PLUGS IN SIMPLY-CONNECTED 4 MANIFOLDS WITH BOUNDARIES

By

Wei Fan

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics - Doctor of Philosophy

2015

ABSTRACT

PLUGS IN SIMPLY-CONNECTED 4 MANIFOLDS WITH BOUNDARIES

By

Wei Fan

In 1986, S. Boyer generalized Freedman's result to simply-connected topological 4 manifolds with boundaries. He proved in many cases, the intersection form and the boundary determine the homeomorphism types of the 4 manifolds. In this thesis, we will study simply-connected smooth 4 manifolds with boundaries by using handlebody techniques. We will show that: there do exist simply-connected smooth 4 manifolds with the same intersection form and the same boundary but not homeomorphic to each other, and the cause of this phenomenon is "Plug".

ACKNOWLEDGMENTS

I would like gratefully and sincerely thank my advisor Selman Akbulut for his guidance, support, patience and encouragement during my PhD program at Michigan State University. His unique perspective on research and sharp insight on almost any issue have been an invaluable resource to me in my studies.

I would also like to thank Matt Hedden and Effie Kalfagianni for many enlightening conversations on many topics. I am indebted Luke Williams for uncountably many inspirational discussions and his continued support for Latex. I am thankful to my fellow graduate students at MSU. I have learned a great deal from many of you. I appreciate the atmosphere of mathematical learning during my years here.

Last but not least, I am grateful to my dad Zhenfeng Fan, my mom Rongmei Yang and my girlfirend Caijin Huang for always having faith in me.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

This thesis will focus on the study of homeomorphism type of simply-connected smooth 4-manifolds with boundaries. One of the main difficulty in studying 4-manifold theory arises from the failure of h-cobordism theorem. In 1960, S. Smale [41] proved for dimension higher than 4, if two simply-connected smooth manifolds are h-cobordant, then they are diffeomorphic to each other.

Unfortunately, in dimension 4, one of the key ingredient of Smale's proof: the Whitney's trick does not work. In 1973, A. Casson looked into the 4-dimensional h-cobordism model, and introduced a new idea "self-pluming handle" which is called "Casson Handle" nowadays (Casson himself called it "flexible handle"). In 1982, M. Freedman [24] proved that the Casson handles are topologically standard handles, and hence the h-cobordism between two simply-connected smooth 4-manifolds actually induces a homeomorphism. By combining with Wall's theorem [44], [45] which claims that any two closed simply-connected 4-manifolds with isomorphic intersection forms are h-cobordant, he derived the following celebrated theorem:

Theorem 1.1. (Freedman, [24]) *For every unimodular symmetric bilinear form Q there exists a simply-connected, closed, topological 4-manifold X such that $Q_X \cong Q$. If Q is even, this manifold is unique (up to homeomorphism). If Q is odd, there are exactly two different homeomorphism types of manifolds with the given intersection form. At most one of these* *homeomorphism types carries a smooth structure.*

As a consequence, the famous Poincaré Conjecture was proved (the 4-dimensional topological case)

Corollary 1.1. (Freedman, [24]) *If a topological 4-manifold X is homotopy equivalent to S^4, then X is homeomorphic to S^4.*

Freedman's Theorem says the homeomorphism type of closed, simply-connected, smooth 4-manifolds are completely classified by their intersection forms. However, the classification of the diffeomorphism type of simply-connected, smooth 4-manifolds is still a mystery and far from achieving.

In dimension 4, homeomorphism type is different from diffeomorphism type; actually, dimension 4 is the lowest dimension that such phenomena happens. Manifolds which are homeomorphic to each other but not diffeomorphic to each other are usually called "exotic manifolds". In the following decades, people invented many different methods to produce exotic 4 manifolds: Logrithmic Transformation, Rational Blow-down [21], Fintushel-Stern Knot Surgery [22], etc. "Cork Twist" [9] is one of the methods that attracts people's attention in recent years. Corks are contractible submanifolds; cork twist means there is an involution map on the boundary of the cork; we cut off the cork and glue it back by the involution map. Many Corks are fairly simple 4-manifolds, while by operating twists on them, abundant illuminating and interesting exotic 4-manifolds were constructed [10], [11], [12], [13].

In 1996 [32], [18], it was proved that any two simply-connected smooth 4-manifolds which are homeomorphic differ by some cork twists. Indeed, the failure of smooth h-cobordism theorem in dimension 4 is due to the cork twists. So, cork twist plays a fundamental and important role in changing the diffeomorphism type (smooth structure) of the 4-manifolds.

Freedman's theorem is also true for simply-connected 4-manifolds whose boundaries are homology 3 spheres, since in this case, Q_X is still ular and hence Wall's theorem still holds. In 1986, S. Boyer [15] studied the homeomorphism type of simply-connected, topological 4-manifolds whose boundaries are arbitrary closed, oriented, connected 3-manifolds. He gave a necessary and sufficient condition under which a given homeomorphism between the boundaries of the two 4-manifolds can be extended into the interior. His result can be considered as a generalization of Freedman's Theorem in the following way:

Theorem 1.2. (Boyer, [15]) *Let M denote a closed, oriented, connected 3-manifold;* $(\mathbb{Z}^n, Q)$ *denote a bilinear form space. If Q presents*[1] $H_*(M)$*, we will denote them by a pair* (Q, M)*. Then:*

(i) if $H_1(M, \mathbb{Q}) \cong 0$*, there are at most 2 homeomorphism types of simply-connected topological 4-manifolds* (X, M) *whose intersection form is* Q_X *such that* (Q_X, M) *is isomorphic*[2] *to* (Q, M)*. At most one of these homeomorphism types carries a smooth structure.*

(ii) if Q is odd, there are at most 2 homeomorphism types of simply-connected topological 4-manifolds (X, M) *whose intersection form is* Q_X *such that* (Q_X, M) *is isomorphic to* (Q, M)*. At most one of these homeomorphism types carries a smooth structure.*

(iii) if Q is even, there are at most Spin$(M)/\mathcal{H}_+(M)$ *many different homeomorphism types of simply-connected topological 4-manifolds* (X, M) *whose intersection form is* Q_X *such that* (Q_X, M) *is isomorphic to* (Q, M)*.*

Moreover, If $H_1(M)$ *is free, then:*

(i) if Q is odd, there are exactly 2 homeomorphism types of simply-connected topological 4-manifolds which are isomorphic to (Q, M) *and they differ by their Kirby-Siebenmann invariant.*

(ii) if Q is even, there are exactly Spin$(M)/\mathcal{H}_+(M)$ *many homeomorphism types of simply-*

connected topological 4-manifolds which are isomorphic to (Q, M).

Although Boyer works within the topological category, his theorem has a very nice implication in the smooth case:

Corollary 1.2. *two simply-connected odd smooth 4-manifolds which have isomorphic* (Q, M) *are homeomorphic.*

Now one may wonder what happens to the even smooth 4 manifolds. Boyer gave a general algorithm on constructing $\mathrm{Spin}(M)/\mathcal{H}_+(M)$ many non-homeomorphic simply-connected even topological 4-manifolds which have isomorphic (Q, M), when $H_1(M)$ is free. However, his construction does not work in the smooth case, since his proof uses Freedman's result [24]: *Every homology 3-sphere bounds a contractible topological 4-manifold* which fails in the smooth case.

Whether there exist non-homeomorphic simply-connected even smooth 4-manifolds which have isomorphic (Q, M)? In the last chapter, we will give a positive answer to this question by presenting a concrete example. We will hunt through all possible orientation preserving self-homeomorphisms of a haken manifold M. M bounds two simply-connected smooth 4-manifolds X_1 and X_2, which induce spin structures s_1 and s_2 respectively on M. Then we will easily see that none of these self-homeomorphisms interchange the two spin structures. So X_1 and X_2 must be non-homeomorphic to each other.

In chapter 3, we will prove that if (X_1, M_1) and (X_2, M_2) are two simply-connected even smooth 4-manifolds such that (Q_{X_1}, M_1) is isomorphic to (Q_{X_2}, M_2), then the homeomorphism types of (X_1, M_1) and (X_1, M_2) differ by a single "Plug Twist". "Plug Twist" is a similar operation as Cork Twist. It was defined and studied in [9]. It comes naturally from many examples. For instance, Gluck Twist [25] can be considered as a simplest Plug Twist.

The key difference between a Cork and a Plug is that a Plug contains a framing 1 (or -1) 2-handle: twisting around this 2-handle may change the homeomorphism type; while since the boundary of a cork is a homology 3-sphere, cork twisting never changes the homeomorphism type.

By combining with the theorem of Cork, we would draw the following conclusion:
If two simply-connected smooth 4-manifolds have isomorphic (Q, M), then the plug twist is responsible for the difference of their homeomorphism types; cork twists is responsible for the difference of their diffeomorphism types (smooth structures).

In chapter 2, we will introduce the necessary terminologies and background.

Chapter 2

Background and Known Results

2.1 Whitney's Trick and Finger Move

2.1.1 Whitney's trick

Whitney's trick is a method for removing a pair of double points of opposite sign (each intersection point has a sign from comparing orientations); it works perfectly in dimension $\geqslant 5$. Since it this thesis, we will only concentrate on 4-dimensional case, let us consider the following Whitney's trick model in dimension 4.

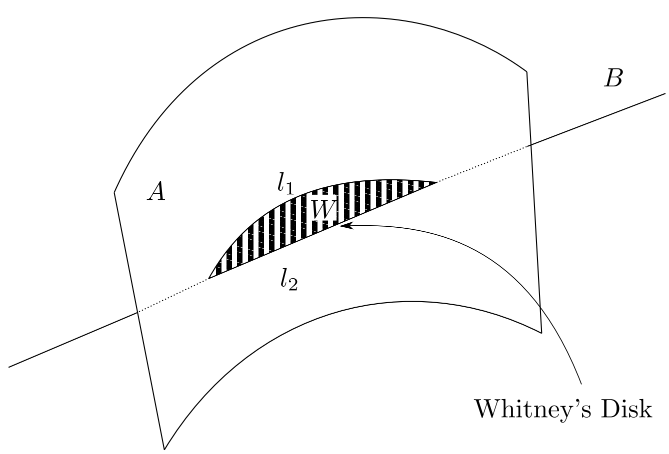


Figure 2.1: Whitney's disk

Suppose A and B are two embedded surfaces in X^4 which intersect in two points with opposite sign (by dimension counting, we may assume all intersections are transverse). Choose a path that links the two intersection points inside A and choose another path linking the two intersection points in B. The union of the two paths form a circle, which is called *Whitney's circle*; we also assume that this circle bounds an embedded 2-disk in the complement of A and B, as shown in the following picture. The interior of this disk is called *Whitney's disk*.

The Whitney's circle consists of two parts: l_1 lies on A; l_2 lies on B. There is a 1-vector bundle λ_1 over l_1 which is tangent to A and normal to W; there is also a 1-vector bundle λ_2 over l_2 which is normal to both W and B. λ_1 and λ_2 agree at their common points. Denote λ the union of λ_1 and λ_2 as a 1-vector bundle over the Whitney's circle. The obstruction of extending λ over W is $\pi_1 SO(2)$. It is called the *framing obstruction*. For this moment, let us assume the framing obstruction is trivial (intuitively, this mean when we push off the Whitney's disk parallelly, the Whitney's circles are not linked on the boundary of the complement). Now, under all these assumptions, we see that there is an ambient isotopy, called *Whitney's move*, supported in a neighbourhood of W which moves A to a surface A' disjoint from B. A' can be described as constructed from A by cutting out a neighbourhood of the arc on A, glueing in two parallel copies of the Whitney's disk, and a parallel of a neighbourhood of the arc on B.

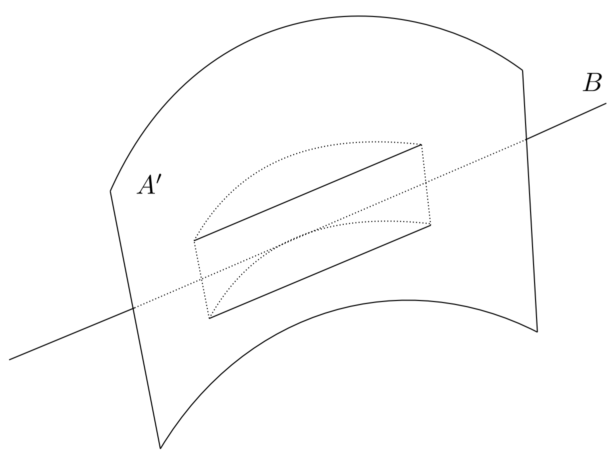


Figure 2.2: Whitney's move

To make the Whitney's trick work in dimension 4, we made 3 assumptions:

1. the Whitney's circle bounds a disk in the complement of A and B.

2. the Whitney's disk is embedded.

3. the framing obstruction is trivial.

In general situations, none of these assumptions can be guaranteed. It seems there are too many issues. However, we may reduce the number of difficulties by using the following tricks [38], [27]:

(i) Interior twisting:

This operation creates a self-intersection of W, while changes the framing obstruction by ± 2.

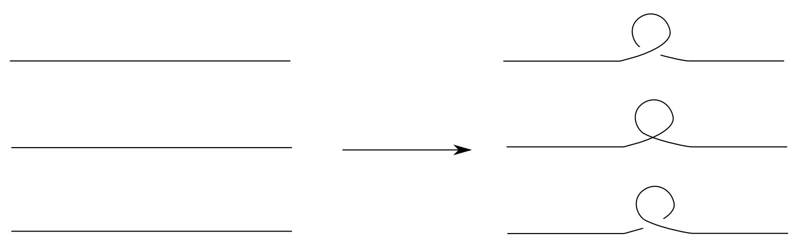


Figure 2.3: Interior twisting

(ii) Boundary twisting:

This operation creates a new intersection point between W and A, while changes the framing obstruction by ± 1.

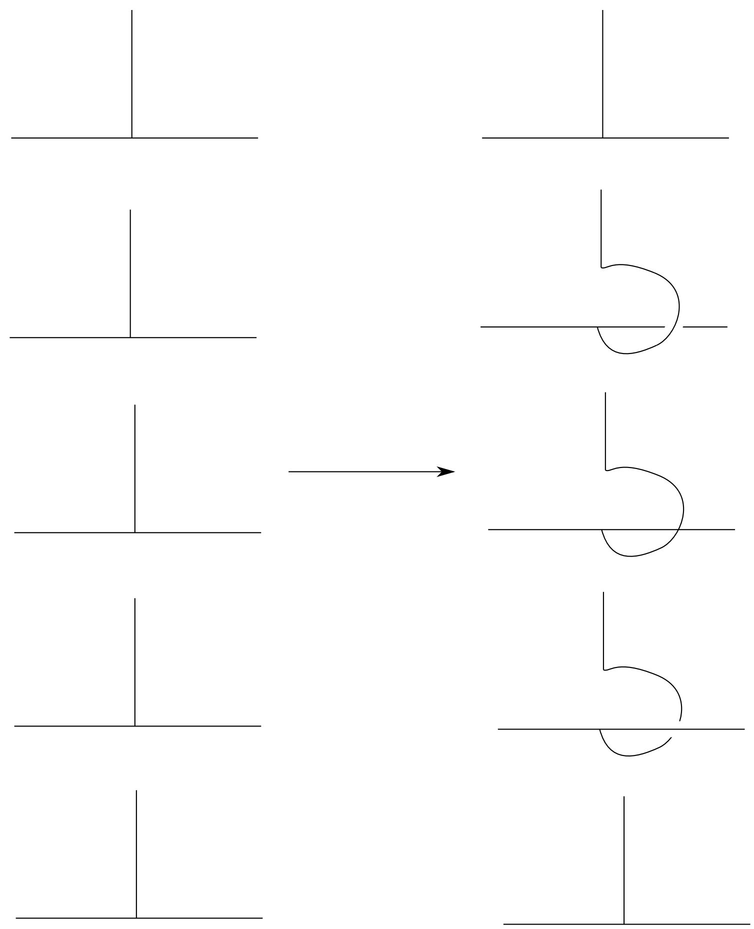


Figure 2.4: Boundary twisting

(iii) Push off interior self-intersections:

This operation eliminates one self-intersection of W; creates two new intersection points between W and A. It does not change the framing obstruction.

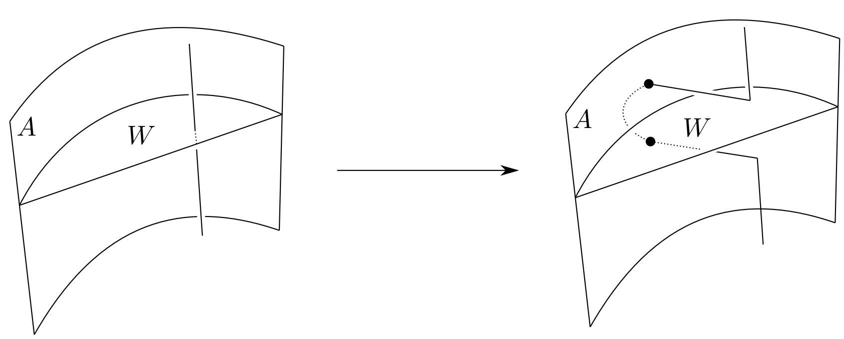


Figure 2.5: Push off interior self-intersections

So, if we can find a Whitney's disk W which is disjoint form B, even if it is immersed, the framing is wrong, we can modify it by (ii) and (iii) such that the new Whitney's disk is embedded, and the framing obstruction becomes trivial. The price to pay is we obtain more intersection points between W and A. Now we can still do the Whitney's trick by dragging A along the new Whitney's disk. The effect is that we get a surface A' which is disjoint from B, but A' is immersed.

2.1.2 Finger move

To use Whitney's trick in dimension 4 (to eliminate intersection points between A and B without getting self-intersections), we need to find a Whitney's disk in the complement of A and B first. This suggests us the complement of A and B better be simply-connected. A brilliant idea of reducing the fundamental group of the complement of A and B is due to Casson: Imagine we push our finger through A following a loop γ in the complement, as shown in the following figure.

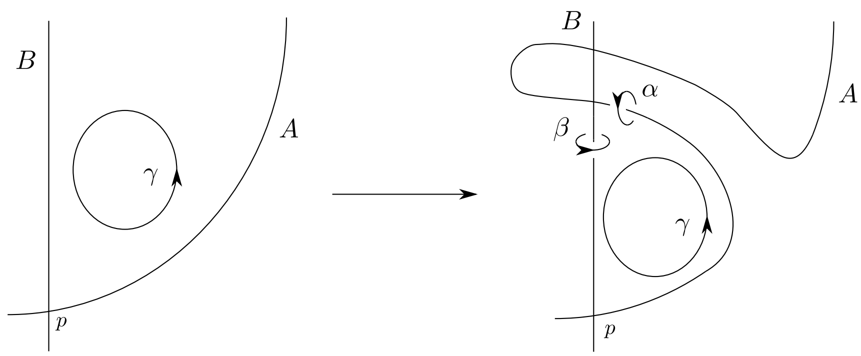


Figure 2.6: Finger move changes the fundmental group of the complement

This method is called *Finger move*. By doing the finger move, we create two new intersection points between A and B, so it can be regarded as "inverse Whitney's trick", and we will denote this Whitney's disk by W. Now let us consider how does this process affect the fundamental group of the complement:

A neighbourhood of p is locally diffeomorphic to $(\mathbb{R}^4 \cong \mathbb{R}^2_{xy} \times \mathbb{R}^2_{zt}, \mathbb{R}^2_{xy} \cup \mathbb{R}^2_{zt})$. There is a torus $T^2 = S^1 \times S^1$ on the boundary of the complement of $\mathbb{R}^2_{xy} \cup \mathbb{R}^2_{zt}$ in $\mathbb{R}^4$ which links $\mathbb{R}^2_{xy} \cup \mathbb{R}^2_{zt}$. ($T^2 = \{(x, y, z, t) \in \mathbb{R}^1 \mid x^2 + y^2 = 1 = z^2 + t^2\}$). This torus is called *the linking torus*. Its fundamental group is generated exactly by α and β. So by introducing new intersection points, we add a relation induced by the linking torus to the fundamental group of the complement. Thus finger move kills the commutator $[\beta, \alpha]$.

Pushing off the interior self-intersections can be also viewed as an application of finger move. In general, suppose A, B, C are surfaces in X^4, part of the boundary of C lies on B, and there is an embedded arc from an intersection point $A \cap C$ to this boundary, as shown. Then we can push A off C through B (along the arc). This gives a surface A' with one fewer intersection with C, but two new intersections with B.

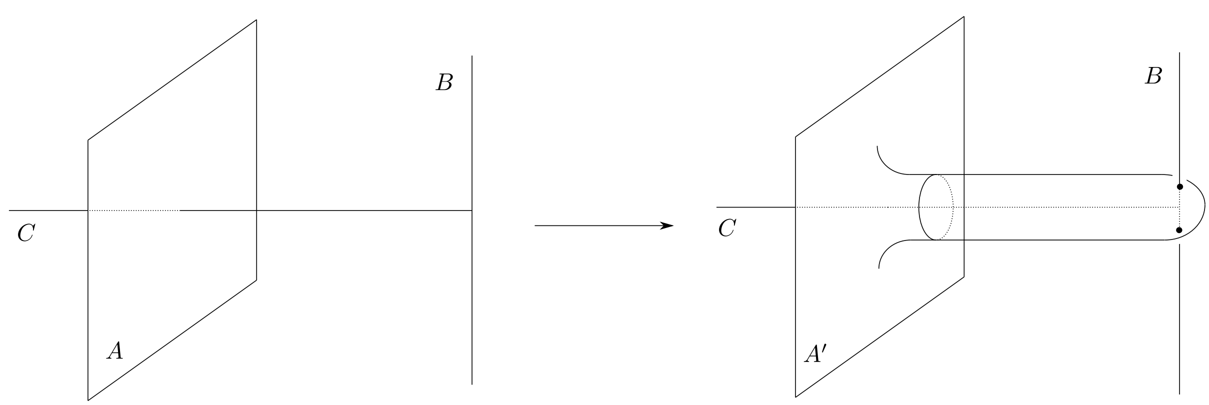


Figure 2.7: Finger move

If B is a sphere whose framing is 0 (means a parallel push off of B in X does not intersect B) and intersect C in exactly one point. Then all the intersections between A and C can be removed, by pushing A along the arc in C and adding parallel copy of B to A. In this situation, B is called a *transverse sphere* of C.

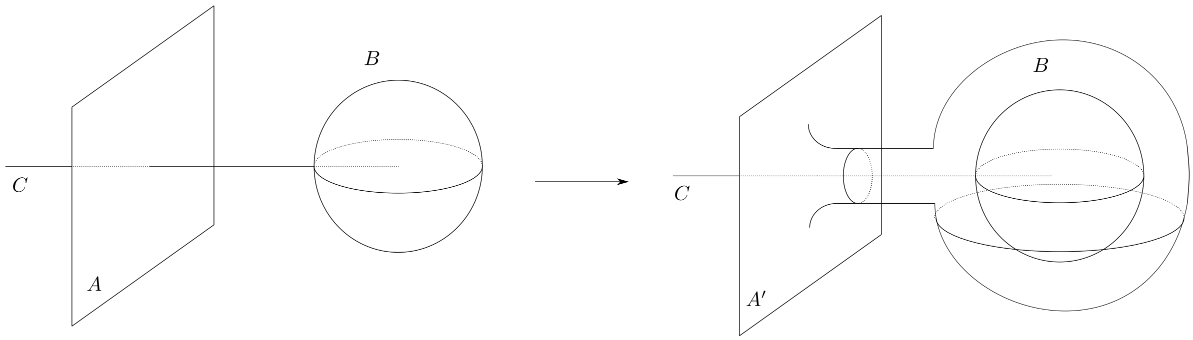


Figure 2.8: Eliminating the intersection point by a transverse sphere

2.1.3 Casson handles

Now let us assume that A, B are embedded surfaces in X, such that $A \cap B = \{p, q\}$ with opposite sign and the $X \setminus A \cup B$ is simply-connected, then we are able to find a Whitney's disk W disjoint from A and B. If W is embedded such that the framing obstruction is trivial (we may think of it as the core of a 0 framing 2-handle as in the following picture), then by Whitney's trick (we may think of it as handle slides over this 0-framing 2-handle), we can remove the two intersection points with opposite sign.

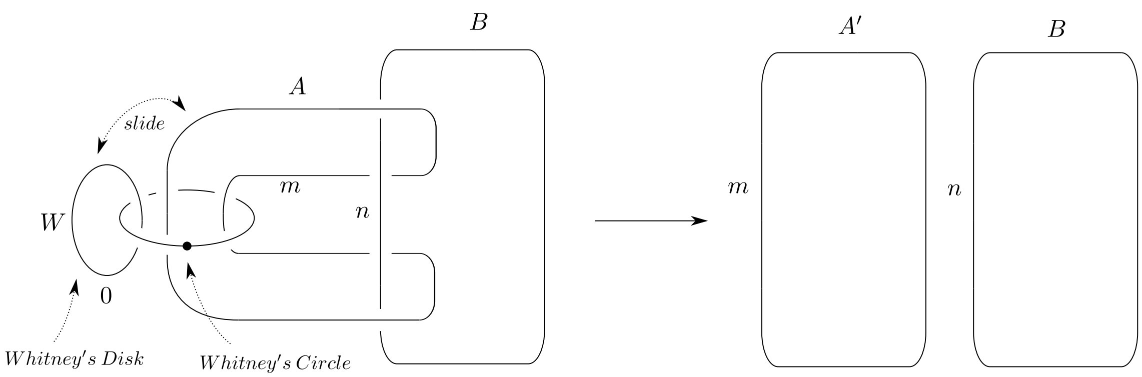


Figure 2.9: Handlebody picture of Whitmey's move

However, if the W is just an immersed disk, it would define a *kinky handle*, i.e., the core of this 2-handle has self-intersections. In this case, we can always adjust its framing by interior twisting such that after introducing more self-intersection points, the framing obstruction is trivial (since for a pair of intersection points of opposite sign, the framing obstruction is $\equiv 0 \mathrm{mod}(2)$). For each self-intersection point of W, choose a loop in W, based at the self-intersection, leaving along one branch and returning along the other. We want to choose these loops pairwise disjoint. There is no framing issue for these loops, since they are entirely contained in W. The linking torus T_p^2 at p intersects W at a single point, so by M-V sequence, we can easily check that $H_1(X \setminus A \cup B \cup W) = 0$. Thus $\pi_1(X \setminus A \cup B \cup W)$ is a perfect group and generated by the conjugates of the meridian circle of W. Now, by using finger moves on W (we push W through itself), we can kill $\pi_1(X \setminus A \cup B \cup W)$. Therefore, every loop bounds a disk in $X \setminus A \cup B \cup W$. If one of these loops actually bounds an embedded disk, then the corresponding self-intersection of W can be eliminated, by handle slide or Whitney's trick (we can create a new self-intersection with opposite sign near the boundary of this embedded disk ([40], 2.1)). In general, we can merely find an immersed disk, with its own self-intersections, which can be viewed as a kinky handle attached to a kinky handle. We iterate the process; add third stage kinky handles and then fourth stage kinky handles, and so on. We carry out the process for infinitely many steps, then take the union of all these kinky handles. The result is called *Casson handle*.

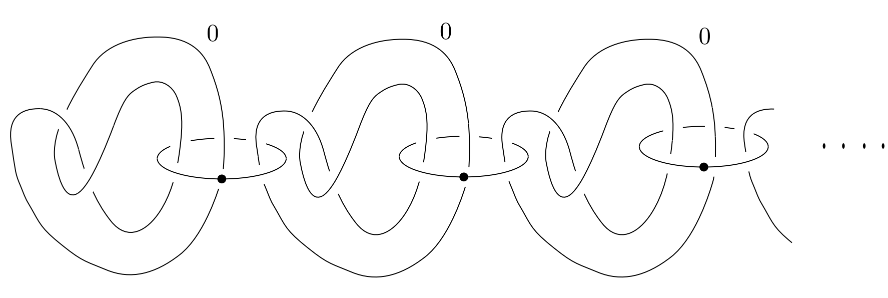


Figure 2.10: Casson handle

From the construction, we see that the Casson handles are very complicated subjects, but the miracle is they are actually topologically the same as standard handles.

