PROPERTIES OF 3-MANIFOLDS WHICH ADMIT A FREE CYCLIC GROUP ACTION

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
JEFFREY LYNN TOLLEFSON
1968

This is to certify that the

thesis entitled

PROPERTIES OF 3-MANIFOLDS
WHICH ADMIT A FREE CYCLIC GROUP ACTION

presented by

Jeffrey Lynn Tollefson

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Kyung Whan Kwun
Major professor

Date July 10, 1968

O-169

ABSTRACT

PROPERTIES OF 3-MANIFOLDS
WHICH ADMIT A FREE CYCLIC GROUP ACTION

by Jeffrey Lynn Tollefson

This thesis is a study of the nature of compact 3-manifolds which admit a free action by a finite cyclic group Z_k of order k. Some interesting results are obtained on manifolds M for which there exist a k-sheeted covering map $p: M \longrightarrow M$.

A proper Z_k action is one with the property that a generator of the action is homotopic to the identity. The orbit space M/Z_k is denoted M^*. The main results center around the following theorems.

Theorem 1: Let M be a compact, connected, orientable, irreducible 3-manifold with Bd M either empty or connected. If M admits a proper free Z_k action, for some prime $k \geq 2$, such that $H_1(M^*;Z)$ has no element of order k then M can be fibered over the circle.

Theorem 2: Let $k \geq 2$ be any integer. A closed, connected, non-prime 3-manifold M is a k-sheeted covering of itself if and only if $M = P_3 \# P_3$.

Theorem 3: Let $k \geq 2$ be any integer, and let M be a compact connected 3-manifold with connected boundary. If M covers itself k times then M is a Δ-prime irreducible manifold and Bd M is either $S^1 \times S^1$ or a Klein bottle K.

PROPERTIES OF 3-MANIFOLDS WHICH ADMIT A FREE CYCLIC GROUP ACTION

By

Jeffrey Lynn Tollefson

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1968

ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude to Professor K. W. Kwun for suggesting the problem and for his stimulating guidance during the research.

To Carolee

TABLE OF CONTENTS

INTRODUCTION

This thesis is an investigation of compact connected 3-manifolds which admit a free action by a finite cyclic group. The study was motivated by K. W. Kwun [7] who considered the class of closed connected orientable 3-manifolds (without boundary) which double-cover themselves. Kwun classified all non-prime manifolds in this class and showed that the prime manifolds, with certain technical restrictions, fiber over the circle in the sense of Stallings [18].

3-Manifolds that fiber over the circle have some very nice properties. Neuwirth [12] has shown that closed irreducible 3-manifolds fibering over the circle are completely classified by their fundamental group. Recently Burde and Zieschang [2] have shown a similar result for irreducible 3-manifolds with boundary. In particular, such manifolds are completely classified by their fundamental group and a peripheral system. Moreover, if a 3-manifold M fibers over the circle then it can be obtained from the product $T \times I$ of a compact surface T and the unit interval I by identifying $(T \times 0)$ and $(T \times 1)$ by a homeomorphism h of T. We write $M = T \times I/h$.

Notation and Terminology

A continuous function between two topological spaces will be called a map. A topological group G is said to act effectively on a space M if G is a group of homeomorphisms of M onto itself, $(g,x) \longrightarrow g(x)$ is a map $G \times M \longrightarrow M$, and $g(x) = x$ for all $x \in M$ implies $g = e$, the identity element of G. The orbit of x is the space $G(x) = \{g(x) | g \in G\}$. Two orbits are either equal or disjoint so that M is partitioned by its orbits. Let M/G denote the set of orbits and $\pi : M \longrightarrow M/G$ the projection assigning each point of M to its orbit. The orbit space of M with respect to G is M/G together with the quotient topology, i.e. the largest topology such that π is continuous. The subgroup $G_x = \{g \in G | g(x) = x\}$ is the isotropy group at x. An effective action of G on M is said to be free if $G_x = \{e\}$ for all $x \in M$.

An n-manifold (n-dimensional manifold, $n \leq 3$) is a connected separable metric space, each of whose points has a closed neighborhood homeomorphic to a closed n-cell. We consider both manifolds with boundary and manifolds without boundary. A closed n-manifold is a compact n-manifold without boundary. In view of the work of Bing [1] and Moise [11] we may suppose without any loss of generality that an n-manifold ($n \leq 3$) has a combinatorial triangulation whenever convenient.

We say that a set Q in a manifold M is <u>tame</u> if there is a triangulation of M and a homeomorphism of M onto itself that throws Q onto a subpolyhedron of M. A 3-manifold M is <u>irreducible</u> if every tame 2-sphere in M bounds a 3-cell. The <u>connected sum</u> $M_1 \# M_2$ of two closed 3-manifolds is obtained by removing a tame open 3-cell from each and then matching the boundaries of the resulting spaces by a homeomorphism (orientation reversing if both manifolds are orientable). The sphere S^3 serves as identity element. A 3-manifold is <u>non-trivial</u> if it is not homeomorphic to S^3. A closed non-trivial 3-manifold is <u>prime</u> if it cannot be written as the connected sum of two non-trivial manifolds.

For manifolds with boundary we define a similar operation. The <u>disk sum</u> $M_1 \Delta M_2$ of compact 3-manifolds with connected boundary is obtained by pasting a tame closed 2-cell on the boundary of M_1 onto a tame closed 2-cell on the boundary of M_2 by a homeomorphism. The closed 3-cell D^3 serves as the identity for the Δ operation. A 3-manifold with connected boundary is <u>Δ-prime</u> if it cannot be written as the Δ-sum of two manifolds different from D^3.

CHAPTER I

3-MANIFOLDS WHICH ADMIT FREE Z_k ACTIONS

In this chapter we show that under certain conditions orientable 3-manifolds admitting a free Z_k action fiber over the circle. Z_k will denote a cyclic topological group of order k with the discrete topology.

Lemma 1.1: Let M be a compact orientable irreducible 3-manifold such that $\text{Bd } M$ is either empty or connected. If either $\pi_1(M)$ is infinite or the genus of $\text{Bd } M$ is positive then M is a $K(\pi,1)$ space and $\pi = \pi_1(M)$ has no elements of finite order.

Proof: If the genus of $\text{Bd } M$ is greater than zero, it is easy to see from the exact homology sequence for the pair $(M, \text{Bd } M)$ that the rank of $H_1(M;Z)$ is greater than zero. Hence $\pi_1(M)$ is infinite under either hypothesis. But, if $\pi_1(M)$ is infinite, the universal covering space $\tilde{M}$ is not compact. Thus by Poincaré duality we have $H_3(\tilde{M};Z) = H^0_c(\tilde{M};Z) = 0$. By the sphere theorem of Whitehead [20], $\pi_2(M) = 0$ since M is irreducible. The Hurewicz theorem then implies that $\tilde{M}$ is contractible. Therefore M is a $K(\pi,1)$ space, i.e. $\pi_1(M) = \pi$ and $\pi_i(M) = 0$ for $i \geq 2$. It follows from a theorem due to P. A. Smith [4, page 287] that $\pi_1(M)$ has no elements of finite order.

It will be convenient to introduce some basic concepts now that will be used in this chapter. A <u>bundle</u> ζ is a triple (E,p,B), where $p : E \longrightarrow B$ is a map, with the property that there exists a space D such that for every $b \in B$ there is an open neighborhood U of b in B and a homeomorphism $\phi_U : U \times D \longrightarrow p^{-1}(U)$ with $p\phi_U(u,d) = u$, $u \in U$, $d \in D$. For each $b \in B$, the space $p^{-1}(b)$ is called the <u>fiber</u> of the bundle over b. A <u>bundle map</u> $\bar{h} : \zeta \longrightarrow \zeta'$, where $\zeta' = (E', p', B')$, is a map $\bar{h} : E \longrightarrow E'$ such that $\bar{h}$ carries each fiber in E onto a fiber in E', thus inducing a map $h : B \longrightarrow B'$ such that $p'\bar{h} = hp$. Let $f : B_1 \longrightarrow B$ be a map. The <u>induced bundle</u> of ζ and f, denoted $f^*(\zeta)$, is the bundle (E_1,p_1,B_1), where $E_1 = \{(b,x) \in B_1 \times E : f(b) = p(x)\}$ and p_1 is the map $(b,x) \longrightarrow b$.

Given a topological group G, a <u>free right G-space</u> is a space M together with a free group action $M \times G \longrightarrow M$, defined by $(x,s) \longrightarrow xs$.

<u>Definition 1.2</u>: A <u>principal G-bundle</u> ζ is a bundle (E,p,B) satisfying the following conditions.

(1) E is a free right G-space with a continuous translation function $t : E^* \longrightarrow G$, where E^* is the set $\{(x,xs) \in E \times E : x \in E,\ s \in G\}$ and t is the function with the property that $x\, t(x,x') = x'$.

(2) There is an equivariant bundle isomorphism $\bar{h} : \zeta \longrightarrow (E,\pi,E/G)$ such that $h : B \longrightarrow E/G$ is an homeomorphism.

It follows that each fiber $p^{-1}(b)$ of a principal G-bundle is homeomorphic to G. The induced bundle of a principal G-bundle is also a principal G-bundle. If M admits a free G action, denote the orbit space M/G by M^*. The bundle (M,p,M^*) is a principal G-bundle, where for convenience we now let p denote the projection map. Further properties of bundles are presented in [5] and [19].

Lemma 1.3: Let M be a compact 3-manifold admitting a free Z_k action, where $k \geq 2$ is prime. If $H_1(M^*;Z)$ has no k torsion then there is a bundle map

$$\bar{g} : (M,p,M^*) \longrightarrow (S^1,p',S^1),$$

where p' is the standard k to 1 covering projection of S^1.

