1.....n. Figure 9 depicts these inputs.

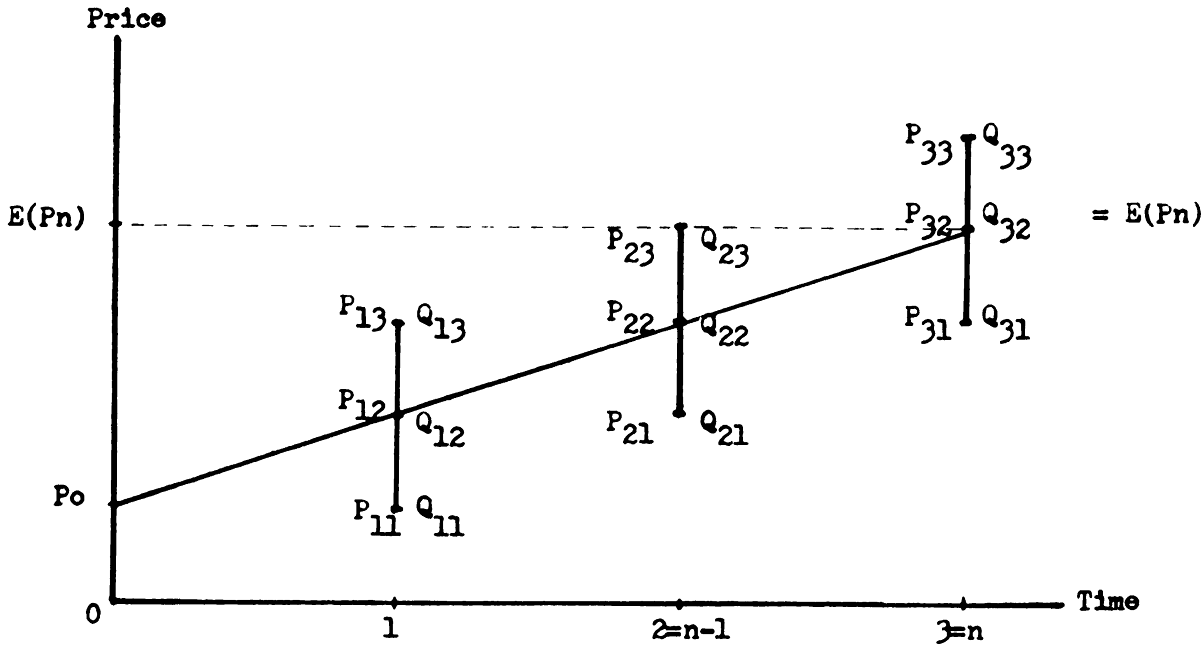

Figure 9

Ptx are possible prices in period t:

$t = 1.....n$ periods $\quad n = 3$
$x = 1.....z$ possible prices $\quad z = 3$

Qtx is the probability of Px occurring in period t:

$t = 1.....n$ periods $\quad n = 3$
$x = 1.....z$ probabilities $\quad z = 3$

Assume "Price" is a random variable and Ptx is a particular price x in period t. The probability function of "Price", f(Ptx) is defined as a rule that assigns to each real price, Ptx, the probability that the real random variable "Price" is equal to the value of Ptx, as shown in Table 1.

$$f(Ptx) = Q(\text{"Price"} = Ptx) \qquad t = 1.....n, \quad x = 1.....z$$

Table 1

t	f(Ptx) = Qtx
t = 1	f(P1x) = Q13 if "Price" = P13 f(P1x) = Q12 if "Price" = P12 f(P1x) = Q11 if "Price" = P11
t = 2 = n-1	f(P2x) = Q23 if "Price" = P23 f(P2x) = Q22 if "Price" = P22 f(P2x) = Q21 if "Price" = P21
t = 3 = n	f(P3x) = Q33 if "Price" = P33 f(P3x) = Q32 if "Price" = P32 f(P3x) = Q31 if "Price" = P31

(a) Terminal Expected Price and Price Variance

The expected terminal price and terminal price variance are determined in the same manner as in the conventional model. At this point, the expected price and variance of price are not converted to expected return and variance of return as in the conventional model. For period n, the calculations are:

$$E(Pn) = \sum_{x=1}^{z} (Pnx) \cdot f(Pnx) \qquad (2\text{-}1)$$

$$= \sum_{x=1}^{3} (P3x) \cdot f(P3x)$$

$$= ((P33) \cdot (Q33)) + ((P32) \cdot (Q32)) + ((P31) \cdot (Q31))$$

The variance of price in period n is calculated as follows:

$$V(Pn) = \sum_{x=1}^{z} (Pnx - E(Pn))^2 \cdot f(Pnx) \qquad (2\text{-}2)$$

$$= \sum_{x=1}^{3} (P3x - E(P3))^2 \cdot f(P3x)$$

$$= (P33 - E(Pn))^2 Q33 + (P32 - E(Pn))^2 Q32 + (P31 - E(Pn))^2 Q31$$

(b) Risk of Exposure to Terminal Risk

Next the investor must determine the probability of facing the terminal price variance, i.e. Kn for the subintervals t = 1.....n-1.

First, the investor will determine Lt which is defined as the probability of liquidation in subinterval t. The investor will liquidate in period t = 1.....n-1 only if Ptx is equal to or greater than $\beta = E(Pn)$.

$$Lt = Q(Ptx \geq \beta = E(Pn)) = \sum_{Ptx \geq \beta = E(Pn)} f(Ptx) \qquad (2\text{-}3)$$

<u>Period 1</u> - There is zero probability of liquidating in period 1.

$$L1 = \sum_{P1x \geq \beta = E(Pn)} = 0$$

<u>Period 2 = n-1</u> - The probability of liquidating in period 2 is Q23.

$$L2 = \sum_{P2x \geq \beta = E(Pn)} = Q23$$

St is defined as the probability of not liquidating in period t (t = 1.....n-1).

$$St = 1 - Lt \tag{2-4}$$

<u>Period 1</u> $S1 = 1 - L1 = 1 - 0 = Q13 + Q12 + Q11$

<u>Period 2</u> $S2 = 1 - L2 = 1 - Q23 = Q22 + Q21$

Kn is the probability of reaching period n and facing the terminal price variance V(Pn).

Kn is defined as the product of the St's t = 1.....n-1.

$$Kn = \prod_{t=1}^{n-1} St \tag{2-5}$$

$$Kn = (S1) \cdot (S2)$$

$$Kn = (1 - L1) \cdot (1 - L2)$$

$$Kn = (Q13 + Q12 + Q11) \cdot (Q22 + Q21)$$

(c) Total Risk and Expected Return

The investor will now determine the expected price and variance of price over the interval. The new probabilities are calculated in Table 2.

Table 2

Px	Probability	= f*(Px)
P3	(Kn) • (Q33)	= Q3*
P2 = β = E(Pn)	(Kn) • (Q32) + (1 - Kn)	= Q2*
P1	(Kn) • (Q31)	= Q1*

(1 - Kn) is the probability of not reaching period n which means that the security was sold at an earlier point in time at the price β = E(Pn).

The expected price, E(P)*, and the variance of price, V(P)*, over the time interval are calculated as follows:

$$E(P)^* = \sum_{x=1}^{z} (Px) \cdot (f^*(Px)) \tag{2-6}$$

$$= ((P3) \cdot (Q3^*)) + ((P2) \cdot (Q2^*)) + ((P1) \cdot (Q1^*))$$

$$V(P)^* = \sum_{x=1}^{z} (Px - E(P)^*)^2 \cdot (f^*(Px)) \tag{2-7}$$

$$= (P3 - E(P)^*)^2 \cdot (Q3^*) + (P2 - E(P)^*)^2 \cdot (Q2^*) + (P1 - E(P)^*)^2 \cdot (Q1^*)$$

From the expected interval price and the interval price variance, the investor can determine the expected interval return and interval return variance.

$$E^* = \frac{E(P)^* - Po}{Po} \tag{2-8}$$

$$V^* = \frac{V(P)^*}{Po^2} \tag{2-9}$$

E* and V* will be employed as the two parameters of the model, where E* is the expected return of the interval and V* is the variance of return of the interval. E* and V* would be calculated for the m securities, and the investor would select based on his risk return preference.

2. Continuous Case

Again, the investor will specify his holding period, (n) and the number of subintervals (1.....n-1). The possible prices in period t and their probabilities will be stated in terms of a continuous probability density function, f(Ptx), Figure 10, where:

f(P1,x) = Probability density function in period 1.

. .
. .
. .

f(Pn-1,x) = Probability density function in period n-1.

f(Pn,x) = Probability density function in period n.

