This is to certify that the

thesis entitled

APPLICATION OF OPTIMIZATION THEORY IN BIOMECHANICS

presented by

Diane Marie Pietryga

has been accepted towards fulfillment
of the requirements for

M.S. degree in Mechanics

[signature]
Major professor

Date 2/20/87

APPLICATION OF OPTIMIZATION THEORY IN BIOMECHANICS

By

Diane Marie Pietryga

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE

Department of Metallurgy, Mechanics and Materials Science

1987

ABSTRACT

APPLICATION OF OPTIMIZATION THEORY IN BIOMECHANICS

By

Diane Marie Pietryga

The purpose of this research was to investigate optimization criteria for the redundant biomechanical problem, in which the muscle forces act as the unknowns. Three optimization problems of the lumbar spine were formulated: the linear problems of minimizing the upper bound of muscle stress and of minimizing the spinal compression and the nonlinear problem of minimizing the summation of muscle stress to the n_ith power. The nonlinear problem is based on maximum endurance of musculoskeletal function, where the parameter n_i is based on the percentage of slow twitch fibers.

The linear criterion of minimizing the upper bound of muscle stress predicted a more even distribution of muscle stress among the synergistic muscles and a greater distribution of muscle activity compared to the other two objective functions. The criterion of minimizing the spinal compression was examined, and it was noted that the upper bounds of muscle stress seemed to limit the solution more than spinal compression.

ACKNOWLEDGEMENTS

The author wishes to express her appreciation to the following people for making the completion of her Master of Science degree possible:

To Dr. Robert Wm. Soutas-Little, her major professor, for his friendship, encouragement and invaluable guidance in this project.

To Dr. Alejandro Diaz, for his assistance in this project and for serving on her committee.

To Dr. George E. Mase, for serving on her committee and for his warm friendship.

To Dr. James J. Rechtien, for serving on her committee.

To Brooks Shoe, Incorporated, for their generous funding of this project.

To Brenda K. Miller, for her warm friendship and patience in the seemingly endless chore of typing this manuscript.

To Andrew Hull, for his assistance in computer program development.

To Daniel D. Lauderback, her best friend, for his endless support and encouragement and, most of all, for his priceless friendship.

To her family, especially her parents, for their understanding, encouragement and support in her academic pursuits.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

INTRODUCTION

The methods of optimization theory are now being applied to the analysis of the redundant muscular system, in which the muscles are active elements. The skeletal system, connected through ligaments and muscles, provides vital structural support for the human body. Load-sharing among the structures of the musculoskeletal system has not been subjected to extensive study. The lumbar spine, due to the widespread problem of low-back pain, is one structure requiring investigation of the muscle interactions.

Most adults will suffer from some form of low-back pain during the course of their lives. According to some estimates, as much as one quarter of the population will lose time from the job or will have to curtail recreational activities. Many people will become permanently disabled by low-back problems. Since low-back conditions are a leading cause of compensation costs to industry, it becomes not only a serious physical problem to the individual sufferer, but a major socioeconomic disability as well.

Far too little is known about what causes low-back pain and how it can be prevented or effectively treated. Although muscle may not be the primary participant in the common low-back pain syndrome, it may have significant influence on its onset and outcome. For this reason, engineers and physicians are collaborating in the effort to understand the mechanical basis of musculature in low-back pain and to design programs for its prevention and treatment.

As stated earlier, optimization methods are now being used to determine muscular activity. The study of a practical optimization problem requires a realistic representation of the physical system by means of a suitable mathematical model and the formulation of an appropriate performance criterion. The mathematical model must describe correctly, at least, the qualitative features of the practical system in the range of operating conditions, and the performance criterion must represent an optimal characteristic of the system. The concept of optimization is well rooted as a principle underlying the analysis of many complex decision problems. It offers a certain degree of philosophical elegance that is hard to dispute, and it now offers an indispensible degree of operational simplicity.

Over the last few decades there has been a steady shift in applied optimization from the status of an art to that of a scientific discipline. In the past, most of the theory of optimization concentrated on the subject of optimality conditions, and practical methods of computation were rarely investigated. Today, due to the interaction between mathematicians and engineers, theory and practice are better integrated. To a large degree, this trend has been fostered by the development of high-speed computers with which large-scale problems can be solved with an exactness that previously was unapproachable. Computer availability has given rise to new optimization techniques and has enhanced previously developed ones. Consequently, practitioners of many disciplines are building large scale optimization models and solving them routinely with linear and nonlinear programming.

Linear programming is a mechanism for formulating a vast array of problems with modest effort. A linear programming problem is characterized, as the name implies, by linear functions of the unknowns; the objective function is linear in the unknowns and the constraints are linear equalities or linear inequalities in the unknowns. The linear structure insures that the extremum will lie at the intersection of two or more constraints. This greatly reduces the number of possible locations for the extremum. Efficient algorithms that inspect this limited region of solution space have been developed with the most significant of these being the simplex method. This method can be referenced in Appendix A.

Alternatively, nonlinear programming pertains to optimization problems in which the objective function and/or the constraints have nonlinear mathematical forms. The constraints, which are classified either as equalities or inequalities, define the solution space from which an optimal solution is to be obtained. Characteristic of nonlinear problems, there are no general techniques for solving a problem, but only special ones, each covering a particular class of practical problems. The generalized reduced gradient method, which can be referenced in Appendix B, was used for solving the nonlinear optimization problem presented in this literature.

SURVEY OF LITERATURE

The human musculoskeletal system can be considered as a system of rigid articulating segments on which known external forces (weight, ground reaction, external load) and unknown muscle, ligament and joint forces are acting. Relationships between known external forces and the unknown musculoskeletal or internal forces can be obtained from force and moment equilibrium equations. Since more muscles than are mechanically necessary normally cross a joint, the number of unknown forces will in general exceed the number of equilibrium equations. This mechanical redundancy yields the problem statically indeterminate.

In statically determinate problems, internal and external forces can be determined by the use of free body diagrams and equilibrium equations. However, in statically indeterminate problems the equilibrium equations must be complemented by relations involving deformations. These deformations are obtained by considering the geometry of the problem and they must be compatible with the external supports. By considering engineering structures as deformable and analyzing the deformations in their various members, it is possible to compute forces which are statically indeterminate.

Unfortunately, the aforementioned method cannot be applied to the indeterminate musculoskeletal problem. Since muscular load and deformation depend on the amount of muscle contraction the exact muscular load cannot readily be determined from load-deformation diagrams. For example, in isometic contraction, no overall length change exists between muscle origin and insertion.

A method for solving the problem of indeterminancy is reduction of the excess number of unknown variables. This is accomplished by either grouping functionally similar muscles together, or by eliminating individual muscles based on electromyographic observation. However, these anatomical simplifications may induce considerable error and the mechanical action of individual muscles is obscured.

Alternatively, optimization methods have also been used to obtain a unique solution. By using an optimization method, not only can a solution be obtained, but possible physiologically based rationales for the solution can be associated. This approach employs a model of the inherent muscle selection process. The model is based upon the assumption that the selection process represents an optimal behavior of the biomechanical system. The optimal response approach provides a consistent basis for a tractable mathematical formulation of the problem and suggests an interesting qualitative picture of muscle response.

Various optimization criteria have been developed over the last twenty years. These criteria include minimization of:

1. Summation of muscle force,

   ΣF_i

2. Summation of ratios,

   $\Sigma (F_i/F_{imax})$, $\Sigma(F_i/A_i)$

3. Weighted summation of muscle force, ligament moments and joint reactions,

   $\Sigma F_i + C_1*(M_{jx} + M_{jy} + M_{jz}) + C_2*R_{joint}$

4. Spinal compression,

   C

5. Squares of muscular forces, ratios and vertebral stresses,

$\Sigma(F_i)^2$, $\Sigma(F_i/A_i)^2$, $\Sigma(F_i/F_{imax})^2$, $\Sigma(\tau_i)^2$

6. Muscular fatigue (maximize activity endurance),

$(\Sigma(S_i)^n)^{1/n}$, and maximize the minimum of T_{iend}

where $T_{iend} = a_i*((F_i/F_{imax})*100)^{n_i}$

7. The upper bound of muscle stress,

S_i

8. The free energy input to the muscles,

E

Each of these criteria will now be discussed in detail.

In 1967, MacConaill defined the "Principle of Minimal Total Muscular Force", which postulates that no more total muscular force than is both necessary and sufficient to maintain a posture or perform a motion would be used. Accordingly, this would minimize the sum of the muscle forces, namely ΣF_i (15).

This criterion was used by several investigators to analyze muscle force in static situations. In 1972, Barbenel calculated the muscular forces at the temporomandibular joint (2). He concluded that the suggested minimum muscle force principle did not apply. In 1973, Seireg and Arvikar analyzed the forces in the lower extremities in standing, leaning and stooping postures (22). Other investigators have studied muscles of the upper limbs. Penrod presented, in 1974, a biomechanical analysis of a simplified biaxial model of the wrist (18). In 1976, Yeo used a study of elbow flexion to examine the validity of the minimum force criterion. His theoretical results contradicted the experimental results; therefore, it was concluded that MacConaill's hypothesis of minimal total muscular force was invalid (28). The minimum force criterion was also used for the analysis of forces in the leg during level walking by Hardt and Pedotti et al. in 1978, and Patriarco et al. in 1981 (12, 16, 17). Pedotti et al. and Patriarco et al. employed additional, physiologically based constraints to improve the muscle force predictions.

Pedotti et al. also used a criterion consisting of the sum of ratios of muscular force to maximum possible muscle force, $\Sigma(F_i/F_{imax})$, and applied this to the analysis of forces in the leg during level walking (17). This criterion was employed because it enhances the total muscular force criterion by utilizing the muscles more efficiently by demanding larger force production from the larger muscles; moreover, it takes into account the instantaneous state of each muscle, since F_{imax} depends upon the instantaneous length of muscle as well as its velocity. Crowninshield and fellow investigators employed a total muscle tensile stress criterion, $\Sigma(F_i/A_i)$ (6). The physiological cross-sectional area, A_i, was determined by muscle volume divided by its length. They studied forces in the arm muscles during isometric and isokinetic elbow flexion and forces at the hip duing level walking, climbing, descending stairs and rising from a sitting position.

Another type of linear objective function was employed by Seireg and Arvikar in 1973 and 1975 (22, 23). They used a weighted sum of muscle forces and ligament moments for analysis of forces in the legs in standing, leaning and stooping postures and quasi-static walking. The weighting factors can be different for each problem and were chosen in order to get reasonable results. A weighting factor between four and infinity was found to be applicable to all the investigated postures. It is difficult to make a physiological interpretation of this kind of empirically adjustable objective function. However, Williams and Seireg also used this type of criterion in 1977 and 1979 for the prediction of muscle forces in the jaw and in the leg during bicycling and by Yettram and Jackman in 1980 for the analysis of forces in the vertebral column (26, 27, 29).

In 1978, Hardt concluded that the minimum force criterion yielded a purely geometric optimization, whereby the set of muscle moment arm vectors which produce the lowest muscle forces will be choosen over all other possibilities. Consequently, the only representation of the muscles in the mathematics is in the form of their moment arms, ignoring the physiology of the system. To incorporate some physiological properties into the problem, Hardt proposed to define a cost function that would minimize the instantaneous energy requirements of the muscles. This formulation was used for the prediction of muscle forces during walking and it revealed an increased number of muscles participating in the movement (12). Patriarco et al. supported this formulation in 1981 (16).