Proof: Since $\zeta = (M,p,M^*)$ is a principal Z_k bundle, there is a map $f : M^* \longrightarrow L_\infty$, L_∞ a classifying space for Z_k, such that ζ is the induced bundle of the universal bundle $(S^\infty, p_\infty, L_\infty)$ and f [5]. We will take S^∞ and L_∞ as the following. Consider the commutative diagram

$$\begin{array}{ccccc} S^1 & \subset & S^3 & \subset & S^5 \quad \cdots \\ p_1 \downarrow & & p_3 \downarrow & & p_5 \downarrow \\ L_1 & \subset & L_3 & \subset & L_5 \quad \cdots \end{array}$$

where (S^i,p_i,L_i) is the standard k to 1 covering of the generalized lens space $L_i = L(k;1,\cdots,1)$ by S^i [17, page 88]. S^{i+2} is to be considered as the double suspension of S^i. Define S^∞ and L_∞ to be the unions $\cup S^i$ and $\cup L_i$ respectively, with the weak topology. S^∞ is contractible so L_∞ is a $K(Z_k,1)$ space. $\eta = (S^\infty, p_\infty, L_\infty)$ is a principal Z_k bundle and hence by [19, page 101] is a universal Z_k bundle.

Since L_∞ is a $K(Z_k,1)$ space, the set of homotopy classes $[M^*, L_\infty]$ is in a one to one correspondence with the hom omorphisms of $\pi_1(M^*)$ into Z_k [4, page 198]. The homotopy class of f is completely determined by a hom omorphism $\Phi : \pi_1(M^*) \longrightarrow Z_k$. Φ is onto since ζ is not the trivial bundle. Φ must factor through $H_1(M^*;Z)$ since Z_k is an abelian group. We get the commutative diagram

$$\begin{tikzcd} \pi_1(M^*) \arrow[rr, "\Phi"] \arrow[dr, "\alpha"'] & & Z_k \\ & H_1(M^*;Z) \arrow[ur] & \end{tikzcd}$$

where α is the projection to the abelianization of $\pi_1(M^*)$. $H_1(M^*;Z)$ is finitely generated and has no k torsion by hypothesis. It follows that $H_1(M^*;Z)$ has a free part mapping onto Z_k. Thus α can be factored through Z and we get the commutative diagram

$$\begin{tikzcd} \pi_1(M^*) \arrow[rr, "\Phi"] \arrow[dr, "\beta"'] & & Z_k \\ & Z \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Because S^1 and L_∞ are $K(\pi,1)$ spaces there are maps $c_\gamma : S^1 \longrightarrow L_\infty$ and $g : M^* \longrightarrow S^1$ corresponding to γ and β respectively. We may suppose that f was chosen so that $f = c_\gamma g$. Consider the following diagram.

$$\begin{tikzcd}
E(f^*(\eta)) = M \arrow[rr] \arrow[dr, "\bar{g}"] \arrow[ddd, "p"'] & & S^\infty \arrow[ddd, "p_\infty"] \\
& S^1 = E(c_\gamma^*(\eta)) \arrow[ur] \arrow[d, "p'"] & \\
& S^1 \arrow[dr, "c_\gamma"] & \\
M^* \arrow[ur, "g"] \arrow[rr, "f"'] & & L_\infty
\end{tikzcd}$$

The induced Z_k bundle $c_\gamma^*(\eta)$ is equivalent to the standard k-sheeted covering of S^1. There is a bundle map $\bar{g} : M \longrightarrow S^1$ since $\zeta = f^*(\eta) = g^*(c_\gamma^*(\eta))$ [5, page 19]. Hence we have shown the existence of the required commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\bar{g}"] \arrow[d, "p"'] & S^1 \arrow[d, "p'"] \\
M^* \arrow[r, "g"] & S^1
\end{tikzcd}$$

where p' is the standard k to 1 covering of S^1 and $\bar{g}$ is a bundle map.

<u>Corollary 1.4</u>: Let M be a compact 3-manifold admitting a free Z_k action, $k \geq 2$ prime. If $H_1(M^*;Z)$ has no k torsion then $H_1(M^*;Z)$ has a nontrivial free part.

Definition 1.5: Suppose M admits a free Z_k action generated by the homeomorphism h. Call (U,T) an equivariant h-partition of M if T is the disjoint union of connected two-sided 2-manifolds regularly embedded in M (i.e. $\mathrm{Bd}\, T = T \cap \mathrm{Bd}\, M$) such that the following conditions are satisfied:

(1) $h^i(T) \cap h^j(T) = \Phi$ if $i \not\equiv j \pmod{k}$,

(2) U is open in M,

(3) $h^i(U) \cap h^j(U) = \Phi$ if $i \not\equiv j \pmod{k}$,

(4) $\mathrm{Fr}\, U = T \cup h(T)$,

(5) $M = \bigcup_{i=1}^{k} (h^i(U) \cup h^i(T))$.

Lemma 1.6: Let M be a compact orientable 3-manifold admitting a free Z_k action and such that if $\mathrm{Bd}\, M \neq \Phi$ then $\mathrm{Bd}\, M$ is connected. If there is a bundle map from (M,p,M^*) to (S^1,p',S^1), where p' is the standard k to 1 covering projection of S^1, then there is a compact connected orientable polyhedral 2-manifold T such that T determines an equivariant h-partition (U,T) of M with the properties that U is connected, and if $\mathrm{Bd}\, M \neq \Phi$ then $\mathrm{Bd}\, T$ is connected and does not separate $\mathrm{Bd}\, M$.

Proof: Consider the diagram below whose existence is given by hypothesis.

$$\begin{array}{ccc} M & \xrightarrow{\quad \bar{g} \quad} & S^1 \\ \Big\downarrow p & & \Big\downarrow p' \\ M^* & \xrightarrow{\quad g \quad} & S^1_0 \end{array}$$

We may suppose there is a point $a \in S^1_0$ such that $g^{-1}(a)$ is the disjoint union of polyhedral two-sided orientable compact 2-manifolds regularly embedded in M^*. Let $p'^{-1}(a) = \{a_i\}_{i=1}^{k} \subset S^1$. By suitable choice of labeling we may let A be the arc in S^1 with endpoints a_1 and a_2 such that no other a_i lies on A. Let $U = \bar{g}^{-1}(A - \{a_1, a_2\})$ and $T = \bar{g}^{-1}(a_1)$. Since $\bar{g}$ is a bundle map, (U,T) is an equivariant h-partition. The difficulty comes in trying to adjust (U,T) to satisfy the conclusions of the lemma.

First we need to adjust the frontiers of the components of U. Suppose a component U_1 of U has at least two components T_1 and T_2 of T on its frontier. Let C be a polyhedral arc in $\mathrm{Cl}\, U_1$ with one end on T_1 and the other end on T_2 but otherwise lying in $\mathrm{Int}\, U_1$. Let $N(C)$ be a closed tubular neighborhood of C in $\mathrm{Cl}\, U_1$ such that $N(C) \cap T_1$ and $N(C) \cap T_2$ are closed disks. Take as a new U the set

$$(U - N(C)) \cup h[\mathrm{Int}_M N(C) \cup \mathrm{Int}_T((T_1 \cup T_2) \cap N(C))]$$

and as a new T one obtained from the old one by replacing $T_1 \cup T_2$ by $\mathrm{Cl}[T_1 \cup T_2 \cup \mathrm{Bd}\, N(C) - \mathrm{Int}(N(C) \cap (T_1 \cup T_2))]$. The number of components of T decreases and the number of components of U does not increase. Repeating this process and a similar one for $h(T)$ we obtain an equivariant h-partition (U,T) such that each component of U has at most one component of T and one component of $h(T)$ on its frontier.

We adjust (U,T) further. Suppose a component U_1 of U is such that $\mathrm{Fr}\, U_1 = T_1 \subset T$, where T_1 is a component of T. Then crossing T_1 from U_1, one enters $h^{k-1}(U)$. Whenever this case occurs take as a new U the set $(U - U_1) \cup h(\mathrm{Cl}\, U_1)$. Notice that $\mathrm{Fr}\, h(U_1) \subset \mathrm{Cl}\, U$. Obtain a new T by dropping T_1 from the old one. Repeating this process and a similar one for $h(T)$, i.e. replace U by $(U - U_i) \cup h^{-1}(\mathrm{Cl}\, U_1)$ where necessary, we obtain a new (U,T) partition where each component of U has exactly one component of T and one component of $h(T)$ on its frontier.

Let T_1 be a component of T. T_1 lies on the frontier of a unique component U_1 of U. Let $T_1^{(1)} \subset h(T)$ be the component of $h(T)$ on the frontier of U_1. $T_1^{(1)}$ lies on the frontier of an unique component $U_1^{(1)}$ of $h(U)$. Continuing the construction of this chain, let $T_1^{(2)} \subset h^2(T)$ be the component of $h^2(T)$ on $\mathrm{Fr}\, U_1^{(1)}$. As before, let $U_1^{(2)}$ be the unique component of $h^2(U)$ with $T_1^{(2)}$ on its frontier. Repetition of this process yields a sequence of components

$$T_1,\ U_1,\ T_1^{(1)},\ U_1^{(1)},\ T_1^{(2)},\ \ldots,\ T_1^{(k-1)},\ U_1^{(k-1)},$$

$$T_2,\ U_2,\ T_2^{(1)},\ U_2^{(1)},\ T_2^{(2)},\ \ldots,\ T_2^{(k-1)},\ U_2^{(k-1)},$$

$$\ldots,\ T_i^{(j)},\ U_i^{(j)},\ \ldots,\ T_n^{(k-1)},\ U_n^{(k-1)},$$

where $T_i \subset T$, $U_i \subset U$, $T_i^{(j)} \subset h^j(T)$, $U_i^{(j)} \subset h^j(U)$, $\mathrm{Fr}\, U_i^{(j)} = T_i^{(j)} \cup T_i^{(j+1)}$, and $T_i^{(k)}$ is identified with

T_{i+1}. This sequence must return to T_1, say at the n^{th} stage, and we identify $T_n^{(k)}$ with T_1 completing the cycle.

All of the components of $h^i(U)$, $i = 1, \ldots, k$, must have appeared as M is connected. From the construction it is clear that $h(T_1)$ will appear $1/k^{th}$ of the way through the cycle and from there on the sequence is nothing more than the first $1/k^{th}$ under repeated application of h. Let K be the union of everything appearing in the first $1/k^{th}$ of the sequence. Then the new U is to be chosen as the set $K - (T_1 \cup h(T_1))$ and the new T as T_1. This version of (U,T) is an equivariant h-partition with U and T connected, U open, and T regularly embedded in M. So if $\mathrm{Bd}\, M = \Phi$ we are done.