Ptx is a possible price in period t. The probability of Ptx occurring in period t is:

$$Q(Pt,x-1 \leq Ptx \leq Pt,x+1) = \int_{Pt,x-1}^{Pt,x+1} f(Ptx)\, dPtx \qquad (2\text{-}10)$$

where f(Ptx) is the probability density function for period t.

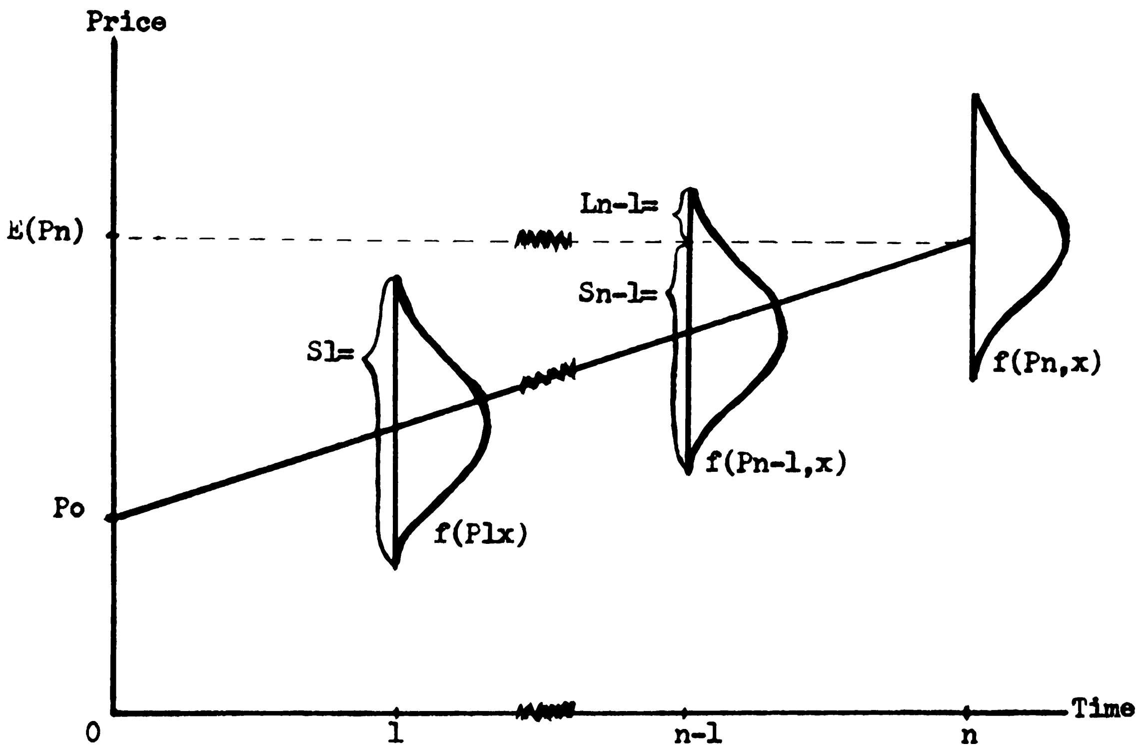

Figure 10

(a) Terminal Expected Price and Price Variance

The expected terminal price and terminal price variance are calculated as follows:

$$E(Pn) = \int_{0}^{\infty} (Pnx)\ f(Pnx)\ dPnx \qquad (2\text{-}11)$$

$$V(Pn) = \int_{0}^{\infty} (Pnx - E(Pn))^2\ f(Pnx)\ dPnx \qquad (2\text{-}12)$$

Again, as in the discrete case, the investor does not convert expected price and variance of price to expected return and variance of return.

(b) Risk of Exposure to Terminal Risk

The risk of exposure to terminal price variance is a function of the probability density functions of the subintervals, $t = 1.\dots.n-1$. As in the discrete case, the investor must determine Lt which is defined as the probability of liquidating the security in subintervals $t = 1.\dots.n-1$, which he will do if Ptx is equal to or greater than $\beta = E(Pn)$.

$$Lt = Q(Ptx \geq \beta = E(Pn)) = \int_{\beta = E(Pn)}^{\infty} f(Ptx)\ dPtx \qquad (2\text{-}13)$$

where: $t = 1.\dots.n-1$

The probability of liquidating in subperiod 1, L1 is determined by the subinterval cumulative probability distribution of price in period 1 being equal to or greater than $\beta = E(Pn)$.

L1 = The probability of liquidating in period 1.

$$L1 = \int_{\beta = E(Pn)}^{\infty} f(P1x)\ dP1x$$

The probability of liquidating in the following subperiods up to n-1 is calculated in the same manner. The probability of liquidating in period n-1 is:

$$Ln-1 = \int_{\beta = E(Pn)}^{\infty} f(Pn-1,x)\ dPn-1,x$$

Next, the investor must determine St which is the probability of not liquidating the security in subinterval t = 1.....n-1.

$$St = 1 - Lt \qquad (2\text{-}14)$$

where: t = 1.....n-1

The probability of not liquidating, (St), in the subintervals 1.....n-1, is calculated as follows:

Period 1 : S1 = (1 - L1)

.

.

.

Period n-1 : Sn-1 = (1-Ln-1)

The probability of reaching period n and being exposed to terminal price variance, (Kn) is the product of the St's t=1.....n-1.

$$Kn = \prod_{t=1}^{n-1} St \qquad (2\text{-}15)$$

$$Kn = (S1)\cdots(Sn\text{-}1)$$

$$Kn = \left(1 - \int_{\beta = E(Pn)}^{\infty} f(P1x)\, dP1x\right)\cdot\cdot\cdot\left(1 - \int_{\beta = E(Pn)}^{\infty} f(Pn\text{-}1,x)\, dPn\text{-}1,x\right)$$

Therefore:

$$Kn = \prod_{t=1}^{n-1} \left(1 - \int_{\beta = E(Pn)}^{\infty} f(Ptx)\, dPtx\right)$$

The probability of not reaching the terminal period, i.e. the probability of selling at β = E(Pn) in an earlier subinterval is (1 - Kn).

(c) Total Risk and Expected Return

Next, the investor must determine the probability density function for prices over the time interval t = 1.....n.

It may be expressed as follows:

$$f^*(Px) = \left[Kn \cdot f(Pnx) \left| \frac{dPnx}{dPx} \right| \right] + \left[(1-Kn)\cdot \alpha(\beta) \right] \qquad (2\text{-}17)$$

The Jacobian, $\left|\frac{dPnx}{dPx}\right|$ is introduced because of a change of variables.[4]

β is defined as the liquidation price or the price at which the investor would sell the security in the subintervals 1.....n-1. In this model β was assigned the value E(Pn), i.e. $\beta = E(Pn)$.

Alpha is defined as a Kronicker delta function and may only have two values, zero or one. In the interval price density function, when the random variable Px is equal to β, alpha is equal to one. When the random variable Px is greater than or less than β, alpha is equal to zero, i.e.

$$\alpha = 1 \text{ if } Px = \beta = E(Pn)$$

$$\alpha = 0 \text{ if } Px \neq \beta = E(Pn)$$

The expected price and variance of price for the interval may now be determined from this new interval probability density function of price $f^*(Px)$.

$$E(P)^* = \int_0^\infty (Px)\ f^*(Px)\ dPx$$

$$V(P)^* = \int_0^\infty (Px - E(P)^*)^2\ f^*(Px)\ dPx \qquad (2\text{-}19)$$

The expected return and variance of return for the interval may be determined from the expected price and variance of price for the interval i.e.

$$E^* = \frac{E(P)^* - Po}{Po} \qquad (2\text{-}20)$$

$$V^* = \frac{V(P)^*}{Po^2} \qquad (2\text{-}21)$$

The new expected return and variance of return will be employed as measures of investment return and risk in the two parameter model.

4. See Appendix B.

CHAPTER III INTERNAL CONSISTENCY AND ANALYSIS OF THE NEW MODEL

The development of the model in the previous chapter appears to be internally consistent, given the assumptions that were introduced or changed. However, before the model may be developed further, internal consistency[1] must be proven along with the techniques employed to determine the parameters.

A secondary analysis will also be made of the relationship between utility and price variance with the introduction of different assumptions about price probability density functions over time. This analysis will show that under certain conditions, the conclusion that price variance always has disutility may not be true.

A. Proof of Internal Consistency

Both the conventional model and the new model specify investments in terms of their expected rates of return and variance of returns. The basic difference between the two models is that the former derives its parameters from the terminal price density function while the latter derives its parameters from the interval price density function.

1. Analysis of Interval Price Density Function

It was shown earlier that the interval price density function must be derived from the intraperiod and terminal period density functions. The derived interval price density function will now be analyzed in order to prove that it is a valid density function.

