

ON THE AUTOMORPHISM GROUP OF
AN INTEGRAL GROUP RING

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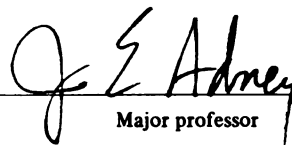
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ABSTRACT

ON THE AUTOMORPHISM GROUP OF AN INTEGRAL GROUP RING

By

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Let G be a finite group and let $Z(G)$ denote the integral group ring of G . The primary purpose of this dissertation is to study $A(G)$, the group of ring automorphisms of $Z(G)$.

If $f \in A(G)$, we say that f is a normalized automorphism if $f(g)$ has augmentation one for all $g \in G$. The set of normalized automorphisms of $Z(G)$ form a subgroup of $A(G)$ denoted by $NA(G)$. Further, little generality is lost by studying $NA(G)$ over $A(G)$.

In Chapter I, we develop some basic facts about $NA(G)$. It is noted that the elements of $NA(G)$ act as a permutation group on the class sums of G , the representations of G , and the characters of G . Next, some subgroups of $NA(G)$ are introduced. First we let $CP(G)$ denote the subgroup of elements of $NA(G)$ which fix every class sum of G . It is well-known that the action of an element of $CP(G)$ is equal to conjugation by a unit in the group ring of G over the rationals. We also let $EA(G)$ denote the subgroup $CP(G)Aut(G)$, $Aut(G)$ the automorphism group of G , and let $I(G)$ denote the group of inner automorphisms of $Z(G)$. In addition, a normalized automorphism which lies in $EA(G)$ is said to have an elementary representation and if $EA(G) = NA(G)$ we say that G is an E.R. group.

The questions of when $NA(G)$, $EA(G)$, $CP(G)$ are each equal to $Aut(G)$ are considered in Chapter I. It was previously known that $NA(G) = Aut(G)$ is equivalent to (1) $I(G) \leq Aut(G)$, (2) G is either abelian or a Hamiltonian 2-group, and (3) $Z(G)$ has only trivial units of finite order. Here we extend the list of equivalences to (4) G is the only group basis of $Z(G)$, (5) $[NA(G) : Aut(G)]$ is finite, and (6) $I(G)$ is periodic. Necessary and sufficient conditions are determined for $Aut(G)$ to equal $EA(G)$, $CP(G)$, and $I(G)$.

Another topic considered in Chapter I is the following. In his Ph.D. Thesis (Michigan State University, 1971), C.F. Brown showed that $Aut(G)$ has a normal complement in $NA(G)$ when G is metabelian. For $N \triangleleft G$, let $\Delta(N)$ denote the kernel of the natural map from $Z(G)$ to $Z(G/N)$ and set

$$W(G,N) = \{f \in NA(G) \mid f(g) \equiv g \bmod \Delta(N)\Delta(G)\}.$$

Then $W(G,N)$ is a subgroup of $NA(G)$. Further, if G is metabelian, $W(G,G')$ is exactly the complement Brown obtained. This led to the question of when is $W(G,N)$ a complement for $Aut(G)$ in $NA(G)$? It is shown that if one wants $W(G,N) \cap Aut(G) = 1$, one in general needs N abelian. Then, if N is abelian, we obtain $W(G,N)$ is a complement for $Aut(G)$ in $NA(G)$ if and only if G/N is either abelian or a Hamiltonian 2-group.

Finally in Chapter I, we consider the following. Let $N \triangleleft G$. If $f(\Delta(N)) = \Delta(N)$ for all f in a subset S of $NA(G)$, we say N is S -admissible. If N is $NA(G)$ -admissible we say that N is NA -characteristic. The admissibility of some subgroups of G under

various subgroups of $NA(G)$ is studied. We also obtain that $N \triangleleft G$ is NA-characteristic if and only if it is characteristic in G provided that G is either an E.R. group or contains an abelian normal subgroup A such that $W(G,A)$ is a complement for $Aut(G)$ in $NA(G)$.

One of the major goals of this dissertation was to determine E.R. groups. That is, when is $EA(G) = NA(G)$, or alternately, when can every element f of $NA(G)$ be written in the form $f(x) = u(\sigma(x))u^{-1}$ for all $x \in Z(G)$ where $\sigma \in Aut(G)$ and u is a unit in the group ring of G over the rationals? In Chapter II, part of Chapter IV, and in Chapter V, we consider this problem. It was previously known that (1) class ≤ 2 nilpotent groups, (2) groups containing a cyclic normal subgroup of index p , (3) groups G where $|G'| = 2$ or 3 , (4) groups with at most one non-linear character, and (5) S_n for $n = 1, \dots, 10$ are all E.R. groups. Some of the E.R. groups obtained in Chapter II and IV are (6) groups G of the form $G = AB$ where A is a cyclic normal subgroup of G and B is an abelian subgroup of G , (7) groups G where $G/Z(G)$ is metacyclic, $Z(G)$ the center of G , (8) groups G where $|G'| = p$, (9) p -groups containing a maximal abelian normal subgroup which is cyclic, and (10) p -groups G of the form $G = AB$ where A is an abelian normal subgroup of G with $A \geq Z(G)$ and $A/Z(G)$ elementary abelian of order p^2 and B is an abelian subgroup of G . Most of these results are obtained by studying the action of $W(G,G')$ on the irreducible characters of G . Included in this is a crucial lemma on the faithful irreducible characters of a metabelian group developed in Chapter II.

Finally, in Chapter V, we show that S_n is an E.R. group for any positive integer n .

One other problem considered in Chapter IV is when is $W(G, G') \leq CP(G)$ for a metabelian group G ? This was known to be true in cases (1), (3), and (4) of the previous paragraph. In Chapter IV, we extend this to cases (6), (7), and (9) of the previous paragraph provided that G is a p -group and $p > 2$.

Chapter III is concerned with normalized automorphisms of direct products. Suppose $G = G_1 \times \dots \times G_n$. It is shown that $NA(G_1) \times \dots \times NA(G_n)$ has a normal complement in $NA(G)$ which lies in $CP(G)$ provided $(|G_i|, |G_j|) = 1$ for $i \neq j$. A similar result is obtained for $W(G, A)$ if $W(G, A)$ is a complement for $Aut(G)$ in $NA(G)$ for an abelian normal subgroup A of G . In the above two cases we also obtain that G is an E.R. group if each G_i is an E.R. group.

In the final chapter, Chapter VI, a technique is presented for extending the known groups for which the integral group ring problem holds. Using this technique we solve the group ring problem for S_4 . However, the technique used here involves lifting automorphisms. The remainder of the chapter involves a discussion of the problem of lifting automorphisms.

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INDEX OF NOTATION

I. Relations:

$\leq$	Is a subset of
$<$	Is a proper subset of
$\triangleleft$	Is a normal subgroup of
$\cong$	Is isomorphic to
$h \equiv g \pmod{S}$	Means $h + S = g + S$ where h and g can either be elements or subsets of a ring and S is an ideal in the ring.

II. Operations:

S/T	Quotient ring or group.
$\times$	Direct product of groups.
$\oplus \Sigma$	Direct sum of rings.
$\langle \rangle$	The group generated by
(a,b)	Greatest common divisor of a and b .
$a b$	a divides b .
$ S $	Number of elements of a set S .
$ g $	$ \langle g \rangle $
$[G:N]$	Index of N in G .
$[x,y]$	$x^{-1}y^{-1}xy$
x^u	$u^{-1}xu$
$\tau_u(x)$	The map $x \rightarrow uxu^{-1}$.

$\iota(x)$ The augmentation map defined by

$$\iota(x) = \sum_{g \in G} a_g \quad \text{where } x = \sum_{g \in G} a_g \text{ is an element of the group ring } R(G).$$

III. Groups, Rings, and Modules:

G A finite group.

C_g The conjugacy class of $g \in G$.

$\overline{C}_g$ The class sum of $g \in G$ defined by

$$\overline{C}_g = \sum_{x \in C_g} x.$$

$[N, M]$ $\langle [n, m] \mid n \in N, m \in M \rangle$

N' $[N, N]$

N'' $[N', N']$

$G^{(n)}$ The n^{th} term of the derived series of G starting with $G^{(1)} = G$.

$\Gamma_n(G)$ The n^{th} term of the lower central series of G starting with $\Gamma_1(G) = G$.

$Z(G)$ The center of G .

$Z_n(G)$ The n^{th} term of the upper central series of G starting with $Z_1(G) = Z(G)$.

$\Phi(G)$ The Frattini subgroup of G .

$C_G(N)$ The centralizer of a subset N of G .

S_n The symmetric group on n letters.

$R(G)$ The group ring of G over the ring R .

$\Delta(S)$ The 2-sided ideal of $R(G)$ generated by $s^{-1}, s \in S$, for a subset S of $R(G)$.

$R(\epsilon)$ The ring R with ϵ adjoined.

$\mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}_n$ $\mathbb{Z}/p^n\mathbb{Z}$	The integers, rationals, complexes, and $\mathbb{Z}/p^n\mathbb{Z}$ respectively.
M^G	The induced $\mathcal{C}(G)$ -module where M is a $\mathcal{C}(N)$ -module for a subgroup N of G .
M_N	M viewed as a $\mathcal{C}(N)$ -module where N is a subgroup of G and M is a $\mathcal{C}(G)$ -module.
Γ^G	The induced representation of G where Γ is a representation of a subgroup N of G .
$\ker \varphi$	The kernel of a homomorphism φ .
$\ker \chi$	The kernel of a character χ of G .

INTRODUCTION

The study of the automorphism group of an integral group ring $Z(G)$ has previously received attention by Brown in [3], Hughes and Pearson in [7], Hughes and Wei in [8], and Sehgal in [14]. The primary purpose of this dissertation is to study the automorphism group of $Z(G)$.

Another problem in integral group rings that has received considerable attention is the group ring problem. In Chapter VI, we will see how knowledge of automorphisms of $Z(G)$ may play a role in solving this problem.

In Chapter I, we obtain some preliminary results about automorphisms of $Z(G)$. In studying the automorphism group of $Z(G)$, it suffices to study the group of normalized automorphisms of $Z(G)$ denoted by $NA(G)$. It was first noted by Sehgal in [14] that if $f \in NA(G)$ and $g \in G$, then $f(\bar{C}_g) = \bar{C}_{g_1}$ for some $g_1 \in G$. Hence $NA(G)$ acts as a permutation group on the class sums of G . Further, $f_1(\bar{C}_g) = f_2(\bar{C}_g)$ for all $g \in G$ if and only if $f_1 = \tau_u f_2$ for some unit u in $Z(G)$. In Section 2 of Chapter I, we note that $NA(G)$ also acts as a permutation group on the representations and characters of G . Finally, we use $CP(G)$ to denote the subgroup $\{f \in NA(G) \mid f(\bar{C}_g) = \bar{C}_g \text{ for all } g \in G\}$.

In Section 3 of Chapter I, we introduce some subgroups of $NA(G)$. First, the automorphism group of G , $Aut(G)$, is naturally

embedded in $NA(G)$ and hence is a subgroup of $NA(G)$. Next, we say that $f \in NA(G)$ has an elementary representation if $f = \tau_u \sigma$ for some unit u in $Z(G)$ and some $\sigma \in \text{Aut}(G)$. We use $EA(G)$ to denote the set of all elements of $NA(G)$ which have an elementary representation. Then $EA(G)$ is a subgroup of $NA(G)$ and in fact $EA(G) = CP(G)\text{Aut}(G)$. Also, if $EA(G) = NA(G)$, we say that G is an E.R. group. Finally, we let $I(G)$ denote the group of inner automorphisms of $Z(G)$.

The question of when $NA(G)$ equals $\text{Aut}(G)$ was previously studied by Brown in [3] and Hughes and Wei in [8]. In Section 3 of Chapter I, this question is again studied and the previously known results are extended. We are also able to obtain necessary and sufficient conditions for $\text{Aut}(G)$ to be equal to $EA(G)$, $CP(G)$ and $I(G)$.

Sections 4 and 6 of Chapter I focus on the question of when $f(\Delta(N)) = \Delta(N)$ for $f \in NA(G)$ and $N \triangleleft G$. This question received some attention by Brown in [3]. One of the major reasons for considering this question here is that if $f(\Delta(N)) = \Delta(N)$, then f induces a normalized automorphism on $Z(G/N)$. For example, the above fact is useful in cases where one wishes to use induction on $|G|$.

In Section 5 of Chapter I, we consider generalizing another result of Brown's. Brown was able to show that $\text{Aut}(G)$ has a normal complement in $NA(G)$ when G is metabelian. Here we are able to generalize Brown's result by introducing some subgroups of $NA(G)$, one of which is exactly Brown's complement when G is metabelian. Further, we are able to determine necessary and sufficient conditions for one of these subgroups to be a complement.

Perhaps one of the outstanding questions in integral group rings is what groups are E.R. groups? This question was first studied by Sehgal in [14] and later by Brown in [3]. In Chapter II, part of Chapter IV, and Chapter V we study this question and determine several types of E.R. groups. For the most part, the previously known E.R. groups were obtained by studying the action of normalized automorphisms on class sums. In a similar manner, we use the action of $NA(G)$ on class sums to show that S_n is an E.R. group in Chapter V. However, it is in general very hard to study the action of $NA(G)$ on class sums. An alternate method is to study the action of $NA(G)$ on the irreducible characters of G . This is the technique that we will use in Chapter II and part of Chapter IV in determining our list of E.R. groups. Included in this is a crucial lemma on the faithful irreducible characters of metabelian groups in Section 3 of Chapter II. It should be noted at this time that I know of no example of a group which fails to be an E.R. group.

The other topic covered in Chapter IV consists of some cases in which the normal complement that we have for $Aut(G)$ in $NA(G)$ when G is metabelian lies in $CP(G)$.

Chapter III is devoted to studying normalized automorphisms in direct products. For the most part, the results of Chapter III were motivated by the following question: if $G = G_1 \times \dots \times G_n$ where each G_i is an E.R. group, then is G an E.R. group? Using the results of this chapter we can obtain a positive answer to the question when either $(|G_i|, |G_j|) = 1$ for $i \neq j$ or when $Aut(G)$ has a normal complement of the form described in Section 5 of Chapter I. We also show that the converse to this question always holds.

Finally, we mention some open questions not considered in this dissertation: the relationships between the results concerning $NA(G)$ and integral representations. For example, $I(G)$ is always contained in $CP(G)$, but there are examples where they are not equal. In [7], Hughes and Pearson obtain $I(S_3) = CP(S_3)$ using integral representations. Perhaps integral representations may be used to study the question of how $I(G)$ and $CP(G)$ are related. Answers to this question may also have some connection with our remarks in Section 4 of Chapter VI towards solving the group ring problem. Conversely, there may also be some applications of our results on $NA(G)$ which would be useful in studying integral representations.

CHAPTER I

PRELIMINARY RESULTS ABOUT AUTOMORPHISMS OF $Z(G)$

Section 1. Facts about Group Bases of $Z(G)$. H is called a group basis of $Z(G)$ if H is a group of units in $Z(G)$ whose elements freely generate $Z(G)$ and if $\iota(h) = 1$ for all $h \in H$. In this section we record some results concerning group bases.

The first result is due to Glauberman and its proof can be found in [12] or [16].

Theorem 1.1.1: Let H be a group basis of $Z(G)$. Then for each $h \in H$ there exists a $g \in G$ such that $\bar{C}_h = \bar{C}_g$.

The next corollary follows easily from Theorem 1.1.1.

Corollary 1.1.2: If H is a group basis of $Z(G)$, then $Z(H) = Z(G)$.

Theorem 1.1.1, as shown in [12] and [16], yields a 1-1 correspondence between the normal subgroups of G and those of a group basis H in the following manner. Let $N \triangleleft G$. Set $K = \bigcup_{g \in N} \{h \in H \mid \bar{C}_h = \bar{C}_g\}$. Then it can be shown that K is a normal subgroup of H .

We state some facts concerning this correspondence from [16].

Theorem 1.1.3: Let H be a group basis of $Z(G)$, N a normal subgroup of G , and let K be the corresponding normal subgroup of H .

(i) Let π be the natural map from $Z(G)$ to $Z(G/N)$. Then

$K = \{h \in H \mid \pi(h) = 1\}$ and $\pi(H)$ is a group basis of $Z(G/N)$.

(ii) $\Delta(N) = \Delta(K)$.

(iii) If M is another normal subgroup of G and if L is the normal subgroup of H corresponding to M , then $[L, K]$ corresponds to $[M, N]$.

Another result concerning class sums, whose proof can be found in [3] or [12], is the following:

Theorem 1.1.4: Let H be a group basis of $Z(G)$. Let $h \in H$ and let $g \in G$ such that $\bar{C}_h = \bar{C}_g$, then $\bar{C}_h n = \bar{C}_g n$ for every integer n .

Finally, we state more results from [16].

Theorem 1.1.5: Let K be a normal subgroup of a group basis H of $Z(G)$, then

$$K/K' \simeq \Delta(K)/\Delta(K)\Delta(H)$$

under the mapping $\varphi_K(\bar{k}) = k - 1 + \Delta(K)\Delta(H)$ where $\bar{k} = kK'$. Hence, if A is an abelian normal subgroup of G and if B is the corresponding normal subgroup of a group basis H of $Z(G)$, then $A \simeq B$ under the mapping $\varphi_B^{-1}\varphi_A$.

Theorem 1.1.6: Let H be a group basis of $Z(G)$, A an abelian normal subgroup of G , and let π denote the natural map from $Z(G)$ to $Z(G/A)$. If $\pi(H) = \pi(G)$, then for each $h \in H$ there exists a unique $g_h \in G$ such that $h \equiv g_h \pmod{\Delta(A)\Delta(G)}$. Further, the mapping $h \rightarrow g_h$ defines an isomorphism from H onto G .

In the process of proving Theorem 1.1.6, Whitcomb actually showed the following result.

Theorem 1.1.7: Let A be an abelian normal subgroup of G and let H be a group basis of $Z(G)$. Suppose $h_1, h_2 \in H$ such that $h_1 \equiv h_2 \pmod{\Delta(A)\Delta(G)}$, then $h_1 = h_2$.

Section 2. Basic Facts about Automorphisms of $Z(G)$. We begin this section by reviewing some facts about automorphisms of $Z(G)$. Let $A(G)$ denote the group of automorphisms of $Z(G)$. By $NA(G)$ we will mean the subgroup of $A(G)$ consisting of all $f \in A(G)$ such that $\iota(f(g)) = 1$ for all $g \in G$. $NA(G)$ is called the group of normalized automorphisms of $Z(G)$. As remarked in [3], little generality is lost by studying $NA(G)$ instead of $A(G)$, since if $f \in A(G)$, the mapping $g \rightarrow \iota(f(g))f(g)$ for $g \in G$ extended linearly to $Z(G)$ is in $NA(G)$.

Let $f \in NA(G)$. It then follows that $f(G)$ is a group basis of $Z(G)$ and that $f(\bar{C}_g) = \bar{C}_{f(g)}$. Thus, by Theorem 1.1.1, $f(\bar{C}_g) = \bar{C}_{g_1}$ for some $g_1 \in G$. Hence, we have that $NA(G)$ acts as a permutation group on the class sums of G .

One of the basic facts about this permutation representation of $NA(G)$ is the following result which is generalized from [14].

Theorem 1.2.1: Let $f_1, f_2 \in NA(G)$, then $f_1(\bar{C}_g) = f_2(\bar{C}_g)$ for all $g \in G$ if and only if $f_1 = \tau_u f_2$ for some unit u in $Z(G)$.

