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By

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ABSTRACT

NORMAL p-SUBGROUPS IN THE AUTOMORPHISM GROUP OF A FINITE p-GROUP

By

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Let G be a finite p-group, p an odd prime. The primary purpose of this dissertation is to study the non-trivial normal p-subgroups of the group $\mathrm{Out}\ G = \mathrm{Aut}\ G/\mathrm{Inn}\ G$ of outer automorphisms of the group G.

Given a chain of subgroups $s: G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$, define $\mathrm{Stab}(s)$ by

$$\mathrm{Stab}(s) = \{\alpha \in \mathrm{Aut}\ G \mid (g_i G_{i+1})^{\alpha} = g_i G_{i+1}, \text{ for all } g_i \in G_i, \ i = 0,1,2,\ldots,n-1\}.$$

In a finite p-group G, the stability group of a characteristic chain is a normal p-subgroup of $\mathrm{Aut}\ G$. Also, if A is a normal p-subgroup of $\mathrm{Aut}\ G$, then A determines a unique characteristic chain in G which it stabilizes. When $B \leq \mathrm{Stab}(s)$, $\bar{B}$ denotes the closure of B, where $\bar{B} = \mathrm{Stab}(G \geq [G,B] \geq [G,B,B] \geq \cdots \geq 1)$, and B is said to be closed if $B = \bar{B}$. Moreover, if $B \trianglelefteq \mathrm{Aut}\ G$, then $\bar{B} \trianglelefteq \mathrm{Aut}\ G$. We use these facts in our efforts to determine normal p-subgroups of $\mathrm{Aut}\ G$.

In Chapter I, we prove the following embedding theorems:

(1) Let G be a finite group, $G = HK$ where $H \leq Z(G)$ is cyclic of order p^n, p a prime, and K is normal in G. Suppose $H \cap K$ is characteristic in K. Then $\operatorname{Aut} K$ embeds isomorphically in $\operatorname{Aut} G$.

(2) Let G be a finite group. Let H and K be characteristic subgroups of G such that $H \geq K$. There exists a homomorphism $\alpha \to \tilde{\alpha}$ mapping $\operatorname{Aut} G$ to $\operatorname{Aut} H/K$ such that

$\alpha \in C_{\operatorname{Aut} G}(H/K)$ if and only if
$\tilde{\alpha} = 1_{\operatorname{Aut} H/K}$.

We also show that an extra-special p-group of exponent p, p an odd prime, contains no proper characteristic extra-special subgroups.

Chapter II deals with p-groups of Hall type, where p is an odd prime. First we derive some basic properties of G, then we show that the maximal normal p-subgroup of $\operatorname{Aut} G$, $O_p(\operatorname{Aut} G)$, is the group of central automorphisms of G. Finally, we determine necessary and sufficient conditions for a non-trivial normal p-subgroup of $\operatorname{Aut} G$ to be closed as a stability group and, when G is not extra-special, show that a non-trivial normal p-subgroup of $\operatorname{Aut} G$ which is closed as a stability group, properly contains the group of inner automorphisms of G.

In the final chapter, Chapter III, we show that under certain conditions, lifting a stability group of a chain in a normal subgroup, will produce a normal and even closed stability group in the automorphism group of the whole group. Included in this chapter are results which can be used to obtain closed abelian stability groups.

To My Parents,

Theodore and Dorcas Rickard

and My Husband,

Mike Shapiro

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INDEX OF NOTATION

I. Relations:

$\leq$	Is a subgroup of
$\lneq$	Is a proper subgroup of
$\trianglelefteq$	Is a normal subgroup of
$\triangleleft_p$	Is a normal p-subgroup of
$\approx$	Is isomorphic to
$\in$	Is an element of
$\equiv$	Is congruent to

II. Operations:

$\langle\ \rangle$	The subgroup generated by
X^α	The image of X under the mapping α
$[x,y]$	$x^{-1}y^{-1}xy$
$[x,\alpha]$	$x^{-1}x^\alpha$
$[H,A]$	Subgroup generated by all $[h,\alpha]$, $h \in H$, $\alpha \in A$.
$[H,\overbrace{A,\ldots,A}^{n}]$	$[[H,\overbrace{A,\ldots,A}^{n-1}],A]$
$[H:K]$	The index of K in H
H/K	The cosets of K in H
HK	$\{hk \mid h \in H$ and $k \in K\}$

$H \times K$	The direct product of H and K
$H \rtimes K$	The split extension of H by K
$H \setminus K$	The elements of H not in K
$\lvert h \rvert$	The order of the element h
$\lvert H \rvert$	The number of elements in H
Exp H	The exponent of H
cl(H)	The class of H
1_H	The mapping $h \to h$ for all $h \in H$
$\alpha\vert_H$	The mapping α restricted to H
(a,b)	The greatest common divisor of a and b
$a \vert b$	a divides b
π	Product, not necessarily direct product

III. <u>Groups and Sets</u>:

Aut G	The automorphism group of G
Out G	The outer automorphism group of G
Inn G	The inner automorphism group of G
$C_H(K)$	The centralizer of K in H
$N_H(K)$	The normalizer of K in H
$C_{\text{Aut } G}(H/K)$	$\{\alpha \in \text{Aut } G \mid h^{-1}h^{\alpha} \in K \text{ for all } h \in H\}$
G'	The commutator subgroup of G
$\Phi(G)$	The Frattini of G
$Z(G)$	The center of G
Hom(H,K)	The set of homomorphisms of H into K
Ker φ	The kernel of the homomorphism φ

INTRODUCTION

In 1966, Gaschütz [3] showed that the outer automorphism group of any finite non-abelian p-group, p a prime, possesses a non-trivial p-subgroup. Later, in 1976, Schmid [7] determined all finite p-groups for which the group of outer automorphisms contains a non-trivial normal p-subgroup. This dissertation is a direct result of efforts to characterize the normal p-subgroups of Aut G, and thus of Out G, when G is a non-abelian p-group.

Stability groups and subgroups of stability groups play an important role in this search for normal p-subgroups. The examples of normal p-subgroups in the outer automorphism group of a finite p-group which were obtained by Schmid in [7] are stability groups. Moreover, in 1974, Bertelsen [2] showed that stability groups and their subgroups offer a viable means by which we may obtain normal p-subgroups in the automorphism group of a finite p-group. He shows that if a normal p-subgroup A of Aut G is a subgroup of a stability group then $\bar{A}$, the closure of A, is a stability group which is a normal p-subgroup of Aut G. When A is a subgroup of a stability group and $A = \bar{A}$ then A is said to be closed. We ask how the non-trivial normal p-subgroups of Out G and the normal closed stability groups of Aut G are related.

Chapter I contains definitions and results which are used in later chapters. We derive some embedding theorems which prove to be useful in the study of p-groups of Hall type, where p is an odd prime, and show that an extra-special p-group of exponent p, where p is odd, contains no proper characteristic extra-special subgroups.

In Chapter II we consider p-groups G of Hall type, where p is an odd prime. After deriving some basic facts about G, we show that the maximal normal p-subgroup of Aut G is the group of central automorphisms. We obtain a partial answer to our question about the relationship between non-trivial normal p-subgroups of Out G and the normal closed stability groups of Aut G. When G is of Hall type but not extra-special, we determine necessary and sufficient conditions for a non-trivial normal p-subgroup of Aut G to be closed as a stability group and show that a non-trivial normal p-subgroup of Aut G which is closed as a stability group properly contains the group of inner automorphisms of G.

The final chapter, Chapter III, consists primarily of two topics: (1) extending (lifting) a group of automorphisms of a normal subgroup to a group of automorphisms of the whole group and (2) closed stability groups. We consider groups G which are split extensions and show that under certain conditions, a stability group of a chain in a normal subgroup can be lifted to a normal and even closed stability group in the automorphism group of the whole group. We also include results which can be used to obtain closed abelian stability groups.

CHAPTER I

PRELIMINARY DEFINITIONS AND RESULTS

This chapter contains definitions and results which are used in later chapters. All groups are assumed to be finite.

<u>Definition 1.1</u>. Let H and K be subgroups of an arbitrary group G such that $H \geq K$. Define $C_{\text{Aut } G}(H/K)$ by

$$C_{\text{Aut } G}(H/K) = \{\alpha \in \text{Aut } G \mid h^{-1} h^{\alpha} \in K \text{ for all } h \in H\}.$$

<u>Definition 1.2</u>. Let $s: G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$ be a chain of subgroups for an arbitrary group G. Define the stability group of s, denoted $\text{Stab}(s)$, by

$$\text{Stab}(s) = \{\alpha \in \text{Aut } G \mid (g_i G_{i+1})^{\alpha} = g_i G_{i+1} \text{ for all } g_i \in G_i,\ i = 0,1,\ldots,n-1\}.$$

With the above definitions, $\text{Stab}(s) = \bigcap_{i=0}^{n-1} C_{\text{Aut } G}(G_i/G_{i+1})$.

<u>Result 1.3</u> ([2],p.5). If each G_i is characteristic in G then $\text{Stab}(s) \trianglelefteq \text{Aut } G$.

<u>Definition 1.4</u>. Let $A \leq \mathrm{Aut}\, G$. Set $\gamma GA^0 = G$, and $\gamma GA^{i+1} = [\gamma GA^i, A]$, for $i \geq 0$.

<u>Result 1.5</u>. Using the notation of 1.2, let $A \leq \mathrm{Stab}(s)$. Then

(i) $\gamma GA^i \leq G_i$, $i = 0,1,\ldots,n$.

(ii) The prime divisors of A are the same as the prime divisors of $[G,A]$.

<u>Proof</u>: See Schmid [8].

The following remark is an immediate consequence of this result.

<u>Remark 1.6</u>. If G is a finite p-group and $s{:}G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$, then $\mathrm{Stab}(s)$ is a p-group.

<u>Result 1.7</u> ([2],p.16). Let G be a group. Let $A \leq Z(G)$ the center of G, $\bar{g} = gA$ and $\alpha \in \mathrm{Stab}(G \geq A \geq 1)$. Then

$$\mathrm{Stab}(G \geq A \geq 1) \approx \mathrm{Hom}(G/A, A)$$

by the mapping $\alpha \leftrightarrow f_\alpha\colon \bar{g} \rightarrow g^{-1}\, g^{\alpha}$.

<u>Result 1.8</u> ([4],p.200). Let H be an abelian subgroup of an arbitrary group G. Let $A = \mathrm{Stab}(G \geq H \geq 1)$. Then A is abelian.