Theorem 2.1. (Casson) *Let X^4 be simply-connected and let $W_1, ..., W_n$ be smoothly immersed transverse 2-disks in X^4 with boundaries, $\partial W_1, ..., \partial W_n$, embedded disjointly in ∂X^4, and $W_i \cdot W_j = 0$ for $i \neq j$. Assume that there exist $\beta_1, ..., \beta_n \in H_2(X)$ such that $\beta_i \cdot \beta_i$ is even and $W_i \cdot \beta_j = \delta_{ij}$. Then the W_i's can be regularly homotoped (rel ∂) to be disjoint, and then kinky handles may be added disjointly so as to build n disjoint, smoothly embedded Casson handles with $W_1, ..., W_n$ as their first stages. Furthermore, these Casson handles satisfy:*

Property 1: Each Casson handle is proper homotopy equivalent, rel $\partial = S^1 \times$ intB^2, to $B^2 \times$ int B^2.

Property 2: Each Casson handle is a smooth submanifold of $B^2 \times B^2$ with $S^1 \times$ int$B^2 = \partial B^2 \times$ int B^2.

The existence of $\beta_i, i = 1, .., n$ in the above theorem is to guarantee the complement of $W_i, i = 1, ..., n$ in X^4 is simply-connected after necessary finger moves.

Theorem 2.2. (Freedman, [24])*Each Casson handle is homeomorphic to $B^2 \times$ intB^2, rel ∂.*

2.2 Useful Results in Handlebody Theory

2.2.1 Pluming 2-handles

The main objective of this section is to find the handlebody picture of the neighbourhood of two immersed spheres A, B in X^4 with intersection points. A detailed description can be found in [26].

At the neighbourhood of each intersection point, the neighbourhood of A is diffeomorphic to $D^2 \times D^2$, call it A'; the neighbourhood of B is diffeomorphic to $D^2 \times D^2$, call it B'. The neighbourhood of $A \cup B$ can be thought as plumbing two normal bundles $D^2 \times D^2$ and $D^2 \times D^2$. We can plumb them as follows: Staring with the relative handle decomposition of the core $(D, \partial D)$ of A' with a single handle, introduce a cancelling 0- and 1-handle.

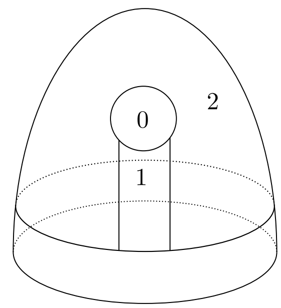


Figure 2.11: Handle decomposition of $(D, \partial D)$

We identify the 0-handle with a cocore of B' (realized as a disk in X bounded by a meridian of the attaching circle of B'). Then we attach A' by attaching the 1-handle and 2-handle.

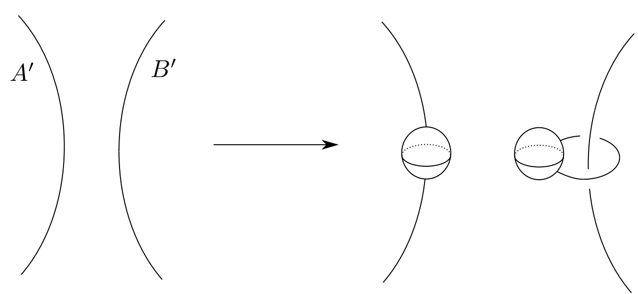


Figure 2.12: Pluming 2-handles

If we switch to "dot" notation, it is shown as in the following figure.

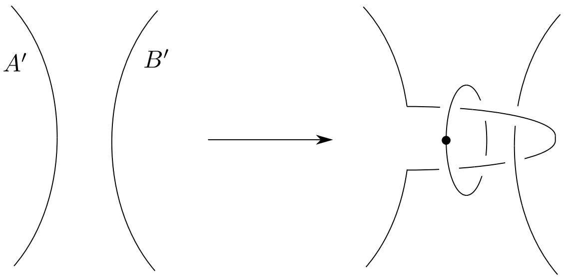


Figure 2.13: Pluming 2-handles in “dot” notation

Example 1: A framing 2 sphere intersects with a framing 3 sphere geometrically 3 times, algebraically once.

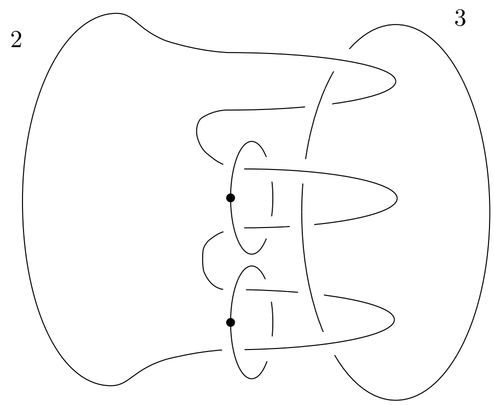


Figure 2.14: Two spheres intersecting geometrically 3 times, algebraically once

Note that at the first clasp, the 1-handle links the two 0-handles coming from each handlebody picture of A and B, so it is cancelled. After the first clasp, we will associate a 1-handle to each clasp.

Example 2: A framing 0 sphere with a self-intersection. (called *a fishtail*)

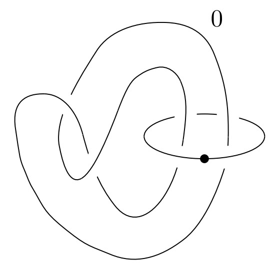


Figure 2.15: A finshtail

2.2.2 Doing Surgery

Definition 1. Let $\varphi : S^k \to M^n(-1 \leq k \leq n)$ be an embedding of a k-sphere in an n-manifold, with a normal framing f on $\varphi(S^k)$ (which we assume lies in int M). Then the pair (φ, f) determines an embedding $\hat{\varphi} : S^k \times D^{n-k} \to M$ (unique up to isotopy), and *surgery* on (φ, f) is the procedure of removing $\hat{\varphi}(S^k \times \text{int} D^{n-k})$ and replacing it by $D^{k+1} \times S^{n-k-1}$, with gluing map $\hat{\varphi} \mid S^k \times S^{n-k-1}$.

Attaching a handle to $(X, \partial_- X)$ has the effect of surgery on $\partial_+ X$, and conversely any surgery on a closed manifold M is realized as $\partial_+(I \times M \cup h)$ where h is attached by (φ, f). Surgery on M produces a manifold with a canonical embedding of $D^{k+1} \times S^{n-k-1}$. If we surger on this framed S^{n-k-1}, we recover M. This corresponds to turning the relative handlebody $I \times M \cup h$ upside down. We call this procedure *reverse the surgery.*

Now, let us consider the particular case when $n = 4$. If $k = 1$, we cut off a framed embedding $S^1 \times D^3$ from M, and glue in $S^2 \times D^2$. As explained above, this is the same as attaching a 5-dimensional 2-handle to $I \times M$. If we start building the handlebody of M from $S^1 \times D^3$, it can be viewed as a 4-dimensional 1-handle; Replacing it by $S^2 \times D^2$ by using the canonical framing is equivalent to switching the 1-handle to a 0 framing 2-handle. Clearly, switching the 0 framing 2-handle to a 1-handle (a circle with a dot) is the reverse surgery.

Proposition 2.1. *Suppose X is a simply-connected 4-manifold, then any 1-surgery ($k = 1$) on X yields either $X\sharp S^2 \times S^2$ or $X\sharp S^2\tilde{\times}S^2$.*

Proof. Suppose we do 1-surgery on a circle $C \subset X$. Write $X = X\sharp S^4$, and let $C_0 \subset X\sharp S^4$ be the circle $\partial D^2 \times 0 \subset \partial(D^2 \times D^3) = S^4$. $\pi_1(X) = 1$, so C must be homotpic to C_0 and hence isotopic to C_0 in X, as dim $X = 4$. Therefore, doing 1-surgery on C_0 would give us the same result. Since $\pi_1(SO(2)) \cong \mathbb{Z}_2$, we can view S^4 as a 1-handle linking with a 0 framing 2-handle once or a 1-handle linking with a 1 framing 2-handle once. 1-surgery switches the 1-handle to a 0 framing 2-handle, which yields $X\sharp S^2 \times S^2$ or $X\sharp S^2\tilde{\times}S^2$ respectively. □

2.2.3 Eliminating 1-handles

Let W be an oriented compact 5-manifold with boundary ∂W decomposed as a disjoint union $\partial_+W \coprod \overline{\partial_-W}$ of two compact submanifolds (either of which may be empty). Let $\partial_-W = X_-$, $\partial_-W = X_+$, we say (W, X_-) is a relative handlebody if W is obtained from $I \times X_-$ by attaching 1,2,3,4-handles (if X_- is empty, we start from a 0-handle; if X_+ is empty, we end up with a 5-handle).

Theorem 2.3. *If W is simply-connected and X_-, X_+ are connected, then we can modify the handle decomposition of W such that it contains no 1-handles and 4-handles.*

Proof. For each 1-handle, we can introduce a cancelling pair of 2 and 3-handles. We hope each new introduced 2-handle would cancel with a 1-handle. This is true if the attaching circle K of the 2-handle is isotopic to the core K_0 of the 1-handle in ∂_+W_2, where W_2 is the 2 skeleton of W, i.e., it is the union of $I \times X_-$ and all the 1-handles and 2-handles. Since dim ∂_+W_2=4, it is enough to show they are homotopic in ∂_+W_2. $\pi_1(W) = 1$ implies that W_2 is also simply-connected. Therefore, there is a homotopy between K and K_0 in W_2. As dim $W_2 = 5$, dimension of this homotopy is 2, and dimensions of the cores of the 1- and 2-handles are 1 and 2 respectively, we can make the homotopy to be disjoint from 1- and 2-handles by small perturbation in W_2. Therefore, this homotopy can be pushed into $\partial_+ W_2$. Thus, all the 1-handles are cancelled with the new introduced 2-handles. The new introduced 3-handles are left, so we can think this process as "trading" 1-handles for 3-handles. By turning W upside-down, we can trade each 4-handle for a 2-handle. In the end, we get a handle decomposition of W involving no 1-handles and 4-handles. □

From the proof, we can easily see that the same result is true for dimension $\geqslant 5$.

The proof clearly fails in dimension 4. Indeed, whether a simply-connected closed 4-manifold always admits a handle decomposition involving no 1-handle is an open question. However, we can use the same idea as in the proof of last theorem to get a weaker result in dimension 4.

Theorem 2.4. *Suppose that $(X, \partial_- X)$ is a 4-dimensional relative handlebody with $\partial_- X$ connected and $\pi_1(X) = 1$. For every 1-handle h of this decomposition, one can introduce a cancelling 2-3 handle pair such that the 2-handle cancels h algebraically (but not necessarily geometrically). In fact, one can arrange the attaching circles of the new 2-handles to represent the canonical basis for the free factor of $\pi_1(X_1)$ determined by the 1-handles (suitably attached to the base point).*

Proof. If we go through the proof of last theorem for dimension 4, we will realize the same technique fails in two places: 1. In dimension 4, homotopy does not imply isotopy; 2. We can not assume the homotopy misses the cores of the 2-handles.

We deal with the second problem first. Use the same notation as in the proof of last theorem: K_0 denote the core of a 1-handle h; K denote the attaching circle of the cancelling 2-handle; H denote the homotopy between K and K_0. Suppose H intersects the core of a 2-handle at a point P, then we can form the band-sum of the boundary circle of a small disk around P in H with the circle K_0 using a band contained in H; the resulting circle will be denoted by K_1. Now, K_1 is homotopic to K_0 and this homotopy can be assumed to be disjoint form the 2-handles and hence can be pushed into $\partial_+ W_2$. Denote the image of K_1 on $\partial_+ W_2$ by K_2. Note that the band might run over 1-handles and 2-handles, so, K_2 might run over 1-handles and 2-handles as well, but it will run over h algebraically once, other 1-handles algebraically 0 times.

Now we deal with the first problem. Dim $\partial_+ W_2 = 3$, the homotopy in $\partial_+ W_2$ fails to be an isotopy at finitely many times when the knot crosses through itself. Each crossing change can be realized by band-summing K_2 with a meridian K_2 along a suitable band in $\partial_+ W_2$. Note again, this band might run over 1-handles and 2-handles, but it does not change the algebraic linking number between K_2 and the core of each 1-handle. Let K_3 be the knot obtained from K_2 by such crossing changes. K_3 run over h algebraically once, other 1-handles algebraically 0 times and it is isotopic to K in $\partial_+ W_2$. So, the 2-handle K algebraically cancels h. $\square$

2.3 Spin Structures

Let X denote a n-manifold $n \geq 3$, T_X denote its tangent bundle. The second Stiefel-Whitney class $\omega_2(T_X)$ measures the obstruction to trivializing T_X over the 2-skeleton of X.

When $n = 4$, Wu's formula says: *For all oriented surfaces S embedded in X, $\omega_2(T_X) \cdot S = S \cdot S$ (mod 2).*

A nice consequence of Wu's formula is: *If* $\omega_2(T_X) = 0$*, then the intersection form of* X *is even.*

By universal coefficient theorem, the converse is true whenever $H_1(X;\mathbb{Z})$ has no 2-torsion.

A *spin structure* on X is a choice of trivialization of T_X over 1-skeleton that can be extended over the 2-skeleton, considered up to homotopies. A manifold endowed with a spin structure is called a *spin manifold*.

Note: When $n \leq 2$, we define a spin structure on X as a trivialization of T_X over the 1-skeleton such that $T_X \oplus \xi^{3-n}$ can be extended over 2-skeleton, where ξ^{3-n} is a trivialized bundle.

By Wu's formula, *Any 4-manifold without 2-torsion, for example simply-connected, admits spin structures if and only if its intersection form is even.*

Let s be a spin structure on X^n, given by a trivialization of T_X over the 1-skeleton of X (for some fixed triangulation of X). For any $\alpha \in H^1(X;\mathbb{Z}_2)$, α is isomorphic to a map $f : X \to \mathbb{RP}^\infty$. We homotope f into a finite skeleton, say $\mathbb{RP}^N$, then the preimage of $\mathbb{RP}^{N-1}$ is a $n-1$ submanifold Y of X, representing α. Now we change the trivialization of s over every 1-cell by a 2π twist each time the 1-cell intersects Y. If a 1-cell bounds a 2-cell, then this 1-cell must intersect Y even number of times, and thus we change the trivialization over this 1-cell by even number of 2π twists. Since $\pi_1(SO(4)) \cong \mathbb{Z}_2$, the new trivialization can also be extended over the 2-skeleton. Therefore, we get a new spin structure on X; denote it by $\alpha \cdot s$. Since s and α are arbitrarily chosen, this can be viewed as an action of $H^1(X;\mathbb{Z}_2)$ on the set of all pin structures of X. This action is free and transitive. Therefore, after fixing a spin structure on X, the action establishes a bijective correspondence between the elements of $H^1(X;\mathbb{Z}_2)$ and the set of all spin structures on X.

For example, $H^1(S^1;\mathbb{Z}_2) = \mathbb{Z}_2$, so there are two spin structures on S^1. One can be extended into the interior; one can not. The latter one is also called *the Lie group spin structure* on S^1.

The 3-torus $T^3 = S^1 \times S^1 \times S^1$ has 8 different spin structures since $H^1(T^3; \mathbb{Z}_2) = \mathbb{Z}_2^3$, which can be also viewed as the products of spin structures on S^1.

For a closed, spin 3-manifold (M, s), the *Rohlin invariant* $\mu(M, s) \in \mathbb{Z}_{16}$ is the signature $\sigma(X)$ reduced modulo 16, where X is any smooth, compact, spin 4-manifold with spin boundary (M, s)

This is a well-defined invariant because if X and Y are two smooth compact spin 4-manifolds which induce the same spin boundary (M, s), then $X \cup_M \overline{Y}$ is a smooth closed spin 4-manifold. The rest of the argument is due to Rohlin's Theorem:

Theorem 2.5. (Rohlin, [37]) *If X is a smooth, closed, spin 4-manifold, then $\sigma(X) \equiv 0$ (mod 16).*

Chapter 3

Corks and Plugs

3.1 Corks

In this chapter and the next chapter, when we talk about a manifold, we always assume it is compact and oriented. "$\approx$" stands for diffeomorphic; "$\simeq$" stands for homotopic equivalent. The h-Cobordism Theorem is one of the most important theorem in modern topology.

Theorem 3.1. (Smale, [41]) *If W is an h-cobordism between the simply-connected n-dimensional smooth manifolds X_1 and X_2, and $n \geqslant 5$, then W is diffeomorphic to the product $I \times X_1$. In particular, X_1 is diffeomorphic to X_2.*

However, this theorem fails in dimension 4, the first counterexample was brought to light by S.K. Donaldson [19], and many others followed. The following theorem which was proved independently by Matveyev [32] and Curtis-Freedman-Hsiang-Stong [18] tells us that the failure of h-cobordism theorem can be localized on a contractible piece.

Theorem 3.2. ([32], [18]) *X_1 and X_2 are simply-connected smooth 4-manifolds (not necessarily closed). If W is an (relative) h-cobodism between X_1 and X_2, then there is a subcobordism $V \subset W$ between submanifolds $C_i \subset X_i$ $(i = 1, 2)$, such that $W - intV$ is the product cobordism, and V, C_i are contractible.*

This theorem says if X_1 is h-cobordant to X_2, then X_1, X_2 can be decomposed as $X_1 = X_0 \cup_\partial C_1$, $X_2 = X_0 \cup_\partial C_2$, where C_1, C_2 are contractible manifolds. In many cases,

$C_1 \approx C_2$. Actually, in [32], Matveyev proved that $C_1 \cup_\partial C_1 \approx S^4$; $C_1 \cup_\partial C_2 \approx S^4$ and so that $X_1 \approx (X_0 \natural C_1) \cup_\partial (C_1 \natural C_2)$ and $X_2 \approx (X_0 \natural C_1) \cup_\partial (C_2 \natural C_1)$. Thus if let $C_1 \natural C_2$ and $C_2 \natural C_1$ play the role of C_1 and C_2, one can always assume that $C_1 \approx C_2$. By Freedman and Quinn's theorem [24], [36], X_1 and X_2 are h-cobordant implies they are homeomorphic. Therefore, we can conclude any exotic smooth structure of a simply-connected 4-manifold arises from cutting off a contractible submanifold and gluing it back by a non-trivial map τ. From Matveyev's proof, we can easily see that this non-trivial map τ is an involution, i.e., $\tau \circ \tau = id$. Furthermore, in [1], Akbulut and Matveyev proved that one can always make C_i stein.

Let C be a contractible stein 4-manifold with boundary and $\tau : C \to C$ an involution on the boundary. We call (C, τ) a *Cork* if τ extends to a self-homeomorphism of C, but can not extend to any self-diffeomorphism of C. The procedure of cutting off C and gluing it back by τ is called *Cork Twist*.

The first example of cork twist changing the smooth structure is due to S. Akbulut [3]:

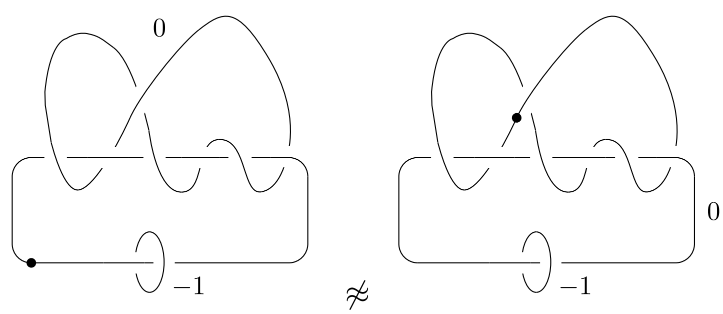


Figure 3.1: An example of being homeomorphic but not diffeomorphic obtained by doing cork twist

Many other interesting exotic manifolds were then constructed using cork twists by Akbulut and Yasui [9], [10], [11], [12], [13].

Note that most of these examples were constructed by a "simple" cork twist, which means we only exchange "dot" and "0" once. Therefore, connecting sum with $S^2 \times S^2$ stabilizes these manifolds, i.e., $X_1 \sharp S^2 \times S^2 \approx X_2 \sharp S^2 \times S^2$. Easy to see that if X_1 and X_2 differ by n disjoint simple cork twist, it is still true that $X_1 \sharp S^2 \times S^2 \approx X_2 \sharp S^2 \times S^2$. So to construct exotic manifolds which can not be stabilized by a single $S^2 \times S^2$, one has to use "linked corks". The following is a potential example:

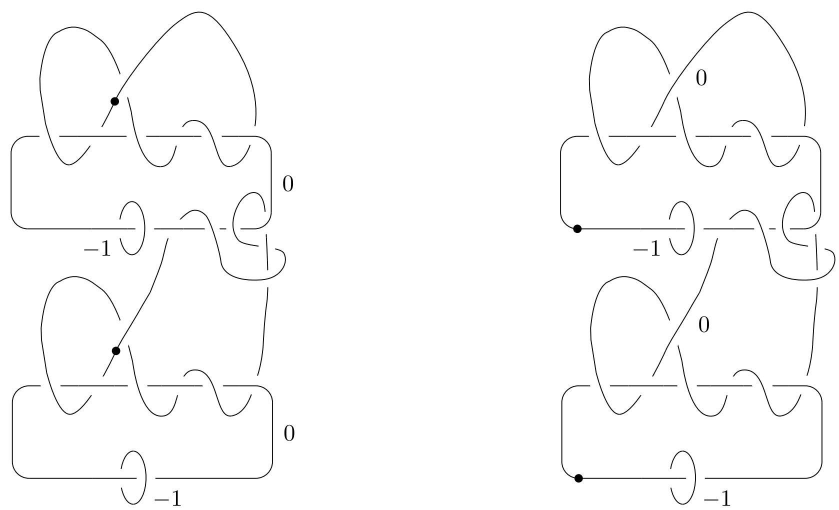


Figure 3.2: A potential example of being homeomorphic but can not be stablized by connecting sum with $S^2 \times S^2$ by doing cork twist

3.2 Boyer's theorem

In this section, we will interpret Boyer's work, which will be needed in the next section.

Let M denote a closed, oriented, connected 3-manifold. A bilinear form space $(\mathbb{Z}^n, Q)$ is said to *present* $H_*(M)$ if there is an exact sequence

$$0 \to H_2(M) \xrightarrow{h} \mathbb{Z}^n \xrightarrow{\text{ad}(Q)} [\mathbb{Z}^n]^* \xrightarrow{\partial} H_1(M) \to 0$$

such that

(i) if $\text{ad}(Q)(\xi_i) = m_i \eta_i$ $(i = 1, 2)$ where $m_1 m_2 \neq 0$, then

$$l_M(\partial\eta_1, \partial\eta_2) \equiv -\frac{1}{m_1 m_2} Q(\xi_1, \xi_2);$$

(ii) if $\beta \in H_2(M)$ and $\eta \in [\mathbb{Z}^n]^*$, then $\partial(\eta) \cdot \beta = \eta(h(\beta))$.

It is well known that, if M bounds a topological 4-manifold X, $(H_2(X), Q_X)$ presents $H_*(M)$, where Q_X is the intersection form of X. If Q presents M, we denote them by (Q, M).

Given two simply-connected 4-manifolds (X_1, M_1) and (X_2, M_2), we say (Q_{X_1}, M_1) is *isomorphic (through* (Λ, f)*)* to (Q_{X_2}, M_2) if there is a homeomorphism $f : M_1 \to M_2$ and an isomorphism $\Lambda : H_2(X_1) \to H_2(X_2)$ preserving the intersection form such that the following diagram commutes:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_2(M_1) & \longrightarrow & H_2(X_1) & \longrightarrow & H_2(X_1, M_1) & \longrightarrow & H_1(M_1) & \longrightarrow & 0 \\ & & f_* \downarrow & & \Lambda \downarrow & & \Lambda^* \uparrow & & f_* \downarrow & & (*) \\ 0 & \longrightarrow & H_2(M_2) & \longrightarrow & H_2(X_2) & \longrightarrow & H_2(X_2, M_2) & \longrightarrow & H_1(M_2) & \longrightarrow & 0 \end{array}$$

Note: Λ^* is the adjoint of Λ with respect to the identification of $H_2(X_i, M_i)$ with $\mathrm{Hom}(H_2(X_i); \mathbb{Z})$; If $F : X_1 \to X_2$ is a homeomorphism which induces f on the boundary, $F_* : H_2(X_1, M_1) \to H_2(X_2, M_2)$ is not the same as the inverse of adjoint of $F_* : H_2(X_1) \to H_2(X_2)$ in general, but (*) is commutative due to Lefschetz duality. Therefore, (*) is a necessary condition for X_1, X_2 to be homeomorphic.

The following theorem is due to Boyer:

Theorem 3.3. (Boyer, [15])(X_1, M_1), (X_2, M_2) *are simply-connected topological 4-manifolds with boundaries. If* $f : M_1 \to M_2$ *is an orientation preserving homeomorphism and* $\Lambda : H_2(X_1) \to H_2(X_2)$ *is an isomorphism which preserves the intersection form, such that* (Q_{X_1}, M_1) *is isomorphic to* (Q_{X_2}, M_2) *through* (Λ, f)*, then* f *can be extended to a homeomorphism* $F : X_1 \to X_2$ *if:*

(i) when $H_1(M_i, \mathbb{Q}) \cong 0$ $(i = 1, 2)$ *and* $\Delta(X_1) \equiv \Delta(X_2)$*(mod 2).*

(ii) when X_i $(i = 1, 2)$ are even manifolds, and $X = X_1 \cup_f \overline{X_2}$ is also an even manifold.

(iii) when X_i $(i = 1, 2)$ are odd manifolds and $\Delta(X_1) \equiv \Delta(X_2)$(mod 2).

In case (i) and (ii), F_*, the isomorphism on $H_2(X_1)$ induced by F agrees with Λ; in case (iii) F_* may not agree with Λ.

From this theorem, one can easily derive the classification theorem we stated in chapter 1.

In this thesis, we are only interested in smooth 4-manifolds, so let us assume X_1 and X_2 are smooth and drop the assumption $\Delta(X_1) \equiv \Delta(X_2)$(mod 2).

To make his theorem more transparent, let us introduce some notations: As stated in the theorem, (X_1, M_1), (X_2, M_2) are simply-connected smooth 4-manifolds with the closed connected boundaries. $f : M_1 \to M_2$ is an orientation preserving homeomorphism. $\Lambda : H_2(X_1) \to H_2(X_2)$ is an isomorphism which preserves the intersection form, such that (Q_{X_1}, M_1) is isomorphic to (Q_{X_2}, M_2) through (Λ, f). Denote $X_1 \cup_f \overline{X_2}$ by X. If $\Psi : A \to B$ is a homomorphism of abelian groups. Let $G(\Psi)$ denote the subgroup of $A \oplus B$ corresponding to the graph of Ψ : $G(\Psi) = \{(a, \Psi(a)) | a \in A\}$. k_*, h_* and ∂_1 are defined by the following diagram:

$$\begin{array}{ccccccc}
 & & H_2(M_1) & & H_2(M_1, M_1) = 0 & & \\
 & & \downarrow & & \downarrow & & \\
H_2(M_1) \bigoplus H_2(M_2) & \xrightarrow{h_{1*} \oplus h_{2*}} & H_2(X_1) \bigoplus H_2(X_2) & \xrightarrow{j_{1*} \oplus j_{2*}} & H_2(X_1, M_1) \bigoplus H_2(X_2, M_2) & & \\
\downarrow{\scriptstyle 1+f_*^{-1}} & & \downarrow{\scriptstyle i_* = i_{1*} + i_{2*}} & & \| & & \\
H_2(M_1) & \xrightarrow{h_*} & H_2(X) & \xrightarrow{k_*} & H_2(X, M_1) & \xrightarrow{\partial_1 + f_*^{-1} \circ \partial_2} & H_1(M_1) \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow & & \\
 & & H_1(M_1) & & H_1(M_1, M_1) = 0 & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

Boyer's theorem is essentially a consequence of the following theorem:

Theorem 3.4. (Boyer, [15]) *Assume rank of $H_2(M_1) = k$. There exists $\{\mu_i'\} \in H_2(X)$*

$(i = 1, 2, ..., k)$, such that $\partial(\mu_i') = \nu_i$, where $\{\nu_i\}$ is a basis of the free part of $H_1(M_1)$, and $k_*(\mu_i') \in G(-\Lambda^*)$. f extends to a homeomorphism $F : X_1 \to X_2$ if $\mu_i' \cdot \mu_i'$ is even for all $i = 1, 2, ..., k$.