In 1981, Schultz and Andersson presented a model for internal force estimation of the lumbar trunk. They chose to minimize the compression on the lumbar vertebra (20). This criterion was applied to several physical activities including nonsymetric weight-holding, resisting a push to the left and resisting a longitudinal twist moment. Schultz, et al., in 1982, used linear programming to investigate the load on lumbar trunk structures during various physical tasks including flexion-extension, lateral bending, and torsion (21). Two different objective functions were applied, the first minimized the compressive load on the lumbar vertebra and the second minimized the largest muscle force crossing the lumbar vertebra. Myoelectric measurements did not reveal much difference between the cost functions.

Unfortunately, the results of a linear criterion are not always physiologically consistent, and this has been noted by most investigators. When muscle force is the variable used to formulate the load sharing criterion, there is a preference for muscles with large moment arms. When muscle stresses, or ratios of muscle force to maximum muscle force are used as the variable in the criterion, there is preference for muscles with the largest product of moment arm and cross-sectional area. Investigators improved the predictions of muscle forces with linear criteria by formulating additional physiologically based constraints. This enforced synergism between the muscles.

Nonlinear objective functions can predict synergism, even without the formulation of additional constraints. It is thought that linear optimization was used more for reasons of mathematical convenience than for reasons of physiological requirement. Investigators are now emphasizing the importance of selecting muscle prediction criteria based on sound physiological bases rather than on an arbitrary or mathematically convenient basis. Unfortunately, nonlinear optimization convergence on a global minimum is not assured.

In 1977, Gracovetsky et al. defined an objective function of the sum of squared shear stresses in the vertebral column and predicted forces during weight lifting (11). This criterion was developed based on a study finding compression to have relatively minor effects on the spine compared to shear effects. This result can be explained by considering that the spine is built to take a compression load but that any shear effect cannot readily be compensated. This criterion was modified to a quadratic objective function, that minimized shear and penalized excessive muscle power, in 1981 by Gracovetsky et al. (10). For the analysis of walking, Pedotti et al., in 1978, used the sum of squared muscle forces, which is a sort of power criterion. This criterion not only minimizes total muscular force, it also penalizes large individual muscle forces. They also used the sum of squared ratios of muscle force to maximum muscle force, namely $\Sigma\ (F_i/F_{imax})^2$. This criterion was selected as the most feasible since it used the muscles most efficiently while keeping their level of activation as low as possible (17).

In 1981, Crowninshield and Brand presented an optimization method which uses a criterion of maximum endurance of musculoskeletal function (4). The method is based on the inversely nonlinear relationship of muscular force and contraction endurance. This relationship was proposed to be of the form:

$$\ln T = -n^*(\ln f) + c$$

where T is the maximum time of contraction, f is the contractile force, and n and c are experimentally obtainable constants. They suggested that the muscle selection to maximize activity endurance is physiologically reasonable during many normal activities, particularly prolonged and repetitive activities, such as normal gait. This criterion is not applicable to all forms of locomtion such as activities occurring to maximize speed or to minimize pain.

Based on several reports, Crowninshield and Brand assumed that, in an approximate manner, the muscle force-endurance relationship is a basic property of muscle tissue (4). They suggested that the maximum endurance of a muscle contraction is thus related to the magnitude of the average stress within the muscle tissue. The determination of muscle force during body function may then be formulated as a nonlinear optimization problem with an objective to minimize the summation of muscle stress to the nth power. The parameter, n, is dependent on the percentage of slow twitch fibers. Muscle forces predicted in this manner will tend to keep individual muscle stresses low. Low individual muscle stresses are achieved by predicting force activity in numerous muscles and preferentially predicting force in muscles with large cross-sectional areas and long moment arms. Since individual muscle stresses are low their potential for prolonged contraction will be high.

The actual value of n may vary between individual subjects and individual muscle due to fiber type and fiber orientation. Since accurate and detailed experimental data were not available n = 3 was selected as a reasonable value, as it is the average value reported in literature. To reduce the magnitude of the objective function, thereby avoiding numerical problems in large scale optimization, the function is normalized. The criterion has the form $[\Sigma(F_i/A_i)^3]^{1/3}$, where m is the number of muscles. This method was demonstrated at the elbow during isometric contraction and in the lower extremity during locomotion. During gait, the observed muscle activity pattern in the lower extremities, as determined by EMG, shows substantial agreement with that activity pattern predicted when endurance is used as the optimization criterion. In addition, since this problem has a continuous convex character of the objective function and the linear constraints it falls into the category of convex programming. This convexity assures that the only minimum is a global or absolute minimum.

Dul et al., in 1984, presented a similar criterion which is based on the hypothesis that muscular fatigue is minimized during learned endurance activities (7). An endurance type of activity such as constrained sitting posture or walking, involves sustained or repetitive muscular contractions. These contractions are fatiguing, and after a specific period of time, the endurance time, the required mechanical output cannot be maintained anymore. It is assumed that the neuromuscular system anticipates this by selecting a load sharing between the muscles such that endurance time of the activity is maximized, hence muscular fatigue minimized. Again, this concept may be less useful for other types of activity where quick contractions are involved.

Dul's criterion is to maximize the minimum of T_i where $T_i = a_i (F_i * 100 / F_{imax})^{n_i}$. T_i is the endurance time and F_i is the force for the ith muscle. The constants a_i and n_i are muscle parameters depending on the percentage of slow twitch fibers for the respective muscle. The criterion was used to determine forces in the lower extremities during static-isometric knee flexion. The predicted muscular load sharing was in good agreement with direct force measurement data. In comparison with Crowninshield and Brand, the general pattern of load sharing is similar, yet the predicted magnitude of the muscle force is not the same. The cubic criterion predicted linear synergism, whereas the minimum-fatigue criterion predicts non-linear synergism. Both criteria predict that there is relatively more force in muscles with large cross-sectional areas. For the cubic criterion more force is also allocated to muscles that have large moment arms. The load sharing predicted with the minimum fatigue criterion does not depend on moment arm, although the absolute force levels do depend on this variable. Instead, relatively more force is allocated to muscles with a high percentage of slow-twitch fibers. This reveals the pertinence of incorporating muscle fiber types into the problem.

A new optimization approach, based on minimizing the upper bound of muscle stress, was introduced in 1984 by An et al. (1). The concept of this new optimization approach is quite different from those previously used in summations of muscle force or stress, or their nonlinear combinations. Optimization procedures used to minimize the sum of unknown force variables have been more or less based on consideration of overall efforts of the system. However, from an energy storage and transport viewpoint, each muscle bundle has its own storage and blood supply. Therefore, in constructing the optimization criteria for this new technique, individual muscle effort was considered. Since this technique allows a solution which considers more even distribution of muscle stress among all synergistic muscles, it will favor the muscular response with the largest endurance for the task.

The criterion of minimizing the upper bound of muscle stress was applied to a simplified model of the elbow joint. For comparison purposes, this problem was also solved using other previously mentioned objective criteria. These included minimizing the summation of muscle forces and summation of muscle stress using the linear optimiztion method, as well as minimizing the summation of the square of muscle force and summation of the square of muscle stress using nonlinear optimization. The solution of muscle force distribution based on the proposed approach predicted the same number of active muscles for the same given loading condition as that using either of the nonlinear criterion. The accuracy of the results obtained by this new technique was further verified by its compatibility with physiological considerations.

From the mathematical point of view, the formulation of this criterion has a major advantage as well. Since the entire system of constraints and objective functions consists of linear terms of unknown variables, the well-established linear programming algorithm, the simplex method, can be used to obtain the solution efficiently. In contrast, the algorithm obtained by nonlinear optimization is usually more involved and less efficient than linear programming. In addition, convergence of the solution to a global minimum is not always guaranteed.

The criteria of minimizing the upper bound of muscle stress and of maximizing the activity endurance time are the least disputed criteria in the biomechanical problem. In other words, these two criteria seem to represent an optimal characteristic of various activities performed by the human musculoskeletal system. For this reason, both of these criteria will be developed for a nonsymmetric weight holding task.

ANALYTICAL METHODS AND RESULTS

Optimization can be used to determine the muscle and joint loads on any part of the body, but it will be applied here to determine the loads on the lumbar spine. Loads on the lumbar spine should be kept as light as possible since it is suspected that heavy loads have a role in both causing back pain and aggravating pre-existing lumbar spine conditions. Since heavy spinal loads are to be avoided, it is necessary to know under what circumstances they arise. By determining the muscle and joint forces that occur during various physical activities, these circumstances can be defined.

To compute the loads on the lumbar spine created by a quasi-static physical activity, the body is first divided into upper and lower parts by an imaginary transverse cutting plane. This cutting plane is passed through the level of the lumbar spine at which the loads are to be determined. In this case, it is at the L3 vertebra. The upper part is considered as a free body subject to the laws of Newtonian mechanics, and a two-stage calculation procedure is carried out (Figures 1 and 2). In the first stage, the external force is considered as an equivalent force and moment acting at the origin; in the second, the internal forces are estimated to place the system in equilibrium. The equivalent external system consists of six components: three force components and three moment components.

To calculate the external equivalent system, a coordinate system must be established. The origin is placed at the center of the L3 vertebral body and it is assumed that the equivalent system acts at that point. Coordinate directions are selected as follows: the x axis is positive to the right, the y axis is positive anteriorly, and the z axis is positive superiorly. The x and y axes lie in the transverse cutting plane and the z axis is perpendicular to the cutting plane, as illustrated in Figures 1 and 2.

The body segment weights and the mass-center locations were obtained from Clauser et al. and Eycleshymer and Schoemaker respectively, as referenced by Schultz and Andersson (3,8,20). The coordinates of the mass center locations are specified as (x_i, y_i, z_i), where i represents the specific body part.

The action of a force on a body can be separated into two effects, external and internal. The weight of a body, an external effect, is the gravitational force distributed over its volume which may be taken as a concentrated force acting through the mass center location.