<u>Definition 1.9</u>. Let $A \leq \mathrm{Aut}\, G$. We say A stabilizes a chain s if $A \leq \mathrm{Stab}(s)$. Following Schmid [8], we denote by τ_G the collection of subgroups of $\mathrm{Aut}\, G$ stabilizing a chain.

We now define a concept introduced by Bertelsen in [2].

<u>Definition 1.10</u>. For $A \in \tau_G$, define $\bar{s}$ by $\bar{s}: \gamma GA^0 \geq \gamma GA^1 \geq \ldots \geq \gamma GA^{n(A)}$, where $n(A)$ is the first integer such that $\gamma GA^{n(A)} = 1$. Define $\bar{A}$, the closure of A, by $\bar{A} = \mathrm{Stab}(\bar{s})$. We say a stability group A is closed if $A = \bar{A}$.

<u>Result 1.11</u> ([2],p.6). Let $A \in \tau_G$. Then

(i) $A \leq \bar{A}$.

(ii) If $A \leq B \leq \bar{A}$, then $\gamma GB^i = \gamma GA^i$ for all i.

(iii) $\bar{\bar{A}} = \bar{A}$.

(iv) If $\beta \in N_{\mathrm{Aut}\,G}(A)$, then $(\gamma GA^i)^\beta = \gamma GA^i$.

(v) If $A \trianglelefteq \mathrm{Aut}\,G$, then $\bar{A} \trianglelefteq \mathrm{Aut}\,G$.

(vi) A and $\bar{A}$ have the same prime divisors.

<u>Remark 1.12</u>. By 1.11 (iv), if $A \in \tau_G$ and $A \trianglelefteq \mathrm{Aut}\,G$ then γGA^i is characteristic in G for all i.

<u>Result 1.13</u> ([2], p.9). Let $H \leq \Phi(G)$, the Frattini subgroup of G. If $B = C_{\mathrm{Aut}\,G}(G/H)$ then B is a closed stability group.

<u>Definition 1.14</u>. Let G be a p-group and $A \leq \mathrm{Aut}\,G$. A is said to be of K-type if

(i) A is a p-group.

(ii) A is normal in every p-Sylow of $\mathrm{Aut}\,G$ that contains A.

(iii) A is the intersection of all the p-Sylows of Aut G that contain A.

<u>Result 1.15</u> ([2],p.21). If G is a p-group and A is of K-type then A is a closed stability group.

<u>Definition 1.16</u>. Let G be any group. $O_p(G)$ will denote the unique maximal normal p-subgroup of G.

$O_p(G)$ may be obtained by intersecting all p-Sylows of G. Hence $O_p(\text{Aut } G)$ is the smallest K-type stability group.

<u>Corollary 1.17</u>. If G is a p-group and $A = O_p(\text{Aut } G)$ then A is a closed stability group.

<u>Definition 1.18</u>. A finite non-abelian p-group G is extra-special if the center $Z(G)$ is cyclic of order p and $G/Z(G)$ is elementary abelian.

<u>Definition 1.19</u>. If G is a p-group, $\Omega_i(G) = \langle x \in G | x^{p^i} = 1 \rangle$ and $\mho^i(G) = \langle x^{p^i} | x \in G \rangle$.

<u>Result 1.20</u>. Let G be a finite non-abelian p-group containing no non-cyclic characteristic abelian subgroups. Then the Frattini subgroup $\Phi(G)$ of G is cyclic. Moreover:

(i) If p is odd, $\Omega_1(G)$ is extra-special of exponent p and $G = \Omega_1(G)Z(G)$.

(ii) If $p = 2$, $U = C_G(\Phi(G))$ is of index at most 2 in G and one of the following holds:

(a) G is extra-special.

(b) G is a dihedral, semi-dihedral or generalized quaternion group.

(c) $\Phi(G) = \Phi(U)$ is of index 2 in the cyclic subgroup $Z(U)$.

This result is an unpublished theorem of P. Hall and can be verified by using the details given in the proof of Satz III.13.10 in [5].

Definition 1.21. A group G satisfying the hypotheses of 1.20 will be called a p-group of Hall type.

Result 1.22 (J. Thompson, [9]). Let G be a finite p-group. Given any maximal characteristic abelian subgroup Z of G, there exists a characteristic subgroup K of G such that $C_G(K) = Z(K) = Z$ and K/Z is an elementary abelian central factor of G.

This is a modification of Lemma 3.7 in [9]. A proof may be found in [4], pages 185-186.

Definition 1.23. A characteristic subgroup K of a p-group G is called a critical subgroup of G if $C_G(K) = Z(K)$ and $K/Z(K)$ is an elementary abelian central factor of G.

<u>Corollary 1.24</u> (Schmid, [7]). A finite non-abelian p-group G is of Hall type if and only if the center of any critical subgroup of G is cyclic.

We will later show that if G is of Hall type and p is odd then the only critical subgroup of G is G itself.

<u>Result 1.25</u> (Schmid, [7]). Suppose that G is a finite non-abelian p-group having a subgroup N such that $\Phi(G) \leq N \leq Z(G)$ then $C = \mathrm{Stab}(G \geq N \geq 1)$ is equal to $\mathrm{Inn}\, G$ if and only if $N = Z(G)$ is cyclic. In any case, C contains $\mathrm{Inn}\, G$.

<u>Theorem 1.26</u>. Let $G = HK$ where $K \trianglelefteq G$ and $H = \langle x \rangle$ is cyclic of order p^n, p a prime. Suppose $H \leq Z(G)$. If $H \cap K$ is characteristic in K then there exists a homomorphism which embeds $\mathrm{Aut}\, K$ isomorphically in $\mathrm{Aut}\, G$.

<u>Proof</u>: If $H \cap K = H$ then $G = K$ and we are done.

Suppose $H \cap K \neq H$. Then $|H \cap K| = p^i$ for some i such that $0 \leq i < n$. Thus $H \cap K = \langle x^{p^{n-i}} \rangle$. Let $K = \langle y_1, \ldots, y_t \rangle$. Then $G = \langle x, y_1, \ldots, y_t \rangle$.

Let $\alpha \in \mathrm{Aut}\, K$. Since $H \cap K$ is characteristic in K, $(x^{p^{n-i}})^\alpha$ is a generator of $H \cap K$. Thus $(x^{p^{n-i}})^\alpha = (x^{p^{n-i}})^\ell$ where $(\ell, p) = 1$. If $i = 0$, we will take $\ell = 1$.

Let g be an arbitrary element of G. Since $G = HK$, we can write $g = x^m k$ where $0 \leq m < p^n$ and $k \in K$. Given

α and ℓ as above, we define a map $\bar{\alpha}$ from G to G by

$$(x^m k)^{\bar{\alpha}} = x^{m\ell} k^{\alpha}.$$

The map $\bar{\alpha}$ is well-defined and $\bar{\alpha}|_K = \alpha$.

<u>Claim</u>: $\bar{\alpha}$ is an automorphism of G.

(i) Let $x^r k_1$ and $x^s k_2$ be elements of $G = HK$.

Since $H \leq Z(G)$ and $\alpha \in \mathrm{Aut}\, K$,

$$\begin{aligned}(x^r k_1 x^s k_2)^{\bar{\alpha}} &= (x^{r+s} k_1 k_2)^{\bar{\alpha}} \\ &= x^{r\ell + s\ell}(k_1 k_2)^{\alpha} \\ &= x^{r\ell} x^{s\ell} k_1^{\alpha} k_2^{\alpha} \\ &= x^{r\ell} k_1^{\alpha} x^{s\ell} k_2^{\alpha} = (x^r k_1)^{\bar{\alpha}} (x^s k_2)^{\bar{\alpha}}.\end{aligned}$$

Hence $\bar{\alpha}$ is a homomorphism.

(ii) Since $(\ell, p) = 1$, x^{ℓ} generates H; and since $\alpha \in \mathrm{Aut}\, K$, $\{y_j^{\alpha}\}_{j=1}^{t}$ generates K. Therefore $\bar{\alpha}$ is onto.

(iii) If $1 = (x^r k)^{\bar{\alpha}} = x^{r\ell} k^{\alpha}$, then $x^{r\ell} = (k^{-1})^{\alpha}$ $\in H \cap K = \langle x^{p^{n-i}} \rangle$. Thus $x^{r\ell} = (x^{p^{n-i}})^m$ where $1 \leq m \leq p^i$. It follows that $r\ell \equiv m(p^{n-i}) \pmod{p^n}$. Since $\ell | r\ell$ and $(\ell, p) = 1$, $\ell | m$. Let $m = \ell q$. Then $r \equiv q \cdot p^{n-i} \pmod{p^n}$ and $x^{r\ell} = (x^{p^{n-i}})^{q\ell}$ $= (x^{q \cdot p^{n-i}})^{\alpha} = (x^r)^{\alpha}$. Thus $x^r = (x^{r\ell})^{\alpha^{-1}} =$ $((k^{-1})^{\alpha})^{\alpha^{-1}} = k^{-1}$ and $x^r k = 1$. Therefore $\bar{\alpha}$ is one-to-one.

The mapping $\alpha \to \bar{\alpha}$ maps $\mathrm{Aut}\, K$ to $\mathrm{Aut}\, G$. Let $\alpha, \beta \in \mathrm{Aut}\, K$. Then $\alpha\beta \in \mathrm{Aut}\, K$ and $\bar{\alpha}, \bar{\beta}, \overline{\alpha\beta}$, as defined above, are automorphisms of G such that $\bar{\alpha}|_K = \alpha$, $\bar{\beta}|_K = \beta$ and $\overline{\alpha\beta}|_K = \alpha\beta$.

First we show $\alpha \to \bar{\alpha}$ is a homomorphism.

Clearly $y_j^{\overline{\alpha\beta}} = y_j^{\bar{\alpha}\bar{\beta}}$, $j = 1,\ldots,t$.

Suppose $(x^{p^{n-i}})^\alpha = (x^{p^{n-i}})^s$ and $(x^{p^{n-i}})^\beta = (x^{p^{n-i}})^r$ where $(s,p) = 1 = (r,p)$ and if $H \cap K = 1$, then $r = s = 1$.

$$
\begin{aligned}
(x^{p^{n-i}})^{\alpha\beta} &= ((x^{p^{n-i}})^\alpha)^\beta \\
&= ((x^{p^{n-i}})^s)^\beta \\
&= ((x^{p^{n-i}})^\beta)^s \\
&= ((x^{p^{n-i}})^r)^s = (x^{p^{n-i}})^{rs}.
\end{aligned}
$$

Clearly $(rs,p) = 1$. Thus $x^{\overline{\alpha\beta}} = x^{rs} = (x^{\bar{\beta}})^s = (x^s)^{\bar{\beta}} = (x^{\bar{\alpha}})^{\bar{\beta}}$. $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$ on the generators so $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$.

