RADICALS IN STANDARD RINGS

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ABSTRACT

RADICALS IN STANDARD RINGS

By

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A standard ring is a non-associative ring, $\mathfrak{A}$, which satisfies:

$(x, y, z) + (z, x, y) - (x, z, y) = 0;$

$(x, y, wz) + (w, y, xz) + (z, y, wx) = 0;$ and

$(x, y, x^2) = 0$

for all w, x, y, z in $\mathfrak{A}$, where $(a, b, c) = (ab)c - a(bc)$.

We consider standard rings $\mathfrak{A}$ for which if $x \in \mathfrak{B}$, a subring of $\mathfrak{A}$, then there exists a unique $y \in \mathfrak{B}$ for which $2y = x$.

In this paper we consider various radicals for standard rings.

A radical property is a property $\mathcal{P}$ which a ring may possess which satisfies the following:

1. A homomorphic image of a ring which has property $\mathcal{P}$ also has property $\mathcal{P}$.
2. Every ring contains a $\mathcal{P}$-ideal which contains all other $\mathcal{P}$-ideals. This ideal is called the $\mathcal{P}$-radical.
3. Any ring modulo its $\mathcal{P}$-radical has zero $\mathcal{P}$-radical.

We begin by deriving the basic identities which we will use throughout the paper. We then prove theorems which are applicable for any choice of a radical property. Finally, we list known results for the nil, Behrens, and Smiley radicals for general non-associative rings.

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In chapter 2, we define a prime ideal and show the existence of a prime radical in a standard ring. We characterize the prime radical of a standard ring $\mathfrak{A}$ as the minimal ideal $\mathfrak{Q}$ such that $\mathfrak{A}/\mathfrak{Q}$ has no non-zero nilpotent ideals. We show that if the standard ring has a maximum nilpotent ideal, then it is equal to the prime radical.

In chapter 3, we prove the equivalence of local solvability and local nilpotence in a standard ring. This leads to the existence of a maximal locally nilpotent ideal, called the Levitski radical. We show that the Levitski radical contains the prime radical and is contained in the nil radical. Also, we show that the Levitski radical contains all locally nilpotent one-sided ideals.

In chapter 4, we give two generalizations of the Jacobson radical for associative rings. The first stems from the definition of an element, x, as being quasi-regular if there exists an element y such that $x + y - xy = 0$, and follows that for Jordan rings. The radical obtained by this process is called the Jacobson-MacCrimmon radical. The second generalization is that of Brown's, and is true for any non-associative ring. This is called the Jacobson-Brown radical. We show that in a standard ring, it makes no difference if the Jacobson-Brown radical is defined in terms of left or right ideals. We show that the Jacobson-MacCrimmon radical contains the nil radical, and is contained in both the Behrens and Jacobson-Brown radicals. Finally, we show that the Smiley radical contains all other

radicals considered.

In chapter 5, we consider the relationship of the different radicals under various chain conditions. If the ring satisfies the descending chain condition on right ideals, then the prime and Levitski radicals are equal, as well as the Jacobson-Brown and Smiley radicals. If every subring of the ring satisfies the descending chain condition on right ideals, then in addition to the above, the nil, Jacobson-MacCrimmon, and Behrens radicals are equal. If the ring satisfies an ascending chain condition on subrings, then the prime, Levitski, and nil radicals are equal.

RADICALS IN STANDARD RINGS

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INTRODUCTION

A standard ring $\mathfrak{A}$ is a non-associative ring satisfying:

1) $(x, y, z) + (z, x, y) - (x, z, y) = 0$;

2) $(x, y, wz) + (w, y, xz) + (z, y, wx) = 0$; and

3) $(x, y, x^2) = 0$

for all x, y, z, w in $\mathfrak{A}$, where $(x, y, z) = (xy)z - x(yz)$.

It is easy to compute that in any non-associative ring, the following identity holds:

$$(x,y,z) + (z,x,y) - (x,z,y) = [xy,z] - [x,z]y - x[y,z]$$

where $[x,y] = xy - yx$. Thus identity (1) is equivalent to:

4) $[xy,z] = [x,z]y + x[y,z]$.

That is, for each z in $\mathfrak{A}$, the mapping $x \rightarrow [x,z]$ is a derivation of $\mathfrak{A}$.

Identity (3) is known as the Jordan identity, for a commutative ring which satisfies identity (3), is called a Jordan ring. Also, identity (3) is equivalent to identity (2) unless $\mathfrak{A}$ possesses an element $a \neq 0$ such that $a + a + a = 0$. Thus it is clear from the definition that any associative ring and any Jordan ring are standard rings.

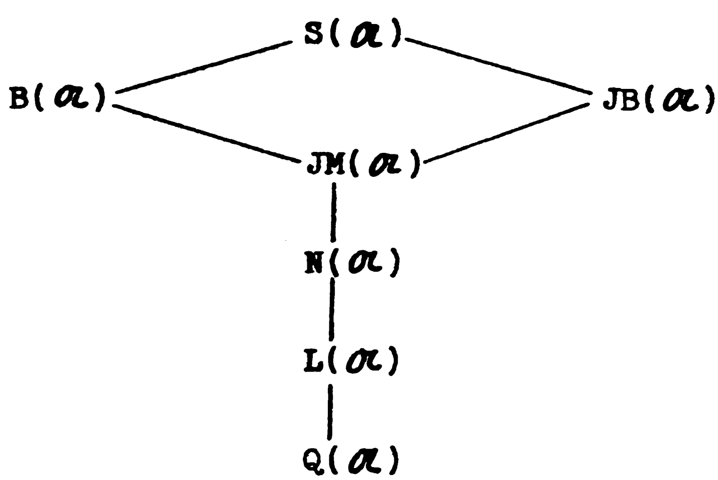
In addition throughout the paper we shall assume that if $\mathfrak{B}$ is any subring of $\mathfrak{A}$ and $x \in \mathfrak{B}$, then there exists a unique $y \in \mathfrak{B}$ such that $2y = x$.

The concept of a standard ring was first introduced by Albert [1] where he considered a finite dimensional standard algebra over a field. He showed that a finite dimensional standard nil algebra is nilpotent and in a finite dimensional standard algebra over a field of characteristic $\neq 2$, an ideal is solvable if and only if it is nilpotent. Then he called the maximal solvable ideal of $\mathfrak{A}$, the radical of $\mathfrak{A}$ and showed that if $\mathfrak{A}$ is any finite dimensional standard algebra over a field of characteristic $\neq 2$, then the quotient algebra $\mathfrak{A}/R$ where R is the radical of $\mathfrak{A}$, is semisimple. Furthermore, any semisimple standard algebra is a finite direct sum of simple ideals.

Kleinfeld [7] showed that a simple standard ring is either associative or commutative Jordan. Thus, combining these, we see that a finite dimensional standard algebra modulo its radical is the finite direct sum of simple associative or commutative Jordan algebras.

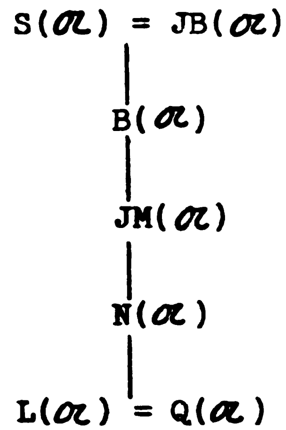
The problem we consider is derived by removing the condition of an algebra over a field. With the relaxation of this condition we have various choices for the radical of the ring. All of our choices will be made so as to furnish information about the structure of the ring in the following manner. We introduce the radical as an ideal such that the ring modulo its radical has zero radical. Then we characterize rings which have zero radical. Since we have dropped Albert's condition of a finite dimensional algebra, we can no longer hope for a result such as a finite direct sum of simple rings. However, we are able to represent rings with zero radicals as subdirect sums of certain types of rings. Thus we study the structure of a standard ring $\mathfrak{A}$, by characterizing an ideal I, then investigating $\mathfrak{A}/I$.

Several radicals have been studied for associative rings and we can show the existence of similar ones in a standard ring. The prime radical, $Q(\mathfrak{A})$, is the intersection of all prime ideals (Chapter 2). The Levitski radical, $L(\mathfrak{A})$, is the maximal locally nilpotent ideal (Chapter 3). Corresponding to the Jacobson radical of associative rings, we have two radicals for standard rings. One, $JM(\mathfrak{A})$, follows that considered by MacCrimmon [6] and Tsai [11] for Jordan rings, and the other, $JB(\mathfrak{A})$, has been introduced by Brown [3] for a general non-associative ring. In addition, there are other radicals which have been studied for arbitrary rings: $S(\mathfrak{A})$, the Smiley radical [10]; $B(\mathfrak{A})$, the Behrens radical [2]; and $N(\mathfrak{A})$, the maximal nil ideal or nil radical [2]. For an arbitrary standard ring, the relationship of these is shown by the following diagram where each radical is contained in those above it in the diagram.

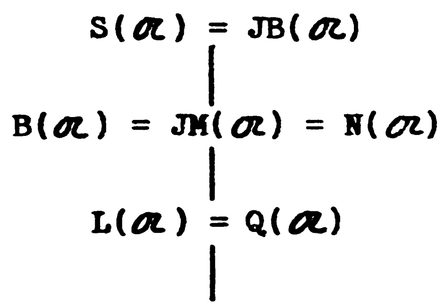


When various chain conditions on the ideals of $\mathfrak{A}$ are present, we have the following.

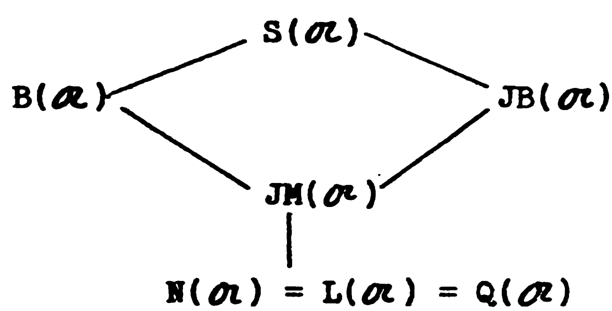
If $\mathfrak{A}$ satisfies a descending chain condition on right ideals, then:



If $\mathfrak{A}$ satisfies a strong descending chain condition [9], that is each subring of $\mathfrak{A}$ satisfies a descending chain condition on right ideals, then:



If $\mathfrak{A}$ satisfies an ascending chain condition on subrings, then:



Chapter 1. Foundations

A. Basic Identities

We begin by deriving some identities which will be used in the remainder of the paper. Recalling the defining identities, (1-3), we see that by setting $x = z$ in (1) we obtain that a standard ring is flexible and this combined with (2) gives that a standard ring is a non-commutative Jordan ring, in particular it is power-associative.

Using flexibility, we may rewrite (2) as

5) $(wx,y,z) - (w,y,xz) + (wz,y,x) = 0$ or as

6) $(wx,y,z) + (xz,y,w) - (x,y,wz) = 0$.

Interchanging x and z in (2) and subtracting, we see that

7) $(w,y,[x,z]) = 0$.

Using (7) and flexibility on (6), we see that

8) $(x,y,zw) - (xz,y,w) + (z,y,xw) = 0$.

Now by considering (6) and (8) as operations of left and right multiplications, we have:

9) $R_y(xz) = R_yR_{xz} + R_x(R_{yz} - R_yR_z) + R_z(R_{yz}-R_yR_x)$ and

10) $L(xz)y = L_yL_{xz} + L_x(L_{zy} - L_zL_y) + L_z(L_{xy}-L_yL_x)$.

Identity (1) considered as operating on z gives:

11) $L_{xy} - L_yL_x + R_xR_y - R_{xy} - L_xR_y + R_yL_x = 0$.

B. General Radical Theory

Before studying standard rings, we discuss some general properties of radicals. Let $\mathcal{P}$ be a property which a ring may possess. A ring is a $\mathcal{P}$-ring if it possesses property $\mathcal{P}$. An ideal I of $\mathfrak{A}$ is a $\mathcal{P}$-ideal if I is a $\mathcal{P}$-ring. A ring which does not contain any non-zero ρ-ideals is ρ-semisimple. ρ is called a radical property if the following hold:

1) A homomorphic image of a ρ-ring is a ρ-ring.
2) Every ring R contains a ρ-ideal S which contains every other ρ-ideal of the ring.
3) The quotient ring, R/S, is ρ-semisimple.