Referring to Figure 1, the following equations were written for the external force and moment components of the upper body segment:

(1) $F_x = 0$

(2) $F_y = 0$

(3) $F_z = -(Q + W_h + W_l + W_r + W_t)$

(4) $M_x = -(y_q Q + y_t W_t + y_r W_r + y_l W_l + Y_h W_h)$

(5) $M_y = -(x_l W_l - x_r W_r)$

(6) $M_z = 0$

where: Q = weight of object held in right hand

W_h = weight of the head and upper neck

W_l = weight of the left upper limb

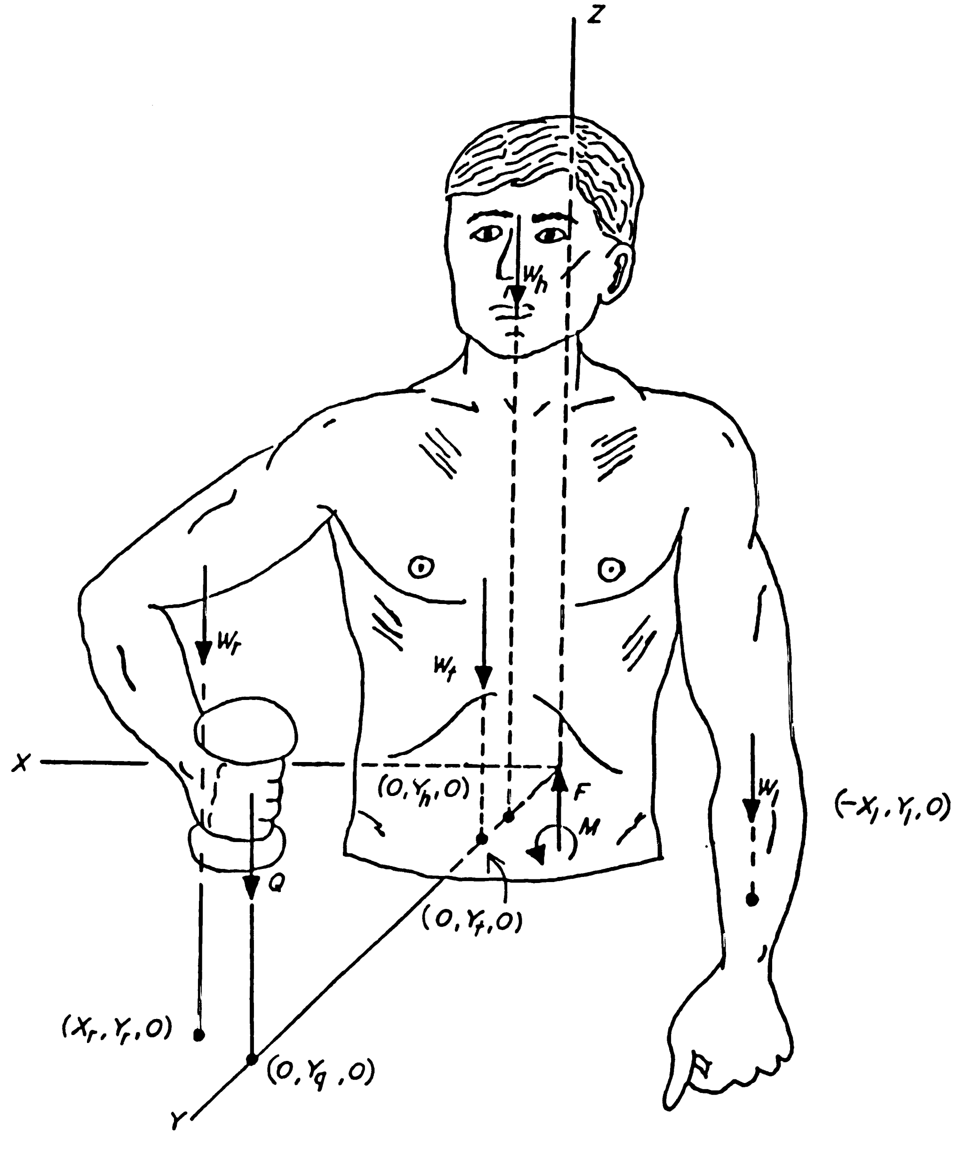

Figure 1. External System

W_r = weight of the right upper limb

W_t = weight of the trunk above the cutting plane

F_x = equivalent external force component in the x direction at the origin

F_y = equivalent external force component in the y direction at the origin

F_z = equivalent external force component in the z direction at the origin

M_x = equivalent external moment component in the x direction at the origin

M_y = equivalent external moment component in the y direction at the origin

M_z = equivalent external moment component in the z direction at the origin

If the numerical values of Table 1 are assumed, then the three nonzero components of the net reaction are:

$$F_z = -391 \text{ N}$$

$$M_x = -3130 \text{ Ncm} \qquad (7)$$

$$M_y = -160 \text{ Ncm}$$

The external system must be balanced by the internal force between the lower body segment and the upper body segment, in order to keep the upper body in equilibrium. However, the external force system is not affected by the material properties of the body tissues. Anatomical variables affect the external gravitational forces only in so far as they influence mass distributions and moment arms.

The trunk model in Figure 2 can be used to identify the internal forces. Since this model incorporates ten muscle equivalents, which represent most of the major muscle groups spanning the lumbar region, it can be used in a variety of physical activities.

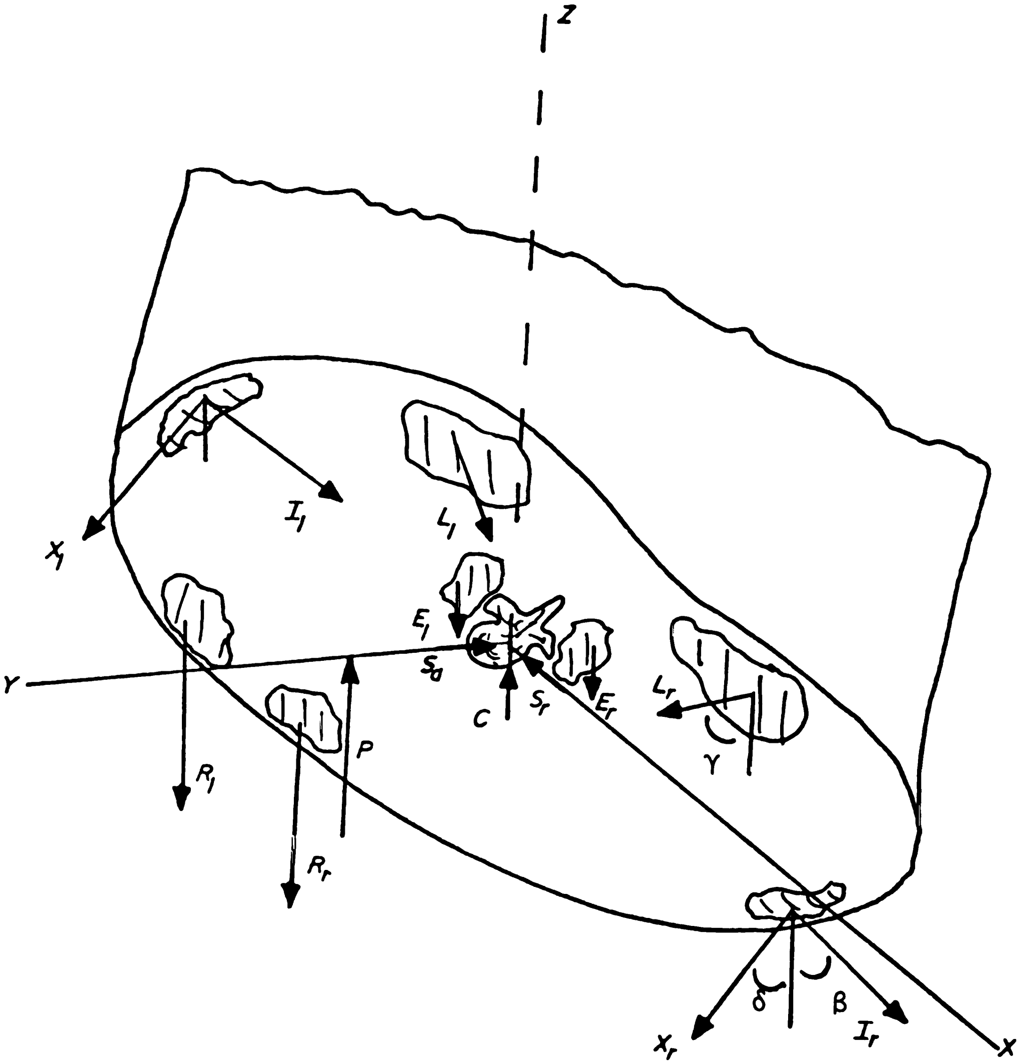

Figure 2. Equivalent Internal System

The three spinal segment loads, C, S_a and S_r are assumed to act atthe coordinate system origin. The muscle orientation angles are defined as: β , the angle between the internal obliques and the z axis; δ , the angle between the external obliques and the z axis; and γ , the angle between the latissimus dorsi and the z axis.

TABLE 1. Body Segment Weights and Mass-Center Locations

Weight (N)	Coordinate Locations (cm)	
$Q = 40$	$x_q = 0$	$y_q = 45$
$W_h = 35$	$x_h = 0$	$y_h = 8$
$W_1 = 32$	$x_1 = 20$	$y_1 = 1$
$W_r = 32$	$x_r = 15$	$y_r = 24$
$W_t = 252$	$x_t = 0$	$y_t = 1$

Solving the equilibrium equations written from Figure 2 for the equivalent external forces and moments generates the following equations:

(8) $-F_x = -S_r + \sin \gamma * (L_1-L_r)$

(9) $-F_y = -S_a - \sin \beta * (I_1 + I_r) + \sin \delta * (X_1 + X_r)$

(10) $-F_z = P + C - E_1 - E_r - \cos \beta * (I_1 + I_r) - \cos \gamma * (L_1+ L_r) - R_1 - R_r - \cos \delta * (X_1 + X_r)$

(11) $-M_x = y_e * (E_1 + E_r) - \cos \beta * y_o * (I_1 + I_r) + \cos\gamma * y_1 * (L_1 + L_r) - y_r * (R_r + R_1) - \cos\delta * y_o * (X_1 + X_r) + y_p P$

(12) $-M_y = -x_e * (E_1 - E_r) - \cos\beta * x_o * (I_1 - I_r) - \cos\gamma * x_1 * (L_1 - L_r) - x_r * (R_1 - R_r) - \cos\delta * x_o * (X_1 - X_r)$

(13) $-M_z = - \sin\beta * x_o * (I_r - I_1) - \sin\gamma * y_1 * (L_r - L_1) - \sin\delta * x_o * (X_1 - X_r)$

where: E_l = left erector equivalent force

E_r = right erector equivalent force

I_l = left internal oblique force

I_r = right internal oblique force

L_l = left latissimus dorsi equivalent force

L_r = right latissimus dorsi equivalent force

R_l = left rectus abdominis force

R_r = right rectus abdominis force

X_l = left external oblique force

X_r = right external oblique force

P = intra-abdominal pressure force

C = compressive spinal force

S_r = right-lateral spinal shear

S_a = anterior spinal shear

The intra-abdominal pressure resultant can be determined from experimental measurements. The maximum calculated intra-abdominal pressure in stance is 25 mmHg as referenced from Gracovetsky, et al. (10). With the abdominal cavity area, 278.4 cm^2, this resultant force from intra-abdominal pressure can be calculated to be 92.8 N (9).

The trunk cross-sectional geometrical data given in Table 2, are representative of a person who has a trunk width of 30 cm and a trunk depth of 20 cm at the L3 vertebral level (20). Substituting the values in Table 2 and the values for the intra-abdominal pressure and the equivalent external system into equations (8) through (13) yields the following simplifications:

(14) $0 = -S_r + 0.707 * (L_l - L_r)$

(15) $0 = -S_a - 0.707 * (I_l + I_r) + 0.707 * (X_l + X_r)$

(16) $298.2 = C - E_l - E_r - 0.707 * (I_l + I_r) - 0.707 * (L_l + L_r) - R_l - R_r - 0.707 * (X_l + X_r)$

(17) $2684.57 = 4.4 * (E_l + E_r) - 2.69 * (I_l + I_r) + 3.96 * (L_l + L_r) - 10.8 * (R_l + R_r) - 2.69 * (X_l + X_r)$

(18) $160 = - 5.4 * (E_l - E_r) - 9.54 * (I_l - I_r) - 4.45 * (L_l - L_r) - 3.6 * (R_l - R_r) - 9.54 * (X_l - X_r)$

(19) $0 = 9.54 * (I_l - I_r) + 3.96 * (L_l - L_r) - 9.54 * (X_l - X_r)$

The thirteen internal forces are unknown. Since only six equations are available to find them, it is obvious that the use of this model leads to a statically indeterminate problem. The examples following will illustrate two different optimization methods for solving this problem.