If $\bar{\alpha} = \bar{\beta}$ then $\alpha = \beta$. Thus the mapping $\alpha \to \bar{\alpha}$ embeds Aut K isomorphically in Aut G.

<u>Corollary 1.27</u>. Let G be as in 1.26 and $C \leq K$ such that C is characteristic in G. Then C is characteristic in K.

<u>Proof</u>: Let $\alpha \in \text{Aut } K$. Extend α to $\bar{\alpha} \in \text{Aut } G$ as above. Then $C^\alpha = C^{\bar{\alpha}} = C$.

<u>Theorem 1.28</u>. Let G be an extra-special p-group of exponent p, p odd. If $H \lneq G$ is an extra-special subgroup of G then H is not characteristic in G.

Our proof depends upon the following results.

<u>Lemma 1.29</u> ([4], p.195). Let C be an extra-special subgroup of the p-group P such that $[P,C] \leq Z(C)$. Then $P = CC_P(C)$.

In the following results, H^m denotes the central product of m copies of the group H.

<u>Lemma 1.30</u> ([4], p.204). An extra-special p-group P is the central product of $r \geq 1$ non-abelian subgroups of order p^3. Moreover, if p is odd, P is isomorphic to $N^k M^{r-k}$ where

$$N = \langle x,y \mid x^{p^2} = y^p = 1,\ x^y = x^{1+p}\rangle \text{ and}$$

$$M = \langle x,y,z \mid x^p = y^p = z^p = 1,\ [x,z] = [y,z] = 1, \text{ and } [x,y] = z\rangle.$$

and $|P| = 2^{r+1}$.

From the proof of this lemma it follows that the r factors involved commute elementwise. Thus we have the following corollary.

<u>Corollary 1.31</u>. Let P be an extra-special p-group, p odd, such that P is isomorphic to the central product of $n \geq 1$ non-abelian subgroups M_j such that

$$M_j = \langle x_j,y_j \mid [x_j,y_j] = z,\ x_j^p = y_j^p = z^p = 1,\ [x_j,z] = [y_j,z] = 1\rangle \quad j = 1,\dots,n.$$

Then

$$P = \langle x_j, y_j \mid [x_j,y_j] = z, \quad [x_i,y_j] = 1 \ \text{ if } \ i \neq j,$$
$$[x_j,z] = [y_j,z] = 1, \quad [x_i,x_j] = [y_i,y_j] = 1,$$
$$x_j^p = y_j^p = z^p = 1, \quad i,j = 1,\ldots,n\rangle.$$

<u>Proof of 1.28</u>: By 1.30, $H \approx N^k M^{r-k}$ where N and M are as in the lemma and $r \geq 1$. Since $H \leq G$ and G has exponent p, $\exp H = p$ and thus $k = 0$.

Since H is extra-special, we may choose x in $H \setminus Z(H)$ and y in H such that $[x,y] = z \neq 1$. Thus $\langle z\rangle = Z(H) = Z(G)$ since $H' = Z(H)$ and $G' = Z(G)$ are cyclic of order p and $H' \leq G'$. Also, $[G,G] \leq Z(G) = Z(H)$ so $[G,H] \leq Z(H)$. Lemma 1.29 implies $G = HR$ where $R = C_G(H)$. Since R centralizes H and R is a p-group, $Z(R) = Z(H)$. It follows that R is non-abelian for otherwise, $G = H$. Thus $R' = G' = Z(G)$ and $R/Z(G) \leq G/Z(G)$ is elementary abelian so R is extra-special. Exp $R = p$. Thus $R \approx M^s$ by 1.30, where $s \geq 1$.

Applying 1.31, H and R have the following presentations:

$$H = \langle x_j, y_j \mid [x_j,y_j] = z, \ [x_i,y_j] = 1 \ \text{ if } \ i \neq j,$$
$$x_j^p = y_j^p = z^p = 1, \ [x_j,z] = [y_j,z] = [x_i,x_j]$$
$$= [y_i,y_j] = 1, \quad i,j = 1,\ldots,r\rangle \ \text{ and}$$

$$R = \langle x_j, y_j \mid [x_j,y_j] = z, \ [x_i,y_j] = 1 \ \text{ if } \ i \neq j,$$
$$x_j^p = y_j^p = z^p = 1, \ [x_j,z] = [y_j,z] = [x_i,x_j]$$
$$= [y_i,y_j] = 1, \quad i,j = r+1,\ldots,r+s\rangle.$$

Since $G = HK$ and $K = C_G(H)$,

$$G = \langle x_j, y_j \mid [x_j, y_j] = z,\ [x_i, y_j] = 1 \text{ if } i \neq j,$$
$$x_j^p = y_j^p = z^p = 1,\ [x_j, z] = [y_j, z] = [x_i, x_j]$$
$$= [y_i, y_j] = 1,\ \ i,j = 1,\ldots,r,r+1,\ldots,r+s\rangle.$$

Let $t = r + s$.

Let $g \in G$. Then we have $g = (\prod_{j=1}^{t} x_j^{c_j} y_j^{d_j}) z^a$ where $0 \leq c_j, d_j, a < p$, $j = 1,\ldots,t$.

Define a mapping $\alpha: G \to G$ by

$$g^\alpha = (\prod_{j=1}^{t} (x_j^\alpha)^{c_j} (y_j^\alpha)^{d_j}) z^a$$

where

$$\begin{cases} (x_j)^\alpha = x_j,\ (y_j)^\alpha = y_j & \text{if } j \neq 1,t \\ (x_1)^\alpha = x_t,\ (y_1)^\alpha = y_t \\ (x_t)^\alpha = x_1,\ (y_t)^\alpha = y_1. \end{cases}$$

The mapping α is a homomorphism which fixes $Z(G)$ elementwise.

Suppose $g \in \operatorname{Ker} \alpha$. Then $g^\alpha = 1$ and hence

$$\text{(A)} \qquad x_t^{c_1} y_t^{d_1} (\prod_{j=2}^{t-1} x_j^{c_j} y_j^{d_j}) x_1^{c_t} y_1^{d_t} z^a = 1.$$

It follows that $x_t^{c_1} y_t^{d_1} (\prod_{j=2}^{t-1} x_j^{c_j} y_j^{d_j}) x_1^{c_t} y_1^{d_t} \in Z(G)$ and is left fixed by α. Therefore

$$x_t^{c_1} y_t^{d_1} (\prod_{j=2}^{t-1} x_j^{c_j} y_j^{d_j}) x_1^{c_t} y_1^{d_t} = \prod_{j=1}^{t} x_j^{c_j} y_j^{d_j}.$$

Hence, by substituting this result into (A) we see that $g = 1$. Thus α is one-to-one. Since G is finite and α is one-to-one, α is an automorphism of G.

By the way we have defined α it follows that $H^{\alpha} \neq H$. Thus H is not characteristic in G.

Since the original draft of this thesis it has been brought to our attention that Theorem 1.28 may be generalized as follows.

If G is an extra-special p-group of exponent p then $Z(G)$ is the only non-trivial characteristic subgroup of G. (See [5], Aufgabe 32, p.360).

This generalization directly affects Theorem 2.3 and 2.4.

<u>Theorem 1.32</u>. Let G be an arbitrary group. Let H and K be characteristic subgroups of G such that $H \geq K$. Then there exists a homomorphism $\alpha \to \tilde{\alpha}$ mapping $\operatorname{Aut} G$ to $\operatorname{Aut} H/K$ such that $\alpha \in C_{\operatorname{Aut} G}(H/K)$ if and only if $\tilde{\alpha} = 1$ in $\operatorname{Aut}(H/K)$. Here $(xK)^{\tilde{\alpha}} = x^{\alpha}K$, $x \in H$.

<u>Proof</u>: Define $\tilde{\alpha}: H/K \to H/K$ by

$$(xK)^{\tilde{\alpha}} = x^{\alpha}K \qquad x \in H.$$

The map $\tilde{\alpha}$ is well-defined for if $xK = yK$ then $xy^{-1} \in K$. Since K is characteristic in G, $x^{\alpha}(y^{\alpha})^{-1} = (xy^{-1})^{\alpha} \in K$. Thus $x^{\alpha}K = y^{\alpha}K$. Also, if $x \in H$ then $x^{\alpha} \in H$ since H is characteristic in G and thus $\tilde{\alpha}$ maps H/K to H/K.

The map $\tilde{\alpha}$ is an automorphism of H/K. Furthermore,

$$\begin{aligned}\alpha \in C_{\mathrm{Aut}\,G}(H/K) &\Leftrightarrow x^{-1}x^{\alpha} \in K \quad \text{for all } x \in H\\ &\Leftrightarrow xK = x^{\alpha}K \quad \text{for all } x \in H\\ &\Leftrightarrow xK = (xK)^{\tilde{\alpha}} \quad \text{for all } x \in H\\ &\Leftrightarrow \tilde{\alpha} = 1 \quad \text{in } \mathrm{Aut}\,H/K.\end{aligned}$$

Let $\alpha,\beta \in \mathrm{Aut}\,G$. Then $\alpha\beta \in \mathrm{Aut}\,G$ and $\tilde{\alpha},\tilde{\beta}$ and $\widetilde{\alpha\beta}$, as defined above, are automorphisms of H/K. Let $x \in H$. Then $(xK)^{\widetilde{\alpha\beta}} = x^{\alpha\beta}K = (x^{\alpha}K)^{\tilde{\beta}} = ((xK)^{\tilde{\alpha}})^{\tilde{\beta}} = (xK)^{\tilde{\alpha}\tilde{\beta}}$. Hence $\alpha \to \tilde{\alpha}$ is a homomorphism.

<u>Corollary 1.33</u>. Let G,H,K and the map $\alpha \to \tilde{\alpha}$ be as in 1.32. Suppose $[H:K] = p$ and $\alpha \in \mathrm{Aut}\,G$ has order a power of p. Then $\alpha \in C_{\mathrm{Aut}\,G}(H/K)$.

<u>Proof</u>: Let $\tilde{\alpha}$ in $\mathrm{Aut}\,H/K$ be as defined in 1.32. Since $\alpha \to \tilde{\alpha}$ is a homomorphism, $|\tilde{\alpha}| \,\Big|\, |\alpha|$. Also, since $[H:K] = p$, H/K is cyclic of order p and thus $|\mathrm{Aut}\,H/K| = p-1$. Therefore $|\tilde{\alpha}| \,\Big|\, p-1$. Since $(p,p-1) = 1$, $\tilde{\alpha} = 1$ in $\mathrm{Aut}\,H/K$. Hence by 1.32, $\alpha \in C_{\mathrm{Aut}\,G}(H/K)$.