Because if $H_1(M_i, \mathbb{Q}) \cong 0$ $(i = 1, 2)$, the free part of $H_1(M_1)$ is trivial, $k = 0$, so $\mu_i' \cdot \mu_i' = 0$ mod(2) is automatically true; if X is even, $\mu_i' \cdot \mu_i'$ must be even for any i; if X is odd, $\mu_i' \cdot \mu_i'$ might be odd for some i. In this case, Boyer showed that one can always find another Λ' and $\{\lambda_i'\} \in H_2(X)$ such that $\partial(\lambda_i') = \nu_i$, $k_*(\lambda_i') \in G(-\Lambda'^*)$ and $\lambda_i' \cdot \lambda_i'$ is even for every i. We will investigate this case carefully and figure out how to construct Λ' from Λ in the next section. Actually, finding an appropriate Λ' is the key to prove the existence of the plug.

For completeness, we outline Boyer's proof of the above theorem.

Proof. The proof consists of two steps.

1. We construct a maximal isotropic subgroup $J \subseteq H_2(X)$ ("isotropic" means for $\forall x, y \in J$, $x \cdot y = 0$) such that (i) $i_*\big(G(-\Lambda)\big) \subseteq J$; (ii) $\partial(J) = H_1(M_1)$.

2. We apply Wall's method [45]: replacing W by another 5-manifold W_1 with $\partial W_1 = X$, $W_1 \backsimeq \bigvee_{i=1}^{n} S^2$ and $J = ker(H_2(X) \to H_2(W_1))$. Then by diagram chasing, we will show that $\psi_i : H_2(X_i) \to H_2(W_1)$ $(i = 1, 2)$ are isomorphisms.

Step 1: J will be built as the sum of two isotropic subgroups J_1 and J_2 of $H_2(X)$ which satisfy (i) $i_*(G(-\Lambda)) \subset J_1$, $\partial(J_1) = T_1(M_1)$ (torsion of $H_1(M_1)$) and rank(J_1) = rank$(H_2(X_1))$ − rank$(H_1(M_1))$; (ii) the composition $J_2 \xrightarrow{\partial|J_2} H_1(M_1) \to H_1(M_1)/T_1(M_1)$ is an isomorphism; (iii) $J_1 \cap J_2 = \{0\}$ and $J_1 \cdot J_2 = \{0\}$.

Assuming we have found such subgroups, we let J be the smallest direct summand of $H_2(X)$ containing $J_1 + J_2$. As rank$(H_2(X))$=2rank$(H_2(X_1))$, J is evidently the desired subgroup of $H_2(X)$.

Construction of J_1:

Let J_1 be the smallest direct summand of $H_2(X)$ containing $i_*(G(-\Lambda))$ which is isotropic in $H_2(X)$ as Λ is an isometry. Thus J_1 is also.

Next we prove $\partial(J_1) = T_1(M_1)$. If $\mu \in J_1$, there is an integer $m > 0$ such that $m\mu \in i_*(G(-\Lambda))$. But then $m\partial(\mu) = 0$, as $\partial \circ i_* = 0$. Hence $\partial(\mu) \in T_1(M_1)$ which shows $\partial(J_1) \subseteq T_1(M_1)$.To derive the opposite inclusion, let $\nu \in T_1(M_1)$.

Claim 3.1. *We can find a class $\mu \in H_2(X)$ with $\partial(\mu) = \nu$ and $k_*(\mu) \in G(-\Lambda^*)$.*

Proof. Fix any $\eta \in H_2(X_2, M_2)$ such that $\partial_2(\eta) = f_*(\nu)$. Now as $(\partial_1 + f_*^{-1} \circ \partial_2)(G(-\Lambda^*)) = 0$, $G(-\Lambda^*) \subset \text{image}(k_*)$, so we may choose a $\mu \in H_2(X)$ for which $k_*(\mu) = (-\Lambda^*(\eta), \eta)$. Further, $\partial(\mu) = f_*^{-1} \circ \partial_2 \circ pr_2 \circ k_*(\mu) = f_*^{-1} \circ \partial_2(\eta) = \nu$ by the choice of ν. □

Let $k_*(\mu) = (-\Lambda^*(\eta), \eta)$. Since $\nu \in T_1(M_1)$, we may find an $m > 0$ and $\xi \in H_2(X_1)$ such that $j_{1*}(\xi) = -m\Lambda^*(\eta)$. Then $k_* \circ i_*(\xi, -\Lambda(\xi)) = m(-\Lambda^*(\eta), \eta)$ and thus $i_*(\xi, -\Lambda^*(\xi)) = m\mu + h_*(\beta)$ for some $\beta \in H_2(M_1)$.

Claim 3.2. *The class β is divisible by m in $H_2(M_1)$*

Proof. It suffices to show that $\beta \cdot \nu \equiv 0 (\text{mod} m)$ for each $\nu \in H_1(M_1)$. But from the properties of J_2, for any such ν there is some $\mu' \in J_2$ with $\partial(\mu') - \nu \in T_1(M_1)$. Then $\beta \cdot \nu = h_*(\beta) \cdot \mu' = i_*(\xi, -\Lambda^*(\xi)) \cdot \mu' - m\mu \cdot \mu' = -m\mu \cdot \mu'$. □

Now since $\mu + h_*(\beta/m) \in J_1$, $\nu = \partial(\mu) = \partial(\mu + h_*(\beta/m)) \in \partial(J_1)$ and as ν was chosen arbitrarily, we conclude $\partial(J_1) = T_1(M_1)$.

Finally, to calculate $\text{rank}(J_1)$, note $ker(i_*) = \{(h_{1*}(\beta), -h_{2*}f_*(\beta) | \beta \in H_2(M_1))\} \subseteq G(-\Lambda)$. Thus,

$\text{rank}(J_1) = \text{rank}(i_*(G(-\Lambda))) = \text{rank}(G(-\Lambda)) - \text{rank}(ker(i_*)) = \text{rank} H_2(X_1) - \text{rank}(H_1(M_1))$.

Construction of J_2:

Set $F_1(M_1) = H_1(M_1)/T_1(M_1)$ and choose $\nu_1, \nu_2, ..., \nu_k \in H_1(M_1)$ which projects to a basis of this group. By Claim 1, there are classes $\mu'_1, \mu'_2, ..., \mu'_k \in H_2(X)$ such that (i) $\partial(\mu'_i) = \nu_i, 1 \leq i \leq k$, (ii) $k_*(\mu'_i) \in G(-\Lambda^*), 1 \leq i \leq k$.

Let $\beta_1, \beta_2, ..., \beta_k \in H_2(M_1)$ be the basis dual to $\nu_1, ..., \nu_k$. That is $\beta_i \cdot \nu_j = \delta_{ij}$. Set $\bar{\beta}_i = h_*(\beta_i)(1 \leq i \leq k)$ and note that (i) $\bar{\beta}_i \cdot \bar{\beta}_j = 0, 1 \leq i, j \leq k$; (ii) $\bar{\beta}_i \cdot \mu'_j = \delta_{ij}, 1 \leq i, j \leq k$. Define $\mu''_i = \mu'_i - \sum_{j=i+1}^{k}(\mu'_i \cdot \mu'_j)\bar{\beta}_j, 1 \leq i \leq k$, and observe that $k_*(\mu''_i) = k_*(\mu'_i) \in G(-\Lambda^*)$. Thus for each i, $\mu''_i \cdot \mu''_i = \mu'_i \cdot \mu'_i \equiv 0(\text{mod}2)$ as $\mu'_i \cdot \mu'_i$ is even. Thus we may form $\mu_i = \mu''_i - \frac{1}{2}(\mu''_i \cdot \mu''_i)\bar{\beta}_i \in H_2(X), 1 \leq i \leq k$.

Now it can be checked that $\mu_i \cdot \mu_j = 0(1 \leq i, j \leq m)$, $\partial(\mu_i) = \nu_i(1 \leq i \leq k)$, and $k_*(\mu_i) \in G(-\Lambda^*)$. Thus if we set $J_2 = Span(\mu_1, \mu_2, ..., \mu_k) \subseteq H_2(X)$, (i) J_2 is isotropic; (ii) the composition $J_2 \xrightarrow{\partial|J_2} H_1(M_1) \to F_1(M_1)$ is an isomorphism; (iii) $k_*(J_2) \subseteq G(-\Lambda^*)$. Thus J_2 satisfies the desired properties.

Observe that under the composition $H_2(X) \xrightarrow{\partial} H_1(M_1) \to F_1(M_1)$, J_1 maps to zero while J_2 maps monomorphically. Thus, $J_1 \cap J_2 = 0$. To see that $J_1 \cdot J_2 = 0$, choose $\mu_i \in J_i (i = 1, 2)$. Now by the construction of J_1 and J_2, we may choose an integer $m > 0$, and elements $\xi \in H_2(X_1)$ and $\eta \in H_2(X_2, M_2)$ such that $m\mu_1 = i_*(\xi, -\Lambda(\xi))$, $k_*(\mu_2) = (-\Lambda^*(\eta), \eta)$. Then $\mu_1 \cdot \mu_2 = \frac{1}{m} i_*(\xi, -\Lambda(\xi)) \cdot \mu_2 = \frac{1}{m}[\xi \cdot (-\Lambda^*(\eta)), \Lambda(\xi) \cdot \eta] = 0$. As μ_1, μ_2 were arbitrary, $J_1 \cdot J_2 = 0$.

Step 2: We replace W by another 5-manifold W_1 with $\partial W_1 = X$, $W_1 \backsimeq \bigvee_{i=1}^{n} S^2$ and $J = ker(H_2(X) \to H_2(W_1))$. Now consider the commutative diagram:

$$\begin{array}{ccccccc} & & J & & & & \\ & & \downarrow & & & & \\ H_2(X_1) \bigoplus H_2(X_2) & \xrightarrow{i_*} & H_2(X) & \xrightarrow{\partial} & H_1(M_1) & \longrightarrow & 0 \\ & & {\scriptstyle j_*}\downarrow & & & & \\ & & H_2(W_1) & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

By assumption, $\partial|J$ is surjective and quick diagram chase shows that $\psi = j_* \circ i_*$ is also. Thus if $\xi \in H_2(W_1)$, there are elements $\xi_i \in H_2(X_i)$ $(i = 1, 2)$ such that $\xi = \psi(\xi_1, \xi_2)$. But as $i_*(G(-\Lambda)) \subseteq J = Ker(j_*)$, $\psi(-\xi_1, \Lambda(\xi_1)) = \psi(\Lambda^{-1}(\xi_2), -\xi_2) = 0$. Thus,

$$\xi = \psi(\xi_1, \xi_2) = \begin{cases} \psi(\xi_1, \xi_2) + \psi(-\xi_1, \Lambda(\xi_1)) \\ \\ \psi(\xi_1, \xi_2) + \psi(\Lambda^{-1}(\xi_2), -\xi_2) \end{cases} = \begin{cases} \psi(0, \xi_2 + \Lambda(\xi_1)) \\ \\ \psi(\xi_1 + \Lambda^{-1}(\xi_2), 0) \end{cases}.$$

Clearly this implies that $\xi \in \text{image}(H_2(X_i) \to H_2(W_1))$ $(i = 1, 2)$, and so both homomorphism $H_2(X_i) \to H_2(W_1)$ are surjective. Since rank of $H_2(X_i)$ = rank of $H_2(W_1)$, $\psi_i = \psi|H_2(X_i)$ $(i = 1, 2)$ are isomorphisms. Therefore, W_1 is a relative h-cobordism. As $G(-\Lambda) \subseteq ker(\psi)$, $\varphi_*|_{H_2(X_1)} = \psi_2^{-1} \circ \psi_1$ is precisely Λ. By Quinn's relative h-cobordism theorem [36], there exists a homeomorphism $F : X_1 \to X_2$ such that $F|_{M_1} = f$ and $F_* = \Lambda$. $\square$

3.3 Plugs

If X_1 and X_2 are closed simply-connected smooth 4-manifolds, and $Q_{X_1} \cong Q_{X_2}$, then Wall's theorem implies X_1 and X_2 are h-cobordant, so the cork theorem applies. Now let us consider simply-connected smooth 4 manifolds with boundaries. From Boyer's proof, we know that if X_i are odd and (Q_{X_1}, M_1) is isomorphic to (Q_{X_2}, M_2), then X_1 is relative h-cobordant to X_2, so they differ by cork twists. What happens when X_i are even? The example in the last chapter will show that X_1 may not be relative h-cobordant to X_2. In this section, we will prove the failure of X_1 being relative h-cobordant to X_2 can be localized on a submanifld Y_1 of X_1 which is homotopic equivalent to S^2. Such a submanifold is called a *Plug*. Plugs naturally appear in many exotic manifolds [9], [7], [42]; it was first introduced and studied in [9] by Akbulut and Yasui.

Plug was originally defined in [9] as a Stein 4-manifold Y with boundary, homotopic equivalent to S^2 and $\tau : \partial Y \to \partial Y$ an involution on the boundary such that τ can not extend to any self-homeomorphism of Y. The procedure of cutting off Y and gluing it back by τ is called *Plug Twist*. Plug is a similar object as cork. The main difference between them is plug twist may change the homeomorphism type, while cork twist never changes the homeomorphism type. We want to prove a theorem for plugs analogous to the cork theorem. To make our theorem work, we need to work with a weaker version of plug. See the paragraph below the statement of the theorem.

Theorem 3.5. *(X_1, M_1), (X_2, M_2) are simply-connected smooth 4-manifolds with diffeomorphic boundaries such that (Q_{X_1}, M_1) is isomorphic to (Q_{X_2}, M_2). Then, there exists submanifolds $Y_i \subset X_i$, $(i = 1, 2)$ such that:*

(1) Y_i are homotopic equivalent to S^2, $\partial Y_1 \approx \partial Y_2 \approx$ a homology $S^1 \times S^2$.

(2) $X_1 \setminus Y_1$ is homoemorphic to $X_2 \setminus Y_2$ and $i_* H_2(Y_i) \subset i_{\partial *} H_2(M_i)$, $(i = 1, 2)$, where i_* and $i_{\partial *}$ are the homomorphisms induced by the inclusion map $i : Y_i \to X_i$ and $i_\partial : M_i \to X_i$.

(3) $Y_i \bigcup_{id} \overline{Y_i} = S^2 \times S^2$, $(i = 1, 2)$; $Y_1 \bigcup_\tau \overline{Y_2} = S^2 \widetilde{\times} S^2$, where τ is an obvious diffeomorphism as we shall see in the proof.

(4) Y_i can be made Stein.

In our theorem, we allow Y_1 and Y_2 being different manifolds, because as we have seen in the cork theorem, C_1 is not necessarily diffeomorphic to C_2. We can make them diffeomorphic by using $C_1 \cup_{id} C_1 \approx S^4$; $C_1 \cup_\tau C_2 \approx S^4$. However, this trick does not work for plugs, since plugs are not contractible.

Proof. Glue X_1, X_2 along their boundaries by f. $X_1 \bigcup_f \overline{X_2} = X$. By Novikov Additivity, $\sigma(X) = 0$; therefore, X bounds a 5 dimensional manifold W. Do 1-surgery to kill all nontrivial elements of $\pi_1(W)$. Thus, we may assume W is simply connected.

Then, we can use the "handle trading" trick to cancel 1-handles and 4-handles. In the end, we get a cobordism between X_1 and X_2, which has only 2- and 3-handles and induces the trivial cobordism between M_1 and M_2 (It can be considered as the collar of M_2 in X_2). We still call this cobordism W. Consider the middle level which is between the 2-handles and the 3-handles, call it X_0, $X_0 \overset{\varphi_1}{\approx} X_1 \sharp nS^2 \times S^2 \sharp mS^2 \widetilde{\times} S^2 \overset{\varphi_2}{\approx} X_2 \sharp nS^2 \times S^2 \sharp mS^2 \widetilde{\times} S^2$. We cannot proceed the proof like Cork Theorem, because we cannot eliminate the existence of $S^2 \widetilde{\times} S^2$. However, we can assume $m = 1$, since $S^2 \times S^2 \sharp 2S^2 \widetilde{\times} S^2 \approx 2S^2 \times S^2 \sharp S^2 \widetilde{\times} S^2$.

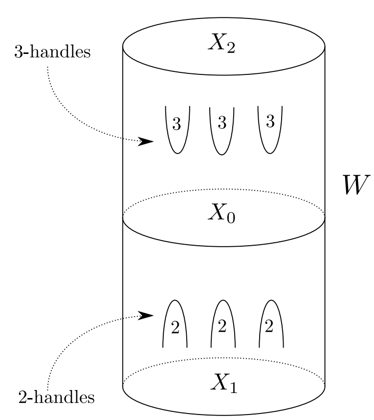


Figure 3.3: Handle decomposition of the cobordism

In $X_1 \sharp nS^2 \times S^2 \sharp S^2 \widetilde{\times} S^2$, we let α_i ($i = 1, ..., n$) denote the belt spheres of 0 framed 5-dimensional 2-handles; β_i denote the geometric dual spheres of α_i, i.e., the attaching spheres generated by the cores of the 0 framed 2-handles and the disks bounded by their attaching circles. $[\alpha_i]\cdot[\alpha_i]=0$, $[\beta_i]\cdot[\beta_i]=0$ in $H_2(X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2)$; α_i, β_j are disjoint when $i\neq j$; α_i intersects with β_i geometrically once ($i=1,2,...n$). We let α_0 denote the belt sphere of the 1 framed 5-dimensional 2-handle; β_0 denote the geometric dual sphere of α_0, i.e., the sphere generated by the core of the 1 framing 2-handle and the disk bounded by its attaching circle. $[\alpha_0]\cdot[\alpha_0]=0$, $[\beta_0]\cdot[\beta_0]=1$ in $H_2(X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2)$; α_0 intersects with β_0 geometrically once. By turning the handlebody upside down, we denote α'_i, β'_i, α'_0, β'_0 in a similar manner. Denote by $\varphi: X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2\to X_2\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2$ the composition of $\varphi_1\circ\varphi_2^{-1}$, so $\varphi^{-1}(\alpha'_i)$ are the attaching spheres of the 5-dimensional 3-handles $i=0,1,...,n$. We wish to prove $\varphi_*([\alpha_0])\cdot[\alpha'_0]=0$; $\varphi_*([\alpha_0])\cdot[\beta'_0]=1$ and $\varphi_*([\alpha_i])\cdot[\alpha'_i]=1$ for $i=1,...,n$. Without ambiguity, let us not distinguish between α_i, β_i and the homology classes $[\alpha_i]$,$[\beta_i]$ in the following context.

When X_i are closed manifolds, by Wall's Theorem, we can find a self-diffeomorphism $\phi: X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2\to X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2$ such that $\varphi_*\circ\phi_*=\Lambda\oplus\Omega$, where $\Lambda\oplus\Omega$ agrees with Λ on $H_2(X_1)$; $\Lambda\oplus\Omega(\alpha_i)=\beta'_i$ for $i=1,...,n$ and $\Lambda\oplus\Omega(\alpha_0)=\alpha'_0$. Therefore, we can assume that $\varphi_*(\alpha_i)=\beta'_i$, $\varphi_*(\alpha_i)\cdot\alpha'_i=1$, $\varphi_*(\alpha_0)\cdot\alpha'_0=0$. When X_i have boundaries, Q_{X_i} are not unimodular, so we can not apply Wall's Theorem directly. We shall prove: $\varphi_*(\alpha_0)=\alpha'_0+\delta$ for some $\delta\in H_2(M_2)$ and $\varphi_*(\alpha_i)=\beta_i$, $i=1,...n$.

If $H_1(M_1)$ is free, consider the long exact sequence:

$$0\to H_2(M_1)\xrightarrow{i_*}H_2(X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2)\xrightarrow{j_*}H_2(X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2,M_1)\to H_1(M_1)\to 0.$$

Assume the rank of $H_2(X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2)$=m, the rank of $H_1(M_1)$=k. Let $(\mathbb{Z}^m,Q)$ represent $H_2(X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2)$, then $(\mathbb{Z}^m,Q)$ splits as $(\mathbb{Z}^{m-k},Q_1)\bigoplus(\mathbb{Z}^k,0)$, where $(\mathbb{Z}^k,0)$ represents $i_*H_2(M_1)$; $(\mathbb{Z}^{m-k},Q_1)$ represents $coker(j_*)$. Easy to check $i_*(a)\cdot b=0$ for any $a\in H_2(M_1)$ and $b\in H_2(X_1\sharp nS^2\times S^2\sharp S^2\widetilde{\times}S^2)$, so the split is orthogonal, i.e.,

$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}$, and Q_1 is unimodular. The same is true for $H_2(X_2\sharp nS^2 \times S^2\sharp S^2\widetilde{\times}S^2)$.

φ induces an isometry (an isomorphism which preserves the intersection form) $\varphi_* : H_2(X_1\sharp nS^2 \times S^2\sharp S^2\widetilde{\times}S^2) \to H_2(X_2\sharp nS^2 \times S^2\sharp S^2\widetilde{\times}S^2)$. Note that $\varphi_* \mid_{i_*H_2(M_1)}$ is an isometry on $(\mathbb{Z}^k, 0)$. Now we want to define an isometry $\tilde{\varphi}_* : (\mathbb{Z}^{m-k}, Q_1) \to (\mathbb{Z}^{m-k}, Q_1)$. Suppose $\varphi_*(\gamma_i) = \sum_{j=1}^{m-k} p_j \cdot \gamma_j + \sum_{j=1}^{k} q_j\eta_j$, where $\{\gamma_i, i = 1, ..., m-k\}$ is a basis of $(\mathbb{Z}^{m-k}, Q_1)$; $\{\eta_i, i = 1, ..., k\}$ is a basis of $(\mathbb{Z}^k, 0)$. Let $\tilde{\varphi}_*(\gamma_i) = \sum_{j=1}^{m-k} p_j \cdot \gamma_j$. Clearly, $\tilde{\varphi}_*\varphi_*^{-1}(\gamma_i) = \gamma_i$, so $\tilde{\varphi}_*$ is surjective, and therefore it is an isomorphism on $coker(j_*)$. Moreover, $\tilde{\varphi}_*(\gamma_i) \cdot \tilde{\varphi}_*(\gamma_j) = \varphi_*(\gamma_i) \cdot \varphi_*(\gamma_j) = \gamma_i \cdot \gamma_j$. Thus, $\tilde{\varphi}_*$ is an isometry on $(\mathbb{Z}^{m-k}, Q_1)$. Let $\Lambda \oplus \Omega$ denote the isometry on $H_2(X_1\sharp nS^2 \times S^2\sharp S^2\widetilde{\times}S^2)$ which agree with Λ on $H_2(X_1)$, and $(\Lambda \oplus \Omega)(\alpha_i) = \beta_i'$, $(\Lambda \oplus \Omega)(\beta_i) = \alpha_i'$, $i = 1, 2, ...m$; $(\Lambda \oplus \Omega)(\alpha_0) = \alpha_0'$, $(\Lambda \oplus \Omega)(\beta_0) = \beta_0'$. Since Q_1 is unimodular, by Wall's Theorem, we are able to find a self-diffeomorphism $\phi : X_1\sharp nS^2 \times S^2\sharp S^2\widetilde{\times}S^2 \to X_1\sharp nS^2 \times S^2\sharp S^2\widetilde{\times}S^2$ such that $\phi_*(x) = x$ for any $x \in i_*H_2(M_1)$ and $\tilde{\phi}_* = \tilde{\varphi}_*^{-1} \circ (\Lambda \oplus \Omega)$ on $coker(j_*)$ ($\tilde{\phi}_*$ is defined in a similar way as $\tilde{\varphi}_*$). This can be proved by working on the relative handlebody pictures, similar to the proof in [30] (Chparter X), which deals with handlebody pictures of closed manifolds. Therefore, $(\varphi_* \circ \phi_*)(\alpha_0) = \alpha_0' + \delta$ for some $\delta \in i_*H_2(M_1)$. Thus, $(\varphi_* \circ \phi_*)(\alpha_0) \cdot \alpha_0' = 0$; $(\varphi_* \circ \phi_*)(\alpha_0) \cdot \beta_0' = 1$; and $(\varphi_* \circ \phi_*)(\alpha_i) \cdot \alpha_i' = 1$, $i = 1, ..., m$, we get what we need.

If $H_1(M_1)$ is not free, we apply Boyer's result.

Let us denote $X_i\sharp S^2\widetilde{\times}S^2$ by $\widetilde{X_i}$ $(i = 1, 2)$, $\widetilde{X_1} \cup_f \overline{\widetilde{X_2}}$ by $\widetilde{X}$; let $\Lambda \oplus id$ denote the isomorphism from $H_2(\widetilde{X_1})$ to $H_2(\widetilde{X_2})$ that agrees with Λ on $H_2(X_1)$ and $\Lambda \oplus id(\alpha_0) = \alpha_0'$, $\Lambda \oplus id(\beta_0) = \beta_0'$. We now consider the cobordism $\widetilde{W}$ between $\widetilde{X_1}$ and $\widetilde{X_2}$.

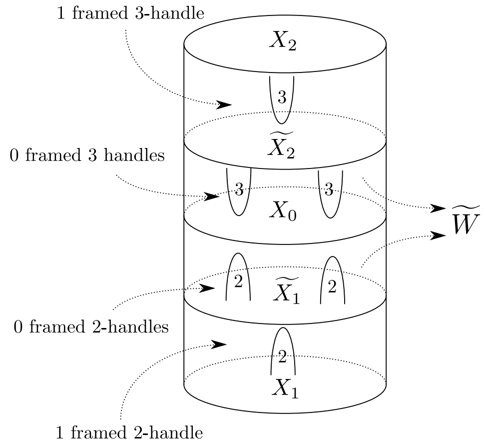


Figure 3.4: Finer handle decomposition of the cobordism

By Boyer's theorem, we can find $\mu'_i \in H_2(\widetilde{X})$ such that $\partial\mu'_i = \nu_i$, where ν_i is a basis of the free part of $H_1(M_1)$ $(i = 1, 2, ..., k)$ and $k_*(\mu'_i) \in G(-(\Lambda \oplus id)^*)$. If $\mu'_i \cdot \mu'_i$ is even for all i, we get $\varphi_*(\alpha_0) = (\Lambda \oplus id)(\alpha_0) = \alpha'_0$. We are done in this case.

If $\mu'_p \cdot \mu'_p$ is odd for some p, assume $k_*(\mu'_p) = (a, b)$, where $b \in H_2(\widetilde{X_2}, M_2)$, $a = (-(\Lambda \oplus id)^*)(b) \in H_2(\widetilde{X_1}, M_1)$. Both $H_2(\widetilde{X_1}, M_1)$ and $H_2(\widetilde{X_1})$ are free.

$$\underset{a}{H_2(\widetilde{X_1}, M_1)} \xrightarrow[\approx]{HOM} \underset{a^*}{H^2(\widetilde{X_1}, M_1)} \xrightarrow[\approx]{PD} \underset{PD(a^*)}{H_2(\widetilde{X_1})}.$$

$\partial\mu'_p = \nu_p \Rightarrow \partial_1 a = \nu_p$, so $a^* \in \delta_1(H^1(M_1))$ where δ is the coboundary map, $PD(a^*) \in i_*(H_2(M_1))$ and it is a primitive element by Lefschetz duality.

Let $\{x_1 = \alpha_0, x_2 = \beta_0, x_3 = PD(a^*), x_4, ..., x_{m-n}\}$ be a basis of $H_2(\widetilde{X_1})$. Note that $PD(a^*) \neq \alpha_0$ or β_0, because we assumed $\mu'_p \cdot \mu'_p$ is odd. We consider an automorphism $\phi_* : H_2(\widetilde{X_1}) \to H_2(\widetilde{X_1})$ which is defined by: $\phi_*(\alpha_0) = \alpha_0 + PD(a^*)$; $\phi_*(x_i) = x_i$ for $i \neq 1$. Easy to check that ϕ_* preserves the intersection form.

Let $\{\bar{x}_1, \bar{x}_2, \bar{x}_3 = a, \bar{x}_4, ..., \bar{x}_{m-n}\}$ be the dual basis for $H_2(\widetilde{X_1}, M_1)$, i.e., $\bar{x}_i \cdot x_j = \delta_{ij}$. Then $\bar{\beta}_0 = j_{1*}(\alpha_0)$; $\bar{\alpha}_0 = j_{1*}(\beta_0) - j_{1*}(\alpha_0)$, because $j_{1*}(\alpha_0) \cdot \alpha_0 = 0, j_{1*}(\alpha_0) \cdot \beta_0 = 1, (j_{1*}(\beta_0) - j_{1*}(\alpha_0)) \cdot \alpha_0 = 1, (j_{1*}(\beta_0) - j_{1*}(\alpha_0)) \cdot \beta_0 = 0$.