We will use two methods for specifying a radical property. The first is to explicitly define the property. This will be done in Chapters 3 and 4. The second is to begin with a class of rings N and construct a new class of rings $\overline{N}$ as follows. A ring is said to be of degree one over N if it is zero or a homomorphic image of some ring in N. A ring, R, is said to be of degree two over N if every homomorphic image of R contains a non-zero ideal which is a ring of degree one over N. For any ordinal number β, if β-1 exists, a ring is said to be of degree β over N if every non-zero homomorphic image of R contains a non-zero ideal which is a ring of degree β-1 over N. If β is a limit ordinal, then R is of degree β over N if it is of degree over N for some $\alpha<\beta$. If $\overline{N}$ is the class of all rings which are of any degree over N, then define a ring to be a $\rho_{\overline{N}}$ ring if it belongs to $\overline{N}$. It can be shown [4, p. 12] that $\rho_{\overline{N}}$ is a radical property. The radical determined by this property is called the lower radical determined by the class N. We shall use this approach at the end of Chapter 2 with N as the class of nilpotent rings. We shall show that this process yields the same result as the prime radical and thus in this special case is a radical property.

Using the concept of a radical property, we can avoid reproving the same theorems several times.

Theorem 1.1: If Rad($\mathfrak{A}$) is the $\mathcal{P}$-radical of $\mathfrak{A}$ and if I is any ideal of $\mathfrak{A}$ such that $\mathfrak{A}/I$ is $\mathcal{P}$-semisimple, then $I \supseteq$ Rad($\mathfrak{A}$).

PROOF:

Let R = Rad($\mathfrak{A}$). Since $(I + R)/I$ is isomorphic to $R/(I \cap R)$, it is a $\mathcal{P}$-ring by property 1) and thus is a $\mathcal{P}$-ideal of $\mathfrak{A}/I$. However, since $\mathfrak{A}/I$ is $\mathcal{P}$-semisimple, $(I + R)/I = 0$, that is, $I \supseteq R$.

We will characterize semi-simple rings as subdirect sums. A subdirect sum is a subring of the complete direct sum such that the natural projections are onto each coordinate summand. The following theorem which we use to obtain our representations is well-known [4, p. 64].

Theorem 1.2: A ring $\mathfrak{A}$ is isomorphic to a subdirect sum of rings $\mathfrak{A}_\alpha$ if and only if $\mathfrak{A}$ contains a class of ideals $\{\mathfrak{B}_\alpha\}$ such that $\cap \mathfrak{B}_\alpha = 0$ and $\mathfrak{A}/\mathfrak{B}_\alpha \cong \mathfrak{A}_\alpha$.

Theorem 1.3: A subdirect sum of $\mathcal{P}$-semisimple rings is $\mathcal{P}$-semisimple.

PROOF:

Let $\mathfrak{A}$ be a subdirect sum of $\mathfrak{A}_\alpha$ where each $\mathfrak{A}_\alpha$ contains no non-zero $\mathcal{P}$-ideals. Let π_α be the projection map onto $\mathfrak{A}_\alpha$. Let R be the $\mathcal{P}$-radical of $\mathfrak{A}$. Then $\pi_\alpha(R)$ is the homomorphic image of a $\mathcal{P}$-ideal and thus is a $\mathcal{P}$-ideal of $\mathfrak{A}_\alpha$. Thus $\pi_\alpha(R) = 0$ for each α and so $R = 0$ and $\mathfrak{A}$ is ρ-semisimple.

C. Known Results

Behrens [2] showed that being nil is a radical property, that is, if $N(\mathfrak{A})$ is the sum of all nil ideals of a ring $\mathfrak{A}$, then $N(\mathfrak{A})$ is a nil ideal and $N(\mathfrak{A}/N(\mathfrak{A})) = 0$.

One method of obtaining radical properties is to assign to each element a of a ring $\mathfrak{A}$ an ideal F(a) for which $F(a+b) \subseteq F(a) + F(b)$ for each a and b in $\mathfrak{A}$. An element, a, is called F-regular if $a \in F(a)$ and an ideal is F-regular if all its elements are F-regular. For each choice of F(a), there is an F-radical $R(\mathfrak{A})$, which consists of all elements, a, of $\mathfrak{A}$ for which the ideal (a) is F-regular, where (A) is the smallest ideal of containing the set A.

Behrens showed that the choice of $F(a) = (a^2 - a)$ satisfies the desired property and leads to the Behrens radical $B(\mathfrak{A})$, which is the set of all elements, a, such that $b \in (b^2 - b)$ for each $b \in (a)$. He also showed that a ring $\mathfrak{A}$ is B-semisimple if and only if it is a subdirect sum of rings $\mathfrak{A}_\alpha$ where $\mathfrak{A}_\alpha$ has a non-zero idempotent generating its minimal ideal. In his paper, it follows easily that $B(\mathfrak{A})$ contains no non-zero idempotents.

Smiley [10] chose for F(a), the ideal $(\{ax - x + ya - y \mid x, y \in \mathfrak{A}\})$. Thus $S(\mathfrak{A})$, the Smiley radical of $\mathfrak{A}$ is the set of all elements a of $\mathfrak{A}$ such that if $b \in (a)$, then $b \in (\{bx - x + yb - y \mid x, y \in \mathfrak{A}\})$. He showed the following:

Theorem 1.4:

a) $S(\mathfrak{A}/S(\mathfrak{A})) = 0$.

b) $\mathfrak{A}$ is S-semisimple if and only if it is isomorphic to a subdirect sum of simple rings with identity.

c) If the descending chain condition holds for ideals of an S-semisimple ring $\mathfrak{A}$, then $\mathfrak{A}$ is isomorphic to the full direct sum of a finite number of simple rings with identity.

Behrens also showed [2] that for any non-associative ring $N(\mathfrak{A}) \subseteq B(\mathfrak{A}) \subseteq S(\mathfrak{A})$.

Chapter 2. The Prime Radical

In this chapter, we shall first introduce the concept of prime ideals which generalizes that of associative rings and that of Jordan rings given in [13]. The prime radical of a standard ring $\mathfrak{A}$, which is defined to be the intersection of all prime ideals of $\mathfrak{A}$, will be investigated. Finally, a characterization of the prime radical will be given.

Lemma 2.1: If A and B are ideals of $\mathfrak{A}$, $A*B = AB^2 + B^2A + B(BA) + B(AB)$, and $A\#B = AB^2 + B^2A + (AB)B + (BA)B$, then $A*B = A\#B$.

PROOF:

It suffices to show that B(BA) and B(AB) are contained in A#B, and that (AB)B and (BA)B are contained in A*B.

Let $a \in A$, $b, b' \in B$. Then by the flexible law, $(b, b', a) + (a, b', b) = 0$, we have $b(b'a) \in B^2A + (AB)B + AB^2 \subseteq A\#B$, i.e. $B(BA) \subseteq A\#B$. Furthermore, by identity (1), $(b, a, b') + (b', b, a) - (b, b', a) = 0$, thus we have $b(ab') \in (BA)B + B^2A + B(BA) + (AB)B \subseteq A\#B$. Thus B(AB) is also contained in A#B.

The same identities also show that (AB)B and (BA)B are contained in A*B.

Theorem 2.1: If A and B are any ideals of $\mathfrak{A}$, then A*B is an ideal of $\mathfrak{A}$.

PROOF:

We first show that A*B is a left ideal of $\mathcal{A}$. Let $x \in \mathcal{A}$, $a \in A$, and $b, b' \in B$. Then $x(a(bb')) = xR_{a(bb')} = x(R_aR_{bb'} + R_bR_{ab'} - R_bR_aR_{b'} + R_{b'}R_{ab} - R_{b'}R_aR_b)$ by identity (9). Hence, $x(a(bb')) = \bar{a}(bb') + \bar{b}(ab') - (\bar{b}a)b' + \bar{b}'(ab) - (\bar{b}'a)b$ A*B where $\bar{a} = xa$, $\bar{b} = xb$, and $\bar{b} = xb'$. Thus $x(AB^2) \subseteq A*B$. Using the same identity with the appropriate substitutions, we obtain $x(B(AB)) \subseteq A*B$ and $x(B(BA)) \subseteq A*B$.

In order to show that A*B is a left ideal, it remains to show that $x(B^2A) \subseteq A*B$. The following two steps are used for that purpose:

(a) $x(R_{bb'}R_a - L_{bb'}R_a) \in A*B$; and

(b) $xL_{(bb')a} \in A*B$ for all $b, b' \in B$ and $a \in A$.

We can verify (a) by direct computation using identity (1), for $-x(R_{bb'}R_a - L_{bb'}R_a) = -[x(bb') - (bb')x]a = [+ (x, b, b') - (xb)b' + (b, b', x) + b(b'x)]a = +[(b, x, b') - (xb)b' + b(b'x)]a = +[(bx)b' - b(xb') - (xb)b' + b(b'x)]a \in B^2A \subseteq A*B$. Identity (b) can be verified using identity (10), for $xL(bb')a = x(L_aL_{bb'} + L_bL_{b'a} - L_bL_{b'}L_a + L_{b'}L_{ba} - L_{b'}I_aL_b) = (bb')\bar{a} + (b'a)\bar{b} - a(b'\bar{b}) + (ba)\bar{b}' - b(a\bar{b}') \in A*B$ where $\bar{a} = ax$, $\bar{b} = bx$, and $\bar{b}' = b'x$.

We now return to showing that $x(B^2A) \subseteq A*B$. Let $b, b' \in B$ and $a \in A$. Then, by identity (11), we have $x((bb')a) = xR_{(bb')a} = x\, L_{(bb')a} - L_aL_{bb'} + R_{bb'}R_a - L_{bb'}R_a + R_aL_{bb'} = xL_{(bb')a} + x(R_{bb'} - L_{bb'}R_a) - (bb')\bar{a} + (bb')a^*$ where $\bar{a} = ax$ and $a^* = xa$. Thus by

(a) and (b) we see that $x((bb')a) \in A*B$.

$A*B$ can be shown to be a right ideal by similar techniques.

Corollary 2.1: If A is an ideal of $\mathfrak{A}$, then $A^3 = A*A$ and A^3 is an ideal of $\mathfrak{A}$.

We now proceed to define a prime ideal and the prime radical for a standard ring.

Lemma 2.2: Let P be an ideal of $\mathfrak{A}$. Then the following are equivalent:

a) Whenever A and B are ideals of $\mathfrak{A}$ with $A*B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

b) Whenever A and B are ideals of $\mathfrak{A}$ with $A \cap c(P) \neq \emptyset$ and $B \cap c(P) \neq \emptyset$, then $(A*B) \cap c(P) \neq \emptyset$ where $c(P)$ is the complement of P.

c) If $a, b \in c(P)$, then $((a)*(b)) \cap c(P) \neq \emptyset$, where (x) is the smallest ideal of $\mathfrak{A}$ containing x.

PROOF:

a)$\Leftrightarrow$b): One is merely the contrapositive of the other.

b)$\Rightarrow$c): Let $A = (a)$, $B = (b)$. Then $a \in A \cap c(P) \neq \emptyset$ and $b \in B \cap c(P) \neq \emptyset$. Thus $((a)*(b)) \cap c(P) = (A*B) \cap c(P) \neq \emptyset$.

c)$\Rightarrow$b): Let A and B be ideals with $A \cap c(P) \neq \emptyset$ and $B \cap c(P) \neq \emptyset$. Thus we may choose $a \in A \cap c(P)$, $b \in B \cap c(P)$. Then by c) $\emptyset \neq ((a)*(b)) \cap c(P) \subseteq A*B \cap c(P)$. Thus b) holds.

Definition 2.1: An ideal P of a standard ring is a prime ideal if it satisfies any one of the statements in Lemma 2.2. An ideal P of an associative ring $\mathfrak{A}$ is an A-prime ideal of $\mathfrak{A}$ if whenever A and B are ideals of $\mathfrak{A}$ such that $AB \subsetneq P$, then $A \subsetneq P$ or $B \subsetneq P$. An ideal P of a Jordan ring $\mathfrak{A}$ is J-prime if whenever A and B are ideals of $\mathfrak{A}$ such that $AU_B \subsetneq P$, then $A \subsetneq P$ or $B \subsetneq P$ where AU_B is the set of all finite sums of elements of the form $a_iU_{b_i}$ for $a_i \in A$ and $b_i \in B$ and U_x is the quadratic operator defined by $U_x = 2R_x{}^2 - R_{x^2}$.

Theorem 2.2: If $\mathfrak{A}$ is an associative ring, then an ideal P is a prime ideal if and only if it is an A-prime ideal. If $\mathfrak{A}$ is a Jordan ring, then an ideal P is a prime ideal if and only if it is a J-prime ideal.

PROOF:

a) Let $\mathfrak{A}$ be an associative ring.

Assume that P is a prime ideal of $\mathfrak{A}$ and that A and B are ideals of $\mathfrak{A}$ such that $AB \subsetneq P$. Now $(BA)*(BA) = (BA)^3 = B(AB)ABA \subsetneq BPABA \subsetneq P$. Thus $BA \subsetneq P$. So $A*B \subsetneq P$ and hence $A \subsetneq P$ or $B \subsetneq P$. That is, P is an A-prime ideal of $\mathfrak{A}$.