CHAPTER II

p-GROUPS OF HALL TYPE

In this chapter we show that every non-trivial closed normal stability group of a finite non-abelian p-group G of Hall type, p an odd prime, properly contains the group of inner automorphisms, Inn G, of G, provided G is not extra-special.

<u>Lemma 2.1</u>. Let G be a finite non-abelian p-group of Hall type, p odd. Then

(i) $\Omega_1(Z(G)) = Z(\Omega_1(G))$.

(ii) $G' = [\Omega_1(G)]' = Z(\Omega_1(G)) = \Phi(\Omega_1(G))$
$= \Omega_1(G) \cap Z(G)$.

(iii) $G/Z(G)$ is elementary abelian.

(iv) $\mho^i(G) = \mho^i(Z(G))$ for $i \geq 1$.

(v) $\Phi(G) = \begin{cases} \mho^1(Z(G)) & \text{if } |Z(G)| > p \\ Z(G) & \text{if } |Z(G)| = p. \end{cases}$

<u>Proof</u>: From result 1.20 (i), $\Omega_1(G)$ is extra-special of exponent p and $G = \Omega_1(G)Z(G)$. Let $x = w_x z_x$ and $y = w_y z_y$ be elements of G where $w_x, w_y \in \Omega_1(G)$ and $z_x, z_y \in Z(G)$.

(i) Since G is a p-group, $Z(G) \neq 1$, whence $1 \neq \Omega_1(Z(G)) \leq Z(\Omega_1(G))$. Now $\Omega_1(G)$ is extra-special and so $|Z(\Omega_1(G))| = p$ and it follows that $\Omega_1(Z(G)) = Z(\Omega_1(G))$.

(ii) For $x, y \in G$, $[x,y] = [w_x z_x, w_y\ z_y] = [w_x, w_y]$. Thus $G' \leq [\Omega_1(G)]' \leq G'$. Therefore $G' = [\Omega_1(G)]'$. Also, $[\Omega_1(G)]' = Z(\Omega_1(G)) = \Phi(\Omega_1(G))$ because $\Omega_1(G)$ is extra-special. Hence by part (i), $G' = \Omega_1(Z(G))$ and thus $1 \neq G' \leq Z(G) \cap \Omega_1(G)$ which is cyclic of exponent p. Therefore $G' = \Omega_1(G) \cap Z(G)$.

(iii) By part (ii), $G' \leq Z(G)$. Therefore $G/Z(G)$ is abelian. Moreover, since $\exp \Omega_1(G) = p$ and $G = \Omega_1(G)Z(G)$, $|xZ(G)| = |w_x z_x Z(G)| = |w_x Z(G)| = p$ for any $x \in G$. Thus $G/Z(G)$ is elementary abelian.

(iv) It suffices to show that $\mho^1(G) = \mho^1(Z(G))$. For any $x \in G$, $x^p = (w_x z_x)^p = w_x^p\ z_x^p = z_x^p$. Therefore $\mho^1(G) \leq \mho^1(Z(G)) \leq \mho^1(G)$.

(v) Since G is a p-group, $\Phi(G) = G'\mho^1(G)$. (See [5], p.272). Thus by part (iv), $\Phi(G) = G'\mho^1(Z(G))$. If $|Z(G)| = p$ then $G' = Z(G)$ since by part (ii), $1 \neq G' \leq Z(G)$. Also, $\mho^1(Z(G)) = 1$. Thus $\Phi(G) = G' = Z(G)$. If $|Z(G)| = p^k$ where $k > 1$ then $Z(G) = \langle z \rangle$

where z has order p^k since G is of Hall type. But then $G' = \langle z^{p^{k-1}} \rangle = \langle (z^{p^{k-2}})^p \rangle \leq \mho^1(Z(G))$. Thus $\Phi(G) = \mho^1(Z(G))$.

In view of Lemma 2.1 (ii), we obtain the following corollary of Theorem 1.26.

Corollary 2.2. Let G be a finite non-abelian p-group of Hall type, p odd. Then $\operatorname{Aut} \Omega_1(G)$ can be embedded isomorphically in $\operatorname{Aut} G$.

Proof: By Result 1.20 (i), $G = Z(G)\Omega_1(G)$. Since $Z(G)$ is an abelian characteristic subgroup of G, it is cyclic of prime power order. By 2.1 (ii), $Z(G) \cap \Omega_1(G) = [\Omega_1(G)]'$ and thus is characteristic in $\Omega_1(G)$. Set $H = Z(G)$ and $K = \Omega_1(G)$ and apply Theorem 1.26.

Theorem 2.3. Let G be a finite non-abelian p-group of Hall type, p odd. If C is a non-abelian subgroup of $\Omega_1(G)$ such that C is characteristic in G then C is extra-special.

Proof: Since C is non-abelian, $1 \neq C' \leq [\Omega_1(G)]'$. But $|[\Omega_1(G)]'| = p$ so $C' = [\Omega_1(G)]'$ is cyclic of order p. Furthermore, $\Phi(C) = C'\mho^1(C) = C'$ because C is a p-group and $\exp C = p$. Since $Z(C)$ is characteristic in C and C is characteristic in G, $Z(C)$ is a characteristic abelian subgroup of G and thus is cyclic. $\operatorname{Exp} C = p$, whence $|Z(C)| = p$. This in turn implies $Z(C) = C'$ since $1 \neq C' \trianglelefteq C$ and $Z(C) \geq Z(C) \cap C' \neq 1$. Thus C is extra-special.

<u>Theorem 2.4</u>. Let G be a finite non-abelian p-group of Hall type, p odd. Then the only subgroups of $\Omega_1(G)$ which are characteristic in G are $\Omega_1(G)$, $\Phi(\Omega_1(G))$ and 1.

<u>Proof</u>: Let $1 \neq C \leq \Omega_1(G)$, where C is characteristic in G. Then $1 \neq C \cap Z(G)$. Moreover, by 2.1 (ii), $C \cap Z(G) \leq \Omega_1(G) \cap Z(G) = \Phi(\Omega_1(G))$. Thus, since $|\Phi(\Omega_1(G))| = p$, $C \cap Z(G) = \Phi(\Omega_1(G))$. Hence $\Phi(\Omega_1(G)) \leq C$.

If C is abelian then C is a characteristic abelian subgroup of G and thus C is a cyclic subgroup of $\Omega_1(G)$. It follows that $|C| = p$ because $\exp C = p$. We now have $1 \neq \Phi(\Omega_1(G)) \leq C$ and $|C| = p$. Hence $C = \Phi(\Omega_1(G))$.

Suppose C is non-abelian. Corollary 2.2 states that $\operatorname{Aut} \Omega_1(G)$ can be embedded isomorphically in $\operatorname{Aut} G$. Thus C is characteristic in $\Omega_1(G)$ as $C \leq \Omega_1(G)$ and C is characteristic in G. By 1.20 (i), $\Omega_1(G)$ is extra-special of exponent p. Since C is a non-abelian subgroup of $\Omega_1(G)$ and C is characteristic in G, Theorem 2.3 applies so C is extra-special. This in turn, by 1.28 implies that C is not characteristic in $\Omega_1(G)$ unless $C = \Omega_1(G)$. We conclude that if C is non-abelian then $C = \Omega_1(G)$.

Therefore, the only subgroups of $\Omega_1(G)$ which are characteristic in G are 1, $\Phi(\Omega_1(G))$ and $\Omega_1(G)$.

<u>Corollary 2.5</u>. Let G be a finite non-abelian p-group of Hall type, p odd. If C is a critical subgroup of G then $C = G$.

<u>Proof</u>: Since C is critical, $Z(G) \leq C_G(C) = Z(C) \leq C$. Let $K = \Omega_1(G) \cap C \neq 1$. Since K is the intersection of characteristic subgroups of G, K is characteristic in G. Hence by 2.4, either $K = \Phi(\Omega_1(G))$ or $K = \Omega_1(G)$.

If $K = \Phi(\Omega_1(G))$ and $c = w_c z_c$ is any element of C where $w_c \in \Omega_1(G)$ and $z_c \in Z(G) \leq C$ then $w_c \in K = \Omega_1(G) \cap C$. Since $K = \Phi(\Omega_1(G))$, 2.1 (ii) implies $K \leq Z(G)$ whence $w_c \in Z(G)$ which in turn implies $c \in Z(G)$. But then $C = Z(G)$. This is a contradiction since $Z(G)$ is not critical as $C_G(Z) = G$ and $G \not\leq Z(G) = Z(Z(G))$. Hence $K = \Omega_1(G)$ and $G = Z(G)\Omega_1(G) \leq C$. Thus $G = C$.

<u>Remark 2.6</u>. Essentially the same proof can be used to show that if C is a characteristic subgroup of G and $Z(G) \leq C$ then either $C = G$ or $C = Z(G)$.

<u>Theorem 2.7</u>. Let G be a finite non-abelian p-group of Hall type, p odd. If $|Z(G)| = p^{\ell}$ then $\Omega_k(G) = \mho^{\ell-k}(G)\Omega_1(G)$, where $\ell \geq k \geq 1$.

<u>Proof</u>: Let $Z(G) = \langle z \rangle$ be cyclic of order p^{ℓ}. By 2.1 (iv), $\mho^j(G) = \mho^j(Z(G))$, $j \geq 1$. Clearly $\mho^j(Z(G)) = \langle z^{p^j} \rangle$.

Let $k \geq 1$. Let $x = w_x z_x$ where $w_x \in \Omega_1(G)$ and $z_x \in Z(G)$. Then

$$\begin{aligned} & x \text{ is a generator of } \Omega_k(G) \\ \Leftrightarrow\ & (w_x z_x)^{p^k} = 1 \\ \Leftrightarrow\ & (z_x)^{p^k} = 1 \Leftrightarrow z_x \in \langle z^{p^{\ell-k}} \rangle = \mho^{\ell-k}(Z(G)). \end{aligned}$$

Thus if $\ell > k \geq 1$, then x is a generator of $\Omega_k(G)$

$$\begin{aligned} \Leftrightarrow\ & x = z_x w_x \text{ for some } z_x \in \mho^{\ell-k}(Z(G)),\ w_x \in \Omega_1(G) \\ \Leftrightarrow\ & x \in \mho^{\ell-k}(Z(G))\Omega_1(G) \\ \Leftrightarrow\ & x \in \mho^{\ell-k}(G)\Omega_1(G) \text{ since } \ell - k \geq 1 \text{ whence 2.1 (iv)} \\ & \qquad \text{implies } \mho^{\ell-k}(Z(G)) = \mho^{\ell-k}(G). \end{aligned}$$

Hence $\Omega_k(G) = \mho^{\ell-k}(G)\Omega_1(G)$ if $\ell - k \geq 1$.