TABLE 2. Cross-Sectional Geometric Data

Coordinate Locations (cm)		Angles (degrees)
$x_r = 3.6$	$y_r = 10.8$	$\beta = 45$
$x_p = 0.0$	$y_p = 4.8$	$\delta = 45$
$x_o = 13.5$	$y_o = 3.8$	$\gamma = 45$
$x_e = 5.4$	$y_e = 4.4$	
$x_l = 6.3$	$y_l = 5.6$	

The major question that arises when optimization is used for solution, is the choice of the objective function. Considering the diversity of musculoskeletal function, it is likely that distinctly different criteria for muscle selection may be utilized for different activities. For the physical activity previously discussed, the following optimization problems will be developed: first, minimization of the upper bound of muscle stress and second, minimization of the summation of muscle stress to the n_ith power, which is based on maximum endurance of musculoskeletal function.

Previously, most optimization procedures have been more or less based on consideration of overall efforts of the system. The criterion of minimizing the upper bound of muscle stress was developed to take individual muscle effort into consideration. This idea stems from the energy storage and transport viewpoint that each muscle bundle has its own storage and blood supply. This technique predicts a solution with a more even distribution of muscle stress among all synergistic muscles, thereby favoring the largest endurance for the task (1).

With this particular criterion, the core of the problem formulation lies within defining the constraints, as opposed to explicitly defining the objective function. The most important of these constraints being those that define the domain where muscle stress S_i is greater than or equal to any other muscle stress S_j. Once this domain is defined, the upper bound of muscle stress S_i is minimized. In order to define this space, nine inequality constraints were developed for each individual muscle, M_i. Mathematically these constraints are represented by:

(20) $S_i \geq S_j$

The muscle cross-sectional areas, given in Table 3, were used to calculate the respective muscle stresses (9).

Additional constraints incorporated into this problem were the equilibrium equations (16) through (19). These equations were obviously formulated as the four equality constraints. Equations (14) and (15)

TABLE 3. Transverse Section Through the Abdomen at L3

Muscle	Area (cm^2) Right	Left
Erector Equivalent	20.202	20.121
Latissimus Dorsi	2.129	2.258
External Oblique	7.032	7.610
Internal Oblique	9.615	10.582
Rectus Abdominis	3.549	4.323

were used to find S_a and S_r after the optimization solution was obtained. This changes the optimization problem to one of eleven unknowns.

Also included in this problem were the ten inequality constraints that the muscle forces must be greater than or equal to zero. This arises from the fact that muscle contractions always produce tensile forces. In the simplex method, these constraints are automatically assumed.

Finally, the problem statement for minimizing the upper bound of muscles stress is:

(21) minimize S_i

subject to $S_i \geq S_j$

$M_i \geq 0$

equations (16) through (19)

Since n-dimensional space is impossible to envision, the solution space for this type of problem will be illustrated in three-dimensional muscle space. Each axis represents the force in one of the three muscles, M_i where i = 1, 2, 3. For simplicity, the respective areas, A_i, will all be unity.

The domain formed by the inequalities:

(22) $S_3 \geq S_1$

(23) $S_3 \geq S_2$

represents the area where S_3 is greater than or equal to S_1 and S_2. Both inequalities, (22) and (23), form a 45 degree plane between the respective variables, M_i. The problem is limited to the positive quadrant due to the constraints that the variables must be greater than or equal to zero. These constraints are represented by the expression:

(24) $M_i \geq 0, \qquad i = 1, 2, 3$

The area above the 45 degree planes and bounded by the perpendicular planes is the domain where S_3 is the greatest. This domain is illustrated in Figure 3. It is important to note that if the areas were not unity and equal, ($A_1 = A_2 = A_3 = 1$), the planes would not be at 45 degree angles.

In addition, two equality constraints, Figure 4, were incorporated into this problem. These constraints are represented by the equations:

(25) $3.0 * M_1 + M_2 + 1.5 * M_3 = 9$

(26) $M_1 + 0.8 * M_2 + 2.0 * M_3 = 8$

The line of intersection formed by these two planes defines the coordinate values that satisfy both constraints. This limits the optimization solution to the values on this line.

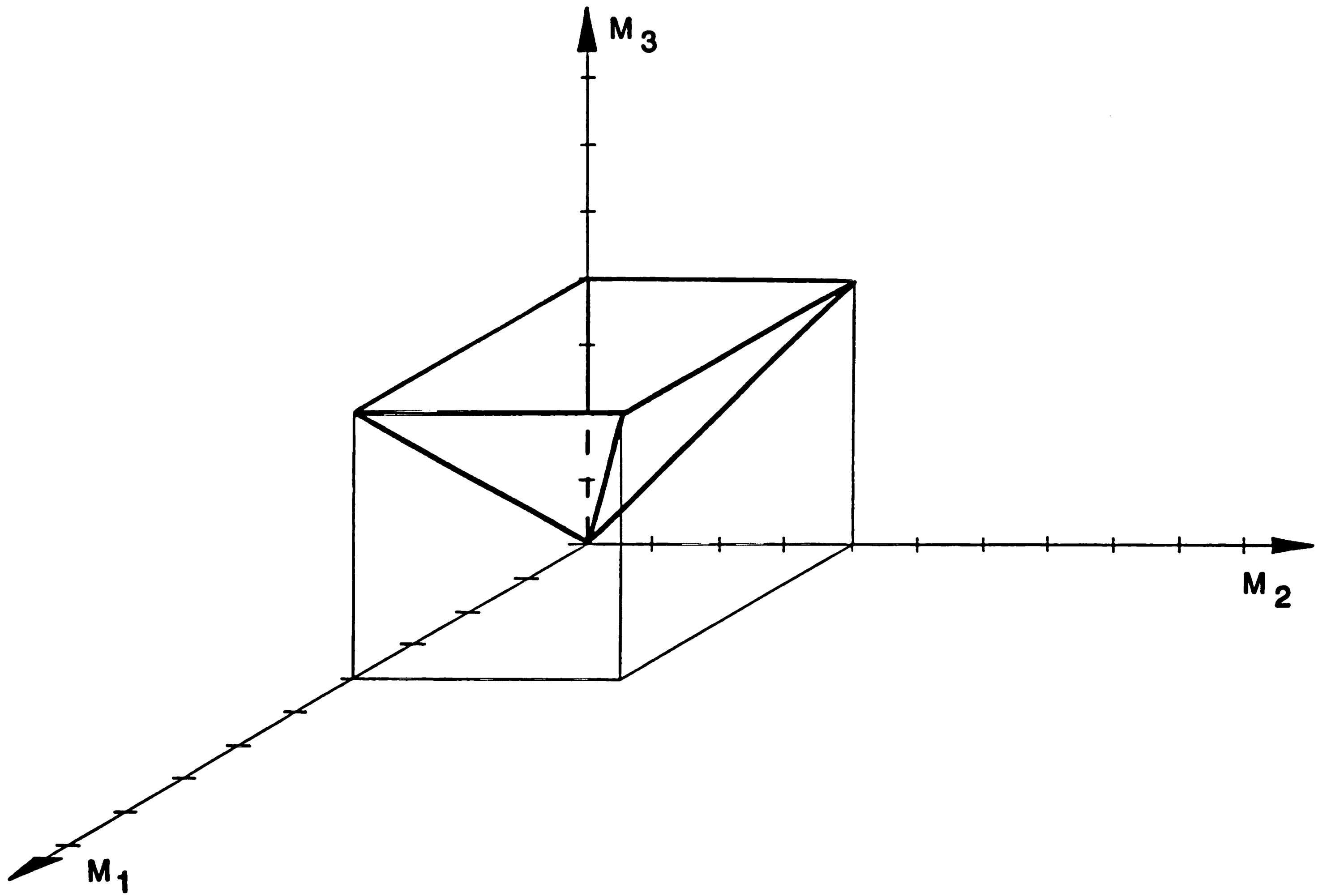

Figure 3. Domain Formed by the Inequality Constraints

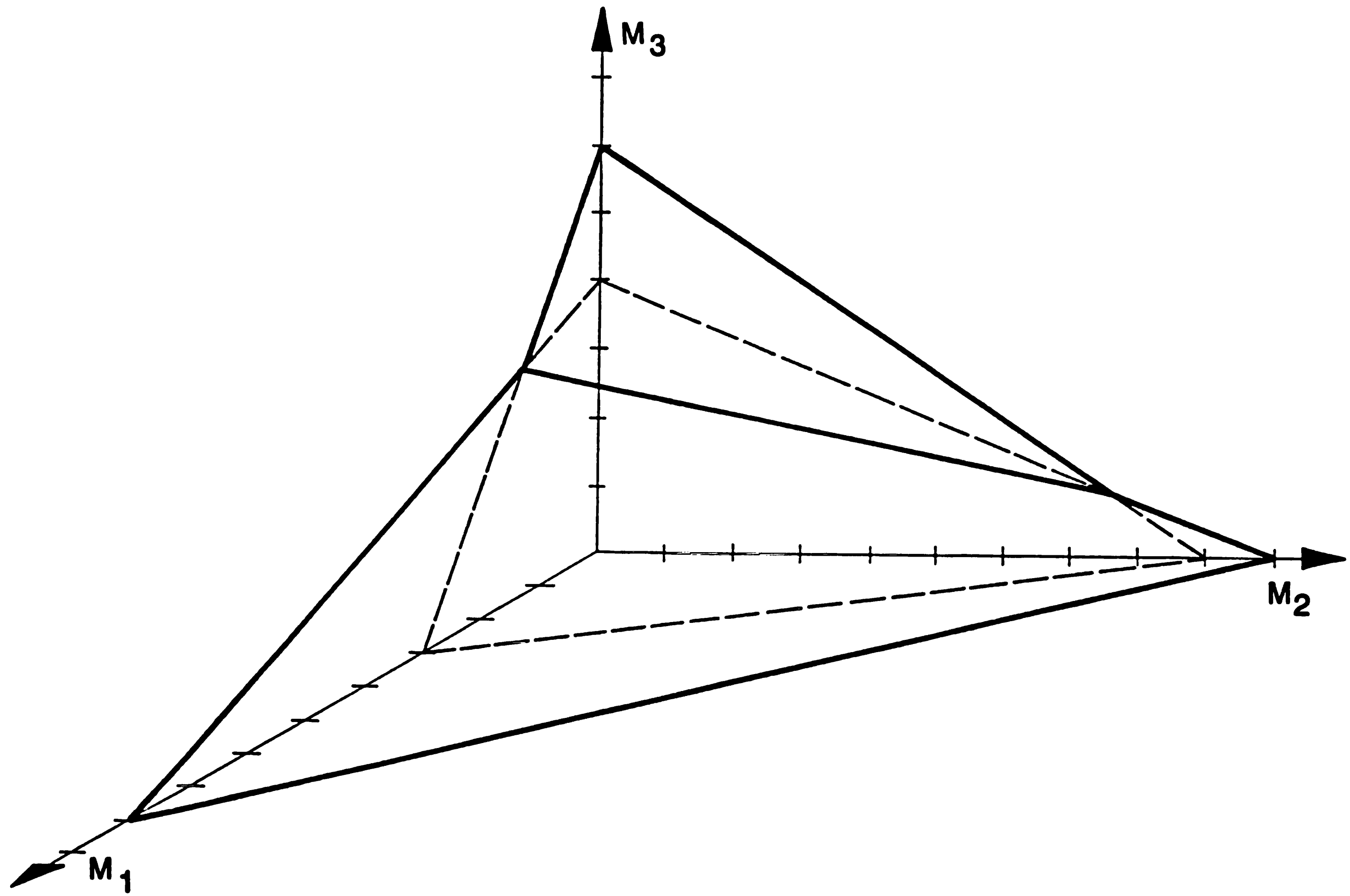

Figure 4. Equality Constraint Planes

Combining all of the constraints together defines the complete solution space. The only possible solution space lies along the line of intersection formed by the equality constraints that is contained in the domain where S_3 is the greatest. This space is illustrated in Figure 5. Note that the line of intersection formed by the equality constraints intersects the 45 degree plane between M_2 and M_3. This point of intersection is the solution to minimizing the upper bound of M_3 muscle stress.