1. The term internal consistency is synonymous with the term valid. All that is required for a deductive argument to be valid is that the premise or premise set, whether true or false, imply the conclusion. From William H. Halberstadt, An Introduction to Modern Logic, (New York: Harper & Brothers), 1960, pp.11.

In order to prove that the interval price density function is valid, it must be shown that the function is always positive and the integral of the function equals unity. The interval price density function is:

$$f^*(Px) = \left[Kn \cdot f(Pnx) \left| \frac{dPnx}{dPx} \right| \right] + \left[(1-Kn) \cdot \alpha(\beta) \right] \quad \text{(Equation 2-17)}$$

where: β = Liquidation value in periods 1.....n-1

and: $\alpha = 1$ if $Px = \beta$

$\alpha = 0$ if $Px \neq \beta$

The integral of the function is:

$$\int_0^\infty f^*(Px)\, dPx = \int_0^\infty \left[Kn \cdot f(Pnx) \left| \frac{dPnx}{dPx} \right| + (1-Kn) \cdot \alpha(\beta) \right] dPx$$

$$= Kn \cdot \int_0^\infty f(Pnx) \left| \frac{dPnx}{dPx} \right| dPx + (1-Kn) \cdot \int_0^\infty \alpha(\beta)\, dPx$$

where: $\alpha = 1$ if $Px = \beta$

$\alpha = 0$ if $Px \neq \beta$

and: $0 \leq Kn \leq 1$

and: $\int_0^\infty f(Pnx)\, dPnx = 1$

Therefore substituting and integrating.

$$\int_0^\infty f^*(Px)\, dPx = Kn \cdot 1 + (1-Kn)$$

$$= Kn + 1 - Kn$$

$$= 1 > 0$$

It may, therefore, be concluded that the interval price density function $f^*(Px)$ and its parameters are valid. i.e.

$$E(P)^* = \int_0^\infty Px \; f^*(Px)\, dPx \quad \text{(Equation 2-18)}$$

$$V(P)^* = \int_0^\infty (Px - E(P)^*)^2 \; f^*(Px)\, dPx \quad \text{(Equation 2-19)}$$

2. Analysis of Terminal Expected Price and Interval Expected Price

In the development of the model, the assumption of independent subinterval distributions was made so that the law of dominance could be employed. This assumption allows the expected price of the terminal price density function to equal the expected price of the interval price density function. This in turn allows the ranking of investments by price variance. Another necessary assumption was that β, the intraperiod selling price would be the expected price of the terminal price density function. Given these two assumptions, it may be shown that the expected terminal price E(Pn), is equal to the expected interval price $E(P)^*$. i.e.

$$f^*(Px) = \left[Kn \cdot f(Pnx) \left| \frac{dPnx}{dPx} \right| \right] + \left[(1-Kn) \cdot \alpha(\beta) \right] \qquad \text{(Equation 2-17)}$$

$$E(P)^* = \left[Kn \cdot \int_0^\infty Pnx \; f(Pnx) \; dPnx \right] + \left[(1-Kn) \cdot \alpha(\beta) \right]$$

where: $\beta = E(Pn)$

and: $E(Pn) = \int_0^\infty Pnx \; f(Pnx) \; dPnx$

therefore substituting:

$$E(P)^* = \left[Kn \cdot E(Pn) \right] + \left[(1-Kn) \cdot \alpha(E(Pn)) \right]$$

$$= (1-Kn + Kn) \; E(Pn)$$

$$= E(Pn) \qquad (3\text{-}1)$$

3. Analysis of the Measure of Risk

It was shown earlier that the interval price variance could be determined from the interval price density function. i.e.

$$V(P)^* = \int_0^\infty (Px - E(P)^*)^2 \; f^*(Px) \; dPx \qquad \text{(Equation 2-19)}$$

However, the interval price variance may also be expressed as the product of the probability of exposure to terminal risk and the terminal price variance, i.e.

$$V(P)^* = Kn \cdot V(Pn) \qquad (3\text{-}2)$$

A proof that this calculation is valid follows:

$$f^*(Px) = \left[Kn \cdot f(Pnx) \left|\frac{dPnx}{dPx}\right|\right] + \left[(1-Kn) \cdot \alpha(\beta)\right] \qquad \text{(Equation 2-17)}$$

$$V(P)^* = Kn \cdot \int_0^\infty (Pnx - E(Pn))^2 f(Pnx)\, dPnx + (\beta - E(P)^*)^2 \cdot (1-Kn)$$

where: $\beta = E(Pn)$

and: $V(Pn) = \int_0^\infty (Pnx - E(Pn))^2 f(Pnx)\, dPnx$

Therefore substituting:

$$V(P)^* = Kn \cdot V(Pn) + (E(Pn) - E(P)^*)^2 \cdot (1-Kn)$$

Substituting Equation (3-1): $E(P)^* = E(Pn)$

$$V(P)^* = Kn \cdot V(Pn) + (E(P)^* - E(P)^*)^2 \cdot (1-Kn)$$

where: $(E(P)^* - E(P)^*)^2 \cdot (1-Kn) = 0$

therefore: $V(P)^* = Kn \cdot V(Pn)$

B. Analysis Employing the New Measure of Risk

With the internal consistency of the model proven, it may be concluded that the new measure of risk is valid, given the assumptions of the model. Of course, the actual value of the risk parameter is a characteristic of the price probability density functions specified. Therefore, an analysis employing the new model along with different assumptions about price probability density functions will follow.

1. Objective of the Analysis

The development of a new measure of risk makes a new approach to the analysis of investment risk possible and introduces new relationships between price variance and utility.

(a) Incremental Analysis

The new measure of risk is divided into two components, terminal risk and the risk of exposure to terminal risk. It will be shown that an incremental analysis of the new total risk function may be necessary in order to select an investment with a minimum level of risk for a specific expected rate of return.

An incremental analysis of risk is unnecessary when the risk parameter has only one component, as in the conventional model. A model employing such a risk measure would select an optimal investment with a specific expected rate of return by minimizing this component, i.e. the model would attempt to minimize price variance.

It will be shown that, in some cases, the two components of the new measure of risk vary inversely with respect to each other when there is an incremental change in price variance. If this relationship is shown to exist, in any case, then an incremental analysis of price variance is required.

(b) Utility and Price Variance

A further analysis will be made with respect to the relationship between price variance and utility. As shown earlier, the conclusion of the conventional model is that price variance will always have disutility, i.e.

$$\frac{\partial U}{\partial V(Pn)} < 0 \qquad \text{(Equation 1-17)}$$

With the assumptions of the new model, its measure of risk, and a proof that an incremental analysis may be required, it will also be shown that in some cases, this conclusion is false.

All that is necessary to refute this conclusion is to show that in at least one case, price variance may not have the effect of decreasing the investors utility.

2. The Deductive Analysis

(a) Assumptions and Determination of the Parameters

Before the analysis proceeds, some simplifying assumptions will be introduced, and the determination of the parameters with these assumptions will be discussed.

(1) Holding Period

The analysis will assume that the holding period is two, (n=2) which will require that there be only one subinterval (n-1 =1), (Figure 11). This assumption will eliminate the problem of manipulating the product function, Kn, i.e.

$$Kn = \prod_{t=1}^{n-1} St \qquad \text{(Equation 2-15)}$$

When n=2, Kn = Sn-1.

It will however, not affect the validity of the analysis.

