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thesis entitled

DYNAMIC PROPERTIES OF COMBINED MDOF PRIMARY AND MDOF SECONDARY SYSTEMS

presented by

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has been accepted towards fulfillment
of the requirements for

M.A degree in Civil Engineering

B S Harichandra

Major professor

Date 2/25/88

# DYNAMIC PROPERTIES OF COMBINED MDOF PRIMARY AND MDOF SECONDARY SYSTEMS

By

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A THESIS

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

MASTER OF SCIENCE

Department of Civil and Environmental Engineering

1988

**ABSTRACT**

DYNAMIC PROPERTIES OF COMBINED

MDOF PRIMARY AND MDOF SECONDARY SYSTEMS

By

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In nuclear power plants and other structures, light equipment is often attached to the primary structure. The dynamic analysis of the combined structure-equipment system is prone to numerical problems, because of the combination of the large mass and stiffness matrices of the primary structure with the much smaller matrices corresponding to the secondary equipment. The increase in the size of the problem due to the addition of equipment is also undesirable in practice. Perturbation techniques that overcome these hurdles have recently been proposed.

A new perturbation method, that has some advantages over existing methods, has been employed to determine the dynamic properties of structural systems composed of multi-degree-of-freedom (MDOF) primary and MDOF secondary systems. As in previous methods, the dynamic properties of the individual subsystems are utilized to estimate the properties of the combined system. High order perturbations of the mode shapes and frequencies are developed, and numerical results can be obtained to any order of accuracy by considering higher-order terms. Sharp error bounds for the estimated mode shapes and frequencies are also derived. The technique is extended to nonclassically damped systems.

**ACKNOWLEDGEMENTS**

This research was supported by Michigan State University through an All-University Research Initiation Grant (Account No. 11-6473). This support is gratefully acknowledged.

The author wishes to thank Professor Ronald Harichandran for his guidance during this work. Gratitude is also expressed to Professors Frank Hatfield and Robert Wen for serving on the thesis committee.

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# 1. INTRODUCTION

## 1.1 General Remarks

Composite systems composed of light secondary system attached to heavier primary system are frequently encountered in civil engineering practice. There are many instances in which the secondary system must always remain operational since its failure may compromise the safety of the whole system. Piping in industrial structures, drilling and exploration equipment on offshore platforms, and communication and control devices on space vehicles are examples of such systems.

## 1.2 Problem Statement

Composite systems consists of different subsystems with vastly different characteristics. The most fundamental property is that the mass of the secondary system is considerably smaller than that of the primary system. There are two major kinds of composite systems, they are distinguished by their damping properties:

1. Classically-Damped System: The primary and secondary systems are viscously and classically damped. The composite system is also assumed to be viscously and classically damped. This assumption is quite true when the two subsystems are made of the same material with approximately the same damping ratio.
2. Nonclassically-Damped System: The primary and secondary systems are viscously and nonclassically damped. The composite system is also

viscously and nonclassically damped. It is known (Igusa and Kiureghian, 1985) that the composite system composed of two classically damped subsystems may be nonclassically damped if there exists a significant difference between the damping ratios of the two subsystems.

The dynamic properties of composite systems can be complex. The main dynamic properties are as follows:

1. Tuning (resonance) Characteristics: Any number of the frequencies of one subsystem may be arbitrarily close to or coincident with the frequencies of the other subsystem. This condition is know as tuning.
2. Complex Eigenvectors and Eigenvalues of the Composite System: This occurs when the system is nonclassically damped (Hurty and Rubinstein, 1964). Under this condition, the composite system will vibrate freely in a set of "modes" in which all points in the system undergo exponentially damped motion at the same frequency, but at differing phase angles.

For stochastic seismic analysis of composite systems, the cross-correlations between modal responses is significant, especially when tuned modes are encountered. Also, when the composite system is nonclassically damped, the modal displacements and velocities are cross related. This may significantly influence the mean and extreme values of the responses.

## 1.3 Common Restrictions

In principle, the exact response of a general secondary system can be obtained by using standard methods of analysis on the composite

system. However, this procedure presents a number of numerical difficulties. The large number of degrees-of-freedom, and the vast differences in magnitudes of the stiffness, damping, and mass terms pose serious problems. Exact solutions are therefore practically impossible to obtain and various approximation techniques have been developed in recent years. These are reviewed in the next section.

## 1.4 Literature Survey

A simple method used often is the floor response spectrum method. In this the motions of the support points of the secondary system are calculated by time history analysis of the primary system. Descriptions of these motions are then used to design the secondary systems. This method, however, is very costly and inefficient. Several researchers have developed more direct methods of finding floor response spectra using the modal properties of the primary system and the ground response spectrum. But neither method takes into account the interaction between the secondary and primary systems.

Realizing the importance of interaction, the perturbation method (Nayfeh, 1981) has been introduced to estimate the modal properties of composite systems. Sackman and Kelly (1979) were among the first to use this approach in the analysis of primary-secondary systems. In their study, the natural frequency of the equipment modeled as a single-degree-of-freedom system (SDOF), is considered to be close to or equal to one of the natural frequencies of the N-degree-of-freedom primary structure. Considering the secondary system and the tuned mode of the primary system as an equivalent two-DOF system, the closed form expressions for the frequencies are obtained. Sackman and Kiureghian (1983) employ-

ed perturbation methods to determine the dynamic properties of a composite system composed of a MDOF structure and a light SDOF equipment. Closed-form expressions were derived for natural frequencies, mode shapes, and modal dampings. The effect of tuning and interaction were included in the analysis. The expressions for detuned modes were relatively complicated, and only the lowest order of perturbation was obtained. These expressions were directly reduced to apply to tuned modes, resulting in very rough approximations. Igusa and Kiureghian (1985) extensively studied a two-DOF equipment-structure system. Three important characteristics of the system were identified: tuning, interaction, and nonclassical damping. The results obtained were extended to a more general P-S system composed of MDOF secondary system and MDOF primary system (1985). For the detuned case, the first order of mode shapes are obtained, but the perturbations of natural frequencies are ignored which result in relatively large errors when the natural frequencies are closely spaced but not tuned. For singly tuned modes, the results obtained from the two-DOF equipment-structure system are used by neglecting the effect of all other modes. The analysis of multiply tuned modes are proposed. A small eigenvalue equation is formulated by neglecting the effect of all other modes. Suarez and Singh (1987) obtained closed-form expressions to calculate the approximate complex eigenvalues and eigenvectors of a system composed of a nonclassically damped primary structure and a SDOF oscillator. The perturbation method they used is general and rigorous which is readily extendable for more general P-S systems composed of MDOF secondary systems and MDOF primary systems. For tuned modes, the authors assume the lowest order of perturbation is $\epsilon^{1/2}$. This approach is practically invalid for multiply tuned cases.

All the studies mentioned above have not given too much attention to error analysis. Error bounds, which are highly desirable in practice, were not derived for the approximate solutions. The tuning criterion was based only on two modes and the effect of other modes was not taken into account. Higher order perturbations have also not received much attention.

## 1.5 Scope of Investigation

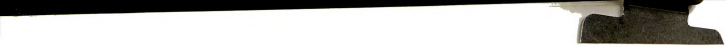
In this study the perturbation method is employed to determine the dynamic properties of structural system composed of MDOF primary and MDOF secondary systems. The effects of nonclassical damping and multiple tuning are considered in detail. Error bounds for the approximate solutions are derived. A synopsis of the approach of the analysis follows:

In Chapter 2 the perturbation method for classically damped systems is derived.

In Chapter 3 nonclassically damped systems are investigated.

Numerical examples for each of the cases considered are presented in Chapter 4.

The report ends with a summary of the main conclusions of the study.



# 2. CLASSICALLY DAMPED SYSTEMS

## 2.1 Modal Synthesis

A n+m-degree-of-freedom (n+m-DOF) composite system is composed of two subsystems: a m-DOF light secondary subsystem supported on a n-DOF primary subsystem. Figure 2-1 shows the individual subsystems and composite system.

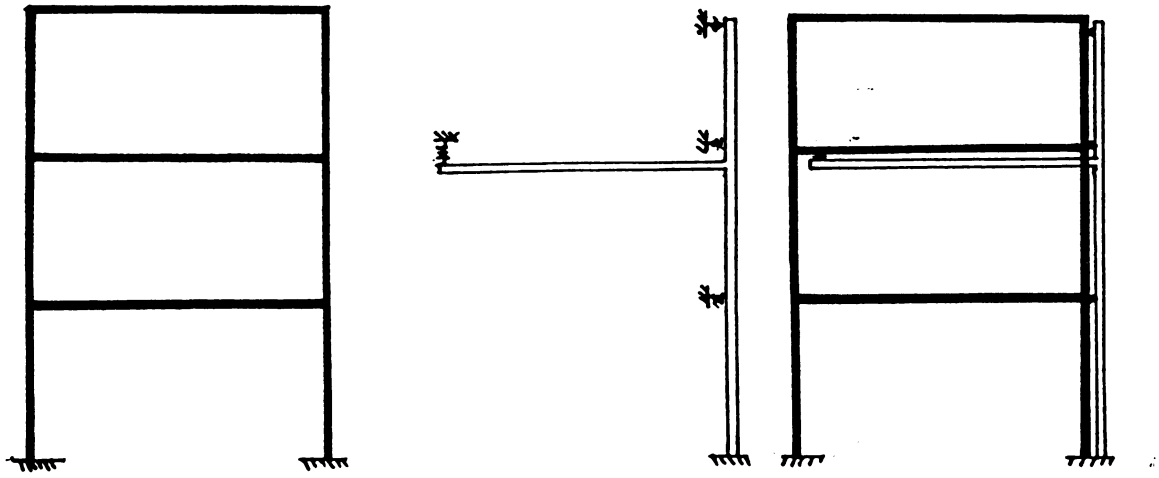



Figure 2-1 : Individual Subsystems and Combined System

The dynamic properties (frequencies and mode shapes) of individual subsystems are assumed to be known. The elements connecting the two systems is to be included with the secondary system as shown in Figure 2-1. The method of mode synthesis is used to formulate the (m+n)-DOF

model of the composite system for dynamic analysis. By this method, only dynamic properties of individual subsystems are involved.

The equation of free motion for the composite system (C-system) is

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{0\} \tag{2-1}$$

where [M], [C] and [K] are mass, damping and stiffness matrices of C-system. For a classically damped system, the corresponding eigenequation may be written as

$$\lambda_i[M]\{\phi_i\} = [K]\{\phi_i\} \ , \quad i=1,2,\ldots,n+m \tag{2-2}$$

where $\lambda_i$ are the square of the circular undamped natural frequencies and $\{\phi_i\}$ are the mode shapes. The matrices [M] and [K] may be written in partitioned form as

$$[M] = \begin{bmatrix} [M_p] & \\ & [M_s] \end{bmatrix} \tag{2-3}$$

$$[K] = \begin{bmatrix} [K_p] & \\ & [K_s] \end{bmatrix} + [K_{ps}] \tag{2-4}$$

where $[M_p]$ and $[K_p]$ are the physical properties of the primary system (P-system) and $[M_s]$ and $[K_s]$ are those of secondary system (S-system).

The matrix $[K_{ps}]$ is a coupling stiffness matrix which depends on the configuration of the attachments between the two systems. $[K_{ps}]$ is a (n+m) x (n+m) matrix partitioned as

$$[K_{ps}] = \begin{bmatrix} [K_{ps}^{(11)}]_{n \times n} & [K_{ps}^{(12)}]_{n \times m} \\ [K_{ps}^{(12)}]^T & [0] \end{bmatrix}_{(n+m) \times (n+m)} \tag{2-5}$$

$[K_{ps}^{(11)}]$ contains only diagonal elements. In some studies (Igusa & Kiureghian, 1985), $[K_{ps}^{(11)}]$ is ignored. Since the S-system is light in relation to the P-system, $[K_{ps}^{(11)}]$ is of small order and is not needed in

the lowest order perturbation analysis. However, $[K_{ps}^{(11)}]$ could substantially influence higher order perturbation analysis. For a C-system as shown in Figure 2-2, the $5^{th}$ and $6^{th}$ DOF of the S-system are attached to the $1^{st}$ and $2^{nd}$ DOF of the P-system, respectively.