It should be noted that the unit u in Theorem 1.2.1 must necessarily normalize $Z(G)$. That is, $Z(G)^u \leq Z(G)$. We also note that u need not be a unit in $Z(G)$ as Whitcomb has given an example in [16] of an $f \in NA(G)$ such that $f = \tau_u$ where u is a unit in $Z(G)$ but u cannot be taken to be a unit in $Z(G)$ when G is the dihedral group of order 8.

Let $CP(G)$ denote the kernel of this permutation representation of $NA(G)$ on the class sums of G , then by Theorem 1.2.1,

$$\begin{aligned} \text{CP}(G) &= \{f \in \text{NA}(G) \mid f(\bar{C}_g) = \bar{C}_g\} \\ &= \{\tau_u \mid u \text{ is a unit in } \mathbb{Z}(G) \text{ normalizing } \mathbb{Z}(G)\}. \end{aligned}$$

Also, note that $|\text{NA}(G)/\text{CP}(G)|$ is finite since $\text{NA}(G)/\text{CP}(G)$ is isomorphic to a subgroup of S_k where k is the number of conjugacy classes of G .

We next state two lemmas on how $\text{NA}(G)$ acts on class sums.

Lemma 1.2.2: Suppose $f \in \text{NA}(G)$ and that $f(\bar{C}_{g_1}) = \bar{C}_{g_2}$ and $f(\bar{C}_{g_3}) = \bar{C}_{g_4}$ where g_1, g_2, g_3 , and $g_4 \in G$, then there exists $x \in G$ such that $f(\bar{C}_{g_1 g_2}) = \bar{C}_{g_3 g_4} x$.

Proof: We have that $f(\bar{C}_{g_1} \bar{C}_{g_2}) = \bar{C}_{g_3} \bar{C}_{g_4}$.

Now, the class sum $f(\bar{C}_{g_1 g_2})$ will appear as a summand in $f(\bar{C}_{g_1} \bar{C}_{g_2})$ when $f(\bar{C}_{g_1} \bar{C}_{g_2})$ is written as a linear combination of class sums. Also, $\bar{C}_{g_3} \bar{C}_{g_4}$ is a linear combination of class sums of the form $\bar{C}_{g_3^{-1} g_4 x_2}$ where $x_1, x_2 \in G$. Hence, there exist $x_1, x_2 \in G$ such that

$$f(\bar{C}_{g_1 g_2}) = \bar{C}_{g_3^{-1} g_4 x_2} = \bar{C}_{g_3 g_4} x_2 x_1^{-1}$$

and so we have the result.

The second lemma appears in [3] and follows directly from Theorem 1.1.4.

Lemma 1.2.3: Let $f \in \text{NA}(G)$ and suppose that $f(\bar{C}_g) = \bar{C}_{g_1}$ where $g, g_1 \in G$, then $f(\bar{C}_g^n) = \bar{C}_{g_1^n}$ for every integer n . Further, $|g| = |g_1|$.

$\text{NA}(G)$ can also be viewed as a permutation group on the representations or characters of G . For if Γ is a representation of G , we can then define another representation Γ^f of G by

setting $\Gamma^f(g) = \Gamma(f(g))$. (For the purposes of this dissertation we will assume that our representations are over the field of complex numbers.) Similarly, if χ is a character, we can define another character χ^f by setting $\chi^f(g) = \chi(f(g))$. Further, it follows that Γ^f or χ^f is irreducible if and only if Γ or χ is irreducible.

Lemma 1.2.4: If $f(\bar{C}_g) = \bar{C}_{g_1}$, then $\chi^f(g) = \chi(g_1)$ for any character χ of G .

Proof: $\chi^f(g) = \frac{1}{|C_g|} \chi^f(\bar{C}_g) = \frac{1}{|C_g|} \chi(\bar{C}_{g_1}) = \chi(g_1)$.

Lemma 1.2.5: If χ is a faithful character of G , then χ^f is also faithful.

Proof: If $g \in \ker \chi^f$, then $\chi^f(g) = \chi^f(1)$. Let $f(\bar{C}_g) = \bar{C}_{g_1}$, then $\chi(g_1) = \chi^f(1) = \chi(1)$. Hence $g_1 \in \ker \chi$, so $g_1 = 1$. Thus $g = 1$ and we are done.

Lemma 1.2.6: Let $f_1, f_2 \in \text{NA}(G)$, then

(i) $f_1(\bar{C}_g) = f_2(\bar{C}_g)$ if and only if $\chi^{f_1}(g) = \chi^{f_2}(g)$ for every irreducible character χ and

(ii) $f_1(\bar{C}_g) = f_2(\bar{C}_g)$ for all $g \in G$ if and only if $\chi^{f_1} = \chi^{f_2}$ for every irreducible character χ .

Proof: (i) Let $f_1(\bar{C}_g) = \bar{C}_{g_1}$ and $f_2(\bar{C}_g) = \bar{C}_{g_2}$, then $\chi^{f_1}(g) = \chi^{f_2}(g)$ for every irreducible character χ if and only if $\chi(g_1) = \chi(g_2)$ for every irreducible character χ . But $\chi(g_1) = \chi(g_2)$ for every irreducible character χ if and only if $C_{g_1} = C_{g_2}$ and so we have (i).

(ii) follows directly from (i).

Section 3. Some Subgroups of $NA(G)$ and Conditions for Equality with $Aut(G)$. Let $Aut(G)$ denote the automorphism group of G , then $Aut(G)$ is naturally embedded in $NA(G)$ by extending every group automorphism linearly to $Z(G)$. We will henceforth use $Aut(G)$ to denote the image of $Aut(G)$ in $NA(G)$ under this embedding.

Another type of normalized automorphism which will play an important role in what follows is contained in the following definition first used in [3].

Definition: Let $f \in NA(G)$. We say that f has an elementary representation if $f = \tau_u \sigma$ where $\sigma \in Aut(G)$ and u is a unit in $Z(G)$ normalizing $Z(G)$.

Let $EA(G)$ denote the set of $f \in NA(G)$ whose elements have an elementary representation. Then $EA(G)$ is a subgroup of $NA(G)$ and $EA(G) = CP(G)Aut(G)$. If $EA(G) = NA(G)$ we will say that G is an E.R. group.

One basic result concerning elementary representations and group bases is contained in the following lemma.

Lemma 1.3.1: Let H be a group basis of $Z(G)$ such that $H \simeq G$.

Then the following are equivalent:

- (i) Every $f \in NA(G)$ such that $f(G) = H$ is in $EA(G)$.
- (ii) There exists $f \in EA(G)$ such that $f(G) = H$.
- (iii) There exists a unit u in $Z(G)$ such that $G^u = H$.

Proof: (i) $\Rightarrow$ (ii) is clear.

For (ii) $\Rightarrow$ (iii), let $f \in EA(G)$ such that $f(G) = H$. Write $f = \tau_u \sigma$ where u is a unit in $Z(G)$ and $\sigma \in Aut(G)$, then $f(G) = \tau_u \sigma(G) = \tau_u(G) = G^{u^{-1}} = H$.

For (iii) $\Rightarrow$ (i), let $f \in \text{NA}(G)$ such that $f(G) = H$. Since $G^u = H$ we have $\tau_u f(G) = G$. Hence, $\tau_u f \in \text{Aut}(G)$ and so $f = \tau_u^{-1}(\tau_u f) \in \text{EA}(G)$.

One final subgroup of $\text{NA}(G)$ that we will introduce here is the group of inner automorphisms of $Z(G)$ which we will denote by $I(G)$. That is, $I(G) = \{\tau_u \mid u \text{ is a unit in } Z(G)\}$. Then $I(G) \leq \text{CP}(G)$. But in general they are not equal, as we have seen in the remarks after Theorem 1.2.1.

We now turn our attention to the question when are the subgroups of $\text{NA}(G)$ that we have so far defined equal to $\text{Aut}(G)$.

The first question that we consider is when does $\text{Aut}(G) = \text{NA}(G)$? This question was first partially solved in [3] and later completely solved in [8]. In fact, an even stronger statement can be made concerning this question than was made in [8]. In the next theorem, parts (1) - (4) appear in [8].

Theorem 1.3.2: The following are equivalent.

- (1) $\text{NA}(G) = \text{Aut}(G)$.
- (2) $I(G) \leq \text{Aut}(G)$.
- (3) G is either abelian or a Hamiltonian 2-group.
- (4) $\pm g, g \in G$, are the only finite units of $Z(G)$.
- (5) G is the only group basis of $Z(G)$.
- (6) $[\text{NA}(G) : \text{Aut}(G)]$ is finite.
- (7) $I(G)$ is a periodic subgroup of $\text{NA}(G)$.

Since (1) - (4) are equivalent and since clearly (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7), it suffices to show (7) $\Rightarrow$ (3). To do this, we use basically the same proof as was used in [8] to show (2) $\Rightarrow$ (3).

We begin with the following lemma. The technique used in this lemma first appeared in [6].

Lemma 1.3.3: Let $g_1, g_2 \in G$ and n be a positive integer. Let $R = 1 + g_1 + g_1^2 + \dots + g_1^{k-1}$ where $k = |g_1|$ and let $P = g_2(1 - g_1)$. Then $1 - nRP$ is a unit in $Z(G)$. Further, if $\tau_{1-nRP} \in \text{Aut}(G)$ for $n \geq 2$, then $g_1^{g_2}$ is a power of g_1 .

Proof: Since $PR = 0$, it follows that $1 - nRP$ is a unit with inverse $1 + nRP$.

Now, suppose $\tau_{1-nRP} \in \text{Aut}(G)$ and $n \geq 2$, then

$(1 - nRP)g_1(1 + nRP) \in G$. But since

$(1 - nRP)g_1(1 + nRP) = g_1 + n(RP - RPg_1)$, it follows that

$RP - RPg_1 = 0$ since $g_1 + n(RP - RPg_1) \in G$ and since $n \geq 2$. Thus

$$\begin{aligned} RP &= (g_2 + g_1g_2 + \dots + g_1^{k-1}g_2) - (g_2g_1 + \dots + g_1^{k-1}g_2g_1) \\ &= RPg_1 \\ &= (g_2g_1 + g_1g_2g_1 + \dots + g_1^{k-1}g_2g_1) - (g_2g_1^2 + \dots + g_1^{k-1}g_2g_1^2). \end{aligned}$$

It follows from the above equation that $g_2 = g_1^i g_2 g_1$ for some i . Thus $g_2 g_1 g_2^{-1} = g_1^{-i}$ and we are done.

We now prove Theorem 1.3.2 by showing (7) $\Rightarrow$ (3). To do this, we assume G is not abelian and show that G is a Hamiltonian 2-group.

We first show G is Hamiltonian. To do this it suffices to show that $g_1^{g_2}$ is a power of g_1 for every $g_1, g_2 \in G$. Let P and R be as in Lemma 1.3.3. Since we are assuming $I(G)$ is periodic, we have $(\tau_{1-RP})^n \in \text{Aut}(G)$ for some integer n . Further, we may assume $n \geq 2$.

Since $PR = 0$, it follows that $(1 - RP)^n = 1 - nRP$. Hence $(\tau_{1-RP})^n = \tau_{1-nRP} \in \text{Aut}(G)$ and so $g_1^{g_2^2}$ is a power of g_1 by Lemma 1.3.3. Thus G is Hamiltonian.

Now, suppose that G is Hamiltonian but is not a 2-group. Let a and b be generators of the quaternion group of order 8 where $a^4 = b^4 = 1$ and $a^b = a^3$. Let $g \in G$ of the form $g = as$ where s is an element of G of odd prime power order p . Finally, for every positive integer d let ϵ_d denote a primitive d th root of unity.

We have that $Z(\langle g \rangle)$ is isomorphic to $\bigoplus_{d|4p} Z(\epsilon_d)$ under the mapping θ where $\theta(g) = \sum_{d|4p} \epsilon_d$. Let $R = \bigoplus_{d|4p} Z(\epsilon_d)$. It follows that $\theta(Z(\langle g \rangle)) \leq R$ and we can find a positive integer m so that $mR \leq \theta(Z(\langle g \rangle))$.

By the Dirichlet Unit Theorem ([11], p. 128), we can find a unit v in $Z(\epsilon_{4p})$ such that v^i is not in $Z(\epsilon_{4p}^2)$ for any integer i . Also, we can find an integer k so that $1 + 1 + \dots + 1 + v^k$ is a unit in $\theta(Z(\langle g \rangle))$ since the ring R/mR is finite.

Let $u = \theta^{-1}(1 + \dots + 1 + v^k)$, then $\tau_u \in I(G)$. Since $I(G)$ is periodic, we can find an integer n so that τ_u^n is the identity. We will now show that this forces v^{kn} to be in $Z(\epsilon_{4p}^2)$, and hence obtain a contradiction which will prove the result.

Since $u^n \in Z(\langle g \rangle)$, we may write $u^n = \alpha + g\beta$ where $\alpha, \beta \in Z(\langle g^2 \rangle)$. But then $b(\alpha + g\beta) = (\alpha + g\beta)b$. It follows that $(1 - a^2)\beta = 0$. Hence $\beta = (1 + a^2)\sigma$ where $\sigma \in Z(\langle g^2 \rangle)$. Thus, $u^n = \alpha + a(1 + a^2)\sigma = f(g^2) + (g^p + g^{-p})h(g^2)$ where f and h are polynomials over Z .

Thus, if ψ denotes the projection from $\bigoplus_{d|4p} 2(\epsilon_d)$ onto $2(\epsilon_{4p})$, we have

$$\begin{aligned}\psi \theta(u^n) &= f(\epsilon_{4p}^2) + (\epsilon_{4p}^p + \epsilon_{4p}^{-p})g(\epsilon_{4p}^2) \\ &= f(\epsilon_{4p}^2) = v^{nk}.\end{aligned}$$

Hence $v^{nk} \in Z(\epsilon_{4p}^2)$ and we have our contradiction.

We next treat the question of when does $EA(G) = \text{Aut}(G)$?

Clearly, if $NA(G) = \text{Aut}(G)$, then $EA(G) = \text{Aut}(G)$. Conversely, if

$EA(G) = \text{Aut}(G)$, then (2) of Theorem 1.3.2 holds and hence

$NA(G) = \text{Aut}(G)$. Thus we have

Corollary 1.3.4: $EA(G) = \text{Aut}(G)$ if and only if $NA(G) = \text{Aut}(G)$.

Finally, we answer the question of when $\text{Aut}(G)$ equals $CP(G)$ or $I(G)$. In fact, we can state the following.

Corollary 1.3.5: The following are equivalent.

- (1) $CP(G) = \text{Aut}(G)$.
- (2) $I(G) = \text{Aut}(G)$.
- (3) $|G| = 1$ or 2 .

Proof: We first note that (1) $\Rightarrow$ (3). For if $CP(G) = \text{Aut}(G)$, then $EA(G) = \text{Aut}(G)$. But then G is either abelian or a Hamiltonian 2-group by Corollary 1.3.4 and Theorem 1.3.2. But all abelian groups of order greater than 2 and all Hamiltonian 2-groups have non-class preserving automorphisms. Hence $|G| = 1$ or 2 .

Similarly (2) $\Rightarrow$ (3), since if $I(G) = \text{Aut}(G)$, (7) of Theorem 1.3.2 holds. Thus G is abelian or a Hamiltonian 2-group. But then again, $|G| = 1$ or 2 since $\text{Aut}(G)$ has non-class preserving automorphisms except when $|G| = 1$ or 2 .

Conversely, if (3) holds, then both (1) and (2) hold since $\text{Aut}(G) = 1$, $I(G) = 1$, and $\text{CP}(G) = 1$.

Section 4. Admissibility. The purpose of this section is to define what we mean by a normal subgroup of a group being admissible under a set of normalized automorphisms.

If S is a subset of $\text{Aut}(G)$ and if N is a normal subgroup of G admissible under S , then every automorphism of S induces an automorphism on the quotient G/N . In defining admissibility for normal subgroups of G under a set of normalized automorphisms, we define it in a manner so that we induce normalized automorphisms on the quotient.

Definition: Let $N \triangleleft G$ and let S be a subset of $\text{NA}(G)$. We say that N is admissible under S or S -admissible if $f(\Delta(N)) = \Delta(N)$ for all $f \in S$. If N is admissible under $\text{NA}(G)$, we will say that N is NA -characteristic.

Thus, if $N \triangleleft G$ and if N is admissible under S where S is a subset of $\text{NA}(G)$, we have that every element $f \in S$ induces a normalized automorphism $\bar{f}$ of $Z(G/N)$ by setting $\bar{f}(\pi(x)) = \pi(f(x))$ where π is the natural map from $Z(G)$ to $Z(G/N)$ and $x \in Z(G)$.

We will give some NA -characteristic subgroups of a group G and some admissible subgroups under particular sets of automorphisms. However, before doing this, we state the following result.

Lemma 1.4.1: Let $N \triangleleft G$ and let $f \in \text{NA}(G)$. Then $f(\Delta(N)) = \Delta(N)$ if and only if $f(N)$ is the normal subgroup of $f(G)$ corresponding to N .

Proof: If $f(\Delta(N)) = \Delta(N)$, then $\Delta(f(N)) = \Delta(N)$. Hence (i) of Theorem 1.1.3 tells us that $f(N)$ is the normal subgroup of $f(G)$ corresponding to N .

Conversely, if $f(N)$ is the normal subgroup of $f(G)$ corresponding to N , then

$$f(\Delta(N)) = \Delta(f(N)) = \Delta(N)$$

by (ii) of Theorem 1.1.3.

Using the above lemma, it follows that the terms of the derived series are all NA-characteristic. For if $f \in \text{NA}(G)$, then $f(G^{(n)}) = f(G)^{(n)}$. But $f(G)^{(n)}$ is the normal subgroup of $f(G)$ corresponding to $G^{(n)}$ by Theorem 1.1.3(iii). Hence, by Lemma 1.4.1, $f(\Delta(G^{(n)})) = \Delta(G^{(n)})$ and we have the result.

Similarly, one can use the same process to obtain that the terms of the lower central series are all NA-characteristic.

We also mention at this time that the terms of the upper central series are also NA-characteristic. Let $f \in \text{NA}(G)$, then $f(Z(G)) = Z(G)$ by Corollary 1.1.2, so certainly $f(\Delta(Z(G))) = \Delta(Z(G))$. Since f induces an automorphism on $Z(G/Z(G))$, we can again use Corollary 1.1.2 to obtain

$$f(Z_2(G)) \equiv Z_2(G) \pmod{\Delta(Z(G))}$$

and so

$$f(\Delta(Z_2(G))) \leq \Delta(Z_2(G)) + \Delta(Z(G)) = \Delta(Z_2(G)).$$

Hence $f(\Delta(Z_2(G))) = \Delta(Z_2(G))$. Continuing in the above fashion,

$f(\Delta(Z_n(G))) = \Delta(Z_n(G))$ for any integer n .

Thus we can state

Theorem 1.4.2: $G^{(n)}$, $\Gamma_n(G)$, and $Z_n(G)$ are all NA-characteristic.

One result on admissibility that we can state is the following restated from [3].

Theorem 1.4.3: Every normal subgroup of G is $CP(G)$ -admissible.

Further, a normal subgroup of G is $EA(G)$ -admissible if and only if it is a characteristic subgroup of G .