Conversely, assume that P is an A-prime ideal of $\mathfrak{A}$ and that A and B are ideals of $\mathfrak{A}$ such that $A*B \subsetneq P$. Then, since $AB^2 \subsetneq A*B$ and both A and B^2 are ideals, we have $A \subsetneq P$ or $B^2 \subsetneq P$. But if $B^2 \subsetneq P$ then $B \subsetneq P$, so $A \subsetneq P$ or $B \subsetneq P$. That is, P is a prime ideal.

b) Let $\mathfrak{A}$ be a Jordan ring.

Assume that P is a prime ideal of $\mathfrak{A}$ and A and B are ideals of $\mathfrak{A}$ such that $AU_B \subseteq P$. Then $C = A \cap B$ is an ideal of $\mathfrak{A}$ contained in P or else $C^3 = C*C = CU_C \subseteq AU_B \subseteq P$ contradicts the fact that P is a prime ideal. However, if $C = A \cap B \subseteq P$, then $A*B \subseteq C \subseteq P$ so either $A \subseteq P$ or $B \subseteq P$. Thus P is a J-prime ideal.

Now, assume that P is a J-prime ideal of $\mathfrak{A}$ and that A and B are ideals such that $A*B \subseteq P$. Then $AU_B \subseteq A*B \subseteq P$ and so either $A \subseteq P$ or $B \subseteq P$. Hence P is a prime ideal of $\mathfrak{A}$.

<u>Definition 2.2</u>: A non-empty subset, M, of $\mathfrak{A}$ is a Q-system if whenever A and B are ideals of $\mathfrak{A}$ such that $A \cap M$ and $B \cap M$ are non-empty, then $(A*B) \cap M$ is non-empty.

<u>Theorem 2.3</u>: If P is an ideal of $\mathfrak{A}$, then c(P) is a Q-system if and only if P is a prime ideal.

<u>PROOF</u>:

This is part b) of Lemma 2.2.

<u>Definition 2.3</u>: Let A be an ideal. The radical of A is $A^Q = (r \mid$ any Q-system containing r meets A).

<u>Theorem 2.4</u>: Let A be an ideal of $\mathfrak{A}$. The radical of A is the intersection of all prime ideals containing A.

<u>PROOF</u>:

Let $b \in A^Q$ and let P be a prime ideal containing A. Then c(P) is a Q-system missing A and so $b \notin c(P)$.

Thus $b \in P$ and hence b is in the intersection of all prime ideals containing A.

Now assume $b \notin A^Q$. Thus there exists a Q-system M with $b \in M$ and $M \cap A = \emptyset$. Let $\mathcal{S}$ be the family of ideals of $\mathfrak{A}$ containing A but disjoint from the set M. Now A is an ideal with $A \subseteq A$ and $A \cap M = \emptyset$ so $A \in \mathcal{S}$ and $\mathcal{S}$ is non-empty. Next, let $I_1 \subseteq I_2 \subseteq \cdots$, be a chain of elements of $\mathcal{S}$. Let $I = \bigcup I_j$. Since $A \subseteq I_1$, $A \subseteq I$. Also if $x \in I \cap M$, then there exists a j such that $x \in I_j$ and the $I_j \cap M \neq \emptyset$, so $I_j \notin \mathcal{S}$ which is a contradiction. Thus $I \in \mathcal{S}$ and $\mathcal{S}$ is an inductive set, and Zorn's Lemma may be applied to obtain a maximal element P. We claim that P is a prime ideal. Let B and C be ideals with $B \nsubseteq P$ and $C \nsubseteq P$, thus $P \subsetneq B + P$ and $P \subsetneq C + P$. So, by the maximality of P, $(B + P) \cap M \neq \emptyset$, and $(C + P) \cap M \neq \emptyset$. Then, since M is a Q-system, $(B + P)*(C + P) \cap M \neq \emptyset$. But $((B + P)*(C + P)) \subseteq (B*C) + P$, thus $(B*C) \nsubseteq P$ or else $P \notin \mathcal{S}$. But then P is a prime ideal and $b \in M$ and $M \cap P = \emptyset$ means that $b \notin P$. Thus b is not in the intersection of all prime ideals containing A.

<u>Definition 2.4</u>: An ideal P is semi-prime if for any ideal A with $A*A \subseteq P$, we must have $A \subseteq P$. A non-empty subset S of $\mathfrak{A}$ is an SQ-system if for any ideal A with $A \cap S \neq \emptyset$, then $(A*A) \cap S \neq \emptyset$.

Note that any prime ideal is also semi-prime.

Lemma 2.3: If P is an ideal, the following are equivalent:

a) P is semi-prime.

b) c(P) is an SQ-system.

c) If $a \in c(P)$, then $(a)*(a) \cap c(P) \neq \emptyset$.

PROOF:

Similar to Lemma 2.2.

Definition 2.5: Let A be an ideal, $A_Q = \{ r \in \mathcal{O}\mathcal{C} /$ any SQ-system containing r meets $A \}$.

Theorem 2.6: Let A be an ideal of $\mathcal{O}\mathcal{C}$. Then

a) A_Q is the intersection of all semi-prime ideals containing A.

b) A_Q is a semi-prime ideal.

c) A is semi-prime if and only if $A = A_Q$.

PROOF:

a) If $b \in A_Q$ and P is a semi-prime ideal with $A \subseteq P$, then c(P) is an SQ-system missing A and thus $b \notin c(P)$ or $b \in P$. Thus b is in the intersection of all semi-prime ideals containing A.

If $b \notin A_Q$, let S be an SQ-system missing A with $b \in S$. Now let $\mathcal{S}$ be the family of all ideals containing A which are disjoint from S. $A \in \mathcal{S}$ so $\mathcal{S}$ is non-empty and we can also show as in Theroem 2.4 that Zorn's Lemma may be applied to get a maximal element P. Since $b \in S$ and $P \cap S = \emptyset$, $b \notin P$. To show that P is a semi-prime ideal, suppose that B is an ideal with $B \nsubseteq P$. Thus $P \subsetneq B + P$, so by the maximality of P, $(B + P) \cap S \neq \emptyset$. Now since S is an SQ-system, $(B + P)*(B + P) \cap S \neq \emptyset$. If $(B*B) \subseteq P$, we would have $\emptyset \neq (B + P)*(B + P) \cap S \subseteq ((B*B) + P) \cap S \subseteq P \cap S$, which would imply that $P \notin \mathcal{S}$. Thus P is semi-prime.

b) By part a), A_Q is an ideal and if B is an ideal with $B*B \subseteq A_Q = \cap P_*$ where P_* runs over all semi-prime ideals containing A, then $B*B \subseteq P_*$ and thus $B \subseteq P_*$ for each P_*. Thus $B \subseteq \cap P_* = A_Q$, that is, A_Q is semi-prime.

c) Since A_Q is semi-prime, A_Q is the smallest semi-prime ideal containing A. So A is semi-prime if and only if $A = A_Q$.

<u>Lemma 2.4</u>: If $a \in \mathfrak{a}$ and S is an SQ-system with $a \in S$, then there is a Q-system M with $a \in M$ and $M \subseteq S$.

<u>PROOF</u>:

Construct $M = \{a_1, a_2, \cdots\}$ as follows. Let $a_1 = a$. Choose $a_2 \in (a_1)*(a_1) \cap S$, $\cdots$, $a_{k+1} \in ((a_k)*(a_k)) \cap S$. This is always possible since S is an SQ-system. Clearly, $a = a_1 \in M$ and $M \subseteq S$. We now show that M is a Q-system, that is, $((a_i)*(a_j)) \cap M \neq \emptyset$. Now $a_{t+1} \in ((a_t)*(a_t)) \subseteq (a_t)$, so that $(a_{t+1}) \subseteq (a_t)$ and more generally, $(a_k) \subseteq (a_t)$ whenever $t \leq k$. Now let $r = \max(i, j)$. $a_{r+1} \in ((a_r)*(a_r)) \cap S \subset ((a_i)*(a_j)) \cap S$ since $(a_r) \subseteq (a_i)$ and $(a_r) \subseteq (a_j)$.

<u>Theorem 2.7</u>: For any ideal A of $\mathfrak{a}$, $A^Q = A_Q$.

PROOF:

Each prime ideal is also semi-prime so, $A_Q = \bigcap P_*$ $\subseteq \bigcap P^* = A^Q$ where P_* runs over all semi-prime ideals containing A and P^* runs over all prime ideals which contain A.

Also, if $x \in A^Q$ and if S is an SQ-system containing x, then by Lemma 2.4, there is a Q-system, M, with $x \in M$ and $M \subseteq S$. But $x \in A^Q$ implies that $\phi \neq A \cap M$ $\subseteq A \cap S$, that is, $x \in A_Q$. Therefore, $A^Q = A_Q$.

Definition 2.6: For any ideal A, A^Q ($=A_Q$) is the prime radical of A. The prime radical, $Q(\mathfrak{a})$, of a standard ring $\mathfrak{a}$ is the prime radical of the zero ideal in $\mathfrak{a}$. A standard ring is Q-semisimple if and only if $Q(\mathfrak{a}) = 0$.

Theorem 2.8: If $\mathfrak{a}$ is a standard ring, and I is an ideal of $\mathfrak{a}$, the quotient ring, $\mathfrak{a}/I$ is Q-semisimple only if $I \supseteq Q(\mathfrak{a})$.

PROOF:

Let $\eta : \mathfrak{a} \rightarrow \bar{\mathfrak{a}} = \mathfrak{a}/I$ be the natural homomorphism. Consider the correspondence between the ideals of $\mathfrak{a}/I$ and the ideals of $\mathfrak{a}$ which contain I. Suppose P is a prime ideal of $\mathfrak{a}$ containing I. Let $\bar{P} = P/I$. Suppose $\bar{A}$ and $\bar{B}$ are ideals of $\bar{\mathfrak{a}}$ with $\bar{A}*\bar{B} \subseteq \bar{P}$. Then $A = \eta^{-1}(\bar{A})$, $B = \eta^{-1}(\bar{B})$ are ideals of $\mathfrak{a}$ with $A*B =$ $\eta^{-1}(\bar{A})*\eta^{1}(\bar{B}) = \eta^{-1}(A*B) \subseteq \eta^{-1}(\bar{P}) = \eta^{-1}(\eta(P)) \subseteq P +$ $I \subseteq P$. Now since P is prime, either $A \subseteq P$ or $B \subseteq P$, but then either $\bar{A} \subseteq \bar{P}$ or $\bar{B} \subseteq \bar{P}$, that is, $\bar{P} = P/I$ is a prime ideal of $\mathfrak{a}/I$. Now suppose P/I is a prime ideal

of $\mathfrak{A}$/I and A and B are ideals of $\mathfrak{A}$ with A*B $\subseteq$ P. Then (A + I)/I * (B + I)/I $\subseteq$ ((A*B) + I)/I $\subseteq$ P/I and so by the primeness of P/I either (A + I)/I $\subseteq$ P/I or (B + I) $\subseteq$ P/I, so either A $\subseteq$ P that is, (A + I)$\subseteq$P or (B + I)$\subseteq$P, or B $\subseteq$ P. Thus we have shown that if I $\subseteq$ P $\subseteq \mathfrak{A}$, then P/I is a prime ideal in $\mathfrak{A}$/I if and only if P is a prime ideal of $\mathfrak{A}$.

Now, $\mathfrak{A}$/I is Q-semisimple if and only if $\bigcap$ P/I = 0 where the intersection runs over all P/I, prime ideals of $\mathfrak{A}$/I, if and only if $\bigcap$ P = I where the intersection is over all prime ideals P which contain I. Thus, if $\mathfrak{A}$/I is Q-semisimple, Q($\mathfrak{A}$), the intersection of all prime ideals of $\mathfrak{A}$ is contained in I.

<u>Corollary 2.2</u>: The quotient ring, $\mathfrak{A}$/Q($\mathfrak{A}$), is Q-semisimple, that is, Q($\mathfrak{A}$/Q($\mathfrak{A}$)) = 0.

<u>PROOF</u>:

In the previous proof, it was shown that $\mathfrak{A}$/I is Q-semisimple if and only if I contains the intersection of all prime ideals which contain I. Now let I = Q($\mathfrak{A}$). Q($\mathfrak{A}$) is the intersection of all prime ideals, so each prime ideal of $\mathfrak{A}$ contains Q($\mathfrak{A}$) and thus $\mathfrak{A}$/Q($\mathfrak{A}$) is Q-semisimple.

<u>Definition 2.7</u>: A ring is a prime ring if and only if (0) is a prime ideal.

<u>Theorem 2.9</u>: A prime ring is Q-semisimple.

<u>PROOF</u>:

If (0) is a prime ideal, then Q($\mathfrak{A}$), the intersection of all prime ideals is zero, that is, $\mathfrak{A}$ is Q-semisimple.

Theorem 2.10: An ideal P of $\mathfrak{A}$ is prime if and only if $\mathfrak{A}/P$ is a prime ring.