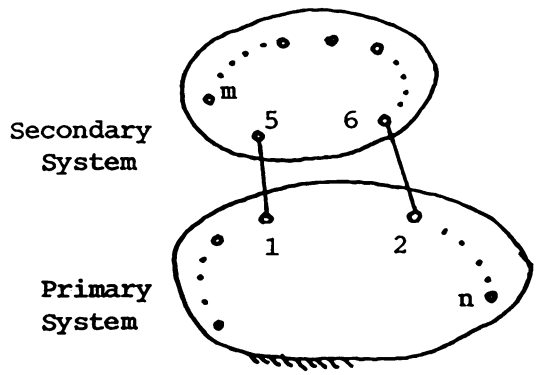



Figure 2-2 : Attachment of Secondary System

The matrix $[K_{ps}]$ will be

$$[K_{ps}] = \begin{array}{c} \\ 1 \\ 2 \\ n \\ n+1 \\ \\ n+m \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & & n & n+1 & & n+5 & n+6 & & n+m \end{array} \\ \left[\begin{array}{cccc|cccc} k_1' & & & & 0 & & -k_1' & 0 & & 0 \\ & k_2' & & & 0 & & 0 & -k_2' & & 0 \\ & & 0 & & 0 & & & & & 0 \\ & & & 0 & & & & & & \\ \hline & & \text{Symmetric} & & & & [0] & & & \end{array}\right] \end{array} \tag{2-6}$$

To facilitate analysis, we introduce the transformation

$$\{u\} = [\psi]\{q\} \tag{2-7}$$

$$\{\phi_i\} = [\psi]\{\varphi_i\} \tag{2-8}$$

where $\{q\}$ is the vector of displacements in the transformed coordinate system. The vector $\{\varphi_i\}$ is the mode shape of the C-system in the transformed coordinate system. The transformation matrix is

$$[\psi] = \left[[\phi_p] \quad [\phi_s]\right] \tag{2-9}$$

where $[\phi_p]$ and $[\phi_s]$ are eigenvector matrices of the P- and S-systems, respectively. The eigenvectors $\{\phi_{p_i}\}$ and $\{\phi_{s_i}\}$ are assumed to be normalized such that their largest value is unity. Substitution of eq. (2-8) into eq. (2-2) and premultiplication by $[\psi]^T$ yields

$$\lambda_i \left[\text{diag } (M_i)\right]\{\varphi_i\} = \left[\text{diag } (K_i) + [\psi]^T[K_{ps}][\psi]\right]\{\varphi_i\} \qquad (2\text{-}10)$$

where

$$\left[\text{diag } (M_i)\right] = \left[\text{diag } (M_1, M_2, \ldots, M_{n+m})\right] = \left[\text{diag } (M_{p_1}, \ldots M_{p_n}, M_{s_1}, \ldots, M_{s_m})\right]$$

and $M_{p_i}$, i=1,..,n, are the modal masses of the P-system, and $M_{s_i}$, i=1, ..,m, are those of the S-system. Eq. (2-10) is a generalized eigenequation. We can convert from the generalized eigenequation to the standard form but must exercise care as noted below.

Premultiplication of eq. (2-10) by $[\text{diag } (M_i)]^{-1}$ yields a standard form of the eigenequation. But the matrix $[\text{diag } (M_i)]^{-1}[\text{diag } (K_i) + [\psi]^T[K_{ps}][\psi]]$ is usually not symmetric. This problem can be overcome by decomposing $[\text{diag } (M_i)]$ into $[\text{diag } (M_i)] = [\text{diag } (\sqrt{M_i})][\text{diag } (\sqrt{M_i})]$ and introducing another vector $\{y_i\}$ defined through the transformation

$$\{\varphi_i\} = \left[\text{diag } (\sqrt{M_i})\right]^{-1}\{y_i\} \qquad (2\text{-}11)$$

where $[\text{diag } (\sqrt{M_i})] = \text{diag } (\sqrt{M_1}, \ldots, \sqrt{M_{n+m}})$ and $[\text{diag } (\sqrt{M_i})]^{-1} = [\text{diag } (1/\sqrt{M_1}, \ldots, 1/\sqrt{M_{n+m}})]$. Substitution of eq. (2-11) into eq. (2-10) and premultiplication by $[\text{diag } (1/\sqrt{M_i})]$ yields the standard form

$$\lambda_i \{y_i\} = [P]\{y_i\} \qquad (2\text{-}12)$$

where [P] is now a symmetric matrix given by

$$[P] = \left[\text{diag } (\omega_i^2)\right] + [E] \qquad (2\text{-}13)$$

$$\left[\text{diag } (\omega_i^2)\right] = \left[\text{diag } (\omega_{p_1}^2, \ldots, \omega_{p_n}^2, \omega_{s_1}^2, \ldots, \omega_{s_m}^2)\right], \text{ and}$$

$$[E] = [\text{diag } (1/\sqrt{M_i})][\psi]^T[K_{ps}][\psi][\text{diag } (1/\sqrt{M_i})].$$

The theory of linear algebra can be used to show that the transformations described in equations (2-8) and (2-11) do not change the eigenvalues. The eigenvectors of eq. (2-10) are recovered from the vectors $\{y_i\}$ by using equations (2-8) and (2-11).

The matrix [P] contains two matrices. The first one, $[\text{diag } (\omega_i^2)]$, fully represents the eigenproperties of subsystems, while the second matrix, [E], represents the deviation of the dynamic properties of the C-system from those of the subsystems. The matrices $[\text{diag } (\omega_i^2)]$ and [E] have vastly different orders of magnitude. We can use the norm

$$\|[A]\| = \max_{i,j} |a_{ij}| \tag{2-14}$$

to measure the magnitude, or size, of matrix [A]. By this norm we have

$$\left\|\left[\text{diag } (\omega_i^2)\right]\right\| = \max_i \omega_i^2 = 0(\omega_i^2) \tag{2-15}$$

where $0(\omega_i^2)$ stands for the order of $\omega_i^2$. For the matrix [E] we have

$$\|[E]\| = \|[\text{diag } (1/\sqrt{M_i})][\psi]^T[K_{ps}][\psi][\text{diag } (\sqrt{M_i})]\|$$

The matrices $[\text{diag } (1/\sqrt{M_i})]$, $[\psi]$ and $[K_{ps}]$ can be partitioned, so that

$$[E] = \begin{bmatrix} \left[\text{diag } (1/\sqrt{M_{p_i}})\right] & \\ & \left[\text{diag } (1/\sqrt{M_{s_i}})\right] \end{bmatrix} \begin{bmatrix} [\phi_p]^T & \\ & [\phi_s]^T \end{bmatrix} \begin{bmatrix} [K_{ps}^{(11)}] & [K_{ps}^{(12)}] \\ [K_{ps}^{(12)}]^T & [0] \end{bmatrix}$$

$$\times \begin{bmatrix} [\phi_p] & \\ & [\phi_s] \end{bmatrix} \begin{bmatrix} \left[\text{diag } (1/\sqrt{M_{p_i}})\right] & \\ & \left[\text{diag } (1/\sqrt{M_{s_i}})\right] \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \left[\text{diag } (1/\sqrt{M_{p_i}})\right]\left[\phi_p\right]^T\left[K_{ps}^{(11)}\right]\left[\phi_p\right]\left[\text{diag } (1/\sqrt{M_{p_i}})\right] & \left[\text{diag } (1/\sqrt{M_{p_i}})\right]\left[\phi_p\right]^T\left[K_{ps}^{(12)}\right]\left[\phi_s\right]\left[\text{diag } (1/\sqrt{M_{s_i}})\right] \\ \hline \text{symmetric} & [0] \end{array}\right]$$

Thus we have

$$\|[E]\| = \max\left[\left\|\left[\text{diag } (1/\sqrt{M_{p_i}})\right][\phi_p]^T[K_{ps}^{(11)}][\phi_p]\left[\text{diag } (1/\sqrt{M_{p_i}})\right]\right\|, \left\|\left[\text{diag } (1/\sqrt{M_{p_i}})\right][\phi_p]^T[K_{ps}^{(12)}][\phi_s]\left[\text{diag } (1/\sqrt{M_{s_i}})\right]\right\|\right]$$

There is an order of difference between the two terms in the above expression. For the first term:

$$\left\|\left[\text{diag } (1/\sqrt{M_{p_i}})\right][\phi_p]^T[K_{ps}^{(11)}][\phi_p]\left[\text{diag } (1/\sqrt{M_{p_i}})\right]\right\|$$

$$\leq \left\|\left[\text{diag } (1/\sqrt{M_{p_i}})\right]\right\| \left\|[\phi_p]^T\right\| \left\|[K_{ps}^{(11)}]\right\| \left\|[\phi_p]\right\| \left\|\left[\text{diag } (1/\sqrt{M_{p_i}})\right]\right\|$$

$$= 0\left(1/\sqrt{M_{p_i}}\right)\ 0\left(\omega_i^2\ M_{s_i}\right)\ 0\left(1/\sqrt{M_{p_i}}\right) = 0\left(\frac{M_{s_i}}{M_{p_i}}\ \omega_i^2\right)$$

The last reduction is due to $[\phi_p]$ being a normalized matrix, for which $\|[\phi_p]\| = 1.0$. Similarly, for the second term:

$$\left\|\left[\text{diag } (1/\sqrt{M_{p_i}})\right][\phi_p]^T[K_{ps}^{(12)}][\phi_s]\left[\text{diag } (1/\sqrt{M_{s_i}})\right]\right\|$$

$$\leq 0\left(1/\sqrt{M_{p_i}}\right)\ 0\left(\omega_i^2\ M_{s_i}\right)\ 0\left(1/\sqrt{M_{s_i}}\right) = 0\left(\left[\frac{M_{s_i}}{M_{p_i}}\right]^{1/2}\omega_i^2\right)$$

Since the S-system is much lighter than the P-system the following statement holds true:

$$\frac{M_{s_i}}{M_{p_i}} \ll 1.0 \tag{2-16}$$

Hence the order of the second term is larger than that of the first, and

$$\|[E]\| \leq 0\left(\left[\frac{M_{s_i}}{M_{p_i}}\right]^{\frac{1}{2}} \omega_i^2\right) \tag{2-17}$$

The difference in the order of magnitude between [diag $(\omega_i^2)$] and [E] can be expressed through

$$0\left(\frac{\|[E]\|}{\left\|\left[\text{diag } (\omega_i^2)\right]\right\|}\right) \leq 0(\epsilon), \quad \text{in which } \epsilon = \left[\frac{M_{s_i}}{M_{p_i}}\right]^{\frac{1}{2}} \tag{2-18}$$

Letting

$$[E] = \epsilon\ [B] \tag{2-19}$$

and substitution eq. (2.19) into eq. (2.13) yields

$$[P] = \left[\text{diag } (\omega_i^2)\right] + \epsilon\ [B] \tag{2-20}$$

where matrices [diag $(\omega_i^2)$] and [B] have the same magnitude. It is now quite clear that matrix [P] contains elements with vastly different orders of magnitude. This poses a major numerical problem when performing an eigen-analysis of matrix [P]. Conventional eigen-analysis schemes give poor or erroneous solutions. Intuitively, the dynamic properties of the C-system should be quite close to those of the P-system, since the S-system is much lighter. This motivates the use of perturbation techniques to determine the dynamic properties of the C-system. The use of such techniques avoids the numerical pitfalls of conventional eigen-analysis of matrix [P].

## 2.2 Perturbation Approach for Detuned Modes

In this section, we discuss approximate solution of eq. (2-12). The eigenvalues and eigenvectors of $[P] = [\text{diag } (\lambda_i)] + \epsilon [B]$ depend on the small parameter $\epsilon$. When $\epsilon = 0$, [P] reduces to

$$[P] = [\text{diag } (\lambda_i)] \tag{2-21}$$

$[\text{diag } (\lambda_i)]$ is called the unperturbed matrix whose eigenvalues are $\lambda_i$, $i=1,\ldots,n+m$, and whose corresponding eigenvectors are $\{e_i\}$, $i=1,\ldots,n+m$, which are elementary vectors. When $\epsilon$ is small but finite, we expect the eigenvalues of [P] to deviate slightly from the $\lambda_i$'s. If $\lambda_i$ is a simple eigenvalue which is well separated from other eigenvalues, the corresponding perturbed eigenvalue, $\lambda_i^*$, is assumed to be of the form of a convergent power series in $\epsilon$:

$$\lambda_i^* = \lambda_i + \epsilon\, k_i^{(1)} + \epsilon^2 k_i^{(2)} + \epsilon^3 k_i^{(3)} + \ldots \tag{2-22}$$

where $k_i^{(j)}$ is a constant, and the superscript (j) denotes the coefficient of the $j^{th}$ order term. Clearly $\lambda_i^* \to \lambda_i$ as $\epsilon \to 0$. Note that

$$|\lambda_i^* - \lambda_i| = 0(\epsilon) \tag{2-23}$$

A similar form can be extended to the corresponding perturbed eigenvector $\{e_i^*\}$. We assume each element of $\{e_i^*\}$ is represented by a convergent power series in $\epsilon$;

$$\{e_i^*\} = \{e_i\} + \epsilon\, \{z_i^{(1)}\} + \epsilon^2 \{z_i^{(2)}\} + \ldots \tag{2-24}$$

where each component of the vector series is a convergent power series in $\epsilon$. Corresponding to the result of eq. (2-23) for the eigenvalues, for the eigenvectors we have the result

$$|\{e_i^*\} - \{e_i\}| = 0(\epsilon) \tag{2-25}$$

In eq. (2-24) $\{z_i^{(k)}\}$ is a n+m dimensional vector (where n = no. of P-system DOF and m = no. of S-system DOF). Since the vectors $\{e_i\}$, i=1,2, ..,n+m, form a basis in the n+m dimensional space, we can express each vector $\{z_i^{(k)}\}$ in the form

$$\{z_i^{(k)}\} = \sum_{j=1}^{n+m} a_{ji}^{(k)} \{e_j\} \tag{2-26}$$

we have

$$\{e_i^*\} = \{e_i\} + \epsilon \sum_{j=1}^{n+m} a_{ji}^{(1)}\{e_j\} + \epsilon^2 \sum_{j=1}^{n+m} a_{ji}^{(2)}\{e_j\} + \ldots \tag{2-27}$$

and collecting together the terms in $\{e_i\}$

$$\{e_i^*\} = (1 + \epsilon a_{ii}^{(1)} + \epsilon^2 a_{ii}^{(2)} + \ldots)\{e_i\} + \epsilon \sum_{\substack{j=1 \\ j\neq i}}^{n+m} a_{ji}^{(1)}\{e_j\}$$

$$+ \epsilon^2 \sum_{\substack{j=1 \\ j\neq i}}^{n+m} a_{ji}^{(2)}\{e_j\} + \ldots \tag{2-28}$$

The relative scaling of $\{e_i^*\}$ is arbitrary and for convenience we redefine it by dividing by $(1 + \epsilon\, a_{ii}^{(1)} + \epsilon^2 a_{ii}^{(2)} + \ldots)$. This gives

$$\{e_i^*\} = \{e_i\} + \epsilon \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(1)}\{e_j\} + \epsilon^2 \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(2)}\{e_j\} + \ldots \tag{2-29}$$

where $\{e_i^*\}$ is not normalized.