If $k = \ell$, $\Omega_k(G) = \langle x \in G \mid x^{p^\ell} = 1 \rangle$. Now $x^{p^\ell} = 1 \Leftrightarrow$ $(w_x z_x)^{p^\ell} = 1 \Leftrightarrow (z_x)^{p^\ell} = 1$. But $|Z(G)| = p^\ell$ whence $(z_x)^{p^\ell}$ $= 1$ for any $z_x \in Z(G)$. Thus $\Omega_\ell(G) = G$. Since $\mho^{\ell-\ell}(G)\Omega_1(G) = \mho^0(G)\Omega_1(G) = G$, we also have $\mho^{\ell-k}(G)\Omega_1(G) =$ $G = \Omega_k(G)$ for $k = \ell$.

<u>Theorem 2.8</u>. Let G be a finite non-abelian p-group of Hall type, p odd. If $|Z(G)| = p^\ell$ then

$$O_p(\mathrm{Aut}\ G) = \mathrm{Stab}(G > \Omega_{\ell-1}(G) > \Omega_{\ell-2}(G) > \cdots > \Omega_1(G) > G' > 1).$$

<u>Proof</u>: By Theorem 2.7, $\Omega_k(G) = \mho^{\ell-k}(G)\Omega_1(G)$ for $\ell \geq k \geq 1$. Thus

$$|\Omega_k(G)| = \frac{|\mho^{\ell-k}(G)|\,|\Omega_1(G)|}{|\mho^{\ell-k}(G) \cap \Omega_1(G)|} = \begin{cases} \dfrac{p^k|\Omega_1(G)|}{p} & \text{if } 1 \leq k < \ell \\ |G| & \text{if } k = \ell \end{cases}$$

$$= \begin{cases} p^{k-1}|\Omega_1(G)| & \text{if } 1 \le k < \ell \\ p^{\ell-1}|\Omega_1(G)| & \text{if } k = \ell \end{cases}$$

$$= \quad p^{k-1}|\Omega_1(G)| \quad \text{if } 1 \le k \le \ell.$$

Therefore if $2 \le k \le \ell$ then

$$[\Omega_k(G) : \Omega_{k-1}(G)] = \frac{|\Omega_k(G)|}{|\Omega_{k-1}(G)|} = \frac{p^{k-1}|\Omega_1(G)|}{p^{k-2}|\Omega_1(G)|} = p$$

Let $\alpha \in \operatorname{Aut} G$ have order a power of p. By Corollary 1.33, $\alpha \in C_{\operatorname{Aut} G}(\Omega_k(G)/\Omega_{k-1}(G))$ for $2 \le k \le \ell$ and $\alpha \in C_{\operatorname{Aut} G}(G')$ since $[\Omega_k(G) : \Omega_{k-1}(G)] = p$ and $|G'| = p$.

Let $s : G > \Omega_{\ell-1}(G) > \Omega_{\ell-2}(G) > \ldots > \Omega_1(G) > G' > 1$. $\operatorname{Stab}(s) = \bigcap_{k=2}^{\ell} C_{\operatorname{Aut} G}(\Omega_k(G)/\Omega_{k-1}(G)) \cap C_{\operatorname{Aut} G}(\Omega_1(G)/G') \cap C_{\operatorname{Aut} G}(G')$ is a normal p-subgroup of $\operatorname{Aut} G$ since s is a characteristic chain and G is a p-group. (See 1.3 and 1.6). Thus $\operatorname{Stab}(s) \le O_p(\operatorname{Aut} G)$. On the other hand, $O_p(\operatorname{Aut} G) \le \bigcap_{k=2}^{\ell} C_{\operatorname{Aut} G}(\Omega_k(G)/\Omega_{k-1}(G)) \cap C_{\operatorname{Aut} G}(G')$. Hence $\operatorname{Stab}(s) = O_p(\operatorname{Aut} G) \cap C_{\operatorname{Aut} G}(\Omega_1(G)/G')$. Let $A = O_p(\operatorname{Aut} G)$. By 1.17, A is a stability group. Therefore by 1.12, γGA^i is characteristic in G for all i. Let j be the least integer such that $\gamma GA^j \le \Omega_1(G)$. (We know $j \le \ell - 1$ by 1.5 (i).) Since $\gamma GA^j \le \Omega_1(G)$ is a characteristic subgroup of G, Theorem 2.4 implies that $\gamma GA^j = \Omega_1(G)$ or $\Phi(\Omega_1(G))$ or 1.

If $\gamma GA^j = 1$ then $A \le \operatorname{Stab}(s)$.

If $\gamma GA^j \ne 1$ then $\gamma GA^{j+1} < \gamma GA^j$, since there exists a smallest integer $n(A)$ such that $\gamma GA^{n(A)} = 1$. It follows that $\gamma GA^{j+1} = \Phi(\Omega_1(G))$ or 1. Thus $A \leq C_{\mathrm{Aut}\,G}(\Omega_1(G)/G')$ since $\Phi(\Omega_1(G)) = G'$ by 2.1 (ii). Hence $A = O_p(\mathrm{Aut}\ G) \leq \mathrm{Stab}(s)$.

We conclude that $O_p(\mathrm{Aut}\ G) = \mathrm{Stab}(s)$.

In [1], Adney and Yen investigated $C_{\mathrm{Aut}\,G}(G/Z(G))$ which they called A_c. They defined a purely non-abelian p-group to be a group which does not have an abelian direct factor and obtained the following result.

<u>Result 2.9</u>. If G is a p-group which is purely non-abelian then the group A_c of central automorphisms is also a p-group.

<u>Result 2.10</u> ([2], p.24). Let G be a p-group and Z be the center of G. If $G' \geq Z$ then

$$A_c = \mathrm{Stab}(G \geq Z \geq 1) \approx \mathrm{Hom}(G/G', Z).$$

<u>Theorem 2.11</u>. Let G be a non-abelian p-group of Hall type, p odd. Then G is purely non-abelian.

<u>Proof</u>: Suppose not. Then $G = A \times B$ where A is abelian and $A \cap B = 1$. Since $G/B \approx A$ and A is abelian, $G' \leq B$. Thus $G' \cap A = 1$. However, $A \leq Z(G)$, $G' \leq Z(G)$, $|G'| = p$ and $Z(G)$ is cyclic. Hence $G' \leq A$. This is a contradiction. Therefore G is purely non-abelian.

<u>Corollary 2.12</u>. Let G be a non-abelian p-group of Hall type, p odd. Then the group A_c of central automorphisms is a p-group.

<u>Proof</u>: The conclusion follows immediately from 2.9 and 2.11.

<u>Theorem 2.13</u>. Let G be a non-abelian p-group of Hall type, p odd, such that $|Z(G)| = p^{\ell}$. Let $s : G \geq Z = \mho^0(Z) \geq \mho^1(Z) \geq \cdots \geq G' = \mho^{\ell-1}(Z) > 1$, where $Z = Z(G)$. Then $A_c = \mathrm{Stab}(s)$.

<u>Proof</u>: Since $Z(G)$ is cyclic of order p^{ℓ}, $\mho^i(Z)$ is cyclic of order $p^{\ell-i}$, $0 \leq i \leq \ell$. Therefore $[\mho^i(Z) : \mho^{i+1}(Z)] = p$ for $0 \leq i \leq \ell - 1$. By 1.33, if $\alpha \in \mathrm{Aut}\, G$ has order a power of p then $\alpha \in C_{\mathrm{Aut}\, G}(\mho^i(Z)/\mho^{i+1}(Z))$ where $0 \leq i \leq \ell - 1$. Hence since A_c is a p-group by 2.12, we have $\mathrm{Stab}(s) = A_c$.

<u>Theorem 2.14</u>. Let G be a non-abelian p-group of Hall type, p odd. Then $A_c = C_{\mathrm{Aut}\, G}(\Omega_1(G)/G')$.

<u>Proof</u>: Let $\alpha \in A_c$ and $w \in \Omega_1(G)$. Then $w^{-1}w^{\alpha} \in Z(G) \cap \Omega_1(G) = G'$. Therefore $\alpha \in C_{\mathrm{Aut}\, G}(\Omega_1(G)/G')$. It follows that $A_c \leq C_{\mathrm{Aut}\, G}(\Omega_1(G)/G')$.

Let $\alpha \in C_{\mathrm{Aut}\, G}(\Omega_1(G)/G')$ and $g = w_g z_g \in G$ where $w_g \in \Omega_1(G)$ and $z_g \in Z(G)$. Then $g^{-1}g^{\alpha} = z_g^{-1}z_g^{\alpha}\, w_g^{-1}w_g^{\alpha} \in Z(G) \cdot G' = Z(G)$. Therefore $\alpha \in A_c$ and $C_{\mathrm{Aut}\, G}(\Omega_1(G)/G') \leq A_c$.

Thus $A_c = C_{\mathrm{Aut}\, G}(\Omega_1(G)/G')$.

Corollary 2.15. With G as above, $C_{\text{Aut } G}(\Omega_1(G)/G')$ is a p-group.

Proof: The conclusion follows immediately from 2.12 and 2.14.

Theorem 2.16. Let G be a non-abelian p-group of Hall type, p odd, such that $|Z(G)| = p^{\ell}$. Then $A_c = O_p(\text{Aut } G)$.

Proof: Let $s : G > \Omega_{\ell-1}(G) > \ldots > \Omega_1(G) > G' > 1$. Then by 2.8, $O_p(\text{Aut } G) = \text{Stab}(s)$ and by 1.33 and 2.15, $C_{\text{Aut } G}(\Omega_1(G)/G') = \text{Stab}(s)$. Thus 2.14 implies $O_p(\text{Aut } G) = A_c$.

Theorem 2.17. Let G be a non-abelian p-group of Hall type, p odd. Then for $i, j \geq 1$, $C_{\text{Aut } G}(G/\mho^j(G)) = C_{\text{Aut } G}(\mho^i(G)/\mho^{i+j}(G)) \cap O_p(\text{Aut } G)$.

Proof: Let $i, j \geq 1$.

Since $\mho^j(G)$ is characteristic in G, $C_{\text{Aut } G}(G/\mho^j(G)) \trianglelefteq \text{Aut } G$. By 1.11 and 1.6, $C_{\text{Aut } G}(G/\mho^j(G))$ is a p-group since G is a p-group and $\mho^j(G) \leq \Phi(G)$. Thus $C_{\text{Aut } G}(G/\mho^j(G)) \leq O_p(\text{Aut } G)$.