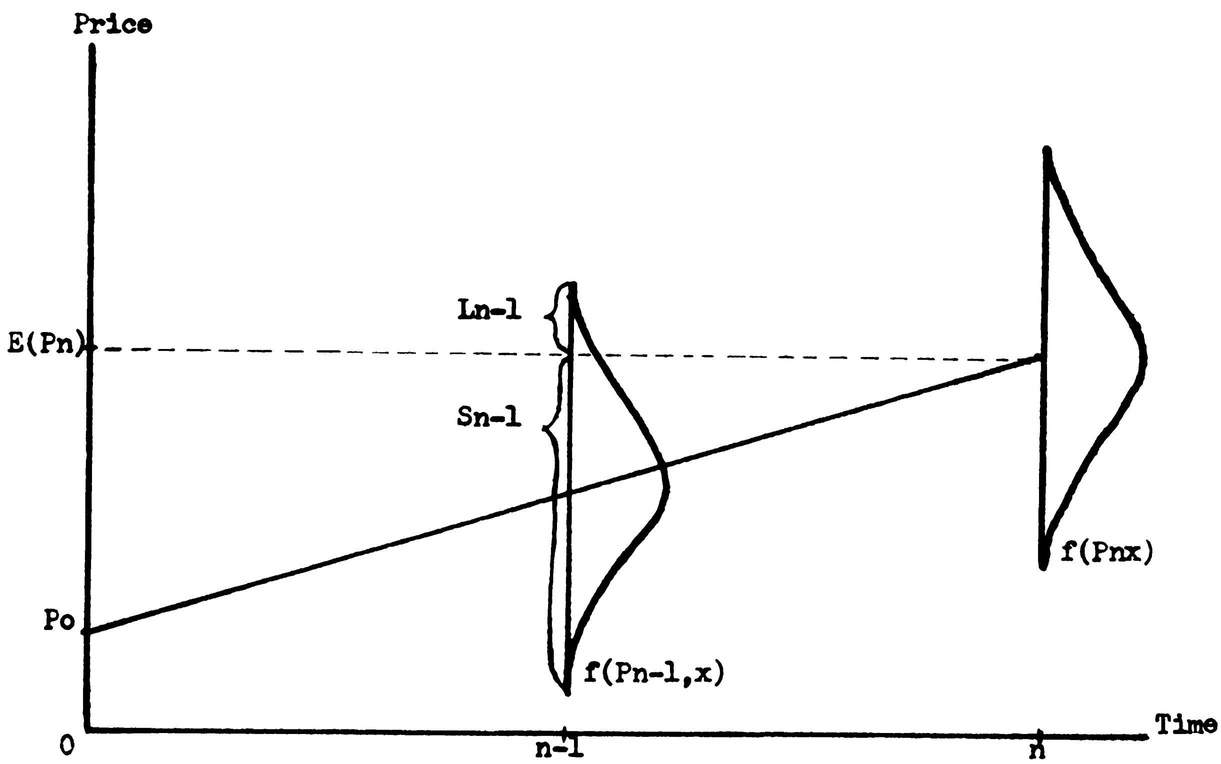

Figure 11

The price probability density function in periods n-1 and n are f(Pn-1,x) and f(Pnx) respectively.

2. Expected Price

It will also be assumed that the expected price of the terminal price density function is equal to the expected price of the interval price density function. i.e.

$$E(P)^{*} = E(Pn) \qquad \text{(Equation 3-1)}$$

This assumption was validated earlier. It will also be assumed that all securities have the same expected rate of return, which will allow concentration on the analysis of the risk parameter.

3. Determination of the Parameters

The parameters, given the above assumptions and the assumptions of the new model are:

a. Terminal Risk = V(Pn)

$$V(Pn) = \int_{o}^{\infty} (Pnx - E(Pn))^{2} f(Pnx)\, dPnx \qquad \text{(Equation 2-12)}$$

b. Risk of Exposure to Terminal Risk = Kn

$$Kn = \prod_{t=1}^{n-1} St \qquad \text{(Equation 2-15)}$$

where: n = 2 n - 1 = 1

$$\text{therefore: } Kn = Sn\text{-}1 = \left(1 - \int_{\beta = E(Pn)}^{\infty} f(Pn\text{-}1,x)\, dPn\text{-}1,x\right) \qquad (3\text{-}3)$$

c. Total Risk = $V(P)^{*}$

$$V(P)^{*} = Kn \cdot V(Pn) \qquad \text{(Equation 3-2)}$$

4. Variance

The variance of the subinterval price density function, f(Pn-1,x) will be expressed in terms of the variance of the terminal price density function, f(Pnx) i.e.

$$V(Pn-1) = f\left[V(Pn)\right]$$

The relationship is stated in this manner in order to maintain terminal price variance as the independent variable so that the measure of risk of the new model may be compared with the conventional model which employs only this parameter as the measure of risk.

The variances of the distributions may have the following characteristics.

a. Homoscedasticity

$$\frac{d\ V(Pt)}{d\ V(Pn)} = 1 \qquad t = 1.....n-1$$

b. Heteroscedasticity

1. Direct: $\frac{d\ V(Pt)}{d\ V(Pn)} > 0 \qquad t = 1.....n-1$

2. Inverse: $\frac{d\ V(Pt)}{d\ V(Pn)} < 0 \qquad t = 1.....n-1$

3. Functional:

$$\frac{d\ V(Pt)}{d\ V(Pn)} = f\left[V(Pn)\right] \qquad t = 1.....n-1$$

With each of these possible assumptions about the distributions, the risk components, total risk and utility may be analyzed in terms of their relationship to variance, of price V(Pn).

(b) Terminal Risk

Under all assumptions about the variance of the distributions, terminal price variance will have disutility, when only considering this one component of risk. By definition, terminal risk is defined as the terminal price variance V(Pn). Therefore, a risk averter would always attempt to minimize terminal price variance for any expected rate of return. i.e. $\frac{\partial U}{\partial V(Pn)} < 0$

(c) Risk of Exposure to Terminal Risk

However, given the assumptions of the new model, the total risk function has two components of which the risk of exposure to terminal risk is the second. This measure of risk is denoted by Kn, and it also would be minimized by an investor who is risk averse. i.e. $\frac{\partial U}{\partial Kn} < 0$ The risk of exposure to terminal risk is a function of the variance of price in periods t = 1.....n-1, and is minimized when the subinterval price variances are maximized. Therefore, in the two period model, the variance of f(Pn-1,x) should be maximized among securities with the same expected rate of return. This would be true under all of the assumptions about the relationships of price density functions. However, in this analysis, the subinterval price variances and, therefore, Kn are stated as functions of the terminal price variance. Therefore, an analysis of the relationship between Kn and the terminal price variance must be made with the different assumptions about the relationships of the price density functions. All that will be necessary to show is the direction of change (+ or -), and not absolute changes in order to determine whether an incremental analysis is necessary.[2]

1. Homoscedasticity:

(a) Assume: $\frac{dV(Pt)}{dV(Pn)} = 1 > 0 \qquad t = 1.....n-1$

Therefore: $\frac{dV(Pn-1)}{dV(Pn)} > 0$

<u>Premise a:</u> $\left[dV(Pn) >(<) 0\right] \supset \left[dV(Pn-1) >(<) 0\right]$

(b) Assume: $\frac{dLt}{dV(Pt)} > 0 \qquad t = 1.....n-1$

Therefore: $\frac{dLn-1}{dV(Pn-1)} > 0$

<u>Premise b:</u> $\left[dV(Pn-1) >(<) 0\right] \supset \left[dLn-1 >(<) 0\right]$

(c) Assume: $St = 1 - Lt$ and, $\frac{dSt}{dLt} < 0 \qquad t = 1.....n-1$

2. "Modus Ponens", A Principle of Logic, is symbolized by $\supset$ and represents the statement, "if.....then----".

Therefore: $Sn-1 = (1 - Ln-1)$ and, $\frac{dSn-1}{dLn-1} < 0$

Premise c: $[dLn-1 \; >(<) \; 0] \supset [dSn-1 \; <(>) \; 0]$

(d) Assume: $Kn = \prod_{t=1}^{n-1} St$

Where: $n = 2$

Therefore: $Kn = Sn-1$

Therefore: $\frac{dKn}{dSn-1} > 0$

Premise d: $[dSn-1 \; >(<) \; 0] \supset [dKn \; >(<) \; 0]$

If homoscedasticity is assumed, it may be concluded that Kn, the risk of exposure to terminal risk is decreased as the terminal price variance is increased and is increased when terminal price variance is decreased. i.e.

$$
\begin{array}{lllllll}
\text{Premise a:} & [dV(Pn) & >(<) & 0] & \supset & [dV(Pn-1) & >(<) & 0] \\
\text{Premise b:} & [dV(Pn-1) & >(<) & 0] & \supset & [dLn-1 & >(<) & 0] \\
\text{Premise c:} & [dLn-1 & >(<) & 0] & \supset & [dSn-1 & <(>) & 0] \\
\text{Premise d:} & [dSn-1 & <(>) & 0] & \supset & [dKn & <(>) & 0] \\
\text{Therefore:} & [dV(Pn) & >(<) & 0] & \supset & [dKn & <(>) & 0]
\end{array}
$$

Therefore: $\frac{dKn}{dV(Pn)} < 0$ (3-7)

2. Heteroscedasticity - Direct:

By heteroscedasticity direct it is assumed that the variance of the subinterval density functions are positively related to the variance of the terminal density function. i.e.

$$\frac{dV(Pt)}{dV(Pn)} > 0 \qquad t = 1.....n-1$$

The analysis for heteroscedasticity direct is identical to that of homoscedasticity. It may, therefore, be concluded that:

$$\frac{dKn}{dV(Pn)} < 0 \qquad (3\text{-}5)$$

3. Heteroscedasticity - Inverse:

The assumption of inverse heteroscedasticity states that the variances of the subinterval price density functions are inversely related to the variance of the terminal density function. i.e.