Proof: Let $N \triangleleft G$ and let $\tau_u \in CP(G)$. Since the class sums of elements of $\tau_u(N)$ are all equal to class sums of elements of N , $\tau_u(N)$ must be the normal subgroup of $\tau_u(G)$ corresponding to N . Hence by Lemma 1.4.1, $\tau_u(\Delta(N)) = \Delta(N)$ so that N is $CP(G)$ -admissible.

Now suppose $\tau_u \sigma \in EA(G)$ where $\sigma \in \text{Aut}(G)$, then $\tau_u \sigma(\Delta(N)) = \tau_u \Delta(\sigma(N)) = \Delta(\sigma(N))$. Thus $\tau_u \sigma(\Delta(N)) = \Delta(N)$ if and only if $\sigma(N) = N$ and the second part of the theorem follows.

It should be noted that NA-characteristic implies characteristic. For if $N \triangleleft G$ which is NA-characteristic, then for all $\sigma \in \text{Aut}(G)$,

$$\sigma(\Delta(N)) = \Delta(\sigma(N)) = \Delta(N) .$$

Hence $\sigma(N) = N$, so N is characteristic in G . Theorem 1.4.3 tells us that the converse is true for E.R. groups. In Section 6 of this chapter we will see an additional case where the converse holds.

We conclude this section with the following lemma which will be useful later.

Lemma 1.4.4: Let $G = G_1 \times G_2$ where $(|G_1|, |G_2|) = 1$. Then G_1 and G_2 are NA-characteristic.

Proof: Let $f \in \text{NA}(G)$ and let $H = f(G)$. Let H_1 and H_2 be the normal subgroups of H corresponding to G_1 and G_2 respectively. Then

$$H = H_1 \times H_2 = f(G_1) \times f(G_2) .$$

But then $H_1 = f(G_1)$ and $H_2 = f(G_2)$ since $|H_i| = |f(G_i)|$ for $i = 1, 2$ and since $(|H_1|, |H_2|) = 1$. Then 1.4.1 yields the result.

Section 5. On a Complement for $\text{Aut}(G)$ in $\text{NA}(G)$. In [3], Brown showed that $\text{Aut}(G)$ has a normal complement in $\text{NA}(G)$ when G is metabelian. In this section we will generalize Brown's result by taking a slightly different approach.

The technique Brown used for constructing a normal complement goes as follows. Let G be a metabelian group. Then, if $f \in \text{NA}(G)$ and if $g \in G$, there exists a unique $g_f \in G$ such that

$$f(g) \equiv g_f \pmod{\Delta(G')\Delta(G)}$$

by Theorem 1.1.6. Then Brown showed that the mapping σ_f defined by $\sigma_f(g) = g_f$ is an element of $\text{Aut}(G)$. Further, he showed the mapping β defined by $\beta(f) = \sigma_f$ is a homomorphism of $\text{NA}(G)$ onto $\text{Aut}(G)$ such that $\beta i = 1$ where i is the injection of $\text{Aut}(G)$ into $\text{NA}(G)$. Thus, $\ker \beta$ is a normal complement for $\text{Aut}(G)$ in $\text{NA}(G)$.

Note that

$$\ker \beta = \{f \in \text{NA}(G) \mid f(g) \equiv g \pmod{\Delta(G')\Delta(G)} \text{ for all } g \in G\}.$$

Because of the above characterization of $\ker \beta$, we make the following definition.

Definition: Let $N \triangleleft G$. Then we define

$$W(G, N) = \{f \in \text{NA}(G) \mid f(g) \equiv g \pmod{\Delta(N)\Delta(G)}\}.$$

Thus when G is metabelian, $W(G, G')$ is a normal complement for $\text{Aut}(G)$ in $\text{NA}(G)$. We will now determine when $W(G, N)$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$ for $N \triangleleft G$.

Lemma 1.5.1: Let $N \triangleleft G$. Then N is admissible under $W(G, N)$ and $W(G, N)$ is a subgroup of $\text{NA}(G)$. In addition, if N is NA-characteristic, $W(G, N)$ is normal in $\text{NA}(G)$.

Proof: Let $f \in W(G, N)$. To show N is $W(G, N)$ -admissible, it suffices to show $f(n - 1) \in \Delta(N)$ for all $n \in N$. But

$$f(n - 1) = n - 1 + x$$

where $x \in \Delta(N)\Delta(G)$. Hence,

$$f(n - 1) \in \Delta(N) + \Delta(N)\Delta(G) = \Delta(N).$$

To show that $W(G, N)$ is a subgroup of $\text{NA}(G)$, we first show that $W(G, N)$ is closed under multiplication. Let $f_1, f_2 \in W(G, N)$. Since G is clearly NA-characteristic, $f_i(\Delta(N)\Delta(G)) = \Delta(N)\Delta(G)$ for $i = 1, 2$. Hence,

$$f_1 f_2(g) = f_1(g + x) \equiv f_1(g) \equiv g \pmod{\Delta(N)\Delta(G)}$$

where $x \in \Delta(N)\Delta(G)$. Thus, $f_1 f_2 \in W(G, N)$.

Next, let $f \in W(G, N)$. First, note that $f^{-1}(\Delta(N)\Delta(G)) = \Delta(N)\Delta(G)$ by applying f^{-1} to the equation $f(\Delta(N)\Delta(G)) = \Delta(N)\Delta(G)$. Let $g \in G$ and write $f(g) = g + x$ where $x \in \Delta(N)\Delta(G)$. Then,

$$f^{-1}(g) = g - f^{-1}(x) \equiv g \pmod{\Delta(N)\Delta(G)}.$$

Hence, $f^{-1} \in W(G, N)$ and so $W(G, N)$ is a subgroup.

To show $W(G, N) \triangleleft NA(G)$ when N is NA-characteristic, let $f_1 \in W(G, N)$ and $f \in NA(G)$. Then there exists an $x \in \Delta(N)\Delta(G)$ such that

$$f^{-1}f_1f(g) = f^{-1}(f(g) + x)$$

since f_1 induces the identity on $Z(G)/\Delta(N)\Delta(G)$. Hence,

$$f^{-1}f_1f(g) = g + f^{-1}(x) \equiv g \pmod{\Delta(N)\Delta(G)}$$

since $f^{-1}(\Delta(N)\Delta(G)) = \Delta(N)\Delta(G)$ and we are done.

Lemma 1.5.2: Let $N \triangleleft G$. Then

$$W(G, N) \cap \text{Aut}(G) = \{\sigma \in \text{Aut}(G) \mid \sigma(gN') = gN' \text{ for all } g \in G\}.$$

Proof: Suppose $\sigma \in W(G, N) \cap \text{Aut}(G)$. Then if $g \in G$,

$\sigma(g) \equiv g \pmod{\Delta(N)\Delta(G)}$, so that $\sigma(g) \equiv g \pmod{\Delta(N)}$. Thus

$\sigma(g)g^{-1} \equiv 1 \pmod{\Delta(N)}$, so $\sigma(g)g^{-1} \in N$. But then, since

$\sigma(g)g^{-1} \equiv 1 \pmod{\Delta(N)\Delta(G)}$, $\sigma(g)g^{-1} \in N'$ by Theorem 1.1.5. Hence

$\sigma(gN') = gN'$ for all $g \in G$.

Conversely, suppose $\sigma(gN') = gN'$ for all $g \in G$. Then,

$\sigma(g)g^{-1} \in N'$ for all $g \in G$. Thus again by Theorem 1.1.5,

$\sigma(g)g^{-1} \equiv 1 \pmod{\Delta(N)\Delta(G)}$. Hence, $\sigma(g) \equiv g \pmod{\Delta(N)\Delta(G)}$ and we

have the result.

As an immediate corollary to Lemma 1.5.2 we have

Corollary 1.5.3: Let A be an abelian normal subgroup of G . Then

(i) $W(G, A) \cap \text{Aut}(G) = 1$,

(ii) if $f_1, f_2 \in W(G, A)$ such that $f_1(G) = f_2(G)$, then $f_1 = f_2$.

Proof: (i) is clear since $A' = 1$.

(ii) Suppose $f_1, f_2 \in W(G, A)$ such that $f_1(G) = f_2(G)$.
Then $f_2^{-1}f_1 \in \text{Aut}(G) \cap W(G, A) = 1$. Hence $f_1 = f_2$.

Thus, if we want $W(G, N)$ to be a complement for $\text{Aut}(G)$ in $NA(G)$, we will need N abelian. Further, we will also have to know when $W(G, N)\text{Aut}(G) = NA(G)$. In order to answer this question, we first prove a theorem similar in nature to Theorem 1.3.2.

Theorem 1.5.4: Let $N \triangleleft G$ and suppose for every group basis H of $Z(G)$ isomorphic to G , $\pi(H) = \pi(G)$ where π is the natural map from $Z(G)$ to $Z(G/N)$. Then G/N is either abelian or a Hamiltonian 2-group.

Proof: We assume that G/N is not abelian and argue that G/N is a Hamiltonian 2-group.

We first show G/N is Hamiltonian. For $x \in Z(G)$ let $\bar{x}$ denote $\pi(x)$. It suffices to show $\bar{g}_2$ is a power of $\bar{g}_1$ for any g_1 and $g_2 \in G$.

Let $R = 1 + g_1 + g_1^2 + \dots + g_1^{k-1}$ where $k = |g_1|$, $P = g_2(1 - g_1)$, and let $H = \tau_{1-2RP}(G)$. Then, $1 - 2\bar{R}\bar{P} = 1 - 2\left(\frac{k}{m}\right)(1 + \bar{g}_1 + \dots + \bar{g}_1^{m-1})\bar{g}_2(1 - \bar{g}_2)$ where $m = |\bar{g}_1|$.

Since $\pi(H) = \pi(G) = \pi(G)^{1+2\bar{R}\bar{P}}$, we have $\tau_{1-2\bar{R}\bar{P}} \in \text{Aut}(G/N)$ and so $\bar{g}_2$ is a power of $\bar{g}_1$ by Lemma 1.3.3. Thus G/N is Hamiltonian.

We now show that G/N is a Hamiltonian 2-group. Suppose it is not. Choose $g \in G$ such that $\bar{g} = as$ where s has odd prime power order p and where a is the element of the quaternion group of order 8 generated by a and b where $a^4 = b^4 = 1$ and $a^b = a^3$.

Let ϵ_d denote a primitive d th root of unity. Then $\mathcal{Z}(\langle g \rangle)$ is isomorphic to $\bigoplus \sum_{d|n} \mathcal{Z}(\epsilon_d)$ where $n = |g|$ under the mapping φ defined by $\varphi(g) = \sum_{d|n} \epsilon_d$. Similarly, $\mathcal{Z}(\langle \bar{g} \rangle)$ is isomorphic to $\bigoplus \sum_{d|4p} \mathcal{Z}(\epsilon_d)$ under the mapping θ defined by $\theta(\bar{g}) = \sum_{d|4p} \epsilon_d$.

Extend π to a mapping from $\mathcal{Z}(G)$ to $\mathcal{Z}(G/N)$. Then $\pi(\mathcal{Z}(\langle g \rangle)) = \mathcal{Z}(\langle \bar{g} \rangle)$. Further, it follows that

$$\theta\pi\varphi^{-1}(\mathcal{Z}(\epsilon_d)) = \begin{cases} \mathcal{Z}(\epsilon_d) & \text{if } d \leq 4p \\ 0 & \text{if } d > 4p \end{cases} \quad (*)$$

To see (*), first note that $\theta\pi\varphi^{-1}(\mathcal{Z}(\epsilon_d))$ is either some $\mathcal{Z}(\epsilon_k)$ where $k|4p$ or is 0 since it must be a minimal ideal in $\bigoplus \sum_{d|4p} \mathcal{Z}(\epsilon_d)$ and since $\mathcal{Z}(\epsilon_d)$, $d|4p$, are the unique minimal ideals of $\bigoplus \sum_{d|4p} \mathcal{Z}(\epsilon_d)$. Further, if $\theta\pi\varphi^{-1}(\mathcal{Z}(\epsilon_d)) \neq 0$, $\theta\pi\varphi^{-1}$ restricted to $\mathcal{Z}(\epsilon_d)$ is an isomorphism since $\mathcal{Z}(\epsilon_d)$ is a minimal ideal. Thus, $\theta\pi\varphi^{-1}(\mathcal{Z}(\epsilon_d)) = 0$ if $d > 4p$. Also, $\theta\pi\varphi^{-1}$ restricted to $\bigoplus \sum_{d|4p} \mathcal{Z}(\epsilon_d)$ is then an onto isomorphism. Then, since

$$\theta\pi\varphi^{-1}\left(\sum_{d|n} \epsilon_d\right) = \theta\pi(g) = \theta(\bar{g}) = \sum_{d|4p} \epsilon_d,$$

it must be the case that $\theta\pi\varphi^{-1}(\mathcal{Z}(\epsilon_d)) = \mathcal{Z}(\epsilon_d)$ when $d \leq 4p$.

Now, let v be a unit in $\mathcal{Z}(\epsilon_{4p})$ such that no power of v lies in $\mathcal{Z}(\epsilon_{4p}^2)$. Let $u = \sum_{d|n} \delta_d$, where $\delta_d = 1$ if $d \neq 4p$ and $\delta_d = v$ if $d = 4p$. Then, as in the proof of Theorem 1.3.2, we can find an integer m so that u^m is a unit in $\varphi(\mathcal{Z}(\langle g \rangle))$.

Then $\theta\pi\varphi^{-1}(u^m)$ will be an element of the form $1 + \dots + 1 + v'$, where v' is a unit in $\mathcal{Z}(\epsilon_{4p})$ such that no power of v' lies in $\mathcal{Z}(\epsilon_{4p}^2)$. Further, we have that $1 + \dots + 1 + v'$ will lie in

$\theta(Z(\langle \bar{g} \rangle))$. Let $\bar{w} = \theta^{-1}(1 + \dots + 1 + v')$. Then, as in the proof of Theorem 1.3.2, no power of $\tau_{\bar{w}}$ can be the identity or else some power of v' would lie in $Z(e_{4p}^2)$.

Let $w = \varphi^{-1}(u^m)$ and set $H = G^w$. Then $\pi(H) = \pi(G^w) = \pi(G)^{\bar{w}} \neq \pi(G)$, a contradiction. Hence, G must be a Hamiltonian 2-group.

We now can state

Corollary 1.5.5: If $W(G, N)\text{Aut}(G) = \text{NA}(G)$ where $N \triangleleft G$, then G/N is either abelian or a Hamiltonian 2-group.

Proof: Let H be a group basis of $Z(G)$ such that H is isomorphic to G . Let $f \in \text{NA}(G)$ such that $f(G) = H$. Then, $f = f_1\sigma$ where $f_1 \in W(G, N)$ and $\sigma \in \text{Aut}(G)$. Thus, $f_1(G) = H$.

Since $f_1(g) \equiv g \pmod{\Delta(N)\Delta(G)}$, $\pi(H) = \pi(G)$ where π is the natural map from $Z(G)$ to $Z(G/N)$. Thus, G/N is either abelian or a Hamiltonian 2-group.

We can now state when $W(G, A)$ is a complement for $\text{Aut}(G)$ when A is abelian and normal.

Corollary 1.5.6: Let A be an abelian normal subgroup of G . Then $W(G, A)$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$ if and only if G/A is either abelian or a Hamiltonian 2-group.

Proof: If $W(G, A)$ is a complement for $\text{Aut}(G)$, then G/A is either abelian or a Hamiltonian 2-group by the previous corollary.

Conversely, if G/A is either abelian or a Hamiltonian 2-group, let $f \in \text{NA}(G)$. Set $H = f(G)$. Then by Theorem 1.3.2, $\pi(H) = \pi(G)$ where π is the natural map from $Z(G)$ to $Z(G/A)$. Hence by Theorem 1.1.6, there exists $f_1 \in \text{NA}(G)$ with $f_1(G) = H$ such that $f_1(g) \equiv g \pmod{\Delta(A)\Delta(G)}$. Then $f_1 \in W(G, A)$ and

$f_1^{-1}f \in \text{Aut}(G)$. Hence $f = f_1(f_1^{-1}f) \in W(G,A)\text{Aut}(G)$. Thus $W(G,A)$ is a complement by Corollary 1.5.3.

Another question that arises here is when is $W(G,N)$ a supplement for $\text{Aut}(G)$ in $\text{NA}(G)$? That is, when is $W(G,N)\text{Aut}(G) = \text{NA}(G)$? Corollary 1.5.5 gives us necessary conditions on G/N for this to occur. However, I have been unable to determine necessary and sufficient conditions for $W(G,N)$ to be a supplement for $\text{Aut}(G)$, although we will see one sufficient condition in Section 3 of Chapter VI.

Section 6. Some Further Results on Admissibility. In Section 4 of this chapter we remarked that NA-characteristic implies characteristic and that the converse is true for E.R. groups. In this section we will see that the converse is also true if G contains an abelian normal subgroup A such that $W(G,A)$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$.

Suppose G is a group containing an abelian normal subgroup A such that $W(G,A)$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$. Let $f \in \text{NA}(G)$ and write $f = f_1\sigma$ where $\sigma \in \text{Aut}(G)$ and $f_1 \in W(G,A)$. If N is a characteristic subgroup of G ,

$$f(\Delta(N)) = f_1\sigma(\Delta(N)) = f_1(\Delta(\sigma(N))) = f_1(\Delta(N)) \quad . \quad (*)$$

Hence, if we could show that every normal subgroup of G is $W(G,A)$ -admissible, it would follow that every characteristic subgroup is NA-characteristic by equation (*), and so we show this. In fact, we will see that every solvable normal subgroup is $W(G,A)$ -admissible regardless of whether or not $W(G,A)$ is a complement for $\text{Aut}(G)$.

We begin by showing that every solvable minimal normal subgroup is $W(G,A)$ -admissible.

Lemma 1.6.1: Let A be an abelian normal subgroup of G and let M be a solvable minimal normal subgroup of G , then M is $W(G,A)$ -admissible.

Proof: Since M is solvable, M is abelian. Further, $M \cap A = 1$ or $M \leq A$ and hence MA is abelian.

Now, let $f \in W(G,A)$, $g \in M$, and let B be the abelian normal subgroup of G corresponding to $f(M)$ in $f(G)$. By Theorem 1.1.5, there is a $b \in B$ such that

$$f(g) - 1 \equiv b - 1 \pmod{\Delta(B)\Delta(G)}.$$

Also, B is a minimal normal subgroup of G and so BA is abelian. Thus, since

$$b \equiv f(g) \pmod{\Delta(BA)\Delta(G)} \equiv g \pmod{\Delta(BA)\Delta(G)},$$

$b = g$ by Theorem 1.1.7. Hence, $M = B$ and so

$$\Delta(M) = \Delta(B) = \Delta(f(M)) = f(\Delta(M)).$$

This completes the proof.

We now can prove

Lemma 1.6.2: Let A be an abelian normal subgroup of G and let N be a solvable normal subgroup of G . Then N is $W(G,A)$ -admissible.

Proof: Let $f \in W(G,A)$ and let M be a minimal normal subgroup of G contained in N . Set $\bar{G} = G/M$, $\bar{A} = AM/M$, and $\bar{N} = N/M$. Then by the previous lemma, f induces an automorphism $\bar{f}$ of $Z(\bar{G})$ with $\bar{f} \in W(\bar{G}, \bar{A})$. Hence $\bar{f}(\Delta(\bar{N})) = \Delta(\bar{N})$ by induction on $|G|$. Therefore,

$$f(\Delta(N)) \equiv \Delta(N) \pmod{\Delta(M)} ,$$

or
$$f(\Delta(N)) \leq \Delta(N) + \Delta(M) .$$

Thus, $f(\Delta(N)) = \Delta(N)$ since $\Delta(M) \leq \Delta(N)$.