Finally suppose $\mathrm{Bd}\, M \neq \Phi$. We must adjust (U,T) so that $\mathrm{Bd}\, T$ is connected. $\mathrm{Bd}\, T$ consists of disjoint simple closed curves $\{C_i\}_{i=1}^n$ lying on $\mathrm{Bd}\, M$. If $n \geq 2$ a polygonal arc A in $\mathrm{Bd}\, M$ can be found with endpoints a and b lying on different curves C_i and C_j such that $A - \{a, b\}$ lies entirely in either U or $h^{-1}(U)$. Suppose $A - \{a, b\} \subset U$. (If it lies in $h^{-1}(U)$ a similar operation with the obvious modifications is used). Let $N(A)$ be a closed tubular neighborhood lying in $\mathrm{Cl}\, U$, containing A in its boundary, and meeting T in two closed disks D_i and D_j which contain arcs of C_i and C_j in their respective boundaries. In particular, $D_i \cap \mathrm{Bd}\, M \subset C_i$ and $D_j \cap \mathrm{Bd}\, M \subset C_j$. $N(A)$ is chosen so that there is a homeomorphism f mapping $A \times [0,1]$ onto $N(A) \cap \text{Bd } M$ with $f(A \times 1/2) = A$. For our new U we take the set

$$(U - N(A)) \cup h[\text{Int } N(A) \cup f(A \times (0,1)) \cup \text{Int } (D_i \cup D_j)].$$

Replace T by the set

$$T \cup \text{Bd } N(A) - [f(A \times (0,1)) \cup \text{Int } (D_i \cup D_j)].$$

This operation decreases the number of components of $\text{Bd } T$ by one but does not disturb the connectedness of U and T. Eventually this gives rise to an equivariant h-partition such that $\text{Bd } T$, U, and T are all connected.

$\text{Cl } U \cap \text{Bd } M$ is an oriented 2-manifold with boundary $\text{Bd } T \cup h(\text{Bd } T)$. It is easy to see that $\text{Bd } M - \text{Bd } T$ is connected already without further adjustment.

Remark 1.7: If $\text{Bd } M \neq \Phi$, then (U,T) in the above lemma is such that $U \cap \text{Bd } M$ is connected.

Definition 1.8: Let M admit a free Z_k action and suppose that the action is generated by the homeomorphism h. The Z_k action is said to be proper if h is homotopic to 1_M.

Theorem 1.9: If M is an irreducible closed orientable 3-manifold such that for some prime $k \geq 2$ M admits a proper free Z_k action and $H_1(M^*;Z)$ has no element of order k, then M can be fibered over the circle.

Proof: From Lemma 1.3 and Lemma 1.6 we get the existence of an equivariant h-partition (U,T) of M with T a connected orientable closed polyhedral 2-manifold. Let A be an arc with one endpoint $x_0 \in T$ and the other endpoint $h(x_0) \in h(T)$ but otherwise lying in U.

There is a retraction r_0 of $\mathrm{Cl}\ U$ onto A such that $r_0^{-1}(x_0) = T$ and $r_0^{-1}(h(x_0)) = h(T)$. Define a retraction r_i of $h^i(\mathrm{Cl}\ U)$ onto $h^i(A)$ by $r_i = h^i r_0 h^{-i} = h r_{i-1} h^{-1}$ for $i = 1, \ldots, k-1$. The r_i's define a retraction $r : M \longrightarrow F$, where $F = \bigcup_{i=1}^{k} h^i(A)$, such that $rh^i = h^i r$ for all i. Hence r is equivariant and induces a retraction $r_0 : M^* \longrightarrow p(F)$. Let f be r_0 followed by a homeomorphism of $p(F)$ onto S^1. Then there exists a bundle map $\bar{f}$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\bar{f}} & S^1 \\ \downarrow p & & \downarrow p' \\ M^* & \xrightarrow{f} & S_0^1 \end{array}$$

where p' is the standard k to 1 covering of S^1.

Since the Z_k action is proper we have that the generator h of Z_k is homotopic to the identity homeomorphism on M. Restricting the homotopy we get a map $G_0 : T \times [0,1] \longrightarrow M$ such that $G_0(x,0) = x$ and $G_0(x,1) = h(x)$. Define the maps $G_i : T \times [i, i+1] \longrightarrow M$ by $G_i(x,t) = h^i G_0(x,t-i)$, for $i = 1, \ldots, k-1$. Then $G_i(x,i+1) = G_{i+1}(x,i+1) = h^{i+1}(x)$ and $G_{k-1}(x,k) = G_0(x,0) = h^k(x)$. Hence the G_i's define a map $G : T \times S^1 \longrightarrow M$, where S^1 is considered as $[0,k]/\{0,k\}$.

We show that the degree of G is not zero. Let $\deg G = d$. Let $i : T \longrightarrow M$ be the inclusion map. Look at the exact homology sequence of the pair (M,T). Since $H_2(M,T;Z)$ is free, it follows that the sequence $0 \longrightarrow H_2(T;Z) \xrightarrow{i^*} H_2(M;Z)$ is split exact. From this we see that $H^2(M;Z) \xrightarrow{i^*} H^2(T;Z) \longrightarrow 0$ is exact using the universal coefficient theorem for cohomology which is functorial. In the sequel, singular homology and cohomology with integer coefficients will be used exclusively. Let $\alpha \in H_3(T \times S^1)$ and $\beta \in H_3(M)$ be generators such that $G_*\alpha = d\beta$. Using Poincaré duality we get the following commutative diagram:

$$\begin{tikzcd}
H^2(T \times x_0) \arrow[rr, equal] & & H^2(T) & \\
H^2(T \times S^1) \arrow[u, "i'^*"] \arrow[rr, "G^*"] \arrow[d, "\alpha\cap"] & & H^2(M) \arrow[u, "i^*"] \arrow[d, "d\beta\cap"] & \\
H_1(T \times S^1) \arrow[rr] & & H_1(M) \arrow[r, "\bar{f}_*"] & H_1(S^1)
\end{tikzcd}$$

where $i' : T \times x_0 \longrightarrow T \times S^1$ is the inclusion for $x_0 \in S^1$.

Using the universal coefficient theorems [17], write

$$H^2(T \times S^1) = P \oplus Q,$$

$$H_1(T \times S^1) = R \oplus S,$$

where $P = H^2(T) \otimes H^0(S^1)$, $Q = H^1(T) \otimes H^1(S^1)$, $R = H_0(T) \otimes H_1(S^1)$, and $S = H_1(T) \otimes H_0(S^1)$. Since i^* is an epimorphism, for each $u \in P$ there is a $u' \in Q$ such that $u + u' \in \operatorname{Im} G^*$. Under the Poincaré duality isomorphism P maps onto R. Since $\bar{f}^*G^*(S) = 0$ we have that

$\bar{f}_*G_*(R) = \bar{f}_*G_*(\alpha \cap P) = \bar{f}_*G_*(\alpha \cap G^*(H^2(M)) = \bar{f}_*(d\beta \cap H^2(M))$. Hence $\bar{f}_*G_*(R) = \bar{f}_*(dH_1(M))$.

We can identify R with $\pi_1(x_0 \times S^1)$ for a fixed $x_0 \in T$. Choose the point so that $\{x_0\} = F \cap T$. R is generated by an element represented by a loop μ going around $x_0 \times S^1$ exactly once. $G\mu$ is a loop starting from $x_0 \in T$ and passing through $h(x_0) \in h(T)$ $1/k^{th}$ of the way around, and thereafter repeating itself $k-1$ times under the action of h^i, for $i = 1, \ldots, k-1$. Hence $\bar{f}G\mu$ will start from some point s_0 in S^1, wrap around some number of times, and then reach $s_0 + e^{i(2\pi j/k)}$ when $G\mu$ is j/k^{th} of the way around, for $j = 1,2, \ldots, k$. Therefore under $\bar{f}G$, the first $1/k^{th}$ part of μ is wrapped around S^1 $m + 1/k$ times. Hence under $\bar{f}G$, μ is wrapped around S^1 $k(m + 1/k) = km + 1$ times. Thus $\bar{f}_*G_*([\mu]) \in \pi_1(S^1)$ is an integer congruent to 1 modulo k. Since $\bar{f}_*G_*(R) = \bar{f}_*(dH_1(M))$, after identifying $H_1(S^1)$ and $\pi_1(S^1)$, it follows that $km + 1$ is a multiple of $d = \deg G$. Therefore $\deg G \not\equiv 0 \pmod k$ and in particular $\deg G \neq 0$.

Let $\tilde{M} \xrightarrow{\tilde{p}} M$ be a covering corresponding to the subgroup π' of $\pi_1(M)$, where $\pi' = \operatorname{Im}(\pi_1(T \times S^1) \xrightarrow{G\#} \pi_1(M))$. If G' is a lifting of G we get the commutative diagram

$$\begin{tikzcd} & \tilde{M} \arrow[d, "\tilde{p}"] & \\ T \times S^1 \arrow[ur, "G'"] \arrow[r, "G"] & M \arrow[r, "\bar{f}"] & S^1 \end{tikzcd}$$

$\tilde{p}$ is a finite to one covering projection since $\deg G$ is finite and $\deg G = (\deg G^*)(\deg \tilde{p})$. Hence $\pi' = \pi_1(\tilde{M})$ has a finite index in $\pi_1(M)$. $G'_{\#} : \pi_1(T \times S^1) \longrightarrow \pi_1(M)$ is an epimorphism, so every loop in $\tilde{M}$ will circle around S^1 under the map $\bar{f}\tilde{p}$ some n number of times, where n is a multiple of $km + 1$.

Let the circle C be the component of $p^{-1}(F)$ that contains the basepoint (we may assume that basepoints have been chosen nicely). Let c be a non-singular loop on C. $t = \deg p$ divides $d = \deg G$ and $\tilde{p}c$ circles around F at most t times (in absolute value). Thus $\bar{f}\tilde{p}c$ circles around S^1 at most t times. It follows from these remarks that $t \leq d \leq km + 1 \leq n \leq t$, and hence $t = d$. In other words, $\deg G = \deg \tilde{p}$ and $[\pi_1(M), \pi'] = d$. So $\tilde{p}$ is a d to 1 covering projection.

Let γ be the element of $\pi_1(M)$ represented by a loop circling around F exactly once. Identify $\pi_1(S^1)$ with the additive group of integers Z. Consider the cosets $\pi', \gamma\pi', \ldots, \gamma^{d-1}\pi'$. These cosets are distinct since $\bar{f}_{\#}(\gamma^i\pi') \equiv i \pmod d$, so that their disjoint union is all of $\pi_1(M)$. Moreover π' is the kernel of a homomorphism from $\pi_1(M)$ onto Z_d and therefore is a normal subgroup.