We now write eq.(2-12) in terms of $\lambda_i^*$ and $\{e_i^*\}$ as

$$\Big[[\text{diag } (\lambda_i)] + \epsilon\, [B]\Big]\{e_i^*\} = \lambda_i^*\, \{e_i^*\} \tag{2-30}$$

Substituting eq. (2-22) and eq. (2-29) into eq. (2-30) and collecting terms in the same power of $\epsilon$ on each side of this equation enables the coefficients $k_i^{()}$ and $\alpha_{ji}^{()}$ to be determined. Collecting terms in $\epsilon$ yields

$$[\text{diag}\ (\lambda_i)] \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(1)}\{e_j\} + [B]\{e_i\} = \lambda_i \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(1)}\{e_j\} + k_i^{(1)}\{e_i\} \qquad (2\text{-}31)$$

Pre-multiplying eq. (2-31) by $\{e_i\}^T$ yields

$$k_i^{(1)} = \{e_i\}^T[B]\{e_i\} = b_{ii} \qquad (2\text{-}32)$$

Pre-multiplying eq. (2-31) by $\{e_\ell\}^T$, $\ell\neq i$ results in

$$\alpha_{\ell i}^{(1)} = \frac{\{e_\ell\}^T[B]\{e_i\}}{\lambda_i - \lambda_\ell} = \frac{b_{\ell i}}{\lambda_i - \lambda_\ell}, \qquad \begin{matrix}\ell=1,\ldots,n+m \\ \ell\neq i\end{matrix} \qquad (2\text{-}33)$$

The first order perturbation is therefore determined for $\lambda_i$ and the corresponding $\{e_i\}$. For sufficiently small $\epsilon$, the main term in the perturbation is the first order term which can give an approximate solutions to reasonable accuracy. It is of interest to take a closer look at the first order term in the perturbation. Assume $\lambda_1$ is a simple eigenvalue. Then from eqs. (2-22) and (2-32) the first order term is $\epsilon b_{11}$. From eqs. (2-29) and (2-33) the first order term in the perturbation of $\{e_1\}$ is given by

$$\epsilon \left[\frac{b_{21}\{e_2\}}{\lambda_1 - \lambda_2} + \frac{b_{31}\{e_3\}}{\lambda_1 - \lambda_3} + \ldots + \frac{b_{n+m,1}\{e_{n+m}\}}{\lambda_1 - \lambda_{n+m}}\right] \qquad (2\text{-}34)$$

Notice that we have the factors $(\lambda_1 - \lambda_i)$ in the denominators. This shows that if $\lambda_1$ is close to any of the other eigenvalues the perturbation could be very substantial. The perturbation of $\lambda_1$ is therefore strongly influenced by close modes. When $\lambda_1$ is well separated from the other eigenvalues, the eigenvector $\{e_1\}$ is comparatively insensitive to the perturbation. In practice, the first order perturbation is often not sufficiently accurate, and higher order perturbations are required. The higher order perturbations are obtained by collecting

terms of the higher powers of $\epsilon$ in eq. (2-30). Collecting coefficients in $\epsilon^2$ yields

$$[\text{diag}\ (\lambda_i)] \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(2)}\{e_j\} + [B] \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(1)}\{e_j\}$$

$$= \lambda_i \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(2)}\{e_j\} + k_i^{(1)} \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(1)}\{e_j\} + k_i^{(2)}\{e_i\} \qquad (2\text{-}35)$$

Premultiplying by $\{e_\ell\}^T$, $\ell=1,2,\ldots,n+m$ gives

$$k_i^{(2)} = \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(1)} b_{ij} \qquad (2\text{-}36)$$

$$\alpha_{\ell i}^{(2)} = \frac{\left(k_i^{(1)}\alpha_{\ell i}^{(1)} - \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(1)} b_{ij}\right)}{\lambda_\ell - \lambda_i}, \qquad \ell=1,\ldots,n+m,\ \ell\neq i \qquad (2\text{-}37)$$

By the same manner, we have the coefficients of $\epsilon^3$:

$$k_i^{(3)} = \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(2)} b_{ij}$$

$$\alpha_{\ell i}^{(3)} = \frac{\left(k_i^{(1)}\alpha_{\ell i}^{(2)} + k_i^{(2)}\alpha_{\ell i}^{(1)} - \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(2)} b_{ij}\right)}{\lambda_\ell - \lambda_i}, \qquad \ell\neq i \qquad (2\text{-}38)$$

For the coefficients of any order perturbation, the general formulas are:

$$k_i^{(h)} = \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(h-1)} b_{ij} \quad \text{and}$$

$$\alpha_{\ell i}^{(h)} = \frac{\sum_{j=1}^{h-1} k_i^{(j)} \alpha_{\ell i}^{(h-j)} - \sum_{\substack{j=1 \\ j\neq i}}^{n+m} \alpha_{ji}^{(\ell-1)} b_{ij}}{\lambda_\ell - \lambda_i} \tag{2-39}$$

It can be seen that the approximate solutions of any order perturbation can be obtained recursively. The formulas obtained here are quite simple for practical implementation. (In practice, the requirement of higher order perturbations beyond the third is rare.)

The above formulas can be applied to any detuned mode independent of the tuning of other modes. However, the formulas are invalid for tuned modes themselves. It is easy to see that from eq. (2-34) if $\lambda_i$ and $\lambda_j$ are tuned, then as $\lambda_i \to \lambda_j$, $\alpha_{ij}^{(1)} \to \infty$, implying that the perturbation of $\{e_i\}$ or $\{e_j\}$ goes to infinity. This is not really true, but the fact is that the perturbation of tuned mode involves fractional powers of $\epsilon$, and hence the basic assumption in eq. (2-25) no longer holds. A special scheme for the tuned case is developed later.

## 2.3 Identification of Tuned Modes by Gerschgorin's Theorem

Gerschgorin's theorem plays an very important role in this study. It gives an accurate estimation of the distribution of eigenvalues of a diagonally dominated matrix such as $[P] = [\text{diag}\,(\lambda_i)] + \epsilon[B]$.

**Gerschgorin's thereom**: Every eigenvalue of the matrix [A] lies in at least one of the circular discs $D_i$ with centers $a_{ii}$ and radii $\sum_{j\neq i} |a_{ij}|$.

Applying the thereom for matrix [P] yields quite accurate locations of the eigenvalues, which lie in the union of all discs. (For a real matrix [P], Gerschgorin's discs degenerate to line segments.)

$$D_i \text{ - center: } (\lambda_i + \epsilon\, b_{ii}), \quad \text{radius: } \epsilon \sum_{j\neq i} |b_{ij}|$$

The radii are small since $\epsilon$ is very small. Quite often, we can further reduce the radii of Gerschgorin discs by the following simple scheme.

If we multiply the $i^{th}$ column of [P] by $\beta$ and its $i^{th}$ row by $1/\beta$, then its eigenvalues are unaltered. Let us apply this for i=1 to matrix [P], taking $\beta > 1.0$. We have

$$\left[\text{diag } (\lambda_i)\right] + \epsilon \begin{bmatrix} b_{11} & \frac{1}{\beta} b_{12} & \frac{1}{\beta} b_{13} & \cdots & \frac{1}{\beta} b_{1,n+m} \\ \beta\, b_{21} & b_{22} & b_{23} & \cdots & b_{2,n+m} \\ \beta\, b_{n+m,1} & b_{n+m,2} & b_{n+m,3} & \cdots & b_{n+m,n+m} \end{bmatrix} \tag{2-40}$$

The first Gerschgorin disc $D_1$ becomes

$$D_1 \text{ - center: } (\lambda_1 + \epsilon b_{11}), \quad \text{radius: } \frac{\epsilon}{\beta} \sum_{j \neq i} |b_{ij}|.$$

We wish to choose $\beta$ so as to make the first Gerschgorin disc as small as possible while keeping the other discs sufficiently small to avoid overlapping the first. This will be true for all small $\epsilon$ if we choose $\beta$ to have the largest value consistent with the inequalities

$$|\beta b_{i1}| \leq \gamma\, |\lambda_1 - \lambda_i| , \quad i=2,\ldots,n+m \tag{2-41}$$

where $\gamma$ is a number less than unity.

For the matrix [P], some of the Gerschgorin discs can overlap each other. We could reduce their radii to some extent. Assume the first S Gerschgorin discs are overlapped, we write [P] in partitioned form

$$\left[\text{diag } (\lambda_i)\right] + \epsilon \begin{bmatrix} [B_{11}] & [B_{12}] \\ [B_{12}]^T & [B_{22}] \end{bmatrix} \tag{2-42}$$

where $[B_{11}]$, $[B_{12}]$ and $[B_{22}]$ are s x s, s x (n+m-s) and (n+m-s) x (n+m-s) submatrices, respectively. Multiplying the first s rows by $1/\beta$ and the first s columns by $\beta$ yields

$$\left[\text{diag } (\lambda_i)\right] + \epsilon \begin{bmatrix} [B_{11}] & \frac{1}{\beta}[B_{12}] \\ \beta[B_{12}]^T & [B_{22}] \end{bmatrix} \tag{2-43}$$

The first s discs are reduced by this procedure, and sometimes each one is isolated.

We now turn to the main problem of the identification of tuned modes. The identification is important in practice. Since we know from the last section that perturbation schemes cannot be applied directly to tuned modes, special treatment is required. Therefore, first we wish to identify the tuned modes.

We apply Gerschgorin's theorem to matrix [P]. If the Gerschgorin disc $D_i$ is isolated from other discs, then $\lambda_i$ is a detuned mode. When discs overlap, we can try to separate these discs as outlined earlier. If overlapping discs cannot be separated by choosing a suitable scaling factor $\beta$, we define them as tuned modes. Sometimes we can find a scaling factor $\beta$ which separates joint discs and keeps others from overlapping. Then these modes are still defined as detuned modes and perturbation schemes can be applied directly. The physical meaning is clear here. Two closely spaced modes (Gerschgorin discs are overlapped) may or may not be tuned modes. If they are well separated from all other modes, they will often be detuned modes. Consider, for example the matrix [P] defined by

$$\begin{bmatrix} 0.1206 & & & & \\ & 1.000 & & & \\ & & 2.347 & & \\ & & & 3.532 & \\ & & & & 0.200 \end{bmatrix} + 0.2236 \begin{bmatrix} .0193 & .0170 & .0126 & -.007 & -.1313 \\ & .0149 & -.0111 & -.0059 & -.1155 \\ & & .0082 & .0044 & .0857 \\ & & & .0023 & .0456 \\ \text{sym} & & & & 0.000 \end{bmatrix}$$

whose Gerschgorin discs are specified in Table 2-1. The discs $D_2$, $D_3$ and $D_4$ are well separated but $D_1$ and $D_5$ overlap. We multiply the first and the fifth columns by 5 and the first and the fifth rows by 1/5. The new Gerschgorin discs are all isolated from each other as is apparent

from Table 2-2. Hence all modes may now be considered to be detuned modes.

Table 2-1 : Original Gerschgorin Disks of Matrix [P]

| Disk | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
|---|---|---|---|---|---|
| Center | 0.1249 | 1.0033 | 2.3491 | 3.5326 | 0.2000 |
| Radius | 0.0375 | 0.0334 | 0.0254 | 0.0140 | 0.0845 |

Table 2-2 : Gerschgorin Disks of [P] After Scaling $5^{th}$ Row and Column

| Disk | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ |
|---|---|---|---|---|---|
| Center | 0.1249 | 1.0033 | 2.3491 | 3.5326 | 0.2000 |
| Radius | 0.0310 | 0.1518 | 0.1133 | 0.0608 | 0.0404 |

## 2.4 Tuned Modes

As noted in previous sections, the analysis of tuned modes requires special treatment. The solution of a small eigenequation is required. This eigenequation, which is related to the tuned modes only, is specially formulated to avoid numerical problems.