Henceforth $\mho^j$ will denote $\mho^j(G)$ which by 2.1 (iv) equals $\mho^j(Z(G))$. Since $Z(G)$ is a characteristic abelian subgroup of G, it is cyclic. Let $Z(G) = \langle x \rangle$ and suppose $|x| = p^{\ell}$.

If $\alpha \in C_{\mathrm{Aut}\,G}(G/\mho^j)$ then $x^{-1}x^{\alpha} \in \mho^j$. Thus $(x^{p^i})^{-1}(x^{p^i})^{\alpha} = (x^{-1}x^{\alpha})^{p^i} \in \mho^{i+j}$. It follows that $\alpha \in C_{\mathrm{Aut}\,G}(\mho^i/\mho^{i+j})$.

Now let $\alpha \in C_{\mathrm{Aut}\,G}(\mho^i/\mho^{i+j}) \cap O_p(\mathrm{Aut}\ G)$. Theorem 2.8 implies $\alpha \in C_{\mathrm{Aut}\,G}(\Omega_1(G)/G')$. Furthermore, $(x^{-1}x^{\alpha})^{p^i} = (x^{p^i})^{-1}(x^{p^i})^{\alpha} \in \mho^{i+j}$ so $x^{-1}x^{\alpha} \in \mho^j$ whence $\alpha \in C_{\mathrm{Aut}\,G}(Z/\mho^j)$. If $g = z_g w_g \in G$ where $z_g \in Z(G)$ and $w_g \in \Omega_1(G)$ then $g^{-1}g^{\alpha} = z_g^{-1}z_g^{\alpha}\, w_g^{-1}w_g^{\alpha} \in \mho^j G' = \mho^j$. Hence $\alpha \in C_{\mathrm{Aut}\,G}(G/\mho^j)$.

It follows that $C_{\mathrm{Aut}\,G}(G/\mho^j) = C_{\mathrm{Aut}\,G}(\mho^i/\mho^{i+j}) \cap O_p(\mathrm{Aut}\ G)$.

<u>Corollary 2.18</u>. If G is as in 2.17 and $i + j = k$ then $C_{\mathrm{Aut}\,G}(G/\mho^{k-i}(G)) = C_{\mathrm{Aut}\,G}(\mho^i(G)) \cap O_p(\mathrm{Aut}\ G)$.

<u>Theorem 2.19</u>. Let G be a finite non-abelian p-group of Hall type, p odd. Let $A = \mathrm{Stab}(s)$ where $s : G > \mho^{i_1} > \mho^{i_2} > \ldots > \mho^{i_\ell} = 1$ and $\ell \geq 2$. Then $A = C_{\mathrm{Aut}\,G}(G/\mho^{\max\{i_1, j\}})$ where $j = \max\limits_{s=1,\ldots,\ell-1} \{i_{s+1} - i_s\}$. Here $\mho^{i_k} = \mho^{i_k}(G)$, $k = 1,\ldots,\ell$.

<u>Proof</u>: Let $t \in \{1,\ldots,\ell-1\}$ such that $i_{t+1} - i_t = j$. Then for any $s \in \{1,\ldots,\ell-1\}$, $i_{s+1} \leq j + i_s$ and it follows that $\mho^{i_s} > \mho^{i_{s+1}} \geq \mho^{i_s+j}$. Thus $C_{\mathrm{Aut}\,G}(\mho^{i_s}/\mho^{i_s+j}) \leq C_{\mathrm{Aut}\,G}(\mho^{i_s}/\mho^{i_{s+1}})$ for $s = 1,\ldots,\ell-1$, whence by 2.17, $C_{\mathrm{Aut}\,G}(G/\mho^j) \leq C_{\mathrm{Aut}\,G}(\mho^{i_s}/\mho^{i_{s+1}}) \cap O_p(\mathrm{Aut}\ G)$ for $s = 1,\ldots,\ell-1$. Thus

$$C_{\text{Aut } G}(G/\mho^j) \leq \bigcap_{s=1}^{\ell-1} C_{\text{Aut } G}(\mho^{i_s}/\mho^{i_{s+1}}) \cap O_p(\text{Aut } G)$$

$$\leq C_{\text{Aut } G}(\mho^{i_t}/\mho^{i_{t+1}}) \cap O_p(\text{Aut } G)$$

$$= C_{\text{Aut } G}(G/\mho^j) \text{ by 2.18, since } i_{t+1} - i_t = j.$$

It follows that $A = C_{\text{Aut } G}(G/\mho^{i_1}) \cap C_{\text{Aut } G}(G/\mho^j)$

$$= C_{\text{Aut } G}(G/\mho^{\max\{i_1, j\}}).$$

Theorem 2.20. Let G be a finite non-abelian p-group of Hall type, p odd. Then $C_{\text{Aut } G}(G/\mho^j) = C_{\text{Aut } G}(Z(G)/\mho^j) \cap O_p(\text{Aut } G)$ for $j \geq 1$ and $\mho^j = \mho^j(G)$.

Proof: Let $j \geq 1$.

By 1.11 and 1.6, $C_{\text{Aut } G}(G/\mho^j)$ is a p-group since G is a p-group and $\mho^j \leq \Phi(G)$. Thus $C_{\text{Aut } G}(G/\mho^j) \leq C_{\text{Aut } G}(Z(G)/\mho^j) \cap O_p(\text{Aut } G)$.

Let $\alpha \in C_{\text{Aut } G}(Z(G)/\mho^j) \cap O_p(\text{Aut } G)$. By Theorem 2.8, $\alpha \in C_{\text{Aut } G}(\Omega_1(G)/G')$. If $g = z_g w_g \in G$ where $z_g \in Z(G)$ and $w_g \in \Omega_1(G)$ then $g^{-1}g^{\alpha} = z_g^{-1} z_g^{\alpha} w_g^{-1} w_g^{\alpha} \in \mho^j \cdot G' = \mho^j$. Hence $\alpha \in C_{\text{Aut } G}(G/\mho^j)$.

It follows that $C_{\text{Aut } G}(G/\mho^j) = C_{\text{Aut } G}(Z(G)/\mho^j) \cap O_p(\text{Aut } G)$.

Theorem 2.21. Let G be a finite non-abelian p-group of Hall type, p odd. Let $A = \text{Stab}(s)$ where $s : G > Z(G) > \mho^{i_1} > \mho^{i_2} > \ldots > \mho^{i_\ell} = 1$ and $\ell \geq 2$. Then $A = C_{\text{Aut } G}(G/\mho^{\max\{i_1, j\}})$ where $j = \max_{s=1,\ldots,\ell-1} \{i_{s+1} - i_s\}$. Here $\mho^{i_k} = \mho^{i_k}(G)$ $k = 1, \ldots, \ell$.

Proof: As in 2.19, $C_{\mathrm{Aut}\,G}(G/\mho^{j}) = \bigcap_{s=1}^{\ell-1} C_{\mathrm{Aut}\,G}(\mho^{i_s}/\mho^{i_{s+1}}) \cap O_p(\mathrm{Aut}\,G)$. Therefore

$$\begin{aligned} A &= C_{\mathrm{Aut}\,G}(G/Z(G)) \cap C_{\mathrm{Aut}\,G}(Z(G)/\mho^{i_1}) \cap C_{\mathrm{Aut}\,G}(G/\mho^{j}) \\ &= C_{\mathrm{Aut}\,G}(G/Z(G)) \cap C_{\mathrm{Aut}\,G}(G/\mho^{i_1}) \cap C_{\mathrm{Aut}\,G}(G/\mho^{j}), \quad \text{by 2.20} \\ &= C_{\mathrm{Aut}\,G}(G/\mho^{i_1}) \cap C_{\mathrm{Aut}\,G}(G/\mho^{j}), \quad \text{since } \mho^{i_1} \leq Z(G) \\ &= C_{\mathrm{Aut}\,G}(G/\mho^{\max\{i_1, j\}}). \end{aligned}$$

Remark 2.22. If G is a finite non-abelian p-group of Hall type, p odd, and G is not extra-special then $\mathrm{Inn}\,G$ is not a closed stability group.

Proof: Let $A = \mathrm{Inn}\,G$. Since $Z(G)$ is cyclic, 1.25 implies $A = \mathrm{Stab}(G > Z(G) > 1)$. Thus

$$\begin{aligned} \bar{A} &= \mathrm{Stab}(G > [G,A] = G' > [G,A,A] = 1) \\ &= \mathrm{Stab}(G > G' > 1) = C_{\mathrm{Aut}\,G}(G/G') \quad \text{by 2.19.} \end{aligned}$$

Let $G = \langle x, w_1, \ldots, w_t \rangle$ where $Z(G) = \langle x \rangle$ and $\Omega_1(G) = \langle w_1, \ldots, w_t \rangle$. Since G is not extra-special, $Z(G) > G'$ thus if $|x| = p^k$ then $k \geq 2$.

Define a mapping α from G to G by

$$\begin{cases} x \to x x^{p^{k-1}} \\ w_i \to w_i . \end{cases}$$

Then as in the proof of 1.26, α is an automorphism of G. Furthermore, $\alpha \in C_{\mathrm{Aut}\,G}(G/G') \setminus \mathrm{Inn}\,G = \bar{A} \setminus A$. Thus $\mathrm{Inn}\,G$ is not closed as a stability group.

Theorem 2.23. Let G be a finite non-abelian p-group of Hall type, p odd, G not extra-special. Let A be a non-trivial normal p-subgroup of $\mathrm{Aut}\, G$. Then $\bar{A} = C_{\mathrm{Aut}\, G}(G/\mho^j)$ for some j such that $k-1 \geq j \geq 1$ where $|Z(G)| = p^k$.

Proof: Since A is a normal p-subgroup of $\mathrm{Aut}\, G$, $A \leq O_p(\mathrm{Aut}\, G)$. Thus by 2.14, and 2.16, $A \leq C_{\mathrm{Aut}\, G}(\Omega_1(G)/G')$. Let $g = z_g w_g \in G$ and $\alpha \in A$ where $z_g \in Z(G)$ and $w_g \in \Omega_1(G)$. Then $g^{-1}g^{\alpha} = z_g^{-1}z_g^{\alpha}\, w_g^{-1}w_g^{\alpha} \in Z(G) \cdot G' = Z(G)$. Thus $[G,A] \leq Z(G)$.