We now can prove

Theorem 1.6.3: Suppose A is an abelian normal subgroup of G such that $W(G,A)$ is a complement for $\text{Aut}(G)$ in $NA(G)$ and let $N \triangleleft G$. Then N is NA -characteristic if and only if N is a characteristic subgroup of G .

Proof: By our previous remarks we only have to show that N is $W(G,A)$ -admissible. Now, by Corollary 1.5.6, G is solvable so N is solvable. Hence, N is $W(G,A)$ -admissible by Lemma 1.6.2.

As an immediate corollary one should note

Corollary 1.6.4: Let G be a metabelian group and let $N \triangleleft G$. Then N is NA -characteristic if and only if N is a characteristic subgroup of G .

CHAPTER II

SOME METABELIAN E.R. GROUPS

Section 1. Introduction. To my knowledge the previously known list of metabelian E.R. groups are class ≤ 2 nilpotent groups from [14] and from [3] groups with a cyclic normal subgroup of index p , groups with at most one non-linear irreducible character, and groups G in which $|G'| = 2$ or 3 . In this chapter we will extend this list.

We remark, as noted in [3], that in showing a metabelian group is an E.R. group it suffices to show that every element of $W(G, G')$ has an elementary representation since $W(G, G')$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$.

Section 2. The Action of $W(G, G')$ on Class Sums and Characters.

In this section we develop some lemmas which will be useful later concerning the action of $W(G, G')$ on class sums and characters. We also point out that the results do not depend on G being metabelian, unless specifically stated.

Lemma 2.2.1: Let G be a metabelian group and let $f \in W(G, G')$.

Then

- (i) $f(z) = z$ for all $z \in Z(G)$,
- (ii) $f(\overline{C}_g) = \overline{C}_g$ for all $g \in Z_2(G)$.

Proof: (i) By Corollary 1.1.2, $f(Z(G)) = Z(G)$. Thus, if $z \in Z(G)$, $f(z) \in G$. Since $f \in W(G, G')$, $f(z) \equiv z \pmod{\Delta(G')\Delta(G)}$ so that $f(z) = z$ by Theorem 1.1.7.

(ii) If $g \in Z(G)$, we are done by (i). Now suppose $g \in Z_2(G) - Z(G)$. Let χ be an irreducible character of G . By Lemma 1.2.6, it suffices to show $\chi^f(g) = \chi(g)$.

If $\ker \chi > 1$, let $N = \ker \chi$. Since N is $W(G, G')$ -admissible, f induces an automorphism on $Z(\bar{G})$ where $\bar{G} = G/N$ which is in $W(\bar{G}, \bar{G}')$. It then follows that $\chi^f(g) = \chi(g)$ by induction on $|G|$.

If $\ker \chi = 1$, let $x \in G$ such that $g^x \neq g$. Then,

$$\chi(g) = \chi(g^x) = \chi(g)\chi([g, x])/\chi(1)$$

since $[g, x] \in Z(G)$. Hence $\chi(g) = 0$. Similarly, $\chi^f(g) = 0$ since χ^f is also faithful. Thus, $\chi^f(g) = 0 = \chi(g)$ and we are done.

It should be noted at this point that Lemma 2.2.1 immediately gives us a result of [16]; namely that $W(G, G') \leq CP(G)$ when G has nilpotence class ≤ 2 . Further, we then also have the result of [14] that class ≤ 2 nilpotent groups are E.R. groups and so we state this as a corollary.

Corollary 2.2.2: Let G be a class ≤ 2 nilpotent group. Then $W(G, G') \leq CP(G)$ and so G is an E.R. group.

We also remark that if χ is a linear character of G , then $\chi^f = \chi$ for all $f \in W(G, G')$ since elements of $W(G, G')$ induce the identity on $Z(G/G')$ and since $G' \leq \ker \chi$.

From this observation, we get an easy proof of the following result of [3].

Theorem 2.2.3: If G has at most one non-linear irreducible character, then G is an E.R. group.

Proof: By the results of [15], G is metabelian. Thus, if χ is an irreducible character and if $f \in W(G, G')$, $\chi^f = \chi$ since all linear characters are fixed by f and since the non-linear character must be fixed if one exists. Thus, $W(G, G') \leq \text{CP}(G)$ by Lemma 1.2.6 and we are done.

We conclude this section with two lemmas, the first of which also appears in [3].

Lemma 2.2.4: Let $f \in W(G, G')$. If $f(\bar{C}_g) = \bar{C}_{g_1}$, then $g_1 \in gG'$.

Proof: Since $f(\bar{C}_g) = \bar{C}_{g_1} \equiv \bar{C}_g \pmod{\Delta(G')\Delta(G)}$, we have $\bar{C}_{g_1} \equiv \bar{C}_g \pmod{\Delta(G')}$. But then, $g_1 \equiv g^x \pmod{\Delta(G')}$ for some $x \in G$ and so $g_1 = g^x y$ for some $y \in G'$. Hence, $g_1 = g[g, x]y \in gG'$.

The second lemma tells us that we can pick the g_1 of Lemma 2.2.4 so that $g_1 \in \Gamma_3(G)$.

Lemma 2.2.5: Let $f \in W(G, G')$. Then there exists an $x_g \in \Gamma_3(G)$ such that $f(\bar{C}_g) = \bar{C}_{gx_g}$.

Proof: Let $\bar{G} = G/\Gamma_3(G)$. Since $\Gamma_3(G)$ is NA-characteristic, f induces an automorphism of $Z(\bar{G})$ which will be in $W(\bar{G}, \bar{G}')$.

Let $f(\bar{C}_g) = \bar{C}_{g_1}$. Since $\bar{G}$ has nilpotence class ≤ 2 , we have

$$f(\bar{C}_g) = \bar{C}_{g_1} \equiv \bar{C}_g \pmod{\Delta(\Gamma_3(G))}.$$

Therefore, $g_1 = g^x y$ for some $y \in \Gamma_3(G)$ and $x \in G$. Then

$$\bar{C}_{g_1} = \bar{C}_{g^x y} = \bar{C}_{gy} x^{-1},$$

and so we have the result with $x_g = y x^{-1}$.

Section 3. A Lemma on Faithful Characters of Metabelian Groups.

In this section we will obtain a useful lemma about normal subgroups from which a faithful character is induced in a metabelian group.

Let $A \triangleleft G$, let χ be an irreducible character of G , and let M be an irreducible module affording χ . As shown in §50 of [4], if M_1 is a homogeneous component of M_A and if $A^* = \{g \in G \mid gM_1 = M_1\}$, then M_1 is an irreducible $\mathcal{C}(A^*)$ -module and $M_1^G = M$.

If in addition, A is abelian and contains G' we can state the following result about A^* when χ is faithful.

Lemma 2.3.1: Let χ be a faithful irreducible character of G and let A be an abelian normal subgroup of G containing G' . Then $A^* \leq C_G(A)$.

Proof: Let Γ denote the irreducible representation of A^* afforded by M_1 . Since M_1 is a direct sum of isomorphic $\mathcal{C}(A)$ -modules and since A is abelian, it follows that $\Gamma(a)$ is a scalar matrix for all $a \in A$. Thus if $g \in A^*$, $\Gamma(ag) = \Gamma(ga)$ for any $a \in A$.

Since $A^* \geq G'$, $A^* \triangleleft G$ and it follows that $\Gamma^G(ga) = \Gamma^G(ag)$ for all $g \in A^*$ and $a \in A$. But Γ^G is a faithful representation of G . Hence, $A^* \leq C_G(A)$.

Section 4. Groups Containing a Cyclic Normal Subgroup with an Abelian

Supplement. In this section we begin by showing that if $G = BA$ where B is abelian and A is a cyclic normal subgroup of G , then G is an E.R. group. This result was originally motivated by trying to extend a result of [3], that any group with a cyclic normal subgroup of index p is an E.R. group, to metacyclic groups.

Theorem 2.4.1: Suppose $G = BA$ where A is a cyclic normal subgroup of G and B is abelian. Then G is an E.R. group.

Proof: Let $A = \langle a \rangle$. Note that $G' \leq A$, and hence G is metabelian.

Let $f \in W(G, G')$, then $f(\bar{C}_a) = \bar{C}_a^s$ where $a^s \in aG'$ by Lemma 2.2.4 and $(s, |a|) = 1$ by Lemma 1.2.3. Let $g \in G$ and write $g = ba^i$, $b \in B$. We define a mapping σ of G by setting $\sigma(ba^i) = ba^{is}$.

To see that σ is well-defined, we first note that if $a^k \in Z(G)$, then $a^{ks} = a^k$. This follows since $f(a^k) = a^k$ by Lemma 2.2.1 and so

$$f(\bar{C}_a^k) = \bar{C}_a^{ks} = f(a^k) = a^k.$$

Hence, if $b_1 a^i = b_2 a^j$ where $b_1, b_2 \in B$, we have $b_1 b_2^{-1} = a^{j-i} \in B \cap A \leq Z(G)$. Therefore,

$$\begin{aligned} \sigma(b_1 a^i) &= b_1 a^{is} = b_2 a^{j-i} a^{is} \\ &= b_2 a^{js-is} a^{is} = b_2 a^{js} = \sigma(b_2 a^j) \end{aligned}$$

and so σ is well-defined.

σ is also a homomorphism, since if $b_1, b_2 \in B$ and if $a^2 = a^k$,

$$\begin{aligned} \sigma(b_1 a^i b_2 a^j) &= \sigma(b_1 b_2 a^{ik+j}) = b_1 b_2 a^{(ik+j)s} \\ &= b_1 a^{is} b_2 a^{js} = \sigma(b_1 a^i) \sigma(b_2 a^j). \end{aligned}$$

Finally, σ is an automorphism of G , since if $\sigma(ba^i) = 1$, then $ba^{is} = 1$. But then $a^{is} \in B \cap A \leq Z(G)$, so $a^i \in Z(G)$ since $(s, |a|) = 1$. Thus, $a^{is} = a^i$ and hence $ba^i = 1$.

If we can show that for any σ defined in the above manner, $\chi^f = \chi^\sigma$ for every irreducible character χ we will be done, so we show this. We also remark at this time that σ is an automorphism of G which has been constructed to agree with f on the class sums of elements of A by Lemma 1.2.3.

Suppose $\ker \chi > 1$. Let M be a minimal normal subgroup of G contained in $\ker \chi$. Set $\bar{G} = G/M$. It follows that f induces an automorphism on $Z(\bar{G})$ by Lemma 1.6.2 which will be in $W(\bar{G}, \bar{G}')$.

We claim that σ also induces an automorphism on $\bar{G}$. To show this we show $\sigma(M) = M$. If $M \leq A$, then clearly $\sigma(M) = M$. Suppose $M \not\leq A$, then $M \cap A = 1$. Further, $[M, G] \leq M \cap A = 1$ and so $M \leq Z(G)$. Thus, if $ba^i \in M$ where $b \in B$, we have a^i must commute with every element of B and hence $a^i \in Z(G)$. Thus, $a^i = a^{is}$ whence $\sigma(ba^i) = ba^{is} = ba^i$. Hence, $\sigma|_M = 1$ and so $\sigma(M) = M$.

Thus, f and σ both induce automorphisms on $Z(\bar{G})$. Further, if $\bar{f}$ and $\bar{\sigma}$ denote these induced automorphisms on $Z(\bar{G})$, we will have $\bar{f}(\bar{C}_{\bar{a}}) = \bar{C}_{\bar{a}^s}$ where $\bar{a} = aM$ and that $\bar{\sigma}$ is defined in the same manner as σ is on G . Hence, by induction on $|G|$, $\bar{f}(\bar{C}_{\bar{g}}) = \bar{\sigma}(\bar{C}_{\bar{g}})$ for all $\bar{g} \in \bar{G}$ and so $\chi^f = \chi^\sigma$.

Now, suppose $\ker \chi = 1$. Let A^* be as in the setting of Lemma 2.3.1. Then χ is induced from an irreducible character of A^* and $A^* \leq C_G(A)$.

Note that if $g \in G - A^*$, then $\chi(g) = 0$. Also, $\chi^f(g) = 0$ for $g \in G - A^*$ for let $f(\bar{C}_g) = \bar{C}_{g_1}$. Then $g_1 \in gG'$ and hence $g_1 \notin A^*$. Therefore,

$$\chi^f(g) = \chi(g_1) = 0.$$

Similarly, one sees that $\chi^\sigma(g) = 0$ for $g \in G - A^*$ since $\sigma(g) \in gA$, so $\sigma(g) \notin A^*$. Hence, $\chi^f = \chi^\sigma$ on $G - A^*$.

Finally, suppose $g \in A^*$. If $g \in A$, then $\chi^f(g) = \chi^\sigma(g)$ since f and σ agree on class sums of elements of A . If $g \in A^* - A$, write $g = ba^i$ where $b \in B$. Since $A^* \leq C_G(A)$, we have $[b, a] = 1$ and hence $b \in Z(G)$. Thus,

$$f(\overline{C}_{ba^i}) = f((\overline{C}_{a^i})b) = (f(\overline{C}_{a^i}))b = (\overline{C}_{a^{is}})b = \sigma(\overline{C}_{ba^i}).$$

Therefore $\chi^f(g) = \chi^\sigma(g)$, and the proof is complete.

In the following corollary, we mention some groups which satisfy the hypothesis of Theorem 2.4.1.

Corollary 2.4.2: The following are E.R. groups.

- (1) Any metacyclic group.
- (2) Any group which contains a cyclic Hall subgroup containing G' .
- (3) Any group which contains a cyclic Hall subgroup A such that $C_G(A) = A$.
- (4) Any Frobenius group with cyclic Frobenius kernel.

Proof: (1) clearly satisfies the hypothesis of Theorem 2.4.1.

For (2), let B be a complement of the cyclic Hall subgroup $A \geq G'$. Then, $B' \leq B \cap G' = 1$ and hence B is abelian. Thus, one can apply Theorem 2.4.1.

In (3), let B be a complement for A . Then B is isomorphically contained in $\text{Aut}(A)$ since $C_G(A) = A$. But $\text{Aut}(A)$ is abelian since A is cyclic. Hence B is abelian, so $G' \leq A$. Now apply (2).

Finally in (4), one merely notes that if A is the Frobenius kernel, $A = C_G(A)$. Thus one can apply (3).

In [3] Brown showed that if $|G'| = 2$ or 3 , then G is an E.R. group. This would lead one to wonder whether G is an E.R. group when $|G'| = p$. By combining Corollary 2.2.2, Theorem 2.4.1, and the following lemma from [10], whose proof we include for the sake of completeness, we can obtain this result.

Lemma 2.4.3: Suppose G' is a p -group. Let K be a p' -Hall subgroup of G . Then

- (i) K is abelian and
- (ii) there exists a subgroup Y of G such that $G = KYG'$, Y is a p -group, and $[K, Y] = 1$.

Proof: (i) is easy since $K' \leq K \cap G' = 1$.

(ii) Since K is a Hall subgroup of KG' ,

$$G = N_G(K)KG' = N_G(K)G' = KP$$

where P is the p -Sylow of G . Also, $N_G(K) = K(N_G(K) \cap P)$. Let $Y = N_G(K) \cap P$. Then, $G = KYG'$ and $[K, Y] \leq K \cap P = 1$.

Corollary 2.4.4: Suppose $|G'| = p$. Then G is an E.R. group.

Proof: Let K and Y be as in Lemma 2.4.3. If $Y \cap G' = 1$, apply Theorem 2.4.1 with $B = KY$, $A = G'$. If $Y \cap G' \neq 1$, then $K \leq Z(G)$. It then follows that G is nilpotent of class ≤ 2 , so apply Corollary 2.2.2.

Section 5. Groups in which $G/Z(G)$ is Metacyclic. We have already seen that any metacyclic group is an E.R. group. In this section we will show that if $G/Z(G)$ is metacyclic, then G is an E.R. group.

We begin with the following lemma in which parts (2) and (3) are generalized from [5] and part (4) appears in [5]. We will not need part (4) in this section, although it will be used in Chapter IV.

Lemma 2.5.1: Let A be an abelian normal subgroup of G containing $Z(G)$.

(1) If $G = AB$ where B is abelian, then $C_G(A) = A$.

(2) If G/A is cyclic, then $|G'| = |A/Z(G)|$.

(3) If G/A and $A/Z(G)$ are both cyclic, then $G' = \langle [x, a] \rangle$

where x generates G/A and a generates $A/Z(G)$.

(4) If, in addition to the hypothesis of (3), G is a p -group where $p > 2$, then

$$|G'| = |A/Z(G)| = |G/A|.$$

Proof: In (1), let $g \in C_G(A)$. Write $g = ba$ where $b \in B$, $a \in A$. Then $b \in C_G(A)$, so $b \in Z(G)$. Hence $b \in A$, so $g \in A$.

For (2), first note that if x generates G/A , the mapping $g \rightarrow [g, x]$ is a homomorphism of A onto G' ([9], Aufgabe 2, S.259). Moreover, the kernel of this map is $Z(G)$. Hence $|A/Z(G)| = |G'|$.

In (3), one first notes that since a commutes with $[x, a]$, $[x, a]^m = [x, a^m]$ for every integer m . Thus,

$$|[x, a]| = |A/Z(G)| = |G'|$$

and we are done.

Finally, for (4), let $p^n = |G/A|$ and $p^m = |A/Z(G)|$. Since G' is cyclic, G is a regular p -group ([9], Satz 10.2c.), S.322). Thus, $[x, a]^{p^m} = 1 = [x^{p^m}, a]$ ([9], Satz 10.6b.), S.326) and hence

and hence $p^n \leq p^m$. Again by Satz 10.6b.), we have $[x^{p^n}, a] = 1 = [x, a]^{p^n}$. Thus $p^m \leq p^n$ and we are done.

Theorem 2.5.2: Suppose $G/Z(G)$ is metacyclic. Then G is an E.R. group.

Proof: Let $x, a \in G$ such that x generates G/A and a generates $A/Z(G)$ where A is a normal subgroup of G containing $Z(G)$ with $A/Z(G)$ and G/A cyclic. Also, suppose $a^x = a^r z_x$ where $z_x \in Z(G)$. Then, $[a, x] = a^{r-1} z_x$.

Let $f \in W(G, G')$ and suppose $f(\bar{C}_a) = \bar{C}_{aa_1}$ where $a_1 \in G'$.

Then,

$$a_1 = (a^{r-1} z_x)^k = a^{k(r-1)} z_x^k$$

for some integer k by Lemma 2.5.1(3). Also, note that if $a^n \in Z(G)$, then $a_1^n = 1$ since

$$f(\bar{C}_{a^n}) = \bar{C}_{a^n a_1^n} = f(a^n) = a^n.$$

Further,

$$f(\bar{C}_{a_1 z}) = f((\bar{C}_{a_1})z) = (\bar{C}_{a_1 a_1})z = \bar{C}_{a_1 a_1 z}$$

for any $z \in Z(G)$.

Let $g \in G$. Write $g = x^i a^j z$ where $z \in Z(G)$. We define a mapping σ on G by setting $\sigma(g) = x^i a^j a_1^j z$ and claim that this is a well-defined automorphism of G .