We have $\pi_1(T \times S^1) = \pi_1(T) \times \pi_1(S^1)$ naturally split, so $\pi_1(\tilde{M}) = G'_{\#}(\pi_1(T)) \times G'_{\#}(\pi_1(S^1))$. If we let $K = \mathrm{Ker}(\bar{f}_{\#} : \pi_1(M) \longrightarrow \pi_1(S^1))$, it is clear that $K \subset \pi'$. Since $\tilde{p}_{\#}$ is a monomorphism, we have that $K = \tilde{p}_{\#}(\mathrm{Ker}((\bar{f}\tilde{p})_{\#} : \pi_1(\tilde{M}) \longrightarrow \pi_1(S^1)))$. But considering the effect of $(\bar{f}\tilde{p})_{\#}$ and $G_{\#}$ it is clear that $K = \tilde{p}_{\#}G'_{\#}(\pi_1(T))$. Hence K is finitely generated and $\pi_1(M)/K = Z$. Moreover, by Lemma 1.1, $\pi_1(M)$ has no elements of finite order. Therefore we can apply the following thoerem due to Stallings [18] to complete the proof of the theorem.

<u>Theorem 1.10</u>: If M is a compact irreducible 3-manifold, and if $\pi_1(M)$ has a finitely generated normal subgroup K different from Z_2, whose quotient group is Z, then M is the total space of a fiber space with base space a circle and with fiber a 2-manifold T embedded in E whose fundamental group is K.

<u>Remark 1.11</u>: In Theorem 1.9 the hypothesis that Z_k be a proper action may be replaced by the weaker condition that for some equivarient h-partition (U,T) such as in Lemma 1.6 and some non-trivial $h \in Z_k$ we have that $h|T$ is homotopic to the inclusion $i : T \longrightarrow M$.

<u>Remark 1.12</u>: Let M be a closed orientable 3-manifold satsifying the hypothesis of Theorem 1.9. It follows from the proof that M is covered by $\tilde{M}$ in a finite to one manner. The multiplicity of the covering is relatively prime to k. By Neuwirth [12], $\tilde{M} = \bar{T} \times S^1$, where $\bar{T}$ is a closed orientable 2-manifold. Moreover, $\pi_1(\bar{T}) = G_{\#}\pi_1(T)$, which is isomorphic to the fundamental group of the fiber of a fibration of M over S^1 [18].

<u>Theorem 1.13</u>: If M is an irreducible compact orientable 3-manifold with connected nonempty boundary such that for some prime $k \geq 2$ M admits a proper free Z_k action and $H_1(M^*;Z)$ has no element of order k, then M can be fibered over the circle and $\text{Bd } M = S^1 \times S^1$.

<u>Proof</u>: The proof of this theorem follows that of Theorem 1.9 very closely, so the obvious overlap will be omitted here. There exists an equivariant h- partition (U,T) of M with T a connected orientable compact polyhedral 2-manifold such that $\text{Bd } T = T \cap \text{Bd } M$ is connected and does not separate $\text{Bd } M$.

Let A be an arc with endpoints $x_0 \in T$ and $h(x_0) \in h(T)$ such that $A - \{x_0, h(x_0)\} \subset U$. As before there is an equivariant retraction $r : M \longrightarrow F$, where $F = \bigcup_{i=1}^{k} h^i(A)$, which induces a retraction $r_0 : M^* \longrightarrow p(F)$. Let f be r_0 followed by a homeomorphism of $p(F)$ onto S^1. We get the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\bar{f}"] \arrow[d, "p"] & S^1 \arrow[d, "p'"] \\ M^* \arrow[r, "f"] & S_0^1 \end{tikzcd}$$

where $\bar{f}$ is a bundle map and p' is the standard k to 1 covering of S^1. In view of $h \simeq 1_M$ we get a map $G : T \times S^1 \longrightarrow M$.

We require a slightly different diagram this time to show that $\deg G = d$ is not zero.

$H^2(\mathrm{Bd}\ M, \mathrm{Bd}\ T) = Z$ since $\mathrm{Bd}\ T$ is a nonseparating simple closed curve in $\mathrm{Bd}\ M$. Consider the following commutative diagram:

$$\begin{tikzcd}
H^1(\mathrm{Bd}\ M) \arrow[r, "j^*"] \arrow[d] & H^1(\mathrm{Bd}\ T) \arrow[r] \arrow[d] & H^2(\mathrm{Bd}\ M, \mathrm{Bd}\ T) \arrow[r, "="] & H^2(\mathrm{Bd}\ M) \arrow[r] & 0 \\
H^2(M, \mathrm{Bd}\ M) \arrow[r, "i^*"] & H^2(T, \mathrm{Bd}\ T) \arrow[d] & & & \\
& 0 & & &
\end{tikzcd}$$

where i and j are inclusion maps. It follows from the fact that j^* is an epimorphism that i^* is an epimorphism.

Let $\alpha \in H_3(T \times S^1, \mathrm{Bd}\ T \times S^1)$ and $\beta \in H_3(M, \mathrm{Bd}\ M)$ be generators such that $G_*\alpha = d\beta$. Consider the following commutative diagram:

$$\begin{tikzcd}
H^2(T \times x_0, \mathrm{Bd}\ T \times x_0) \arrow[rr, equal] & & H^2(T, \mathrm{Bd}\ T) & \\
H^2(T \times S^1, \mathrm{Bd}\ T \times S^1) \arrow[u, "i'^*"] \arrow[rr, "G^*"] \arrow[d, "\alpha\cap"] & & H^2(M, \mathrm{Bd}\ M) \arrow[u, "i^*"] \arrow[d, "d\beta\cap"] & \\
H_1(T \times S^1) \arrow[rr, "G_*"] & & H_1(M) \arrow[r, "\bar{f}_*"] & H_1(S^1)
\end{tikzcd}$$

where $i' : (T \times x_0, \mathrm{Bd}\ T \times x_0) \longrightarrow (T \times S^1, \mathrm{Bd}\ T \times S^1)$ is the inclusion map for a fixed $x_0 \in S^1$. Write $H^2(T \times S^1, \mathrm{Bd}\ T \times S^1) = P \oplus Q$, where $P = H^2(T, \mathrm{Bd}\ T) \otimes H^0(S^1)$ and $Q = H^1(T, \mathrm{Bd}\ T) \otimes H^1(S^1)$. Let $H_1(T \times S^1) = R \oplus S$ as before. Then $\bar{f}_* G_*(R) = \bar{f}_*(dH_1(M))$. The remainder of the proof to show that M fibers over the circle is the same as that of Theorem 1.9.

<u>Theorem 1.14</u>: If a closed compact 3-manifold M fibers over the circle, then M is a prime manifold.

<u>Proof</u>: Denote the fiber of a fibering of M over S^1 by T (we may assume that T is connected). From previous remarks it follows that there is a homeomorphism $h : T \longrightarrow T$ such that $M = T \times I/h$. If $T = S^2$ or P_2, then M is either an S^2 or P_2 bundle over S^1, hence prime (in fact irreducible in the second instance). Otherwise, the universal covering space of T is R^2, and hence the universal covering space of M is R^3. Therefore M is irreducible in this case, and prime in either case.

CHAPTER II

3-MANIFOLDS THAT COVER THEMSELVES

In this chapter we consider 3-manifolds M that admit a free Z_k action such that the orbit space is homeomorphic to M. Kwun's results [7] on closed orientable 3-manifolds (without boundary) which double-cover themselves are contained here as special cases of Theorem 2.8 and Theorem 2.16.

A bundle (E,p,B) is called a <u>covering space of B</u> if every $b \in B$ has an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets in E each of which is mapped homeomorphically onto U by p. The map p is called the <u>covering projection</u>. Let E be path-connected. The <u>number of sheets of p</u> is the cardinal number of the discrete set $p^{-1}(b)$, which is independent of $b \in B$. The covering projection is <u>regular</u> if for some $x_0 \in E$, $p_{\#}\pi_1(E,x_0)$ is a normal subgroup of $\pi_1(B,p(x_0))$. The <u>group of covering transformations</u> $G(E|B)$ of p is the group of homeomorphisms $f : E \longrightarrow E$ such that $pf = p$.

Let M be an n-manifold. If (M,p,B) is a regular covering space, then $G = G(M|B)$ acts freely on M and the orbit space $M^* = M/G$ is homeomorphic to B. On the other hand, if a finite group G acts freely on M, the bundle (M,p,M^*) is a regular covering space, where p denotes the projection. In either case

$G = \pi_1(M^*, p(x_0))/p_{\#}(\pi_1(M, x_0))$, and the number of sheets of p equals the order of G. For more details on covering spaces see [17, Chapter 2].

In the introduction we defined the concept of a connected sum. Closely related to this is the operation of <u>adding a handle</u> to a 3-manifold M. Remove the interiors of two disjoint tame 3-cells in M and match the resulting boundaries by a homeomorphism. If M is orientable, the result is homeomorphic to $M \# S^1 \times S^2$ when the attaching homeomorphism is orientation reversing, and homeomorphic to $M \# N$ when it is orientation preserving. By N we mean the non-orientable locally trivial S^2 bundle over S^1. If M is non-orientable to start with, the resulting space is homeomorphic in either case to $M \# S^1 \times S^2$ since $M \# S^1 \times S^2 = M \# N$.

Milnor [10] has shown that every closed orientable 3-manifold M is homeomorphic to a sum $P_1 \# P_2 \# \cdots \# P_k$ of prime manifolds, where the summands P_i are uniquely determined up to order and homeomorphism. It has been observed by Raymond [16] that Kneser [6] actually proved, modulo the truth of Dehn's lemma, a unique decomposition theorem for closed 3-manifolds, orientable or not. Kneser's theorem states that every closed 3-manifold can be written uniquely in "normal form" as the sum of prime manifolds. In normal form means as the sum of irreducible manifolds and handles, where the number of non-orientable handles is minimal (i.e. 1 or 0, depending on whether the irreducible summands are all orientable or not).