PROOF:

If $\mathfrak{A}/P$ is a prime ring, suppose A and B are ideals with $A*B \subseteq P$. Then $\bar{A}*\bar{B} \subseteq \bar{P} = 0$, so $\bar{A} = 0$ or $\bar{B} = 0$, that is, $A \subseteq P$ or $B \subseteq P$.

If P is a prime ideal and $\bar{A}$ and $\bar{B}$ are ideals of $\mathfrak{A}/P$ with $\bar{A}*\bar{B} = 0$, then since $\overline{A*B} = \bar{0} = \bar{P}$, we have $A*B \subseteq P$ and thus $A \subseteq P$ or $B \subseteq P$, that is, $\bar{A} = 0$ or $\bar{B} = 0$.

Theorem 2.11: A ring is Q-semisimple if and only if it is a subdirect sum of prime rings.

PROOF:

Apply Theorem 1.2.

Lemma 2.5: If A is an ideal of $\mathfrak{A}$ and $r \in A_Q$, then $r^k \in A$ for some positive integer k.

PROOF:

We will show that $M = \{r, r^3, r^{3^2}, \ldots, r^{3^k} \ldots\}$ is an SQ-system. If C is an ideal and $r^{3^j} \in C \cap M$, then $r^{3^{j+1}} \in (C*C) \cap M$. So M is an SQ-system containing r and thus $M \cap A \neq \emptyset$, so $r^k \in A$ for some k.

Theorem 2.12: Q($\mathfrak{A}$) is a nil ideal of $\mathfrak{A}$.

PROOF:

If $r \in Q(\mathfrak{A}) = (0)_Q$, then for some k, $r^k \in (0)$, that is $r^k = 0$.

Definition 2.8: An ideal I of $\mathfrak{A}$ is nilpotent if there exists a positive integer k such that the product of any k elements of I in any association is equal to zero, that is, $I^k = 0$.

Theorem 2.13: $\mathfrak{A}$ is Q-semisimple if and only if contains no non-zero nilpotent ideals.

PROOF:

$\mathfrak{A}$ is Q-semisimple if and only if $(0) = (0)_Q$ if and only if (0) is semi-prime. Suppose K is a non-zero nilpotent ideal. Then there is a t such that $K3^t = 0$, but $K3^{t-1} \neq 0$. Hence $K3^{t-1} * K3^{t-1} = (K3^{t-1})3 = K3^t = 0$, which shows that (0) is not semi-prime. Now if (0) is not semi-prime, then there exists a non-zero ideal K with $K3 = K*K = 0$.

Corollary 2.3: $Q(\mathfrak{A})$ is the smallest ideal I of $\mathfrak{A}$ such that $\mathfrak{A}/I$ has no non-zero nilpotent ideals.

PROOF:

Since $Q(\mathfrak{A}/Q(\mathfrak{A})) = 0$, $\mathfrak{A}/Q(\mathfrak{A})$ has no non-zero nilpotent ideals. If $\mathfrak{A}/I$ has no non-zero nilpotent ideals, then $\mathfrak{A}/I$ is Q-semisimple, and by Theorem 2.8, $I \supseteq Q(\mathfrak{A})$.

Theorem 2.14: A prime standard ring $\mathfrak{A}$ is either associative or Jordan.

PROOF:

If in identity (2), we interchange x and z and then subtract, we obtain: $(w, y, [x, z]) = 0$. Thus each standard ring is an accessible ring as investigated by Kleinfeld [7], and thus we have:

(a) $[(w, x, y), z] = 0$ for all w, x, y, z in $\mathfrak{A}$.

Now, following Kleinfeld, let A be the set of all finite sums of elements of the form (x, y, z) or of the form w(x, y, z). Let B consist of all finite sums of elements of the form [x, y] or of the form [x, y]z. Then in a standard ring A and B are ideals such that $\mathfrak{A}/A$ is associative and $\mathfrak{A}/B$ is Jordan. Kleinfeld showed that if $\mathfrak{A}$ possesses no non-zero nilpotent ideals, then AB = 0.

Now suppose that AB = 0. We claim that BA = 0. Let a represent any associator, that is, a = (x, y, z). Then A consists of sums of elements of the form either a or wa. By (a) ba = ab for any $b \in B$, and since AB = 0, we have Ba = 0. Also, $b(wa) = [b, wa] + (wa)b = [b, wa] = -[wa, b] = -w[a, b] - [w, b]a \in Ba = 0$. Thus if AB = 0, BA = 0, and hence A*B = 0.

Now, since a prime ring is Q-semisimple, it has no non-zero nilpotent ideals and thus AB = 0 = A*B. But then the primeness of $\mathfrak{A}$ means that A = 0 or B = 0. If A = 0, then $\mathfrak{A}$ is associative, and if B = 0, $\mathfrak{A}$ is Jordan.

We now proceed to give another characterization of the prime radical. Given a ring $\mathfrak{A}$, let $N_0 = \sum I_a$ such that I_a

is a nilpotent ideal of $\mathfrak{A}$. Next choose N_1 such that $N_1 N_0 = \sum I_\sigma'/N_0$ where I_σ' is a nilpotent ideal of $\mathfrak{A}/N_0$. In general, suppose that a is any ordinal number. If a is not a limit ordinal, select N_a such that N_a/N_{a-1} is the sum of all nilpotent ideals of $\mathfrak{A}/N_{a-1}$. If a is a limit ordinal, let $N_a = \sum N_\beta$ for all $\beta < a$. If $\mathfrak{A}$ has ordinal number γ, then this process stops in at most γ steps. Let τ be the smallest ordinal such that $N_\tau = N_{\tau+1}$ ($= N_{\tau+2} = \cdots$). Let $N_\tau = N$. $\mathfrak{A}/N$ has no non-zero nilpotent ideals and is the smallest ideal in this chain with this property.

Lemma 2.6: $N = \bigcap Q_i$ where Q_i is such that $\mathfrak{A}/Q_i$ has no non-zero nilpotent ideals.

PROOF:

$\mathfrak{A}/N$ has no non-zero nilpotent ideals, so $\bigcap Q_i \subseteq N$. Let Q_i be an ideal such that $\mathfrak{A}/Q_i$ has no non-zero nilpotent ideals. Thus $N_0 \subseteq Q_i$. We will show by transfinite induction that $N_\alpha \subseteq Q_i$. If a is a limit ordinal, $N_a = \sum_{\beta < a} N_\beta \subseteq Q_i$. If a is not a limit ordinal, by induction $N_{a-1} \subseteq Q_i$ and if $N_a \nsubseteq Q_i$, then there is a nilpotent ideal $I/N_{a-1} \subseteq \mathfrak{A}/N_{a-1}$ with $I/N_{a-1} \nsubseteq Q_i/N_{a-1}$, so $I \nsubseteq Q_i$ and $(I + Q_i)/Q_i \neq 0$. Now I is nilpotent and thus so is $I \cap Q_i$, but then $(I + Q_i)/Q_i$ ($\cong I/(I \cap Q_i)$) is nilpotent which contradicts the choice of Q_i. Thus $N_a \subseteq Q_i$ for each ordinal number a. Hence $N \subseteq Q_i$ for each i, and thus $N \subseteq \bigcap Q_i$.

Theorem 2.15: The prime radical of $\mathfrak{A}$ is equal to N.

PROOF:

$Q(\mathfrak{A})$ is the intersection of all prime ideals of $\mathfrak{A}$. If P is a prime ideal, then $\mathfrak{A}/P$ is a prime ring and hence Q-semisimple. Thus by Theorem 2.13, $\mathfrak{A}/P$ has no non-zero nilpotent ideals. So by the preceeding lemma, $N \subseteq P$. Thus $N \subseteq Q(\mathfrak{A}) = \bigcap P$.

Also, if Q_i is such that $\mathfrak{A}/Q_i$ has no non-zero nilpotent ideals, then by Theorem 2.13, $\mathfrak{A}/Q_i$ is Q-semisimple and then by Theorem 2.8, $Q_i \supseteq Q(\mathfrak{A})$. Thus $Q(\mathfrak{A}) \subseteq \bigcap Q_i = N$.

Corollary 2.4: The prime radical of $\mathfrak{A}$ contains all nilpotent ideals of $\mathfrak{A}$.

PROOF:

By definition, each nilpotent ideal of $\mathfrak{A}$ is contained in $N_0 \subseteq Q(\mathfrak{A})$.

Corollary 2.5: If $\mathfrak{A}$ possesses a maximal nilpotent ideal, $W(\mathfrak{A})$, then $Q(\mathfrak{A}) = W(\mathfrak{A})$.

PROOF:

$W(\mathfrak{A}) = N_0$ and $\mathfrak{A}/N_0$ has no non-zero nilpotent ideals, that is, $N_1 = N_0 = Q(\mathfrak{A})$. So $W(\mathfrak{A}) = Q(\mathfrak{A})$.

Chapter 3. The Levitski Radical

In this chapter, we show first that a finitely generated standard ring is nilpotent if and only if it is solvable. This implies the existence of a maximal locally nilpotent ideal called the Levitski radical. Thus local nilpotence is a radical property in the sense of chapter 1. We characterize rings with zero Levitski radical and show the relationship between the Levitski and prime radicals. Finally, it is shown that the maximal locally nilpotent ideal contains all locally nilpotent one-sided ideals.

Let X be an ordered subset of a standard ring $\mathfrak{A}$, and let $\bar{X}$ be the set of all words generated by X. If $a \in \bar{X}$, then let deg a be the X-length of a. Order $\bar{X}$ as follows: If $a, a' \in \bar{X}$, then $a \prec a'$ if

1) deg a < deg a'; or

2) deg a = deg a', $a = a_1a_2$' $a' = a_1'a_2'$, and $a_1 \prec a_1'$ or $a_1 = a_1'$ and $a_2 \prec a_2'$.

e.g. $((x_1x_2)x_3)x_4 \succ (y_1y_2)(y_3y_4)$ for all x_i, y_i in X since deg $(x_1x_2)x_3 \succ$ deg (y_1y_2). Let Y be the set of right and left multiplications by elements of $\bar{X}$, that is, $Y = \{ R_x, L_x \mid x \in \bar{X}$ and let $\bar{Y}$ be the set of all (associative) words generated by Y. For $a \in \bar{X}$, let T_a denote either R_a or L_a. If $W = T_{y_1}T_{y_2}T_{y_3}\cdots T_{y_n}$ is an element of $\bar{Y}$, then deg $W = \sum \deg y_i$ and $t(W) = n$ = the T-length of W. Order Y and $\bar{Y}$ as follows:

1) If $y_1 < y_2$ are elements of $\bar{X}$, then $R_{y_1} < L_{y_1} < R_{y_2}$;

2) If $W = T_{y_1}T_{y_2}\cdots T_{y_n}$ and $W' = T_{z_1}T_{z_2}\cdots T_{z_m}$, then $W < W'$ if

a) $n < m$, or

b) $m = n$, $T_{y_1} = T_{z_1} = T_{z_1}$, $T_{y_2} = T_{z_2}$, $\cdots$, $T_{y_k} = T_{y_{k+1}}$ and $T_{y_{k+1}} < T_{z_{k+1}}$ for some $k = 0, 1, 2, \cdots, n-1$.

A word W in $\bar{Y}$ is normal if $W = T_{y_1}T_{z_1}T_{y_2}T_{z_2}\cdots T_{y_n}T_{z_n}$ where $T_{y_1} < T_{y_2} < \cdots < T_{y_n}$ and $T_{z_1} < T_{z_2} < \cdots < T_{z_n}$ (possibly T_{y_1} does not appear.) A word W in $\bar{Y}$ can be normalized if $\pm 2^k W$ can be written as a sum of normal words each of degree equal to deg W, for some non-negative integer k.

For the following, let X be a generating set for $\mathfrak{A}$.

<u>Lemma 3.1</u>: If a' is a product of at least 2n + 1 elements of X, then $a' = a\sum W_i$ for some $a \in \mathfrak{A}$, $W_i \in \bar{Y}$ where $t(W_i) \geq n$.

<u>PROOF</u>:

It is possible to choose a such that a' = aW where deg W = 2n. Thus it suffices to show that if b is a product of three elements of X, then Tb can be written as a combination of words of T-length 2 or 3, each of degree equal to deg b.

Rewriting identities (9) and (10), we have:

$R_{x(yz)} = R_xR_{yz} + R_y(R_{xz} - R_xR_z) + R_z(R_{xy} - R_xR_y)$ and

$L_{(xy)z} = L_zL_{xy} + L_x(L_{yz} - L_zL_y) + L_y(L_{xz} - L_zL_x)$.

From (11) we obtain: $L_{xy} - R_{xy} = (L_x - R_x)R_y + (L_y - R_y)L_x$.