Consider the matrix $[P] = [\text{diag}\ (\lambda_i)] + \epsilon[B]$. The case may be sufficiently illustrated by assuming the first two modes to be tuned, i.e., $\lambda_1$ and $\lambda_2$ must be very close and separated by a quantity of order $\epsilon$. We partition [P] in the form

$$\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \Lambda_{n-2} \end{bmatrix} + \epsilon \begin{bmatrix} [B_{11}] & [B_{12}] \\ [B_{12}] & [B_{22}] \end{bmatrix} \qquad (2\text{-}44)$$

If we write

$$[G] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} + (\lambda_2 - \lambda_1)/\epsilon \end{bmatrix} \tag{2-45}$$

then eq. (2-44) may be expressed as

$$\left[\begin{array}{cc|c} \lambda_1 & & \\ & \lambda_1 & \\ \hline & & [\Lambda_{n-2}] \end{array}\right] + \epsilon \left[\begin{array}{cc|c} \multicolumn{2}{c|}{[G]} & [B_{12}] \\ \hline \multicolumn{2}{c|}{[B_{12}]^T} & [B_{22}] \end{array}\right] \tag{2-46}$$

The order of elements in the second matrix of eq. (2-46) is uniform. The eigenequation for [G] is

$$[X]^T[G][X] = \begin{bmatrix} \mu_1 & \\ & \mu_2 \end{bmatrix} \tag{2-47}$$

where [X] is the eigenvector matrix of [G] and $\mu_1$, $\mu_2$ are the eigenvalues of [G]. Letting

$$[T] = \begin{bmatrix} [X] \\ [I] \end{bmatrix} \tag{2-48}$$

We obtain

$$\begin{aligned} &[T]^T\Big[[\text{diag } (\lambda_i)] + \epsilon[B]\Big][T] \\ &= \left[\begin{array}{cc|c} \lambda_1 + \epsilon\,\mu_1 & & \\ & \lambda_1 + \epsilon\mu_2 & \\ \hline & & [\Lambda_{n+m-2}] \end{array}\right] + \epsilon \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & [X]^T\,[B_{12}] \\ \hline \multicolumn{2}{c|}{[B_{12}]^T\,[X]} & [B_{22}] \end{array}\right] \\ &= [\text{diag } (\lambda'_i)] + \epsilon[B'] \end{aligned} \tag{2-49}$$

The Gerschgorin discs of $[\text{diag } (\lambda'_i)] + \epsilon[B']$ are now disjoint and the case becomes detuned. By the perturbation scheme which has been developed in the Section 2.2 we can obtain the eigenvalues $\lambda'^*_i$ and the eigenvectors $(e'^*_i)$ of $[\text{diag } (\lambda'_i)] + \epsilon[B']$. Since the matrices $[\text{diag } (\lambda_i)] + \epsilon[B]$ and $[\text{diag } (\lambda'_i)] + \epsilon[B']$ are similar, the eigenvalues and eigenvectors of $[\text{diag } (\lambda_i)] + \epsilon[B]$ can be easily obtained by

$$\lambda^*_i = \lambda'^*_i \tag{2-50}$$

and $$\{e_i^*\} = [T]\{e'^*_i\} \quad , \quad i=1,\ldots,n+m$$

In general when a group of r modes are tuned, we will need to solve an eigenequation of order r using the same precision of computation as in the main problem. Hence numerical problems are not encountered. For the case where two or more modes are actually coincident the corresponding discs will continue to overlap, but the radii will be reduced sufficiently enabling us to compute the mode shapes and frequencies of the tuned modes to required accuracy. For example, if the first two modes of eq. (2-44) are actually coincident in the C-system, the submatrix $[X]^T[B_{12}]$, which represents the radii of the first two modes, will be very small.

In general the solution of a small eigenequation does not give results of sufficient accuracy, but it separates the joint discs by shifting the centers and reducing the radii of joint discs which gives rise to an ideal model for applying perturbation schemes.

## 2.5 Error Bounds for Approximate Solutions

In previous sections, the fundamental theories have been established and technical schemes were developed. It is of practical importance to evaluate the accuracy of the approximate solutions. From the structural response point of view, we wish the accuracy of eigenvalues and that of eigenvectors to be consistent. When high order spectral moments are required, we need a higher degree of accuracy in the eigenvalues than in the eigenvectors. Let us examine the perturbation formulas of (2-22) and (2-29). For the first order approximation for the eigenvalues eq. (2-22) reduces to

$$\lambda_i(\epsilon) = \lambda_i + \epsilon k_i^{(1)} \qquad (2\text{-}51)$$

and we know $k_i^{(1)} = b_{ii}$ from eq. (2-32). The first order approximation for the eigenvalues does not improve the accuracy at all, since $b_{ii} = 0$ for i=n+1,...,n+m. This reflects the fact is that eigenvectors are more sensitive to perturbation than eigenvalues, so that the lowest order term of eigenvectors is $\epsilon$ while the lowest order term of eigenvalues is $\epsilon^2$. To speed up the convergence of eigenvalues, we introduce the well-known Rayleigh's quotient, defined as

$$R(\{e_i^*\}) = \frac{\{e_i^*\}^T[P]\{e_i^*\}}{\{e_i^*\}^T\{e_i^*\}} \tag{2-52}$$

Hence, if $\{e_i^*\}$ is the first order approximation of the eigenvector then

$$\lambda_i^* = R(\{e_i^*\}) \tag{2-53}$$

will be the second order approximation of the eigenvalue (Meirovitch, 1986). The convergence of eigenvalues is greatly improved. With the low order eigenvectors we can obtain high order eigenvalues.

In practice sharp error bounds are highly desirable. For eigenvalues, we employ Gerschgorin's discs. We compute the perturbated eigenvectors $\{e_i^*\}$, i=1,..,n+m, then form the perturbated eigenvector matrix

$$[I^*] = [\{e_1^*\}, \{e_2^*\}, \ldots, \{e_{n+m}^*\}] \tag{2-54}$$

By the following computation we obtain

$$[S] = [I^*]^T[P][I^*] \tag{2-55}$$

It is clear that the diagonal elements of [S] are identical to that of eq. (2-53). The off diagonal elements are very small. The Gerschgorin discs can be easily obtained and the radii serve as error bounds. Of course, we can use the schemes mentioned in Section 2.3 to

reduce the radius of each disc, then even sharper error bounds can be obtained.

The derivation of error bounds for the eigenvectors $\{e_i^*\}$ is only slightly more difficult. If the eigenvectors of [S] are $\{u_i\}$, then the eigenvectors of [P] are $[I^*]\{u_i\}$. Hence we only need to find the error bounds of $\{u_i\}$ to obtain those of $\{e_i^*\}$. We illustrate this technique by means of a numerical example. Suppose we have

$$[S] = \begin{bmatrix} 0.1174 & & & & \\ & 1.014 & & & \\ & & 2.352 & & \\ & & & 3.533 & \\ & & & & 0.456 \end{bmatrix}$$

$$+ 10^{-2} \begin{bmatrix} 0.0 & -1.40 & .915 & .476 & -.016 \\ -1.40 & .0 & -.707 & -.325 & -.226 \\ .915 & -.707 & .0 & .972 & .076 \\ .476 & -.325 & .972 & .0 & .036 \\ -.016 & -.226 & .076 & .036 & .0 \end{bmatrix} \qquad (2\text{-}56)$$

By the Gerschgorin's theorem we may show that the first eigenvalue $\mu$ satisfies

$$|\mu - 0.1174| \leq 0.028 \qquad (2\text{-}57)$$

Let $\{u\}$ be the corresponding eigenvector normalized so that its largest element (the first one) is unity. We write $\{u\} = [1, u_2, u_3, u_4, u_5]^T$ and the eigenequation as

$$[S]\{u\} = \mu \{u\} \qquad (2\text{-}58)$$

From the second equation of eq. (2-58) we have

$$-1.4\text{x}10^{-2} + 1.014u_2 - 0.707\text{x}10^{-2}u_3 - 0.325\text{x}10^{-2}u_4 - 0.226\text{x}10^{-2}u_5 = \mu u_2 \qquad (2\text{-}59)$$

Since $|u_2|$, $|u_3|$, $|u_4|$ and $|u_5|$ are all less than unity, the above equation together with eq. (2-57) gives

$$|u_2| < \frac{(1.4 + 0.707 + 0.325 + 0.226)\times 10^{-2}}{|\mu - 1.014|}$$

$$< \frac{(1.4 + 0.707 + 0.325 + 0.226)\times 10^{-2}}{|0.1174 - 0.028 - 1.014|} = 0.029 \qquad (2.60)$$

From the other equations of eq. (2-58), we can similarly obtain crude bounds for the other elements of {u}

$$|u_3| < 0.008, \ |u_4| < 0.003, \ |u_5| < 0.011 \qquad (2\text{-}61)$$

Using these crude bounds we may now return to eq. (2-59) and the other equations of eq. (2-58) to recursively obtain much closer bounds for $u_2$, $u_3$, $u_4$ and $u_5$. In this study, however, the crude bounds from (2-60) and (2-61) are sufficient.

# 3. NONCLASSICALLY DAMPED SYSTEMS

## 3.1 General Theory

Modal superposition is a common method used to solve dynamic equations of motion of linear systems, such as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{p\} \tag{3-1}$$

If the system is classically damped, we can uncouple eq. (3-1) by the undamped real-valued mode shapes. But if the system is nonclassically damped, the undamped mode shapes can no longer uncouple the damping matrix. To solve eq. (3-1) by modal superposition for the nonclassically damped system, we need to find more general mode shapes which are complex-valued (Hurty & Rubinstein, 1964). Each component of the mode shape is distinguished not only by its amplitude but also by its phase.

Equation (3-1) can always be transformed into an equivalent system of first order differential equations with double size:

$$\begin{bmatrix} 0 & [M] \\ [M] & [C] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}\} \\ \{\dot{u}\} \end{Bmatrix} + \begin{bmatrix} -[M] & 0 \\ 0 & [K] \end{bmatrix} \begin{Bmatrix} \{\dot{u}\} \\ \{u\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix} \tag{3-2}$$

This equation can be written as

$$[A]\{\dot{y}\} + [B]\{y\} = \{p\} \tag{3-3}$$

where

$$[A] = \begin{bmatrix} 0 & [M] \\ [M] & [C] \end{bmatrix}, \quad [B] = \begin{bmatrix} -[M] & 0 \\ 0 & [K] \end{bmatrix}, \quad \{y\} = \begin{Bmatrix} \{\dot{u}\} \\ \{u\} \end{Bmatrix}, \text{ and } \{P\} = \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix}$$

Equation (3-3) implies a linear eigenvalue problem of the form

$$\lambda[A]\{\phi\} = -[B]\{\phi\} \tag{3-4}$$

Therefore, to solve equation (3-4) techniques very similar to those used for classically damped systems may be used. It should be noted, however, that the matrices [A] and [B] are not positive definite; hence, the eigenvalues and corresponding eigenvectors are generally complex-valued.

The complex eigenvalues will occur in conjugate pairs. For example, if the $i^{th}$ and $j^{th}$ eigenvalues are complex conjugate, they may be written as

$$\begin{aligned} \lambda_i &= u_i + iv_i \\ \lambda_j &= \bar{\lambda}_i = u_i - iv_i \end{aligned} \tag{3-5}$$

and their corresponding eigenvectors will also be conjugate pairs, i.e.,

$$\{\phi_i\} = \{\bar{\phi}_j\} \tag{3-6}$$

The mode shape $\{\phi_i\}$ has 2n components which may be partitioned as

$$\{\phi_i\} = \begin{Bmatrix} \{\phi_i^u\} \\ \{\phi_i^d\} \end{Bmatrix} \tag{3-7}$$

The n component eigenvectors $\{\phi_i^u\}$ and $\{\phi_i^d\}$ are related through

$$\{\phi_i^u\} = \lambda_i \{\phi_i^d\} \tag{3-8}$$

Eq. (3-4) leads to a set of 2n eigenvalues and corresponding eigenvectors. In a manner similar to that used for the classically damped system we transform eq. (3-3) to a new coordinate system spanned by the eigenvectors of eq. (3-4). The transformation is given by

$$\{y\} = [\Phi]\{z\} \tag{3-9}$$

The transformation matrix $[\Phi]$ is constructed by 2n eigenvectors

$$[\Phi] = [\{\phi_1\}\{\phi_2\} \ \ldots \ \{\phi_{2n}\}] \tag{3-10}$$

Substituting eq. (3-9) into eq. (3-3) and then premultiplying it by $[\Phi]^T$ yields

$$[\Phi]^T[A][\Phi]\{\dot{z}\} + [\Phi]^T[B][\Phi]\{z\} = [\Phi]^T\{P\}$$

which may be written as

$$[\text{diag } (\alpha_i)]\{\dot{z}\} + [\text{diag } (\beta_i)]\{z\} = \{q\} \tag{3-11}$$

where

$$[\text{diag } (\alpha_i)] = [\Phi]^T[A][\Phi]$$

$$[\text{diag } (\beta_i)] = [\Phi]^T[B][\Phi]$$

and $$\{q\} = [\Phi]^T\{P\}$$

Eq. (3-11) is a set of uncoupled equations which may be written in scalar form as

$$\alpha_i\dot{z}_i + \beta_i z_i = q_i \qquad i=1,2,\ldots,2n$$

or

$$\dot{z}_i - \lambda_i z_i = {}^{q_i}/_{\alpha_i} \qquad i=1,2,\ldots,2n \tag{3-12}$$

Since

$$\beta_i = -\lambda_i\alpha_i \qquad i=1,2,\ldots,2n \tag{3-13}$$

the solution to eq. (3-11) is

$$z_i(t) = e^{\lambda_i t} z_i(0) + \frac{1}{\alpha_i}\int_0^t e^{\lambda_i(t-\tau)} z_i(\tau)\, d\tau\,, \quad i=1,2,\ldots,2n \tag{3-14}$$

Modal analysis for nonclassically damped systems is very costly. The matrices [A] and [B] are real and symmetric, but neither of them is positive definite. The presence of complex eigenvalues and eigenvectors increases the amount of computation substantially and the numerical algorithms are not efficient. In practice the 2n x 2n matrices [A] and [B] may be extremely large. A different approach appears highly desirable.