Milnor [10] also proved that, with the exception of S^3 and $S^1 \times S^2$, an orientable closed manifold is prime if and only if it is irreducible. In light of Raymond's observation, the proof given by Milnor can easily be extended to the non-orientable case if N is also excluded.

We are now ready to initiate our investigation of compact manifolds covering themselves. First we need to define a property of covering spaces closely related to proper group actions.

<u>Definition 2.1</u>: Let (M,p,B) be a regular k-sheeted covering space, $k \geq 2$ prime. We say that M <u>properly</u> covers B if the action of the group of covering transformations $G(M|B) = Z_k$ on M is proper.

The next two examples will serve to motivate the discussion in this chapter and to exhibit the general character of manifolds covering themselves.

<u>Example 2.2</u>: Let T be a compact 2-manifold and $k \geq 2$ any integer. Then the map

$$1_T \times p' : T \times S^1 \longrightarrow T \times S^1,$$

where p' is the standard k-sheeted covering projection of the circle, is a proper k-sheeted covering projection.

<u>Example 2.3</u>: Consider the connected sum $P_3 \# P_3$ of two real projective 3-spaces. $P_3 \# P_3$ is homeomorphic to

the sum $P(k) = P_3 \# S^3 \# \cdots \# S^3 \# P_3$, where $P(k)$ contains $k - 1$ summands of S^3 (note that orientation need not be specified since P_3 admits an orientation reversing homeomorphism). Let $p : P(k) \longrightarrow P_3 \# P_3$ be the covering projection by which the sphere summands S^3 of $P(k)$ alternately double cover the P_3 summands of the base space, the first P_3 of $P(k)$ covers the left half of $P_3 \# P_3$, and finally the last P_3 of $P(k)$ covers the left (right) half of $P_3 \# P_3$ if k is even (odd).

<u>Remark 2.4</u>: Let $p : P_3 \# P_3 \longrightarrow P_3 \# P_3$ be the k-sheeted covering just described. Notice that if $k = 2$, then $p_{\#} \pi_1(P_3 \# P_3)$ is a normal subgroup of $\pi_1(P_3 \# P_3)$, and hence p is a regular covering projection in this case. This is the only case in which p is a regular covering projection. Moreover, none of these coverings are proper.

Theorem 2.8 shows that Example 2.3 is the only non-prime closed 3-manifold to cover itself in a non-trivial way. The next three lemmas lead up to this theorem.

Let $H_0 = \{S^3\}$ and let H_1 denote the collection of non-trivial prime closed 3-manifolds. For $j \geq 2$ let H_j denote the collection of closed 3-manifolds which are homeomorphic to connected sums of exactly j elements of H_1. For convenience of notation we will let N^* denote a handle of either type, i.e. either $S^1 \times S^2$ or N.

<u>Lemma 2.5:</u> If $M \in H_2$ then M covers itself k times, $k \geq 2$, if and only if $M = P_3 \# P_3$.

Proof: Suppose $p : \tilde{M} \longrightarrow M$ is a k to 1 covering projection. Let $M = A \# B$, where $A, B \in H_1$. Write $A \# B = A' \cup B'$, where A' (B') is obtained from A (B) by deleting a tame open 3-cell. $A' \cap B' = S$ is a 2-sphere and $p^{-1}(S)$ is a disjoint collection of 2-spheres $\{S_i\}_{i=1}^{k}$.

Case 1. Each S_i separates $A \# B$, for $i = 1, \ldots, k$. Therefore $M - p^{-1}(S)$ has $k + 1$ components with the closure of each component covering either A' or B'. The closure of at least two components, say U_1 and U_2, have connected boundary, say S_1 and S_2 respectively. Let C (D) be obtained from $Cl\ U_1$ $(Cl\ U_2)$ by sewing a 3-cell along S_1 (S_2). C (D) covers either A or B exactly once, so C and D belong to H_1. Since $M \in H_2$, $A \# B = C \# D$ and each of the remaining components must have disconnected boundary of exactly two components lying in $p^{-1}(S)$ and be homeomorphic to S^3 minus two tame open 3-cells. An analysis of the situation reveals that either both A and B are double-covered by S^3, or $A = B$ and one of them is double-covered by S^3. We need now to invoke the following theorem due to Livesay [8].

Theorem: If $T : S^3 \longrightarrow S^3$ is any fixed-point-free homeomorphism of period 2 on the 3-sphere, then there exists a homeomorphism $h : S^3 \longrightarrow S^3$ such that hTh^{-1} is the antipodal map.

It follows that the orbit space of the action on S^3 by any free involution must be P_3. In particular, this proves that $A = B = P_3$.

Case 2. $A \# B - S_i$ is connected for some i, say $i = 1$. Since $A \# B - S_1$ is connected, either A or B must be N^*. Suppose $B = N^*$. The closure of each component of $M - p^{-1}(S)$ covers either A' or B'. If the closure of a component U covers B', Cl U plus some tame open 3-cells sewn along its boundary components is homeomorphic to N^*. This is because $\pi_1(N^*) = \pi_1(B') = Z$ has exactly one subgroup of a given finite index, and hence has a unique connected m-sheeted covering for any given m. It follows that the closure of only one component of $M - p^{-1}(S)$ covers B'. Otherwise we would have at least three handles N^* upstairs which is in violation of M being in H_2. Therefore only Cl U covers B' and hence does so in a k to 1 fashion. The boundary of Cl U must be $p^{-1}(S)$. Since $M - S_1$ is connected there must be another copy of N^* upstairs, in addition to Cl U with k open 3-cells attached. Hence $A = N^*$ and $B = N^*$. By the uniqueness of coverings for N^*, the closure of some component, say Cl V, covers A' k times and is such that Cl $V \cap$ Cl $U = p^{-1}(S)$. But this implies that there are $k - 2$ handles upstairs besides Cl U and Cl V (with appropriate 3-cells attached). Therefore we must have $k = 2$ to avoid contradicting that $M \in H_2$. This case is ruled out by the following argument which is essentially that of Kwun's [Proposition 3.1, Case 2] with a slight modification to extend it to the non-orientable case.

We have then that neither S_1 nor S_2 separates $A \# B$, and so $A \# B - S_1 \cup S_2$ has two components P and Q. Since $p(P \cup Q) = A \# B - S$ is disconnected, $p(P) = A' - B'$ and $p(Q) = B' - A'$ (by proper choice of labeling). Let C and D be manifolds obtained from $\mathrm{Cl}\, P$ and $\mathrm{Cl}\, Q$ respectively by attaching 3-cells. Then $A \# B = C \# N^* \# D$. By the uniqueness of the decomposition, either A or B, say A, is a handle. But this implies that either C or D is N^*, which in turn implies that $B = N^*$, and finally that both C and D are handles. Hence the connected sum of three copies of N^* would be homeomorphic to that of two copies of N^*. This contradiction rules out this case and thus completes the proof of the lemma.

Lemma 2.6: Let $M_1 \in H_m$ and $M_2 \in H_n$, and suppose that M_1 contains no handles. If there exists a k to 1 covering projection $p : M_1 \longrightarrow M_2$, then $m \geq n$ (if $m \geq 1$).

Proof: The proof is by induction on m. Let $m = 1$. Write $M_2 = A \# B$, where $A \in H_1$ and $B \in H_{n-1}$. As usual write $M_2 = A' \cup B'$, with $A' \cap B' = S$, a 2-sphere. Since M_1 has no handles, each component S_i of $p^{-1}(S) = S_1 \cup \cdots \cup S_k$ must separate M_1. There are at least two components of $M_1 - p^{-1}(S)$, each of which covers either A' or B' exactly once. But this is impossible unless $n = 1$ since M_1 is prime.

Now suppose that the lemma is true for $q \leq m$ and let $M_1 \in H_{m+1}$, $M_2 \in H_{n+1}$. We suppose that $n+1 > m+1$

and show this leads to a contradiction. Write $M_2 = A \# B = A' \cup B'$, where $A \in H_1$, $B \in H_n$, and $A' \cap B' = S$, a 2-sphere. Again every component S_i of $p^{-1}(S)$ separates M_1. Hence there must be at least two components of $M_1 - p^{-1}(S)$ such that the closure of each must cover either A' or B' in a one to one manner. Let C' be any component of $p^{-1}(B)$, and let C be the manifold obtained by capping the 2-sphere boundary components of $\mathrm{Cl}\ C'$ with 3-cells. Then $C \in H_q$, where $1 \leq q \leq m$. However, p induces a covering of B by C, which violates our induction hypothesis since $q < n$. This completes the induction.

<u>Lemma 2.7</u>: If $M \in H_n$, $n > 2$, then M does not cover itself k times for any $k \geq 2$.

<u>Proof</u>: Let $M = A_1 \# A_2 \# \ldots \# A_n$, where each $A_i \in H_1$.

Case 1. At least one $A_i = N^*$. Suppose that $A_i = N^*$ for $1 \leq i \leq m$ and that $A_i \neq N^*$ for $i > m$ $(1 \leq m \leq n)$ Write $M = A_1' \cup A_2' \cup \cdots \cup A_n'$, where A_1' and A_n' are obtained from A_1 and A_n respectively by deleting a tame open 3-cell, and A_i' is obtained from A_i by deleting two tame open 3-cells for $1 < i < n$. Since M is connected, $A_i' \cap A_{i+1}' = S_i$, a 2-sphere. A connected t-sheeted covering of A_i', for

$$1 < i < m \text{ (and for } i = m \text{ if } m \neq 1, n),$$

must be homeomorphic to N^* minus $2t$ tame open 3-cells (or t 3-cells if $i = 1$ or $i = m = n$). Hence each component of $M - p^{-1}(S)$ covering A_1', $i \leq m$, is of this form. To cover $A_1 \# A_2 \# \cdots \# A_m$ k times requires at least $k(m-1)$ copies of N^* upstairs. So M must have at least $k(m-1)$ handles. But M has exactly m handles and $k(m - 1) \leq m$ holds only when $k = m = 2$. An analysis of this special situation reveals that it takes more than two copies of N^* upstairs to double-cover M. Since M has only two handles such an M cannot double-cover itself. This rules out Case 1.