$$\frac{dV(Pt)}{dV(Pn)} < 0 \qquad t = 1.....n-1$$

The only change in the analysis is that premise a is altered (a') to reflect inverse heteroscedasticity. i.e.

a'. Assume: $\frac{dV(Pt)}{dV(Pn)} < 0 \qquad t = 1.....n-1$

Therefore: $\frac{dV(Pn-1)}{dV(Pn)} < 0$

Premise a': $[dV(Pn) \;\; >(<) \;\; 0] \supset [dV(Pn-1) \;\; <(>) \;\; 0]$

Therefore, if inverse heteroscedasticity is assumed, it may be concluded that Kn, the risk of exposure to terminal risk is decreased as terminal price variance is decreased and is increased when terminal price variance is increased. i.e.:

Premise a': $[dV(Pn) \;\; <(>) \;\; 0] \supset [dV(Pn-1) \;\; >(<) \;\; 0]$

Premise b : $[dV(Pn-1) \;\; >(<) \;\; 0] \supset [dLn-1 \;\; >(<) \;\; 0]$

Premise c : $[dLn-1 \;\; >(<) \;\; 0] \supset [dSn-1 \;\; <(>) \;\; 0]$

Premise d : $[dSn-1 \;\; <(>) \;\; 0] \supset [dKn \;\; <(>) \;\; 0]$

Therefore : $[dV(Pn) \;\; <(>) \;\; 0] \supset [dKn \;\; <(>) \;\; 0]$

Therefore : $\frac{dKn}{dV(Pn)} > 0 \qquad (3\text{-}6)$

4. Heteroscedasticity - Functional:

The term functional heteroscedasticity describes all other possible relationships between the subinterval price density functions and the terminal price desnity function. For example, the variance of some of the subinterval price density functions might vary directly with the variance of the terminal price density function while the others may vary inversely. Another possibility is that the individual investor would simply specify each subinterval price density function. This classification would also include cases where the means of the subinterval price density functions are not located on the trend line (Po, E(Pn), but vary about it in some other manner. No theoretical proof can be given to show the relationships that exist in these cases. An inductive proof of the new model would be impossible because there exists an infinite number of possible cases. However, based on the previous analysis and intuitive judgment, the author believes that the investor should determine and evaluate the parameters developed in the new model, in order to make an optimal investment decision.

The final evaluation of the investment may or may not be different from that of the conventional model. However, this is a function of the specification of the price probability density functions and the other variables in the model. Because no theoretical proof can be developed for this classification of density functions, the analysis of them will not be further developed. The remaining analysis will concentrate on the first three classifications, homoscedasticity, heteroscedasticity-direct and heteroscedasticity-inverse. All that is necessary to meet the objective of this analysis is to show that in one case, an incremental analysis may be necessary and that price variance may have utility.

5. Elasticity of Risk

It has been shown that with different assumptions about the price probability density functions, some general conclusions may be made about the relationships between Kn and V(Pn). i.e.

Homoscedasticity: $\frac{dKn}{dV(Pn)} < 0$ (Equation 3-4)

Heteroscedasticity - Direct: $\frac{dKn}{dV(Pn)} < 0$ (Equation 3-5)

Heteroscedasticity - Inverse: $\frac{dKn}{dV(Pn)} > 0$ (Equation 3-6)

These relationships show the direction of change, but do not express the relative rate of change in these variables. Therefore, the concept of elasticity of risk will be introduced in order to reflect this concept.

$$\eta = \frac{\frac{dKn}{Kn}}{\frac{dV(Pn)}{V(Pn)}} = \frac{dKn}{Kn} \cdot \frac{V(Pn)}{dV(Pn)}$$

$$= \frac{V(Pn)}{Kn} \cdot \frac{dKn}{dV(Pn)} \qquad (3\text{-}7)$$

a. Homoscedasticity:

(1) Elastic: $\eta < -1$ (% change in Kn > % change in V(Pn))

(2) Unitary: $\eta = -1$ (% change in Kn = % change in V(Pn))

(3) Inelastic: $\eta > -1$ (% change in Kn < % change in V(Pn))

b. Heteroscedasticity - Direct:

(1) Elastic: $\eta < -1$ (% change in Kn > % change in V(Pn))

(2) Unitary: $\eta = -1$ (% change in Kn = % change in V(Pn))

(3) Inelastic: $\eta > -1$ (% change in Kn < % change in V(Pn))

c. Heteroscedasticity - Inverse:

(1) Elastic: $\eta > 1$ (% change in Kn > % change in V(Pn))

(2) Unitary: $\eta = 1$ (% change in Kn = % change in V(Pn))

(3) Inelastic: $\eta < 1$ (% change in Kn < % change in V(Pn))

(d) Total Risk

An analysis of the total risk function may now be made by analyzing the marginal risk function. i.e.

$$V(P)^* = Kn \cdot V(Pn) \qquad \text{(Equation 3-2)}$$

$$\frac{dV(P)^*}{dV(Pn)} = Kn \cdot \frac{dV(Pn)}{dV(Pn)} + V(Pn) \cdot \frac{dKn}{dV(Pn)}$$

$$= Kn + V(Pn) \cdot \frac{dKn}{dV(Pn)}$$

$$= Kn \left[1 + \left(\frac{V(Pn)}{Kn} \cdot \frac{dKn}{dV(Pn)} \right) \right]$$

where: $\eta = \left(\frac{V(Pn)}{Kn} \cdot \frac{dKn}{dV(Pn)} \right)$ (Equation 3-7)

Therefore: $\frac{dV(P)^*}{dV(Pn)} = Kn\ (1 + \eta)$ (3-8)

Equation (3-8) is an expression of the marginal risk function. When marginal risk is negative, total risk is decreasing. When it is zero, total risk is constant or at a minimum and when it is positive, the total risk function is increasing. Therefore, the following relationships may exist.

(1) Homoscedasticity:

a. Elastic: $\eta < -1$ $\quad \frac{dV(P)^*}{dV(Pn)} < 0$

b. Unitary: $\eta = -1$ $\quad \frac{dV(P)^*}{dV(Pn)} = 0$

c. Inelastic: $\eta > -1$ $\quad \frac{dV(P)^*}{dV(Pn)} > 0$

(2) Heteroscedasticity - Direct:

a. Elastic: $\eta < -1$ $\quad \frac{dV(P)^*}{dV(Pn)} < 0$

b. Unitary: $\eta = -1$ $\quad \frac{dV(P)^*}{dV(Pn)} = 0$

c. Inelastic: $\eta > -1$ $\quad \frac{dV(P)^*}{dV(Pn)} > 0$

(3) Heteroscedasticity - Inverse:

a. Elastic: $\eta > 1$ $\qquad \frac{dV(P)^*}{dV(Pn)} > 0$

b. Unitary: $\eta = 1$ $\qquad \frac{dV(P)^*}{dV(Pn)} > 0$

c. Inelastic: $\eta < 1$ $\qquad \frac{dV(P)^*}{dV(Pn)} > 0$

This analysis of risk fulfills the first objective of this section. It shows that an incremental analysis of the total risk function may be necessary in order to select an investment with a minimum level of risk.

(e) Utility and Price Variance

The second objective of this analysis was to show that variance of price may not always have disutility for the investor. It is assumed that the investor is risk averse and that risk is defined as the interval price variance $V(P)^*$. i.e.

$$\frac{dU}{dV(P)^*} < 0$$

The relationship between utility and price variance (V(Pn)), may now be expressed (Table 3) by employing the chain rule of derivatives and the information previously developed. i.e.

$$\frac{dU}{dV(Pn)} = \frac{dU}{dV(P)^*} \cdot \frac{dV(P)^*}{dV(Pn)}$$

Table 3

(1) Homoscedasticity:	$\frac{dU}{dV(P)^*}$	$\cdot$	$\frac{dV(P)^*}{dV(Pn)}$	=	$\frac{dU}{dV(Pn)}$
a. Elastic:	(-)	.	(-)	=	(+)
b. Unitary:	(-)	.	(0)	=	(0)
c. Inelastic:	(-)	.	(+)	=	(-)

Table 3 continued:

(2) Heteroscedasticity - Direct:	$\frac{dU}{dV(P)^*}$	$\cdot$	$\frac{dV(P)^*}{dV(Pn)}$	=	$\frac{dU}{dV(Pn)}$
a. Elastic:	(-)	.	(-)	=	(+)
b. Unitary:	(-)	.	(0)	=	(0)
c. Inelastic:	(-)	.	(+)	=	(-)

(3) Heteroscedasticity - Inverse:	$\frac{dU}{dV(P)^*}$	$\cdot$	$\frac{dV(P)^*}{dV(Pn)}$	=	$\frac{dU}{dV(Pn)}$
a. Elastic:	(-)	.	(+)	=	(-)
b. Unitary:	(-)	.	(+)	=	(-)
c. Inelastic	(-)	.	(+)	=	(-)

Thus, it has been shown that given the assumptions of the new model and assumptions about the relationships of price density functions, there are cases where price variance does not have disutility.