3.2 Modal Synthesis

A n+m-DOF C-system is composed of two subsystems - a m-DOF light secondary subsystem supported on a n-DOF primary subsystem. Figure 2-1 shows the individual subsystems and the C-system. Generally the individual subsystems are assumed to be classically damped in which case the damped eigenproperties are directly related to the undamped eigenproperties. For example, solving the undamped eigenequation for the P-system

$$\omega_{p_i}^2 [M_p]\{\phi_{p_i}\} = [K_p]\{\phi_{p_i}\} , \quad i=1,..,n \tag{3-15}$$

we obtain the undamped eigenvalues $\omega_{p_i}^2$ and eigenvectors $\{\phi_{p_i}\}$, i=1,2,..,n. The eigenvectors are normalized with respect to the mass matrix

$$[\Phi_p]^T[M_p][\Phi_p] = [I] \tag{3-16}$$

thus

$$[\Phi_p]^T[C_p][\Phi_p] = \left[\text{diag } (2\omega_{p_i}\beta_{p_i})\right] \tag{3-17}$$

and

$$[\Phi_p]^T[K_p][\Phi_p] = \left[\text{diag } (\omega_{p_i}^2)\right] \tag{3-18}$$

where $\beta_{p_i}$ is known as the modal damping ratio. The damped eigenvalues $\lambda_{p_i}$ and eigenvectors $\{\psi_{p_i}\}$, i=1,2,...,2n are obtained as follows

$$\begin{aligned} \lambda_{p_i} &= -\beta_{p_i}\omega_{p_i} + i\omega_{p_i}\sqrt{1-\beta_{p_i}^2} , && i=1,2,...,n \\ \lambda_{p_{i+n}} &= \bar{\lambda}_{p_i} && i=1,2,...,n \end{aligned} \tag{3-19}$$

$$\begin{aligned} \{\psi_{p_i}\} &= \begin{Bmatrix} \lambda_{p_i}\{\phi_{p_i}\} \\ \{\phi_{p_i}\} \end{Bmatrix} , && i=1,2,...,n \\ \{\psi_{p_{i+n}}\} &= \{\bar{\psi}_{p_i}\} , && i=1,2,...,n \end{aligned} \tag{3-20}$$

$\lambda_{P_i}$ and $\{\psi_{P_i}\}$ automatically satisfy the damped eigenequation (3-4)

$$\lambda_{P_i}\begin{bmatrix} 0 & [M_p] \\ [M_p] & [C_p] \end{bmatrix}\{\psi_{P_i}\} = -\begin{bmatrix} -[M_p] & 0 \\ 0 & [K_p] \end{bmatrix}\{\psi_{P_i}\}, \quad i=1,\ldots,2n \tag{3-21}$$

Exactly the same statements apply to the S-system. The damped eigenproperties and the undamped eigenproperties are related as

$$\begin{aligned} \lambda_{s_i} &= -\beta_{s_i}\omega_{s_i} + i\omega_{s_i}\sqrt{1-\beta_{s_i}^2}, && i=1,2,\ldots,m \\ \lambda_{s_{i+m}} &= \bar{\lambda}_{s_i}, && i=1,2,\ldots,m \end{aligned} \tag{3-22}$$

$$\begin{aligned} \{\psi_{s_i}\} &= \begin{Bmatrix} \lambda_{s_i}\{\phi_{s_i}\} \\ \{\phi_{s_i}\} \end{Bmatrix} && i=1,2,\ldots,m \\ \{\psi_{s_{i+m}}\} &= \{\bar{\psi}_{s_i}\}, && i=1,2,\ldots,m \end{aligned} \tag{3-23}$$

Now let us consider the C-system in a manner quite similar to that used in Section 2.1. The physical properties of the C-system are represented by the matrices [M], [C] and [K] which can be written in partitioned form as

$$[M] = \begin{bmatrix} [M_p] & \\ & [M_s] \end{bmatrix} \tag{3-24}$$

$$[C] = \begin{bmatrix} [C_p] & \\ & [C_s] \end{bmatrix} + [C_{ps}] = [C_c] + [C_{ps}] \tag{3-25}$$

$$[K] = \begin{bmatrix} [K_p] & \\ & [K_s] \end{bmatrix} + [K_{ps}] = [K_c] + [K_{ps}] \tag{3-26}$$

where $[K_{ps}]$ and $[C_{ps}]$ are coupling matrices. $[K_{ps}]$ has been mentioned in Section 2.1. $[C_{ps}]$ is similar and partitioned as

$$[C_{ps}] = \begin{bmatrix} [C_{ps}^{(11)}]_{n \times n} & [C_{ps}^{(12)}]_{n \times m} \\ [C_{ps}^{(12)}]^T & [0] \end{bmatrix} \tag{3-27}$$

Even though the subsystems are classically damped the C-system may be nonclassically damped (Igusa and Kiureghian, 1983). Thus, in general we need to solve the damped eigenequation

$$\lambda_i^* [A]\{\psi_i^*\} = - [B]\{\psi_i^*\} \quad i=1,2,..,2(n+m) \tag{3-28}$$

where

$$[A] = \begin{bmatrix} 0 & [M] \\ [M] & [C] \end{bmatrix} = \begin{bmatrix} 0 & [M] \\ [M] & [C_c] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [C_{ps}] \end{bmatrix}$$

$$[B] = \begin{bmatrix} -[M] & \\ & [K] \end{bmatrix} = \begin{bmatrix} -[M] & \\ & [K_c] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [K_{ps}] \end{bmatrix}$$

In the theory of linear algebra, $\lambda_i$ and $\{\psi_i\}$ are defined as the eigenvalue and eigenvector of the pair ([A],-[B]). Two pairs, for example $([A_1],[B_1])$ and $([A_2],[B_2])$, are said to be equivalent if there exist invertible matrices [E] and [F] such that (Parlett, 1980)

$$[A_2] = [E][A_1][F], \quad [B_2] = [E][B_1][F] \tag{3-29}$$

The eigenvalues of two equivalent pairs are the same and the eigenvectors are simply related through

$$\{\phi_2\} = [F]\{\phi_1\} \tag{3-30}$$

To find a simple pair equivalent to ([A], -[B]), we construct an invertible matrix [T]

$$[T] = \begin{bmatrix} [\Phi][\text{diag } (\lambda_i)] & [\bar{\Phi}][\text{diag } (\bar{\lambda}_i)] \\ [\Phi] & [\bar{\Phi}] \end{bmatrix}_{2(n+m) \times 2(n+m)} \tag{3-31}$$

in which

$$\left[\text{diag } (\lambda_i)\right] = \begin{bmatrix} \lambda_{p_1} & & & & & & \\ & \lambda_{p_2} & & & & & \\ & & \ddots & & & & \\ & & & \lambda_{p_n} & & & \\ & & & & \lambda_{s_1} & & \\ & & & & & \lambda_{s_2} & \\ & & & & & & \ddots \\ & & & & & & & \lambda_{s_m} \end{bmatrix} \tag{3-32}$$

$$[\Phi] = \begin{bmatrix} [\Phi_p] & \\ & [\Phi_s] \end{bmatrix} \left[ \text{diag} \left( \frac{1 - i}{2(\omega_i \sqrt{1-\beta_i^2})} \right) \right] \tag{3-33}$$

where

$$\left[ \text{diag} \left( \frac{1 - i}{2(\omega_i \sqrt{1-\beta_i^2})} \right) \right]$$

$$= \begin{bmatrix} \left( \frac{1 - i}{2(\omega_i \sqrt{1-\beta_i^2}_{p_i})} \right) & & & \\ & \left( \frac{1 - i}{2(\omega_i \sqrt{1-\beta_i^2}_{p_n})} \right) & & \\ & & \left( \frac{1 - i}{2(\omega_i \sqrt{1-\beta_i^2}_{s_i})} \right) & \\ & & & \left( \frac{1 - i}{2(\omega_i \sqrt{1-\beta_i^2}_{s_n})} \right) \end{bmatrix} \tag{3-34}$$

Then an equivalent pair $([A^*],[B^*])$ may be obtained as follows:

$$[A^*] = [T]^T[A][T]$$

$$= [T]^T \begin{bmatrix} [0] & [M] \\ [M] & [C_c] \end{bmatrix} [T] + [T]^T \begin{bmatrix} [0] & [0] \\ [0] & [C_{ps}] \end{bmatrix} [T] = [I] + [F] \tag{3-35}$$

where

$$[F] = \begin{bmatrix} [\Phi]^T[C_{ps}][\Phi] & [\Phi]^T[C_{ps}][\bar{\Phi}] \\ [\bar{\Phi}]^T[C_{ps}][\Phi] & [\bar{\Phi}]^T[C_{ps}][\bar{\Phi}] \end{bmatrix} \tag{3-36}$$

and

$$[B^*] = -[T]^T[B][T] = [\text{diag}\ (\lambda_i)] + [E] \tag{3-37}$$

where

$$[E] = - \begin{bmatrix} [\Phi]^T[K_{ps}][\Phi] & [\Phi]^T[K_{ps}][\bar{\Phi}] \\ [\bar{\Phi}]^T[K_{ps}][\Phi] & [\bar{\Phi}]^T[K_{ps}][\bar{\Phi}] \end{bmatrix} \tag{3-38}$$

The eigenequation may now be written as

$$\lambda_i^* \left\{ [I] + [F] \right\} (x_i^*) = \left\{ [\text{diag}\ (\lambda_i)] + [E] \right\} (x_i^*) \tag{3-39}$$

The eigenvectors of eq. (3-28) can be easily recovered through

$$\{\psi_i^*\} = [T]\{x_i^*\} \ , \qquad i=1,..,2(n+m) \tag{3-40}$$

From eq. (3-39) we can see that [E] and [F] represent the deviation of the dynamic properties of the C-system from those of the subsystems. We use the same norm as the one used in Section 2.1 to measure the magnitude of [E] and [F]. From eq. (3-33) and (3-38) we have

$$\|[E]\| = \|[\Phi]^T[K_{ps}][\Phi]\|$$

$$= \left\| \left[ \text{diag} \left( \frac{1 - i}{2(\omega_i\sqrt{1-\beta_i^2})} \right) \right] \begin{bmatrix} [\Phi_p]^T & \\ & [\Phi_s]^T \end{bmatrix} \begin{bmatrix} [K_{ps}^{(11)}] & [K_{ps}^{(12)}] \\ \text{sym} & [0] \end{bmatrix} \begin{bmatrix} [\Phi_p] & \\ & [\Phi_s] \end{bmatrix} \right.$$

$$\left. \times \left[ \text{diag} \left( \frac{1 - i}{2(\omega_i\sqrt{1-\beta_i^2})} \right) \right] \right\|$$

$$\leq \left\| \left[ \text{diag} \left( \frac{1 - i}{2(\omega_i\sqrt{1-\beta_i^2})} \right) \right] \right\|^2 \left\| \begin{bmatrix} [\Phi_p]^T[K_{ps}^{(11)}][\Phi_p] & [\Phi_p][K_{ps}^{(12)}][\Phi_s] \\ \text{sym} & [0] \end{bmatrix} \right\|$$

Since $[\Phi_p]$ and $[\Phi_s]$ are normalized as in eq. (3-16), $\|[\Phi_p]\| = 0(1/\sqrt{M_p})$ and $\|[\Phi_s]\| = 0(1/\sqrt{M_s})$. We also have $\|[K_{ps}^{(12)}]\| = \|[K_{ps}^{(11)}]\| = 0\left(M_s\omega_{s_i}^2\right)$. The following results are therefore obtained:

$$\|[\Phi_p][K_{ps}^{(12)}][\Phi_s]\| = 0\left(\left[\frac{M_s}{M_p}\right]^{\frac{1}{2}}\omega_{s_i}^2\right) > \|[\Phi_p][K_{ps}^{(11)}][\Phi_p]\| = 0\left(\frac{M_s}{M_p}\,\omega_{s_i}^2\right)$$

$$\left\| \left[ \text{diag} \left( \frac{1 - i}{2(\omega_i\sqrt{1-\beta_i^2})} \right) \right] \right\|^2 = 0\left(\frac{1}{\omega_i}\right)$$

thus

$$\|[E]\| \leq 0\left(\frac{1}{\omega_i}\right) \cdot 0\left(\left[\frac{M_s}{M_p}\right]^{\frac{1}{2}}\omega_{s_i}^2\right)$$

or

$$\|[E]\| = 0\left(\epsilon\omega_i\right) \tag{3-41}$$

where

$$\epsilon = \left[\frac{M_s}{M_p}\right]^{\frac{1}{2}} << 1.0$$

Since $\|[\text{diag } (\lambda_i)]\| = 0(\omega_i)$, we have

$$\frac{\|[E]\|}{\|[\text{diag } (\lambda_i)]\|} = 0(\epsilon) \tag{3-42}$$

Therefore, the two matrices on the right side of the eq. (3-39) have different orders of magnitude.