To see that σ is well-defined, suppose $x^i a^j z_1 = x^l a^m z_2$ where $z_1, z_2 \in Z(G)$. Then $x^{i-l} \in A$, so $x^{i-l} \in Z(G)$. It then follows that a^{m-j} is also in $Z(G)$. Thus, $a_1^{m-j} = 1$ and so

$$\sigma(x^i a^j z_1) = x^i a^j a_1^j z_1 = x^i a^j a_1^m z_1 = x^l a^m a_1^m z_2 = \sigma(x^l a^m z_2) .$$

It follows that σ is a homomorphism, for let $x^i a^j z_1$ and $x^l a^m z_2$, z_1 and $z_2 \in Z(G)$, be any elements of G . Then,

$$\begin{aligned} \sigma(x^i a^j z_1 x^l a^m z_2) &= \sigma(x^{i+l} a^{jr^l} z_x^{j(r^{l-1} + \dots + r+1)} z_1^m z_2^m) \\ &= x^{i+l} a^{jr^l} a_1^{jr^l} z_x^{j(r^{l-1} + \dots + r+1)} z_1^m a_1^m z_2^m \\ &= x^i a^j (x^l a_1^{jr^l} x^{-l}) z_1 x^l a^m a_1^m z_2 . \end{aligned}$$

Since

$$\sigma(x^i a^j z_1) \sigma(x^l a^m z_2) = x^i a^j a_1^j z_1 x^l a^m a_1^m z_2 ,$$

we see that we need $x^l a_1^{jr^l} x^{-l} = a_1^j$, or $x^{-l} a_1^j x^l = a_1^{jr^l}$ for σ to be a homomorphism. But

$$\begin{aligned} x^{-l} a_1^j x^l &= x^{-l} a^{jk(r-1)} z_x^{jk} x^l \\ &= a^{jk(r-1)} r^l z_x^{jk(r-1)} (r^{l-1} + \dots + r+1) z^{jk} \\ &= a^{jk(r-1)} r^l z_x^{jkr^l} = a_1^{jr^l} , \end{aligned}$$

so σ is indeed a homomorphism.

Finally, σ is an automorphism, for suppose

$\sigma(x^i a^j z) = x^i a^j a_1^j z = 1$. Then $x^i \in A$, so $x^i \in Z(G)$. Thus,

$$a^j a_1^j = (a^{1+k(r-1)} z_x^k)^j \in Z(G) ,$$

so $a^{(1+k(r-1))j} \in Z(G)$. But $(1 + k(r-1), |A/Z(G)|) = 1$, for let

$\bar{f}$ denote the automorphism f induces on $Z(G/Z(G))$. We have

$$\bar{f}(\bar{C}_a^-) = \bar{C}_a^-(1 + k(r-1))$$

where $\bar{a} = aZ(G)$. Thus, $(1 + k(r-1), |A/Z(G)|) = 1$ by Lemma 1.2.3. Hence $|A/Z(G)| \nmid j$, so that $a^j \in Z(G)$. Therefore, $a_1^j = 1$ and hence $\sigma(x^i a^j z) = x^i a^j z = 1$. Thus, σ is 1-1 and we are done.

Next, we claim that for any σ defined in the above manner, $\chi^f = \chi^\sigma$ for every irreducible character χ , from which it follows that G is an E.R. group.

If $\ker \chi > 1$, let M be a minimal normal subgroup of G contained in $\ker \chi$ and let $\bar{G} = G/M$. Then f induces an automorphism on $Z(\bar{G})$ by Lemma 1.6.2 which we will denote by $\bar{f}$. It also follows that $\sigma(M) = M$. For if $M \leq G'$, then M is characteristic in G since G' is a cyclic characteristic subgroup of G . If $M \not\leq G'$, then $M \cap G' = 1$ and hence $[M, G] \leq M \cap G' = 1$. Thus, $M \leq Z(G)$ so that $\sigma|_M = 1$. Hence $\sigma(M) = M$. Therefore, σ also induces an automorphism of $\bar{G}$ which we will denote by $\bar{\sigma}$.

It will follow that $\chi^f = \chi^\sigma$ by induction on $|G|$ provided that $\bar{\sigma}$ is defined in the same manner as σ is on G . Let $\bar{g}$ denote gM for $g \in G$ and let $\bar{A} = \langle \bar{a}, Z(\bar{G}) \rangle$. Since $\bar{f}(\bar{C}_{\bar{a}}) = \bar{C}_{\bar{a}a_1}$, it suffices to show $\bar{\sigma}(x^{-i-j}\bar{g}) = x^{-i-j}a_1^{-j}\bar{g}$ where $\bar{g} \in Z(\bar{G})$ in order to have $\bar{\sigma}$ defined on $\bar{G}$ as σ is on G .

Write $\bar{g} = x^l a^m zM$ where $z \in Z(G)$. Then $a^m \in Z(\bar{G})$ so that

$$(a^m)^x_M = a^{rm} z^m_M = a^m_M.$$

Thus,

$$a_1^m_M = a^{km(r-1)} z^m_M = M.$$

Hence,

$$\begin{aligned}\bar{\sigma}(\bar{x}^{-i}\bar{a}^{-j}\bar{g}) &= \bar{\sigma}(\bar{x}^{-i}\bar{a}^{-j}\bar{x}^{-l}\bar{a}^{-m}\bar{z}) \\ &= \bar{x}^{-i}\bar{a}^{-j}\bar{a}_1^{-j}\bar{x}^{-l}\bar{a}^{-m}\bar{z} = \bar{x}^{-i}\bar{a}^{-j}\bar{a}_1^{-j}\bar{g}\end{aligned}$$

and we have that $\bar{\sigma}$ is defined as σ .

Thus we may assume $\ker \chi = 1$. Since $C_G(A) = A$ by Lemma 2.5.1(1), we have that $A^* = A$ in Lemma 2.3.1. Hence, any faithful character is zero on $G - A$ since the character is induced from A . Therefore, $\chi^f = \chi^\sigma$ on $G - A$. Also, $\chi^f = \chi^\sigma$ on A since $f(\bar{C}_{a_i z}) = \bar{C}_{a_i a_1 z} = \sigma(\bar{C}_{a_i z})$ and the proof is complete.

It is interesting to note at this point that Theorem 2.4.1 follows as a corollary to Theorem 2.5.2 when the cyclic normal subgroup A has odd prime power order by the next lemma.

Lemma 2.5.3: Suppose that $G = BA$ where B is abelian and A is a cyclic normal subgroup of odd prime power order. Then $G/Z(G)$ is metacyclic.

Proof: We have that $\text{Aut}(A)$ is cyclic and that the mapping $b \rightarrow \tau_b$ is a homomorphism from B into $\text{Aut}(A)$. Further, the kernel of this map is $B \cap Z(G)$. Hence, $BZ(G)/Z(G)$ is cyclic and so $G/Z(G) = (BZ(G)/Z(G))(AZ(G)/Z(G))$ is metacyclic.

It should be noted that the conclusion of Lemma 2.5.3 is not true if A does not have odd prime power order. For example, let A be a cyclic group of order 8 and let G be the holomorph of A . Then one can easily verify that $G/Z(G)$ is not metacyclic.

We will also find Lemma 2.5.3 useful in Chapter IV.

CHAPTER III

NORMALIZED AUTOMORPHISMS IN DIRECT PRODUCTS

Section 1. Introduction. One question that naturally arises is the following: suppose G is a direct product of E.R. groups, then is G an E.R. group? Indeed, this would be a useful result to know if one wanted to determine whether nilpotent groups are E.R. groups, since one would then have to only consider p-groups.

In this chapter, we will obtain a positive answer to this question when G contains an abelian normal subgroup A such that $W(G,A)$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$ or when the direct summands have pairwise relatively prime orders. Notice that the latter case will yield the sufficiency of studying only p-groups in the nilpotent case. We will also obtain the converse of this question (that is, if G is a direct product of groups and if G is an E.R. group, then each summand is an E.R. group) without any restriction on G .

When we use the notation $\overline{G}_i$ in this chapter, we will mean the subgroup $G_1 \times \dots \times G_{i-1} \times G_{i+1} \times \dots \times G_n$ of G when $G = G_1 \times \dots \times G_n$.

Section 2. The Containment of $\text{NA}(G_1) \times \dots \times \text{NA}(G_n)$ in $\text{NA}(G)$.

Suppose $G = G_1 \times \dots \times G_n$ and let $f_i \in \text{NA}(G_i)$. We then can extend f_i to an element F_i of $\text{NA}(G)$ by setting

$$F_i(g_1 g_2 \dots g_n) = g_1 g_2 \dots g_{i-1} f_i(g_i) g_{i+1} \dots g_n$$

where $g_j \in G_j$, and then by extending F_i linearly to $Z(G)$. Further, the mapping $f_i \rightarrow F_i$ is an embedding of $NA(G_i)$ into $NA(G)$.

Also, if $f_1 f_2 \dots f_n$, where $f_i \in NA(G_i)$, is an element of $NA(G_1) \times \dots \times NA(G_n)$, it follows that the mapping φ defined by

$$\varphi(f_1 f_2 \dots f_n) = F_1 F_2 \dots F_n$$

defines an embedding of $NA(G_1) \times \dots \times NA(G_n)$ into $NA(G)$.

We summarize these remarks with the following theorem.

Theorem 3.2.1: Suppose $G = G_1 \times \dots \times G_n$, then every f_i in $NA(G_i)$ can be extended to an F_i in $NA(G)$ such that F_i is the identity on $\bar{G}_i$. Further, the mapping φ given by $\varphi(f_1 \dots f_n) = F_1 \dots F_n$ defines an isomorphism of $NA(G_1) \times \dots \times NA(G_n)$ into $NA(G)$.

Because of the embedding, we will also let $NA(G_1) \times \dots \times NA(G_n)$ denote its image under φ .

We conclude this section with the following remark.

Suppose $G = G_1 \times \dots \times G_n$ where $(|G_i|, |G_j|) = 1$ for $i \neq j$.

Then

$$\text{Aut}(G) = \text{Aut}(G_1) \times \dots \times \text{Aut}(G_n) .$$

This might lead one to conjecture that

$$NA(G) = NA(G_1) \times \dots \times NA(G_n) .$$

However, this later equation is not true.

For an example of this, let $G = G_1 \times G_2$ where G_1 is the quaterion group of order 8 and G_2 is the cyclic group of order 3.

Then $NA(G) \neq Aut(G)$ by Theorem 1.3.2. However, also by Theorem 1.3.2, $NA(G_1) = Aut(G_1)$ and $NA(G_2) = Aut(G_2)$. Thus,

$$NA(G_1) \times NA(G_2) = Aut(G) \neq NA(G) .$$

Hence $NA(G_1) \times \dots \times NA(G_n)$ is always contained in $NA(G)$, but they are not in general equal, even if the summands have relatively prime orders.

Section 3. When the Summands have Relatively Prime Orders. In this section we will show that if $G = G_1 \times \dots \times G_n$ where $(|G_i|, |G_j|) = 1$ for $i \neq j$ and if each G_i is an E.R. group, then G is an E.R. group.

To accomplish this we first show the following:

Theorem 3.3.1: Let $G = G_1 \times \dots \times G_n$ where $(|G_i|, |G_j|) = 1$ for $i \neq j$ and let π_i denote the natural map from $Z(G)$ to $Z(G_i)$. Then, for any $f \in NA(G)$, $\pi_i f|_{Z(G_i)}$ is in $NA(G_i)$. Further, the mapping ψ defined by

$$\psi(f) = \pi_1 f|_{Z(G_1)} \pi_2 f|_{Z(G_2)} \cdots \pi_n f|_{Z(G_n)}$$

gives rise to an exact sequence

$$0 \rightarrow \ker \psi \rightarrow NA(G) \xrightarrow{\psi} NA(G_1) \times \dots \times NA(G_n) \rightarrow 0$$

and φ is a splitting map for this sequence. In addition, $\ker \psi \leq CP(G)$.

Proof: In order to show $\pi_i f|_{Z(G_i)}$ is in $NA(G_i)$ we first note that $\pi_i f|_{Z(G_i)}$ is clearly a homomorphism. Further, $\pi_i f|_{Z(G_i)}$ is 1-1, for suppose $x \in Z(G_i)$ such that $\pi_i f(x) = 0$, then

$f(x) \in \Delta(\bar{G}_i)$. But $\bar{G}_i$ is NA-characteristic by Lemma 1.4.4. Hence,

$$x \in f^{-1}(\Delta(\bar{G}_i)) = \Delta(\bar{G}_i)$$

and so $x = 0$. Finally, we show $\pi_i f|_{Z(G_i)}$ is onto $Z(G_i)$. Let $g_i \in G_i$. Since $Z(G)/\Delta(\bar{G}_i) = Z(G)/\Delta(f(\Delta(\bar{G}_i)))$ by Lemma 1.4.4, we can write

$$g_i = \sum_{g \in G_i} a_g f(g) + x$$

where $a_g \in Z$ and $x \in \Delta(\bar{G}_i)$. Hence,

$$\pi_i f(\sum_{g \in G} a_g g) = \pi_i(g_i - x) = g_i$$

and so $\pi_i f|_{Z(G_i)}$ is onto. Thus we have established that

$$\pi_i f|_{Z(G_i)} \in \text{NA}(G_i).$$

Next, we show that the map ψ is a homomorphism. To do this it suffices to show

$$\pi_i f|_{Z(G_i)} \cdot \pi_i f'|_{Z(G_i)} = \pi_i(ff')|_{Z(G_i)}$$

for any f, f' in $\text{NA}(G)$. Let $x \in Z(G_i)$ and write $f'(x) = x_1 + x_2$ where $x_1 \in Z(G_i)$ and $x_2 \in \Delta(\bar{G}_i)$. Then,

$$\pi_i ff'(x) = \pi_i f(x_1 + x_2) = \pi_i f(x_1)$$

since $f(x_2) \in \Delta(\bar{G}_i)$. Similarly,

$$\pi_i f \pi_i f'(x) = \pi_i f \pi_i(x_1 + x_2) = \pi_i f(x_1),$$

whence ψ is a homomorphism.

It now follows that $\psi\varphi = 1$ from the definition of ψ and φ . Hence, ψ is onto and φ splits the sequence.

Finally, we have to show $f \in \text{CP}(G)$ for all $f \in \ker \psi$.

In order to do this it suffices to show $f(\bar{C}_g) = \bar{C}_g$ for any $g \in G_i$. For if $g = g_1 g_2 \dots g_n$ where $g_i \in G_i$, $\bar{C}_g = \bar{C}_{g_1} \bar{C}_{g_2} \dots \bar{C}_{g_n}$ and hence if the class sums of elements of each G_i are fixed, every class sum is fixed.

Suppose $f \in \ker \psi$ and $g \in G_i$. Let $f(\bar{C}_g) = \bar{C}_{g_1}$. Since G_i is NA-characteristic, $f(\Delta(G_i)) = \Delta(G_i)$ and hence $f(G_i)$ is the normal subgroup of $f(G)$ corresponding to G_i . Thus, since $f(g) \in f(G_i)$ and since $\bar{C}_{f(g)} = \bar{C}_{g_1}$, we must have $g_1 \in G_i$. But then $\pi_i f(\bar{C}_g) = \bar{C}_g$ since $\pi_i f|_{Z(G_i)} = 1$ and $\pi_i f(\bar{C}_g) = \pi_i(\bar{C}_{g_1}) = \bar{C}_{g_1}$. Thus $\bar{C}_g = \bar{C}_{g_1}$ and we are done.

Since $\ker \psi$ is then a complement for $\text{NA}(G_1) \times \dots \times \text{NA}(G_n)$ we can state

Corollary 3.3.2: If $G = G_1 \times \dots \times G_n$ where $(|G_i|, |G_j|) = 1$ for $i \neq j$, then $\text{NA}(G_1) \times \dots \times \text{NA}(G_n)$ has a normal complement in $\text{NA}(G)$. Further, this normal complement can be taken to lie inside $\text{CP}(G)$.

As another corollary we get the result we seek concerning E.R. groups.

Corollary 3.3.3: Let $G = G_1 \times \dots \times G_n$ where $(|G_i|, |G_j|) = 1$ for $i \neq j$ and where each G_i is an E.R. group. Then G is an E.R. group.

Proof: First note that if $f_i \in \text{NA}(G_i)$ has an elementary representation, then $\varphi(f_i)$ also has an elementary representation. Thus, $\text{NA}(G_1) \times \dots \times \text{NA}(G_n) \leq \text{EA}(G)$. Therefore,

$$\begin{aligned} \text{NA}(G) &= (\text{NA}(G_1) \times \dots \times \text{NA}(G_n)) \ker \psi \\ &\leq \text{EA}(G) \text{CP}(G) = \text{EA}(G) \end{aligned}$$

and hence $\text{NA}(G) = \text{EA}(G)$. Thus G is an E.R. group.

Section 4. When $W(G,A)$ is a Complement. In the previous section we were successful in showing that a direct product of E.R. groups of relatively prime order is an E.R. group since we could construct the map ψ . If G contains an abelian normal subgroup of G such that $W(G,A)$ is a complement for $\text{Aut}(G)$ in $NA(G)$, we can restrict ourselves to $W(G,A)$ when studying whether G is an E.R. group. In this case we will show that a direct product of E.R. groups is an E.R. group by carrying out the same procedure as in the last section on $W(G,A)$. Here our success will hinge upon the fact that the summands are $W(G,A)$ -admissible when $W(G,A)$ is a complement for $\text{Aut}(G)$.

We first state two lemmas about the map φ .

Lemma 3.4.1: Suppose $G = G_1 \times \dots \times G_n$ and let $N = N_1 \times \dots \times N_n$ where $N_i \triangleleft G_i$, then

$$\varphi(W(G_1, N_1) \times \dots \times W(G_n, N_n)) \leq W(G, N) .$$

Proof: Let $f_i \in W(G_i, N_i)$. It suffices to show $\varphi(f_i) \in W(G, N)$.

If $g \in G_i$, then

$$f_i(g) \equiv g \pmod{\Delta(N_i)\Delta(G_i)}$$

in $Z(G_i)$. Hence by the definition of φ ,

$$\varphi(f_i)(g) \equiv g \pmod{\Delta(N)\Delta(G)} .$$

If $g \in \overline{G_i}$, then $\varphi(f_i)(g) = g$ so

$$\varphi(f_i)(g) \equiv g \pmod{\Delta(N)\Delta(G)} .$$

The result now follows since it is true on a set of generators for G .

Lemma 3.4.2: Let $G = G_1 \times \dots \times G_n$, π_i denote the natural map from G to G_i , and let A be an abelian normal subgroup of G such that $W(G, A)$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$. Then $W(G, A_1 \times \dots \times A_n) = W(G, A)$ and

$$\varphi(W(G_1, A_1) \times \dots \times W(G_n, A_n)) \leq W(G, A)$$

where $A_i = \pi_i(A)$.

Proof: Since $W(G, A)$ is a complement for $\text{Aut}(G)$, G/A is either abelian or a Hamiltonian 2-group. If G/A is abelian, certainly $G/A_1 \times \dots \times A_n$ is abelian. If G/A is a Hamiltonian 2-group, then $G/A_1 \times \dots \times A_n$ is either an abelian 2-group or a Hamiltonian 2-group since every subgroup of $G/A_1 \times \dots \times A_n$ is normal. Thus, $W(G, A_1 \times \dots \times A_n)$ is a complement for $\text{Aut}(G)$ and so $W(G, A_1 \times \dots \times A_n) = W(G, A)$. Also,

$$\varphi(W(G_1, A_1) \times \dots \times W(G_n, A_n)) \leq W(G, A)$$

now follows from the previous lemma.