Claim 3.3. *$(\phi^*)^{-1}(\bar{x}_i) = \bar{x}_i$ for $i \neq 3$; $(\phi^*)^{-1}(a) = a - \bar{\alpha}_0 = a - j_{1*}(\beta_0) + j_{1*}(\alpha_0)$, where $(\phi^*)^{-1}$ is defined in the following commutative diagram:*

$$\begin{array}{ccccc} H_2(\widetilde{X_1}) & \xrightarrow{PD} & H^2(\widetilde{X_1}, M_1) & \xrightarrow{HOM} & H_2(\widetilde{X_1}, M_1) \\ \downarrow \phi_* & & \downarrow & & \downarrow (\phi^*)^{-1} \\ H_2(\widetilde{X_1}) & \xrightarrow{PD} & H^2(\widetilde{X_1}, M_1) & \xrightarrow{HOM} & H_2(\widetilde{X_1}, M_1) \end{array}$$

Proof. $(\phi^*)^{-1}(\bar{x_i}) \cdot \phi(x_j) = \delta_{ij}$, for $i \neq 1, 2, 3$, $\phi_*(x_i) = x_i \Longrightarrow (\phi^*)^{-1}(\bar{x}_i) = \bar{x}_i$;

when $i = 1$, $\bar{\alpha}_0 \cdot \phi_*(\alpha_0) = \bar{\alpha}_0 \cdot (\alpha_0 + PD(a^*)) = 1$, $\bar{\alpha}_0 \cdot \phi_*(\beta_0) = \bar{\alpha}_0 \cdot \beta_0 = 0$, $\bar{\alpha}_0 \cdot \phi_*(PD(a^*)) = \bar{\alpha}_0 \cdot PD(a^*) = 0 \Longrightarrow (\phi^*)^{-1}(\bar{\alpha}_0) = \bar{\alpha}_0$;

when $i = 2$, $\bar{\beta}_0 \cdot \phi_*(\alpha_0) = \bar{\beta}_0 \cdot (\alpha_0 + PD(a^*)) = 0$, $\bar{\beta}_0 \cdot \phi_*(\beta_0) = \bar{\alpha}_0 \cdot \beta_0 = 1$, $\bar{\beta}_0 \cdot \phi_*(PD(a^*)) = \bar{\beta}_0 \cdot PD(a^*) = 0 \Longrightarrow (\phi^*)^{-1}(\bar{\beta}_0) = \bar{\beta}_0$;

when $i = 3$, $(a - \bar{\alpha}_0) \cdot \phi_*(\alpha_0) = (a - \bar{\alpha}_0) \cdot (\alpha_0 + PD(a^*)) = 0$, $(a - \bar{\alpha}_0) \cdot \phi_*(\beta_0) = (a - \bar{\alpha}_0) \cdot \beta_0 = 0$, $(a - \bar{\alpha}_0) \cdot \phi_*(PD(a^*)) = (a - \bar{\alpha}_0) \cdot PD(a^*) = 1 \Longrightarrow (\phi^*)^{-1}(a) = a - \bar{\alpha}_0$. □

ϕ_* satisfies (*), since $j_{1*}(PD(a^*)) = 0$:

$$\begin{array}{ccc} PD(a^*) & \xrightarrow{j_{1*}} & 0 \\ \downarrow \phi_* & & \downarrow (\phi^*)^{-1} \\ PD(a^*) & \xrightarrow{j_{1*}} & 0 \end{array} \qquad \begin{array}{ccc} \alpha_0 & \xrightarrow{j_{1*}} & j_{1*}(\alpha_0) = \bar{\beta}_0 \\ \downarrow \phi_* & & \downarrow (\phi^*)^{-1} \\ \alpha_0 + PD(a^*) & \xrightarrow{j_{1*}} & j_{1*}(\alpha_0) = \bar{\beta}_0 \end{array}$$

Therefore, $\Lambda \circ \phi_*$ satisfies (*).

Now we consider $\widetilde{u_p}' = u'_p - i_{1*}(\beta_0) + i_{1*}(\alpha_0)$, $\widetilde{u_i}' = u'_i$ for $i \neq p$, then $\partial \widetilde{u_i}' = \partial u'_i = v_i$; $k_*(\widetilde{u_i}') = k_*(u'_i) - k_*(i_{1*}(\beta_0)) + k_*(i_{1*}(\alpha_0)) = \big(a, b\big) - \big(j_{1*}(\beta_0), 0\big) + \big(j_{1*}(\alpha_0), 0\big) = \big((\phi^*)^{-1}(a), b\big)$. So, $k_*(\widetilde{u_i}') \in G\big(-(\phi^*)^{-1} \circ (\Lambda \oplus id)^*\big)$ and $\widetilde{u_p}' \cdot \widetilde{u_p}' = u'_p \cdot u'_p - 1$ which is even. Thus, $(\Lambda \oplus id) \circ \phi_*^{-1}$ can be realized geometrically, i.e., φ_* agrees with $(\Lambda \oplus id) \circ \phi_*^{-1}$ on $H_2(\widetilde{X_1})$. $\varphi_*(\alpha_0) = (\Lambda \oplus id) \circ \phi_*^{-1}(\alpha_0) = \alpha'_0 + \delta$ for some $\delta \in H_2(M_2)$, so $\varphi_*(\alpha_0) \cdot \alpha'_0 = 0$ and $\varphi_*(\alpha_0) \cdot \beta'_0 = 1$. Since $\widetilde{W}$ is a relative h-cobordism, $\varphi_*(\alpha_i) = \beta'_i$ for $i = 1, ..., n$. This can be achieved by introducing cancelling pairs of 5 dimensional 2-handles and 3-handles, and sliding handles (see [40] 1.7). Therefore, $\varphi_*(\alpha_i) \cdot \alpha'_i = 1$.

Now we can put our hands on constructing the plug.

Let us first assume $n = 0$, $\widetilde{X_2}$ is diffeomorphic to $\widetilde{X_1}$. Doing 2-surgery on $\widetilde{X_2}$ along α_0' (change "0" on α_0' to "•") gives us X_2; 2-surgery along $\varphi(\alpha_0)$ on the same manifold yields X_1.

We consider the neighborhood of $\varphi(\alpha_0)$, α_0', β_0' in $\widetilde{X_2} = X_2 \sharp S^2 \widetilde{\times} S^2$.

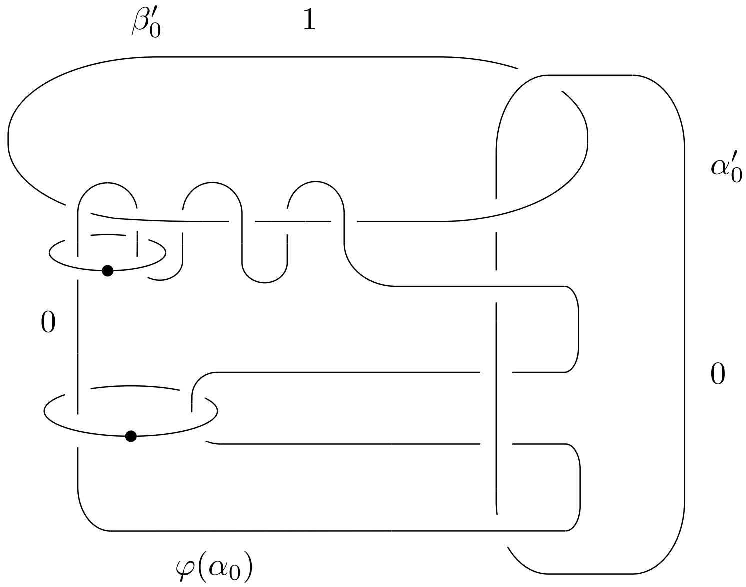


Figure 3.5: The neighborhood of $\varphi(\alpha_0)$, α_0', β_0' in $X_2 \sharp S^2 \widetilde{\times} S^2$

$\widetilde{X_2} \setminus nb(\varphi(\alpha_0))$ is simply-connected, because $\varphi(\beta_0)$ is a transverse sphere of $\varphi(\alpha_0)$; $\widetilde{X_2} \setminus nb(\alpha_0') \cup nb(\beta_0')$ is also simply-connected. By M-V sequence, the first homology of $\widetilde{X_2} \setminus (nb(\varphi(\alpha_0)) \cup nb(\beta_0') \cup nb(\alpha_0'))$ is isomorphic to $\mathbb{Z}$ and generated by a. By Van kampen Theorem, the fundamental group of $\widetilde{X_2} \setminus (nb(\varphi(\alpha_0)) \cup nb(\beta_0') \cup nb(\alpha_0'))$ is generated by the conjugates of a. Any $\gamma \in$ the fundamental group of $\widetilde{X_2} \setminus (nb(\varphi(\alpha_0)) \cup nb(\beta_0') \cup nb(\alpha_0'))$ can be represented by an embedded loop, denote it by Γ. We choose a point $\in \varphi(\alpha_0)$ near any intersection point between $\varphi(\alpha_0)$ and α_0' or β_0', and then we choose an arc τ connecting this point with any point on Γ. Now we do the finger move following τ, and Γ and then coming back by an arc which is parallel and close enough to τ. The effect of this finger move on the fundamental group is killing $[\gamma a\gamma^{-1}, b']$. Assume $b' = r^{-1}ar$, then $[a, (r\gamma)^{-1}ar\gamma] = 1$. Since γ is arbitrary, $[s^{-1}as, t^{-1}at] = [a, (ts^{-1})^{-1}ats^{-1}] = 1$ for any $s, t \in \pi_1(\widetilde{X_2} \setminus (nb(\varphi(\alpha_0)) \cup nb(\beta_0') \cup nb(\alpha_0')))$. Hence, after finger moves (creating more intersections between $\varphi(\alpha_0)$ and α_0' or β_0'), we can kill all the commutators. The fundamental group now becomes abelian, therefore, it is cyclic and generated by a.

For each pair of intersection points between $\varphi(\alpha_0)$ and β_0' with opposite sign, there exists a Whitney's circle l. Since the fundamental group of $\widetilde{X_2} \setminus (nb(\varphi(\alpha_0)) \cup nb(\beta_0') \cup nb(\alpha_0'))$ is generated by a, we can always change l by adding 2π twists around the meridian of $\varphi(\alpha_0)$ so that l represents a trivial element in this fundamental group.

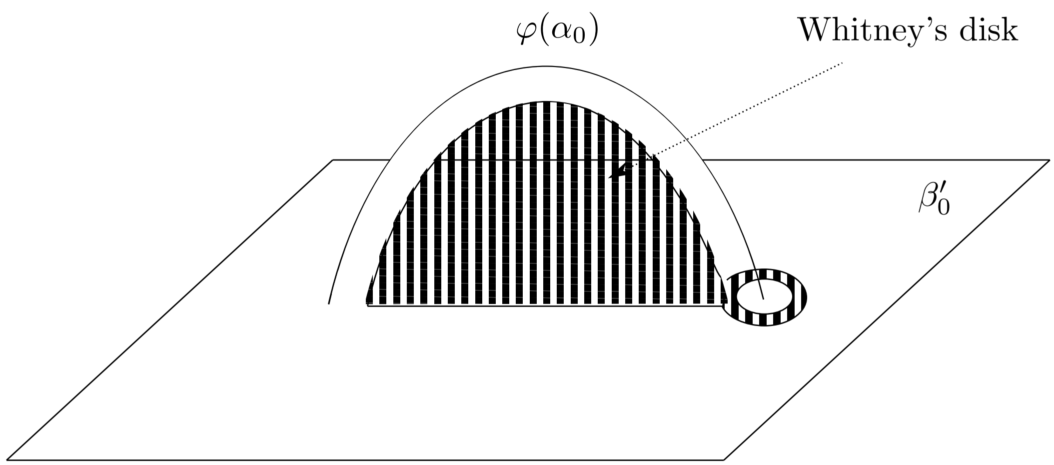


Figure 3.6: Finding a Whitney's disk

Therefore l bounds a Whitney's disk W disjoint from $\varphi(\alpha_0)$, β_0' and α_0'. This Whitney's disk might be immersed, then we can push the interior intersection points of W off at β_0'. If the framing of W is wrong, we can do boundary twists to fix the framing problem by creating more intersections between W and β_0'. Doing Whitney's trick along all these Whitney's disks, we can cancel the intersection points of opposite signs between $\varphi(\alpha_0)$ and β_0', but β_0' becomes an immersed sphere.

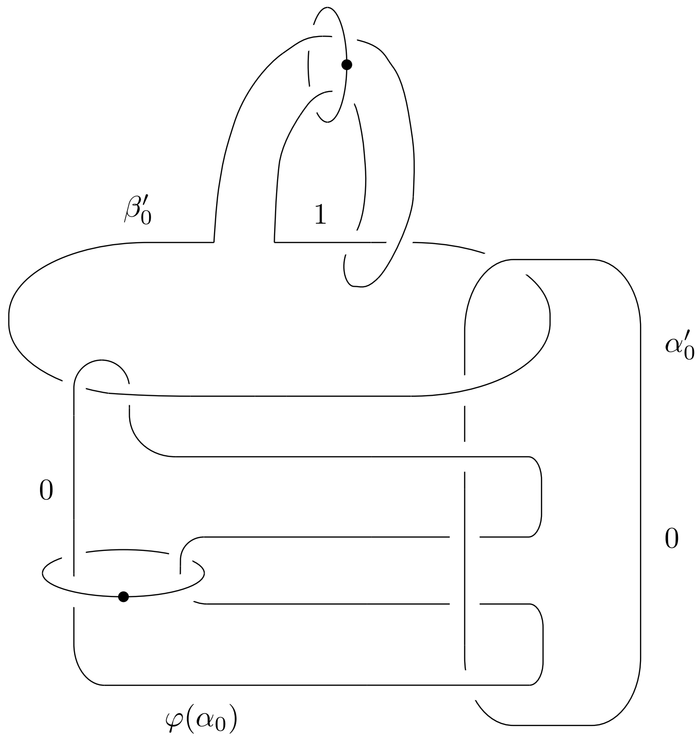


Figure 3.7: The neighborhood of $\varphi(\alpha_0)$, α_0', β_0' in $X_2\sharp S^2\widetilde{\times}S^2$ after Whitney's move

We add all 1-handles of $\widetilde{X_2}$ into this picture and introduce 2- and 3-handle cancelling pairs such that each 2-handle cancels a 1-handle algebraically. We drop all the 3-handles and call the resulting manifold Y_0. Y_0 is simply-connected. Doing 2-surgery on $\varphi(\alpha_0)$ to Y_0 gives us a submanifold of X_1, call it Y_1; Doing 2-surgery on α_0' to Y_0 gives us a submanifold of X_2, call it Y_2. The handle decomposition of Y_1 and Y_2 differ by exchanging "dot" and "0". That does not change the boundary 3-manifold. So, $\partial Y_1 \xrightarrow{\tau} \partial Y_2$ is an obvious homeomorphism. As β_0' is an immersed transverse sphere for α_0' $\big(\varphi(\alpha_0)\big)$, after doing 2-surgery on α_0' $\big(\varphi(\alpha_0)\big)$, Y_2 (Y_1) is still simply-connected. Since $X_1\backslash Y_1$ has identical handle decomposition as $X_2\backslash Y_2$, $X_1 \setminus Y_1$ is diffeomorphic to $X_2 \setminus Y_2$.

We want to show Y_i $(i = 1, 2)$ are homotopic equivalent to S^2. All of the 1-handles and 2-handles of Y_1 (Y_2) are algebraically cancelled except for α_0' $(\varphi(\alpha_0))$, so Y_1 (Y_2) has the same homology group as S^2. By using the long exact sequence, we can easily check that ∂Y_1 (∂Y_2) is a homology $S^1 \times S^2$. If we erase 0 framed 2-handle α_0' ($\varphi(\alpha_0)$) from the picture, we get a cork which is contractible. Now consider the map $g_1 : Y_1 \to S^2$ ($g_2 : Y_2 \to S^2$) such that g_1 (g_2) maps the complement of α_0' ($\varphi(\alpha_0)$) in Y_1 (Y_2) to a base point of S^2 and squeezes α_0' ($\varphi(\alpha_0)$) onto its core. This map induces an isomorphism on the homology groups, therefore, by Whitehead's Theorem, Y_1 (Y_2) is homotopic equivalent to S^2.

Since $H_2(Y_2)$ is generated by $\varphi(\alpha_0)$ and $\varphi_*(\alpha_0) = \alpha_0' + \delta$ for some $\delta \in H_2(M_2)$, $i_* H_2(Y_2) \subset i_{\partial *} H_2(M_2)$. Similarly, $i_* H_2(Y_1) \subset i_{\partial *} H_2(M_1)$.

Since Y_1 (Y_2) does not contain any 3-handles, the handle decomposition of $Y_1 \cup_{id} \overline{Y_1}$ can be constructed from the handle decomposition of Y_1 by attaching each 2-handle a 0 framed 2-handle along its meridian and 3-handles. The number of 3-handles attached is the same as the number of 1-handles of Y_1. Then by handle slides, everything is cancelled except for a Hopf link (α_0' and its meridian), framing 0 on each component. Clearly, this is $S^2 \times S^2$. Similarly, $Y_2 \cup_{id} \overline{Y_2}$ is $S^2 \times S^2$. The handle decomposition of $Y_1 \cup_\tau \overline{Y_2}$ is almost the same as $Y_1 \cup_{id} \overline{Y_1}$, except that the 2-handle which is attached along the meridian of α_0' now is attached along the meridian of $\varphi(\alpha_0)$. By handle slides, everything is cancelled except for β_0' and its meridian, which is clearly a $S^2 \widetilde{\times} S^2$.

By doing the "cut and paste" procedures as in [1], we can make Y_i Stein.

If $n > 0$, after doing 2-surgery on $X_2 \sharp n S^2 \times S^2 \sharp S^2 \widetilde{\times} S^2$ along α_0', we will get $X_2 \sharp n S^2 \times S^2$; doing surgery on the same manifold along $\varphi(\alpha_0)$ will give us $X_1 \sharp n S^2 \times S^2$. If we keep doing surgery along α_i' ($\varphi(\alpha_i)$), $(i = 1, ..., n)$, we will obtain X_2 (X_1). So, X_2 and X_1 differ by $n+1$ "dot" and "0" exchanges. If $\varphi(\alpha_0)$ and α_i' $(i = 1, ..., n)$ are unlinked; α_0' and $\varphi(\alpha_i)$ $(i = 1, ..., n)$ are unlinked , then exchanging "dot" and "0" between $\varphi(\alpha_0)$ and α_0' is a plug twist; exchanging "dot" and "0" between α_i and α_i' $(i = 1, ..., n)$ are n cork twists.

Otherwise, WLOG, we may assume $\varphi(\alpha_0)$ and α'_j $(j = 1, ..., l)$ are linked, then we will consider the neighbourhood of $\varphi(\alpha_0) \bigcup \cup_{j=1}^{l} \varphi(\alpha_j) \bigcup \alpha'_0 \bigcup \cup_{j=1}^{l} \alpha'_j$ in $X_2 \sharp n S^2 \times S^2 \sharp S^2 \widetilde{\times} S^2$. We include all the 1-handles and add algebraic cancelling 2-handles. We call the resulting manifold Y_0. Doing surgery on Y_0 along $\varphi(\alpha_0)$ and $\varphi(\alpha_j)$ $(j = 1, ..., l)$ will give us Y_1; doing surgery on Y_0 along α'_0 and α'_j $(j = 1, ..., l)$ will give us Y_2. $(X_1 \setminus Y_1) \sharp (n-l) S^2 \times S^2$ is diffeomorphic to $(X_2 \setminus Y_2) \sharp (n-l) S^2 \times S^2$ since their handle decomposition are identical. Therefore, $(X_1 \setminus Y_1)$ and $(X_2 \setminus Y_2)$ differ by cork twists and hence they are homeomorphic. It is also easy to check that Y_1 and Y_2 satisfy other desired properties. □

Note that $i_*(H_2(Y_i)) \subset H_2(M_i)$ $(i = 1, 2)$ guarantees that the plug twist does not change the intersection form of X_i, and keep hold of (*). While if $i_*(H_2(Y_2)) = 0$ in $H_2(X_2)$, then $\varphi_*(\alpha_0) = 0$ in $H_2(X_2)$. On the other hand, by using Boyer's theorem, we can find $\lambda'_i \in H_2(X)$ $(X = X_1 \cup_f \overline{X_2})$ such that $\partial \lambda'_i = \nu_i$ and $k_*(\lambda'_i) \in G(-\Lambda^*)$, where ν_i is a basis of the free part of $H_1(M_1)$, $i = 1, ..., m-n-1$. If $\lambda'_i \cdot \lambda'_i$ is even for $\forall i = 1, ..., m-n-1$, then X_1 is homeomorphic to X_2. Otherwise, $\lambda'_p \cdot \lambda'_p$ is odd for some p, then $i_*(\lambda'_p) \cdot i_*(\lambda'_p)$ is also odd, where i_* is the homomorphism induced by the inclusion map $i : X \to \widetilde{X} = X \sharp S^2 \widetilde{\times} S^2 \sharp \overline{S^2 \widetilde{\times} S^2}$. Obviously, $\partial(i_*(\lambda'_i)) = \nu_i$ and $k_*(i_*(\lambda'_i)) \in G(-(\Lambda \oplus id)^*)$. So by the construction in the proof, $\varphi_*(\alpha_0) = \alpha'_0 + \delta$ for some non-trivial element $\delta \in H_2(M_2) \subset H_2(X_2)$. A contradiction. Hence:

Corollary 3.1. *Under the assumption of last theorem, X_1 is homeomorphic to X_2 if Y_2 is null-homologous in X_2, or Y_1 is null-homologous in X_1,.*

By property (3) of the theorem, we could think the plug twist $X_1 \to X_2$ as cutting off a submanifold Y_1 which is a homotopic $S^2 \times D^2$ from X_1 and gluing back the complement of Y_1 in $S^2 \widetilde{\times} S^2$ by the naturally induced boundary homeomorphism $\partial Y_1 \to$

∂(complement of Y_1) in $S^2\widetilde{\times}S^2$. This can be considered as a generalization of "Gluck Twist" which is an operation that cut off $S^2 \times D^2$ from X_1 and glue back the complement of $S^2 \times D^2$ in $S^2\widetilde{\times}S^2$ (which is still a $S^2 \times D^2$) by the naturally induced boundary homeomorphism $S^2 \times S^1 \to S^2 \times S^1$.

[9], [14] give a very easy description of the handle-body picture of Gluck twist. Exchanging "dot" and "0" in the following picture is a Gluck twist.

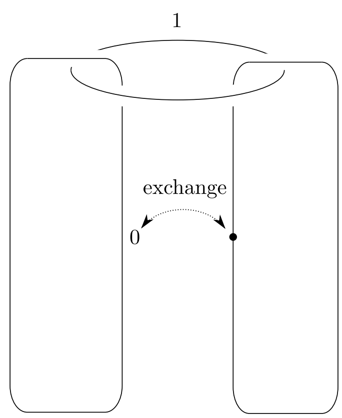


Figure 3.8: Glug twist

Plug twist can be viewed as a generalization of the Gluck twist also from its handle-body picture. Recall that in the proof, assuming $n = 0$, before we do surgery along $\varphi(\alpha_0)$ and α'_0 we have the following submanifold of $\widetilde{X_2}$

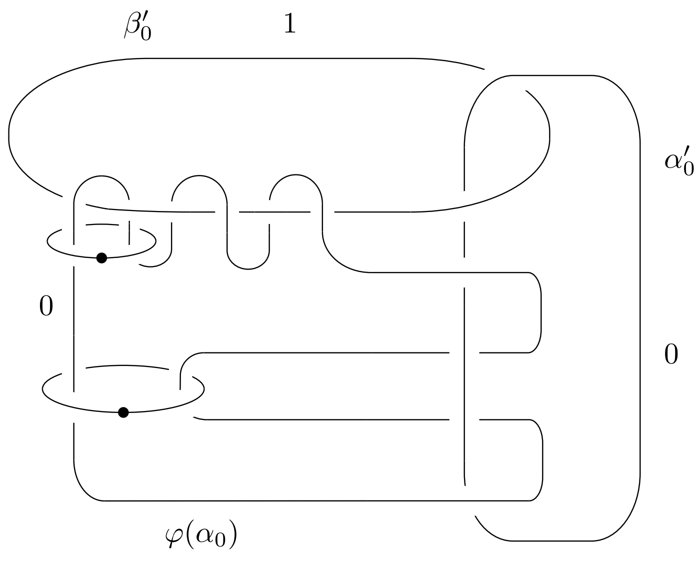


Figure 3.9: The neighborhood of $\varphi(\alpha_0)$, α_0', β_0' in $X_2 \sharp S^2 \widetilde{\times} S^2$

After the finger moves, the fundamental group of $\widetilde{X_2} \setminus (nb(\varphi(\alpha_0)) \cup nb(\beta_0'))$ becomes trivial. So each Whitney's circle connecting a pair of intersection points between $\varphi(\alpha_0)$ and β_0' with opposite sign bounds an immersed Whitney's disk disjoint from $\varphi(\alpha_0)$ and β_0' (it might intersect with α_0'). The linking torus T^2 intersects with this Whitney's disk algebraically once, and $[T^2] \cdot [T^2] = 0$, so by Casson's theorem, there is a Casson handle attached along this Whitney's circle. Then by Freedman's theorem, all these Casson handles are homeomorphic to standard 2-handles relative the boundary. Therefore, there exists a manifold Z homeomorphic to $\widetilde{X_2}$ such that in Z, these Casson handles become standard 2-handles. Since this homeomorphism only change the interior of the Casson handle, $\varphi(\alpha_0)$ and β_0' are preserved by this homeomorphism. α_0' is also preserved because it intersects with the Casson handles only at points. So, in Z, the intersection points between $\varphi(\alpha_0)$ and β_0' with opposite signs can be cancelled, while $\varphi(\alpha_0)$ may have more intersection points with α_0' than in $\widetilde{X_2}$. Doing surgery along $\varphi(\alpha_0)$ on Z gives a manifold X_1' homeomorphic to X_1. Similarly, doing surgery along α_0' on Z gives a manifold X_2' homeomorphic to X_2.

Corollary 3.2. *Under the assumption of the last theorem, there exists X_i' such that X_i' is homeomorphic to X_i $(i = 1, 2)$ and X_1', X_2' are related by doing the following "dot" and "0" exchange.*

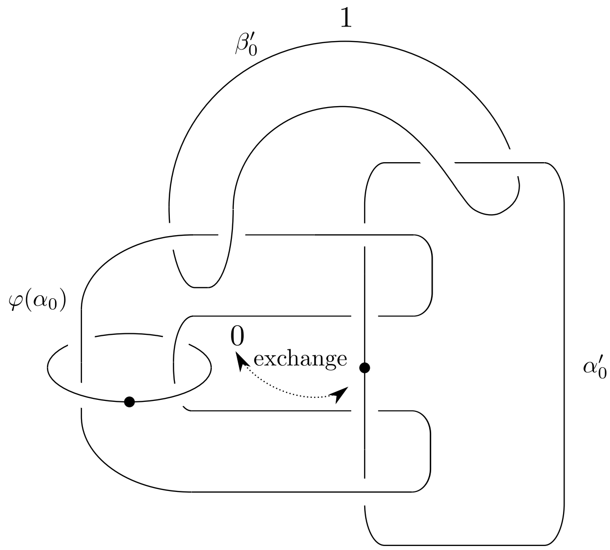


Figure 3.10: Plug twist

The diffeomorphism type change from X_i to X_i' is due to cork twists; the "dot" and "0" exchange in the above picture is the prime cause of changing the homeomorphism type. It is easy to observe that: if $\varphi(\alpha)$ is unlinked with α_0', it is a Gluck twist; if there is one clasp between $\varphi(\alpha)$ and α_0' (as in the above picture), it is a 0 logarithmic transformation. Since we allow any number of clasps between $\varphi(\alpha)$ and α_0'. This operation can be thought of a generalization of Gluck twist and 0 logarithmic transformation.