Since $\|[C_{ps}^{(12)}]\| = 0(\omega_{s_i} \beta_{s_i})$, we can obtain the order of magnitude for [F] in a similar manner:

$$\|[F]\| \leq 0\left(\frac{1}{\omega_i}\right) 0\left(\left[\frac{M_s}{M_p}\right]^{\frac{1}{2}} \cdot \beta_{s_i} \omega_{s_i}\right) = 0(\epsilon \beta_s) \tag{3-43}$$

$$\frac{\|[F]\|}{\|[I]\|} = 0(\epsilon \beta_i) \tag{3-44}$$

Therefore, the two matrices on the left side of eq. (3-39) also have different orders of magnitude. The typical damping ratio $\beta_s$ is generally of order $0(\epsilon)$, so that $\|[F]\| \approx 0(\epsilon^2)$ is generally true. Let

$$[E] = \epsilon\,[E^*]$$

$$[F] = \epsilon^2 [F^*]$$

Substituting into eq. (3-39) yields

$$\lambda_i^* \left[[I] + \epsilon^2 [F^*]\right]\{x_i^*\} = \left[[\text{diag } (\lambda_i)] + \epsilon [E^*]\right]\{x_i^*\} \tag{3-45}$$

The matrices on both sides now have uniform magnitudes. Eq. (3-45) is called the generalized eigenequation which is generally more difficult to solve. We prefer to change the problem into a standard eigenequation, so that all the methods used in the Chapter 2 may also be utilized here. To perform this we premultiply eq. (3-45) by $\left[[I] + \epsilon^2 [F^*]\right]^{-1}$ on both sides, thus

$$\lambda_i^*(x_i^*) = \left[[I] + \epsilon^2[F^*]\right]^{-1} \left[[\text{diag } (\lambda_i)] + \epsilon[E^*]\right](x_i^*) \qquad (3\text{-}46)$$

Conventional numerical algorithms for the computation of $\left|[I] + \epsilon^2[F^*]\right]^{-1}$ are extremely difficult in this case, but an alternative approach can be used by expanding this as a power series. In the theory of matrix algebra we have

$$[I] + [A] + [A]^2 + \ldots \rightarrow ([I] - [A])^{-1} \qquad (3\text{-}47)$$

subject to a sufficiency condition that any norm of [A] is less than unity. The matrix $[F] = \epsilon^2[F^*]$ satisfies this condition because $\epsilon \ll 1.0$ and $\|[F^*]\| \approx 1$. Thus $\left[[I] + \epsilon^2[F^*]\right]^{-1}$ can be expressed in a convergent series

$$\left\{[I] + \epsilon^2[F^*]\right\}^{-1} = [I] - \epsilon^2[F^*] + \epsilon^4[F^*]^2 - \ldots \qquad (3\text{-}48)$$

Substituting eq. (3-48) into eq. (3-46), we have

$$\lambda_i^*(x_i^*) = \left[[\text{diag } (\lambda_i)] + \epsilon[A_1] + \epsilon^2[A_2] + \epsilon^3[A_3] + \epsilon^4[A_4] + \ldots\right](x_i^*)$$

$$i=1,2,..,2(n+m) \qquad (3\text{-}49)$$

where

$$[A_1] = [E^*]$$
$$[A_2] = -[F^*][\text{diag } (\lambda_i)]$$
$$[A_3] = -[F^*]\,[E^*]$$
$$[A_4] = [F]^2[\text{diag } (\lambda_i)]$$

$[A_j]$ are complex matrices. In the next section we use perturbation techniques to solve eq. (3-49).

## 3.3 Perturbation Approach For Detuned Modes

In the Section 2.2 we have discussed about the perturbation theory in details. The fundamental principles apply here too. We rewrite eq. (3-48) as

$$\lambda_i^* \{x_i^*\} = [P^*] \{x_i^*\} , \quad i=1,\ldots,2(n+m) \tag{3-50}$$

where

$$[P^*] = [\text{diag } (\lambda_i)] + \epsilon [A_1] + \epsilon^2 [A_2] + \ldots \tag{3-51}$$

The matrix $[P^*]$, diagonally dominated by $[\text{diag } (\lambda_i)]$, is similar to $[P]$ in eq. (2-13). However, we should note the main differences that $[P^*]$ is a non-Hermitian complex matrix, and the eigensolutions are generally complex.

Letting $\epsilon = 0$ in eq. (3-51) reduces it to $[P^*] = [\text{diag } (\lambda_i)]$ and the eigensolutions of eq. (3-50) are then obviously

$$\lambda_i^* = \lambda_i , \qquad i=1,2,\ldots,2(n+m)$$

and

$$\{x_i^*\} = \{e_i\} , \quad i=1,2,\ldots,2(n+m) \tag{3-52}$$

When $\epsilon \neq 0$, we expect the eigensolutions for the $i^{th}$ detuned mode to be as follows (see Section 2.2)

$$\lambda_i^* = \lambda_i + \epsilon k_i^{(1)} + \epsilon^2 k_i^{(2)} + \epsilon^3 k_i^{(3)} + \ldots \tag{3-53}$$

and

$$\{x_i^*\} = \{e_i\} + \epsilon \sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(1)} \{e_i\} + \epsilon^2 \sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(2)} \{e_j\} + \ldots \tag{3-54}$$

where $k_i^{(1)}$ and $\alpha_{ji}^{(1)}$ are all complex constants.

By substituting eq. (3-53) and (3-54) into eq. (3-50) and collecting the terms in $\epsilon$, $\epsilon^2$, $\epsilon^3$, etc., we obtain the perturbation equations for any order of solution. For the first order eigensolutions, we have

$$[\text{diag } (\lambda_i)] \left( \sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(1)} \{e_j\} \right) + [A_1]\{e_i\} = \lambda_i \left( \sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(1)} \{e_j\} \right) + k_i^{(1)} \{e_i\} \tag{3-55}$$

premultiplying by $\{e_i\}^T$ yields

$$k_i^{(1)} = \{e_i\}^T [A_1] \{e_i\} = a_{1_{ii}} \tag{3-56}$$

Premultiplying by $\{e_j\}^T$, $j \neq i$ yields

$$\alpha_{ji}^{(1)} = \frac{\{e_j\}^T [A_1] \{e_i\}}{\lambda_1 - \lambda_j} = \frac{a_{1_{ji}}}{\lambda_i - \lambda_j} , \quad \begin{matrix} j=1,\ldots,2(n+m) \\ j \neq i \end{matrix} \tag{3-57}$$

The first order eigensolutions for the $i^{th}$ mode are

$$\begin{aligned} \lambda_i^* &= \lambda_i + \epsilon k_i^{(1)} \\ \{x_i^*\} &= \{e_i\} + \epsilon \sum_{\substack{j=1 \\ j \neq i}}^{2(n+m)} \alpha_{ji}^{(1)} \{e_j\} \end{aligned} \tag{3-58}$$

For the second order equations, we have

$$[\text{diag } (\lambda_i)] \left[ \sum_{\substack{j=1 \\ j \neq i}}^{2(n+m)} \alpha_{ji}^{(2)} \{e_j\} \right] + [A_1] \left[ \sum_{\substack{j=1 \\ j \neq i}}^{2(n+m)} \alpha_{ji}^{(1)} \{e_j\} \right] + [A_2] \{e_i\}$$

$$= \lambda_i \left[ \sum_{\substack{j=1 \\ j \neq i}}^{2(n+m)} \alpha_{ji}^{(2)} \{e_j\} \right] + k_i^{(1)} \left[ \sum_{\substack{j=1 \\ j \neq i}}^{2(n+m)} \alpha_{ji}^{(1)} \{e_j\} \right] + k_i^{(2)} \{e_i\} \tag{3-59}$$

Premultiplying by $\{e_i\}^T$ yields

$$k_i^{(2)} = \{e_i\}^T [A_1] \left[ \sum_{\substack{j=1 \\ j \neq i}}^{2(n+m)} \alpha_{ji}^{(1)} \{e_j\} \right] + \{e_i\}^T [A_2] \{e_i\}$$

$$= \sum_{\substack{\ell=1 \\ \ell \neq i}}^{2(n+m)} a_{1_{i\ell}} \alpha_{\ell i}^{(1)} + a_{2_{ii}} \tag{3-60}$$

Premultiplying by $\{e_j\}^T$ $j \neq i$ yields

$$\alpha_{ji}^{(2)} = \frac{\sum_{\ell=1,\ell\neq i}^{2(n+m)} a_{1_{j\ell}} \alpha_{\ell i}^{(1)} + a_{2_{ji}} - k_i^{(1)} \alpha_{ji}^{(1)}}{\lambda_i - \lambda_j}, \quad \begin{matrix} j=1,\ldots,2(n+m) \\ j\neq i \end{matrix} \tag{3-61}$$

For the third order eigensolution, we have

$$[\text{diag } (\lambda_i)]\left[\sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(3)}\{e_j\}\right] + [A_1]\left[\sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(2)}\{e_j\}\right]$$

$$+ [A_2]\left[\sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(1)}\{e_j\}\right] + [A_3]\{e_i\}$$

$$- \lambda_i\left[\sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(3)}\{e_j\}\right] + k_i^{(1)}\left[\sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(2)}\{e_j\}\right]$$

$$+ k_i^{(2)}\left[\sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(1)}\{e_j\}\right] + k_i^{(3)}\{e_i\} \tag{3-62}$$

By the same procedures, we obtain

$$k_i^{(3)} = \sum_{\substack{l=1 \\ l\neq i}}^{2(n+m)} a_{1_{il}} \alpha_{li}^{(2)} + \sum_{\substack{l=1 \\ l\neq i}}^{2(n+m)} a_{2_{il}} \alpha_{li}^{(1)} + a_{3_{ii}} \tag{3-63}$$

$$\alpha_{ji}^{(3)} = \frac{\sum_{\ell=1,\ell\neq i}^{2(n+m)} a_{1_{j\ell}} \alpha_{\ell i}^{(2)} + \sum_{\ell=1,\ell\neq i}^{2(n+m)} a_{2_{j\ell}} \alpha_{\ell i}^{(1)} - k_i^{(1)} \alpha_{ji}^{(2)} - k_i^{(2)} \alpha_{ji}^{(1)}}{\lambda_i - \lambda_j}$$

$$\begin{matrix} j=1,\ldots,2(n+m) \\ j\neq i \end{matrix} \tag{3-64}$$

The third order eigensolutions for the $i^{th}$ mode are

$$\lambda_i^* = \lambda_i + \epsilon\, k_i^{(1)} + \epsilon^2 k_i^{(2)} + \epsilon^3 k_i^{(3)} \tag{3-65}$$

and

$$\{x_i^*\} = \{e_i\} + \epsilon \sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(1)}\{e_j\} + \epsilon^2 \sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(2)}\{e_j\} + \epsilon^3 \sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \alpha_{ji}^{(3)}\{e_j\} \tag{3-66}$$

The higher order solutions can be obtained by a similar manner, but it is barely necessary in practice.

## 3.4 Identification of Tuned Modes

All the methods discussed in the Section 2.3 can be applied here. The so called "Gerschgorin's discs" in Section 2.3 are actually an intervals on the real axis. But in this section, since the matrix $[P^*]$ is complex, Gerschgorin's discs are actual discs and all the eigenvalues lie within the union of all discs.

$$\begin{aligned} D_i\text{- center:} \quad & \lambda_i + \epsilon a_{1_{ii}} + \epsilon^2 a_{1_{ii}} + \ldots \\ \text{radius:} \quad & \sum_{\substack{j=1 \\ j\neq i}}^{2(n+m)} \left| \epsilon a_{1_{ij}} + \epsilon^2 a_{2_{ij}} + \epsilon^3 a_{3_{ij}} + \ldots \right| \\ & i=1,\ldots,2(n+m) \end{aligned} \tag{3-67}$$

Figure 3-1 shows the distribution of Gerschgorin's discs. Any disc, being isolated from others, corresponds to a detuned mode. When discs overlap, we can try to separate them. Only those that cannot be separated correspond to tuned modes.

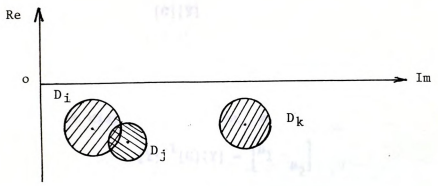



Figure 3-1 : Illustration of Gerschgorin's Discs

3.5 Tuned Modes

Consider the matrix $[P^*] = [\text{diag}\ (\lambda_i)] + \epsilon[A_1] + \epsilon^2[A_2] + \ldots$ and assume that the first two modes are tuned, i.e., $\lambda_1$ and $\lambda_2$ are very close and separated by a quantity of order $\epsilon$ or less. Let $[R] = [A_1] + \epsilon[A_2] + \ldots$, then we have $[P^*] = [\text{diag}\ (\lambda_i)] + \epsilon[R]$. We partition $[P^*]$ in the form

$$\left[\begin{array}{c|c} \begin{matrix}\lambda_1 & \\ & \lambda_2\end{matrix} & \\ \hline & [\Lambda_{n-2}] \end{array}\right] + \epsilon \left[\begin{array}{c|c} [R_{11}] & [R_{12}] \\ \hline [R_{21}] & [R_{22}] \end{array}\right] \tag{3-68}$$

If we write

$$[G] = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} + (\lambda_2 - \lambda_1)/\epsilon \end{bmatrix} \tag{3-69}$$

Then eq. (3-68) may be expressed as:

$$\left[\begin{array}{c|c} \begin{matrix}\lambda_1 & \\ & \lambda_1\end{matrix} & \\ \hline & [\Lambda_{n-2}] \end{array}\right] + \epsilon \left[\begin{array}{c|c} [G] & [B_{12}] \\ \hline [B_{21}] & [B_{22}] \end{array}\right] \tag{3-70}$$