CHAPTER IV POSSIBLE EXTENSIONS OF THE NEW MODEL AND ITS SIGNFICANCE

The development of the new model will lead to the development of a general portfolio model which will allow investors to consider all possible assumptions about intraperiod liquidation i.e. forced and voluntary. The new model is the main component of this general portfolio model. However, before the general portfolio model is developed, the implications of changing or removing some of the simplifying assumptions will be discussed.

A. Possible Extensions of the New Model With Revised Assumptions - Areas For Further Research

The assumptions that were made in the development of the new model may be classified into two catagories, simplifying and critical.

A revision of the simplifying assumptions will make the new model more realistic. Later, a revision of the critical assumptions will be made in order to develop the general portfolio model. The simplifying assumptions of the new model are:

(a) Initial price is given: Po

(b) Time horizon or investment period: t = 1.....n

(c) No tax or transaction cost.

(d) One security portfolios.

(e) No dividends: D1.....Dn = 0.

(f) Normal price distributions.

(g) No time value of money.

(h) Homoscedasticity: The n-1 price density functions all have the same variance about the trend line (Po,E(Pn)). (This assumption has been revised in the previous analysis).

(i) The n price density functions are independent.

(j) β = E(Pn)

The revision of all of these simplifying assumptions will not be considered. However, the assumptions that will be discussed are considered to have the most significant impact on the model and, therefore, may have the most significant change on the model when they are revised.

1. Time Value of Money

The new model was developed without considering the time value of money because reinvestment decisions were not considered. However, the new model allows the investor to liquidate his investment in the subintervals, (1.....n-1). Therefore, reinvestment at a riskless rate of return (i) should be considered, in order to account for the time value of money.[1] A simple approach of displaying the impact of the time value of money is to reanalyze expected return and variance of return as of the terminal time period n. The new model assumed that the subinterval liquidation value β was equal to E(Pn). Of course, if the time value of money was ignored, β was held as cash and not reinvested at (i). The impact of the time value of money on the measure of risk (variance of return) may be analyzed by maintaining the same expected terminal wealth, (E(Pn)) and allowing βt^{*} to be reinvested at the riskless rate of return. βt^{*} is defined as the liquidation value in periods t = 1.....n-1, which when reinvested at the riskless rate i, will equal the value E(Pn) in period n. i.e.

$$E(Pn) = \beta t^{*} \cdot (1 + i)^{(n-t)} \qquad t = 1.....n-1 \qquad (4\text{-}1)$$

Therefore, if (i) is any positive rate of return, βt^{*} will be less than E(Pn) and approaches E(Pn) as t approaches n. Terminal expected wealth will

1. If at time zero, the investor is considering reinvesting in another risky investment in a specific subinterval (1.....n-1), he should redefine his investment period n, to be that specific subinterval. Although the new model assumes period n is a period of forced liquidation, the reason for forced liquidation may be due to a decision to reinvest in another risky investment at that time.

remain the same (E(Pn)), but the measure of risk will decrease. i.e. If,

$$\beta t^{*} < \beta = E(Pn) \qquad t = 1.....n-1$$

the probability of liquidation in a subinterval is increased, (equation 2-13).

$$Lt^{*} = \int_{\beta_t^{*}}^{\infty} f(Ptx)\ dPtx > Lt = \int_{\beta = E(Pn)}^{\infty} f(Ptx)\ dPtx$$

Therefore, substituting in equation (2-16):

$$Kn^{*} = \prod_{t=1}^{n-1} (1 - \int_{\beta_t^{*}}^{\infty} f(Ptx)\ dPtx) < Kn = \prod_{t=1}^{n-1} (1 - \int_{\beta = E(Pn)}^{\infty} f(Ptx)\ dPtx$$

Finally, substituting in the total risk equation (3-2)

$$\left[V(P)^{*}\right]^{*} = Kn^{*} \cdot V(Pn) < V(P)^{*} = Kn \cdot V(Pn)$$

When the time value of money is introduced into the new model, the risk of exposure to terminal risk, (Kn) is decreased and, therefore, the risk of the interval is decreased. Thus, the utility for subinterval price variance is increased, and this result reinforces the conclusions of the previous chapter.

2. Dependent Subinterval Distributions

The assumption of independent subinterval price density functions was made in order that the law of weak dominance could be employed and the use of indifference curves avoided. Independence of subinterval price density functions allowed the new model to maintain the same expected rate of return as the conventional model and, thus, shifted the analysis on the measure of risk.

The introduction of dependent subinterval price density functions will not destroy the internal consistency of the new model but may make the determination of the parameters more complex. However, the explicit consideration of intraperiod voluntary liquidation should still be considered.

The introduction of dependent subinterval distributions makes it impossible to compare the new model with the conventional model with the law of weak dominance because the value of both parameters may change. The direction or magnitude of change cannot be specified in a general model, for they are a function of the specific price density functions of a specific investment. The correlation of price movements over time must be specified for each individual investment in terms of conditional probabilities. Thus, because price movements over time may range from perfect positive correlation to perfect negative correlation, each investment must be considered in terms of its' conditional probabilities. However, when dependent subinterval price density functions are introduced into the new model, the consideration of intraperiod voluntary liquidation will lead to the optimal selection of investments.

The analysis of investments with dependent subinterval price density functions is similar to the analysis of abandonment value in capital budgeting.[2] The conventional model implicitly assumes that an investment is held until, and disposed of in period n and ignores the possibility of voluntary intraperiod liquidation. Therefore, the conventional model determines expected price and variance of price from the terminal joint probability density function. However, when an investment has price movements that are correlated over time, it should be liquidated in a subinterval if the subinterval price exceeds the expected price in subsequent periods and variance of price remains unchanged, or if subinterval price equals expected price in subsequent periods and the interval price variance is reduced. If both the expected price and variance

2. A. Robichek and J. Van Horne, "Abandonment Value and Capital Budgeting", Journal of Finance, XXII, December, 1967.

of price are reduced when subinterval voluntary liquidation is considered, the investor must consider the benefits of liquidation based on his disutility for risk and utility for return. The analysis of the terminal joint probability density function may not reflect the benefits of this type of voluntary intraperiod liquidation.

A simplified example will demonstrate the limitation of ignoring intraperiod voluntary liquidation. Assume there are two investments, A and B, with their respective possible prices and probabilities shown in Table 4 and 5.

Table 4

INVESTMENT A

Period 0	Period 1		Period 2		
Price	Price	Prob.	Price	Prob. (2/1)	Joint Prob.
			130	.40	.10
	100	.25	100	.40	.10
			70	.20	.05
			100	.20	.10
50	70	.50	70	.60	.30
			40	.20	.10
			70	.20	.05
	40	.25	40	.40	.10
			10	.40	.10

Table 5

INVESTMENT B

Period 0	Period 1		Period 2		
Price	Price	Prob.	Price	Prob. (2/1)	Joint Prob.
			130	.40	.10
	100	.25	100	.40	.10
			70	.20	.05
			100	.20	.10
50	70	.50	70	.60	.30
			40	.20	.10
			70	.20	.05
	30	.25	40	.40	.10
			10	.40	.10

When the expected price and variance of price are determined from the terminal joint probability price density function, both investment A and B have the same parameters.

$E(Pn)_A = E(Pn)_B = 70$

$V(Pn)_A = V(Pn)_B = 1080$

Therefore, the investor would be indifferent between holding A or B.

If the analysis allowed for intraperiod voluntary liquidation, the conclusion of indifference would be false. Assume the investor will liquidate the security in period 1 if price is equal to or less than E(Pn). i.e.

$$\beta_1 \leq E(Pn) = 70$$

The new possible prices and their probabilities for investments A* and B* are shown in Table 6 and 7.

Table 6

INVESTMENT A* ($\beta_1 \leq 70$)

Period 0	Period 1		Period 2		
Price	Price	Prob.	Price	Prob. (2/1)	Joint Prob.
			130	.40	.10
	100	.25	100	.40	.10
			70	.20	.05
50	70	.50	70	1.00	.50
	40	.25	40	1.00	.25

Table 7

INVESTMENT B* ($\beta_1 \leq 70$)

Period 0	Period 1		Period 2		
Price	Price	Prob.	Price	Prob. (2/1)	Joint Prob.
			130	.40	.10
	100	.25	100	.40	.10
			70	.20	.05
50	70	.50	70	1.00	.50
	30	.25	30	1.00	.25

When the expected price and variance of price are determined from the interval price density functions, Investment A^* is preferred to Investment B^*, (Figure 12).