Replacing y by yz we obtain:

$L_{x(yz)} = R_{x(yz)} + (L_x - R_x)R_{yz} + (L_{yz} - R_{yz})L_x$ and replacing x by xz:

$R_{(xz)y} = L_{(xz)y} + (L_{xz} - R_{xz})R_y + (L_y - R_y)L_{xz}$.

Thus the first two cases imply the desired result for the remaining two cases.

Lemma 3.2: For every a, b, c $\in \mathfrak{A}$, we have the following:

a) $R_aL_bL_c = -L_cL_bL_a + R_bR_{ca} - R_{b(ca)} + R_aL_{cb} + L_cL_{ab}$;

b) $R_aR_b - R_{ab} = L_aL_b - L_{ba}$;

c) $L_aR_b - R_bL_a = L_{ab} - R_{ab} + R_aR_b - L_bL_a$.

PROOF:

a) This is (2) considered as operating on x with the following substitutions: z = a, y = b, w = c.

b) This is the linearized flexible law.

c) This is (1) operating on z with x = a and y = b.

Lemma 3.3: If $W \in \mathfrak{T}$, then $W = \sum W_i^L + \sum R_{a_j} + \sum S_k$ where W_i^L is a word involving only L_y's and S_k is either a R_yL_z or a L_yR_z.

PROOF:

We shall prove this by induction on the T-length of W.

If $W = R_xR_y$, we use Lemma 3.2b. For all other cases where t(W) equals 1 or 2, the result is immediate.

Now assume the result is true for all words of T-length less than n. Let S denote any word (or sum of words) of T-length less than n, that is, words to which the induction hypothesis may be applied.

Let $W = T_1T_2\cdots T_n$, where $n \geq 3$ and each T_i is either L_{a_i} or R_{a_i}.

CASE 1: $T_{n-1}T_n = L_yL_z$

If each T_i is equal to an L_{x_i}, then we are done immediately. If not, let $T_k = R_a$ be the last right multiplication occuring in W, that is, $T_t = R_d$ implies $t \leq k$, $k < n-1$. Using Lemma 3.2a, we have

$$\begin{aligned} W &= T_1T_2\cdots T_kT_{k+1}T_{k+2}\cdots T_n = \\ &= T_1\cdots T_{k-1}(R_aL_bL_c)T_{k+3}\cdots T_n \\ &= T_1\cdots T_{k-1}(-L_cL_bL_a + R_bR_{ca} - R_{b(ca)} + R_aL_{cb} \\ &\qquad + L_cL_{ab})T_{k+3}\cdots T_n \\ &= T_1\cdots T_{k-1}L_cL_bL_aT_{k+3}\cdots T_n + S \end{aligned}$$

Now repeat the above process on the first term of the above and continue until $k = 0$, that is

$$W = L_{x_1}L_{x_2}\cdots L_{x_n} + S.$$

CASE 2: $T_{n-1}T_n = R_yR_z$

Apply Lemma 3.2b.

$$W = T_1\cdots T_{n-1}(L_yL_z - L_{zy} + R_{yz}) = T_1\cdots T_{n-2}L_yL_z + S.$$

Now apply Case 1.

CASE 3: $T_{n-2}T_{n-1}T_n = R_xR_yL_z$

Apply Lemma 3.2b.

$$\begin{aligned} W &= T_1\cdots T_{n-3}(L_xL_y - L_{yx} + R_{xy})L_z \\ &= T_1\cdots T_{n-3}L_xL_yL_z + S. \end{aligned}$$

Apply Case 1.

CASE 4: $T_{n-2}T_{n-1}T_n = L_xR_yL_z$

Apply Lemma 3.2c.

$$W = T_1\cdots T_{n-3}(R_yR_x + L_{xy} - R_{xy} + R_xR_y - L_yL_x)L_z$$

$= T_1 \cdots T_{n-3}(R_y R_x + R_x R_y - L_y L_x)L_z + S.$

Apply Cases 1 and 3.

CASE 5: $T_{n-2}T_{n-1}T_n = L_x L_y R_z$

Apply Lemma 3.2b.

$W = T_1 \cdots T_{n-3}(R_x R_y - R_{xy} + L_{yx})R_z$

Apply Case 2.

CASE 6: $T_{n-2}T_{n-1}T_n = R_x L_y R_z$

Apply Lemma 3.2c.

$W = T_1 \cdots T_{n-3}(L_y R_x - L_{yx} + R_{yx} - R_y R_x + L_x L_y)R_z$

Apply Cases 2 and 5.

Thus we have considered all possibilities for the final factors of W and have either proven theses cases or reduced them to ones already proven, and therefore the proof is complete.

Lemma 3.4: Any word W in $\overline{Y}$ involving only left multiplications can be normalized.

PROOF:

This follows by induction on t(W) by using:

$L_x L_y L_z = -L_z L_y L_x + L_y L_{xz} + L_x L_{zy} + L_z L_{xy} - L_{(xz)y}$ and

$2L_x L_y L_x = L_y L_{x^2} + 2L_x L_{xy} - L_{x^2 y}$.

These are both merely restatements of identity (10). Note that in the resulting words which are of the same T-length as that we started with, none of the subscripts increase in degree. Thus if $W = L_{a_1} L_{b_1} L_{a_2} L_{b_2} \cdots L_{a_n} L_{b_n}$, suppose that b_N is the largest element of $b_1, b_2, \cdots b_n$ with respect to the ordering of $\overline{X}$. Then using the above identities we obtain $\pm 2^k W = L_{c_1} L_{d_1} \cdots L_{c_n} L_{d_n} + S$ where $c_1, c_2, \cdots, c_n$ is a reordering of $a_1, \cdots, a_n$,

$d_1, \cdots, d_n$ is a reordering of $b_1, \cdots, b_n$, $d_n = b_N$ and S is a sum of words of T-length less than t(W) (perhaps the first word does not appear.) Then the induction hypothesis may be applied to $L_{c_1}L_{d_1}\cdots L_{d_{n-1}}L_{c_n}$. Since d_n is larger than any other d_i, we have the desired result.

<u>Theorem 3.1</u>: Any word in $\bar{Y}$ can be normalized.

<u>PROOF</u>:

Any word of length 1 or 2 is normal, so the result follows from the two preceding lemmas.

<u>Definition 3.1</u>: $\mathfrak{A}_0 = \mathfrak{A}, \mathfrak{A}_1 = \mathfrak{A}_0^3, \cdots \mathfrak{A}_{i+1} = \mathfrak{A}_i^3$. Also, let $\mathfrak{A}_{(0)} = \mathfrak{A}$ and $\mathfrak{A}_{(i+1)} = \mathfrak{A}_{(i)}\mathfrak{A}_{(i)}$. For any integer $k \geq 1$, $\mathfrak{A}^k$ is the set of all sums of elements which are products of k elements of $\mathfrak{A}$. Thus in agreement with Definition 2.8, $\mathfrak{A}$ is nilpotent if there exists a positive integer k such that $\mathfrak{A}^k = 0$. $\mathfrak{A}$ is solvable if there exists a positive integer t such that $\mathfrak{A}_{(t)} = 0$. Clearly, if $\mathfrak{A}$ is nilpotent, it is solvable. Also for each n, $\mathfrak{A}_n \subseteq \mathfrak{A}_{(n)}$.

Note: By Theorem 2.1, each $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}$.

<u>Theorem 3.2</u>: If $\mathfrak{A}$ is a standard ring generated by a finite set X, then for each integer m, there exists an integer f(m), depending on $|X|$, such that $\mathfrak{A}^{f(m)} \subseteq \mathfrak{A}_m$.

<u>PROOF</u>:

By induction on m.

If $m = 1$, let $f(1) = 3$.

Now assume the existence of f(m-1), so that $\mathfrak{A}^{f(m-1)} \subseteq \mathfrak{A}_{m-1}$. Let P_k be the number of distinct words of length k generated by X, and let $M = \sum_1^{f(m-1)} P_k$ and let $N = 2\sum_1^{f(m-1)} kP_k$. Thus M is the number of distinct words of length less than or equal to f(m-1) and N is twice the number of elements of X used in the expression of these M words. Suppose $W \in \bar{Y}$ is a word with $t(W) \geq [2N(M+2) + f(m-1)]$. By inserting parenthesis, write W as $W_0W_1W_2\cdots W_{M+2}$ where $t(W_i) = 2N$, $i = 1, 2, \cdots, M+2$ and $t(W_0) \geq f(m-1)$. Each W_i can be normalized by words of the form $u = T_{y_1}T_{z_1}T_{y_2}T_{z_2}\cdots T_{y_k}T_{z_k}$ where $\deg u = \deg W_i \geq t(W_i) = 2N$. Thus $\sum \deg y_i + \sum \deg z_i \geq 2N$, so either $\sum \deg y_i \geq N$ or $\sum \deg z_i \geq N$. Now $T_{y_1} < T_{y_2} < \cdots < T_{y_k}$ and $T_{z_1} < T_{z_2} < \cdots < T_{z_k}$ and because of the ordering of $\bar{Y}$ each y_i and each z_i can occur at most twice in each sequence. If we assume that each sequence contains only elements of $\mathfrak{A}$ which are of degree less that f(m-1) and that each of these elements actually occurs twice in each sequence, we have $\deg y_i = \sum_1^{f(m-1)-1} 2kP_k < N$ and similarly $\sum \deg z_i < N$, which is a contradiction. Thus either $\deg y_k \geq f(m-1)$ or $\deg z_k \geq f(m-1)$., that is, $y_k \in \mathfrak{A}_{m-1}$ or $z_k \in \mathfrak{A}_{m-1}$. Hence $u = T_{t_1}T_{t_2}\cdots T_{t_s}T_bT_{t_{s+1}}$ with $b \in \mathfrak{A}_{m-1}$ and where $T_{t_{s+1}}$ need not appear. So the subword, $W_1W_2\cdots W_{M+2}$ is a sum of words of the form $\mathcal{J} = 2^{n(\mathcal{J})}T_{t(0,1)}T_{t(0,2)}\cdots T_{t(0,k_0)}T_{b_1}T_{t(1,1)}T_{t(1,2)}\cdots T_{b_2}\cdots T_{b_{M+2}}T_{t(M+2,1)}$ where $t(i, j) \in \mathfrak{A}$, $b_k \in \mathfrak{A}_{m-1}$ and possibly $T_{t(M+2,1)}$ does not appear.

<u>Lemma 3.5</u>: $T_bT_xT_y$, b an element of an ideal B of $\mathfrak{A}$, x, y $\in \mathfrak{A}$ can be written as a sum of words of the form $T_{x'}T_{y'}T_{b'}$, $T_{b'}T_{x'}$, $T_{x'}T_{b'}$, $T_{b'}$ where $x' \in \mathfrak{A}$, $b' \in B$.

<u>PROOF</u>:

This is proven by considering all combinations of left and right multiplications possible in $T_bT_xT_y$. We do this by first considering the case involving only left multiplications or only left multiplications. This is done by using (9) and (10).

$$R_xR_yR_z = -R_zR_yR_x + R_yR_{xz} + R_xR_{yz} + R_zR_{yx} - R_{y(xz)}$$
$$L_xL_yL_z = -L_zL_yL_x + L_yL_{xz} + L_xL_{zy} + L_zL_{xy} - L_{(xz)y}.$$

We check the remaining cases by reducing them to ones already considered by means of the identities used in the proof of Cases 3-6 in Lemma 3.3.

Now apply this lemma repeatedly to J, starting with $T_{b_{M+1}}T_{t(M+1,1)}T_{t(M+1,2)}$, so that J may be written as a sum of words of the type:

I. $J' = ST_bT_{b'}S'$; or

II. $J'' = ST_{b_1}T_{s_1}T_{b_2}T_{s_2}\cdots T_{b_{M+1}}T_{s_{M+1}}T_{b_{M+2}}S'$

where S, $S' \in \overline{Y}$ and b, b', $b_k \in \mathfrak{A}_{m-1}$, $s_k \notin \mathfrak{A}_{m-1}$. Thus $W = W_0(W_i' + W_j'')$ where W_i' are of type I and W_j'' are of type II.

Let J be a word of type II. $J = ST_{b_1}T_{s_1}\cdots T_{b_{M+2}}S'$, $s_i \notin \mathfrak{A}_{m-1}$. By the choice of M and since $\mathfrak{A}^{f(m-1)} \subseteq \mathfrak{A}_{m-1}$, there are less than M possible distinct choices for the s_i, so two of the s_j must be equal.

<u>Lemma 3.6</u>: $T_{b'}T_sT_bT_{s'}T_{b''} = T_{b'}T_{s'}T_bT_sT_{b''}$ plus words

of type I.

PROOF:

Similar to the proof of Lemma 3.5 by considering possibilities for $T_s T_b T_{s'}$.

Lemma 3.7: $T_{b'} T_s T_b T_s T_{b''}$ is a sum of words of type I.

PROOF:

Similar to the above.