The order of elements in the second matrix of eq. (3-70) is uniform. The eigensolution of [G] is

$$[G][Y] = [Y]\begin{bmatrix}\mu_1 & \\ & \mu_2\end{bmatrix} \tag{3-71}$$

where [Y] is the eigenvector matrix and $\mu_1$, $\mu_2$ are the eigenvalues. Since [G] is not a Hermitian matrix, [Y] is not a unitary matrix (i.e. $[Y]^T[G][Y]$ is not diagonal). However, we have

$$[Y]^{-1}[G][Y] = \begin{bmatrix}\mu_1 & \\ & \mu_2\end{bmatrix} \tag{3-72}$$

Constructing a transformation matrix

$$[Q] = \begin{bmatrix}[Y] & \\ & [I]\end{bmatrix} \tag{3-73}$$

We obtain

$$[Q]^{-1}\Big[[\text{diag }(\lambda_i)] + \epsilon[R]\Big][Q]$$

$$= \left[\begin{array}{c|c|c} \lambda_1+\epsilon\mu_1 & \lambda_1+\epsilon\mu_2 & \\ \hline & & \\ & & [\Lambda_{n-2}] \end{array}\right] + \epsilon\left[\begin{array}{c|c|c} [0] & [Y]^{-1}[R_{12}] \\ \hline [R_{21}] & [Y] & [R_{22}] \end{array}\right]$$

$$= [\text{diag }(\lambda'_i)] + \epsilon[R'] \tag{3-74}$$

The Gerschgorin discs of $[\text{diag }(\lambda'_i)] + \epsilon[R']$ are now disjoint and the case becomes detuned. The eigensolutions of $[P^*]$ can be easily obtained from those of $[\text{diag }(\lambda'_i)] + \epsilon\,[R']$, $\lambda'^*_i$ and $\{x'^*_i\}$, since

$$\begin{aligned} \lambda^*_i &= \lambda'^*_i \\ \{x^*_i\} &= [Q]\{x'^*_i\}\ , \quad i=1,2,\ldots,2(n+m) \end{aligned} \tag{3-75}$$

3.6 Error Bounds For Approximate Solutions

As was discussed in the Section 2.5, we can obtain higher order eigenvalues through Rayleigh's quotient

$$\lambda^*_i = \frac{\{x^*_i\}^T[P^*]\{x^*_i\}}{\{x^*_i\}^T\{x^*_i\}} \tag{3-76}$$

The development of error bounds for eigenvalues and eigenvectors are somewhat complicated for the nonclassically damped case. Suppose an approximate eigensolution has been determined from eq. (3-53) and (3-54). We denote all the eigenvalues by [diag $(\lambda_i^*)$] and the eigenvectors by $[X^*]$. If the residual matrix [F] is defined as

$$[P^*][X^*] = [X^*][\text{diag } (\lambda_i^*)] + [F] \tag{3-77}$$

then

$$[X^*]^{-1}[P^*][X^*] = [\text{diag } (\lambda_i^*)] + [E] \tag{3-78}$$

where

$$[E] = [X^*]^{-1}[F] \tag{3-79}$$

The residual matrix [F] can be easily obtained, and we wish to compute [E] as accurately as is convenient. Since each column of $[X^*]$ is from eq. (3-54), we conclude that $[X^*]$ is a diagonal dominant matrix which can be expressed as

$$[X^*] = [I] + \epsilon[Z] \tag{3-80}$$

in which [I] is the identity matrix and $\epsilon[Z]$ contains all the off-diagonal elements of $[X^*]$. $[X^*]^{-1} = ([I] + \epsilon\,[Z])^{-1}$ can be expressed as a convergent series

$$[X^*]^{-1} = [I] - \epsilon\,[Z] + \epsilon^2[Z]^2 - \ldots$$

and therefore

$$[E] = [F] - \epsilon\,[F][Z] + \epsilon^2[F][Z]^2 - \ldots \tag{3-81}$$

Once a sufficiently accurate estimate of [E] is obtained, the analysis of [diag $(\lambda_i^*)$] + [E] will be carried out as discussed in Section 2.5.

# 4. NUMERICAL EXAMPLES

## 4.1 Numerical Example For Classically Damped System

A simple four DOF model representing a shear building is chosen as the P-system. A single DOF S-system is attached to the fourth floor as shown in Figure 4-1. The dynamic properties of the P-system are tabulated in Table 4-1.

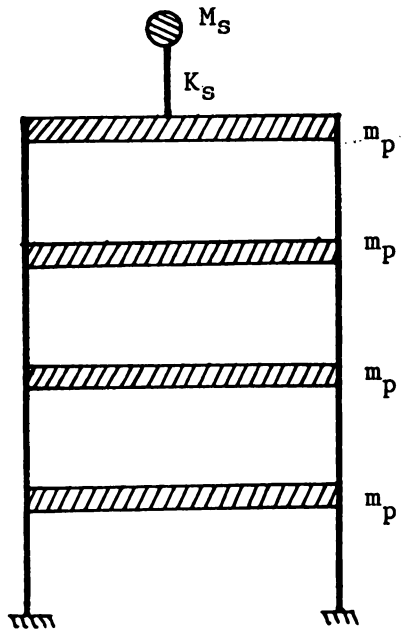



Figure 4-1 : Classically Damped Composite System

Three cases are analyzed. The mass ratios of the S-system to the P-system are the same in all cases, but the frequencies of the S-systems

are different. Table 4-2 lists the properties of the S-system for these cases.

Table 4-1 : Dynamic Properties of the Primary System

| Mode i | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Eigenvalue | 0.12061 | 1.0000 | 2.34730 | 3.53209 |
| Eigenvector | 0.65654 | 0.57735 | -0.42853 | -0.22801 |
| | 0.57735 | 0.00000 | 0.57735 | 0.57735 |
| | 0.42853 | -0.57735 | 0.22801 | -0.65654 |
| | 0.22801 | -0.57735 | -0.65654 | 0.42853 |

Table 4-2 : Dynamic Properties of Secondary Systems

| Case | 1 | 2 | 3 |
|---|---|---|---|
| Mass Ratio ($M_s/M_p$) | 0.05 | 0.05 | 0.05 |
| $\lambda_s = k_s/M_s$ | 0.45 | 0.20 | 1.00 |

It should be noted that this example is idealized and may not necessarily resemble C-systems encountered in practice. Nevertheless, it possesses the essential dynamic properties of such systems and is simple enough to provide a clear demonstration of the method.

The results for Case 1, which is a detuned system, are presented in Tables 4-3 and 4-4. Case 2 is a closely detuned system, where the frequency of the S-system is close to the first frequency of the P-system. Results for this are tabulated in Tables 4-5 and 4-6. Case 3 represents a tuned system, with the S-system being tuned to the second

mode of the P-system. Results for Case 3 are presented in Tables 4-7 and 4-8. Both first order and higher order perturbation results are presented in each example to show the accuracy of the high order perturbation results. The estimate error bounds are compared with exact error bounds. The estimate error bounds are very close to the real error bounds, especially for high order perturbations.

Table 4-3 : Eigenvalues of the Composite System - Case 1

($M_s/M_p = 0.05$, $\lambda_s = k_s/M_s = 0.45$)

| | $1^{st}$ Order Perturbation | | | | |
|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 |
| Eigenvalue | .11736 | 1.01363 | 2.35246 | 3.53344 | 0.45574 |
| Est. Error Bound (%) | 0.28 | 0.27 | 0.00 | 0.00 | 0.00 |
| Exact Error Bound (%) | 0.16 | 0.00 | 0.00 | 0.00 | 0.00 |

| | $3^{rd}$ Order Perturbation | | | | |
|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 |
| Eigenvalue | 0.11719 | 1.01367 | 2.35246 | 3.53344 | 0.45574 |
| Est. Error Bound (%) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Exact Error Bound (%) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Table 4-4 : Eigenvectors of the Composite System - Case 1

| | 1st Order Perturbation | | | | | |
|---|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 | Error Bound * |
| Eigenvector | .48348 | .52593 | -.43013 | -.23000 | -.01323 | 0.1 |
| | .43143 | .00783 | .57168 | .57753 | -.02975 | 0.2 |
| | .32316 | -.51810 | .22921 | -.65550 | -.03288 | 0.4 |
| | .17283 | -.52201 | -.65223 | .42752 | -.02121 | 0.3 |
| | .66769 | -.42710 | .10112 | .03327 | .99870 | 0.8 |
| Exact Error Bounds (%) | 0.95 | 0.64 | 0.52 | 0.39 | 0.48 | |
| Est. Error Bounds (%) | 9.7 | | | | | |

| | 3rd Order Perturbation | | | | | |
|---|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 | Error Bound * |
| Eigenvector | .49231 | .53003 | -.43013 | -.23033 | -.01272 | .0004 |
| | .43041 | .01425 | .57096 | .57760 | -.02969 | .0006 |
| | .31824 | -.51594 | .22979 | -.65538 | -.03312 | .0004 |
| | .16896 | -.52310 | -.65194 | .42739 | -.02145 | .0004 |
| | .66525 | -.42312 | .10195 | .03361 | .99870 | .0003 |
| Exact Error Bounds (%) | 0.03 | 0.01 | 0.02 | 0.02 | 0.00 | |
| Est. Error Bounds (%) | 0.10 | | | | | |

* Note: Estimated absolute error bounds are given for the first eigenvector only.

Table 4-5 : Eigenvalues of the Composite System - Case 2

($M_s/M_p = 0.05$, $\lambda_s = k_s/M_s = 0.20$)

| | $1^{st}$ Order Perturbation | | | | |
|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 |
| $\lambda_i$ | 0.11493 | 1.00418 | 2.34931 | 3.53264 | 0.20906 |
| Est. Error Bound (%) | 1.7 | 0.5 | 0.04 | 0.00 | 0.03 |
| Exact Error Bound (%) | 0.13 | 0.00 | 0.00 | 0.00 | 0.00 |

| | $3^{rd}$ Order Perturbation | | | | |
|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 |
| $\lambda_i$ | 0.11478 | 1.00418 | 2.34931 | 3.53264 | 0.20909 |
| Est. Error Bound (%) | 0.03 | 0.00 | 0.00 | 0.00 | 0.00 |
| Exact Error Bound (%) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Table 4-6 : Eigenvectors of the Composite System - Case 2

| | 1st Order Perturbation | | | | | |
|---|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 | Error Bound* |
| Eigenvectors | 0.33805 | 0.57333 | -0.42988 | -0.22893 | -0.04895 | 0.01 |
| | 0.29921 | 0.00381 | 0.57568 | 0.57752 | -0.04912 | .004 |
| | 0.22299 | -0.56952 | 0.22889 | -0.65618 | -0.03948 | .01 |
| | 0.11892 | -0.57142 | -0.65560 | 0.42814 | -0.02193 | .01 |
| | 0.85576 | -0.14286 | 0.03988 | 0.01368 | 0.99657 | .03 |
| Exact Error Bound (%) | 2.0 | 1.3 | 0.77 | 0.34 | 1.5 | |
| Est. Error Bounds (%) | 3.5 | | | | | |

| | 3rd Order Perturbation | | | | | |
|---|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 | Error Bound* |
| Eigenvectors | 0.36143 | 0.57380 | -0.43004 | -0.22899 | -0.04474 | 0.001 |
| | 0.31446 | 0.00477 | 0.57556 | 0.57753 | -0.04580 | 0.0005 |
| | 0.23181 | -0.56905 | 0.22899 | -0.65616 | -0.03721 | 0.001 |
| | 0.12286 | -0.57144 | -0.65555 | 0.42812 | -0.02079 | 0.001 |
| | 0.83765 | -0.14271 | 0.04002 | 0.01374 | 0.99704 | 0.003 |
| Exact Error Bounds (%) | 0.34 | 0.23 | 0.14 | 0.01 | 0.27 | |
| Est. Error Bounds (%) | 0.35 | | | | | |

* Note: The error bounds are for the first eigenvector.

Table 4-7 : Eigenvalues of the Composite System - Case 3

($M_s/M_p = 0.05$, $\lambda_s = k_s/M_s = 1.00$)

| | $1^{st}$ Order Perturbation | | | | |
|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 |
| $\lambda_i$ | 0.11786 | 0.88771 | 2.36373 | 3.53581 | 1.14525 |
| Est. Error Bound (%) | 0.26 | 0.34 | 0.01 | 0.01 | 0.27 |
| Exact Error Bound (%) | 0.11 | 0.05 | 0.00 | 0.00 | 0.00 |

| | $3^{rd}$ Order Perturbation | | | | |
|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 |
| $\lambda_i$ | 0.11773 | 0.88729 | 2.36377 | 3.53582 | 1.14539 |
| Est. Error Bound (%) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Exact Error Bound (%) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Table 4-8 : Eigenvectors of the Composite System - Case 3

| | 1st Order Perturbation | | | | | |
|---|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 | Error Bound * |
| Eigenvectors | 0.52439 | 0.10015 | -0.41950 | -0.23237 | -0.14767 | 0.005 |
| | 0.46613 | -0.03086 | 0.54323 | 0.57608 | 0.03495 | 0.007 |
| | 0.34169 | -0.13475 | 0.22579 | -0.65173 | -0.11753 | 0.004 |
| | 0.17860 | -0.12020 | -0.61836 | 0.42567 | -0.13630 | 0.007 |
| | 0.59923 | 0.97796 | 0.30910 | 0.09051 | -0.97190 | 0.005 |
| Exact Error Bounds (%) | 1.2 | 0.61 | 1.1 | 1.1 | 0.47 | |
| Est. Error Bounds (%) | 1.3 | | | | | |

| | 3rd Order Perturbation | | | | | |
|---|---|---|---|---|---|---|
| Mode | 1 | 2 | 3 | 4 | 5 | Error Bound * |
| Eigenvectors | 0.52783 | 0.11015 | -0.42177 | -0.23375 | 0.14150 | 0.0002 |
| | 0.46193 | -0.03089 | 0.53863 | 0.57646 | 0.03519 | 0.0002 |
| | 0.34201 | -0.14451 | 0.22580 | -0.65159 | -0.11141 | 0.0001 |
| | 0.18180 | -0.12988 | -0.62076 | 0.42426 | -0.13038 | 0.0002 |
| | 0.59832 | 0.97425 | 0.30929 | 0.09217 | -0.97433 | 0.0002 |
| Exact Error Bounds (%) | 0.04 | 0.02 | 0.04 | 0.04 | 0.02 | |
| Est. Error Bounds (%) | 0.04 | | | | | |

* Note: The error bounds are for the first eigenvector.