We next construct a map η similar to ψ .

Theorem 3.4.3: Let $G = G_1 \times \dots \times G_n$, π_i denote the natural map from $Z(G)$ to $Z(G_i)$, A be an abelian normal subgroup of G , and $A_i = \pi_i(A)$. Further suppose $W(G, A)$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$. Then for every $f \in W(G, A)$, $\pi_i f|_{Z(G_i)} \in W(G, A_i)$. Moreover, the mapping η defined by

$$\eta(f) = \pi_1 f|_{Z(G_1)} \pi_2 f|_{Z(G_2)} \cdots \pi_n f|_{Z(G_n)}$$

for $f \in W(G, A)$ gives rise to an exact sequence

$$0 \rightarrow \ker \eta \rightarrow W(G, A) \xrightarrow{\eta} W(G_1, A_1) \times \dots \times W(G_n, A_n) \rightarrow 0$$

such that φ restricted to $W(G_1, A_1) \times \dots \times W(G_n, A_n)$ is a splitting map. Also $\ker \eta \leq CP(G)$.

Proof: First note that G is solvable by Corollary 1.5.6. Hence every normal subgroup of G is $W(G, A)$ -admissible by Lemma 1.6.2.

Let $f \in W(G, A)$. Then $f(\Delta(\bar{G}_1)) = \Delta(\bar{G}_1)$. Thus, going through the same steps as in the proof of Theorem 3.3.1,

$\pi_i f|_{Z(G_i)} \in NA(G_i)$. Further, if $g \in G_i$, $f(g) = g + x$ where $x \in \Delta(A)\Delta(G)$ and so

$$\pi_i f(g) = g + \pi_i(x) \equiv g \pmod{\Delta(A_i)\Delta(G_i)}$$

in $Z(G_i)$. Thus $\pi_i f|_{Z(G_i)} \in W(G_i, A_i)$. Also, one can again mimic the proof of Theorem 3.3.1 to obtain that η is a homomorphism.

It then follows that $\eta\varphi = 1$ on $W(G_1, A_1) \times \dots \times W(G_n, A_n)$ and hence η is onto and φ is the splitting map.

Finally, it follows that $\ker \eta \leq CP(G)$. For if $f \in \ker \eta$, one can again go through the same steps as in showing $\ker \psi \leq CP(G)$ in Theorem 3.3.1 to conclude $f \in CP(G)$ since $f(\Delta(G_i)) = \Delta(G_i)$.

Corresponding to Corollary 3.3.2 we can state

Corollary 3.4.4: Let $G = G_1 \times \dots \times G_n$ and suppose G contains an abelian normal subgroup A such that $W(G, A)$ is a complement for $\text{Aut}(G)$ in $NA(G)$. Then, if π_i denotes the natural map from $Z(G)$ to $Z(G_i)$ and if $A_i = \pi_i(A)$, $W(G_1, A_1) \times \dots \times W(G_n, A_n)$ has a normal complement in $W(G, A)$. Further, this normal complement can be taken to lie inside $CP(G)$.

Finally corresponding to Corollary 3.3.3, we have

Corollary 3.3.5: Let $G = G_1 \times \dots \times G_n$ and suppose that G contains an abelian normal subgroup A such that $W(G, A)$ is a complement for $\text{Aut}(G)$ in $\text{NA}(G)$. Then, if each G_i is an E.R. group, G is an E.R. group.

Proof: It suffices to show $W(G, A) \leq \text{EA}(G)$. Since each G_i is an E.R. group, $W(G_i, A_i) \leq \text{EA}(G_i)$ where $A_i = \pi_i(A)$ and π_i is the natural map from $Z(G)$ to $Z(G_i)$. Hence in $\text{NA}(G)$, $W(G_i, A_i) \leq \text{EA}(G)$. Thus,

$$\begin{aligned} W(G, A) &= (W(G_1, A_1) \times \dots \times W(G_n, A_n)) \ker \pi \\ &\leq \text{EA}(G) \text{CP}(G) \\ &= \text{EA}(G) \end{aligned}$$

and we are done.

In concluding this section, one should also note that Theorem 3.3.3 and Corollary 3.3.4 will hold under the assumptions that G is solvable and A is equal to $A_1 \times \dots \times A_n$ instead of assuming $W(G, A)$ is a complement.

Section 5. The Converse. We conclude this chapter by showing that if $G = G_1 \times \dots \times G_n$ and if G is an E.R. group then each G_i is an E.R. group.

Theorem 3.5.1: Let $G = G_1 \times \dots \times G_n$ and suppose that G is an E.R. group, then each G_i is an E.R. group.

Proof: Let $f \in \text{NA}(G_i)$ and set $H_i = f(G_i)$. By Lemma 1.3.1, it suffices to show the existence of an f' in $\text{NA}(G_i)$ such that $f'(G_i) = H_i$ and $f'(\bar{C}_g) = \bar{C}_g$ in $Z(G_i)$ for all $g \in G_i$.

Set $H = H_i \overline{G_i}$. Then H is a group basis of $Z(G)$. Further, H is isomorphic to G . Hence by Lemma 1.3.1, there exists $f' \in \text{CP}(G)$ such that $f'(G) = H$ since G is an E.R. group. Then, since H_i is the normal subgroup of H corresponding to G_i and since $f'(\Delta(G_i)) = \Delta(G_i)$ by Theorem 1.4.3, $f'(G_i) = H_i$ by Lemma 1.4.1. Hence f' restricted to $Z(G_i)$ lies in $\text{NA}(G_i)$, $f'(G_i) = H_i$, and $f'(\overline{C_g}) = \overline{C_g}$ for all $g \in G_i$ since the class sum of an element of G_i in $Z(G_i)$ is also its class sum in $Z(G)$. This completes the proof.

CHAPTER IV

ELEMENTARY REPRESENTATIONS IN p -GROUPS

Section 1. Introduction. We consider two problems in this chapter. In Section 2 we will see some p -groups which are E.R. groups, although all the E.R. groups obtained here will also be metabelian. We also emphasize again that knowledge of p -groups which are E.R. groups yields knowledge about nilpotent groups which are E.R. groups by Corollary 3.3.3. In Section 3 we will consider the question of when $W(G, G') \leq CP(G)$ for p -groups.

Section 2. p -Groups which are E.R. Groups. One useful fact about p -groups is the following. Let G be a metabelian p -group and let A be a maximal abelian normal subgroup of G containing G' . Since A is a maximal abelian normal subgroup of G , $C_G(A) = A$. Hence, if χ is a faithful character of G , χ is induced from a linear character on A by Lemma 2.3.1. Also, χ is then zero on $G - A$.

One immediate result we can obtain from the above observation on faithful characters is

Theorem 4.2.1: Suppose G is a p -group such that $Z(G)$ is cyclic, $|\Gamma_3(G)| = p$, and G contains a maximal abelian normal subgroup such that $G' \leq A \leq Z_2(G)$. Then $W(G, G') \leq CP(G)$ and hence G is an E.R. group.

Proof: Let χ be an irreducible character of G and let $f \in W(G, G')$. It suffices to show $\chi^f = \chi$.

If $\ker \chi > 1$, let $N = \ker \chi$. Then $N \geq \Gamma_3(G)$ and N is $W(G, G')$ -admissible by Lemma 1.6.2. Hence, f induces an automorphism on $Z(\bar{G})$ which will be in $W(\bar{G}, \bar{G}')$ where $\bar{G} = G/N$. Further, $\chi^f = \chi$ since $\bar{G}$ has class ≤ 2 .

If $\ker \chi = 1$, then $\chi^f = \chi$ since both χ^f and χ are zero on $G - A$ and since f fixes the class sums of elements of A . This completes the proof.

In attempting to prove that a metabelian p -group is an E.R. group, one might try to construct an automorphism of G that agrees with a given $f \in W(G, G')$ on class sums of a maximal abelian normal subgroup A containing G' , as this would be similar to the technique used in proving Theorems 2.4.1 and 2.5.2. One case in which one might first try the above technique is when A is cyclic. However, by the next lemma we will see that we have already treated this case in Chapter II.

Lemma 4.2.2: Let G be a p -group such that G contains a maximal abelian normal subgroup A which is cyclic. Then $G = BA$ where B is abelian.

Proof: Let $\bar{g}$ denote gA for $g \in G$. Since $C_G(A) = A$, it follows that G/A is isomorphically contained in $\text{Aut}(A)$ under the mapping $\bar{g} \rightarrow \tau_{\bar{g}}$.

We may assume G/A is not cyclic, or else we are done. Hence, we have that $p = 2$ and that $|A| \geq 2^3$ since $\text{Aut}(A)$ is cyclic when p is odd or $|A| \leq 2^2$.

Let $A = \langle a \rangle$ and suppose $|A| = 2^m$, $m \geq 3$. Then $\text{Aut}(A) = \langle \alpha \rangle \times \langle \beta \rangle$ where $\alpha(a) = a^5$ and $\beta(a) = a^{-1}$. Thus since G/A is not cyclic, we can choose d and c in G such that d and c generate G/A and so that τ_d is some power of α and $\tau_c = \beta$.

Claim: There exists $b \in G$ such that b and c generate G/A and $[b, c] = 1$.

Note that once we have established the claim, we are done with $B = \langle b, c \rangle$.

Proof of the claim: Let $[d, c] = a^r$. We first show that $2 \mid r$.

Case 1: If $|dA| = 2$.

Let $d^2 = a^j$. Then $2 \mid j$, or else $\langle d \rangle$ would be an abelian normal subgroup of G contradicting the maximality of A . We also have

$$\begin{aligned} (d^2)^c &= (a^j)^c = a^{-j} \\ &= (d^c)^2 = (da^r)^2. \end{aligned} \tag{1}$$

Further, since $|dA| = 2$ and since $|\alpha| = 2^{m-2}$,

$$a^d = a^{5^{2^{m-3}}} = a^{1+2^{m-1}}. \tag{2}$$

Thus from (1) and (2),

$$a^{-j} = da^r da^r = d^2 a^{r(2+2^{m-1})} = a^{j+r(2+2^{m-1})}.$$

Therefore, $r(2 + 2^{m-1}) \equiv -j \pmod{2^m}$. Since $4 \mid -2j$ and since $4 \nmid 2 + 2^{m-1}$, we have $2 \mid r$.

Case 2: If $|dA| > 2$.

Suppose $2 \nmid r$ in this case. Then $a \in G'$ and hence $\Phi(G) = \langle d^2, a \rangle$. Also, $Z(\Phi(G)) \leq C_G(A) = A$ so that $Z(\Phi(G))$ is cyclic. But then $\Phi(G)$ is cyclic by Satz 7.8 c.), S.306 of [9], a contradiction. Hence $2 \mid r$.

To complete the proof of the claim, let $b = da^{r/2}$. Then b and c generate G/A and b and c commute since $b^c = (da^{r/2})^c = da^r a^{-r/2} = da^{r/2} = b$.

Thus we get as a corollary to Theorem 2.4.1

Corollary 4.2.3: Let G be a p -group containing a maximal abelian normal subgroup which is cyclic. Then G is an E.R. group.

Another type of p -group which is an E.R. group is contained in the following result.

Theorem 4.2.4: Let G be a p -group and suppose that $G = BA$ where B is abelian and A is an abelian normal subgroup of G containing $Z(G)$. Further, suppose $A/Z(G)$ is elementary abelian of order p^2 . Then G is an E.R. group.

Proof: Let x and y be elements of A which generate $A/Z(G)$. We may further assume that y is chosen so that $y \in Z_2(G)$. Since $\langle y, Z(G) \rangle$ lies in the center of $G/\langle y, Z(G) \rangle$, $G/\langle y, Z(G) \rangle$ is abelian and hence G has nilpotence class ≤ 3 . Further, if the class of G is 3 then $x \notin Z_2(G)$.

Let $b \in B$. Since $G/\langle y, Z(G) \rangle$ is abelian, $x^b Z(G) = xy^j Z(G)$ for some $0 \leq j < p$. Thus, if the class of G is 3, we can find $b_x \in B$ such that

$$b_x x^b Z(G) = xy^j Z(G)$$

where $0 < j < p$. In the remainder of the proof we will assume $j = 1$, for if not replace y by y^j .

Let $f \in W(G, G')$. Since $y \in Z_2(G)$, we have $f(\bar{C}_y) = \bar{C}_y$ by Lemma 2.2.1. We also know that $f(\bar{C}_x) = \bar{C}_{xz_x}$ for some $z_x \in \Gamma_3(G) \leq Z(G)$ by Lemma 2.2.5. In addition, $|z_x| \leq p$ since $x^p \in Z(G)$ and so

$$f(\bar{C}_{x^p}) = f(x^p) = x^p = \bar{C}_{x^p z_x^p}.$$

Let $g \in G$. Write $g = bx^i y^j z$ where $b \in B$ and $z \in Z(G)$. We define a mapping σ by setting $\sigma(g) = bx^i y^j z_x^i z$ and claim that σ is a well-defined automorphism of G .

To see that σ is well-defined, suppose

$$b_1 x^i y^j z_1 = b_2 x^k y^l z_2$$

where $b_1, b_2 \in B$, $z_1, z_2 \in Z(G)$. Then, $b_1 A = b_2 A$ and so $b_1 b_2^{-1} \in Z(G)$. Thus, $x^{i-k} y^{j-l} \in Z(G)$ and hence $p \mid i-k$. Therefore $z_x^{i-k} = 1$. Hence

$$\begin{aligned} \sigma(b_1 x^i y^j z_1) &= b_1 x^i y^j z_x^i z_1 = b_1 x^i y^j z_x^k z_1 \\ &= b_2 x^k y^l z_x^k z_2 = \sigma(b_2 x^k y^l z_2), \end{aligned}$$

so σ is well-defined.

σ is a homomorphism, for if $b_1 x^i y^j z_1$ and $b_2 x^k y^l z_2$ are any elements of G , we have $(x^i y^j)^{b_2} = x^i y^m z$ for some integer m and some $z \in Z(G)$. Then

$$\begin{aligned}
\sigma(b_1 x^i y^j z_1 b_2 x^k y^l z_2) &= \sigma(b_1 b_2 x^{i+k} y^{m+l} z_1 z_2 z) \\
&= b_1 b_2 x^{i+k} z_x^{i+k} y^{m+l} z_1 z_2 z \\
&= (b_1 x^i y^j z_1 z_x^i) (b_2 x^k y^l z_x^k z_2) \\
&= \sigma(b_1 x^i y^j z_1) \sigma(b_2 x^k y^l z_2)
\end{aligned}$$

and σ is a homomorphism.

Finally, σ is an automorphism. For if

$\sigma(bx^i y^j z) = bx^i z_x^i y^j z = 1$, then $b \in A$ and hence $b \in Z(G)$. Thus, $x^i y^j \in Z(G)$ so that $x^i \in Z(G)$. Then, $z_x^i = 1$ and so $bx^i y^j z = 1$.

At this time, we remark that A is a maximal abelian normal subgroup by Lemma 2.5.1(1) and that $A \geq G'$. Further, if G has class 3, f and σ agree on class sums of elements of A . For let $g \in A$. Write $g = x^i y^j z$ where $z \in Z(G)$, $0 \leq i < p$, and $0 \leq j < p$. If $i = 0$, then $g \in Z_2(G)$ so that

$$f(\bar{C}_g) = \bar{C}_g = \sigma(\bar{C}_g) .$$

If $j = 0$, then

$$f(\bar{C}_g) = f(\bar{C}_{x^i z}) = \bar{C}_{x^i z_x^i z} = \sigma(\bar{C}_{x^i z}) .$$

Finally, if both $i \neq 0$ and $j \neq 0$, let k be an integer such that $ki \equiv j \pmod{p}$. Then

$$(x^i)^{b^k} x_{Z(G)} = x^i y^{ik} z_{Z(G)} = x^i y^j z_{Z(G)} .$$

Hence $x^i y^j = (x^i)^{b^k} x_{z_1}$ for some $z_1 \in Z(G)$. Hence,

$$\begin{aligned}
f(\bar{C}_{x^i y^j z}) &= f(\bar{C}_{x^i z_1 z}) = \bar{C}_{x^i z_x^i z_1 z} \\
&= \bar{C}_{x^i y^j z_x^i z} = \sigma(\bar{C}_{x^i y^j z}) .
\end{aligned}$$

We claim that $\chi^f = \chi^\sigma$ for every irreducible character χ and for any σ defined as above, from which it will follow that G is an E.R. group.

Suppose our claim is false. Let G be a minimal counter-example and let χ be an irreducible character such that $\chi^f \neq \chi^\sigma$.

We first show that χ is not faithful, for suppose it is. If G has class ≤ 2 , then χ, χ^f , and χ^σ are all zero on $G - Z(G)$ since they are all faithful. From this observation it follows that $\chi^f = \chi^\sigma$ since f and σ agree on $Z(G)$. If G has class 3, then the faithful characters χ, χ^f , and χ^σ are all induced from A . But then $\chi^f = \chi^\sigma$ since χ^f and χ^σ are both zero on $G - A$ and since f and σ agree on class sums of A . Hence, χ cannot be faithful.

Thus we have $\ker \chi \cap Z(G) > 1$. Let $N = \ker \chi \cap Z(G)$. Then f and σ are both the identity on N . Therefore, f and σ induce automorphisms on $Z(\bar{G})$ where $\bar{G} = G/N$. Let $\bar{f}$ and $\bar{\sigma}$ denote the induced automorphisms, $\bar{g}$ denote gN for $g \in G$, $\bar{\chi}$ the character χ induces on $\bar{G}$, $\bar{B} = BN/N$, and let $\bar{A} = \langle \bar{x}, \bar{y}, Z(\bar{G}) \rangle$. Also, note that $\bar{f} \in W(\bar{G}, \bar{G}')$. Since $\chi^f \neq \chi^\sigma$, $\bar{\chi}^{\bar{f}} \neq \bar{\chi}^{\bar{\sigma}}$. We will obtain a contradiction by showing $\bar{\chi}^{\bar{f}} = \bar{\chi}^{\bar{\sigma}}$.

Suppose $\bar{x} \in Z(\bar{G})$. Since

$$\bar{f}(\bar{C}_{\bar{x}}) = \bar{C}_{\bar{x}\bar{z}_x} = \bar{C}_{\bar{x}},$$

$\bar{z}_x = 1$ and so $\bar{\sigma} = 1$. Also, $\bar{G}$ has class 2 since $\bar{G}/Z(\bar{G})$ would have to be abelian. Thus,

$$\bar{\chi}^{\bar{f}} = \bar{\chi} = \bar{\chi}^{\bar{\sigma}}.$$

If $\bar{y} \in Z(\bar{G})$, then again $\bar{G}$ would have class ≤ 2 . Thus, since

$$\bar{f}(\bar{C}_{\bar{x}}) = \bar{C}_{\bar{x}\bar{z}_{\bar{x}}} = \bar{C}_{\bar{x}},$$

there exists a $\bar{b}_1 \in \bar{B}$ such that $\bar{x}^{\bar{b}_1} = \bar{x}\bar{z}_{\bar{x}}$. We also must have $\bar{f}(\bar{C}_{\bar{g}}) = \bar{C}_{\bar{g}}$ for $\bar{g} \in \bar{G}$ since $\bar{G}$ has class ≤ 2 . Hence,

$$\begin{aligned} \bar{f}(\bar{C}_{\bar{b}\bar{x}\bar{y}\bar{z}}^{\bar{i}\bar{j}}) &= \bar{C}_{\bar{b}\bar{x}\bar{y}\bar{z}}^{\bar{i}\bar{j}} = (\bar{C}_{\bar{b}\bar{x}\bar{y}\bar{z}}^{\bar{i}\bar{j}})^{\bar{b}_1} \\ &= \bar{C}_{\bar{b}\bar{x}\bar{z}\bar{y}\bar{z}}^{\bar{i}\bar{i}\bar{j}} = \sigma(\bar{C}_{\bar{b}\bar{x}\bar{y}\bar{z}}^{\bar{i}\bar{j}}), \end{aligned}$$

so again $\bar{f}_{\bar{x}} = \bar{\sigma}_{\bar{x}}$.