Furthermore, imagine the domains defining S_1 and S_2 as the greatest superimposed on Figure 5. This involves a 45 degree plane between the M_1 and M_2 axes, parallel to the M_3 axis. It is easy to visualize that minimizing the upper bound of M_2 muscle stress yields the same optimum solution as that of M_3. However, since the line of intersection formed by the equality constraints does not intersect the domain where S_1 is the greatest, no solution exists when minimizing the upper bound of M_1 muscle stress. Table 4 lists the numerical results obtained when searching each domain.

Table 4. Three-Dimensional Optimum Solutions

Domain Searched	Optimum Solution M_1	M_2	M_3
M_1	---	---	---
M_2	0.881	2.542	2.542
M_3	0.881	2.542	2.542

Referring back to the weight lifting task, all ten muscular domains were searched using the simplex method. Seven of the ten domains searched yielded a numerical solution. However, the optimum

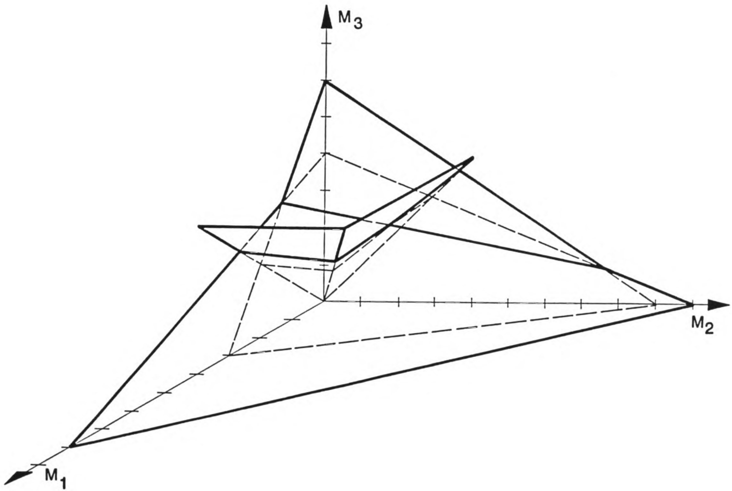

Figure 5. Solution Space Defined by the Constraints

solution was found in only two of these seven domains, the right equivalent erector and the right latissimus dorsi. Note that these two domains generated the identical solution. Table 5 lists the load and stress values that the muscles were predicted to carry in the optimal solution. The stress values of the remaining five domains are listed in Table 6. The three domains that did not have a numerical solution were E_1, I_1 and L_1, in which the solution did not satisfy the constraints.

In comparison, Schultz and Andersson also solved this nonsymmetric weight holding problem of ten muscular unknowns using linear programming (20). They selected the cost function to minimize the compression on the L3 lumbar vertebra. The constraints consisted of the equilibrium equations (16) through (19), the ten requirements that the muscle tensions cannot be negative and the ten requirements that the muscle contraction intensities cannot exceed the reasonable level of 100 N/cm^2. The equations for x and y force equilibrium, (14) and (15), were used to calculate S_a and S_r after the solution was obtained. This problem is formulated as:

(27) minimize C

subject to $0 \leq M_i \leq 100$ N/cm^2

equations (16) through (19)

The solution to this optimization problem is listed in Table 7.

In constructing the optimization criterion for minimizing the summation of muscle stress to the n_ith power, the importance of selecting a criterion based on sound physiological bases was emphasized. It is assumed, that in an approximate manner the muscle force-endurance relationship is a basic property of muscle tissue (4). The maximum

TABLE 5. Optimum Load and Stress Data from the Criterion of Minimizing the Upper Bound of Muscle Stress

Element	Load (N)	Stress (N/cm^2)
E_l	282.013	14.016
E_r	283.153	14.016
I_l	0.000	0.000
I_r	8.859	0.921
L_l	31.646	14.016
L_r	29.840	14.016
R_l	0.000	0.000
R_r	0.000	0.000
X_l	0.000	0.000
X_r	8.110	1.153
C	918.835	---
S_a	0.530	---
S_r	1.277	---

Table 6. Stress Data from the Criterion of Minimizing the Upper Bound of Muscle Stress

Element	Space Searched / Stress (N/cm^2)				
	I_r	R_l	R_r	X_l	X_r
E_l	18.746	18.999	17.262	16.000	17.119
E_r	18.746	18.999	17.262	19.372	17.119
I_l	16.182	0.000	0.245	0.960	0.000
I_r	18.746	2.551	0.000	0.000	0.929
L_l	18.745	18.999	17.262	19.372	17.119
L_r	18.746	18.999	17.262	19.371	17.119
R_l	0.000	18.999	0.000	0.000	0.000
R_r	0.000	0.000	17.262	0.000	0.000
X_l	0.000	0.000	0.461	19.372	14.762
X_r	1.140	3.343	0.000	19.372	17.119

Table 7. Optimum Load and Stress Data from the Criterion of Minimizing the Compression on the L3 Lumbar Vertebra

Element	Load (N)	Stress(N/cm^2)
E_l	98.640	4.902
E_r	128.270	6.349
I_l	0.000	0.000
I_r	0.000	0.000
L_l	212.900	94.293
L_r	212.900	99.999
R_l	0.000	0.000
R_r	0.000	0.000
X_l	0.000	0.000
X_r	0.000	0.000
C	826.150	---
S_a	0.000	---
S_r	0.000	---

endurance of a muscle contraction is thus inversely related to the magnitude of the average stress within the muscle tissue. The determination of muscle force during body function may then be formulated as a nonlinear optimization problem with an objective to minimize the summation of muscle stress to the n_ith power. The constant n_i is related to fiber type and fiber orientation or more specifically to the percentage of slow twitch fibers for the respective muscle, M_i. This criterion is valid only when applied to an endurance activity. An endurance type of activity involves sustained or repetitive muscular contractions. The activity of holding a relatively small weight in front of the body can be considered an endurance activity since it involves sustained muscular contractions.

The muscle force-endurance relationship was proposed to be of the form:

(28) $\ln T_i = -n_i * \ln M_i + c_i$

where:

T_i = maximum time of contraction for the ith muscle

M_i = muscular contractile force for the ith muscle

n_i = constant relating endurance time with muscle force for the ith muscle

c_i = endurance time for a muscle force level of 1% maximum muscle force for the ith muscle

The parameters n_i and c_i are dependent on the percentage of slow-twitch fibers, Z_i. These parameters can be represented by the following equations:

(29) $n_i = 0.25 + 0.036 * Z_i$

(30) $c_i = 3.48 + 0.169 * Z_i$

The percentage of slow twitch fibers for the relative muscles, referenced from Johnson et al., are listed in Table 8 (13).

TABLE 8. Slow Twitch Fiber Parameters

Muscle	z_i	n_i	c_i
Erector Equivalent	58.4	2.4	13.3
Latissimus Dorsi	50.5	2.1	12.0
Internal Obliques	76.4	3.0	16.4
External Obliques	76.4	3.0	16.4
Rectus Abdominis	46.1	1.9	11.3

The values for both the internal and external obliques are the average values estimated from experimental studies since a more accurate value was not obtainable (4).

The constraints involved in this problem consist of four equality constraints and eleven inequality constraints. The equality constraints were the equilibrium equations (16) through (19). As before, the equations (14) and (15) were used to back solve for S_a and S_r after the optimization solution was obtained. The inequality constraints designate the reasonable range each muscle force can be found in. In other words, these constraints form both an upper and lower bound for muscle force. The upper bound is determined by assuming that the muscle contraction intensities do no exceed the reasonable level of 100 N/cm^2.

Hence, the determination of muscle force during body function is formulated as a nonlinear optimization problem with an objective to minimize the summation of muscle stress to the n_i power. The problem statement is summarized as:

$$(31) \quad \text{minimize} \quad \sum_{i=1}^{m} (S_i)^{n_i}$$

$$\text{subject to} \quad 0 \leq S_i \leq 100 \text{ N/cm}^2$$

$$\text{equations (16) through (19)}$$

Predicting muscle forces to minimize this objective function coincides with maximizing endurance for the defined activity. The load and endurance data obtained from this optimization problem are listed in Table 9.

Figures (6) and (7) are referred to as two variable design spaces, where the design variables correspond to the coordinate axes. In general, a design space will be n - dimensional, where n is the number of design variables. The two variable design space is used to help visualize the concepts of optimization techniques. Figures 6 and 7 illustrate the objective function contours (dashed lines) as a function of the erector equivalents, E_l and E_r, and the latissimus dorsi, L_l and L_r, respectively. The solid lines on these figures represent the corresponding numbered constraints.

In general, nonlinear optimization convergence on a global minimum is not assured. However, the present problem due to the continuous convex character of the objective function and the linear constraints, falls into the category of convex programming. This convexity assures that the only minimum is a global minimum (25).