Case 2. No $A_i = N^*$, Write $M = A \# B$, where $A = A_1$ and $B = A_2 \# \cdots \# A_n$. As before write $M = A' \cup B'$ such that $A' \cap B' = S$, a 2-sphere. Since M contains no handles, each component of $p^{-1}(S)$ must separate M. Hence the closure of at least two components of $M - p^{-1}(S)$ must have connected 2-sphere boundary and cover in a one to one manner. Call two of these components U and V. We must have that $\mathrm{Cl}\, U = \mathrm{Cl}\, V = A'$, for otherwise we would have too many summands upstairs (i.e. at least $2(n-1) > n$). Let W' be a component of $\mathrm{Cl}(p^{-1}(B'))$. If we cap the 2-sphere boundary components of W' and B' with 3-cells we get the closed 3-manifolds W and B, $W \in H_q$ and $B \in H_{n-1}$, where $q < n-1$. But $p|W'$ induces a covering of B by W which is impossible in view of Lemma 2.6. This rules out Case 2, thus completing the proof.

The following theorem is an immediate consequence of the above three lemmas.

Theorem 2.8: Let $k \geq 2$ be any integer. A closed non-prime 3-manifold M is a k-sheeted covering of itself if and only if $M = P_3 \# P_3$.

We digress for a moment to organize some of the problems that have arisen in the above discussion. In Lemma 2.6 we have shown that no irreducible 3-manifold can cover a non-prime one. However, it is possible for a prime 3-manifold to cover a non-prime one. For example $S^1 \times S^2$ double-covers $P_3 \# P_3$. We ask if the opposite situation can occur, that is, can a non-prime 3-manifold cover a prime 3-manifold?

Conjecture 2.9: A closed covering space of a prime closed 3-manifold is itself prime.

We now would like to obtain a result for manifolds with connected boundary similar to Theorem 2.8.

Lemma 2.10: Let $k \geq 2$ be an integer. A closed 2-manifold T is a k-sheeted covering of itself if and only if either $T = S^1 \times S^1$ or $T = K$ and k is odd (K denotes the Klein bottle).

Proof: Let $\chi(T)$ denote the Euler characteristic of T. In view of the fact that $\chi(T) = k\chi(T)$ [4, page 277], we must have $\chi(T) = 0$. Therefore T must be either a torus or a Klein bottle K. Since K can only cover itself an odd number of times, the lemma is proved.

Lemma 2.11: If a manifold M covers itself k times for some $k \geq 2$, then $\pi_1(M)$ is infinite.

Proof: It follows that M covers itself k^n times for all integers $n \geq 1$. Hence $\pi_1(M)$ contains subgroups of index k^n for all $n \geq 1$.

Theorem 2.12: Let $k \geq 2$ and let M be a compact 3-manifold with nonempty connected boundary. If M covers itself k times, then M is a prime irreducible 3-manifold and Bd M is either the torus or the Klein bottle.

Proof: Suppose $p : M \longrightarrow M$ is a k to 1 covering projection. Then $p|\text{Bd } M : \text{Bd } M \longrightarrow \text{Bd } M$ is also a k to 1 covering projection. By Lemma 2.10, Bd M is either $S^1 \times S^1$ or K.

Now suppose that $M = A \,\Delta\, B$, where A and B are not closed 3-cells. By proper choice of notation we may suppose that Bd A = Bd M and Bd $B = S^2$. Consider $2M$, the double of M, obtained by sewing two copies of M together along their boundaries by the identity map. It is clear that $2M = 2A \# 2B$, where $2A$ and $2B$ are non-trivial. p induces a k to 1 covering of $2M$ by itself. According to Theorem 2.8, $2M = P_3 \# P_3$. By the uniqueness of the connected sum decomposition we must have that $2B = P_3$. But this would imply that $Z_2 = \pi_1(B) * \pi_1(B)$, where $*$ denotes the free product. This is a contradiction. Thus M must be prime.

We want to show that M is also irreducible. Since $2M$ covers itself k times there are only three cases to consider, namely $2M = P_3 \# P_3$, $2M = N^*$ (i.e. $S^1 \times S^2$ or N), and $2M$ irreducible. For clarity in notation we suppose that $2M = M \cup M'$, where $M = M'$ and $M \cap M' = \mathrm{Bd}\, M = \mathrm{Bd}\, M'$.

First suppose that $2M = P_3 \# P_3$. Let $S \subset \mathrm{Int}\, M \subset 2M$ be a tamely embedded 2-sphere. If S does not bound a 3-cell in M, then S must bound P_3 less a tame open 3-cell. By the symmetry of $2M$, a corresponding tame 2-sphere S' in $\mathrm{Int}\, M'$ also bounds P_3 less a tame open 3-cell. If we let $\bar{M}$ and $\bar{M}'$ denote M and M' respectively with the P_3's removed and replaced by 3-cells, we get $2\bar{M} = S^3$. Since there is a retraction of S^3 onto $\bar{M}$, $\pi_1(\bar{M}) = 0$. It follows that $\pi_1(M) = Z_2$. But this is impossible by Lemma 2.11. Hence this case is ruled out.

Suppose $2M$ is irreducible. Let $S \subset \mathrm{Int}\, M \subset 2M$ be a tamely embedded 2-sphere. Let $2M - S = A \cup B$ and suppose that $A \subset \mathrm{Int}\, M$. Then $\mathrm{Cl}\, A$ must be a 3-cell. For if A were not a 3-cell, $\mathrm{Cl}\, B$ would have to be. But there is a 2-sphere S' in $\mathrm{Int}\, M'$ corresponding to S. $S' \subset \mathrm{Int}\, M' \subset \mathrm{Cl}\, B$ and S' bounds a homeomorphic copy of A in $\mathrm{Cl}\, B$. This is a contradiction since $\mathrm{Cl}\, B$ is irreducible. Hence $2M$ irreducible implies M is irreducible.

The last case, when $2M = N^*$, is taken care of by the next lemma.

<u>Lemma 2.13</u>: Let M be a compact 3-manifold with connected boundary. If $2M = N^*$ then M is irreducible.

<u>Proof</u>: Let $S \subset \text{Int } M$ be a tamely embedded 2-sphere. If S separates N^*, then S must bound a 3-cell in N^* and this 3-cell would lie in $\text{Int } M$. Hence it is sufficient to show that every 2-sphere tamely embedded in $\text{Int } M \subset N^*$ must separate N^*.

Let $i : \text{Bd } M \longrightarrow 2M$ be the inclusion map. Suppose that $S \subset \text{Int } M \subset 2M - \text{Bd } M$ is a non-separating tame 2-sphere. Then $i(\text{Bd } M) \subset 2M - S$ and the induced map $i_\# : \pi_1(\text{Bd } M) \longrightarrow \pi_1(2M)$ is trivial. This is true since $N^* - S = S^2 \times (0,1)$. Consider the commutative diagram obtained using the Van Kampen theorem.

$$\begin{tikzcd}
 & \pi_1(\text{Bd } M) \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] \arrow[dd, "i_\#"] & \\
G = \pi_1(M) \arrow[dr] & & G' = \pi_1(M) \arrow[dl] \\
 & \pi_1(2M) &
\end{tikzcd}$$

If we let $\pi_1(\text{Bd } M) = (\bar{z} : \bar{t})$, $G = (\bar{x} : \bar{r})$, and $G' = (\bar{y} : \bar{s})$, then we can write

$$\pi_1(2M) = (\bar{x}, \bar{y} : \bar{r}, \bar{s}, \{\theta_1(z_k)\theta_2(z_k)^{-1} : z_k \in \bar{z}\}).$$

Since $i_\# = 0$ and M is a retract of $2M$, θ_1 and θ_2 are trivial. Therefore $\pi_1(2M) = G*G'$, where $G = G'$. But this is a contradiction since $\pi_1(2M) = Z$ and Z is not the free product of two isomorphic groups. Therefore every tame 2-sphere S in Int M must separate $2M$ and hence bound a 3-cell in M.

Relating to Lemma 2.13, Kwun observed that if $2M = S^1 \times S^2$ and Bd M connected, then $M = D^2 \times S^1$ (the solid torus). This follows from the above mentioned lemma.

Corollary 2.14: Let M be a compact 3-manifold with connected boundary. If $2M = S^1 \times S^2$ then M is a solid torus. If $2M = N$ then M is the product of the Möbius band with the interval.

Proof: Suppose $2M = N^*$. Then there is a retraction of N^* onto M. Since $\pi_1(N^*) = Z$, it follows that $\pi_1(M)$ is either Z or 0. But $\pi_1(N^*)$ is isomorphic to an amalgamated free product of two copies of $\pi_1(M)$, hence $\pi_1(M) = Z$. M is irreducible by Lemma 2.13. By Theorem 1.10 (Stallings), M fibers over the circle with the closed 2-disk as fiber. There are only two locally-trivial 2-disk bundles over S^1, the solid torus and the non-trivial one. If $2M = S^1 \times S^2$ then M must be the solid torus since a non-orientable 3-manifold cannot be embedded in $S^1 \times S^2$. On the other hand, the double of the non-trivial 2-disk bundle over S^1 is N.

Lemma 2.15: If a compact 3-manifold M properly covers itself k times for some $k \geq 2$, then M is either N^* or an irreducible $K(\pi,1)$ space.

Proof: This follows directly from Lemma 1.1, Theorem 2.8, Lemma 2.11, and Theorem 2.12.

Theorem 2.16: Let $k \geq 2$ be a prime integer. Suppose M is a compact orientable 3-manifold such that $H_1(M;Z)$ has no element of order k and Bd M is either empty or connected. If M properly covers itself k times, then M can be fibered over the circle.

Proof: M admits a proper free action by $G(M|M) = Z_k$, with $M^* = M$. If M is closed, then by Theorem 2.8 and Remark 2.4, M must be a prime closed manifold. Since $S^1 \times S^2$ satisfies the conclusion of the theorem we may assume that M is irreducible. Application of Theorem 1.9 shows that M fibers over the circle. If Bd $M \neq \phi$, then by Theorem 2.12 M is irreducible. Application of Theorem 1.13 completes the proof.

In the next chapter examples are given to show that in general M need not be a product of a 2-manifold with the circle.

CHAPTER III

EXAMPLES OF PROPER Z_k ACTIONS

In this chapter we present examples of 3-manifolds with free Z_k actions that can be extended to effective $SO(2)$ actions. Such Z_k actions will clearly be proper. Closed 3-manifolds admitting effective $SO(2)$ actions and the actions have been classified by Orlik and Raymond [14]. We follow their notation which seems convenient for the presentation of our examples.