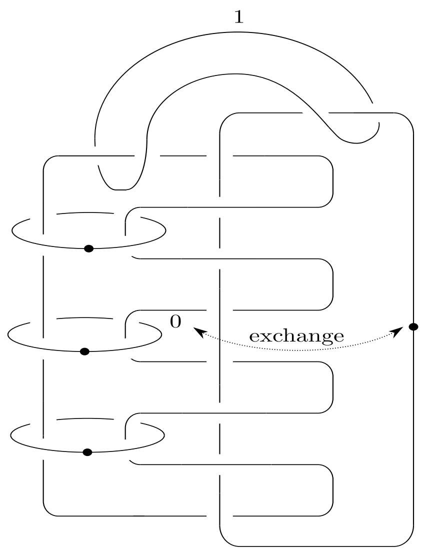


Figure 3.11: A general plug twist

Combine the cork theorem stated in the first section of this chapter, we can summarize the features of cork and plug by the following pictures:

If X_1 and X_2 are simply-connected smooth 4 manifolds such that $(Q_{X_1}, \partial X_1)$ is isomorphic to $(Q_{X_2}, \partial X_2)$, then X_1 and X_2 are related by:

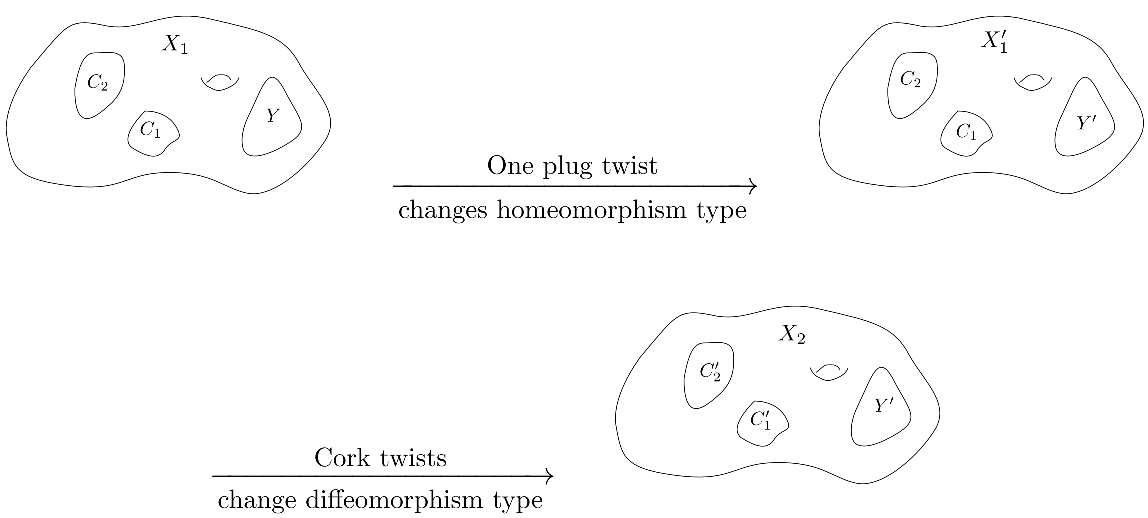


Figure 3.12: Cork twists together with a plug twist

Chapter 4

An Example

4.1 An irreducible 3 manifold

For a closed, connected, orientable 3 manifold M, denote $\mathcal{H}(M)$ the group of isotopy classes of homeomorphisms of M; $\mathcal{H}_+(M)$ the the group of isotopy classes of orientation preserving homeomorphisms of M.

Recall Boyer's result: $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$ $(\partial X_1 \approx \partial X_2 \approx M)$ are simply-connected topological 4-manifolds, if $(Q_{X_1}, \partial X_1)$ is isomorphic to $(Q_{X_2}, \partial X_1)$ through (Λ, f), then they are homeomorphic if one of the following conditions holds:

(i) $H_1(M, \mathbb{Q}) = 0$ and $\Delta(X_1) \equiv \Delta(X_2)$;

(ii) X_i are odd manifolds and $\Delta(X_1) \equiv \Delta(X_2)$;

(iii) X_i are even manifolds and $X_1 \cup_g \overline{X_2}$ is also even for some homeomorphism $g : \partial X_1 \to \partial X_2$.

Note that in the last case, $X_1 \cup_f \overline{X_2}$ might be odd, but if we could find a self-homeomorphism $h : \partial X_2 \to \partial X_2$ such that h interchanges the two spin structures on ∂X_2 induced from $f(\partial X_1)$ and X_2 respectively, then $X_1 \cup_{h \circ f} \overline{X_2}$ is even, and thus X_1 is homeomorphic to X_2. Conversely, if there exists a homeomorphism $F : X_1 \to X_2$, then $F|_{\partial X_1} \circ f^{-1}$ must interchanges the two spin structures on ∂X_1. So this suggests us, for a fixed boundary M and a intersection form Q presenting M, there are at most $\mathrm{Spin}(M)/\mathcal{H}_+(M)$ different homoemorphism types of simply-connected topological 4-manifolds with boundary M and intersection form Q. We denote the set of these manifolds by $\mathcal{V}_Q(M)$.

Theorem 4.1. (Boyer, [15]) *When $H_1(M)$ is free, $\mathcal{V}_Q(M)$ is one-to-one correspondent to:*

(i) Spin$(M)/\mathcal{H}_+(M)$ if Q is even;

(ii) $\mathbb{Z}/2$ if Q is odd.

For instance, when $M = S^1 \times S^2$, for a fixed Q presenting $S^1 \times S^2$,

$$| \mathcal{V}_Q(S^1 \times S^2) | = \begin{cases} 1, & \text{if } Q \text{ is even} \\ 2, & \text{if } Q \text{ is odd} \qquad \text{(the two homeomorphism types differ by } \Delta\text{).} \end{cases}$$

, since $\mathcal{H}_+(S^1 \times S^2)$ acts transitively on Spin$(S^1 \times S^2)$. ([25])

When $M = T^3$, we know that T^3 has 8 different spin structures; $\mathcal{H}_+(T^3)$ is isomorphic to $SL(3;\mathbb{Z})$ which is generated by 6 elements. $\mathcal{H}_+(T^3)$ acts transitively on 7 spin structures, however there exists an exceptional spin structure which is on a different orbit by itself, denoted by s_{Lie} ([30], Chapter IV). This spin structure spin-bounds the complement of a generic fiber in the rational elliptic fibration $E(1) = CP^2 \sharp 9\overline{CP^2}$. ([30], Chapter V)

Therefore, $$| \mathcal{V}_Q(T^3) | = \begin{cases} 2, & \text{if } Q \text{ is even} \\ 2, & \text{if } Q \text{ is odd} \quad \text{(the two homeomorphism types differ by } \Delta\text{).} \end{cases}$$

So, there exist 2 simply-connected even topological 4-manifolds X_1, X_2 such that $\partial X_1 = \partial X_2 = T^3$, (Q_{X_1}, T^3) is isomorphic to (Q_{X_2}, T^3) and X_1 is not homeomorphic to X_2. WLOG, we assume the induced spin structure on ∂X_1 is s_{Lie}, while the induced spin structure on ∂X_2 is one of the other 7 spin structures.

Since Boyer's construction involves Freedman's theorem:

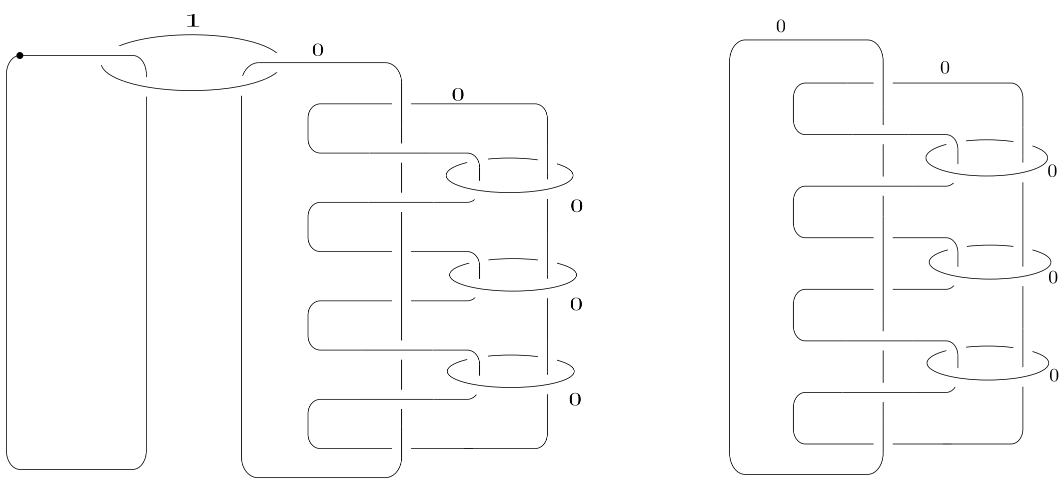
Every homology 3-sphere bounds a contractible topological 4-manifolds.

We know that Freedman's theorem fails for smooth 4-manifolds, so the second part of Boyer's result may not be true in the smooth situation.

Indeed, the Rohlin invariant of (T^3, s_{Lie}) is 8, since $\sigma(CP^2 \sharp 9\overline{CP^2}) = -8$, $\sigma(T^2 \times D^2) = 0$, σ(the complement of $T^2 \times D^2$) $= -8$ by Novikov additivity; the Rohlin invariant of the other 7 spin structures on T^3 is 0, since they all bound $T^2 \times D^2$. So there do not exist smooth 4-manifolds X_1, X_2 such that X_1 induces s_{Lie} on $\partial X_1 \approx T^3$, X_2 induces one of the other 7 spin structures on ∂X_2 and $Q_{X_1} \cong Q_{X_2} \approx T^3$. Thus, if X_1 and X_2 are smooth simply-connected 4-manifolds such that (Q_{X_1}, T^3) is isomorphic to (Q_{X_2}, T^3), then X_1 is homeomorphic to X_2.

It is natural to ask the question: Is it always true that for two simply-connected smooth 4-manifolds X_1 and X_2, (Q_{X_1}, M) isomorphic to (Q_{X_2}, M) implies they are homeomorphic?

We will show that the following example gives a negative answer:



X_1 $\approx$ X_1

Figure 4.1: A simply-connected 4-manifold

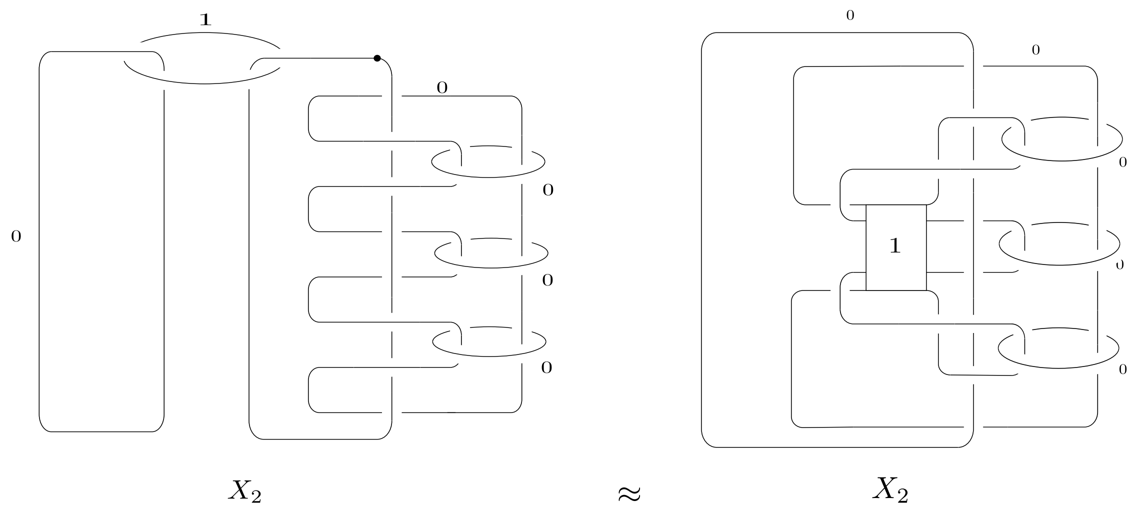
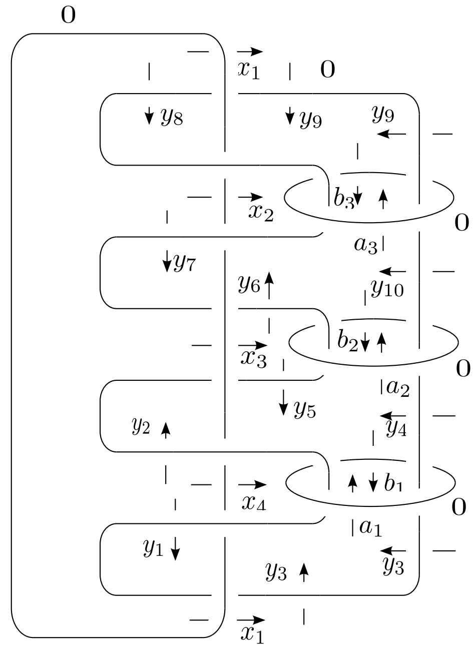


Figure 4.2: Doing Gluck twist gives a non-homeomorphic simply-connected 4-manifold

Clearly, $Q_{X_1} \cong Q_{X_2}$, $\partial X_1 \approx \partial X_2$; denote the boundary by M. We can easily find the plug twist; it is indeed a Gluck twist. $H_2(Y_i) \subset H_2(\partial X_i)$ $(i = 1, 2)$, so easy to check that (*) is commutative, i.e., we have (Q_{X_1}, M) isomorphic to (Q_{X_2}, M). To prove X_1 is not isomorphic to X_2, we only need to prove there does not exist a homeomorphism $M \to M$ interchanging the two spin structures induced from X_1 and X_2 respectively.



M

Figure 4.3: Finding the fundmental group of the boundary

We can easily write the Wirtinger presentation of the fundamental group of M. It has 20 generators: $\{x_1, ..., x_4, y_1, ..., y_{10}, a_1, b_1, a_2, b_2, a_3, b_3\}$ and 25 relations:

$$y_8 = a_3 y_8 b_3^{-1};\ y_9 = a_3 y_9 b_3^{-1};\ y_7 = a_3 y_8 a_3^{-1};\ y_{10} = a_3 y_9 a_3^{-1};$$

$$y_6 = a_2 y_6 b_2^{-1};\ y_{10} = a_2 y_{10} b_2^{-1};\ y_5 = a_2 y_6 a_2^{-1};\ y_4 = a_2 y_{10} a_2^{-1};$$

$$y_2 = a_1 y_2 b_1^{-1};\ y_4 = a_1 y_4 b_1^{-1};\ y_1 = a_1 y_2 a_1^{-1};\ y_3 = a_1 y_4 a_1^{-1};$$

$$x_1 = y_1 x_1 y_3^{-1};\ x_4 = y_1 x_1 y_1^{-1};\ x_4 = y_2 x_3 y_2^{-1};\ x_3 = y_2 x_3 y_5^{-1};$$

$$x_3 = y_7 x_3 y_6^{-1};\ x_2 = y_7 x_3 y_7^{-1};\ x_2 = y_8 x_1 y_8^{-1};\ x_1 = y_8 x_1 y_9^{-1};$$

$$y_9 y_8^{-1} = 1;\ y_{10} y_6^{-1} = 1;\ y_4 y_2^{-1} = 1;\ y_8^{-1} y_7 y_2^{-1} y_1 = 1;\ x_1^{-1} a_3^{-1} x_1 a_2^{-1} x_1^{-1} a_1^{-1} x_1 a_1 a_2 a_3 = 1.$$

The first 3 equations of the last line implies $y_8 = y_9$, $y_2 = y_4$, $y_6 = y_{10}$, substituting into the last equations of the first 3 rows, we get$y_6 = a_3 y_8 a_3^{-1}$, $y_2 = a_2 y_6 a_2^{-1}$, $y_3 = a_1 y_2 a_1^{-1}$. By comparing the second last equations of the first 3 rows, we have $y_7 = y_6, y_2 = y_5, y_1 = y_3$.

So, $y_1 = y_3, y_2 = y_4 = y_5, y_6 = y_7 = y_{10}, y_8 = y_9; x_1 = x_2 = x_3 = x_4$. Hence, after simplification, we get:

$$\pi_1(M) = \langle x_1, y_1, a_1, a_2, a_3 \mid [x_1, y_1] = 1, [x_1, y_2] = 1, [x_1, y_6] = 1, [x_1, y_8] = 1,$$
$$y_8^{-1} y_6 y_2^{-1} y_1 = 1 \quad (\dagger) \quad , x_7^{-1} x_6 x_5^{-1} x_1 = 1 \quad (\ddagger) \quad \rangle,$$

where $y_2 = a_1^{-1} y_1 a_1$; $y_6 = a_2^{-1} a_1^{-1} y_1 a_1 a_2$; $y_8 = a_3^{-1} a_2^{-1} a_1^{-1} y_1 a_1 a_2 a_3$; $x_5 = a_1 x_1 a_1^{-1}$; $x_6 = a_1 a_2 x_1 a_2^{-1} a_1^{-1}$; $x_7 = a_1 a_2 a_3 x_1 a_3^{-1} a_2^{-1} a_1^{-1}$.

Now we denote $\pi_1(M)$ by G, Let

$$G_1 = \langle x, a, b, c \mid c x^{-1} c^{-1} b x b^{-1} a x^{-1} a^{-1} x = 1 \rangle;$$
$$G_2 = \langle y, a, b, c \mid c^{-1} y^{-1} c b^{-1} y b a^{-1} y^{-1} a y = 1 \rangle,$$

where $x = x_1, y = y_1, a = a_1, b = a_1 a_2, c = a_1 a_2 a_3$. Easy to see that $G = \langle G_1, G_2 \rangle$, the group generated by G_1 and G_2, and $G_1 \cap G_2 \neq \{1\}$.

We can check that the relator of G_1, $r_1 = c x^{-1} c^{-1} b x b^{-1} a x^{-1} a^{-1} x$ is of minimal length under $Aut(\langle a, b, c, x \rangle)$ by Whitehead's Theorem; similarly, the relator of G_2, $r_2 = c^{-1} y^{-1} c$ $b^{-1} y b a^{-1} y^{-1} a y$ is of minimal length under $Aut(\langle a, b, c, y \rangle)$.

Theorem 4.2. (Whitehead, [46]) *If w, $v \in F_n$ such that w can be transformed to v by automorphisms of F_n, and v is of minimal length, then there exists a sequence $S_1, S_2, ..., S_m$ of Type I and Type II automorphisms such that $S_m ... S_2 S_1(w) = v$, and for $k \leq m$, $\mid S_k ... S_2 S_1(w) \mid \leq \mid S_{k-1} ... S_2 S_1(w) \mid$, with strictly inequality unless $S_{k-1} ... S_2 S_1(w)$ is minimal.*

If we denote the set of generators and their reverses $\{x_1, x_2, ... x_n, x_1^{-1}, ..., x_n^{-1}\}$ of F_n by L_n, then Whitehead Type I automorphism is a permutation $S \in$ Aut F_n acting on L_n such that $S(x^{-1}) = (S(x))^{-1}$ for $\forall x \in L_n$;

For $x \in L_n$ and $A \in L_n$, Whitehead Type II automorphism $S(A, x)$ is defined as follows:

$$S(y) = \begin{cases} yx, & \text{if } y \in A, y^{-1} \notin A, y \notin \{x, x^{-1}\}, \\ x^{-1}y, & \text{if } y \notin A, y^{-1} \in A, y \notin \{x, x^{-1}\}, \\ x^{-1}yx, & \text{if } y, y^{-1} \in A, \\ y, & \text{otherwise.} \end{cases} \qquad \text{for any } y \in L_n.$$

Now, by using the following theorem,

Theorem 4.3. ([31], Proposition 5.13) *Let $H = \langle x_1, x_2, ..., x_n \mid r \rangle$ where r is of minimal length under $Aut(\langle x_1, x_2, ..., x_n \rangle)$ and contains exactly the generators $x_1, x_2, ..., x_k$ for some $0 \leq k \leq n$. Then $H = H_1 * H_2$ where $H_1 = \langle x_1, x_2, ..., x_k \mid r \rangle$ is freely indecomposable and H_2 is free with basis $x_{k+1}, ..., x_n$.*

A group G is called *freely indecomposable* if G can not be written as $G = A * B$ for nontrivial subgroups $A \leq G, B \leq G$.

we can easily see that:

Lemma 4.1. *G_1 and G_2 are freely indecomposable.*

Proof. By this theorem, G_1 can splits as $G_1 = A_1 * B_1$ where A_1 is freely indecomposable and B_1 is free. Since $r_1 = cx^{-1}c^{-1}bxb^{-1}ax^{-1}a^{-1}x$, $k = n = 4$. Therefore, B_1 is trivial, and $G_1 = A_1$ which is freely indecomposable. Similarly, one can prove that G_2 is also freely indecomposable. □

Next we will show that G is also freely indecomposable by applying Kurosh subgroup theorem.

Theorem 4.4. (Kurosh Subgroup Theorem)*: If G is freely decomposable, i.e., $G = A * B$ where A, B are both nontrivial subgroups, and $H \leq G$ is a subgroup of G, then there exist a* *family* $(A_i)_{i\in I}$ *of subgroups* $A_i \leq A$*, a family* $(B_j)_{j\in J}$ *of subgroups* $B_j \leq B$*, families* $g_i, i \in I$ *and* $f_j, j \in J$ *of elements of* G*, and a subset* $X \subseteq G$ *such that* $H = F(X) * (*_{(i\in I)} g_i A_i g_i^{-1}) * (*_{(j\in J)} f_j B_j f_j^{-1})$*, where* $F(X)$ *is the free group generated by* X*.*

Proposition 4.1. *G is freely indecomposable.*

Proof. Since G_1 is not free and is freely indecomposable, by Kurosh subgroup theorem, $G_1 = gA_1g^{-1}$ or $G_1 = gB_1g^{-1}$ for some $g \in G$ and $A_1 \leq A$, $B_1 \leq B$; similarly, $G_2 = fA_2f^{-1}$ or $G_2 = fB_2f^{-1}$ for some $f \in G$ and $A_2 \leq A$, $B_2 \leq B$. If $G_1 = gA_1g^{-1}$ and $G_2 = fB_2f^{-1}$, then $a = gsg^{-1} = frf^{-1}$ for some $s \in A_1 \leq A$ and $r \in B_2 \leq B$, as $a \in G_1 \cap G_2$. This contradicts to $G = A * B$. Similarly, $G_1 = gB_1g^{-1}$, $G_2 = fA_2f^{-1}$ is not possible.

Now suppose $G_1 = gA_1g^{-1}$ and $G_2 = fA_2f^{-1}$. If both f and $g \in A$, then $G_1 \leq A$; $G_2 \leq A$, so $G \leq A$, B is trivial. If one of f and g is not contained in A, say f involves non-trivial elements of B, but g does not, then $a \in G_1 \leq A$ and $a \in G_2$ so that $a = frf^{-1}$, contradicting with $G = A * B$. Now if neither f nor g is contained in A, we take any non-trivial element $s \in A$, $s \in G = \langle G_1, G_2\rangle$, so s is equal to a product of ft_if^{-1} and gt_jg^{-1} for some $t_i, t_j \in A$. As f and g involve non-trivial elements of B, this contradicts with $G = A * B$. The case where $G_1 = gB_1g^{-1}$ and $G_2 = fB_2f^{-1}$ can be proved in a similar manner. □

This proposition implies that M is a prime manifold, because if $M = M_1 \sharp M_2$, then $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$ by Van kampen Theorem. Clearly, M is not a 2-sphere bundle over S^1, so it is an irreducible 3-manifold. As the rank of $H_1(M)$ is 5, M is sufficiently large.

4.2 Finding $\mathcal{H}_+(M)$ by $Out(\pi_1(M))$

Denote $Out(\pi_1(M))$ the outer automorphism group of the fundamental group of M. Waldhausen [43] proved that:

When M is closed orientable irreducible 3-manifold which is sufficiently large, the naturally induced homomorphism $\Phi : \mathcal{H}(M) \to Out(\pi_1(M))$ is an isomorphism.

We will analyse $Out(\pi_1(M))$ and find the corresponding orientation-preserving homeomorphisms. We will easily see that none of these homeomorphisms can interchange the two spin structures. Therefore X_1 is not homeomorphic to X_2.

Lemma 4.2. *If Φ is an automorphism of G, then at least one of $\Phi(x_1)$ and $\Phi(y_1)$ is contained in $\langle Cl(x_1), Cl(y_1)\rangle$, where $Cl(x_1) = \{g^{-1}x_1g : g \in G\}$; $Cl(y_1) = \{g^{-1}y_1g : g \in G\}$ are the conjugate classes of x_1 and y_1.*

Proof. Denote H the normal subgroup generated by $g^{-1}x_1^{-1}gx_1$ and $g^{-1}y_1^{-1}gy_1$ for $\forall g \in G$, then $G/H = \langle x_1\rangle \oplus \langle y_1\rangle \oplus \langle a_1, a_2, a_3\rangle$. Let $\bar{x}$ denote the image of x in the quotient G/H. In G/H, $\bar{x}_1, \bar{y}_1$ commute with all elements, so $\overline{\Phi(x_1)} = \bar{x}_1^{m_1}\bar{y}_1^{n_1}f_1$; $\overline{\Phi(y_1)} = \bar{x}_1^{m_2}\bar{y}_1^{n_2}f_2$ for some $f_1, f_2 \in \langle a_1, a_2, a_3\rangle$. Assume that $\Phi(x_1) \notin \langle Cl(x_1), Cl(y_1)\rangle$; $\Phi(y_1) \notin \langle Cl(x_1), Cl(y_1)\rangle$, then f_1 and f_2 are not trivial elements.

Now, $\overline{\Phi(x_1)\Phi(y_1)} = \bar{x}_1^{m_1+m_2}\bar{y}_1^{n_1+n_2}f_1f_2$; $\overline{\Phi(x_1)\Phi(y_1)} = \bar{x}_1^{m_1+m_2}\bar{y}_1^{n_1+n_2}f_2f_1$. In G, $x_1y_1 = y_1x_1$, so $\Phi(x_1)\Phi(y_1) = \Phi(y_1)\Phi(x_1)$. Hence $f_1f_2 = f_2f_1$. Consider the subgroup K of $\langle a_1, a_2, a_3\rangle$ generated by f_1, f_2. Any subgroup of a free group is free, so K is a free group. By considering the abelinization of K, it is easy to see that the rank of $K \leqslant 2$, since K is generated by 2 elements. If the rank of K is 2, as a consequence of Grushko's Theorem, f_1, f_2 generate K freely, which contradicts to the fact $f_1f_2 = f_2f_1$. Thus, the rank of $K = 1$, which implies $f_1 = t^{k_1}, f_2 = t^{k_2}$ for some $t \in \langle a_1, a_2, a_3\rangle$ and k_1, $k_2 \in \mathbb{Z}$.

Assume $\overline{\Phi(a_1)} = \bar{x}_1^{m_3}\bar{y}_1^{n_3}f_3$ for some $f_3 \in \langle a_1, a_2, a_3\rangle$, then

$$\overline{\Phi(x_1)\Phi(a_1)^{-1}\Phi(y_1)\Phi(a_1)} = \bar{x}_1^{m_1+m_2+m_3}\bar{y}_1^{n_1+n_2+n_3}f_1f_3^{-1}f_2f_3;$$

while $$\overline{\Phi(a_1)^{-1}\Phi(y_1)\Phi(a_1)\Phi(x_1)} = \bar{x}_1^{m_1+m_2+m_3}\bar{y}_1^{n_1+n_2+n_3}f_3^{-1}f_2f_3f_1.$$

$x_1a_1^{-1}y_1a_1 = a_1^{-1}y_1a_1x_1$ in G, so $\Phi(x_1)\Phi(a_1)^{-1}\Phi(y_1)\Phi(a_1) = \Phi(a_1)^{-1}\Phi(y_1)\Phi(a_1)\Phi(x_1)$, $f_1f_3^{-1}f_2f_3 = f_3^{-1}f_2f_3f_1$. Therefore the rank of $\langle f_1, f_3^{-1}f_2f_3\rangle = 1 \Longrightarrow$ rank of $\langle f_1, f_2, f_3\rangle = 1$. Thus, we can conclude $f_1 = t^{k_1}, f_2 = t^{k_2}, f_3 = t^{k_3}$ for some $t \in \langle a_1, a_2, a_3\rangle$ and integers k_1, k_2, k_3.