In comparison, Crowinshield and Brand used a similar optimization criterion:

$$(32) \quad \sum_{i=1}^{m} [(S_i)^3]^{1/3}$$

TABLE 9. Optimum Load and Stress Data from the Endurance Criterion

Element	Load (N)	Stress (N/cm^2)
E_l	282.700	14.050
E_r	312.300	15.459
I_l	0.000	0.000
I_r	0.000	0.000
L_l	8.377	3.709
L_r	8.377	3.934
R_l	0.000	0.000
R_r	0.000	0.000
X_l	0.000	0.000
X_r	0.000	0.000
C	905.100	---
S_a	0.000	---
S_r	0.000	---

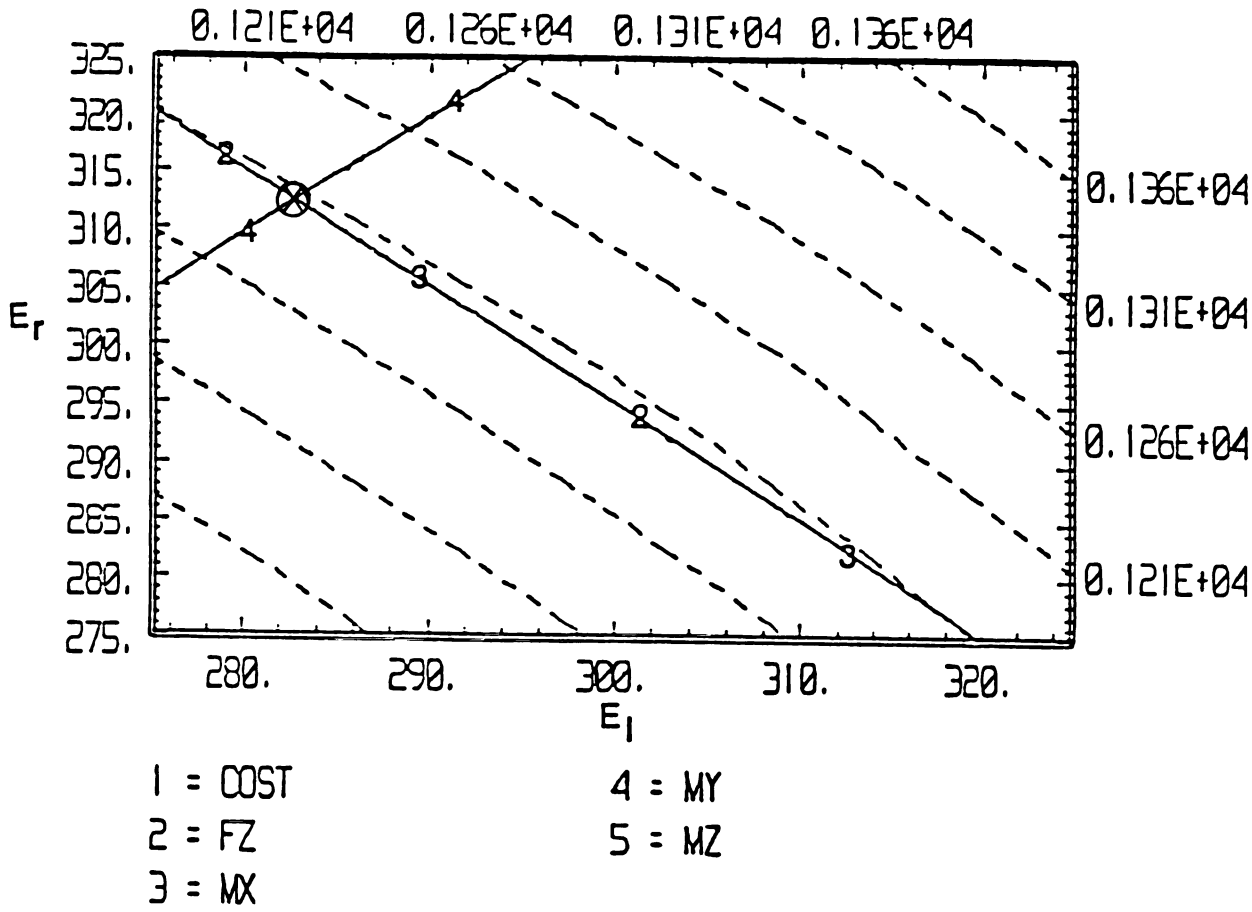

Figure 6. Erector Equivalent Space

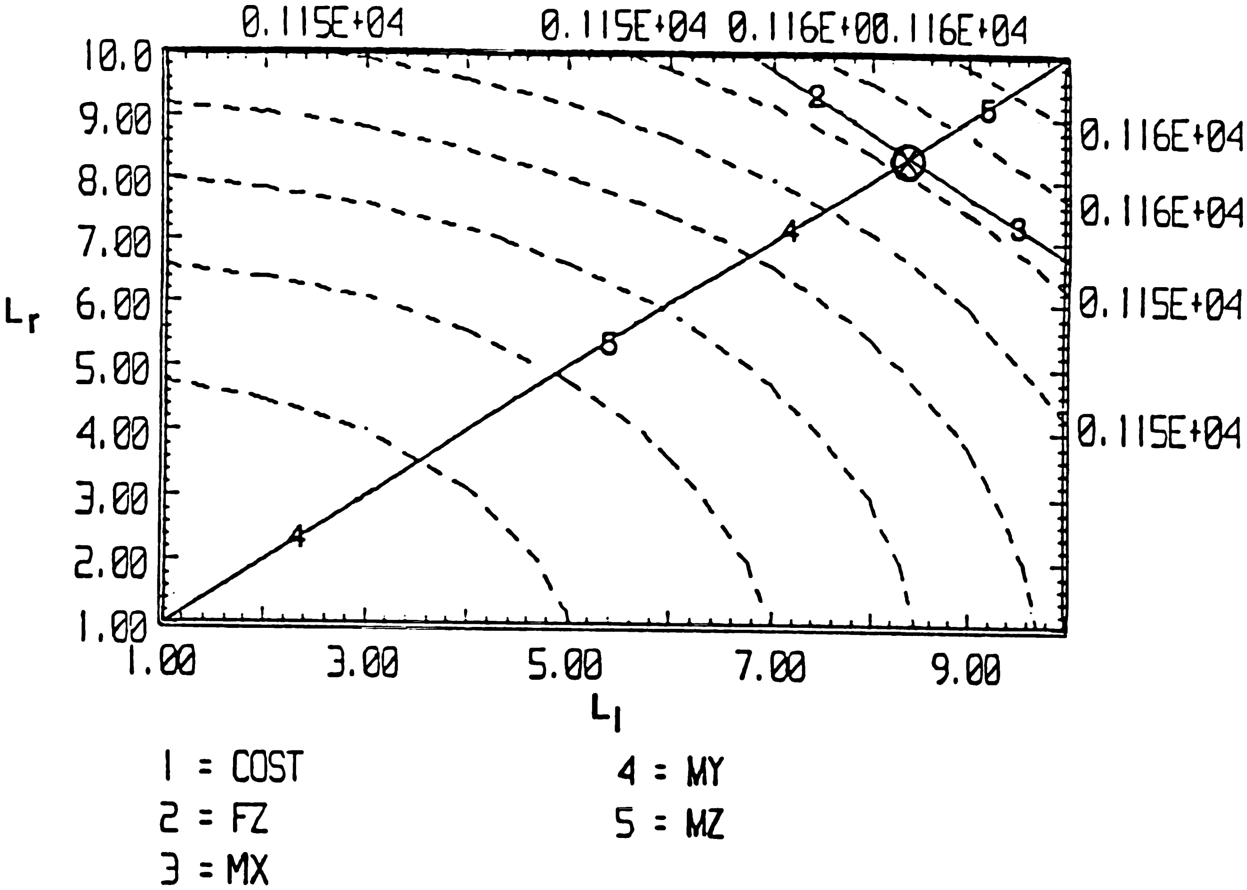

Figure 7. Latissimus Dorsi Space

The cube root of the objective function was taken for two practical reasons: first, so the objective function will have muscle stress units and second, to reduce the magnitude of the objective function thereby avoiding numerical problems. The average experimental value of n=3 is chosen for n. This method was demonstrated at the elbow during isometric contraction and in the lower extremity during locomotion (4).

When this method was applied to the weight lifting task, only a nominal change in the solution occurred. Table 10 lists the load values with n=3, with and without the taking cubed root and taking the cubed root with n varied according to the percentage of slow twitch fibers.

Table 10. Load Values of Nonlinear Criteria

Element	Load (N) Normalized n=3	Not Normalized n=3	Normalized n=variable
E_l	281.200	281.200	282.700
E_r	310.800	310.800	312.300
I_l	0.000	0.000	0.000
I_r	0.000	0.000	0.000
L_l	10.060	10.090	8.375
L_r	10.060	10.110	8.375
R_l	0.000	0.000	0.000
R_r	0.000	0.000	0.000
X_l	0.000	0.000	0.000
X_r	0.000	0.000	0.000
C	904.400	904.400	905.100
S_a	0.000	0.000	0.000
S_r	0.000	0.014	0.000

CONCLUSION

The method of optimization is applied to complex decision or redundant problems, in which a unique solution cannot readily be determined. When formulating an optimization problem, the performance criterion must represent an optimal characteristic of the system. Although electromyographic results were not available to confirm the validity of the criteria presented in this research, the optimization solutions are discussed mathematically.

The linear optimization problem of minimizing the upper bound of muscle stress yielded seven numerical solutions. Two of these seven muscular domains shared the same solution, which was also the optimal solution. The solution did not satisfy the constraints in the remaining three domains. The constraint violations occurred when searching the left internal oblique, the left erector equivalent and the left latissimus dorsi domains. The conflict that developed is that the equality constraints defined a solution space that was not common to the solution space or domain formed by the inequality constraints. This noncommonality did not allow the solution to satisfy the constraints.

The five muscle domains that yielded a numerical solution, but not the optimal solution, selected both different muscle force magnitudes and different active muscles. However, in each of the seven domains, all but two of the calculated stress values have the same stress as that of the muscular domain searched. The optimal solution, muscle spaces E_r

and L_r, has a maximum stress of 14.016 N/cm^2 compared to the largest stress generated of 19.372 N/cm^2 in muscle space X_1.

For the nonlinear problem, only four active muscles were selected. Nonlinear cost functions are assumed to select more synergistic muscles. Here, the linear method of minimizing the upper bound of muscle stress predicted a more even distribution of muscle stress among the synergistic muscles and a greater distribution of muscle activity. However, it is interesting to note that the stress values for both the minimum fatigue and minimum upper bound of muscle stress criteria were in the same range.

The active muscles selected for the minimum fatigue criterion were the same active muscles selected by the linear criterion of minimizing the compression on the L3 vertebra. In comparing the dominant muscles, the erector equivalents were selected by the minimum fatigue criterion, while the latissimus dorsi were chosen by the criterion of minimizing the spinal compression. The method of minimizing the spinal compression drove the latissimus dorsi to its upper bound. When the upper bound was removed, the latissimus dorsi carried 96% of the muscular load with only a 128 N load decrease carried by the spine compared to the other two cost functions. This response reveals that the solution had a greater dependency on the upper bound than on the cost function itself. Neither the criterion of minimizing the upper bound of muscle stress nor the endurance criterion yielded muscle stresses near the upper bound.

The minimum fatigue criterion was modified four times. Only a negligible difference in force magnitude was noticed when the cubed root of the cost functions was taken for both cases of $n=3$ and n_i values. Therefore, taking the cubed root was unnecessary in this problem. Only a minor difference in force magnitude existed when the values of n_i according to the percentage of slow twitch fibers were used compared to the experimental average value of n=3. For the activity of holding a 40 N weight, this response can be expected since there is not a large difference between the n_i parameters of the erector equivalents and the latissimus dorsi. The endurance criterion should be investigated further by increasing the load held in the right hand. Increasing the load would involve more muscles, including the antagonistic rectus abdominis muscles, thereby involving a full range of n_i parameters.