First we need to describe some elementary $SO(2)$ actions on solid tori which will serve as "building blocks" for more complicated actions on arbitrary 3-manifolds admitting an $SO(2)$ action. Parameterize the solid torus $D^2 \times S^1$ by $(\rho e^{i\theta}, e^{i\psi})$, where $0 \leq \rho \leq 1$, $0 \leq \theta, \psi < 2\pi$. Define an <u>ordinary action</u> on $D^2 \times S^1$ by

$$z \times (\rho e^{i\theta}, e^{i\psi}) \longrightarrow (\rho e^{i\theta}, z e^{i\psi}),$$

where z ranges over the complex number of norm 1.

Let μ and ν be relatively prime integers with $0 < \nu < \mu$. Define the <u>standard linear action (μ, ν)</u> of $SO(2)$ by

$$z \times (\rho e^{i\theta}, e^{i\psi}) \longrightarrow (z^{\nu} \rho e^{i\theta}, z^{\mu} e^{i\psi}).$$

The center circle $(0, e^{i\psi})$ will be called an <u>exceptional orbit</u> of type (μ, ν). The isotropy group of any point on such an orbit is $Z_\mu \subset SO(2)$. The principal orbits wind around an exceptional orbit of type (μ,ν) μ times and around a bounding curve $m = (e^{i\theta}, e^{i\psi_0})$, ψ_0 fixed, ν times.

We may choose a closed curve q on the boundary torus such that q is a cross-section to the standard linear action, i.e. q cuts each orbit on the boundary torus in a single point. The group $SO(2)$ is naturally oriented and by choosing an orientation for the solid torus a compatible orientation of q is determined. Orient $m = (e^{i\theta}, e^{i\psi_0})$ such that the homology relation $m \sim \mu q + \beta h$, where $\beta\nu \equiv 1 \pmod{\mu}$, holds. Notice that we can assume q was chosen so that $0 < \beta < \mu$, since q can be modified by $q \sim q' + sh$, where s is an arbitrary integer and h denotes a principal orbit on the boundary torus. In this case we refer to (μ,β) as being in normal form.

Using these building blocks, Orlik and Raymond [14] defined the symbol

$$\{b; (o,g,0,0); (\alpha_1,\beta_1), \cdots, (\alpha_n,\beta_n)\},$$

which describes both a closed orientable 3-manifold and a standard $SO(2)$ action on this manifold. For completeness, a brief description of this notation follows.

Let the integers $g \geq 0$ and b be given and let $\{(\alpha_i,\beta_i)\}_{i=1}^{n}$ be pairs of relatively prime integers such that $0 < \beta_i < \alpha_i$. Let M^+ be a closed orientable 2-manifold of genus g. Define an $SO(2)$ action on $M^+ \times S^1$ by $e^{i\theta} \times (m^+ \times e^{i\psi}) \longrightarrow (m^+ \times e^{i\theta}e^{i\psi})$. The orbit space $(M^+ \times S^1)/SO(2)$ is homeomorphic to M^+. There is a cross-section to this action which can be identified with $(M^+ \times 1)$.

Choose n points $x_1^+, \cdots, x_n^+$ in M^+ and remove the interior of a closed disk neighborhood D_j^+ around each of the points. The resulting manifold $M_0 = M^+ - \mathrm{Int}(D_j^+ \times S^1)$ is a compact 3-manifold with n toral boundary components, T_j. Since we removed only invariant tubular neighborhoods of principal orbits, $SO(2)$ still operates freely on M_0. On T_j, let $q_j = \mathrm{Bd}(D_j^+) \times 1$ and let h be any one of the principal orbits. q_j and h form an orthogonal curve system on T_j.

Now sew a solid torus ${}_jV$ with standard linear action (α_j, ν_j) on T_j by matching the principal orbits on the boundary of ${}_jV$ to the principal orbits on T_j, and matching a cross-section ${}_jq$ (to the action on ${}_jT$) to the cross-section q_j. This is done in an orientation reversing manner. Moreover the cross-section ${}_jq$ is chosen such that $\nu_j \equiv \beta_j^{-1} (\mathrm{mod}\ \alpha_j)$.

If $b \neq 0$ remove one more invariant tubular neighborhood $\mathrm{Int}(D_0^+ \times S^1)$ of a principal orbit over x_0^+. On T_0 we have the cross-section curve $q_0 = \mathrm{Bd}(D_0^+) \times 1$. Equivariantly sew in a solid torus ${}_0V$ on which we have an ordinary action by matching the principal orbits on the corresponding torus boundaries, but sending q_0 to a cross-section curve ${}_0q$ on the boundary of ${}_0V$ satisfying the homology relation ${}_0m \sim {}_0q + bh$, where ${}_0m$ is a bounding curve on ${}_0T$ and h a principal orbit on ${}_0T({}_0T = \mathrm{Bd}\ {}_0V)$. We send $-q_0$ to ${}_0q$ to retain orientation.

The resulting closed 3-manifold M is denoted by the symbol $\{b;(o;g,0,0);(\alpha_1,\beta_1), \cdots, (\alpha_n,\beta_n)\}$. Observe that this symbol also describes an effective $SO(2)$ action on M without any fixed point sets and with exactly n isotropy groups Z_α, $\alpha = \alpha_i$, for $i = 1, \cdots, n$, corresponding to the exceptional orbits of the n solid tori which were sewn in.

Theorem [14, Theorem 2]: Let $SO(2)$ act effectively on a closed, connected, orientable 3-manifold M with no fixed point sets. Then there exists a 3-manifold with standard action $M_1 = \{b;(o,g,0,0);(\alpha_1,\beta_1),\cdots,(\alpha_n,\beta_n)\}$ and a homeomorphism $H : M_1 \longrightarrow M$ together with an automorphism $a : SO(2) \longrightarrow SO(2)$ such that for all $m \in M_1$ and $g \in SO(2)$, $H(g(m)) = a(g)H(m)$.

Notice that any subgroup $Z_k \subset SO(2)$ with $(k,\alpha_i) = 1$ for $i = 1,\cdots, n$, acts on M with a proper free action. If we apply Van Kampen's theorem $n + 1$ times to M_1, keeping in mind how the solid tori were sewn in, the fundamental group of M_1 can be calculated. The calculation yields:

$$\pi_1(M_1) = (a_i,b_i,q_j,h\colon \pi_* h^{-b},[a_i,h],[b_i,h],[q_j,h],q^{\alpha_j}h^{\beta_j}),$$

where $\pi_* = q_1 \cdots q_n[a_1,b_1] \cdots [a_g,b_g]$.

One final observation is needed before we are in a position to consider specific examples. Suppose we have the 3-manifold $M = \{b;(o,g,0,0);(\alpha_1,\beta_1), \cdots, (\alpha_n,\beta_n)\}$. Let $k \geq 2$ be an integer such that $(k,\alpha_i) = 1$ for $i = 1,\ldots,n$.

There is a proper free Z_k action induced on M by the subgroup Z_k of the given standard $SO(2)$ action. The orbit space $M^* = M/Z_k$ admits an effective $SO(2)$ action induced by the original one on M. We wish to determine a set of invariants for M^* and this induced $SO(2)$ action.

Let T be a solid torus with the standard linear action (α, ν), where $s\alpha + tk = 1$ and $0 < s < k$. If we let T^* denote the orbit space T/Z_k, T^* has an induced standard linear $SO(2)$ action $(\alpha, t\nu)$.

Recall the construction of M from the system of invariants. $M_0 = M^+ \times S^1 - \mathrm{Int}(n \text{ solid tori})$. We let $M_0^* = M_0/Z_k$, which is just $M^+ \times (S^1/Z_k)$ less the interior of n solid tori. Under the projection $\pi : M \longrightarrow M^*$, a principal orbit h is wrapped around its image, h^*, k times. A cross-section q on one of the toral boundaries of M_0 is mapped by π bijectively to a cross-section q^*.

Hence the orbit space M^* is obtained from M_0 by sewing in the solid tori ${}_jV^*$ by matching principal orbits with principal orbits as before. The only difference in the construction is that the cross-section ${}_jq^*$ on the boundary ${}_jT^*$ of ${}_jV^*$ which is matched to the meridian on the boundary of M_0 now satisfies the homology relation ${}_jm \sim \alpha_j\,{}_jq^* + k\beta_j h^*$. We do not require α_j and β_j to satisfy the normalization conditions $0 < \beta_j < \alpha_j$ as before. The solid torus ${}_0V$ corresponding to b will come down to one, ${}_0V^*$ in M^*, which is sewn in the same as before except with b replaced by kb. Therefore

$$M^* = \{kb; (o,g,0,0); (\alpha_1, k\beta_1), \cdots, (\alpha_n, k\beta_n)\}^N,$$

where the N indicates that the invariants are not necessarily in normal form (i.e. we do not require $0 < k\beta_i < \alpha_i$). This set of invariants can be put in normal form by reducing the $k\beta_i$ terms modulo α_i for each i and adjusting kb to compensate for the "untwisting" of the solid tori corresponding to the exceptional orbits. This normalization seems very complicated in general, but is is computable in our special cases.

We first consider a class of examples which indicates that we cannot conclude in Theorem 2.16 that the 3-manifolds which satisfy the hypothesis are products of the form $W \times S^1$.

<u>Example 3.1</u>: Let M be the closed, orientable, irreducible 3-manifold $\{-1; (o,g,0,0); (\lambda+1, 1), (\lambda+1, \lambda)\}$, where $\lambda, g > 0$. Then M is a proper k-sheeted covering of itself for every $k \equiv 1 \pmod{\lambda+1}$. Moreover, $H_1(M;Z)$ is a free abelian group of rank $2g + 1$, but M is not a product of the form $W \times S^1$.

<u>Proof</u>: From the above discussion the orbit space M^* of the induced free Z_k action is homeomorphic to $\{-k; (o,g,0,0); (\lambda+1, k), (\lambda+1, k\lambda)\}^N$. Computation of $H_1(M^*;Z)$ from this set of invariants reveals that it is a free abelian group of rank $2g + 1$. Using this information we can determine the normal form for M^*. It is easy to

see what all the invariants will be except the "b" term. Hence we have $M^* = \{b';(o,g,0,0);(\lambda+1,1),(\lambda+1,\lambda)\}$ with b' undetermined. It follows that

$$H_1(M^*;Z) = [\bigoplus_{2g} Z] \oplus (q_1,q_2,h \mathrel{\dot{\vdots}}_A q_1q_2h^{-b'},q_1^{\lambda+1}h^{\lambda},q_2^{\lambda+1}h).$$

The group given by this presentation is free if and only if $b' = -1$. Hence $M = M^*$, i.e. M is a proper k-sheeted covering of itself for $k \equiv 1 \pmod{\lambda+1}$.