$x_1a_2^{-1}a_1^{-1}y_1a_1a_2 = a_2^{-1}a_1^{-1}y_1a_1a_2x_1$; $x_1a_3^{-1}a_2^{-1}a_1^{-1}y_1a_1a_2a_3 = a_3^{-1}a_2^{-1}a_1^{-1}y_1a_1a_2a_3x_1$ in G, so we can similarly prove that

$$\overline{\Phi(a_2)} = \bar{x}_1^{m_4}\bar{y}_1^{n_4}t^{k_4}; \quad \overline{\Phi(a_3)} = \bar{x}_1^{m_5}\bar{y}_1^{n_5}t^{k_5}.$$

Now we consider $G/[G,G] = \langle x_1\rangle \oplus \langle y_1\rangle \oplus \langle a_1\rangle \oplus \langle a_2\rangle \oplus \langle a_3\rangle$, which is a free abelian group of rank 5. The abelinization of G is the same as the abelinization of G/H. G is generated by $\{\Phi(x_1), \Phi(y_1), \Phi(a_1), \Phi(a_2), \Phi(a_3)\}$, but rank of $\dfrac{\langle \Phi(x_1), \Phi(y_1), \Phi(a_1), \Phi(a_2), \Phi(a_3)\rangle}{[G,G]} \leqslant 3$. A contradiction. Therefore, at least one of $\Phi(x_1)$ and $\Phi(y_1) \in \langle Cl(x_1), Cl(y_1)\rangle$. □

Lemma 4.3. *If $I := \{\overline{sxs^{-1}}, s \in \langle a_1, ..., a_n\rangle\}$ is a free group, $u, v \in I$, v is nontrivial, and satisfies: $uv^k = dv^ld^{-1}u$ for some $d \in \langle a_1, ..., a_n\rangle$ and integers k and l, then $d = 1$ and $uv = vu$.*

Proof. If u is trivial, It is obvious. Let us assume u is not trivial.

If v is not cyclically reduced, $v = sv_0s^{-1}$ for some nontrivial element $s \in I$, where v_0 is cyclically reduced, i.e., the first letter of v_0 is not the inverse of the last letter of v_0. Then $usv_0^ks^{-1}u^{-1} = dsd^{-1}dv_0^ld^{-1}ds^{-1}d^{-1}$. Thus, $ds^{-1}d^{-1}usv_0^ks^{-1}u^{-1}d^{-1}dsd^{-1} = dv_0^ld^{-1}$. Denote $ds^{-1}d^{-1}us$ by u_0, we have $u_0v_0^ku_0^{-1} = dv_0^ld^{-1}$. Therefore, WLOG, we can assume that v is cyclically reduced.

Since $uv^ku^{-1} = dv^ld^{-1}$, u or u^{-1} must be completely cancelled out to make dv^ld^{-1} cyclically reduced ($d \in I$). If the last letter of u cancels the first letter of v, then the first letter of u^{-1} can not be cancelled with the last letter of v and vice verse, since v is cyclically reduced. Therefore, $k = l$. Assume $v^k = e_1xe_1^{-1}e_2xe_2^{-1}...e_mxe_m^{-1}$, where $e_i \in \langle a_1, ..., a_n\rangle$, then $u = e_px^{-1}e_p^{-1}e_{p-1}x^{-1}e_{p-1}^{-1}...e_1x^{-1}e_1^{-1}$ for some $p < m$; $uv^ku^{-1} = e_{p+1}xe_{p+1}^{-1}e_{p+2}xe_{p+2}^{-1}...e_mxe_m^{-1}e_1xe_1^{-1}...e_pxe_p^{-1}$. Thus, $de_i = e_{\sigma(i)}$, where σ is the permutation on $\{1, 2, ..., m\}$. Now it is easy to check that d must be identity. Thus, $uv^k = v^ku$. The subgroup generated by u and v is a free group and the rank of this group is 1. So, $u = t^{k_1}$, and $v = t^{k_2}$ for some $t \in I$ and some integers k_1 and k_2. Therefore, $uv = vu$. □

Proposition 4.2. *If Φ is an automorphism of G, then $\Phi(x_1) = f^{-1}x_1^{\varepsilon}f$; $\Phi(y_1) = g^{-1}y_1^{\varepsilon}g$ or $\Phi(x_1) = f^{-1}y_1^{\varepsilon}f$; $\Phi(y_1) = g^{-1}x_1^{\varepsilon}g$ for $\varepsilon = \pm 1$ and $f, g \in G$.*

Proof. By the first lemma, WLOG, we can assume $\Phi(y_1) \in \langle Cl(x_1), Cl(y_1)\rangle$. Denote J the normal subgroup generated by $[x_1, t^{-1}y_1t]$ for $\forall t \in \langle a_1, a_2, a_3\rangle$, note that J also contains $[s^{-1}x_1s, t^{-1}y_1t]$ for $\forall s, t \in \langle a_1, a_2, a_3\rangle$. In G/J, the the conjugates of x_1 commute with the conjugates of y_1, however, $s_1^{-1}x_1s_1$ does not commute with $s_2^{-1}x_1s_2$ unless $s_1 = s_2$. Consider the subgroup I_1 of G, $I_1 := \{s^{-1}x_1s, s \in \langle a_1, a_2, a_3\rangle\}$. It is a free group generated by $\{s^{-1}x_1s\}$, where $s \in \langle a_1, a_2, a_3\rangle$ and the last letter of s is neither a_3 nora_3^{-1}, since $a_3x_1a_3^{-1} = x_1a_2^{-1}a_1^{-1}(a_1x_1a_1^{-1})^{-1}x_1a_1a_2 = x_1(a_2^{-1}x_1a_2)^{-1}a_2^{-1}a_1^{-1}x_1a_1a_2$ and $a_3^{-1}x_1a_3 = x_1a_3^{-1}a^{-2}x_1a_2a_3(a_3^{-1}a_2^{-1}a_1^{-1}x_1a_1a_2a_3)^{-1}$ by (‡). Similarly, The subgroup $I_2 := \{t^{-1}y_1t, t \in \langle a_1, a_2, a_3\rangle\}$ of G is also a free group.

Assume in G/J, $\overline{\Phi(y_1)} = uv$ where $u \in \bar{I_1}, v \in \bar{I_2}$; $\overline{\Phi(x_1)} = wzd$ where $w \in \bar{I_1}, z \in \bar{I_2}, d \in \langle a_1, a_2, a_3\rangle$. $\overline{\Phi(x_1)\Phi(y_1)} = \overline{\Phi(y_1)\Phi(x_1)} \Rightarrow wzduv = uvwzd$. Since $\bar{I_1} \cup \bar{I_2}$ is normal in G/J, we are able to move the elements in $\langle a_1, a_2, a_3\rangle$ after the elements in $\bar{I_1}$ and $\bar{I_2}$.

$wzduv = wzu'v'd = wu'zv'd$, where $u' = dud^{-1}, v' = dvd^{-1}$ and $uvwzd = uwvzd$ in G/J. Since $\bar{I_1} \cap \bar{I_2} = \{1\}$, $wu' = uw$, $zv' = vz$. By last lemma and the freeness of $\bar{I_1}$ and $\bar{I_2}$, d must be trivial and $uw = wu$; $vz = zv$. Thus, $\Phi(x_1) \in \langle Cl(x_1), Cl(y_1)\rangle$.

$\Phi(a_1) \notin \langle Cl(x_1), Cl(y_1)\rangle$, otherwise, we would reach a contradiction by counting the rank of the abelinization of G. So we may assume that in G/J, $\overline{\Phi(a_1)} = pqr$ where $p \in \bar{I_1}, q \in \bar{I_2}, r \in \langle a_1, a_2, a_3\rangle$ and r is nontrivial. $\overline{\Phi(a_1^{-1})\Phi(y_1)\Phi(a_1)} = r^{-1}p^{-1}upq^{-1}vqr$ and $\overline{\Phi(x_1)\Phi(a_1^{-1})\Phi(y_1)\Phi(a_1)} = \overline{\Phi(a_1^{-1})\Phi(y_1)\Phi(a_1)\Phi(x_1)}$, so, $r^{-1}p^{-1}uprw = wr^{-1}p^{-1}upr$; $r^{-1}q^{-1}vqrz = zr^{-1}q^{-1}vqr$. The rank of $\langle u, w\rangle = 1$, so $(r^{-1}p^{-1}r)(r^{-1}t^kr)(r^{-1}pr) = t^l$ for some $t \in \bar{I_1}$ and integers k and l. By last lemma, if r is nontrivial, one of u and w must be trivial; similarly, one of v and z must be trivial. $u = 1, v = 1$ or $w = 1, z = 1$ is not possible since if $\overline{\Phi(y_1)} = 1$ or $\overline{\Phi(x_1)} = 1$, then the image of $\Phi(y_1)$ or $\Phi(x_1)$ would be trivial in the abelinization of G. A contradiction. Thus, $u = 1, z = 1$ or $w = 1, v = 1$, which means $\overline{\Phi(x_1)} \in \bar{I_1}$ or $\bar{I_2}$; if $\overline{\Phi(x_1)} \in \bar{I_1}$, then $\overline{\Phi(y_1)} \in \bar{I_2}$ and vice verse.

Consider $I = \langle I_1, I_2\rangle$, the subgroup of G generated by I_1 and I_2. Φ induces an automorphism on I. $\overline{\Phi(x_1)} \in \bar{I_1}$ or $\bar{I_2}$, so $\Phi(x_1) \in I$ as a product of elements in I_1 and I_2. We can further assume $\Phi(x_1)$ is cyclically reduced by composing Φ with an appropriate inner automorphism. The rank of the ablelianization of the centralizer of x_1 in I is 4 (generated by $\{x_1, y_1, y_2, y_6\}$); the only cyclically reduced elements in I with this property are x_1, y_1, x_1^{-1} and y_1^{-1}. Therefore, $\Phi(x_1) = f^{-1}x_1^{\varepsilon}f$; or $\Phi(x_1) = f^{-1}y_1^{\varepsilon}f$ for some $f \in G$; similarly, $\Phi(y_1) = g^{-1}y_1^{\varepsilon}g$ or $\Phi(y_1) = g^{-1}x_1^{\varepsilon}g$ for some $g \in G$. □

Since we are only interested in the outer automorphism of G, by composing with appropriate innner automorphisms, we can assume that $\Phi(x_1) = x_1$ or $\Phi(x_1) = x_1^{-1}$ or $\Phi(x_1) = y_1$ or $\Phi(x_1) = y_1^{-1}$.

Case I. $\Phi(x_1) = x_1$.

In this case, $\Phi(y_1)$ is contained in the centralizer of x_1, denoted by $C_G(x_1)$, which is generated by $\{x_1, y_1, y_2, y_6, y_8\}$. By last proposition, we know $\Phi(y_1) = g^{-1}y_i^{\varepsilon}g$, where $\varepsilon = \pm 1$, $g \in \dfrac{C_G(X_1)}{\langle x_1 \rangle} = \langle y_1, y_2, y_6, y_8 \rangle$ and i =1 or 2 or 6 or 8.

First, let us assume Φ induces a permutation on $\{y_1, y_2, y_6, y_8, y_1^{-1}, y_2^{-1}, y_6^{-1}, y_8^{-1}\}$. Under this assumption, we consider the possible images of y_1 under Φ. If Φ carries y_1 to y_8, then $\Phi(y_2) = y_6$, $\Phi(y_6) = y_2$, $\Phi(y_8) = y_1$ by (†), thereby $\Phi(y_8^{-1}y_6y_2^{-1}y_1) = y_1^{-1}y_2y_6^{-1}y_8 = (y_8^{-1}y_6y_2^{-1}y_1)^{-1}$. However, $\Phi(x_1) = x_1$. So Φ is induced by an orientation-reversing homeomorphism. For the same reason, we can exclude the other 3 cases: $\Phi(y_1) = y_2$; $\Phi(y_1) = y_1^{-1}$; $\Phi(y_1) = y_6^{-1}$. Now, we have 4 possibilities left: $\Phi(y_1) = y_1$; $\Phi(y_1) = y_6$; $\Phi(y_1) = y_2^{-1}$; $\Phi(y_1) = y_8^{-1}$. Next we consider 4 particular automorphisms corresponding to each possibility.

(S1). Assume that $\Phi(x_i) = x_i$ for $i = 1, 5, 6, 7$ and $\Phi(y_i) = y_i$ for $i = 1, 2, 6, 8$. Now We want to find $\Phi(a_1)$. Since $\Phi(y_2) = y_2$. $\Phi(y_2) = \Phi(a_1^{-1})\Phi(y_1)\Phi(a_1) = \Phi(a_1)^{-1}y_1\Phi(a_1) = y_2 = a_1^{-1}y_1a_1$. So, $\Phi(a_1)^{-1}a_1a_1^{-1}\Phi(y_1)a_1a_1^{-1}\Phi(a_1) = a_1^{-1}y_1a_1$, $[a_1^{-1}\Phi(a_1), y_2] = 1$.

Similarly, $\Phi(x_5) = \Phi(a_1)\Phi(x_1)\Phi(a_1)^{-1} = \Phi(a_1)x_1\Phi(a_1)^{-1} = x_5 = a_1x_1a_1^{-1}$

So, $a_1^{-1}\Phi(a_1)\Phi(x_1)\Phi(a_1)^{-1}a_1 = x_1$, $[a_1^{-1}\Phi(a_1), x_1] = 1$.

The centralizer of x_1 is generated by

$$\{x_1, y_1, y_2, y_6, y_8\}; \qquad (y_8 \text{ is not necessary since } y_8 = y_6y_2^{-1}y_1)$$

the centralizer of y_2 is generated by

$$\{y_2, a_1^{-1}x_1a_1, a_1^{-1}x_5a_1 = x_1, a_1^{-1}x_6a_1, a_1^{-1}x_7a_1\}. \ (a_1^{-1}x_7a_1 = a_1^{-1}x_6a_1x_1^{-1}a_1^{-1}x_1a_1)$$

Since $a_1^{-1}\Phi(a_1)$ commutes with both x_1 and y_2,

$$a_1^{-1}\Phi(a_1) = x^{m_1}y_2^{n_1} = x^{m_1}a_1^{-1}y_1^{n_1}a_1 = a_1^{-1}y_1^{n_1}a_1x^{m_1}, \quad \text{for } m_1, n_1 \in \mathbb{Z}.$$

$$\text{Thus, } \Phi(a_1) = y_1^{n_1}a_1x_1^{m_1}.$$

Similarly, we can derive $[a_2^{-1}a_1^{-1}\Phi(a_1)\Phi(a_2), y_6] = 1$, $[a_2^{-1}a_1^{-1}\Phi(a_1)\Phi(a_2), x_1] = 1$, so

$$a_2^{-1}a_1^{-1}\Phi(a_1)\Phi(a_2) = a_2^{-1}a_1^{-1}y_1^{n_2}a_1a_2x_1^{m_2}, \quad \text{for } m_2, n_2 \in \mathbb{Z}$$

$$\text{Therefore, } \Phi(a_2) = x_1^{-m_1}a_1^{-1}y_1^{n_2-n_1}a_1a_2x_1^{m_2} = y_2^{n_2-n_1}x_1^{-m_1}a_2x_1^{m_1}x_1^{m_2-m_1}.$$

Also $[a_3^{-1}a_2^{-1}a_1^{-1}\Phi(a_1)\Phi(a_2)\Phi(a_3), y_8] = 1$, $[a_3^{-1}a_2^{-1}a_1^{-1}\Phi(a_1)\Phi(a_2)\Phi(a_3), x_1] = 1$,

$$a_3^{-1}a_2^{-1}a_1^{-1}\Phi(a_1)\Phi(a_2)\Phi(a_3) = a_3^{-1}a_2^{-1}a_1^{-1}y_1^{n_3}a_1a_2a_3x_1^{m_3}, \text{ for } m_3, n_3 \in \mathbb{Z}. \text{ So,}$$

$$\Phi(a_3) = x_1^{-m_2}a_2^{-1}a_1^{-1}y_1^{n_1-n_2}a_1x_1^{m_1}x_1^{-m_1}a_1^{-1}y_1^{-n_1}y_1^{n_3}a_1a_2a_3x_1^{m_3} =$$

$$y_6^{n_3-n_2}x_1^{-m_2}a_3x_1^{m_2}x_1^{m_3-m_2}.$$

This kind of automorphisms are induced by the diffeomorphisms of doing the following moves (move 1.1 and 1.2).

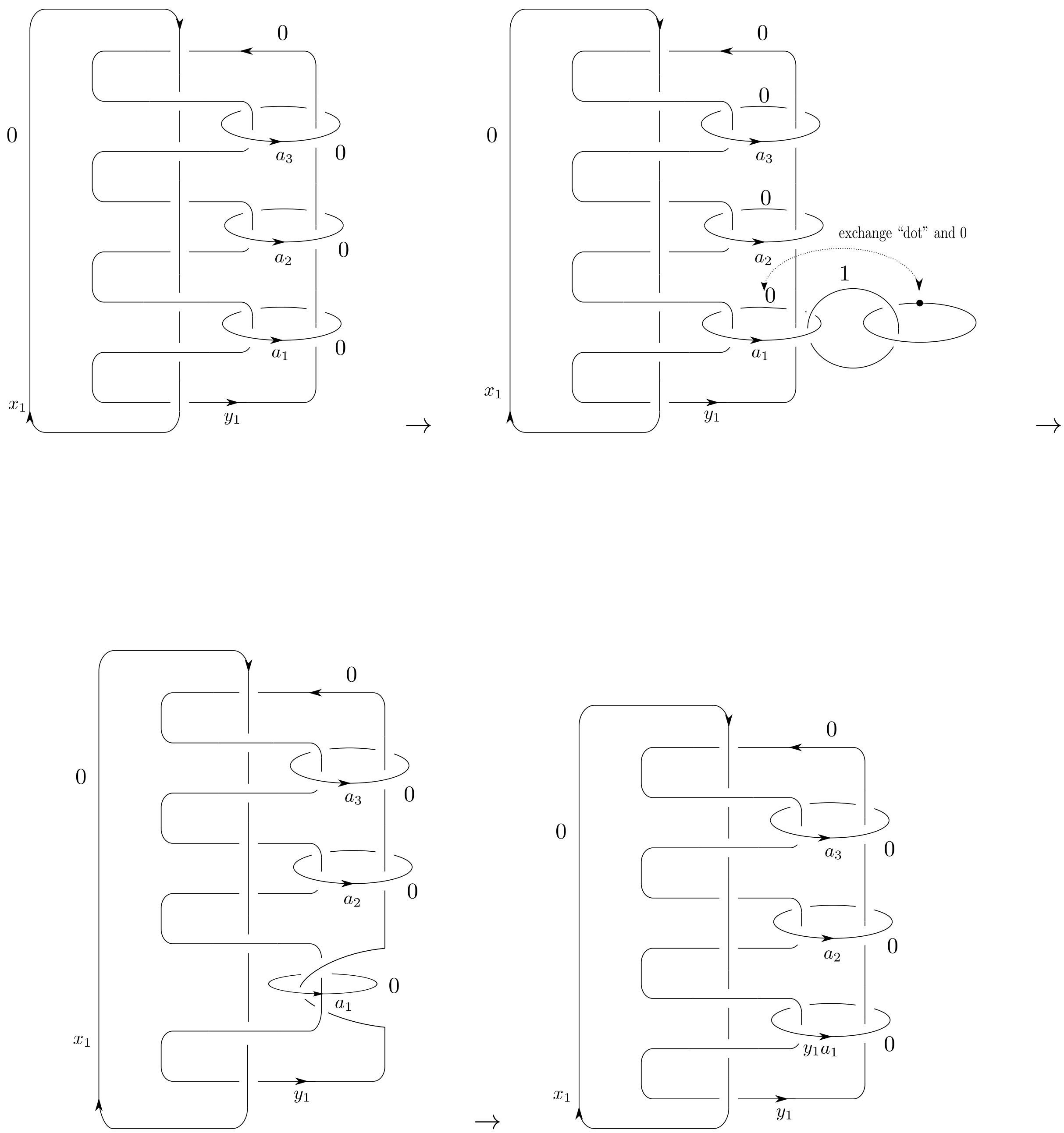


Figure 4.4: Move 1.1

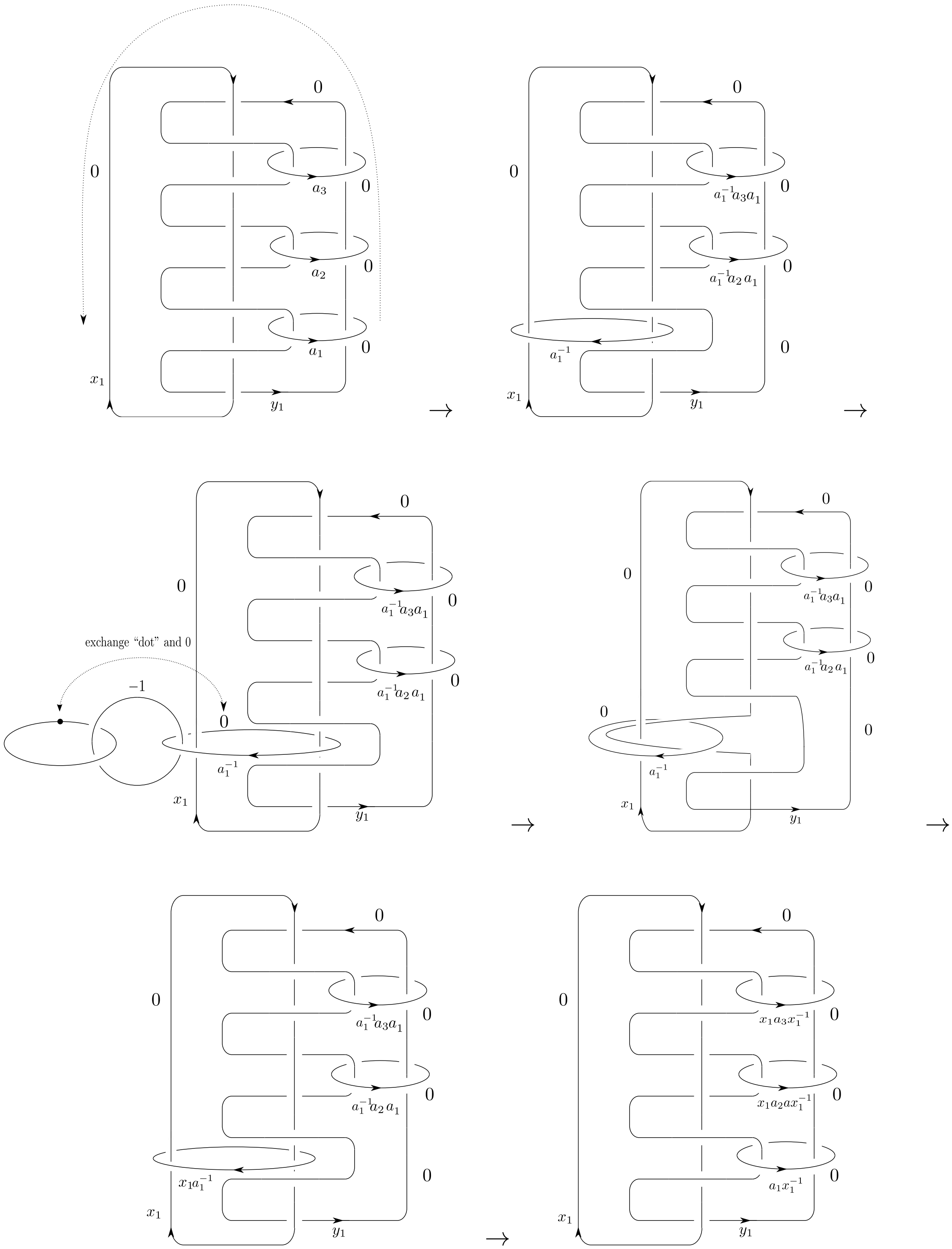


Figure 4.5: Move 1.2

(S2). Assume that $\Phi(y_1) = y_6$, $\Phi(y_2) = y_8, \Phi(y_6) = y_1, \Phi(y_8) = y_2$ and $\Phi(x_1) = x_1$, $\Phi(x_5) = a_3x_1a_3^{-1}, \Phi(x_6) = a_2^{-1}a_1^{-1}x_1a_1a_2, \Phi(x_7) = a_2^{-1}x_1a_2$.

Now, $\Phi(y_2) = \Phi(a_1^{-1})\Phi(y_1)\Phi(a_1) = \Phi(a_1)^{-1}y_6\Phi(a_1) = \Phi(a_1)^{-1}a_3y_8a_3^{-1}\Phi(a_1) = y_8$, so $[a_3^{-1}\Phi(a_1), y_8] = 1$; $\Phi(x_5) = \Phi(a_1)x_1\Phi(a_1)^{-1} = a_3x_1a_3^{-1}$, so $[a_3^{-1}\Phi(a_1), x_1] = 1$. Thus, $a_3^{-1}\Phi(a_1) = y_8^{n_4}x_1^{m_4}$ for $m_4, n_4 \in \mathbb{Z}$; $\Phi(a_1) = y_6^{n_4}a_3x_1^{m_4}$.

Similarly, we can derive $[a_1a_2\Phi(a_1)\Phi(a_2), y_1] = 1$ and $[a_1a_2\Phi(a_1)\Phi(a_2), x_1] = 1$,

$$\text{So} \quad a_1a_2\Phi(a_1)\Phi(a_2) = y_1^{n_5}x_1^{m_5} \quad \text{for } m_5, n_5 \in \mathbb{Z};$$

$$\Phi(a_2) = x_1^{-m_4}a_3^{-1}y_6^{-n_4}a_2^{-1}a_1^{-1}y_1^{n_5}x_1^{m_5} = y_8^{n_5-n_4}x_1^{-m_4}a_3^{-1}a_2^{-1}a_1^{-1}x_1^{m_5}.$$

Also, $[a_2\Phi(a_1)\Phi(a_2)\Phi(a_3), y_2] = 1$ and $[a_2\Phi(a_1)\Phi(a_2)\Phi(a_3), x_1] = 1$,

$$\text{So} \quad a_2\Phi(a_1)\Phi(a_2)\Phi(a_3) = y_2^{n_6}x_1^{m_6} \quad \text{for } m_6, n_6 \in \mathbb{Z};$$

$$\Phi(a_3) = y_1^{n_6-n_5}x_1^{-m_5}a_1x_1^{m_6}.$$

Note that, $\Phi(a_1) = a_3, \Phi(a_2) = a_3^{-1}a_2^{-1}a_1^{-1}, \Phi(a_3) = a_1$ is a special solution which is exactly induce by the diffeomorphism of doing the following move (move 2: wind y_8 around x_1 counter-clockwisely) twice. The general solutions are just induced by the composition of this diffeomorphism and the diffeomorphisms of doing move 1.1 and 1.2.