Finally, suppose neither $\bar{x}$ nor $\bar{y}$ lies in $Z(\bar{G})$. Since $\bar{f}(\bar{C}_{\bar{x}}) = \bar{C}_{\bar{x}\bar{z}_{\bar{x}}}$, if we can show

$$\sigma(\bar{b}\bar{x}\bar{y}\bar{g})^{\bar{i}\bar{j}} = \bar{b}\bar{x}\bar{y}\bar{g}\bar{z}_{\bar{x}}^{\bar{i}\bar{j}} \quad (*)$$

where $\bar{g} \in Z(\bar{G})$, we will again have $\bar{f}_{\bar{x}} = \bar{\sigma}_{\bar{x}}$ since G is a minimal counter-example.

To show (*), it suffices to show that $\bar{\sigma}(\bar{g}) = \bar{g}$. Write $\bar{g} = \bar{b}_1 \bar{x}^{\bar{k}} \bar{y}^{\bar{l}} \bar{z}$ where $\bar{b}_1 \in \bar{B}$, $\bar{z} \in Z(\bar{G})$, $0 \leq \bar{k} < p$, and $0 \leq \bar{l} < p$. Then,

$$\bar{g} = \bar{b}_1 \bar{x}^{\bar{k}} \bar{y}^{\bar{l}} \bar{z} \in \bar{A}$$

and so $\bar{b}_1 \in Z(\bar{G})$. But then $\bar{x}^{\bar{k}} \bar{y}^{\bar{l}} \in Z(\bar{G})$, so $\bar{k} = \bar{l} = 0$. Hence, $\bar{\sigma}(\bar{g}) = \bar{g}$ and so we have (*).

This completes the proof.

Theorem 4.2.4 was originally motivated by the following corollary.

Corollary 4.2.5: Let G be a p -group where $p > 3$. Suppose that every normal abelian subgroup of G is generated by at most two elements. Then G is an E.R. group.

Proof: By Satz 12.4, S.343 of [9], G is one of the following types of groups:

- (1) G is metacyclic.
- (2) $G = \langle x, y, z \mid x^p = y^p = z^{p^n} = [x, z] = [y, z] = 1, y^x = yz^{p^{n-1}} \rangle$.
- (3) $G = \langle x, y, z \mid x^p = y^p = z^{p^n} = [y, z] = 1, y^x = yz^{sp^{n-1}}, z^x = zy \rangle$ where $s = 1$ or s is a quadratic non-residue mod p .

We already know that (1) is an E.R. group. (2) is easily seen to have class 2 and hence is also an E.R. group. In (3), let $B = \langle x \rangle$ and $A = \langle y, z \rangle$. Also in (3), $Z(G) = \langle z^p \rangle$ and A is an abelian normal subgroup. Thus one can apply Theorem 4.2.4.

Finally we mention one more result.

Corollary 4.2.6: Suppose G contains an abelian normal subgroup A such that G/A is cyclic, $Z(G) \leq A$, and $|A/Z(G)| \leq p^2$. Then G is an E.R. group.

Proof: Note that $A/Z(G)$ is either cyclic or elementary abelian of order p^2 . If $A/Z(G)$ is cyclic, apply Theorem 2.5.2. If $A/Z(G)$ is elementary abelian of order p^2 , apply Theorem 4.2.4.

Section 3. When is $W(G, G') \leq CP(G)$? An interesting question related to whether a metabelian group is an E.R. group is when is $W(G, G') \leq CP(G)$? Certainly, if $W(G, G') \leq CP(G)$ where G is metabelian, then G is an E.R. group. In addition, one can obtain

the following generalization of Lemma 2.2.5 by replacing $\Gamma_3(G)$ by N in the proof of Lemma 2.2.5 when G is metabelian.

Lemma 4.3.1: Let N be a normal subgroup of the metabelian group G . Further, suppose that $W(\bar{G}, \bar{G}') \leq CP(\bar{G})$ where $\bar{G} = G/N$. Then for each $f \in W(G, G')$ and for each $g \in G$, there exists an x_g in N such that $f(\bar{C}_g) = \bar{C}_{gx_g}$.

Thus we see that knowledge of when $W(G, G') \leq CP(G)$ will also increase our knowledge on how $W(G, G')$ acts on class sums.

Up to this point, we have seen that $W(G, G') \leq CP(G)$ whenever G has nilpotence class ≤ 2 , G has at most one non-linear character, and when G satisfies the hypothesis of Theorem 4.2.1. In this section we will obtain some metabelian p -groups where this is true.

Before obtaining some p -groups where $W(G, G') \leq CP(G)$, one should note that we do have the following result on direct products as a corollary to Theorem 3.4.3.

Corollary 4.3.2: Let $G = G_1 \times \dots \times G_n$ be a metabelian group. Then, if $W(G_i, G_i') \leq CP(G_i)$ for each i , $W(G, G') \leq CP(G)$.

Proof: If $W(G_i, G_i') \leq CP(G_i)$, then $W(G_i, G_i') \leq CP(G)$. Hence,

$$W(G, G') = (W(G_1, G_1') \times \dots \times W(G_n, G_n')) \ker \eta \leq CP(G) .$$

Then in particular, studying the question of when $W(G, G') \leq CP(G)$ for metabelian nilpotent groups reduces to studying this question for p -groups.

The major case in which we can get p -groups for which $W(G, G') \leq CP(G)$ is the following.

Theorem 4.3.4: Let G be a p -group where $p > 2$ such that $G/Z(G)$ is metacyclic. Then $W(G, G') \leq CP(G)$.

Proof: Let $x, a \in G$ such that $A = \langle a, Z(G) \rangle$ is a normal subgroup of G and such that x generates G/A . By Lemma 2.5.1, we know that $|G/A| = |G'|$. Hence, $|C_a| = |G/C_G(a)| = |G/A| = |G'|$, so that $C_a = aG'$.

In the proof of Theorem 2.5.2, for an $f \in W(G, G')$, we took an $a_1 \in G'$ such that $f(\bar{C}_a) = \bar{C}_{aa_1}$, constructed an automorphism σ by setting $\sigma(x^i a^j z) = x^i a^j a_1^j z$ where $z \in Z(G)$, and showed that $f(\bar{C}_g) = \sigma(\bar{C}_g)$ for all $g \in G$. In this case we can actually take $a_1 = 1$ since $C_a = aG'$. Then $\sigma = 1$ and we are done.

Theorem 4.3.4 gives us some additional cases of when $W(G, G') \leq CP(G)$ which are contained in the following corollary.

Corollary 4.3.5: Let G be a p -group, $p > 2$. Suppose either

(1) $G = AB$ where A is a cyclic normal subgroup of G and B is abelian or

(2) G contains a maximal abelian normal subgroup A where A is cyclic.

Then $W(G, G') \leq CP(G)$.

Proof: (1) follows since Lemma 2.5.3 implies $G/Z(G)$ is metacyclic.

(2) follows since, as we noted in the proof of Lemma 4.2.2, G is metacyclic.

CHAPTER V

ELEMENTARY REPRESENTATIONS IN S_n

In [3], Brown showed that S_n , $n = 1, \dots, 10$, are E.R. groups. In this chapter, we will show that S_n is an E.R. group for any positive integer n .

We begin by recording two ~~lemmas~~ about S_n , the first of which is Exercise 11.4.11 of [13]. The second lemma is a well-known result about the order of conjugacy classes of S_n and can be found, for instance, in [2].

Lemma 5.1.1: Let $n > 2$, $n \neq 6$, then $x \in S_n$ is a 2-cycle if and only if $|x| = 2$ and $\max |xx^y| = 3$, where $y \in S_n$.

Lemma 5.1.2: Let $g \in S_n$ and suppose that g is the product of disjoint α_1 1-cycles, α_2 2-cycles, $\dots$, α_n n -cycles. Then

$$|C_g| = \frac{n!}{\alpha_1! 1^{\alpha_1} \alpha_2! 2^{\alpha_2} \dots \alpha_n! n^{\alpha_n}}.$$

In showing that S_n is an E.R. group, we will show that $NA(S_n) = CP(S_n)$ for $n > 2$, $n \neq 6$. (Note that this would have to be the case if S_n is to be an E.R. group since $\text{Aut}(S_n) = \text{Inn}(S_n)$ for $n \neq 6$.) We first show that every normalized automorphism of $Z(S_n)$ fixes the class sums of elements of order 2 for $n > 2$, $n \neq 6$.

Lemma 5.1.3: Let f be a normalized automorphism of $Z(S_n)$ and suppose that $n > 2$, $n \neq 6$. Let $g \in S_n$ be a product of disjoint transpositions. Then $f(\bar{C}_g) = \bar{C}_g$.

Proof: Let t denote the number of transpositions appearing in g .

We may assume that g has the form $g = (12)(34)\dots(2t-1, 2t)$. Let

$f(\bar{C}_g) = \bar{C}_{g_1}$. To show $\bar{C}_{g_1} = \bar{C}_g$ we proceed by induction on t .

If $t = 1$, let $y \in S_n$. Then by Lemma 1.2.2, we can find $x \in S_n$ such that $f(\bar{C}_{g_1^y}) = \bar{C}_{g_1^x}$. Thus, $|g_1^y| = |g^x| \leq 3$ by Lemma 5.1.1. Next, let $x \in S_n$ such that $|g^x| = 3$. Again by Lemma 1.2.2, there exists $y \in S_n$ such that $f(\bar{C}_{g^x}) = \bar{C}_{g_1^y}$. Hence $|g_1^y| = |g^x| = 3$. Therefore, $\max |g_1^y| = 3$, so g_1 is a transposition by Lemma 5.1.1 and we are done when $t = 1$.

In the general case we have that

$$f(\bar{C}_{(12)(34)\dots(2t-3, 2t-2)}) = \bar{C}_{(12)(34)\dots(2t-3, 2t-2)}$$

and $f(\bar{C}_{(2t-1, 2t)}) = \bar{C}_{(2t-1, 2t)}$. Hence,

$$f(\bar{C}_g) = \bar{C}_{(12)(34)\dots(2t-3, 2t-2)(2t-1, 2t)^x}$$

for some $x \in S_n$ by Lemma 1.2.2.

If $(2t-1, 2t)^x$ is disjoint from $(12)\dots(2t-3, 2t-2)$ we are done. Suppose this is not the case. If $(2t-1, 2t)^x$ has one letter in common with $(12)\dots(2t-3, 2t-2)$, it follows that

$(12)\dots(2t-3, 2t-2)(2t-1, 2t)^x$ is a product of disjoint cycles which are transpositions and a 3-cycle. But then $3 \mid |g_1|$ which is impossible. Next, suppose $(2t-1, 2t)^x$ has two letters in common with one transposition of $(12)\dots(2t-3, 2t-2)$. It follows that g_1

is a product of disjoint transpositions and has one less transposition than g . But then f fixes $\bar{C}_{g_1}$ so that $f(\bar{C}_g) \neq \bar{C}_{g_1}$.

The final possibility would be for $(2t-1, 2t)^x$ to have one letter

in common with two different transpositions of $(12) \dots (2t-3, 2t-2)$. But then $(12) \dots (2t-3, 2t-2)(2t-1, 2t)^x$ is a product of disjoint cycles which are transpositions and a 4-cycle. Therefore $4 \mid |g_1|$ which is impossible. Hence, $(2t-1, 2t)^x$ is disjoint from $(12) \dots (2t-3, 2t-2)$ so that $\bar{C}_g = \bar{C}_{g_1}$.

Theorem 5.1.4: S_n is an E.R. group for every positive integer n .

Proof: By the results of [3] we may assume $n > 2$, $n \neq 6$. Let $f \in \text{NA}(G)$ and let $N = \{g \in S_n \mid f(\bar{C}_g) \neq \bar{C}_g\}$. We will show that $N = \emptyset$, which proves the theorem.

Suppose $N \neq \emptyset$. We pick a "minimal element" g of N , which satisfies the following properties in the order that they are listed:

- (1) Suppose that g has its largest cycle of smallest length among the elements of N . Let h denote the length of its largest cycle.
- (2) Suppose that g has the fewest number of cycles of length h among the elements of N satisfying (1).
- (3) Suppose that g has the fewest number of cycles of length greater than or equal to two among the elements of N satisfying (2).

Note that we have $h \geq 3$ by Lemma 5.1.3.

Write $g = \beta_1 \beta_2 \dots \beta_r$ where the β_i are disjoint cycles and $2 \leq |\beta_i| \leq |\beta_{i+1}|$. Also, assume that $\beta_r = (1, 2, \dots, h)$. Let $\beta'_r = (1, 2, \dots, h-1)$. Then $g = \beta_1 \dots \beta_{r-1} \beta'_r (1, h)$. Also,

$$f(\bar{C}_{\beta_1 \dots \beta_{r-1} \beta'_r}) = \bar{C}_{\beta_1 \dots \beta_{r-1} \beta'_r}$$

by the minimality of g and

$$f(\bar{C}_{(1, h)}) = \bar{C}_{(1, h)}.$$

Hence by Lemma 1.2.2,

$$f(\bar{C}_g) = \bar{C}_{\beta_1 \dots \beta_{r-1} \beta'_r (1,h)^x}$$

for some $x \in S_n$. Let $g_1 = \beta_1 \dots \beta_{r-1} \beta'_r (1,h)^x$. We will now show that $\bar{C}_{g_1} = \bar{C}_g$, from which it follows that $N = \emptyset$ and so we will be done.

Case 1: Suppose $(1,h)^x$ is disjoint from $\beta_1 \dots \beta_{r-1} \beta'_r$.

If $(1,h)^x$ is disjoint from $\beta_1 \dots \beta_{r-1} \beta'_r$, then $\bar{C}_{g_1}$ is fixed by f since it has fewer cycles of length h with its largest cycle of length $\leq h$. But then $f(\bar{C}_g) \neq \bar{C}_{g_1}$, so this case cannot occur.

Case 2: If $(1,h)^x$ has one letter in common with $\beta_1 \dots \beta_{r-1} \beta'_r$.

Suppose that $(1,h)^x$ has one letter in common with $\beta_1 \dots \beta_{r-1} \beta'_r$ and that $(n_1, n_2, \dots, n_t)$ is the cycle of $\beta_1 \dots \beta_{r-1} \beta'_r$ where the common letter occurs. Then $(1,h)^x = (n_j, a)$ where a does not appear in $\beta_1 \dots \beta_{r-1} \beta'_r$. Also, note that

$$(n_1, \dots, n_t)(n_j, a) = (n_1, \dots, n_{j-1}, a, n_j, \dots, n_t) .$$

If $t < h-1$, then g_1 would have fewer cycles of length h with its largest cycle having length $\leq h$. Thus $\bar{C}_{g_1}$ is fixed by f , so $f(\bar{C}_g) \neq \bar{C}_{g_1}$. Hence, we must have $t \geq h-1$.

If $t > h-1$, then $t = h$. But then g_1 has one more cycle of length $h-1$ than g does. Thus, g_1^h has one more cycle of length $h-1$ than g^h does since $(h, h-1) = 1$. But $\bar{C}_{g^h}$ is fixed by f since all cycles in g^h have length less than h . Thus,

$$f(\bar{C}_g^h) = \bar{C}_g^h = \bar{C}_{g_1}^h$$

which is impossible.

Thus we must have $t = h-1$. But then g_1 has the same cycle structure as g does. Hence $f(\bar{C}_g) = \bar{C}_g = \bar{C}_{g_1}$.

Case 3: If $(1, h)^x$ has two letters in common with $\beta_1 \dots \beta_{r-1} \beta'_r$.

First, suppose that $(1, h)^x$ has two letters in common with one cycle of $\beta_1 \dots \beta_{r-1} \beta'_r$. Let $(n_1, n_2, \dots, n_t)$ denote this cycle and suppose $(1, h)^x = (n_j, n_s)$ where $j < s \leq t$. Then

$$(n_1, \dots, n_t)(n_j, n_s) = (n_1, \dots, n_{j-1}, n_s, n_{s+1}, \dots, n_t)(n_j, n_{j+1}, \dots, n_{s-1}).$$

But then g_1 has fewer cycles of length h with its largest cycle having length $\leq h$. Thus, f fixes $\bar{C}_{g_1}$ and so $f(\bar{C}_g) \neq \bar{C}_{g_1}$.

Therefore, $(1, h)^x$ must have its letters in common with two cycles of $\beta_1 \dots \beta_{r-1} \beta'_r$. Let $(n_1, n_2, \dots, n_t)$ and $(n'_1, n'_2, \dots, n'_s)$ denote these cycles where $t \leq s$. Then $(1, h)^x$ has the form (n_i, n'_j) and

$$(n_1, \dots, n_t)(n'_1, \dots, n'_s)(n_i, n'_j) = (n_1, \dots, n_{i-1}, n'_j, n'_{j+1}, \dots, n'_s, n'_1, \dots, n'_{j-1}, n_i, \dots, n_t).$$

If $s + t \leq h$, then g_1 has its largest cycle of length $\leq h$, has at most as many cycles of length h as g , and has fewer cycles of length ≥ 2 than g . Thus f fixes $\bar{C}_{g_1}$ so that $f(\bar{C}_g) \neq \bar{C}_{g_1}$. Hence, we must have $s + t > h$.

If neither s nor t is $h-1$, then g_1 has one more cycle of length $h-1$ than g . Therefore, g_1 has at least one more cycle of length $h-1$ than g does and so $\bar{C}_{g_1}^h \neq \bar{C}_g^h$. But $f(\bar{C}_g^h) = \bar{C}_g^h$

since g^h has its largest cycle of length $< h$. Thus,

$$f(\bar{C}_{g^h}) = \bar{C}_{g^h} = \bar{C}_{g_1^h},$$

a contradiction. Hence either s or t must be $h-1$.

Suppose that g has α_1 1-cycles, α_2 2-cycles, ..., α_h h -cycles.

Then,

$$|C_g| = \frac{n!}{\alpha_1! \alpha_2! 2^{\alpha_2} \dots \alpha_h! h^{\alpha_h}}.$$

If $s = h-1$ and $t < s$, then

$$|C_{g_1}| = n! / (\alpha_1! \alpha_2! 2^{\alpha_2} \dots \alpha_{t-1}! (t-1)^{\alpha_{t-1}} (\alpha_t-1)! t^{\alpha_t-1} \alpha_{t+1}! (t+1)^{\alpha_{t+1}} \dots \\ \alpha_{h-1}! (h-1)^{\alpha_{h-1}} (\alpha_h-1)! h^{\alpha_h-1} (s+t)).$$

Thus since $|C_{g_1}| = |C_g|$, we have

$$s + t = \alpha_t t \alpha_h h.$$

But $th > s + t$, a contradiction.