Applying the last two lemmas and the previous comment about equality of two s_k's, we see that $W = W_0(\sum W_i')$, where each W_i' is of type I.

Now, let $f(m) = [2(2N(M+2) + f(m-1))] + 1$ and let $a' \in \mathfrak{A}^{f(m)}$. Then by Lemma 3.1, a' is a sum of words of the form aW where $t(W) \geq 2N(M+2) + f(m-1)$. Now applying the above, a' is a sum of words of the form aW_0W_i' where the W_i' are of type I and $t(W_0) \geq f(m-1)$. Thus $aW_0 \in \mathfrak{A}_{m-1}$. Now let $W_i' = ST_bT_{b'}S'$, b, $b' \in \mathfrak{A}_{m-1}$. Since $\mathfrak{A}_{m-1}$ is an ideal, $(aW_0)S \in \mathfrak{A}_{m-1}$, so $a' = \sum aW_0W_i' \in \mathfrak{A}_{m-1}T_bT_{b'}S' \subseteq ((\mathfrak{A}_{m-1}\mathfrak{A}_{m-1})\mathfrak{A}_{m-1} + \mathfrak{A}_{m-1}(\mathfrak{A}_{m-1}\mathfrak{A}_{m-1})S' \subseteq \mathfrak{A}_m S' \subseteq \mathfrak{A}_m$.

Thus $\mathfrak{A}^{f(m)} \subseteq \mathfrak{A}_m$.

Theorem 3.3: If $\mathfrak{A}$ is a finitely generated standard ring, then $\mathfrak{A}$ is nilpotent if and only if $\mathfrak{A}$ is solvable.

PROOF:

$$\mathfrak{A}^{f(m)} \subseteq \mathfrak{A}_m \subseteq \mathfrak{A}^{(m)} \subseteq \mathfrak{A}^{2^m}.$$

Definition 3.2: A standard ring $\mathfrak{A}$ is locally nilpotent if every finitely generated subring is nilpotent.

It is locally solvable if every finitely generated subring is solvable. An ideal I of $\mathfrak{A}$ is locally solvable (nilpotent) if it is locally solvable (nilpotent) when considered as a ring.

Lemma 3.8: Let $\mathfrak{A}$ be a finitely generated standard ring. Then for any m, $\mathfrak{A}_m$ is finitely generated.

PROOF:

It suffices to show that $\mathfrak{A}_1$ is finitely generated, for then the result follows easily by induction.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite generating set for $\mathfrak{A}$. By Theorem 3.2, there is an integer f(2) such that $\mathfrak{A}^{f(2)} \subseteq \mathfrak{A}_2 = (\mathfrak{A}_1\mathfrak{A}_1)\mathfrak{A}_1 + \mathfrak{A}_1(\mathfrak{A}_1\mathfrak{A}_1)$. Now let Y be the set of words in $\mathfrak{A}_1$ of X-degree less than f(2). Since X is finite, Y is a finite subset of $\mathfrak{A}_1$. Also, if $a \in \mathfrak{A}_1$, $a = u + v + \cdots + w$ is a finite sum of X-words of $\mathfrak{A}_1$. Suppose deg $u \geq f(2)$, then by the choice of f(2), $u \in \mathfrak{A}_2$, so $u = \sum (r_i s_i)t_i + \sum r_i'(s_i't_i')$, $r_i, s_i, t_i, r_i', s_i'\ t_i' \in \mathfrak{A}_1$ and $\deg r_i + \deg s_i + \deg t_i = \deg r_i' + \deg s_i' + \deg t_i' = \deg u$. Continuing this process, we see that a is an element of the subring of $\mathfrak{A}_1$ generated by the elements of $\mathfrak{A}_1$ which are of X-length less than f(2), that is, $\mathfrak{A}_1$ is finitely generated.

Theorem 3.4: If I is locally solvable ideal of $\mathfrak{A}$ and $\mathfrak{A}/I$ is locally solvable, then $\mathfrak{A}$ is locally solvable.

PROOF:

Let B be a finitely generated subring of $\mathfrak{A}$. $\overline{B} =$

$(B + I)/I$ is a finitely generated subring of $\mathfrak{A}/I$ and hence solvable. Thus, there is an N so that $(\overline{B})_N = \overline{0}$, that is $B_N \subseteq I$. By the preceding lemma, B_N is a finitely generated subring of the locally solvable ideal I. Thus B_N is solvable, so there exists an M so that $(B_N)_M = 0$. Thus $B_{N+M} = 0$.

<u>Corollary 3.1</u>: If I is a locally nilpotent ideal and $\mathfrak{A}/I$ is locally nilpotent, then $\mathfrak{A}$ is locally nilpotent.

<u>Theorem 3.5</u>: The sum of two locally solvable ideals is a locally solvable ideal.

<u>PROOF</u>:

Let I and I' be locally solvable ideals. $I/(I \quad I')$ is the homomorphic image of the locally solvable ring I and hence is locally solvable. Thus $(I + I')/I'$ $(\cong I/I \cap I')$ is locally solvable. Since I' is locally solvable, by the preceding Theorem, we have that $I + I'$ is locally solvable.

<u>Corollary 3.2</u>: A finite sum of locally solvable ideals is locally solvable.

<u>Theorem 3.6</u>: The sum of all locally solvable ideals is the unique maximal locally solvable ideal of $\mathfrak{A}$.

<u>PROOF</u>:

Let $\{I_\alpha\}$ be the collection of locally solvable ideals of $\mathfrak{A}$. Let $L(\mathfrak{A}) = \sum I_\alpha$. Clearly, if $L(\mathfrak{A})$ is locally solvable, then it is the unique maximal such ideal of $\mathfrak{A}$. Let A be a finitely generated subring of $L(\mathfrak{A})$. Since A is finitely generated, $A \subseteq \sum_1^n I_{a_i}$ for some choice of the a_i. But by the above, $\sum_1^n I_{a_i}$ is locally solvable. Thus A is solvable. Therefore, $L(\mathfrak{A})$ is a locally solvable ideal.

Definition 3.3: The maximal locally solvable (nilpotent) ideal of a standard ring $\mathfrak{A}$, $L(\mathfrak{A})$, is called the Levitski radical of $\mathfrak{A}$. $\mathfrak{A}$ is L-semisimple if $L(\mathfrak{A}) = 0$.

Theorem 3.7: The quotient ring, $\mathfrak{A}/L(\mathfrak{A})$ is L-semisimple, that is, $L(\mathfrak{A}/L(\mathfrak{A})) = 0$.

PROOF:

Suppose that $I/L(\mathfrak{A})$ is a locally nilpotent ideal of $\mathfrak{A}/L(\mathfrak{A})$. Since $L(\mathfrak{A})$ is locally nilpotent, I is locally nilpotent by Corollary 3.1. Hence by the maximality of $L(\mathfrak{A})$, $I \subseteq L(\mathfrak{A})$ or $I/L(\mathfrak{A}) = 0$.

Corollary 3.3: Local nilpotence is a radical property.

PROOF:

The only part of the definition given in Chapter 1 which we have not shown is the easily verified fact that the homomorphic image of a locally nilpotent ideal is locally nilpotent.

Corollary 3.4: If $\mathfrak{A}/I$ is L-semisimple, then $I \supseteq L(\mathfrak{A})$.

PROOF:

Since local nilpotence is a radical property, this follows from Theorem 1.1.

Theorem 3.8: The Levitski radical of a standard ring

$\mathfrak{A}$ is equal to the intersection of all prime ideals P_a such that $\mathfrak{A}/P_a$ is L-semisimple.

PROOF:

By the preceding corollary, $L(\mathfrak{A})$ is contained in each P_a, so $L(\mathfrak{A}) \subseteq \bigcap P_a$.

Now assume that $x \notin L(\mathfrak{A})$. We will find a prime ideal P, such that $\mathfrak{A}/P$ is L-semisimple and $x \notin P$. By the maximality of $L(\mathfrak{A})$ and since $x \notin L(\mathfrak{A})$, (x) is not locally solvable, so there is a finitely generated subring $S \subsetneq (x)$ with S not solvable. Now let $\mathcal{S}$ be the collection of all ideals Q such that $S_m \nsubseteq Q$ for all m. Since $(0) \in \mathcal{S}$, $\mathcal{S}$ is non-empty and to see that Zorn's Lemma may be applied, let $A_1 \subseteq A_2 \subseteq \cdots$ be a chain of elements of $\mathcal{S}$. Let $A = \bigcup A_i$. A is an ideal and if $S_m \subseteq A$, then since S_m is finitely generated, $S_m \subseteq A_k$ for some k, which is a contradiction. Thus $A \in \mathcal{S}$. Now let Q be a maximal element of $\mathcal{S}$. $S_0 = S \subseteq (x)$, $x \notin Q$.

We claim that Q is a prime ideal, for if not, there are ideals A and B with $A \nsubseteq Q$ and $B \nsubseteq Q$ but $A*B \subseteq Q$. Then $Q \subsetneq A + Q = A'$ and $Q \subsetneq B + Q = B'$ so by the maximality of Q, $A', B' \notin \mathcal{S}$. Thus for some m and n, $S_m \subseteq A'$ and $S_n \subseteq B'$. Without loss of generality, $m <$ n so $S_n \subseteq S_m \subseteq A'$ and thus $S_n \subseteq A' \cap B'$. Now $S_{n+1} = (S_n)^3 = S_n * S_n \subseteq A'*B' \subseteq (A*B) + Q \subseteq Q$ which contradicts the choice of Q.

We now claim that $\mathfrak{A}/Q$ is L-semisimple. If not, let $(0) \neq \bar{N} = N/Q$ be a locally solvable ideal of $\mathfrak{A}/Q$.

The maximality of Q means that $S_t \subsetneq N$ for some t. S_t is finitely generated by Lemma 3.8 and hence $(S_t + Q)/Q \subsetneq N/Q = \overline{N}$ is finitely generated and thus solvable. So for some m, $(S_t + Q/Q)_m = \overline{0}$, that is, $(S_t + Q)_m \subseteq Q$ so $(S_t)_m \subseteq Q$ or $S_{t+m} \subseteq Q$ which is a contradiction.

Corollary 3.5: For a standard ring $\mathfrak{A}$, the prime radical, $Q(\mathfrak{A})$, is contained in the Levitski radical $L(\mathfrak{A})$.

PROOF:

Since $Q(\mathfrak{A})$ is the intersection of all prime ideals, the result follows from the above.

Corollary 3.6: $\mathfrak{A}$ is L-semisimple if and only if it is a subdirect sum of prime L-semisimple rings.

PROOF:

Theorem 1.2.

Lemma 3.9: If $\mathfrak{A}$ is L-semisimple, then $\mathfrak{A}$ contains no non-zero locally nilpotent one-sided ideals.

PROOF:

Suppose $A \neq 0$ is a locally nilpotent one-sided ideal of $\mathfrak{A}$. Since $L(\mathfrak{A}) = 0$, $\mathfrak{A}$ is a subdirect sum of $\mathfrak{A}/P_a$ where $\mathfrak{A}/P_a$ is L-semisimple and prime with $\cap P_a = 0$. By Theorem 2.14, each $\mathfrak{A}/P_a$ is either associative or Jordan. Now, $(A + P_a)/P_a$ is a locally nilpotent one-sided ideal of $\mathfrak{A}/P_a$. If $\mathfrak{A}/P_a$ is Jordan, then $A + P_a/P_a$ is a locally nilpotent two-sided ideal of the L-semisimple ring $\mathfrak{A}/P_a$. Thus $A \subseteq P_a$. If $\mathfrak{A}/P_a$ is associative, then it contains no locally nilpotent one-sided ideals, [4, p. 127], so $A \subseteq P_a$. Thus for each a, $A \subseteq P_a$ or $A \subseteq \bigcap P_a = 0$.

Theorem 3.9: The Levitski radical, $L(\mathfrak{A})$ contains all locally nilpotent one-sided ideals of $\mathfrak{A}$.

PROOF:

If A is a locally nilpotent one-sided ideal of $\mathfrak{A}$, then $(A + L(\mathfrak{A}))/L(\mathfrak{A})$ is a locally nilpotent one-sided ideal of the L-semisimple ring $\mathfrak{A}/L(\mathfrak{A})$ so $(A + L(\mathfrak{A})/L(\mathfrak{A}) = 0$, or $A \subseteq L(\mathfrak{A})$.

Chapter 4. The Jacobson-MacCrimmon Radical and the Jacobson-Brown Radical

In this chapter, we obtain two generalizations for the Jacobson radical of associative rings. The first follows that for Jordan rings given by MacCrimmon [6], and is true for any non-commutative Jordan ring. The second is that given by Brown [3] for any non-associative ring. We show that the possibility illustrated by Brown for any arbitrary non-associative ring, the inequality of the radical defined in terms of right ideals with that defined by left ideals, cannot hold in a standard ring. Finally, the position of each of these radicals in relation to other radicals is shown.