The given standard $SO(2)$ action is the only one which can be defined on M [14, Theorem 4]. Therefore M cannot be homeomorphic to the product of a 2-manifold and S^1.

It is interesting to note that if we were to let $g = 0$ in this class of manifolds we would get a handle. That is

$$S^1 \times S^2 = \{-1;(o,0,0,0);(\lambda+1,1),(\lambda+1,\lambda)\}.$$

It is natural to ask at this point what conditions of this type are necessary to characterize products of the form $W \times S^1$. Kwun has posed the following question.

Question: If a closed orientable 3-manifold M covers itself properly k times, for every prime k, is M a product of a 2-manifold and S^1?

If we alter the question by only requiring M to properly cover itself k times for every odd prime k, we can answer in the negative with an example. Let M be a manifold from the class given in Example 3.1 with $\lambda = 1$. Then M properly covers itself k times for every odd integer k, $H_1(M;Z)$ is free, and M is not a product.

Another description of M when $g = 1$ may be illustrative. Let T_2 be a closed surface of genus 2. Consider the homeomorphism $h : T_2 \longrightarrow T_2$ obtained by interchanging the holes of T_2 in such a way that h has exactly two fixed points and $h^2 = 1$. Then $M = T_2 \times I/h$.

Example 3.2: Let $k \geq 2$ be given and let M be the closed, orientable, irreducible 3-manifold described by $\{b;(o,g,0,0);(\alpha_1,\beta_1), \cdots,(\alpha_n,\beta_n)\}$, where $b(k+1) + n = 0$, $\alpha_i = (k+1)\beta_i$ for all i, and $g,n > 0$. Then M is a proper k-sheeted covering of itself and is not a product of a 2-manifold and S^1.

Proof: The orbit space $M^* = M/Z_k$ is homeomorphic to $\{kb;(o,g,0,0);(\alpha_1,k\beta_1); \ldots; (\alpha_n,k\beta_n)\}^N$. Reversing the orientation on M^* changes the invariants by replacing kb with $-n-kb = b$ and by replacing $k\beta_i$ with $\alpha_i - k\beta_i = \beta_i$ [14]. Again M is not a product by [14, Theorem 4].

We now wish to consider some examples related to Theorem 1.9. In particular we show that we cannot drop from the hypothesis either that the Z_k action be proper or the condition that $H_1(M^*;Z)$ have no k torsion.

Example 3.3: Let M be the closed, orientable, irreducible 3-manifold $\{b;(o,g,0,0);(\alpha,\beta)\}$. The given $SO(2)$ action has the single isotropy group Z_α. If $(k,\alpha) = 1$ and $k \geq 2$, there is a proper free Z_k action on M such that $H_1(M;Z)$ has k torsion. Moreover, no M in this class fibers over the circle.

Proof: The orbit space corresponding to $Z_k \subset SO(2)$ is homeomorphic to $\{kb;(o,g,0,0);(\alpha,k\beta)\}^N$. From this we note that $H_1(M^*;Z) = [\bigoplus_{2g} Z] \oplus (q,h :_A qh^{-kb}, q^{\alpha}h^{k\beta})$. A simple reduction yields $H_1(M\ ;Z) = [\bigoplus_{2g} Z] \oplus Z_{kp}$, where

$p = |b\alpha + \beta|$.

To show that M does not fiber over the circle for any g, notice that $H_1(M;Z) = [\bigoplus_{2g} Z] \oplus (q,h :_A qh^{-b}, q^{\alpha}h^{\beta})$,

where h is the element generated by a principal orbit of the $SO(2)$ action on M. The order of h in $H_1(M;Z)$, $o(h)$, is always finite. In particular, $o(h) = |b\alpha + b|$. But by the next theorem, manifolds of this type do not fiber over the circle unless $o(h)$ is infinite.

Theorem [15, Theorem 4]: Let M be a closed, orientable 3-manifold such that $M = \{b;(o,g,0,0);(\alpha_1,\beta_1),\cdots,(\alpha_n,\beta_n)\}$. Then either M is a T-bundle over S^1 (T denotes a torus of genus 1) or M fibers over S^1 if and only if the order of the element in $H_1(M;Z)$ generated by a principal orbit h is infinite.

When $g = 0$, the manifolds in Example 3.3 are lens spaces. In particular, $L(\beta,-\alpha(\text{mod } \beta)) = \{0;(o,0,0,0);(\alpha,\beta)\}$. Therefore, if $(k,\alpha) = 1$, the lens space $L(\beta,-\alpha(\text{mod } \beta))$ is a proper k-sheeted covering of $L(k\beta,-\alpha(\text{mod } k\beta))$.

Example 3.4: Let M be the closed, orientable, irreducible 3-manifold $\{0;(o,1,0,0);(2,1),(2,1)\}$. There is a free Z_2 action on M that is not proper and is such that the orbit space $M^* = \{0;(o,1,0,0);(2,1)\}$. M does not fiber over the circle, although $H_1(M^*;Z) = Z \oplus Z$.

Proof: As in the construction of M, let M_0 denote M less two invariant tubular neighborhoods of its exceptional orbits. Then $M_0 = T^+ \times S^1$, where T^+ denotes a torus of genus 1 less the interiors of two disjoint closed disks. These disks can be chosen such that there is a free involution of T^+ which interchanges its two circle boundary components. This involution induces a free Z_2 action on M_0 which can be extended to M. It is clear the M/Z_2 is the manifold M^* given above. To see that M does not fiber over the circle, all we need observe is that the order of the element generated by a principal orbit in $H_1(M;Z) = Z \oplus Z \oplus Z_4$ is two.

In light of [15, Theorem 4], the next theorem indicates the close connection between the properties of fibering over the circle and not having k torsion in $H_1(M/Z_k;Z)$ when in the class of closed orientable 3-manifolds with effective $SO(2)$ actions.

Theorem 3.5: Let $M = \{b;(o,g,0,0);(\alpha_1,\beta_1),\cdots,(\alpha_n,\beta_n)\}$. Let $k \geq 2$ be such that $(k,\alpha_i) = 1$ for $i = 1, \ldots, n$. Denote the elements of $H_1(M;Z)$ and $H_1(M/Z_k;Z)$ generated by a principal $SO(2)$ orbit by h and $\bar{h}$ respectively. Then $o(\bar{h}) = k\, o(h)$, where o denotes the order.

<u>Proof</u>: $M/Z_k = \{kb; (o,g,0,0); (\alpha_1, k\beta_1), \cdots, (\alpha_n, k\beta_n)\}$. The calculation of the first integral homology groups of M and M/Z_k yields:

$$H_1(M;Z) = [\underset{2g}{\oplus} Z] \oplus (q_1, \ldots, q_n, h :_A q_1 \cdots q_n h^{-b}, q_i^{\alpha_i} h^{\beta_i}),$$

$$H_1(M/Z_k;Z) = [\underset{2g}{\oplus} Z] \oplus (q_1, \ldots, q_n, \bar{h}, x :_A q_1 \ldots q_n x^{-b}, q_i^{\alpha_i} x^{\beta_i}, x = \bar{h}^k).$$

It follows by inspection that $o(\bar{h}) = ko(h)$.

<u>Corollary 3.6</u>: Let $M = \{b; (o,g,0,0); (\alpha_1,\beta_1), \cdots (\alpha_n,\beta_n)\}$. If there is an integer $k \geq 2$ such that $(k,\alpha_i) = 1$ for $i = 1, \ldots, n$, and such that $M/Z_k = M$, where $Z_k \subset SO(2)$, then M can be fibered over the circle.

BIBLIOGRAPHY

[1] R. H. Bing, Locally tame sets are tame, Ann. of Math. 59 (1954) 145-158.

[2] G. Burde and H. Zieschang, A topological classification of certain 3-manifolds, Bull. Amer. Math. Soc. 74 (1968) 122-124.

[3] M. J. Greenberg, Lectures on Algebraic Topology, Benjamin, 1967.

[4] S. T. Hu, Homotopy Theory, Academic Press, 1959.

[5] D. Husemoller, Fibre Bundles, McGraw-Hill, 1966.

[6] H. Kneser, Geschlossen Flachen in dreidimensionalen Mannigfaltigkeiten, Jahresbericht der Deutschen Math. Vervinigung, 38 (1929) 248-260.

[7] K. W. Kwun, 3-Manifolds which double-cover themselves, American Journal of Math. (to appear).

[8] G. R. Livesay, Fixed-point-free involutions on the 3-sphere, Ann. of Math. 72 (1960) 603-611.

[9] J. Milnor, Topology from the Differential Viewpoint, University of Virginia Press, 1965.

[10] J. Milnor, A unique decomposition theorem for 3-manifolds, American Journal Math. 84 (1962) 1-7.

[11] E. E. Moise, Affine structures in 3-manifolds, V, Ann. of Math. 56 (1952) 96-114.

[12] L. P. Neuwirth, A topological classification of certain 3-manifolds, Bull. Amer. Math. Soc. 69 (1963) 372-375.

[13] P. Orlik, E. Vogt and H. Zieschang, Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten, Topology 6 (1967) 49-64.

[14] P. Orlik and F. Raymond, Actions of SO(2) on 3-manifolds, Proceedings of the Conference on Transformation Groups (to appear).

[15] ______________________, On 3-manifolds with local SO(2) actions, (to appear).

[16] F. Raymond, Classification of the actions of the circle on 3-manifolds, Trans. Amer. Math. Soc. 131 (1968), 51-78.

[17] E. H. Spanier, Algebraic Topology, McGraw-Hill, 1966.

[18] J. Stallings, On fibering 3-manifolds, Topology of 3-manifolds and Related Topics, Prentice-Hall, 1962, 95-100.

[19] N. Steenrod, The Topology of Fiber Bundles, Princeton University Press, 1951.

[20] J. H. C. Whitehead, On 2-spheres in 3-manifolds, Bull. Amer. Math. Soc. 64 (1958) 161-166.