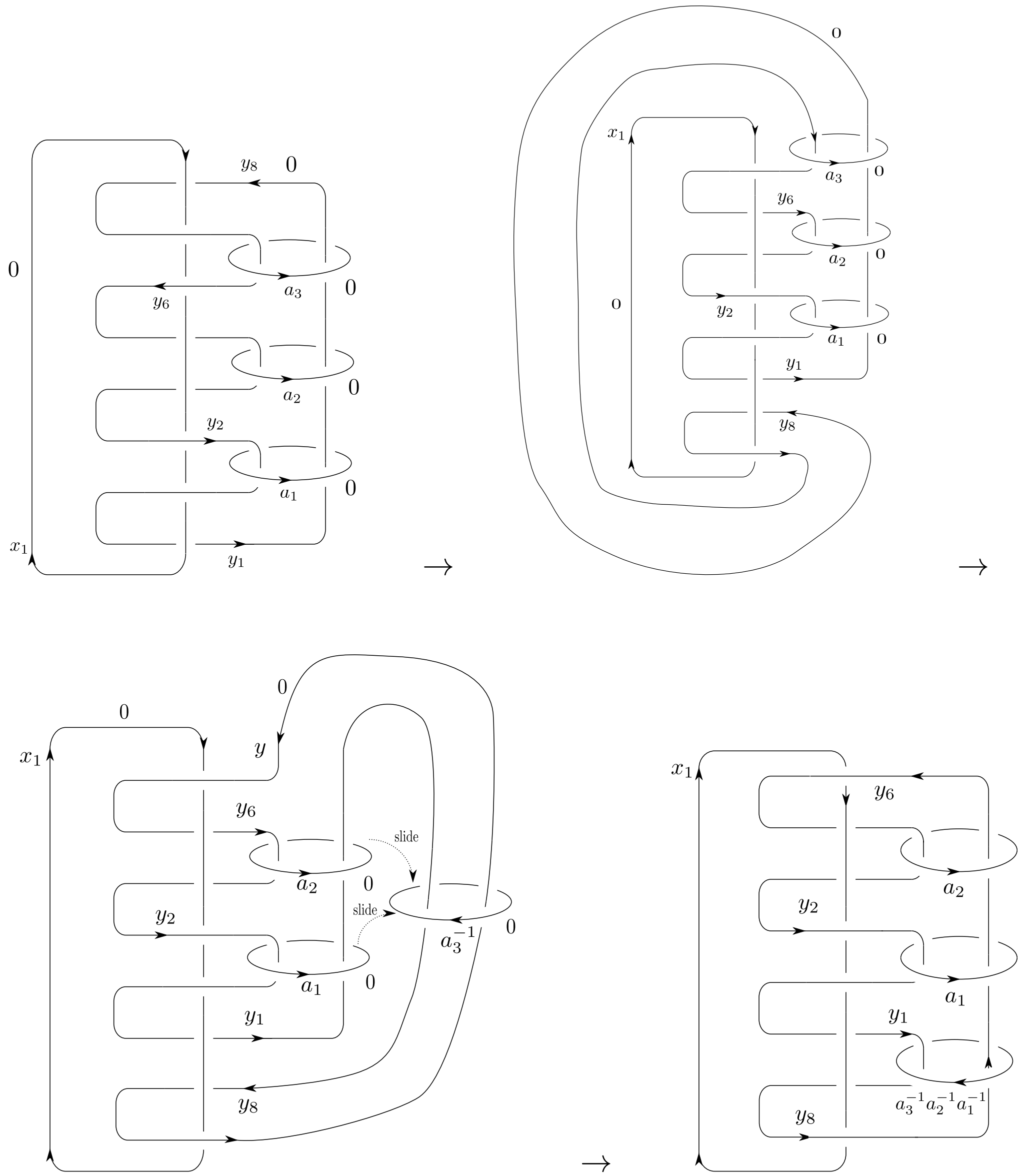


Figure 4.6: Move 2

For simplicity, from now on, we will modular diffeomorphisms induced by move 1.1 and 1.2, i.e., the automorphism Φ is uniquely determined by the image of $\Phi(x_i)$ and $\Phi(y_j)$ for $i = 1, 5, 6, 7$ and $j = 1, 2, 6, 8$ up to composition with Φ_1, the automorphism induced by move 1.1 and 1.2.

(S3). Assume that $\Phi(y_1) = y_2^{-1}$, $\Phi(y_2) = y_6^{-1}, \Phi(y_6) = y_8^{-1}, \Phi(y_8) = y_1^{-1}$ and $\Phi(x_1) = x_1$, $\Phi(x_5) = x_1 a_2 x_1 a_2^{-1} x_1^{-1}, \Phi(x_6) = x_1 a_2 x_1^{-1} a_3 x_1 a_3^{-1} x_1 a_2^{-1} x_1^{-1}, \Phi(x_7) = a_1^{-1} x_1 a_1$. By similar calculations, we can check that $\Phi(a_1) = x_1 a_2 x_1^{-1}$, $\Phi(a_2) = a_3$, $\Phi(a_3) = a_3^{-1} x_1 a_2^{-1} x_1^{-1} a_1^{-1}$. This is induced by the diffeomorphism of doing move 2 then followed by the following move (move 3).

Note that the actual diffeomorphism induced is the inverse of what it is look like from the picture.

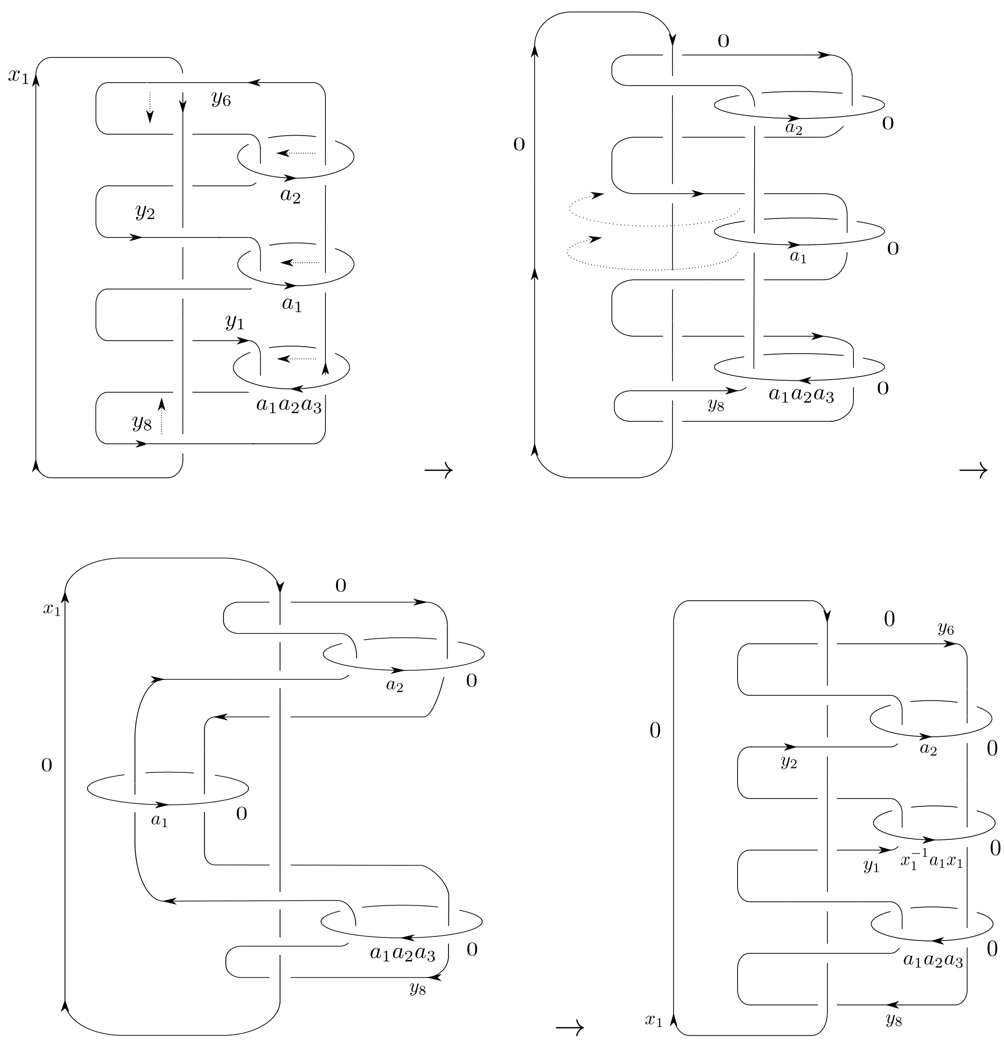


Figure 4.7: Move 3

(S4). Assume that $\Phi(y_1) = y_8^{-1}$, $\Phi(y_2) = y_1^{-1}, \Phi(y_6) = y_2^{-1}, \Phi(y_8) = y_6^{-1}$ and $\Phi(x_1) = x_1$, $\Phi(x_5) = a_3^{-1}x_1a_2^{-1}x_1^{-1}a_1^{-1}x_1a_1x_1a_2x_1^{-1}a_3$, $\Phi(x_6) = a_3^{-1}x_1a_2^{-1}x_1a_2x_1^{-1}a_3$, $\Phi(x_7) = a_3^{-1}x_1a_3$. By similar calculations, we can check that $\Phi(a_1) = a_3^{-1}x_1a_2^{-1}x_1^{-1}a_1^{-1}$, $\Phi(a_2) = a_1$, $\Phi(a_3) = x_1a_2x_1^{-1}$. This is induced by the diffeomorphism of doing the inverse move 2 (i.e., wind y_1 around x_1 clockwisely), then followed by move 3.

Now, let us analyse the general case, i.e., $\Phi(y_1) = g^{-1}y_i^{\varepsilon}g$, for some $g \in \dfrac{C_G(X_1)}{\langle x_1\rangle} = \langle y_1, y_2, y_6, y_8\rangle$ and $\varepsilon = \pm 1$, i =1 or 2 or 6 or 8. By applying the 4 types of diffeomorphisms described above, we can therefore assume that $\Phi(y_1) = g^{-1}y_1g$, $g \in \langle y_1, y_2, y_6, y_8\rangle$.

In the following diffeomorphism (move 4, which we denote by Φ_4), the "arm" y_1 winds around x_1 counter-clockwisely and go over all the other "arms": y_2, y_6, y_8. The effect of this diffeomorphism on G is sending every element α of G to $y_1^{-1}\alpha y_1$ (if y_1 winds around x_1 clockwisely, α would be sent to $y_1\alpha y_1^{-1}$). Similarly, there exists diffeomorphisms that send α to $y_i^{-1}\alpha y_i$ and $y_i\alpha y_i^{-1}$, for each $i = 2, 6, 8$. Thus, we can just assume $\Phi(y_1) = y_1$. Then in $\langle y_1, y_2, y_6, y_8\rangle^{ab}$, Φ induces a permutation on $\{\bar{y_2}, \bar{y_6}, \bar{y_8}\}$.

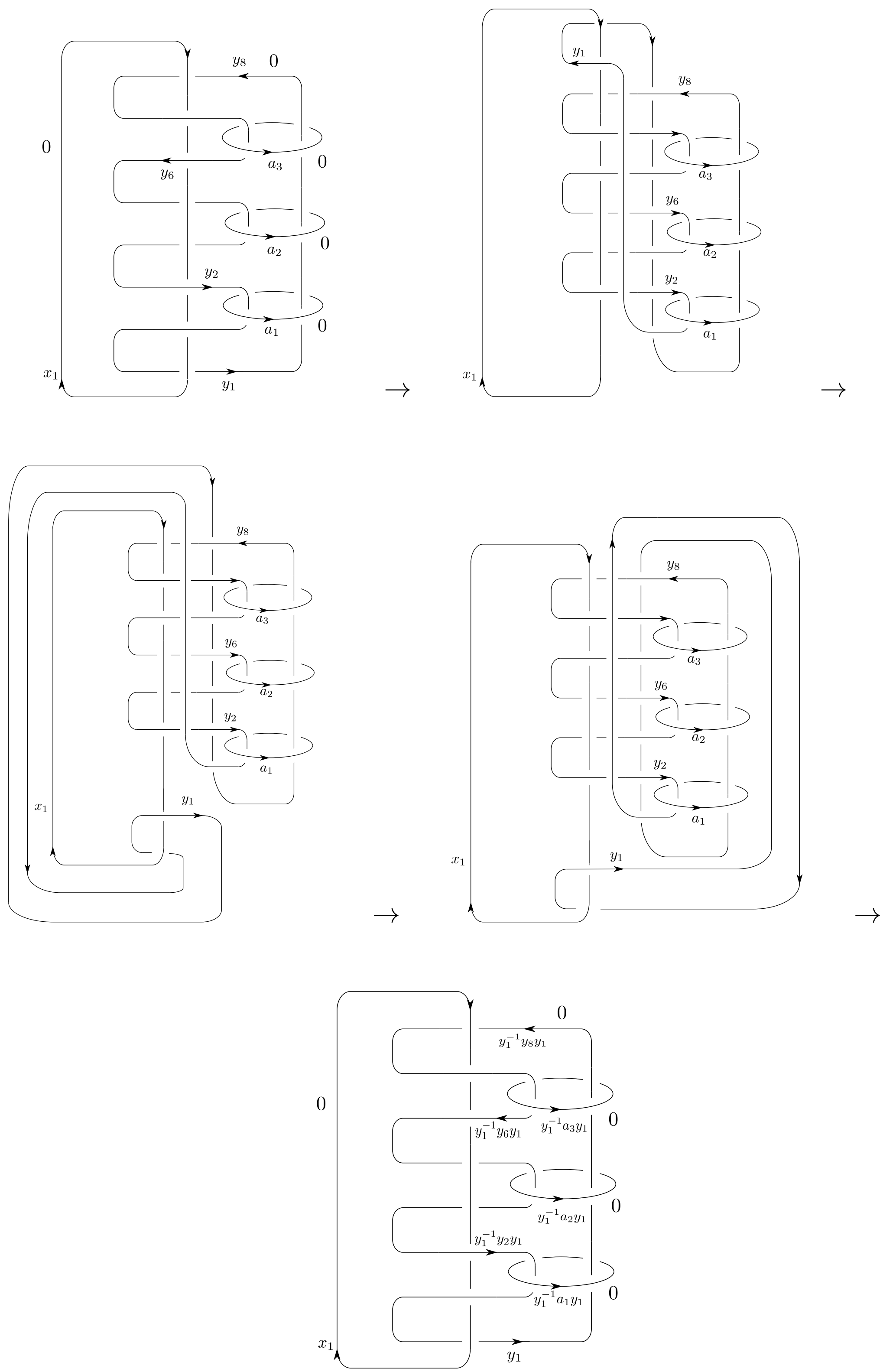


Figure 4.8: Move 4

For simplicity, we denote y_1, y_2^{-1}, y_6 and y_8 by a, b, c and cba respectively. We know that $\Phi(a) = a$ and $\Phi(cba) = \Phi(c)\Phi(b)a$.

(i) If $\Phi(b) = t_1^{-1}bt_1$, $\Phi(c) = t_2^{-1}ct_2$, $\Phi(cba) = t_3^{-1}cbat_3$ for some $t_1, t_2, t_3 \in \langle a, b, c\rangle$, we get $t_3^{-1}cbat_3 = t_2^{-1}ct_2t_1^{-1}bt_1a$ $(\star)$. WOLG, we may assume that $t_1^{-1}bt_1$, $t_2^{-1}ct_2$, t_3 are reduced.

(ii) If $\Phi(b) = t_3^{-1}a^{-1}b^{-1}c^{-1}t_3$, $\Phi(c) = t_2^{-1}ct_2$, $\Phi(cba) = t_1^{-1}b^{-1}t_1$ for some $t_1, t_2, t_3 \in \langle a, b, c\rangle$ (i.e., $\Phi(\bar{y_2}) = \bar{y_8}$, $\Phi(\bar{y_6}) = \bar{y_6}$, $\Phi(\bar{y_8}) = \bar{y_2}$), then by compositing the following diffeomorphism, this is equivalent to the previous case. Note that $\Phi(y_1) = y_1$, so by considering the homology, $\Phi(\bar{y_i})$ can not $=\bar{y_j}^{-1}$.

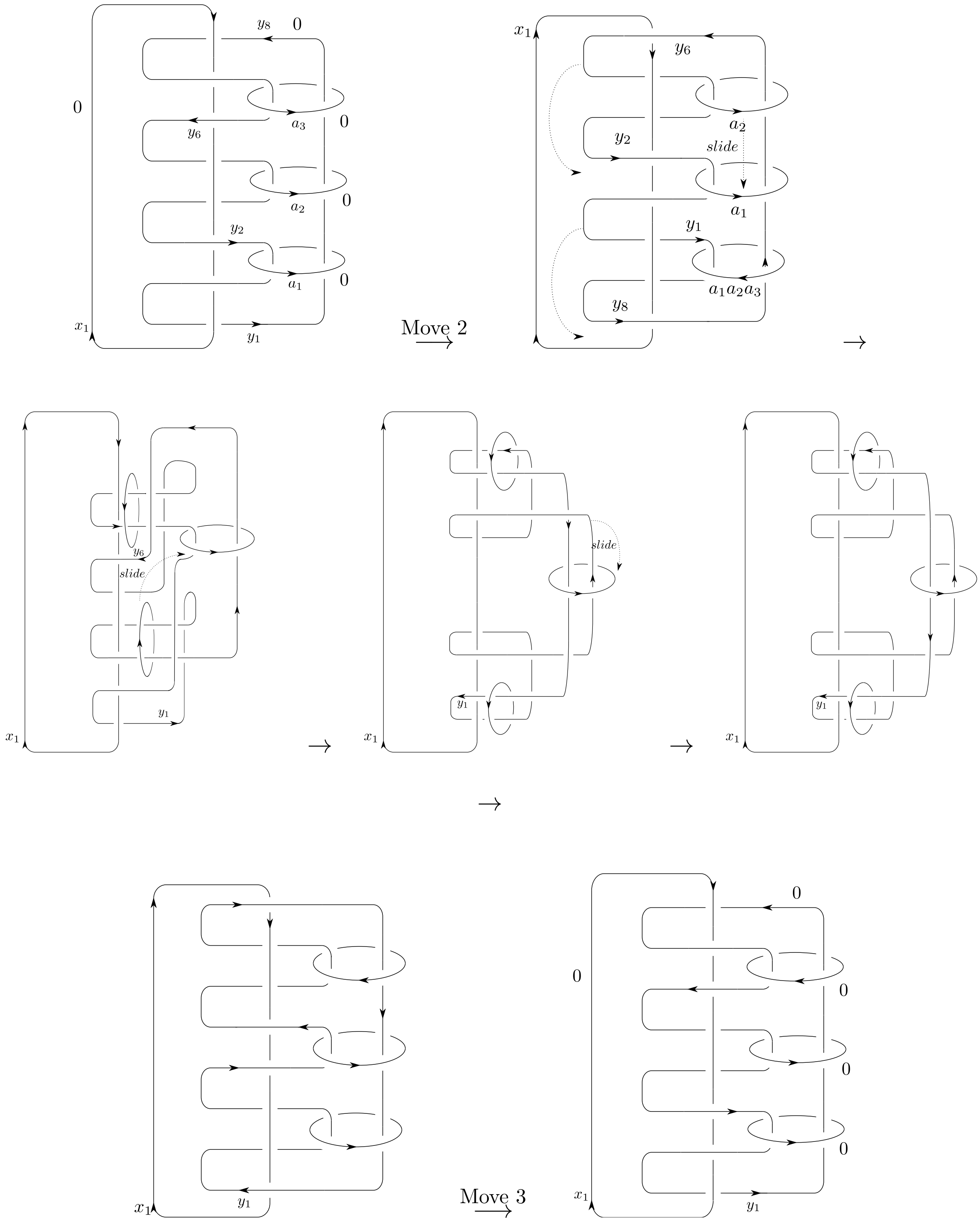


Figure 4.9: Move 5

(iii) If $\Phi(b) = t_1^{-1}bt_1$, $\Phi(c) = t_3^{-1}cbat_3$; $\Phi(cba) = t_2^{-1}ct_2$ (i.e., $\Phi(\bar{y_2}) = \bar{y_2}$, $\Phi(\bar{y_6}) = \bar{y_8}$, $\Phi(\bar{y_8}) = \bar{y_6}$), then we have $t_2^{-1}ct_2 = t_3^{-1}cbat_3t_1^{-1}bt_1a$. Clearly, this is not possible. Similarly, we can check the other 3 permutations are not possible.

Now, our goal is to show that every possible solution of equation $(\star)$ can be obtained by diffeomorphisms. Note that by applying Φ_4, we can assume that the last letter of t_1 is neither a or a^{-1} so that there is no cancellation between t_1 and a. Either $t_2^{-1}ct_2$ is cancelled out by $t_1^{-1}bt_1a$ or vice verse or there is no cancellation between t_2^{-1} and t_1, since if neither $t_2^{-1}ct_2$ or $t_1^{-1}bt_1a$ is cancelled out, the last letter of t_3 must be a, so that the last letter of t_2 is also a. Denote the length of t_i by $l(t_i)$, $i = 1, 2, 3$.

(1) If there is no cancellation between t_2 and t_1^{-1}, then $l(t_2) \neq l(t_1)$. Since if $l(t_2) = l(t_1)$, by comparing both sides of the equation, we know that the last letter of t_3 is a, however, the last letter of t_2 is c and so is t_3. A contradiction. So, in this case, $l(t_3) \geq l(t_2) + l(t_1)$.

(1a) If $l(t_3) \geq l(t_1) > l(t_2)$, then $t_3^{-1} = t_2^{-1}ct_2t_4^{-1}$ for some $t_4 \in \langle a, b, c\rangle$, so $t_4^{-1}cbat_4t_2^{-1}c^{-1}t_2a^{-1} = t_1^{-1}bt_1$. Note that the last letter of t_2 is a, we denote t_2a^{-1} by t_5. Then we have $t_4^{-1}cbat_4a^{-1}t_5^{-1}c^{-1}t_5 = t_1^{-1}bt_1$, and there is no cancellation between t_4^{-1} and a^{-1}. Clearly, $l(t_4) + 1 \neq l(t_5)$, since otherwise the middle letter on the left hand side is a^{-1}, while the middle letter on the right hand side is b.

(1aa) Now, if $l(t_4) + 1 > l(t_5)$, $t_1 = t_6t_5^{-1}c^{-1}t_5$ for some $t_6 \in \langle a, b, c\rangle$, then we have $t_4^{-1}cbat_4 = t_5^{-1}ct_5t_6^{-1}bt_6a$. So we are back to the original from: $t_4 = t_3t_2^{-1}ct_2$ plays the role of t_3; $t_5 = t_2a^{-1}$ plays the role of t_2; $t_6 = t_1at_2^{-1}ct_2a^{-1}$ plays the role of t_1 and we know that $l(t_4) < l(t_3)$; $l(t_5) < l(t_2)$; $l(t_6) < l(t_1)$. This transformation is induced by the the diffeomorphism dragging y_6 around x_1 counter-clockwisely and go over y_8, underneath y_1, and over y_2 as described in the following picture.

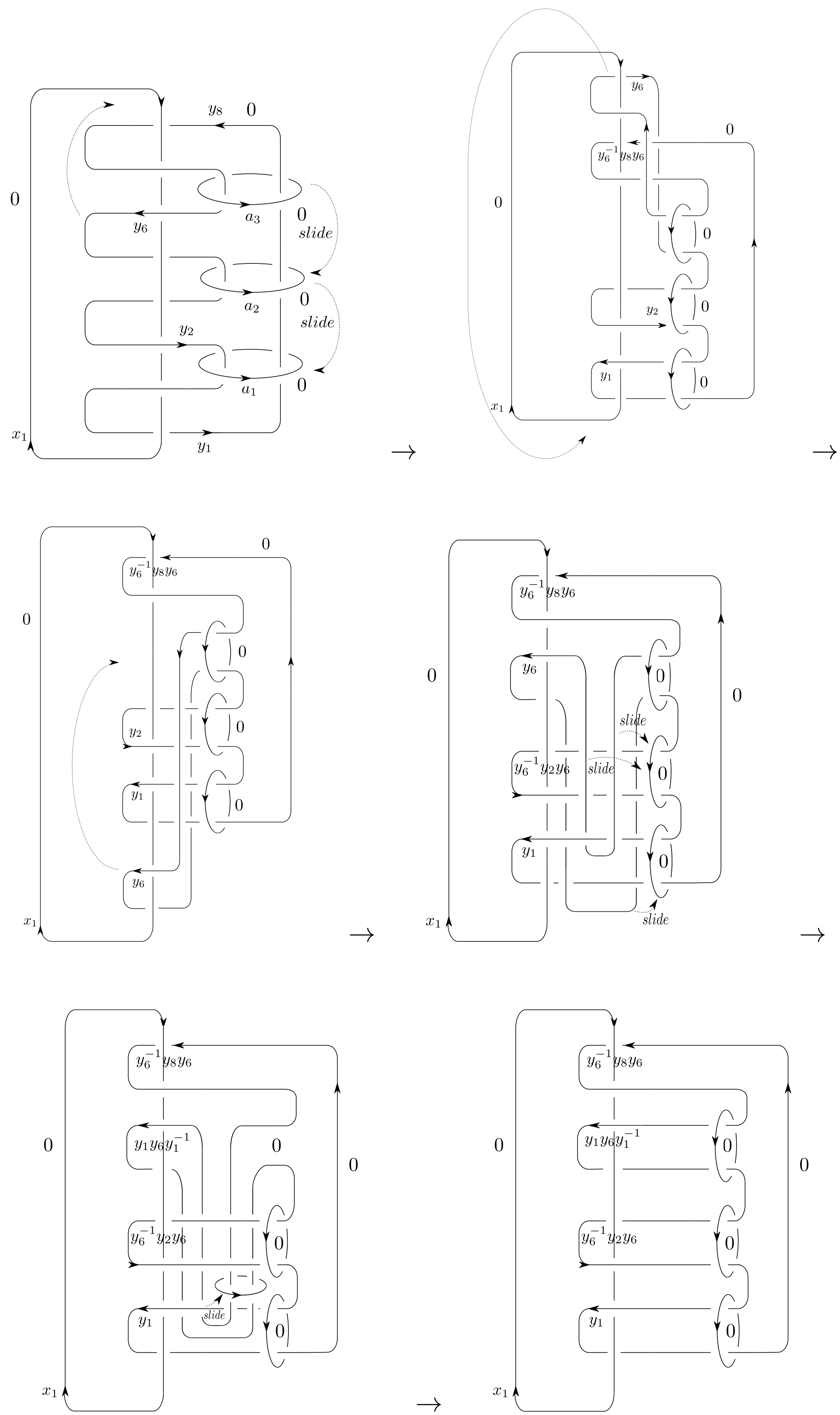


Figure 4.10: Move 6

(1ab) If $l(t_4)+1 < l(t_5)$, $t_1^{-1} = t_4^{-1}cbat_4t_6^{-1}$ for some $t_6 \in \langle a,b,c\rangle$. Therefore,

$$a^{-1}t_5^{-1}c^{-1}t_5 = t_6^{-1}bt_6t_4^{-1}(cba)^{-1}t_4.$$

Note that the first letter of t_6^{-1} must be a^{-1}, so $t_4^{-1}cbat_4 = t_5^{-1}ct_5t_7^{-1}bt_7a$, where $t_7 = t_6a^{-1}$. Now, $t_4 = t_3t_2^{-1}ct_2$ plays the role of t_3; $t_5 = t_2a^{-1}$ plays the role of t_2; $t_7 = t_1t_4^{-1}(cba)^{-1}t_4a^{-1}$ plays the role of t_1 and we also have: $l(t_4) < l(t_3)$; $l(t_5) < l(t_2)$; $l(t_7) < l(t_1)$. This transformation is induced by the the diffeomorphism of winding y_8 around x_1 clockwisely, going underneath y_6, over y_2 and underneath y_1, then followed by Φ_4^{-1}.

(1b) If $l(t_3) \geq l(t_2) > l(t_1)$, then by conjugating both sides with a, $at_3^{-1}cbat_3a^{-1} = at_2^{-1}ct_2t_1^{-1}bt_1$. Now we know $t_3a^{-1} = t_4t_1^{-1}bt_1$ for some $t_4 \in \langle a,b,c\rangle$. Therefore,

$$a^{-1}t_1^{-1}b^{-1}t_1t_4^{-1}cbat_4 = t_2^{-1}ct_2.$$

(1ba) If $l(t_1) \geq l(t_4)+1$, $t_2 = t_5t_4^{-1}cbat_4$ for some $t_5 \in \langle a,b,c\rangle$. Then we have $a^{-1}t_1^{-1}b^{-1}t_1 = t_4^{-1}(cba)^{-1}t_4t_5^{-1}ct_5$, which implies $t_4^{-1}(cba)t_4 = t_5^{-1}ct_5t_1^{-1}bt_1a$. So $t_4 = t_3a^{-1}t_1^{-1}b^{-1}t_1$ plays the role of t_3; $t_5 = t_2t_4^{-1}(cba)^{-1}t_4$ plays the role of t_2; t_1 does not change and $l(t_4) < l(t_3)$; $l(t_5) < l(t_2)$. This transformation is induced by the the diffeomorphism of winding y_8 around x_1 counter-clockwisely, going underneath y_1, y_2 and over y_6.

(1bb) If $l(t_1) < l(t_4)+1$, $t_2^{-1} = a^{-1}t_1^{-1}b^{-1}t_1t_5^{-1}$ for some $t_5 \in \langle a,b,c\rangle$. Then we have $t_4^{-1}cbat_4 = t_5^{-1}ct_5t_1^{-1}bt_1a$. So $t_4 = t_3a^{-1}t_1^{-1}b^{-1}t_1$ plays the role of t_3; $t_5 = t_2a^{-1}t_1^{-1}b^{-1}t_1$ plays the role of t_2, t_1 does not change and $l(t_4) < l(t_3)$; $l(t_5) < l(t_2)$. This transformation is induced by the the diffeomorphism of winding y_2 around x_1 clockwisely, going underneath

y_1 and over y_8, y_6, then followed by Φ_4^{-1}.