Definition 4.1: If $\mathfrak{A}$ is a non-commutative Jordan ring with identity, 1, then an element $a \in \mathfrak{A}$ is regular with inverse $b \in \mathfrak{A}$ if $ab = ba = 1$, and $(a, a, b) = 0$.

Note that is a is regular according to the above, then $a^2b = (a, a, b) + a(ab) = a \cdot 1 = a$, and $ba^2 = (ba)a - (b, a, a) = 1 \cdot a + (a, a, b) = a$. Thus this definition corresponds to the definition given by MacCrimmon [8] where he showed:

Theorem 4.1: If $a \in \mathfrak{A}$, then a is regular with inverse b if and only if a is regular in $\mathfrak{A}^+$ with inverse b.

For each element a of a non-commutative Jordan ring, consider the operator $U_a = R_aL_a + R_a{}^2 - R_{a^2}$. By identity

(11) we also have, $U_a = L_aR_a + L_a^2 - L_{a^2}$. In the commutative Jordan ring $\mathfrak{A}^+$, we have the quadratic operator $U_a^+ = 2R_a^{+2} - R_{a^2}^+$, where R_x^+ is right multiplication by x in $\mathfrak{A}^+$. Since $R_x^+ = \frac{1}{2}(R_x + L_x)$, we have: $U_a^+ = 2R_a^{+2} - R_{a^2}^+ = 2[\frac{1}{4}(R_a + L_a)^2] - \frac{1}{2}(R_{a^2} + L_{a^2}) = \frac{1}{2}(R_a^2 + R_aL_a + L_aR_a + L_a^2 - R_{a^2} - L_{a^2}) = \frac{1}{2}[(R_aL_a + R_a^2 - R_{a^2}) + (L_aR_a + L_a^2 - L_{a^2})] = U_a$. Thus the identities for quadratic operators in Jordan rings are also true in the non-commutative Jordan case.

Theorem 4.2: Let a, b be elements of a non-commutative Jordan ring with 1. Then the following are equivalent:

a) a is regular with inverse b;

b) 1 is in the range of U_a;

c) U_a^{-1} exists;

d) $\frac{1}{2}(ab + ba) = 1$, $bU_a = a$;

e) $bU_a = a$, $b^2U_a = 1$.

PROOF:

This follows since it is true in $\mathfrak{A}^+$ (See Tsai[11]) and each of the stated conditions is valid in $\mathfrak{A}$ if and only if it is valid in $\mathfrak{A}^+$.

Lemma 4.1: Let $\mathfrak{A}$ be a non-commutative Jordan ring with 1. Let a, b $\in \mathfrak{A}$. Then aU_b is regular if and only if both a and b are regular.

PROOF:

Again this is true in $\mathfrak{A}^+$, thus it is true in $\mathfrak{A}$.

Definition 4.2: An element a of $\mathfrak{A}$ is quasi-regular with quasi-inverse b if a + b = ab = ba and (a, a, b).

Lemma 4.2: If $1 \in \mathfrak{A}$, a is quasi-regular with quasi-inverse b if and only if (1 - a) is regular with inverse (1 - b).

PROOF:

$(1-a)(1-b) = 1 - (a + b - ab)$ and $(1-b)(1-a) = 1 - (a + b - ba)$. Thus $(1-a)(1-b) = (1-b)(1-a) = 1$ if and only if $a + b = ab = ba$. Also, $(1-a, 1-a, 1-b) = -(a, a, b)$ since the associator is linear and any associator involving 1 is equal to zero.

Given a standard ring $\mathfrak{A}$, we can embed $\mathfrak{A}$ in a ring with identity, $\mathfrak{A}_1$, as follows: Let $\mathfrak{A}_1 = \mathbb{Z} \oplus \mathfrak{A}$. Define addition component wise and multiplication by $(n,a)(n',a') = (nn', na' + n'a + aa')$. Elements of $\mathfrak{A}_1$ will be denoted by $n + a$ where $n \in \mathbb{Z}$ and $a \in \mathfrak{A}$.

Lemma 4.3: If K is an ideal of $\mathfrak{A}$, $(\mathfrak{A}/K)_1 = (\mathfrak{A}_1/K)$ and $(\mathfrak{A}^+)_1 = (\mathfrak{A}_1)^+$. If $\mathfrak{A}$ is a non-commutative Jordan (standard) ring, then $\mathfrak{A}_1$ is a non-commutative Jordan (standard) ring.

PROOF:

The first isomorphism is given by: $n + (a + K) \rightarrow (n - a) + K$. The second statement is true since both are composed of the same vector space and it can easily be verified that the multiplication is the same. The last part is also easily verified.

Lemma 4.4: If $a \in \mathfrak{A}$ and a is quasi regular in $\mathfrak{A}_1$, then a is quasi-regular in $\mathfrak{A}$.

PROOF:

Since a is quasi-regular in $\mathfrak{A}_1$, 1-a is regular with inverse $x \in \mathfrak{A}_1$. Now $x = n + x'$ where $x' \in \mathfrak{A}$. $1 = (1-a)(n+x') = n + x' - an - ax'$ so $n = 1$ and thus the quasi-inverse of a, $1-x = x'$ is in $\mathfrak{A}$, and so a is quasi-regular in $\mathfrak{A}$.

Theorem 4.3: If $a \in \mathfrak{A}$, then a is quasi-regular in $\mathfrak{A}$ if and only if a is quasi-regular in $\mathfrak{A}^+$.

PROOF:

Since $(\mathfrak{A}_1)^+ = (\mathfrak{A}^+)_1$, the following statements are equivalent:

a is quasi-regular in $\mathfrak{A}$;

a is quasi-regular in $\mathfrak{A}_1$;

(1-a) is regular in $\mathfrak{A}_1$;

(1-a) is regular in $(\mathfrak{A}_1)^+ = (\mathfrak{A}^+)_1$;

a is quasi-regular in $\mathfrak{A}^+{}_1$;

a is quasi-regular in $\mathfrak{A}^+$.

Definition 4.3: An ideal is quasi-regular if each of its elements is quasi-regular.

Theorem 4.4: No non-zero idempotent of $\mathfrak{A}$ is quasi-regular. Every nilpotent element is quasi-regular. If for some n, z^n is quasi-regular, then z is quasi-regular. The sum of two quasi-regular ideals is a quasi-regular ideal.

PROOF:

This follows from the preceding theorem and the fact that they are all known to be true in a Jordan ring [11].

Corollary 4.1: The sum of all quasi-regular ideals of a standard ring $\mathfrak{A}$ is the unique maximal quasi-regular ideal.

Definition 4.4: The maximal quasi-regular ideal of will be denoted by JM($\mathfrak{A}$) and called the Jacobson-MacCrimmon radical. $\mathfrak{A}$ is JM-semisimple if JM($\mathfrak{A}$) = 0.

Lemma 4.5: If $1 \in \mathfrak{A}$ and K is a quasi-regular ideal of $\mathfrak{A}$, suppose that $\bar{w}$ is a quasi-regular element of $\bar{\mathfrak{A}} = \mathfrak{A}/K$. Let w be any pre-image of $\bar{w}$ in $\mathfrak{A}$. Then w is quasi-regular in $\mathfrak{A}$,

PROOF:

Since $\bar{w}$ is quasi-regular, $(\overline{1-w})$ is regular in so there exists $\bar{z} \in \bar{\mathfrak{A}}$ with $(\overline{1-w})U_{\overline{1-z}} = \bar{1}$ or equivalently $(1-w)U_{1-z} = 1 - y$ for some $y \in K$. Since K is a quasi-regular ideal, 1-y is regular and thus by Lemma 4.1, both 1-w and 1-z are regular, thus w is quasi-regular.

Theorem 4.5: $\mathfrak{A}$/JM($\mathfrak{A}$) is JM-semisimple.

PROOF:

Let $\bar{A}$ be a quasi-regular ideal of $\bar{\mathfrak{A}} = \mathfrak{A}$/JM($\mathfrak{A}$). Let $A \subseteq \mathfrak{A}$ be the pre-image of $\bar{A}$. We will show that A $\subseteq$ JM($\mathfrak{A}$) and thus $\bar{A} = 0$. Clearly, A is an ideal so it suffices to show that A is quasi-regular. But $\bar{A}$ = (A + JM($\mathfrak{A}$))/JM($\mathfrak{A}$) is quasi-regular in $\mathfrak{A}$/JM($\mathfrak{A}$) and hence in $(\mathfrak{A}/JM(\mathfrak{A}))_1 \cong \mathfrak{A}_1/JM(\mathfrak{A})$. So by the preceding lemma, A is quasi-regular in $\mathfrak{A}_1$ and thus by Lemma 4.4, quasi-regular in $\mathfrak{A}$.

Corollary 4.2: Quasi-regularity is a radical property of standard rings.

PROOF:

This follows from Corollary 4.1, Theorem 4.5, and the easily verified fact that the homomorphic image of a quasi-regular element is quasi-regular. Thus all conditions for a radical property given in Chapter 1 are satisfied.

Theorem 4.6: If $\mathfrak{A}/I$ is JM-semisimple, then $I \supseteq$ JM $(\mathfrak{A})$.

PROOF:

Corollary 4.2 and Theorem 1.1.

We now turn to the generalization of the Jacobson radical given by Brown [3]. We begin by listing some of his results.

Definition 4.5:

a) Let I be a right ideal of $\mathfrak{A}$. $I' = \{a \in \mathfrak{A} \mid (a) \subseteq I\}$. $I : \mathfrak{A} = \{a \in \mathfrak{A} \mid \mathfrak{A}a \subseteq I\}$.

b) If $e \in \mathfrak{A}$, Q(e) is the minimal right ideal containing $\{ex - x \mid x \in \mathfrak{A}\}$.

c) An element a of $\mathfrak{A}$ is Brown quasi-regular (B.q.r) if $a \in Q(a)$.

d) An ideal (right, left, or two-sided) is Brown quasi-regular if each element is B.q.r.

e) $JB(\mathfrak{A}) = \{a \in \mathfrak{A} \mid (a)$ is B.q.r.$\}$.

f) A right ideal I is modular if there is an element $e \in \mathfrak{A}$ with $ex - x \in I$ for each $x \in \mathfrak{A}$.

g) $\mathfrak{A}$ is primitive if $\mathfrak{A}$ contains a modular maximal right ideal M such that $M' = 0$.

Theorem 4.7:

a) If I is a right ideal of $\mathfrak{A}$, then I' is the largest two-sided ideal of $\mathfrak{A}$ contained in I.

b) If I is a right ideal and $I:\mathfrak{A} \subseteq I$, then $I' = I:\mathfrak{A}$.

c) If I is a modular right ideal, then $I:\mathfrak{A} \subseteq I$ and hence $I:\mathfrak{A} = I'$.

d) $JB(\mathfrak{A}) = \bigcap M'$ such that M is a modular maximal right ideal of $\mathfrak{A}$.

e) $JB(\mathfrak{A})$ is the maximal B.q.r. ideal of $\mathfrak{A}$.

f) $JB(\mathfrak{A}/JB(\mathfrak{A})) = 0$.

g) $JB(\mathfrak{A}) = 0$ if and only if $\mathfrak{A}$ is isomorphic to a subdirect sum of primitive rings.

Theorem 4.8: Brown quasi-regularity is a radical property.

Corollary 4.3: If $JB(\mathfrak{A}/I) = 0$, then $JB(\mathfrak{A}) \subseteq I$.

PROOF:

Theorems 1.1 and 4.9.

Theorem 4.9: A primitive standard ring is either primitive associative or a simple Jordan ring with indentity.

PROOF:

Kleinfeld [7] showed that a primitive standard ring is either associative or commutative. If $\mathfrak{A}$ is associative, then it is primitive associative. If is commutative, it is thus Jordan and primitivity implies that $\mathfrak{A}$ contains a modular maximal right ideal M containing no non-zero two-sided ideals. But under commutativity the concepts of one-sided and two-sided ideals are equivalent and thus (0) is a modular maximal ideal of $\mathfrak{A}$. (0) being maximal means that has no non-trivial ideals, that is, $\mathfrak{A}$ is simple. (0) being modular implies that there exists $e \in \mathfrak{A}$ such that $ex - x = 0$ for all $x \in \mathfrak{A}$, that is, $ex = x = xe$, or e is an identity for .

As Brown states, a similar theory can be derived in terms of left ideals. He presents an example to show that for an arbitrary non-associative ring, the left radical and the right radical may not be equal. We now show that this is not possible for standard rings.

Theorem 4.10: A right (left) JB-semisimple standard ring is a subdirect sum of right (left) primitive standard rings. A right (left) primitive standard ring is either associative or Jordan.

PROOF:

The statement with "right" is merely Kleinfeld's result and the proof for "left" is similar.

Lemma 4.6: Let $\mathfrak{A}$ be a standard ring. $\mathfrak{A}$ is right JB-semisimple if and only if it is left JB-semisimple.