If $t = h-1$ and $s = h-1$, then

$$|C_{g_1}| = n! / (\alpha_1! \alpha_2! 2^{\alpha_2} \dots (\alpha_{h-1}-1)! (h-1)^{\alpha_{h-1}-1} (\alpha_h-1)! h^{\alpha_h-1} (s+t)).$$

Since $|C_{g_1}| = |C_g|$, we see that

$$s + t = \alpha_{h-1} (h-1) \alpha_h h.$$

But $(h-1)h > s + t$, so we have a contradiction.

Finally, if $t = h-1$ and $s = h$, then

$$|C_{g_1}| = n! / (\alpha_1! \alpha_2! 2^{\alpha_2} \dots \alpha_{h-1}! (h-1)^{\alpha_{h-1}} (\alpha_h-2)! h^{\alpha_h-2} (s+t)).$$

Again, since $|C_{g_1}| = |C_g|$, we have

$$s + t = (\alpha_h - 1)\alpha_h h^2 .$$

But $h^2 > s + t$, so we again have a contradiction and case 3 cannot occur.

Thus we have now shown $f(\overline{C}_g) = \overline{C}_g$, so $N = \emptyset$.

CHAPTER VI

A LOOK AT THE GROUP RING PROBLEM

Section 1. Introduction. Let R be a ring and let G and H be two groups. A question which has received considerable attention is when does $R(G) \simeq R(H)$ imply $G \simeq H$? This is known as the group ring problem or isomorphism problem.

In the particular case when $R = \mathbb{Z}$, the best general result known at this time is that $\mathbb{Z}(G) \simeq \mathbb{Z}(H)$ implies $G \simeq H$ when G is a metabelian group. This result was shown by Whitcomb in [16] by using Theorem 1.1.6. It should be noted at this time that in [10], Jackson claims to have obtained a positive answer to the group ring problem when G is either metabelian or nilpotent. However, the nilpotent result depends on Lemma 4 of [10], which is false, and I have been unable to follow Jackson's proof in the metabelian case.¹ In this chapter, we will give some procedures which may prove fruitful in extending Whitcomb's result.

Finally, we remark, as noted in [16], that in order to prove the group ring problem when $R = \mathbb{Z}$, it suffices to show that every group basis of $\mathbb{Z}(G)$ is isomorphic to G .

Section 2. Lifting and a Generalization of Theorem 1.1.6. We begin this section with the following definition.

¹ Jackson uses Lemma 4 in his proof of the metabelian result, but the use of it can be omitted.

Definition: Let $N \triangleleft G$ and let $\bar{f} \in \text{NA}(G/N)$. We say that $\bar{f}$ can be lifted if there exists an $f \in \text{NA}(G)$ which induces $\bar{f}$ on $Z(G/N)$ (that is, $\pi(f(x)) = \bar{f}(\pi(x))$ for all $x \in Z(G)$ where π is the natural map from $Z(G)$ to $Z(G/N)$). We will also say that f is a lift of $\bar{f}$.

Note that in the above definition it is implicit that $f(\Delta(N)) = \Delta(N)$ if f is to be a lift of $\bar{f}$.

Next, we state a generalization of Theorem 1.1.6.

Theorem 6.2.1: Let A be an abelian normal subgroup of G , let π denote the natural map from $Z(G)$ to $Z(G/A)$, and let H be a group basis of $Z(G)$. Suppose that there exists an $\bar{f} \in \text{NA}(G/A)$ such that $\bar{f}(\pi(H)) = \pi(G)$ which possesses a lift f . Then for each $h \in H$, there is a unique $g_h \in G$ such that

$$f(h) \equiv g_h \pmod{\Delta(A)\Delta(G)}.$$

Further, the mapping $h \rightarrow g_h$ defines an isomorphism of H onto G .

Proof: Let $h \in H$, then there exists $g \in G$ such that $f(h) \equiv g \pmod{\Delta(A)}$. Thus, $f(h) = g + \sum_{a \in A} (a-1)t(a)$ where $t(a) \in Z(G)$. Then, computing as Whitcomb did in [16],

$$f(h) = g + \sum_{a \in A} (a-1)t(a) \equiv \prod_{a \in A} a^{\ell(t(a))} g \pmod{\Delta(A)\Delta(G)}.$$

Letting $g_h = \prod_{a \in A} a^{\ell(t(a))} g$, we have g_h is unique by Theorem 1.1.7.

The mapping $h \rightarrow g_h$ will be a homomorphism since f is a homomorphism and since $\Delta(A)\Delta(G)$ is an ideal. Further, $h \rightarrow g_h$ is an isomorphism. For if $g_{h_1} = g_{h_2}$, then $f(h_1) \equiv f(h_2) \pmod{\Delta(A)\Delta(G)}$ so that $f(h_1) = f(h_2)$ by Theorem 1.1.7. But then $h_1 = h_2$.

Theorem 6.2.1 can be used to solve the group ring problem for S_4 . To see this, first note that $S_4 = S_3 V_4$ where $V_4 = \langle (1,2)(3,4), (1,3)(2,4) \rangle$. Thus, $Z(S_4/V_4) \simeq Z(S_3)$.

Now, in [7] it is shown that $NA(S_3) = I(S_3)$. Hence if $\bar{f} \in NA(S_4/V_4)$, we can write $\bar{f} = \tau_{\bar{u}}$ where $\bar{u}$ is a unit in $Z(S_4/V_4)$. Since $Z(S_3) \leq Z(S_4)$, we can then find a unit u in $Z(S_3)$ such that $\pi(u) = \bar{u}$ where π is the natural map from $Z(S_4)$ to $Z(S_4/V_4)$. Therefore, τ_u will be a lift of $\tau_{\bar{u}}$. Thus every element of $NA(S_4/V_4)$ can be lifted.

Finally, let H be a group basis of $Z(S_4)$. Then, $\pi(H) \simeq \pi(S_4)$ since S_3 is metabelian. Hence, since V_4 is abelian and since every element of $NA(S_4/V_4)$ can be lifted, we can apply Theorem 6.2.1 to conclude $H \simeq S_4$.

More generally, one might attempt to use Theorem 6.2.1 in the following manner to solve the group ring problem for solvable groups.

Let G be a solvable group, let $A \neq 1$ be an abelian normal subgroup of G , and let π be the natural map from $Z(G)$ to $Z(G/A)$. Suppose H is a group basis of $Z(G)$. Then by induction on $|G|$, there exists $\bar{f} \in NA(G/A)$ such that $\bar{f}(\pi(H)) = \pi(G)$. Then, if $\bar{f}$ can be lifted, we can apply Theorem 6.2.1 to conclude $H \simeq G$.

Unfortunately, the success of the above process depends on $\bar{f}$ having a lift, and in general it is not true that every normalized automorphism of $Z(G/A)$ can be lifted from $Z(G)$.

For an example of this, let G be the dihedral group of order 8 with generators a and b where $a^4 = b^2 = 1$ and $a^b = a^3$. Let $A = \langle a^2 \rangle$ and let $\bar{f}$ denote the normalized automorphism of $Z(G/A)$ defined by $\bar{f}(\bar{a}) = \bar{b}$ and $\bar{f}(\bar{b}) = \bar{a}$ where $\bar{a} = aA$ and $\bar{b} = bA$.

Then $\bar{f}$ cannot be lifted, for suppose that $\bar{f}$ did have a lift f . Then, $f(a) \equiv b \pmod{\Delta(A)}$, so that $f(a) = b + (a^2 - 1)t(a)$ where $t(a) \in Z(G)$. But then

$$f(a) = b + (a^2 - 1)t(a) \equiv a^{2\mathcal{L}(t(a))} b \pmod{\Delta(A)\Delta(G)}.$$

Hence $f(a^2) \equiv 1 \pmod{\Delta(A)\Delta(G)}$, so that $f(a^2) = 1$ by Theorem 1.1.7. But $|f(a)| = 4$, a contradiction. Thus $\bar{f}$ does not have a lift.

However, it should be noted that we do not have to be able to lift every $\bar{f}$ such that $\bar{f}(\pi(H)) = \pi(G)$ to prove the group ring problem for solvable groups. Rather, it would suffice to have the existence of one $\bar{f}$ mapping $\pi(H)$ to $\pi(G)$ which has a lift.

It is also interesting to note, that although not every normalized automorphism of $Z(G/N)$ can be lifted when $N \triangleleft G$, they can be lifted from $\mathcal{L}(G)$. By this we mean the following. Let $\bar{f} \in \text{NA}(G/N)$. Extend $\bar{f}$ to $\mathcal{L}(G/N)$ and let π be the natural map from $\mathcal{L}(G)$ to $\mathcal{L}(G/N)$. Then, since $\mathcal{L}(G) \simeq \mathcal{L}(G/N) \oplus \ker \pi$, it follows that any automorphism of the form $\bar{f} \oplus f'$, where f' is an automorphism of $\ker \pi$, will induce $\bar{f}$ on $\mathcal{L}(G/N)$.

Let us at this time introduce some notation. For a group G and a normal subgroup N of G , let $L(G/N)$ denote the normalized automorphisms of $Z(G/N)$ which can be lifted.

Section 3. Groups of Solvable Length Three. Let G be a group such that G'' is abelian and set $\bar{G} = G/G''$. Let $\bar{H}$ be a group basis of $Z(\bar{G})$. Then there exists $\bar{f} \in W(\bar{G}, \bar{G}')$ such that $\bar{f}(\bar{H}) = \bar{G}$ by Theorem 1.1.6. Hence, if $W(\bar{G}, \bar{G}') \leq L(G/G'')$, we can apply Theorem 6.2.1 to obtain that the group ring problem holds for G .

Whether every element of $W(\bar{G}, \bar{G}')$ can be lifted is to my knowledge unknown. However, we can state a necessary condition for this to occur.

Theorem 6.3.1: Let G be a group such that G'' is abelian and such that $W(\bar{G}, \bar{G}') \leq L(G/G'')$ where $\bar{G} = G/G''$. Then $W(G, G')\text{Aut}(G) = \text{NA}(G)$.

Proof: Let $f \in \text{NA}(G)$ and set $H = f(G)$. Let $\bar{f}_1 \in W(\bar{G}, \bar{G}')$ such that $\bar{f}_1(\pi(H)) = \pi(G)$ where π is the natural map from $Z(G)$ to $Z(G/G'')$ and let f_1 be a lift of $\bar{f}_1$. Note that $f_1 \in W(G, G')$, for let $g \in G$. Then $\pi(f_1(g)) \equiv \pi(g) \pmod{\Delta(\bar{G}')\Delta(\bar{G})}$. Hence $f_1(g) \in g + \Delta(G')\Delta(G) + \Delta(G'')$. But applying Theorem 1.1.5 with $K = G'$, we see $\Delta(G'') \leq \Delta(G')\Delta(G)$. Thus, $f_1(g) \equiv g \pmod{\Delta(G')\Delta(G)}$ or $f_1 \in W(G, G')$. Also, note that f induces a normalized automorphism $\bar{f}$ on $Z(\bar{G})$ since G' is NA-characteristic. Further, $\pi f_1 f(G) = \pi(G)$.

Now, we have $f_1 f(g) \equiv g_1 \pmod{\Delta(G'')}$ for some $g_1 \in G$, or $f_1 f(g) = g_1 + \sum_{a \in A} (a-1)t(a)$ where $t(a) \in Z(G)$. Again computing as Whitcomb did in [16],

$$f_1 f(g) = g_1 + \sum_{a \in A} (a-1)t(a) \equiv \prod_{a \in G''} a^{t(a)} g_1 \pmod{\Delta(G'')\Delta(G)}.$$

Thus for each $g \in G$, we can find a $g_{f_1 f}$ in G such that $f_1 f(g) = g_{f_1 f} \pmod{\Delta(G'')\Delta(G)}$. Further, $g_{f_1 f}$ is unique by Theorem 1.1.7.

Let σ be the mapping $\sigma(g) = g_{f_1 f}$. Since $g_{f_1 f}$ is unique, $f_1 f$ is an isomorphism, and since $\Delta(G'')\Delta(G)$ is an ideal, it follows that $\sigma \in \text{Aut}(G)$. Also,

$$\sigma^{-1} f_1 f(g) \equiv g \pmod{\Delta(G'')\Delta(G)}.$$

Hence $\sigma^{-1}f_1f \in W(G, G'')$, or

$$f \in f_1^{-1}\sigma W(G, G'') \leq W(G, G')\text{Aut}(G)W(G, G'') .$$

But $W(G, G'') \leq W(G, G')$ so that $f \in W(G, G')\text{Aut}(G)$. Thus we have $W(G, G')\text{Aut}(G) = \text{NA}(G)$.

We remark at this time, that in Section 5 of Chapter I it was mentioned that we would obtain a sufficient condition for $W(G, N)$ to be a supplement for $\text{Aut}(G)$ in $\text{NA}(G)$. Indeed, for any group G satisfying the hypothesis of Theorem 6.3.1, we have $W(G, G')$ is a supplement for $\text{Aut}(G)$ in $\text{NA}(G)$.

Section 4. Lifting Units. In this section, we record another possible procedure for attacking the group ring problem involving E.R. groups. Our approach here is similar in nature to one presented in [16].

Let A be an abelian normal subgroup of G and suppose that G/A is a metabelian E.R. group. Then, if $\bar{H}$ is a group basis of $Z(G/A)$, we can find a unit $\bar{u}$ in $Z(G/A)$ such that $\tau_{\bar{u}}(\bar{H}) = G/A$ by Lemma 1.3.1. Thus, if $\text{CP}(G/A) \leq L(G/A)$, we could apply Theorem 6.2.1 to conclude $H \simeq G$ for any group basis H of $Z(G)$.

Indeed, the proof in Section 2 that the group ring problem holds for S_4 fits the above procedure. Thus, we are led to the question of when do elements of $\text{CP}(G/A)$ lie in $L(G/A)$. In [16], Whitcomb suggested using the ring of p -adic integers to consider this question when G is a p -group. Here we will instead use the ring Z_n for a suitable integer n .

We begin by recalling that

$$J(Z_n(G)) = \Delta(G) + pZ_n(G)$$

when G is a p -group where $J(Z_n(G))$ denotes the Jacobson radical of $Z_n(G)$.

Next, we record the following result from [16].

Lemma 6.4.1: Let A be an abelian normal subgroup of G . Suppose

$\sum_{a \in A} (a-1)t(a) \in p^n Z(G)$ where $t(a) \in Z(G)$ and where the exponent of A divides p^n . Then $\prod_{a \in A} \ell(t(a)) = 1$.

Now, let φ denote the natural projection of $Z(G)$ onto $Z_n(G)$ where n is a positive integer. We next state

Lemma 6.4.2: Let $u \neq 1$ be a finite unit of $Z(G)$ where G is a finite group. Then $\varphi(u) \neq 1$.

Proof: Suppose $\varphi(u) = 1$, then $u \in 1 + p^n Z(G)$. But then the coefficient of the identity element of G in u is nonzero. Hence $u = 1$ by Lemma 7 of [10].

Another fact we can state is

Lemma 6.4.3: Let G be a p -group and let $x \in Z(G)$. Then, if p does not divide $\ell(x)$, $\varphi(x)$ is a unit in $Z_n(G)$.

Proof: Since $\ell(x)$ is not divisible by p , $\varphi(x)$ is not in $J(Z_n(G))$, but then $\varphi(x)$ is a unit.

We now return to the problem of lifting elements of $CP(G/A)$. Although we will not directly lift elements of $CP(G/A)$ here, we can use an alternate procedure to obtain the following result.

Theorem 6.4.4: Let A be an abelian normal subgroup of a p -group G and let π denote the natural map from $Z(G)$ to $Z(G/A)$. Suppose H is a group basis of $Z(G)$ such that $\pi(H)^{\bar{u}} = \pi(G)$ where $\bar{u}$ is a unit in $Z(G)$ and $\bar{u} \in Z(G)$. Then, if $\ell(\bar{u})$ is not divisible by p , $H \simeq G$.

Proof: Let $|G| = p^n$ and let φ be the natural map from $Z(G)$ to $Z_p(G)$. By Lemma 6.4.2, $\varphi(H)$ is isomorphic to H . Hence we will let H denote $\varphi(H)$ in $Z_p(G)$. Also, let us use π to denote the natural map from $Z_p(G)$ to $Z_p(G/A)$ and let $\bar{u}$ denote the image of u in $Z_p(G/A)$.

Then by Lemma 6.4.3, $\bar{u}$ is a unit in $Z_p(G/A)$. But then if u is any element of $Z_p(G)$ such that $\pi(u) = \bar{u}$, u is also a unit since $u \notin J(Z_p(G))$. Since $\pi(H)\bar{u} = \bar{u}\pi(G)$, we have for each $h \in H$ there exists a $g \in G$ such that $hu \equiv ug \pmod{\Delta(A)}$ or $h^u \equiv g \pmod{\Delta(A)}$ in $Z_p(G)$. Writing $h^u = g + \sum_{a \in A} (a-1)t(a)$ where $t(a) \in Z_p(G)$,

$$h^u = g + \sum_{a \in A} (a-1)t(a) \equiv \prod_{a \in A} a^{\ell(t(a))} g \pmod{\Delta(A)\Delta(G)}.$$

Let $g_h = \prod_{a \in A} a^{\ell(t(a))} g$. As in the integral case, we claim g_h is unique. To see this, suppose $g_1 \equiv g_2 \pmod{\Delta(A)\Delta(G)}$ where $g_1, g_2 \in G$. Then $g_1 = g_2 a$ for some $a \in A$ since $g_1 \equiv g_2 \pmod{\Delta(A)}$. Thus, $a \equiv 1 \pmod{\Delta(A)\Delta(G)}$ so that $a-1 = \sum_{x \in A} (x-1)t(x)$ where $t(x) \in \Delta(G)$. Considering the $t(x)$'s in $Z(G)$, we have $(a-1) - \sum_{x \in A} (x-1)t(x) \in p^n Z(G)$. Thus $a^{\prod_{x \in A} \ell(t(x))} = a = 1$ by Lemma 6.4.1. Hence $g_1 = g_2$, so g_h is unique.

It then follows, as in the integral case, that the mapping $h \rightarrow g_h$ defines an isomorphism of H onto G .

It should be noted with regard to Theorem 6.4.4, that if $\tau_u \in NA(G)$, u can be assumed to lie in $Z(G)$ by multiplying u by an integer if necessary. Also, one should note that if $f = \tau_u$, where u is a unit in $Z(G)$, then $\ell(u)$ is not divisible by any

prime dividing $|G|$ since $\ell(u) = \pm 1$. Thus Theorem 6.4.4 can be applied when $\tau_{\bar{u}} \in I(G/A)$.

Theorem 6.4.4 also motivates some further questions on normalized automorphisms which would lend themselves to the study of the group ring problem. One question is what can one say about u if $\tau_u \in CP(G)$. Another question is the following. Let $\bar{f} \in NA(G/N)$ where $N \triangleleft G$. Then $\bar{f}$ induces an automorphism of $Z_n(G/N)$ for any positive integer n . Then, when can the automorphism $\bar{f}$ induce be lifted from an automorphism of $Z_n(G)$? Indeed, our success in proving Theorem 6.4.4 rests upon the fact that the last question has a positive answer when G is a p -group and when $\bar{f}$ has the form $\tau_{\bar{u}}$ where $\bar{u} \in Z(G/N)$ and $p \nmid \ell(\bar{u})$.

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