By 1.17, $O_p(\mathrm{Aut}\, G)$ is a stability group, whence there exists an integer n such that $\gamma\, G\, O_p(\mathrm{Aut}\, G)^n = 1$. This follows from 1.5 (i). Since $A \leq O_p(\mathrm{Aut}\, G)$, $\gamma G A^n = 1$ also. Now $n \neq 1$ since $A \neq 1$. Furthermore, if $[G,A] = Z(G)$ then $n \neq 2$, otherwise $\bar{A} = \mathrm{Stab}(G > Z(G) > 1)$ and 1.25 implies $\bar{A} = \mathrm{Inn}\, G$ since $Z(G)$ is cyclic. But this is a contradiction since $\bar{A}$ is closed by 1.11 (iii) while $\mathrm{Inn}\, G$ is not closed (see 2.22). Therefore

$$\bar{A} = \mathrm{Stab}(G > Z(G) > \mho^{i_1} > \dots > \mho^{i_\ell} = 1) \qquad \ell \geq 2$$

or

$$\bar{A} = \mathrm{Stab}(G > \mho^{i_1} > \dots > \mho^{i_\ell} = 1) \qquad \ell \geq 2.$$

In either case Theorems 2.19 and 2.21 imply $\bar{A} = C_{\mathrm{Aut}\, G}(G/\mho^j)$ for some j such that $1 \leq j \leq k-1$ where $|Z(G)| = p^k$.

Corollary 2.24. Let G be as in 2.23. Let A be a non-trivial normal p-subgroup of $\text{Aut } G$. Then $A = \bar{A}$ if and only if $A = C_{\text{Aut } G}(G/\mho^j)$ for some $j \geq 1$.

Proof: The conclusion follows immediately from 2.23 and 1.13 since $\mho^j \leq \Phi(G)$.

Theorem 2.25. Let G be a finite non-abelian p-group of Hall type, p odd, G not extra-special. If A is a non-trivial normal p-subgroup of $\text{Aut } G$ then $\bar{A} > \text{Inn } G$.

Proof: By 2.23, $\bar{A} = C_{\text{Aut } G}(G/\mho^j)$ for some j such that $k-1 \geq j \geq 1$, where $|Z(G)| = p^k$. Since $G' = \mho^{k-1} \leq \mho^j$, $\text{Inn } G \leq \bar{A}$. Moreover, by 2.22, $\text{Inn } G$ is not closed. Hence $\text{Inn } G < A$.

Corollary 2.26. If G and A are as in 2.25 and A is closed as a stability group then $A > \text{Inn } G$.

Proof: By 2.25 and the definition of closed, $A = \bar{A} > \text{Inn } G$.

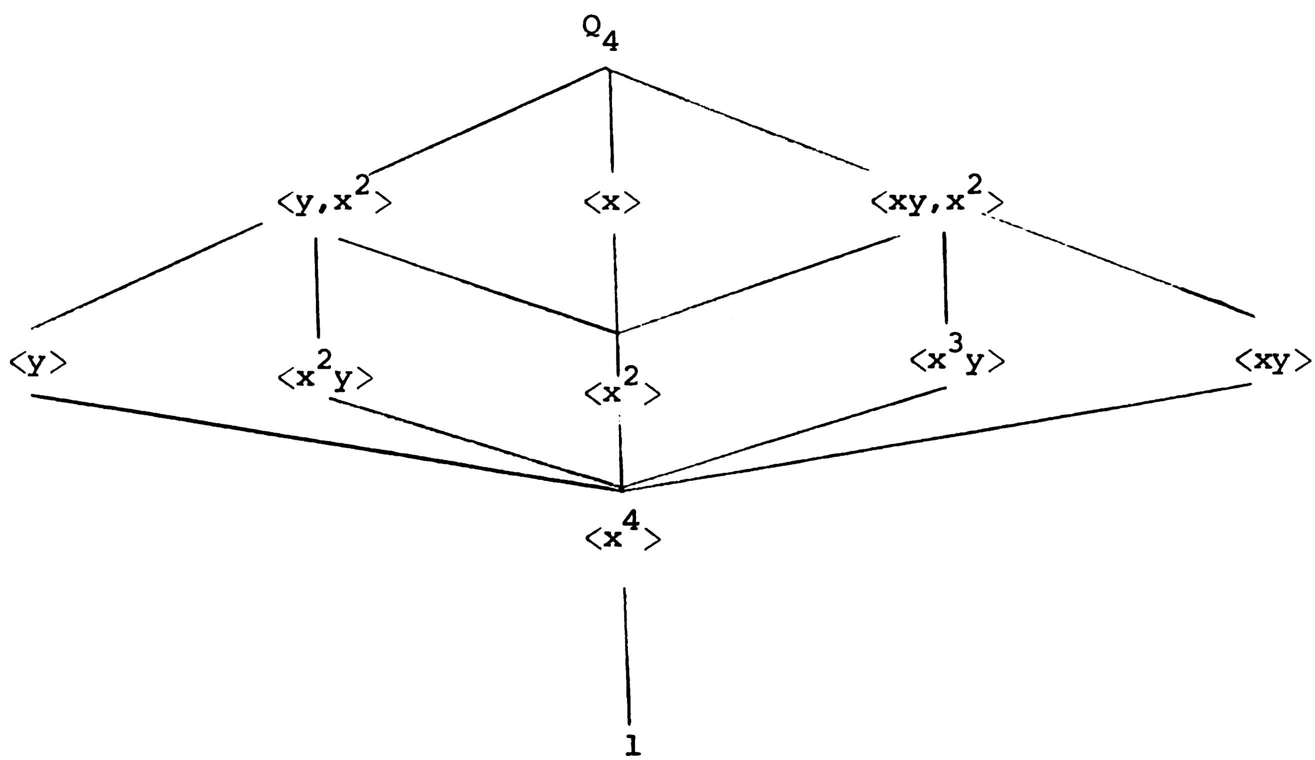
Remark 2.27. Let G be an extra-special p-group of Hall type, p a prime. It follows from the proof of Proposition 3.2 in [7] that $O_p(\text{Out } G) = 1$. Therefore $\text{Aut } G$ possesses no normal p-subgroups properly containing $\text{Inn } G$. Moreover, $O_p(\text{Aut } G) = \text{Inn } G$ whence, by Corollary 1.17, $\text{Inn } G$ is closed as a stability group.

An obvious question remains. What happens when G is a 2-group of Hall type, G not extra-special?

Consider the Generalized Quaternion group of order 16 which is given by

$$Q_4 = \langle x,y \mid x^4 = y^2 = m,\ m^2 = 1,\ y^{-1}xy = x^{-1}\rangle$$

and has the following subgroup lattice.



Since $\Phi(Q_4) = Q_4' = \langle x^2\rangle$ and $Z(Q_4) = \langle x^4\rangle$, Q_4 is not extra-special. Moreover, Q_4 is of Hall type.

An automorphism of Q_4 takes x to one of the four elements of order eight (x,x^3,x^5,x^7) and y to one of the eight elements of order four, $(x^iy$ where $i = 0,1,\ldots,7)$. Thus $|\mathrm{Aut}\ Q_4| = 32$.

Inn $Q_4 \approx Q_4/Z(Q_4) = Q_4/\langle x^4 \rangle \approx D$ where D is the Dihedral group of order eight.

Set $A = \mathrm{Stab}(Q_4 \geq Z(Q_4) \geq 1)$. Then A is a normal abelian 2-group of Aut Q_4. Since Inn $G \approx D$ is non-abelian, Inn $G \nleq A$. The automorphism α such that $x^\alpha = x^5$ and $y^\alpha = y$ is a non-trivial automorphism in A. Moreover, $A = \bar{A}$. Thus $1 \neq A \triangleleft_2$ Aut Q_4 and $A = \bar{A}$ does not guarantee that A properly contains Inn G.

CHAPTER III

CLOSED SUBGROUPS AND LIFTING OF AUTOMORPHISM GROUPS

As was demonstrated in Chapter I, an automorphism of a normal subgroup can sometimes be extended to an automorphism of the whole group. In this chapter we show that under certain conditions, lifting a stability group of a chain in a normal subgroup will produce a normal and even closed stability group in the automorphism group of the whole group.

The material in this chapter stems from our consideration of p-groups which are not of Hall type and our efforts to determine the normal p-subgroups in the automorphism group of such a group.

Let G be a non-abelian p-group, p a prime, and $N \trianglelefteq G$. We consider the group of extensions of N by G/N.

We use the following results from cohomology groups; the notation $H^n(B,A)$ denotes the nth cohomology group of B over A, $n \geq 1$. (See [6], pp.128-130).

<u>Result 3.1</u> ([6], p.131). Let A be an abelian group. The second cohomology group $H^2(B,A)$ coincides with the group

of non-equivalent extensions of A by the group B with a given associated homomorphism θ.

Result 3.2 ([6], p.142). Suppose there exist extensions of the group A by the group B with associated homomorphism θ. Then there exists a one-to-one correspondence between all non-equivalent extensions of A by B associated with θ and all non-equivalent extensions of the center $Z(A)$ of A by B associated with θ.

Schmid, in [7], obtains the following criterion for extending automorphisms of normal subgroups to automorphisms of the whole group.

Result 3.3 (Schmid, [7]). Let $N \trianglelefteq G$ such that $C_G(N) = Z(N)$ and $H^2(G/N, Z(N)) = 1$. Let D denote the group of automorphisms of N induced by G, and let A_N be any automorphism group of N. Then

i) There is an automorphism group A of G normalizing N and inducing A_N on N if and only if A_N normalizes D.

ii) If A_N centralizes $D/\mathrm{Inn}\, N$, then A_N can be lifted to G and any automorphism group of G extending A_N acts trivially on G/N.

Let G be an arbitrary finite group. Suppose $N \trianglelefteq G$ and $A_N \leq \mathrm{Aut}\, N$ such that A_N can be lifted to $A \leq \mathrm{Aut}\, G$. The following question arises. What properties of A_N are inherited by A? As a partial answer to this question when A_N is a stability group, we have the following.

<u>Theorem 3.4</u>. Let $N \trianglelefteq G$ where G is an arbitrary finite group. Let $A_N \leq \mathrm{Stab}(N = N_0 \geq N_1 \geq \ldots \geq N_k = 1)$. Suppose A_N extends to a subgroup A of $\mathrm{Aut}\, G$ such that A centralizes G/N. Then

$$\gamma GA^i \leq \gamma NA^{i-1} \leq N_{i-1} \quad \text{for} \quad i \geq 1.$$

<u>Proof</u>: Induct on i.

If $i = 1$, $\gamma GA \leq N$ since A centralizes G/N. Also, $N = N_0 = \gamma NA_N^0 = \gamma NA^0$ since A extends A_N.