PROOF:

The proof follows from the facts that for commutative rings the two concepts coincide and that for associative rings the result is known [5, p. 13]. Thus if $\mathfrak{A}$ is right JB-semisimple, it is a subdirect sum of commutative and associative right primitive rings which are both right and left JB-semisimple. Thus $\mathfrak{A}$ is a subdirect sum of left JB-semisimple rings and hence left semisimple itself. Similarly, if $\mathfrak{A}$ is left semisimple, it is right JB-semisimple.

Theorem 4.11: Let JB_r be the radical in terms of right ideals and let JB_l be the radical in terms of left ideals, then $JB_r = JB_l$.

PROOF:

$\mathfrak{A}/JB_r$ is right JB-semisimple and thus by the above lemma, it is left JB-semisimple. Hence $JB_r \supseteq JB_l$ by Corollary 4.3. Similarly, $JB_l \supseteq JB_r$. Thus equality holds.

We now show the position of these two generalizations in relation to the radicals we have previously considered. Recall that the nil radical of $\mathfrak{A}$ is the largest ideal of $\mathfrak{A}$ containing only nilpotent elements. Also the Behrens radical is the ideal such that $\mathfrak{A}$ is a subdirect sum of rings in which every ideal contains a non-zero idempotent. Finally, the Smiley radical of a ring is zero if and only if the ring is a subdirect sum of simple rings with identity.

Theorem 4.12: $JM(\mathfrak{A})$ contains $N(\mathfrak{A})$, the nil radical of $\mathfrak{A}$.

PROOF:

By Theorem 4.4, each nilpotent element is quasi-regular. Thus $N(\mathfrak{A})$ is a quasi-regular ideal and contained in the maximal quasi-regular ideal, $JM(\mathfrak{A})$.

<u>Theorem 4.13</u>: $JM(\mathfrak{A})$ is contained in the Behrens radical, $B(\mathfrak{A})$.

<u>PROOF</u>:

We first assume that $\mathfrak{A}$ is B-semisimple. Then is a subdirect sum of $\mathfrak{A}_s$ where $\mathfrak{A}_s$ has a non-zero idempotent generating its minimal ideal. Thus each ideal of $\mathfrak{A}_s$ contains a non-zero idempotent and thus cannot be quasi-regular. Thus $JM(\mathfrak{A}_s) = 0$, and hence $JM(\mathfrak{A}) = 0$.

Now let $\mathfrak{A}$ be an arbitrary standard ring. $\mathfrak{A}/B(\mathfrak{A})$ is B-semisimple and thus by the above, $JM(\mathfrak{A}/B(\mathfrak{A})) = 0$. So by Theorem 4.6, $JM(\mathfrak{A}) \subseteq B(\mathfrak{A})$.

<u>Theorem 4.15</u>: $JB(\mathfrak{A})$ contains $JM(\mathfrak{A})$.

<u>PROOF</u>:

$JM(\mathfrak{A})$ is a quasi-regular ideal and we will show that a quasi-regular element is B.q.r., and thus the result will follow from the maximality of $JB(\mathfrak{A})$.

Let a be a quasi-regular element of $\mathfrak{A}$. Then there exists an element $b \in \mathfrak{A}$, with $a + b - ab = 0$, that is $a = ab - b$. Thus a is an element of $Q(a)$, the right ideal generated by $\{ax - x \mid x \in \mathfrak{A}\}$. Thus a is B.q.r.

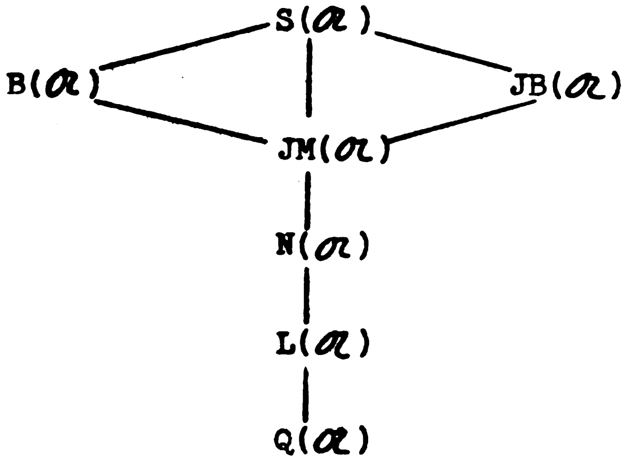
<u>Theorem 4.16</u>: $S(\mathfrak{A})$ contains $JB(\mathfrak{A})$.

PROOF:

First, let $\mathfrak{A}$ be S-semisimple, that is, $S(\mathfrak{A}) = 0$. Then $\mathfrak{A}$ is a subdirect sum of $\mathfrak{A}_\alpha$, simple rings with identity. Since $\mathfrak{A}_\alpha$ is simple, $JB(\mathfrak{A}_\alpha) = 0$ or $\mathfrak{A}_\alpha$. But $Q(1) = (\{1 \cdot x - x\}) = 0$ so 1 is not B.q.r. Thus $JB(\mathfrak{A}_\alpha) = 0$ and each $\mathfrak{A}_\alpha$ is JB-semisimple. Therefore $\mathfrak{A}$ is JB-semisimple.

Now if $\mathfrak{A}$ is an arbitrary standard ring, $JB(\mathfrak{A}/S(\mathfrak{A})) = 0$, and so $JB(\mathfrak{A}) \subseteq S(\mathfrak{A})$.

Thus we have completed all parts of the diagram given in the introduction and reproduced below.



Chapter 5. Chain Conditions

In this chapter, we study the relationship of the radicals we have previously considered when the ring satisfies a chain condition. $\mathfrak{A}$ is said to satisfy the DCC if every descending chain of right ideals of $\mathfrak{A}$ has a minimal element. $\mathfrak{A}$ is said to satisfy the strong DCC [9] if every subring of $\mathfrak{A}$ satisfies the DCC.

<u>Theorem 5.1</u>: If $\mathfrak{A}$ satisfies the DCC, then $Q(\mathfrak{A}) = L(\mathfrak{A})$.

<u>PROOF</u>:

We have that $Q(\mathfrak{A}) \subseteq L(\mathfrak{A})$.

Now let $\bar{\mathfrak{A}} = \mathfrak{A}/Q(\mathfrak{A})$. $\bar{\mathfrak{A}}$ is Q-semisimple and thus is a subdirect sum of $\mathfrak{A}_\alpha = \mathfrak{A}/P_\alpha$ where $\cap P_\alpha = 0$ and each $\mathfrak{A}_\alpha$ is either prime associative or prime Jordan. Now since $\mathfrak{A}_\alpha$ is Q-semisimple and satisfies the DCC, if $\mathfrak{A}_\alpha$ is either associative [4] or Jordan [12] then $L(\mathfrak{A}_\alpha) = Q(\mathfrak{A}_\alpha) = 0$. Thus $\bar{\mathfrak{A}}$ is a subdirect sum of L-semisimple rings and thus L-semisimple itself, i.e. $L(\bar{\mathfrak{A}}) = L(\mathfrak{A}/Q(\mathfrak{A})) = 0$. So, $L(\mathfrak{A}) \subseteq Q(\mathfrak{A})$. Thus equality holds.

<u>Theorem 5.2</u>: If $\mathfrak{A}$ satisfies the DCC, then $JB(\mathfrak{A}) = S(\mathfrak{A})$.

<u>PROOF</u>:

We have $JB(\mathfrak{A}) \subseteq S(\mathfrak{A})$.

Consider $\mathfrak{A}/JB(\mathfrak{A})$. Since this is Brown semisimple, it is a subdirect sum of simple commutative Jordan rings with 1, and primitive associative rings. But since the DCC is preserved under homomorphisms, each of the subdirect summands satisfies the DCC. Also, a primitive associative ring is a simple ring with 1. Thus $\mathfrak{A}/JB(\mathfrak{A})$ is a subdirect sum of simple rings with 1 and thus by Theorem 1.3b, $S(\mathfrak{A}/JB(\mathfrak{A})) = 0$, or $S(\mathfrak{A}) \subseteq JB(\mathfrak{A})$. Thus equality holds.

Thus if $\mathfrak{A}$ satisfies the DCC we have the following diagram.

$$\begin{array}{c} S(\mathfrak{A}) = JB(\mathfrak{A}) \\ | \\ B(\mathfrak{A}) \\ | \\ JM(\mathfrak{A}) \\ | \\ N(\mathfrak{A}) \\ | \\ L(\mathfrak{A}) = Q(\mathfrak{A}) \end{array}$$

<u>Corollary 5.1</u>: If $\mathfrak{A}$ satisfies the strong DCC, then $L(\mathfrak{A}) = Q(\mathfrak{A})$ and $S(\mathfrak{A}) = JB(\mathfrak{A})$.

<u>PROOF</u>:

This is true since if $\mathfrak{A}$ satisfies the strong DCC every subring of $\mathfrak{A}$, including $\mathfrak{A}$ itself satisfies the DCC. Thus we merely apply the above.

<u>Theorem 5.3</u>: If $\mathfrak{A}$ satisfies the strong DCC, then $B(\mathfrak{A}) = JM(\mathfrak{A}) = N(\mathfrak{A})$.

<u>PROOF</u>:

We have $N(\mathfrak{A}) \subseteq JM(\mathfrak{A}) \subseteq B(\mathfrak{A})$.

Now suppose that $x \notin N(\mathfrak{A})$. We will show that $x \notin B(\mathfrak{A})$. $x \notin N(\mathfrak{A})$ implies that (x) is not a nil ideal so there exists $y \in (x)$ which is not nilpotent. Since $\mathfrak{A}$ is power-associative, $\{y^k\}$ generates an associative subring of $\mathfrak{A}$ which according to our hypothesis satisfies the DCC. But then this subring contains a non-zero idempotent, e, [5, p.22]. Since $e \in (y) \subseteq (x)$ and $B(\mathfrak{A})$ is an ideal which contains no non-zero idempotents, $x \notin B(\mathfrak{A})$. Thus $B(\mathfrak{A}) \subseteq N(\mathfrak{A})$, and the stated equality holds.

Thus if $\mathfrak{A}$ satisfies the strong DCC, we have the following.

$$S(\mathfrak{A}) = JB(\mathfrak{A})$$
$$B(\mathfrak{A}) = JM(\mathfrak{A}) = N(\mathfrak{A})$$
$$L(\mathfrak{A}) = Q(\mathfrak{A})$$

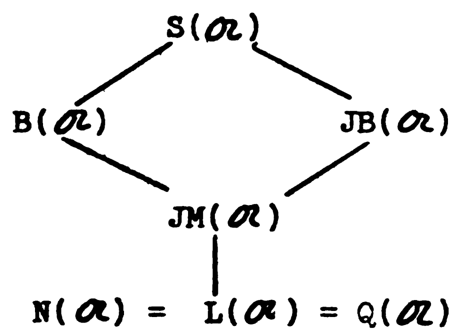
<u>Theorem 5.4</u>: If $\mathfrak{A}$ satisfies an ascending chain condition on subrings, that is, every ascending chain of subrings has a maximal element, then $Q(\mathfrak{A}) = L(\mathfrak{A}) = N(\mathfrak{A})$.

<u>PROOF</u>:

We have $Q(\mathfrak{A}) \subseteq L(\mathfrak{A}) \subseteq N(\mathfrak{A})$.

Let $\bar{\mathfrak{A}} = \mathfrak{A}/Q(\mathfrak{A})$. It suffices to show that $N(\bar{\mathfrak{A}}) = 0$, that is $\bar{\mathfrak{A}}$ has no non-zero nil ideals. Since $\bar{\mathfrak{A}}$ is Q-semisimple, it is a subdirect sum of $\mathfrak{A}_\alpha = \bar{\mathfrak{A}}/P_\alpha$ where $\mathfrak{A}_\alpha$ are either associative or Jordan. Each $\mathfrak{A}_\alpha$ is Q-semisimple and thus possesses no non-zero nilpotent ideals. Let K be a nil ideal of $\bar{\mathfrak{A}}$. $(K + P_\alpha)/P_\alpha$ is a nil ideal of $\mathfrak{A}_\alpha$. If $\mathfrak{A}_\alpha$ is Jordan then $(K + P_\alpha)/P_\alpha$ is nilpotent [14] and thus $K + P = 0$ or $K \subseteq P_\alpha$. If $\mathfrak{A}_\alpha$ is associative then $(K + P)/P$ is nilpotent [4] and so again $K \subseteq P_\alpha$. Thus for each α, $K \subseteq P_\alpha$ or $K \subseteq \bigcap P_\alpha = 0$. Therefore, $N(\bar{\mathfrak{A}}) = N(\mathfrak{A}/Q(\mathfrak{A})) = 0$.

Hence if $\mathfrak{A}$ satisfies an ACC on subrings, we have:



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