		A^*		B^*
$E(P)^*$	=	71.50	>	69.00
$V(P)^*$	=	672.75	<	849.00

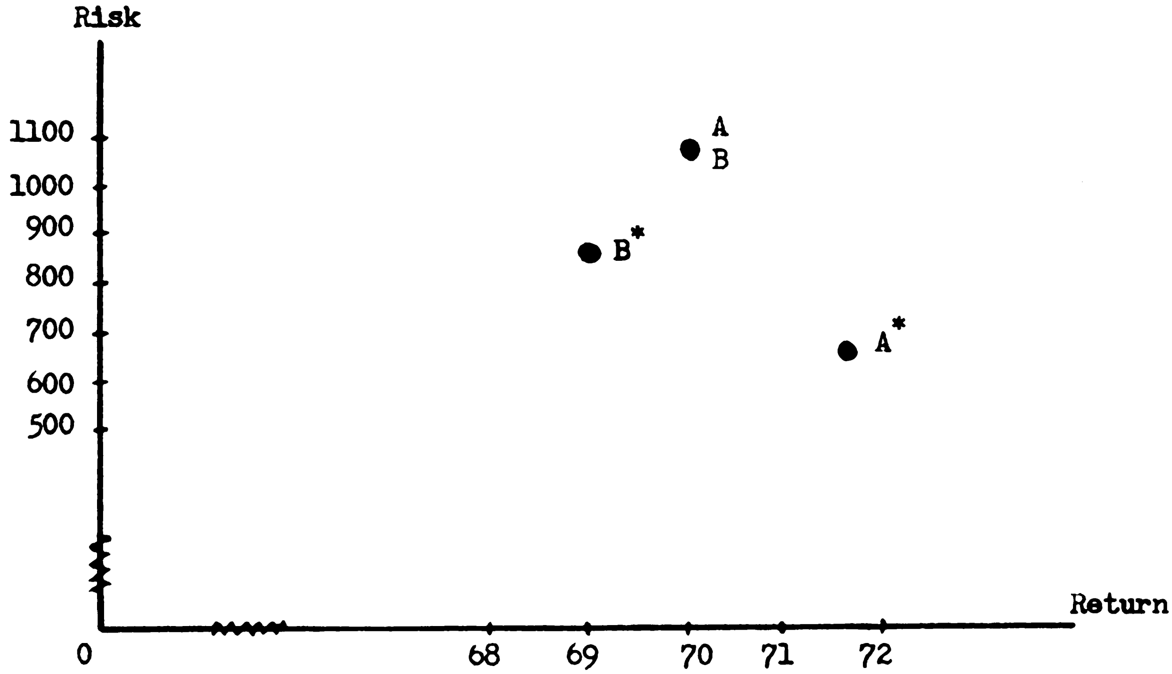

Figure 12

From this example it may be concluded that:

(1) A^* is preferred to B^* - Law of Strict Dominance

(2) A^* is preferred to A - Law of Strict Dominance

(3) A^* is preferred to B - Law of Strict Dominance

(4) It may not be concluded, a priori that B^* is preferred to B or A by simply employing the law of dominance. Additional information about the investors preference structure is necessary to make this comparison.

Although this example is very simplified, it does show that superior investment decisions may be made when intraperiod voluntary liquidation is considered, and when the parameters are determined from the interval probability density function rather than the terminal probability density function.

3. Portfolio Analysis

The simplifying assumption of one security portfolios was made in order to concentrate on the measure of investment risk and to avoid the analysis of covariance between investments. When this assumption is dropped, covariance is reintroduced into the portfolio model. i.e.

$$Ep = \sum_{i=1}^{m} (a_i) \cdot (E_i) \qquad i = 1.....m \qquad \text{(Equation 1-7)}$$

$$Vp = \sum_{i=1}^{m} \sum_{j=1}^{m} (a_i) \cdot (a_j) \cdot (COV_{ij}) \qquad \text{(Equation 1-8)}$$

where:

$$COV_{ij} = (R_{ij}) \cdot (V_i)^{\frac{1}{2}} \cdot (V_j)^{\frac{1}{2}} \qquad \text{(Equation 1-6)}$$

therefore:

$$Vp = \sum_{i=1}^{m} \sum_{j=1}^{m} (a_i) \cdot (a_j) \cdot (R_{ij}) \cdot (V_i)^{\frac{1}{2}} \cdot (V_j)^{\frac{1}{2}} \qquad (4\text{-}2)$$

E_i, V_i, and V_j are the expected returns and variances of return of the individual investments (1.....m) determined by the terminal price density functions. i.e.

$$E_i = \frac{E(Pn)i - (Po)i}{(Po)i} \qquad i = 1.....m \qquad \text{(Equation 1-9)}$$

$$V_i = \frac{V(Pn)i}{(Po)_i^2} \qquad i = 1.....m \qquad \text{(Equation 1-10)}$$

The parameters (E_i), (V_i) and (V_j) should be replaced with the expected returns (E_i^*) and variances of return (V_i^*), (V_j^*) which are determined by the interval price density functions, i.e.

$$E_i^* = \frac{E(P)i^* - (Po)i}{(Po)i} \qquad i = 1....m \qquad \text{(Equation 2-8)} \quad \text{(Equation 2-20)}$$

$$V_i^* = \frac{V(P)_i^*}{(Po)_i^2} \qquad i = 1.....m \qquad \text{(Equation 2-9)} \; \text{(Equation 2-21)}$$

therefore:

$$E_p^* = \sum_{i=1}^{m} (a_i) \cdot (E_i^*) \qquad i = 1.....m \qquad (4\text{-}3)$$

$$V_p^* = \sum_{i=1}^{m} \sum_{j=1}^{m} (a_i) \cdot (a_j) \cdot (R_{ij}) \cdot (V_j^*)^{\frac{1}{2}} \cdot (V_j^*)^{\frac{1}{2}} \qquad (4\text{-}4)$$

An additional problem is created when the parameters from the interval price density functions are introduced into the portfolio model. The determination of the portfolio expected return (E_p^*) will not be altered. However, (V_p^*) may not be the correct measure of portfolio risk because a new question arises in regards to the measurement of covariance.

In the conventional model, when only the terminal period is considered, the covariance is the covariance of period n. However, this may not be the relevant covariance when the total time interval is considered. As with investment risk, it might also prove beneficial to analyze covariance in two components, i.e. terminal covariance and subinterval covariance.

Terminal covariance is similar to the measure of covariance in the conventional model. It would be calculated as the covariance of period n, but considering the probability of being exposed to terminal risk, i.e.

$$COV_{ij}^* = (R_{ij}) \cdot (V_i^*)^{\frac{1}{2}} \cdot (V_j^*)^{\frac{1}{2}} \qquad i = 1.....m \qquad (4\text{-}5)$$

substituting:

$$COV_{ij}^* = (R_{ij}) \cdot \left[\frac{V(P)_i^*}{(Po)_i^2}\right]^{\frac{1}{2}} \cdot \left[\frac{V(P)_j^*}{(Po)_j^2}\right]^{\frac{1}{2}} \qquad i = 1.....m$$

therefore:

$$COV_{ij}^* = (R_{ij}) \cdot \left[\frac{(Kn)i \cdot V(Pn)i}{(Po)_i^2}\right]^{\frac{1}{2}} \cdot \left[\frac{(Kn)j \cdot V(Pn)j}{(Po)_j^2}\right]^{\frac{1}{2}}$$

$$i = 1.....m \qquad (4\text{-}6)$$

where:

R_{ij}	=	Correlation coefficient between return of investment i and j in period n.
$V(Pn)_i$, $V(Pn)_j$	=	Terminal price variances of investments i and j respectively.
$(Kn)_i$, $(Kn)_j$	=	Risk of exposure to terminal risk for investments i and j respectively.

The new measure of terminal covariance may be less than the terminal covariance of the conventional model because Kn is introduced into the calculation and may decrease terminal covariance. The introduction of the new covariance measure may imply that diversification is less important for portfolios where the risk of exposure to terminal risk is low. The new model does not suggest that diversification will have disutility, for an investor would still prefer investments with less positive (or negative) correlation if he did reach period n and was exposed to terminal risk. However, an investor may select a more positively correlated portfolio over a less positively correlated (or negatively correlated) portfolio if the risk of exposure to terminal risk is less in the investments contained in the former portfolio.