(2) If $t_2^{-1}ct_2$ is cancelled out by $t_1^{-1}bt_1a$, then

$$t_1^{-1}bt_1 = t_2^{-1}c^{-1}t_2t_3^{-1}cbat_3a^{-1} = t_2^{-1}c^{-1}t_2a^{-1}t_4^{-1}cbat_4$$

($\star\star$), where $t_3a^{-1} = t_4$ and there is no cancellation between t_2 and t_3^{-1}. Clearly, $l(t_4) + 1 \neq l(t_2)$.

(2a) If $l(t_4)+1 > l(t_2)$, then $t_1^{-1} = t_2^{-1}c^{-1}t_2t_5^{-1}$ for some $t_5 \in \langle a, b, c\rangle$, so $at_5^{-1}bt_5t_2^{-1}ct_2 = t_4^{-1}cbat_4$. Note, as we assume the last letter of t_1 is neither a or a^{-1}, either is t_2 by($\star\star$). So the first letter of t_5^{-1} must be a^{-1}. The equation can written as $t_6^{-1}bt_6at_2^{-1}ct_2 = t_4^{-1}cbat_4$, where $t_6 = t_5a^{-1}$. Clearly, $l(t_2) \neq l(t_6)$.

(2aa) If $l(t_6) > l(t_2)$, $t_4 = t_7t_2^{-1}ct_2$ for some $t_7 \in \langle a, b, c\rangle$. Then we have $t_6^{-1}bt_6a = t_2^{-1}c^{-1}t_2t_7^{-1}cbat_7 \Longrightarrow t_7^{-1}cbat_7 = t_2^{-1}ct_2t_6^{-1}bt_6a$. Now $t_7 = t_3a^{-1}t_2^{-1}c^{-1}t_2$ plays the role of t_3; t_2 does not change; $t_6 = t_1t_2^{-1}c^{-1}t_2a^{-1}$ plays the role of t_1 and $l(t_7) < l(t_3)$; $l(t_6) < l(t_1)$. This transformation is induced by the the diffeomorphism of winding y_6 around x_1 clockwisely, going over y_2, underneath y_1 and over y_8, then followed by Φ_4^{-1}.

(2ab) If $l(t_6) < l(t_2)$, $t_4^{-1} = t_6^{-1}bt_6t_7^{-1}$ for some $t_7 \in \langle a, b, c\rangle$. Then we have $at_2^{-1}ct_2 = t_7^{-1}cbat_7t_6^{-1}b^{-1}t_6$. Note the first letter of t_7^{-1} must be a, so $8t_8^{-1}cbat_8 = t_2^{-1}ct_2t_6^{-1}bt_6a$, where $t_8 = t_7a$. Now $t_8 = t_3a^{-1}t_6^{-1}b^{-1}t_6a = t_3t_5^{-1}b^{-1}t_5$ plays the role of t_3; t_2 does not change; $t_6 = t_1t_2^{-1}c^{-1}t_2a^{-1}$ plays the role of t_1 and $l(t_8) < l(t_3)$; $l(t_6) < l(t_1)$. This transformation is induced by the the diffeomorphism of winding y_2 around x_1 counter-clockwisely, going underneath y_6, over y_8, and underneath y_1.

(2b) If $l(t_4)+1 < l(t_2)$, then $t_1 = t_5 t_4^{-1} cbat_4$ for some $t_5 \in \langle a,b,c\rangle$, so

$$t_4^{-1}(cba)^{-1}t_4 t_5^{-1} bt_5 = t_2^{-1}c^{-1}t_2 a^{-1}.$$

The last letter of t_2 is not a, so the last letter of t_5 must be a^{-1}. Thus, $t_4^{-1}(cba)^{-1}t_4 at_6^{-1}bt_6 = t_2^{-1}c^{-1}t_2$, where $t_6 = t_5 a$. Clearly, $l(t_4)+1 \neq l(t_6)$, since otherwise the middle letter on the left hand side is a, while the middle letter on the right hand side is c^{-1}.

(2ba) If $l(t_4)+1 > l(t_6)$, $t_2 = t_7 t_6^{-1} bt_6$ for some $t_7 \in \langle a,b,c\rangle$, so $t_4^{-1}(cba)^{-1}t_4 a = t_6^{-1}b^{-1}t_6 t_7^{-1}c^{-1}t_7 \Longrightarrow t_3^{-1}cbat_3 = t_7^{-1}ct_7 t_6^{-1}bt_6 a$. Now, t_3 does not change; $t_7 = t_2 t_6^{-1}b^{-1}t_6$ plays the role of t_2; $t_6 = t_1 t_4^{-1}(cba)^{-1}t_4 a$ plays the role of t_1 and $l(t_7) < l(t_2)$; $l(t_6) < l(t_1)$. This transformation is induced by the the diffeomorphism of winding y_2 around x_1 clockwisely, going underneath y_1, y_8 and over y_6.

(2bb) If $l(t_4)+1 < l(t_6)$, $t_2^{-1} = t_4^{-1}(cba)^{-1}t_4 t_7^{-1}$ for some $t_7 \in \langle a,b,c\rangle$, so $at_6^{-1}bt_6 = t_7^{-1}c^{-1}t_7 t_4^{-1}cbat_4 \Longrightarrow t_4^{-1}cbat_4 = t_7^{-1}ct_7 t_5^{-1}bt_5 a$. Now, $t_4 = t_3 a^{-1}$ plays the role of t_3; $t_7 = t_2 t_4^{-1}(cba)^{-1}t_4$ plays the role of t_2; $t_5 = t_1 t_4^{-1}(cba)^{-1}t_4$ plays the role of t_1 and $l(t_4) < l(t_3)$; $l(t_7) < l(t_2)$; $l(t_5) < l(t_1)$. This transformation is induced by the the diffeomorphism of winding y_8 around x_1 counter-clockwisely, going underneath y_1 and over y_2, y_6.

(3) If $t_1^{-1}bt_1 a$ is is cancelled out by $t_2^{-1}ct_2$, then $t_2^{-1}ct_2 = t_3^{-1}cbat_3 a^{-1}t_1^{-1}b^{-1}t_1$ $(\star\star\star)$, and there is no cancellation between t_3 and a^{-1}. Clearly, $l(t_3)+1 \neq l(t_2)$.

(3a) If $l(t_3)+1 > l(t_1)$, $t_2 = t_4 t_1^{-1}b^{-1}t_1$ for some $t_4 \in \langle a,b,c\rangle$, so

$$t_3^{-1}cbat_3 = t_1^{-1}bt_1 t_4^{-1}ct_4 a = t_1^{-1}bt_1 at_5^{-1}ct_5,$$

where $t_5 = t_4 a$. Note that the last letter of t_4 must be a^{-1}, so $l(t_5) < l(t_4)$. It is clear that $l(t_1) \neq l(t_5)$.

(3aa) If $l(t_1) > l(t_5)$, $t_3 = t_6 t_5^{-1} c t_5$ for some $t_6 \in \langle a, b, c \rangle$, so $t_5^{-1} c^{-1} t_5 t_6^{-1} cbat_6 = t_1^{-1} bt_1 a \implies t_6^{-1} cbat_6 = t_5^{-1} ct_5 t_1^{-1} bt_1 a$. Now, $t_6 = t_3 t_5^{-1} c^{-1} t_5$ plays the role of t_3; $t_5 = t_2 t_1^{-1} bt_1 a$ plays the role of t_2; t_1 does not change and $l(t_6) < l(t_3)$; $l(t_5) < l(t_2)$. This transformation is induced by the the diffeomorphism of winding y_6 around x_1 clockwisely, going underneath y_2, y_1 and over y_8.

(3ab) If $l(t_1) < l(t_5)$, $t_3^{-1} = t_1^{-1} bt_1 t_6^{-1}$ for some $t_6 \in \langle a, b, c \rangle$, so $t_6^{-1} cbat_6 t_1^{-1} b^{-1} t_1 = at_5^{-1} ct_5$. Note the first letter of t_6^{-1} must be a, so $t_7^{-1} cbat_7 = t_5^{-1} ct_5 t_1^{-1} bt_1 a$, where $t_7 = t_6 a$. Now, $t_7 = t_3 t_1^{-1} bt_1 a$ plays the role of t_3; $t_5 = t_2 t_1^{-1} bt_1 a$ plays the role of t_2; t_1 does not change and $l(t_7) < l(t_3)$; $l(t_5) < l(t_2)$. This transformation is induced by the the diffeomorphism of winding y_2 around x_1 counter-clockwisely, going over y_6, Y_8, and underneath y_1, then followed by Φ_4.

(3b) If $l(t_3) + 1 < l(t_1)$, $t_2^{-1} = t_3^{-1} cbat_3 t_4^{-1}$ for some $t_4 \in \langle a, b, c \rangle$, so $t_1^{-1} b^{-1} t_1 = at_4^{-1} ct_4 t_3^{-1} (cba)^{-1} t_3 = t_5^{-1} ct_5 at_3^{-1} (cba)^{-1} t_3$, where $t_5 = t_4 a^{-1}$. Note that the first letter of t_4^{-1} must be a^{-1}, so $l(t_5) < l(t_4)$. It is clear that $l(t_3) + 1 \neq l(t_5)$.

(3ba) If $l(t_3) + 1 > l(t_5)$, $t_1^{-1} = t_5^{-1} bt_5 t_6^{-1}$ for $t_6 \in \langle a, b, c \rangle$, so $t_6^{-1} b^{-1} t_6 t_5^{-1} c^{-1} t_5 = at_3^{-1} cbat_3 \implies t_3^{-1} cbat_3 = t_5^{-1} ct_5 t_6^{-1} bt_6 a$. Now, t_3 does not change; $t_5 = t_2 t_3^{-1} cbat_3 a^{-1}$ plays the role of t_2; $t_6 = t_1 t_5^{-1} bt_5$ plays the role of t_1 and $l(t_5) < l(t_2)$; $l(t_6) < l(t_1)$. This transformation is induced by the the diffeomorphism of winding y_6 around x_1 counter-clockwisely, going underneath y_8, y_1 and over y_2.

(3bb) If $l(t_3) + 1 < l(t_5)$, $t_1 = t_6 t_3^{-1} (cba)^{-1} t_3$ for $t_6 \in \langle a, b, c \rangle$, so $t_3^{-1} cbat_3 t_6^{-1} b^{-1} t_6 = t_5^{-1} ct_5 a$. Note, the last letter of t_6 must be a, so $t_3^{-1} cbat_3 = t_5^{-1} ct_5 t_7^{-1} bt_7 a$, where $t_7 = t_6 a^{-1}$. Now, t_3 does not change; $t_5 = t_2 t_3^{-1} cbat_3 a^{-1}$ plays the role of t_2; $t_7 = t_1 t_3^{-1} cbat_3 a^{-1}$ plays the role of t_1 and $l(t_5) < l(t_2)$; $l(t_7) < l(t_1)$. This transformation is induced by the the diffeomorphism of winding y_8 around x_1 clockwisely, going over y_6, y_2 and underneath y_1, then followed by Φ_4^{-1}.

Note that in the above discussion, we can not assume that $t_3^{-1}cbat_3$ is reduced. For instance, if the first letter of t_3 is a^{-1}, then the 3 letters in the middle are acb instead of cba. So if we require that $\Phi(y_8)$ is reduced, there are 3 possibilities for the middle 3 letters: cba, acb and bac. The calculations for acb and bac are parallel to cba. For instance, if $\Phi(y_8) = t_3^{-1}acbt_3$, then $t_3^{-1}acbt_3 = t_2^{-1}ct_2t_1^{-1}bt_1a$. In step (1), we want to show $l(t_1) \neq l(t_2)$ which now is a little troublesome. Note the last letter of t_3 is a, and so is the last letter of t_2, therefore, $t_4^{-1}acbt_4 = t_5^{-1}ct_5at_1^{-1}bt_1$, where $t_3 = t_4a$, $t_2 = t_5a$. If $l(t_1) = l(t_2)$, $l(t_1) = l(t_5) + 1$ so the first two letters of t_1^{-1} are cb. Let $t_1^{-1} = cbt_6^{-1}$, we have $t_4^{-1}acbt_4 = t_5^{-1}ct_5acbt_6^{-1}bt_6b^{-1}c^{-1}$. By comparing both sides, we know $t_5 = b^{-1}t_6$; $c^{-1}t_5 = t_6b^{-1}c^{-1}$. This implies $b^{-1}c^{-1}t_5 = t_5b^{-1}c^{-1}$. So $t_5 = (cb)^k$, $k \in \mathbb{Z}$, $t_1 = b(cb)^{k-1}$. This contradicts with the assumption $t_1^{-1}bt_1$ is reduced. The following steps are very similar to the case $\Phi(y_8) = t_3^{-1}cbat_3$, we omit these tedious calculations.

Remember we are still under the assumption $\Phi(x_1) = x_1$, what we just proved is: by composing appropriate automorphisms of G which are all induced by self-diffeomorphisms of M, we can assume that $\Phi(y_1) = y_1$, $\Phi(y_2) = y_2$, $\Phi(y_6) = y_6$, $\Phi(y_8) = y_8$. Each of these automorphism changes a_1 into a_1y, a_1a_2 into a_1a_2y', $a_1a_2a_3$ into $a_1a_2a_3y''$ for some $y, y', y'' \in \langle y_1, y_2, y_6\rangle$. Now, by rolling over the circles a_1, a_2, a_3 to the other side as shown in the following figure, we are able to modify $\Phi(x_5)$, $\Phi(x_6)$ and $\Phi(x_7)$ in the same manner until $\Phi(x_5) = x_5$, $\Phi(x_6) = x_6$, $\Phi(x_7) = x_7$. Note, in this process, a_1^{-1} is transformed into $a_1^{-1}x$, $a_2^{-1}a_1^{-1}$ is transformed into $a_2^{-1}a_1^{-1}x'$, $a_3^{-1}a_2^{-1}a_1^{-1}$ is transformed into $a_3^{-1}a_2^{-1}a_1^{-1}x''$, where $x, x'x'' \in \langle x_1, x_5, x_6\rangle$. So when we roll over a_1, a_2 and a_3 back, y_2, y_6 and y_8 does not change. Therefore, we are now in the very first particular case (S1): $\Phi(x_i) = x_i$ for $i = 1, 5, 6, 7$ and $\Phi(y_j) = y_j$ for $j = 1, 2, 6, 8$. We've already known that all this kind of automorphisms are induced by self-diffeomorphisms of M. Thus, we are done with the first big case $\Phi(x_1) = x_1$.

Case II: $\Phi(x_1) = x_1^{-1}$.

This case can be transformed into Case I by turning the picture upside-down (rotating the plane containing this paper about the horizontal axis), then doing move 3.

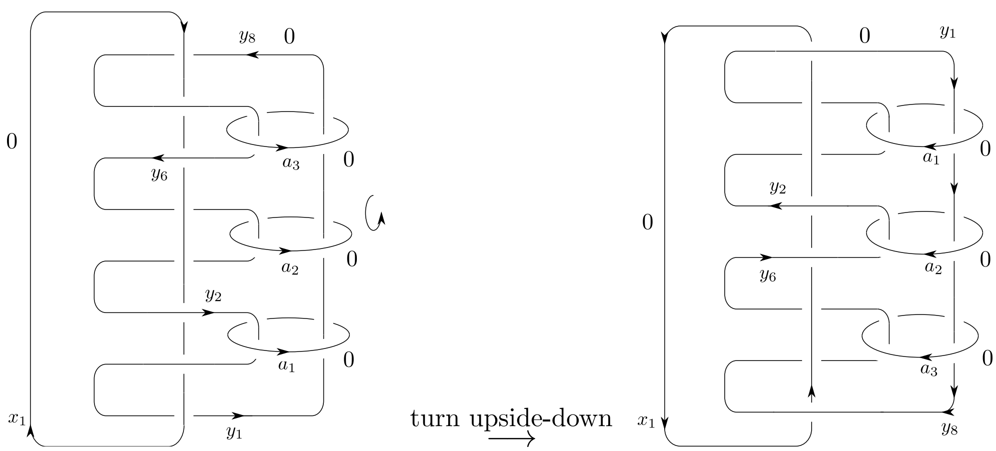


Figure 4.11: Turning upside-down

Case III: $\Phi(x_1) = y_1$.

This case can be transformed into Case I by flipping the paper (rotating the plane containing this paper about the vertical axis).

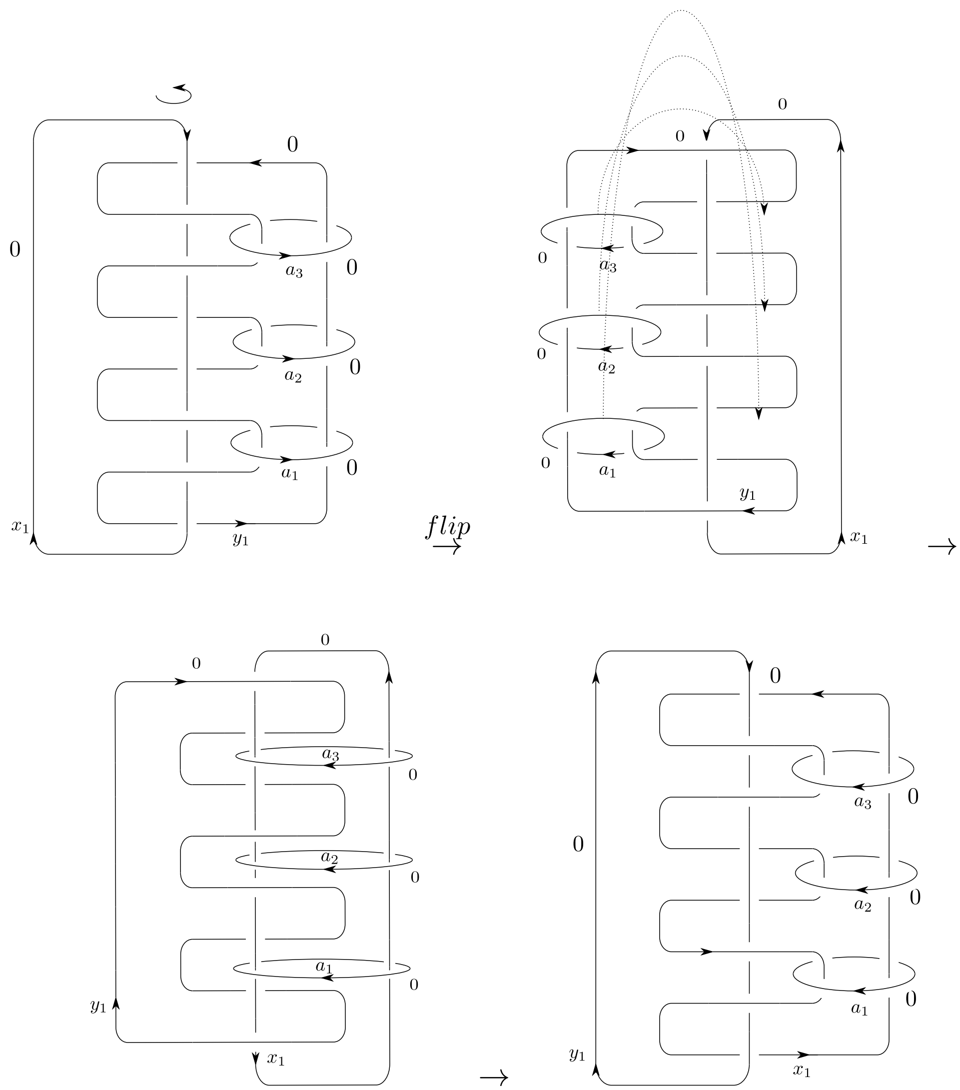


Figure 4.12: Flip over

Case IV: $\Phi(x_1) = y_1^{-1}$.

This is just a composition of Case II and Case III.

We saw that except for the diffeomorphisms induced by move 1.1 and 1.2, all the other diffeomorphisms do not change the spin structures of M. (They are indeed diffeomorphims of 4 manifolds). Move 1.1 and 1.2 clearly do not interchange the spin structures induced from X_1 and X_2, therefore, we obtain:

Theorem 4.5. *There exist simply-connected smooth 4 manifolds* (X_1, M) *and* (X_2, M), *such that* (Q_{X_1}, M) *is isomorphic to* (Q_{X_2}, M)*, but* X_1 *is not homeomorphic to* X_2*. The heomoemorphism type is changed by a Gluck Twist.*

Actually, we can construct four simply-connected smooth 4 manifolds X_1, X_2, X_3, X_4, where X_3 is obtained by doing Gluck Twist on T; X_4 is obtained by doing Gluck Twist on both T and S, such that any two of them are not homeomorphic.

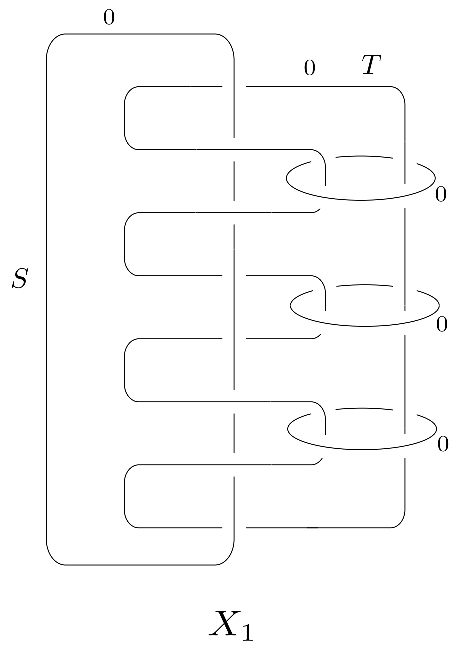


Figure 4.13: Doing Gluck twist on either S, T or both S and T

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] S. Akbulut and R. Matveyev. A convex decomposition theorem for 4-manifolds. *Internat. Math. Res. Notices*, (7):371–381, 1998.

[2] Selman Akbulut. Constructing a fake 4-manifold by Gluck construction to a standard 4-manifold. *Topology*, 27(2):239–243, 1988.

[3] Selman Akbulut. An exotic 4-manifold. *J. Differential Geom.*, 33(2):357–361, 1991.

[4] Selman Akbulut. A fake compact contractible 4-manifold. *J. Differential Geom.*, 33(2):335–356, 1991.

[5] Selman Akbulut. A fake cusp and a fishtail. In *Proceedings of 6th Gökova Geometry-Topology Conference*, volume 23, pages 19–31, 1999.

[6] Selman Akbulut. Variations on Fintushel-Stern knot surgery on 4-manifolds. *Turkish J. Math.*, 26(1):81–92, 2002.

[7] Selman Akbulut. Twisting 4-manifolds along $\mathbb{RP}^2$. In *Proceedings of the Gökova Geometry-Topology Conference 2009*, pages 137–141. Int. Press, Somerville, MA, 2010.

[8] Selman Akbulut. Knot surgery and Scharlemann manifolds. *Forum Math.*, 25(3):639–645, 2013.

[9] Selman Akbulut and Kouichi Yasui. Corks, plugs and exotic structures. *J. Gökova Geom. Topol. GGT*, 2:40–82, 2008.

[10] Selman Akbulut and Kouichi Yasui. Knotting corks. *J. Topol.*, 2(4):823–839, 2009.

[11] Selman Akbulut and Kouichi Yasui. Small exotic Stein manifolds. *Comment. Math. Helv.*, 85(3):705–721, 2010.

[12] Selman Akbulut and Kouichi Yasui. Stein 4-manifolds and corks. *J. Gökova Geom. Topol. GGT*, 6:58–79, 2012.

[13] Selman Akbulut and Kouichi Yasui. Cork twisting exotic Stein 4-manifolds. *J. Differential Geom.*, 93(1):1–36, 2013.

[14] Selman Akbulut and Kouichi Yasui. Gluck twisting 4-manifolds with odd intersection form. *Math. Res. Lett.*, 20(2):385–389, 2013.

[15] Steven Boyer. Simply-connected 4-manifolds with a given boundary. *Trans. Amer. Math. Soc.*, 298(1):331–357, 1986.

[16] Steven Boyer. Realization of simply-connected 4-manifolds with a given boundary. *Comment. Math. Helv.*, 68(1):20–47, 1993.

[17] James Conant, Rob Schneiderman, and Peter Teichner. Universal quadratic forms and Whitney tower intersection invariants. In *Proceedings of the Freedman Fest*, volume 18 of *Geom. Topol. Monogr.*, pages 35–60. Geom. Topol. Publ., Coventry, 2012.

[18] C. L. Curtis, M. H. Freedman, W. C. Hsiang, and R. Stong. A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds. *Invent. Math.*, 123(2):343–348, 1996.

[19] S. K. Donaldson. Irrationality and the h-cobordism conjecture. *J. Differential Geom.*, 26(1):141–168, 1987.

[20] Robert D. Edwards. The solution of the 4-dimensional annulus conjecture (after Frank Quinn). In *Four-manifold theory (Durham, N.H., 1982)*, volume 35 of *Contemp. Math.*, pages 211–264. Amer. Math. Soc., Providence, RI, 1984.

[21] Ronald Fintushel and Ronald J. Stern. Rational blowdowns of smooth 4-manifolds. *J. Differential Geom.*, 46(2):181–235, 1997.

[22] Ronald Fintushel and Ronald J. Stern. Knots, links, and 4-manifolds. *Invent. Math.*, 134(2):363–400, 1998.

[23] Michael H. Freedman and Frank Quinn. *Topology of 4-manifolds*, volume 39 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1990.

[24] Michael Hartley Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[25] Herman Gluck. The embedding of two-spheres in the four-sphere. *Trans. Amer. Math. Soc.*, 104:308–333, 1962.

[26] Robert E. Gompf and András I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[27] Cameron Gordon and Robion Kirby, editors. *Four-manifold theory*, volume 35 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1984.

[28] Sungbok Hong and Darryl McCullough. Mapping class groups of 3-manifolds, then and now. In *Geometry and topology down under*, volume 597 of *Contemp. Math.*, pages 53–63. Amer. Math. Soc., Providence, RI, 2013.

[29] Rob Kirby. Akbulut's corks and h-cobordisms of smooth, simply connected 4-manifolds. *Turkish J. Math.*, 20(1):85–93, 1996.

[30] Robion C. Kirby. *The topology of* 4*-manifolds*, volume 1374 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[31] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[32] R. Matveyev. A decomposition of smooth simply-connected h-cobordant 4-manifolds. *J. Differential Geom.*, 44(3):571–582, 1996.

[33] Darryl McCullough. Topological and algebraic automorphisms of 3-manifolds. In *Groups of self-equivalences and related topics (Montreal, PQ, 1988)*, volume 1425 of *Lecture Notes in Math.*, pages 102–113. Springer, Berlin, 1990.

[34] Darryl McCullough. *3-manifolds and their mappings*, volume 26 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1995.

[35] Darryl McCullough. Homeomorphisms which are Dehn twists on the boundary. *Algebr. Geom. Topol.*, 6:1331–1340 (electronic), 2006.

[36] Frank Quinn. Ends of maps. III. Dimensions 4 and 5. *J. Differential Geom.*, 17(3):503–521, 1982.

[37] V. A. Rohlin. New results in the theory of four-dimensional manifolds. *Doklady Akad. Nauk SSSR (N.S.)*, 84:221–224, 1952.

[38] Rob Schneiderman. Simple Whitney towers, half-gropes and the Arf invariant of a knot. *Pacific J. Math.*, 222(1):169–184, 2005.

[39] Rob Schneiderman and Peter Teichner. Higher order intersection numbers of 2-spheres in 4-manifolds. *Algebr. Geom. Topol.*, 1:1–29 (electronic), 2001.

[40] Alexandru Scorpan. *The wild world of 4-manifolds*. American Mathematical Society, Providence, RI, 2005.

[41] Stephen Smale. Generalized Poincaré's conjecture in dimensions greater than four. *Ann. of Math. (2)*, 74:391–406, 1961.

[42] M. Tange. A plug with infinite order and some exotic 4-manifolds. *ArXiv e-prints*, January 2012.

[43] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. of Math. (2)*, 87:56–88, 1968.

[44] C. T. C. Wall. Diffeomorphisms of 4-manifolds. *J. London Math. Soc.*, 39:131–140, 1964.

[45] C. T. C. Wall. On simply-connected 4-manifolds. *J. London Math. Soc.*, 39:141–149, 1964.

[46] J. H. C. Whitehead. On equivalent sets of elements in a free group. *Ann. of Math. (2)*, 37(4):782–800, 1936.

[47] Masayuki Yamasaki. Whitney's trick for three 2-dimensional homology classes of 4-manifolds. *Proc. Amer. Math. Soc.*, 75(2):365–371, 1979.