Figure 6 illustrates that constraints 2 and 3, equations (16) and (17) respectively, are redundant in the vacinity of the erector space solution. Also, constraint 5, equation (19), does not even exist in the limited erector space shown. All four equality constraints are present in the latissimus dorsi solution space illustrated by Figure 7. Constraints 2 and 3 are again found redundant in this space. In addition, constraints 4 and 5, equations (18) and (19) respectively, are also found redundant in the latissimus dorsi space shown. In both of the muscle spaces illustrated, the optimal solution is found at the intersection of the constraints.

Knowledge of the magnitudes and directions of muscle forces is necessary in the design of preventative and rehabilitative programs. In such applications, the solution to the indeterminate biomechanical problem can be obtained by formulating an objective function and utilizing an optimization technique. The criterion chosen to formulate an optimization problem must represent an optimal characteristic if the problem is to yield an accurate solution. There is a great need to experimentally validate the predictions presented and to continue formulating optimization problems based on physiological reasons.

APPENDIX A

APPENDIX A

SIMPLEX METHOD

The original form of the simplex algorithm was developed by George B. Dantzig in 1947 and was formally published in 1951. Many variations of the original technique have been developed since, but the original simplex algorithm is still the best procedure for the solution of the general linear programming problem when manual computations are used. Certain other revised simplex algorithms are computationally advantageous when the solution is calculated with a digital computer. The following derivation of the simplex method is referenced from David G. Luenberger.

As previously stated, a linear programming problem is a mathematical program in which the objective function is linear in the unknowns and the constraints consist of linear inequalities and/or linear equalities. The exact form of these constraints may vary from one problem to another but any linear program can be transformed into the following standard form:

$$\text{minimize} \quad \overleftarrow{c}\overleftarrow{x}$$

$$\text{(A1)} \quad \text{subject to} \quad \tilde{A}\overleftarrow{x} = \overleftarrow{b}$$

$$\overleftarrow{x} \geq 0$$

where: $\tilde{A}$ = m x n matrix

$\overleftarrow{x}$ = n - dimensional column vector

$\overleftarrow{b}$ = m - dimensional column vector

$\overleftarrow{c}$ = n - dimensional row vector

Various other forms of linear programs can be converted to the standard form by one or more of the following techniques: slack variables, surplus variables, free variables and by eliminating a variable unconstrained in sign.

It is also important to note that all inequality constraints must be in the form of less than inequality constraints. The purpose of this is for the geometric advantage of having the gradients of the cost function and the constraints point away from the optimal solution.

A basic solution is obtained by setting all the independent variables equal to zero and solving for the dependent variables. This matrix is said to be in canonical form. Since optimal solutions are always basic solutions, it is important to understand the concept of basic solutions and the fundamental theorem of linear programming. Consider the system of equalities,

(A2) $\tilde{A}\overleftarrow{x} = \overleftarrow{b}$

where: $\tilde{A}$ = m x n matrix

$\overleftarrow{x}$ = n - dimensional vector

$\overleftarrow{b}$ = m - dimensional vector

From the n columns of $\tilde{A}$, select a set of m linearly independent columns. Such a set exists if the rank of $\tilde{A}$ is m. For notational simplicity, assume that the first m columns of $\tilde{A}$ were selected and denote the m x m matrix determined by these columns by $\tilde{B}$. This matrix is illustrated in equations (A3).

$$(A3)\quad \begin{matrix} 1 & 0 & . & . & . & 0 & a_{1,m+1} & a_{1,m+2} & . & . & . & a_{1,n} & b_{10} \\ 0 & 1 & . & . & . & 0 & a_{2,m+1} & a_{2,m+2} & . & . & . & a_{2,n} & b_{20} \\ 0 & 0 & . & . & . & 0 & . & . & . & . & . & . & . \\ . & . & & & & . & . & . & . & . & . & . & . \\ . & . & & & & . & . & . & . & . & . & . & . \\ . & . & & & & . & . & . & . & . & . & . & . \\ 0 & 0 & & & & 1 & a_{m,m+1} & a_{m,m+2} & . & . & . & a_{m,n} & b_{mo} \end{matrix}$$

The matrix $\tilde{B}$ is referred to as a basis since it consists of m linearly independent columns that can be regarded as a basis for the space E^m. The matrix $\tilde{B}$ is then nonsingular and a unique solution may be obtained from the equation:

$$(A4)\quad \tilde{B}\overleftarrow{x}_B = \overleftarrow{b}$$

where: $\tilde{B}$ = m x m matrix

$\overleftarrow{x}_B$ = m - dimensional vector

$\overleftarrow{b}$ = m - dimensional vector

The vector $\overleftarrow{x}_B$ is composed of the first m components of $\overleftarrow{x}$, the basic variables, and the remaining components of $\overleftarrow{x}$ are equal to zero. That is, $\overleftarrow{x} = (\overleftarrow{x}_B, \overleftarrow{0})$.

In general, equations (A2) may not have any basic solutions. By making certain elementary assumptions regarding the structure of the matrix $\tilde{A}$, this trivial problem may be avoided. First, assume that $n > m$, that is, the number of variables x_j exceeds the number of equality constraints. Second, assume that the rows of $\tilde{A}$ are linearly independent, corresponding to linear independence of the m equations. $\tilde{A}$ linear dependency among the rows of $\tilde{A}$ would lead either to contradictory constraints and hence no solutions to (A2), or to a redundancy that could be eliminated. Now, with the assumption that $\tilde{A}$ has rank m, there exists at least one basic solution to (A2).

Another point worth commenting on is that of a degenerate solution. If one or more of the basic variables in a basic solution has value zero, that solution is said to be a degenerate basic solution. Note that in a nondegenerate basic solution, the basic variables, and hence the basis $\tilde{B}$, can be immediately identified from the positive components of the solution. Since the zero-valued basic and nonbasic variables can be interchanged, ambiguity is associated with a degenerate basic solution.

So far in the discussion of basic solutions, no reference has been made to the positivity constraints on the variables. Similar definitions apply when these constraints are also considered. Thus, consider the system of contraints:

$$\text{(A5)} \quad \tilde{A}\overleftarrow{x} = \overleftarrow{b},$$
$$\overleftarrow{x} \geq 0$$

which represents the constraints of a linear program in standard form. If a vector $\overleftarrow{x}$ satisfies equations (A5) it is said to be feasible for these constraints. A feasible solution to the constraints (A5) that is also basic is said to be a basic feasible solution; if this solution is also a degenerate basic solution, it is called a degenerate basic feasible solution.

The primary importance of basic feasible solutions in solving linear programming problems is represented through the fundamental theorem of linear programming. The theorem itself shows that it is necessary only to consider basic feasible solutions when seeking an optimal solution to a linear program because the optimal value is always achieved at such a solution.

The geometric interpretation of the simplex method is straight forward. The constraints form a polyhedron (multidimensional case) which is either convex from or to the origin, according to whether the "greater than" or the "less than" inequality condition is imposed. The polyhedron defines the feasible solution space. There is a special category of the infinite feasible solutions that is of a finite number. These solutions, called basic feasible solutions, are situated geometrically at the vertices of the polyhedron. The number of basic solutions is determined by the number of variables, n, and the number of constraints, m, according to the combinatorial formulas:

(A6) $C_n^m = n!/m!(n - m)!$

The z-constant equations (evaluated cost functions for each basic solution) form a family of straight lines or hyperplanes. The extremum solution is given by the remotest or the nearest intersecting points between the z-hyperplane and the polyhedron (24).

The first step of the simplex method, referred to as phase I, simply locates a vertex of the feasible set, or establishes that the set is empty. Assuming that a vertex, or basic solution, has been found the procedure continues with phase II. This phase is the heart of the method, which searches from vertex to vertex along the edges (intersections of the planes) of the feasible set. At a typical vertex there are n edges to choose from, some leading away from the optimal solution and others leading gradually toward it. Since linear programming forces the solution to stay in the feasible set, an edge that is guaranteed to decrease (increase) the cost is chosen. Eventually the optimal solution is reached.

Another important point is the convexity of the canonical system forms. During the various linear transformations occurring in linear programming, the convexity of the polyhedron does not change. The properties concerning the vertices therefore remain the same; they guarantee the necessity and sufficiency of the final extremal solution, if it exists.

The simplex method proceeds from one basic feasible solution, that is one extreme point, of the constraint set of a problem in standard form to another, in such a way as to continually decrease the value of the objective function until a minimum is reached. This is accomplished by simple multiplications and additions referred to as pivoting. A pivot operation consists of m elementary operations which replace a standard system by an equivalent canonical system.

To initiate the use of the simplex method, the problem of finding an initial basic feasible solution arises. Except for the cases where the linear constraints are inequalities in which slack and/or surplus variables are used to transform the problem into one of standard form, it is not always possible to easily find an initial basic feasible solution. Therefore, it is necessary to develop a means for determining one so that the simplex method can be initiated. Interestingly, an auxiliary linear program and corresponding application of the simplex method can be used to determine the required initial solution.

Another important point is that many linear programs arising from practical situations involve variables that are subject to both lower and upper bounds. The simplex method is easily modified to accommodate the upper bound.

One final comment pertains to the revised simplex method. Extensive field experience has indicated that the simplex method converges to an optimum solution in about m or 1.5*m pivot operations. If m is much smaller than n, that is, if the matrix $\tilde{A}$ has far fewer rows than columns, pivots will occur in only a small fraction of the columns during the course of optimization. Since the other columns are not explicitly used, the work expended in calculating the elements in these columns after each pivot is wasted effort. The revised simplex method is a scheme for ordering the computations required of the simplex method so that unnecessary calculations are avoided.

APPENDIX B

APPENDIX B

GENERALIZED REDUCED GRADIENT METHOD

From a computational viewpoint, the simplex method is related to the generalized reduced gradient method of nonlinear programming in that the problem variables are partitioned into basic and nonbasic groups. The following description of the generalized reduced gradient method was referenced from Garret N. Vanderplaats (25). However, before beginning this derivation, it is important to mention some basic properties.

Recalling from basic calculus that in order for a function of one variable to have a minimum, its second derivative must be positive. In the general n-dimensional case, this translates into the requirement that the matrix of second partial derivatives of the objective with respect to the design variables must be positive definite. This matrix is called the Hessian matrix. Positive definiteness means that this matrix has all positive eigenvalues.

If the Hessian matrix is positive definite at a given x, this insures that an extremum of the design is at least a relative minimum. This vector, $\overleftarrow{x}$, is special in the sense that it satisfies the first order conditions. If the Hessian matrix is positive definite for all possible values of the design variables, $\overleftarrow{x}$, then a relative minimum of the design is guaranteed to be a global minimum. In this case, the objective function is said to be convex. When the objective function is convex and the constraints form a convex solution space the necessary

Kuhn-Tucker conditions are also sufficient to guarantee that if an optimal solution is obtained, it is the global optimum.