Up to this point, only terminal covariance has been discussed. However, a question arises as to the benefit or detriment of subinterval covariance. This question is beyond the scope of this work, but merits further analysis.

B. Signficance of the New Model

The development of the new model is significant for it will lead to the formulation of a general investment model. It will also add to the development of the theory of risk and suggest some practical applications or investment strategies.

1. General Investment Model

The first part of this chapter discussed the possible effects of changing some of the simplifying assumptions on the new model. A general investment model will be developed which will consider changes in the critical assumptions of the new model. The focus will still remain on the measure of risk while allowing for all possible assumptions about liquidation, i.e. voluntary, forced, terminal and intraperiod.

(a) Case A

Assumption 1: No voluntary liquidation in periods 1.....n-1.

Assumption 2: No risk of forced liquidation in periods 1.....n-1.

Assumption 3: Forced liquidation in period n.[3]

In this case, the new model would yield the same results as the conventional model i.e.

$$V(P)^* = Kn \cdot V(Pn) \quad \text{(Equation 3-2)}$$

where:

$$Kn = 1$$

therefore:

$$V(P)^* = V(Pn)$$

(b) Case B

Assumption 1: Voluntary liquidation in periods 1.....n-1.

Assumption 2: No risk of forced liquidation in periods 1.....n-1.

Assumption 3: Forced liquidation in period n.

As discussed earlier, the optimal measure of risk is:

$$V(P)^* = Kn \cdot V(Pn) \quad \text{(Equation 3-2)}$$

where: $0 \leq Kn \leq 1$

therefore: $V(P)^* \leq V(Pn)$

3. Forced liquidation may also be defined as a point in time when the investor would like to liquidate his investment in order to reinvest in another investment.

(c) Case C

Assumption 1: Voluntary liquidation in periods 1.....n-1.

Assumption 2: Risk of forced liquidation in periods 1.....n-1.

Assumption 3: Forced liquidation in period n.

Case C may be divided into 2 sub cases.

(1) Forced liquidation in every subinterval.

If assumption 2 states that there may be forced liquidation in every subinterval, then each subinterval should be analyzed independently with the approach stated in case A. Each subinterval would be analyzed as if it were an interval n with no voluntary or forced liquidation in its own subintervals. The total risk of the investment would be defined as:

$$V(P)^* = \sum_{t=1}^{n} V(P)_t^*$$

This analysis would yield the same results as analyzing each subinterval with the conventional model. This would require summing the variances of each subinterval in order to determine the variance of the total period n. i.e.

if: $V(P)_t^* = K_t \cdot V(P_t) \qquad t = 1.....n$

and: $K_t = 1$

then: $V(P)_t^* = V(P_t)$

therefore: $V(P)^* = \sum_{t=1}^{n} V(P)_t^* = V = \sum_{t=1}^{n} V(P_t)$

(2) Forced liquidation in specific subintervals.

If there are specific subintervals where there may be risk of forced liquidation, these specific intervals should be analyzed as if they were terminal periods and analyzed by the approach discussed in case B. For example, assume the investor may be forced to liquidate at the end of the specific subintervals x and y which are subintervals in the total time interval n. If subintervals x and y have intervals preceding them where there is no forced liquidation, the measure of risk for the total interval is:

$$V(P)^* = \sum_{\substack{t=x\\t=y\\t=n}} V(P)_t^*$$

$$= V(P)_x^* + V(P)_y^* + V(P)_n^*$$

$$= K_x \cdot V(P_x) + K_y \cdot V(P_y) + K_n \cdot V(P_n)$$

$$= \prod_{t=1}^{x-1} S_t \cdot V(P_x) + \prod_{t=x}^{y-1} S_t \cdot V(P_y) + \prod_{t=y}^{n-1} St \cdot V(P_n)$$

The procedure in case C-1 and C-2 is to define any period of possible forced liquidation as a terminal period and find the variance of the total interval by summing the variances of the subintervals. However, the measure of the subinterval variances should be determined by the new model. Note that the summation of all subinterval variances may lead to a suboptimal measure of risk as shown in the difference between cases B, C-1, and C-2. Thus, the new model may be employed when any of the critical assumptions are changed or introduced into the analysis.

2. The Measure of Risk and Practical Applications of the New Model

The development of the new model may contribute to the theory of risk measurement. First, the new model extends the measure of investment risk because it considers different assumptions about subinterval trading which the conventional model ignores. The analysis of risk in two components allows the new model to account for these different assumptions. Secondly, as stated earlier in the chapter, the new model may have an inpact on the theory of portfolio risk, for it raises some questions about the benefits of diversification.

Its major impact may be on the actual portfolio management of institutions as well as individuals. Institutions such as Life Insurance Companies and

Pension Funds which have little risk of forced liquidation in the sub-intervals of their investment horizon may find the new model beneficial in determining an investment strategy.

The new model may also have an impact on past empirical research which attempted to analyze real world data and explain it in terms of the conventional model. If, in fact, the new model has an improved and different measure of risk, then the conclusions of this empirical research may have to be reevaluated.

Although the new model also employs variance of return as the measure of risk, the possible relationships between return variance and price variance is significant. The possibility that return variance is not always directly related to price variance may have an impact on both theoretical as well as practical investment theory.

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APPENDIX A

AXIOM OF DOMINANCE

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Axiom of Dominance

A commodity space Z is a field of choice consisting of all possible bundles of two commodities (A,B), where each commodity can be obtained in any non-negative amount. With this definition of a commodity space, all bundles become pairs (A,B) of quantities of commodities. Since these are pairs of numbers, it is not immediately obvious what one might mean if one said that a bundle X_0 (consisting of the pair $[A_0, B_0]$) were larger than another bundle X_1 (consisting of $[A_1, B_1]$). Therefore, the axiom of dominance was developed based on the following definitions.

Definition 1. A bundle X_0 is said to dominate a bundle X_1 if either (i) $A_0 \geq A_1$ and $B_0 > B_1$ or (ii) $A_0 > A_1$ and $B_0 \geq B_1$.

Definition 2. A bundle X_0 is said to dominate X_1 <u>strictly</u> if $A_0 > A_1$ and $B_0 > B_1$. Clearly, if X_0 strictly dominates X_1, it dominates X_1.

Definition 3. If X_0 dominates X_1, but does not strictly dominate X_1, then we say that X_0 <u>weakly</u> dominates X_1.

If an individual considers the commodities to be positive goods and not actual nuisances (or discommodities), he would be implicitly saying that if bundle X_0 dominates X_1, he prefers X_0 to X_1. This leads us to formulate the axiom of dominance

Axiom of Dominance: If two bundles X_0, X_1 in Z are such that X_0 dominates X_1, then X_0 is preferred to X_1 (i.e. X_0 p X_1).

From, Peter Newman, <u>The Theory of Exchange</u>, (New Jersey: Prentice-Hall), 1965, pg. 19.

APPENDIX B

CALCULUS OF PROBABILITY DENSITY FUNCTIONS

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Calculus of Probability Density Functions

Let X be a continuous random variable and let $Y = g(X)$. If $g(\cdot)$ is differentiable at every real number x and further, either $g'(x) > 0$ for all x or $g'(x) < 0$ for all x then:

(i) As x goes from $-\infty$ to ∞, $g(x)$ is either monotone increasing or monotone decreasing.

(ii) Limits Exist:

$$a' = \lim_{x \to -\infty} g(x) \quad , \quad b' = \lim_{x \to \infty} g(x)$$

$$a = \min(a', b') \quad , \quad b = \max(a', b')$$

(iii) For every value of y such that $a < y < b$ there exists exactly one value of x such that $y = g(x)$ [this value of x is denoted by $g^{-1}(y)$].

(iv) This inverse function $x = g^{-1}(y)$ is differentiable and its derivative is given by:

$$\frac{dx}{dy} = \frac{dg^{-1}(y)}{dy} = \frac{1}{dy/dx}$$

The formula for the probability density function of $g(x)$ in terms of the probability density function of X is stated in the following theorem.

If $y = g(x)$ is differentiable for all x, and either $g'(x) > 0$ for all x or $g'(x) < 0$ for all x, and if X is a continuous random variable, then $Y = g(x)$ is a continuous random variable with probability density function given by:

$$f_Y(y) = f_X\left[g^{-1}(y)\right] \left|\frac{dg^{-1}(y)}{d(y)}\right| \qquad \text{if } a < y < b$$

$$= 0 \quad \text{otherwise}$$

From, Emanuel Parzen, Modern Probability Theory and Its Applications, (New York: Wiley & Sons), 1960, pp. 312.