## 4.2 Numerical Examples for Nonclassically Damped System

To illustrate application of results presented in Chapter 3, a 7-DOF system shown in Figure 4-2 is analyzed. The P-system is a 3-DOF shear building and the S-system is modeled as a 4-DOF beam (with rotational DOF condensed out) attached to the P-system through two support members.

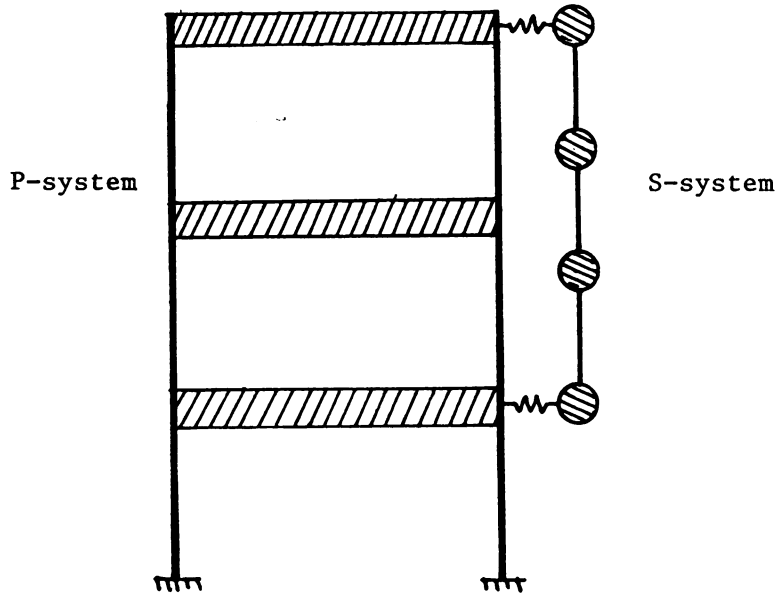



Figure 4-2 : Nonclassically Damped Composite System

Two cases are selected to examine the effects of interaction:

$$\text{Case 1: } M_s/M_p = 0.01$$

$$\text{Case 2: } M_s/M_p = 0.05$$

The other physical and dynamic properties are tabulated in Tables 4-9 and 4-10.

In both cases the $1^{st}$ frequency of the P-system and the $1^{st}$ frequency of the S-system are tuned. Tables 4-11 and 4-12 show the estimated complex eigenvalues of the C-system obtained with the present approach (to the third perturbation term). The results are compared with exact results.

The lower half of the eigenvectors are given in Tables 4-13 and 4-14. Only three eigenvectors are listed. The $1^{st}$ and $4^{th}$ are tuned modes; the $3^{rd}$ is a detuned mode. These are all results of the $3^{rd}$ order perturbation. The results are compared with exact solutions and error bounds are derived by $\|(\Delta\phi)\|/\|(\phi)\|$. The results obtained from the perturbation method agree very well with the exact solutions.

Table 4-9 : Physical Properties of Primary and Secondary Systems

| Systems | Properties |
|---|---|
| Primary | $k_p/M_p = 400$ $rad^2/sec^2$ |
| Secondary | $k_s/M_s = 380$ $rad^2/sec^2$ |
| | $EI/(L^3M_s) = 90.0$ $rad^2/sec^2$ |

Table 4-10: Physical Properties of Primary and Secondary Systems

| Systems | Modes | Frequency $\left(\frac{rad}{sec}\right)$ | Damping Ratio |
|---|---|---|---|
| Primary | 1 | 8.90 | 0.05 |
| | 2 | 24.93 | 0.05 |
| | 3 | 36.04 | 0.05 |
| Secondary | 1 | 8.91 | 0.02 |
| | 2 | 18.26 | 0.02 |
| | 3 | 22.73 | 0.02 |
| | 4 | 42.97 | 0.02 |

It is also of interest to examine the relative importance of the nonclassical damping phenomenon. In order to examine this, two other cases with different damping properties are presented in Table 4-15 and

Table 4-16. The components of any eigenvector differs in magnitude as well as in phase. Note that the larger the difference in phase, the more nonclassically damped the system is. The results of case 3 (Table 4-15) show that tuned modes will be basically classically damped if the damping ratios of the subsystems are the same. (Large phases corresponding to very small modulus values are not physically meaningful. Estimates of the phase are poor when the corresponding modulus is very small, and for this situation the small component could be ignored and the phase is not important anyway.)

It should be noted that the small changes in either $\beta_s$ or $\beta_p$ do not significantly affect the modulus of the eigenvectors.

Overall, the nonclassical damping phenomenon will be influenced by tuning, difference in damping ratios and the magnitude of the damping ratios.

Table 4-11: Complex Eigenvalues - Case 1: $M_s/M_p = 0.01$

| | 1st Order | | | 3rd Order | | |
|---|---|---|---|---|---|---|
| | Eigenvalues | | Exact Error | Eigenvalues | | Exact Error |
| Mode | Real | Imaginary | Bound (%) | Real | Imaginary | Bound (%) |
| 1 | -0.3189 | 9.276 | 0.11 | -0.3174 | 9.286 | 0.00 |
| 2 | -1.246 | 25.05 | 0.01 | -1.246 | 25.05 | 0.00 |
| 3 | -1.804 | 36.03 | 0.00 | -1.804 | 36.03 | 0.00 |
| 4 | -0.3039 | 8.520 | 0.10 | -0.3040 | 8.512 | 0.00 |
| 5 | -0.3718 | 18.17 | 0.01 | -0.3698 | 18.16 | 0.00 |
| 6 | -0.4623 | 22.74 | 0.02 | -0.4583 | 22.74 | 0.00 |
| 7 | -0.8600 | 42.97 | 0.00 | -0.8599 | 42.97 | 0.00 |

Table 4-12: Complex Eigenvalues - Case 2: $M_s/M_p = 0.05$

| Mode | 1st Order | | | 3rd Order | | |
|---|---|---|---|---|---|---|
| | Eigenvalues | | Exact Error Bound (%) | Eigenvalues | | Exact Error Bound (%) |
| | Real | Imaginary | | Real | Imaginary | |
| 1 | -0.3353 | 9.688 | 1.0 | -0.3263 | 9.787 | 0.01 |
| 2 | -1.244 | 25.56 | 0.03 | -1.248 | 25.57 | 0.00 |
| 3 | -1.814 | 36.17 | 0.01 | -1.811 | 36.17 | 0.00 |
| 4 | -0.2855 | 8.067 | 0.90 | -0.2886 | 8.000 | 0.07 |
| 5 | -0.3981 | 17.85 | 0.15 | -0.3819 | 17.83 | 0.01 |
| 6 | -0.4918 | 22.84 | 0.10 | -0.4723 | 22.83 | 0.01 |
| 7 | -0.8623 | 42.97 | 0.00 | -0.8616 | 42.97 | 0.00 |

Table 4-13: Complex Eigenvectors - Case 1

$M_s/M_p = 0.01$, $\beta_s = 0.02$, $\beta_p = 0.05$

| Mode | 1st | | 4th | | 3rd | |
|---|---|---|---|---|---|---|
| DOF | Modulus | Phase | Modulus | Phase | Modulus | Phase |
| 1 | 0.11 | 0.00° | 0.11 | 0.00° | 0.45 | 0.0° |
| 2 | 0.08 | -0.00° | 0.09 | 0.00° | 1.0 | 0.0° |
| 3 | 0.05 | 0.00° | 0.05 | 0.00° | 0.81 | 0.0° |
| 4 | 0.17 | -33.0° | 0.36 | 13.2° | 0.25 | -2.2° |
| 5 | 0.97 | -20.1° | 1.00 | 18.3° | 0.10 | 2.8° |
| 6 | 1.00 | -19.5° | 0.98 | 18.9° | 0.04 | 13.9° |
| 7 | 0.24 | -22.2° | 0.29 | 16.6° | 0.34 | 3.4° |
| % Error | 0.1 | | 0.1 | | 0.0 | |

Table 4-14: Complex Eigenvectors - Case 2

$M_s/M_p = 0.05$, $\beta_s = 0.02$, $\beta_p = 0.05$

| Mode | $1^{st}$ | | $4^{th}$ | | $3^{rd}$ | |
|---|---|---|---|---|---|---|
| DOF | Modulus | Phase | Modulus | Phase | Modulus | Phase |
| 1 | 0.24 | 0.0° | 0.22 | 0.0° | 0.45 | 0.0° |
| 2 | 0.19 | 0.0° | 0.18 | 0.0° | 1.0 | 0.0° |
| 3 | 0.10 | 0.0° | 0.10 | 0.0° | 0.83 | 0.0° |
| 4 | 0.05 | 2.2° | 0.47 | 1.7° | 0.26 | -1.7° |
| 5 | 0.93 | -1.8° | 1.00 | 1.8° | 0.10 | 1.7° |
| 6 | 1.00 | -1.8° | 0.95 | 1.8° | 0.05 | 1.9° |
| 7 | 0.20 | -1.9° | 0.32 | 1.7° | 0.34 | -1.7° |
| % Error | 1.0 | | 1.0 | | 0.01 | |

Table 4-15: Complex Eigenvectors - Case 3

$M_s/M_p = 0.05$, $\beta_s = 0.02$, $\beta_p = 0.05$

| Mode | $1^{st}$ | | $4^{th}$ | | $3^{rd}$ | |
|---|---|---|---|---|---|---|
| DOF | Modulus | Phase | Modulus | Phase | Modulus | Phase |
| 1 | 0.24 | 0.0° | 0.22 | 0.0° | 0.45 | 0.0° |
| 2 | 0.19 | 0.0° | 0.18 | 0.1° | 1.0 | -0.2° |
| 3 | 0.10 | 0.0° | 0.10 | 0.0° | 0.83 | 0.1° |
| 4 | 0.01 | 0.0° | 0.47 | -0.5° | 0.26 | 6.5° |
| 5 | 0.93 | 0.2° | 1.00 | -0.4° | 0.10 | 6.9° |
| 6 | 1.00 | 0.3° | 0.95 | -0.3° | 0.05 | 6.9° |
| 7 | 0.20 | -0.4° | 0.32 | -0.3° | 0.34 | 6.3° |
| % Error | 1.0 | | 1.0 | | 0.0 | |

Table 4-16: Complex Eigenvectors - Case 4

$M_s/M_p = 0.05$, $\beta_s = 0.05$, $\beta_p = 0.08$

| Mode | 1st | | 4th | | 3rd | |
|---|---|---|---|---|---|---|
| DOF | Modulus | Phase | Modulus | Phase | Modulus | Phase |
| 1 | 0.24 | 0.0° | 0.22 | 0.0° | 0.45 | 0.0° |
| 2 | 0.19 | -0.1° | 0.18 | -0.2° | 1.0 | -0.1° |
| 3 | 0.10 | 0.3° | 0.10 | 0.2° | 0.83 | 0.0° |
| 4 | 0.05 | 9.1° | 0.47 | 3.9° | 0.26 | 1.7° |
| 5 | 0.93 | -9.2° | 1.0 | 7.4° | 0.10 | 7.4° |
| 6 | 1.0 | -8.6° | 0.95 | 8.0° | 0.05 | 1.8° |
| 7 | 0.20 | 1.2° | 0.32 | 6.3° | 0.34 | 0.2° |
| % Error | 1.0 | | 0.8 | | 0.0 | |

# 5. CONCLUSIONS

Classically and nonclassically damped primary-secondary systems are studied in detail. The eigenproperties of the C-system are estimated from those of the individual subsystems through a perturbation approach. All cases, including detuned systems, closely detuned systems and tuned system, are studied. By utilizing Gerschgorin's discs, any general system can be easily classified into one of these three cases.

The eigenvectors of the C-system are more sensitive to the interaction between the P-system and the S-system than the eigenvalues. Once a perturbed eigenvector is found, corresponding eigenvalues can be obtained quite accurately through Rayleigh's quotient.

In this study, higher order perturbations are derived, and can be obtained recursively. The computations are not difficult. When closely detuned modes, tuned modes and high frequency interactions exist, higher order perturbations are necessary to obtain satisfactory results.

For tuned modes, a special transformation is introduced to transform the problem into a detuned one. This method greatly facilitates the numerical algorithm.

It is well known that the solution of eigenproblems for nonclassically damped systems involves much greater numerical effort than the solution for classically damped systems. However, by using the perturbation methods presented in this report the numerical effort can be reduced substantially.

Sharp error bounds are derived for the approximate eigenvalues and eigenvectors.

The methods and equations are verified through numerical examples, and sufficiently accurate approximate solutions have been obtained.

In summary, the complex dynamic characteristics of the C-system can be accurately determined by the presented method. The major steps are as follows: (1) mode synthesis; (2) identification of detuned and tuned modes; (3) treatment of tuning modes; (4) perturbation method; and (5) error analysis. The method can be applied to very general C-systems.

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## LIST OF REFERENCES


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