The general nonlinear constrained optimization problem statement can be written mathematically as:

$$\text{minimize} \quad F(\vec{x})$$

$$\text{(B1)} \quad \text{subject to} \quad g_j(\vec{x}) \leq 0 \qquad j = 1,m$$

$$h_k(\vec{x}) = 0, \qquad k = 1, L$$

$$x_i^L \leq x_i \leq x_i^u$$

where:

$F(\vec{x})$ = objective function

$\vec{x}$ = vector of design variables

$g_j(\vec{x})$ = inequality constraints

$h_k(\vec{x})$ = equality constraints

x_i^L = lower bound on x_i

x_i^u = upper bound on x_i

Notice that the bounds on the variables are considered as side constraints. These side constraints could be included in the inequality constraint set, but are usually treated separately since they define the search region. Since the generalized reduced gradient method only solves equality-constrained problems, the problem statement (B1) must be modified by adding slack variables, x_{j+n}, to the inequality constraints to yield the general form:

$$\text{minimize} \quad F(\vec{x})$$

$$\text{(B2)} \quad \text{subject to} \quad g_j(\vec{x}) + x_{j+n} = 0, \qquad j = 1,m$$

$$h_k(\vec{x}) = 0 \qquad k = 1,L$$

$$x_i^L \leq x_i \leq x_i^u \qquad i = 1,n$$

$$x_{j+n} \leq 0 \qquad j = 1,m$$

A total of m slack variables are added, therefore the problem now consists of n+m variables.

The concept of this method is that one dependent design variable can be written for each equality constraint, thereby reducing the number of independent design variables. This creates an unconstrained minimization problem subject only to side constraints on the variables. The independent variables are referred to as the decision variables and the dependent variables are referred to as state variables.

Now, since $\overleftarrow{x}$ contains both the original n variables and the m slack variables, $\overleftarrow{x}$ can be partitioned as:

(B3) $\overleftarrow{x} = (\overleftarrow{z},\overleftarrow{y})^T$

where: $\overleftarrow{z}$ = n-L independent variables

$\overleftarrow{y}$ = m+L dependent variables

Notice that no restrictions as to which variables are contained in $\overleftarrow{z}$ and $\overleftarrow{y}$ are made. Also, since the problem consists only of equality constraints, the problem statement can be simplified to:

$$\begin{array}{llll} & \text{minimize} & F(\overleftarrow{x}) = F(\overleftarrow{z},\overleftarrow{y}) & \\ \text{(B4)} & \text{subject to} & h_j(\overleftarrow{x}) = 0 & j = 1,m+L \\ & & x_i^L \leq x_i \leq x_i^u & i = 1,n+m \end{array}$$

The side constraints for the original variables and the slack variables were combined with the understanding that the upper bounds associated with slack variables are set very large (infinite) and that the lower bound associated with each variable is zero.

Equations (B4) are used to formulate the generalized reduced gradient, $\overleftarrow{G}_R$.

The reduced gradient is used to determine a search direction, $\overleftarrow{S}$, for use in the iterative equation:

(B5) $x_i^q = x_i^{q-1} + \alpha^* S^q$

where: q = iteration number

$\overleftarrow{S}$ = vector search direction

α^*= scalar quantity that defines the distance of travel

In order to improve a design, it is necessary to determine a direction vector which will reduce the objective function without violating any active constraints. Any direction which reduces the objective function is defined as a usable direction. The portion of the design space that is referred to as the usable sector is defined by the hyperplane that is tangent to the objective function. The area in the usable sector that does not violate the active constraints is referred to as the usable feasible sector. Any direction vector $\overleftarrow{S}$, in the usable feasible sector of the design space satisfies the criterion.

During various iterations the dependent variables, $\overleftarrow{y}$, are updated. However, since this equation is a linear approximation to the original nonlinear problem, the constraints may not be zero for a proposed α . Therefore, a new expression for $d\overleftarrow{y}$ must be developed to drive $h(\overleftarrow{x}) = 0$.

One final comment pertains to the process of selecting the dependent variables such that the $\tilde{B}$ matrix is nonsingular and so that a small change in the variables will not violate the side constraints on these variables. The second requirement is easily met by picking dependent variables which are not too close to their side constraints. The first requirement, a nonsingular matrix, is accomplished by performing Gaussian elimination operations on a matrix $\tilde{Q}$, using the pivot search. The matrix $\tilde{Q}$, is an (m+L)*(n+m) matrix with elements $\nabla^T h_j(x)$. Notice that the independent and dependent variables are not already partitioned in $\tilde{Q}$. By invoking the nondegeneracy assumptions that every collection of m columns from $\tilde{Q}$ is linearly independent and that every basic solution to the constraints has m strictly positive variables any feasible solution will have at most n-m variables taking the value zero.

Recalling that after the dependent variables were chosen and the reduced gradient was developed, a search direction, $\overleftarrow{S}$, must be determined. In its simplest form, the search direction $\overleftarrow{S}$ is the negative of the generalized reduced gradient. This is represented by the expression:

$$\text{(B6)} \quad \overleftarrow{S} = -\overleftarrow{G}_R$$

In subsequent iterations a different method may be employed as long as the set of independent variables is not altered.

A first estimate for the step size, α ,can be found by using the distance to the nearest side constraint. Note that by searching in a specified direction, $\overleftarrow{S}$, the problem of n+m variables in $\overleftarrow{x}$ is converted to one variable, α. Hence, this method is referred to as the one-dimensional search.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. An, K.N., B.M. Kwak, E.Y. Chao, B.F. Morrey, "Determination of Muscle and Joint Forces: A New Technique to Solve the Indeterminate Problem," Journal of Biomechanical Engineering, Vol. 106, No. 4, pp. 364-367, 1984.

2. Barbenel, J.L., "The Biomechanics of the Temporomandibular Joint: A Theoretical Study," Journal of Biomechanics, Vol.5, pp. 251-256, 1972.

3. Clauser, C., J. McConville, J. Young, "Weight Volume and Center of Mass of Segments of the Human Body," AMRL-TR-69-70, Wright-Patterson Air Force Base, Ohio, 1969.

4. Crowninshield, R.D., and R.A. Brand, "A Physiologically Based Criterion of Muscle Force Prediction in Locomation," Journal of Biomechanics, Vol. 14, No. 11, pp. 793-801, 1981.

5. Crowninshield, R.D., "Use of Optimization Techniques to Predict Muscle Forces," Journal of Biomechanical Engineering, Vol. 100, pp. 88-92, 1978.

6. Crowninshield, R.D., R.C. Johnston, J.G. Andrews, R.A. Brand, "A Biomechanical Investigation of the Human Hip," Journal of Biomechanics, Vol. 11, pp. 75-85, 1978.

7. Dul, J., G.E. Johnson, R. Shiaui, M.A. Townsend, "Muscular Synergism - II. A Minimum-Fatique Criterion for Load Sharing Between Synergistic Muscles," Journal of Biomechanics, Vol. 17, No. 9, pp. 675-684, 1984.

8. Eycleshymer, A.L., D.M. Schoemaker, A Cross-Section Anatomy. New York, Appleton-Century-Crofts, 1911.

9. Farfan, H.F. Mechanical Disorders of the Low Back. Philadelphia, Lea and Febiger, 1973.

10. Gracovetsky, S., H.F. Farfan, and C. Lamy, "The Mechanism of the Lumbar Spine," Spine, Vol. 6, No. 3, pp. 249-262, 1981.

11. Gracovetsky, S., H.F. Farfan, and C. Lamy, "A Mathematical Model of the Lumbar Spine Using an Optimized System to Control Muscles and Ligaments," Orthopedic Clinics of North America, Vol. 8, No. 1, pp. 135-153, 1977.

12. Hardt, D.E., "Determining Muscle Forces in the Leg During Normal Human Walking - An Application and Evaluation of Optimization

Methods," Journal of Biomechanical Engineering, Vol. 100, pp. 72-78, 1978.

13 . Johnson, M.A., J. Polgar, D. Weightman, D. Appleton, "Data on the Distribution of Fibre Types in Thirty-Six Human Muscles an Autopsy Study," Journal of the Neurological Sciences, Vol. 18, pp. 111-129, 1973.

14. Luenberger, David G. Introduction to Linear and Nonlinear Programming. Massachusetts, Addison-Wesley Publishing Company, 1973.

15. MacConaill, M.A., "The Ergonomic Aspects of Articular Mechanics," Studies on the Anatomy and Function of Bones and Joints, pp. 69-80, Springer, Berlin, 1967.

16. Patriarco, A.G., R.W. Mann, S.R. Simon, and J.M. Mansour, "An Evaluation of the Approaches of Optimization Models in the Prediction of Muscle Forces During Human Gait," Journal of Biomechanics, Vol. 14, No. 8, pp. 513-525, 1981.

17. Pedotti, A., V.V. Krishnan, L. Stark, "Optimization of Muscle-Force Sequencing in Human Locomotion," Mathematical Biosciences, Vol. 38, pp. 57-76, 1978.

18. Penrod, D.D., D.T. Davy and D.P. Singh, "An Optimization Approach to Tendon Force Analysis", Journal of Biomechanics, Vol. 7, pp. 123-129, 1974.

19. Schultz, A.B., K.N. Warwick, M.H. Berkson, and A Nachemson, "Mechanical Properties of Human Lumbar Spine Motion Segments. Part I: Responses in Flexion-Extension, Lateral Bending and Torsion," Journal of Biomechanical Engineering, Vol. 101, pp. 46-52, 1979.

20. Schultz, A.B. and G.B.J. Andersson, "Analysis of Loads on the Lumbar Spine," Spine, Vol. 6, No. 1, pp. 76-82, 1981.

21. Schultz, A.B., G.B.J. Andersson, K. Haderspeck, R. Ortengren, M. Nordin, and R. Bjork, "Analysis and Measurement of Lumbar Trunk Loads in Tasks Involving Bends and Twists," Journal of Biomechanics, Vol. 15, No. 9, pp. 669-675, 1982.

22. Seireg, A., and R.J. Arvikar, "A Mathematical Model for Evaluation of Forces in Lower Extremeties of the Musculo-Skeletal System," Journal of Biomechanics, Vol. 6, No. 3, pp. 313-326, 1973.

23. Seireg, A., and R.J. Arvikar, "The Prediction of Muscular Load Sharing and Joint Forces in the Lower Extremities During Walking," Journal of Biomechanics, Vol. 8, No. 2, pp. 89-102, 1975.

24. Strang, Gilbert. Linear Algebra and Its Applications. New York, Academic Press, Incorporated, 1980.

25. Vanderplaats, Garret N. Numerical Optimization Techniques for Engineering Design; With Application. McGraw-Hill Book Company, 1984.

26. Williams, R.J. and A. Seireg, "Interactive Computer Modeling of the Musculoskeletal System," IEEE Trans. Biomedical Engineering BME-24, pp. 213-219, 1977.

27. Williams, R.J. and A. Seireg, "Interactive Modeling and Analysis of Open or Closed Loop Dynamic Systems with Redundant Actuators," Journal of Mechanical Design, Vol. 101, pp. 407-416, 1979.

28. Yeo, B.P., "Investigations Concerning the Principle of Minimal Total Muscular Force," Journal of Biomechanics, Vol. 9, No. 6, pp. 413-416, 1976.

29. Yettram, A.L., and M.J. Jackman, "Equilibrium Analysis for the Forces in the Human Spinal Column and Its Musculature," Spine, Vol. 5, No. 5, pp. 402-411, 1980.