Suppose $\gamma GA^j \leq \gamma NA^{j-1} \leq N_{j-1}$, for $j \leq i$. Then $\gamma GA^{i+1} = [\gamma GA^i, A] \leq [\gamma NA^{i-1}, A] = \gamma NA^i$. Moreover, since A extends A_N and $\gamma NA_N^{i-1} \leq N_{i-1}$, it follows that $[\gamma NA^{i-1}, A] = [\gamma NA_N^{i-1}, A] = [\gamma NA_N^{i-1}, A_N] = \gamma NA_N^i \leq N_i$.

Let G, N, A and A_N be as in 3.4. We have:

<u>Corollary 3.5</u>. $A \leq \mathrm{Stab}(G \geq N_0 \geq N_1 \geq \ldots \geq N_k = 1)$ and $A \leq \mathrm{Stab}(G \geq \gamma NA_N^0 \geq \gamma NA_N^1 \geq \ldots \geq \gamma NA_N^k = 1)$.

Let $B = \mathrm{Stab}(G \geq \gamma NA_N^0 \geq \gamma NA_N^1 \geq \ldots \geq \gamma NA_N^k = 1)$ and $B_N = B|_N$. Then $\gamma NB_N^i \leq \gamma NA_N^i$ for all $i \geq 0$. Since $A \leq B$, $A_N \leq B_N$. Consequently $\gamma NA_N^i \leq \gamma NB_N^i$ for all $i \geq 0$. Thus $\gamma NA_N^i = \gamma NB_N^i$ for $i \geq 0$. Hence $B_N \leq \mathrm{Stab}(N \geq \gamma NA_N^1 \geq \ldots \geq \gamma NA_N^k = 1) = \overline{A_N}$ and therefore $A_N \leq B_N \leq \overline{A_N}$.

If A_N is closed then $A_N = B_N$ and B is a stability group extending A_N. If, in addition, N is characteristic in G and A_N is normal in $\mathrm{Aut}\, N$ then by 1.3, B is normal in Aut G since for $i \geq 0$, γNA_N^i is characteristic in N and thus in G.

In summary, we have the following theorem.

Theorem 3.6. Let $N \trianglelefteq G$ where G is an arbitrary finite group. Let $A_N \leq \mathrm{Stab}(N = N_0 \geq N_1 \geq \cdots \geq N_k = 1)$. Suppose A_N extends to a subgroup A of Aut G such that A centralizes G/N. If A_N is closed, then A_N extends to a stability group in Aut G, namely $B = \mathrm{Stab}(G \geq \gamma NA_N^0 \geq \gamma NA_N^1 \geq \cdots \geq \gamma NA_N^k = 1)$. Moreover, if N is characteristic in G and A_N is normal in Aut N then A_N extends to a normal stability group in Aut G.

Theorem 3.7. Let $N \trianglelefteq G$ where G is a finite group. Let $A_N \leq \mathrm{Aut}\, N$ such that $A_N \leq \mathrm{Stab}(N = N_0 \geq N_1 \geq \cdots \geq N_k = 1)$. Suppose A_N extends to a subgroup A of Aut G such that $A \leq C_{\mathrm{Aut}\, G}(G/\gamma NA^1)$. Set $B = \bar{A}|_N$. Then $B \leq \overline{A_N}$ and B extends to $\bar{A}$.

Proof: Since $N \leq G$, $\gamma NA \leq \gamma GA$. On the other hand, $\gamma GA \leq \gamma NA$ since $A \leq C_{\mathrm{Aut}\, G}(G/\gamma NA)$. Thus $\gamma GA = \gamma NA$ which in turn implies $\gamma GA^i = \gamma NA^i$. It follows from 3.5 and 1.11 (ii) that for $i \geq 1$, $\gamma N\bar{A}^i \leq \gamma G\bar{A}^i = \gamma GA^i = \gamma NA^i \leq \gamma N\bar{A}^i$ and so $\gamma N\bar{A}^i = \gamma NA^i$.

Let $\bar{A}|_N \leq \mathrm{Aut}\, N$ be the restriction of $\bar{A}$ to N. Then for $i \geq 0$, $\gamma N\bar{A}|_N^i = \gamma N\bar{A}^i$, thus $\gamma N\bar{A}|_N^i = \gamma NA^i = \gamma NA_N^i$ for $i \geq 0$.

It follows that $\bar{A}|_N \leq \mathrm{Stab}(N \geq \gamma NA_N \geq \cdots \geq \gamma NA_N^k = 1) = \overline{A_N}$.
Thus $B = \bar{A}|_N \leq \overline{A_N}$ and B extends to $\bar{A}$.

Corollary 3.8. Let G, A_N, A and B be as in 3.7. If A_N is closed then A is closed.

Proof: From 3.7, $A_N \leq \bar{A}|_N = B \leq \overline{A_N}$. Thus if A_N is closed then $B = A_N$ and therefore $A = \bar{A}$. Hence A is closed.

Remark 3.9. Let N be a normal subgroup of a finite group G such that $C_G(N) = Z(N)$ and $H^2(G/N, Z(N)) = 1$. Suppose $A_N \leq \tau_N$ such that A_N centralizes $D/\mathrm{Inn}\ N$ where D is the group of automorphisms of N induced by G. Then $A_N = \overline{A_N}$ in Aut N implies A_N extends to a stability group in Aut G. Moreover, if N is characteristic in G and A_N is normal in Aut N then A_N extends to a normal stability group in Aut G.

Proof: By 3.3, A_N can be lifted to an automorphism group of G which centralizes G/N. Now apply Theorem 3.6.

Remark 3.10. It should be noted that:

(i) If G is a p-group then a critical subgroup K of G is characteristic in G and has the property that $C_G(K) = Z(K)$. (See [4], p.185). Thus a proper critical subgroup of a p-group would be a candidate for N in 3.9.

(ii) If N is a proper normal subgroup of a finite group G such that there is only one non-equivalent extension of N by G/N associated with a given homomorphism θ then $H^2(G/N, Z(N)) = 1$. (See Results 3.1 and 3.2).

The following results were motivated by my search for examples where the above theorems hold and my efforts to produce normal p-subgroups in the automorphism group of a finite p-group.

Remark 3.11. Let G be a semi-direct product of $K \trianglelefteq G$ and H. Let $A_K \leq \mathrm{Aut}\, K$. If the automorphisms of K induced by H are contained in $C_{\mathrm{Aut}\, K}(A_K)$ then A_K can be extended to a subgroup A of $\mathrm{Aut}\, G$ such that $A \leq C_{\mathrm{Aut}\, G}(G/K)$.

Proof: Let $\alpha \in A_K$. Define φ_α from G to G by

$$(hk)^{\varphi_\alpha} = hk^\alpha.$$

In [2], Lemma 1.23, Bertelsen showed that φ_α, as defined above, is an automorphism of $G = HK$. Also α centralizes G/K. Let $A = \{\varphi_\alpha | \alpha \in A_K\}$. A has the desired properties.

Corollary 3.12. Let K be a finite group. If $A_K \leq \mathrm{Aut}\, K$ is an abelian group and $G = K \rtimes A_K$ then A_K can be extended to a subgroup A of $\mathrm{Aut}\, G$ centralizing G/K.

Proof: Let $\alpha \in A_K$, $k \in K$. The automorphism which α induces on K is α itself since $\alpha^{-1}k\alpha = k^{\alpha}$ for every $k \in K$. Thus the group of automorphisms of K induced by A_K is just A_K. Since A_K is abelian, $A_K \le C_{\mathrm{Aut}\,K}(A_K)$. Hence, by Remark 3.11, A_K extends to a subgroup A of $\mathrm{Aut}\,G$ such that $A \le C_{\mathrm{Aut}\,G}(G/K)$.

In the same vein, we have the following.

Remark 3.13. Let K be a non-abelian p-group such that $K' \le Z(K)$. Let $G = K \rtimes A_K$ where $A_K = \mathrm{Stab}(K = K_0 \ge K_1 \ge \cdots \ge K_n = 1)$ and $K_1 \le Z(K)$. Then $[G,K] \le Z(K)$.

Proof: Let $k \in K$ and $g = x\alpha \in G$ where $x \in K$ and $\alpha \in A_K$. Then $g^{-1}kg = \alpha^{-1}x^{-1}kx\alpha = (k^x)^{\alpha} = k^x \cdot z$ for some $z \in K$, since $\alpha \in A_K$. It follows that $k^{-1}g^{-1}kg = k^{-1}k^x z \in K'K_1 \le Z(K)$.

Theorem 3.14. Let G be a non-abelian p-group. Let $A = \mathrm{Stab}(G \ge N \ge 1)$ where $N \trianglelefteq G$ and $\exp N = p$. Then A is elementary abelian.

Proof: Since N is abelian, 1.8 implies A is abelian. Let $\alpha \in A$ and $g \in G$. Then $g^{\alpha} = gn$ for some $n \in N$.

We now show that $g^{\alpha^i} = gn^i$ for all $i \ge 1$.

If $i = 1$, $g^{\alpha^1} = gn^1$.

Suppose $g^{\alpha^k} = gn^k$. Then

$$g^{\alpha^{k+1}} = (g^{\alpha^k})^\alpha$$

$$= (gn^k)^\alpha \quad \text{by the induction hypothesis}$$

$$= gn \cdot n^k = gn^{k+1}.$$

It follows that $g^{\alpha^p} = gn^p = g$ since $\exp N = p$. Hence $\alpha^p = 1$ so $|\alpha| \,\big|\, p$.

Thus A is elementary abelian.

<u>Remark 3.15</u>. Let K be a non-abelian p-group such that $\Phi(K) \leq Z(K)$, $|Z(K)| > p$, and $\exp Z(K) = p$. Let $A_K = \mathrm{Stab}(K \geq Z(K) \geq 1)$. By 1.25, with $G = K$ and $N = Z(K)$, $A_K > \mathrm{Inn}\, K$. Let $\alpha \in A_K \setminus \mathrm{Inn}\, K$. Set $G = K \rtimes \langle\alpha\rangle$. Then

(i) $C_G(K) = Z(K)$

(ii) $[G,K] \leq Z(K)$ and

(iii) $cl(K) \leq 2$ and $K/Z(K)$ is elementary abelian.

<u>Proof</u>: Let $g = x\beta \in G$ where $x \in K$ and $1 \neq \beta \in \langle\alpha\rangle$. Let $k \in K$.

(i) By 3.14, α has order p. Thus $\beta = \alpha^i$ for some i, $0 < i < p$. Also, since $\alpha \notin \mathrm{Inn}\, K$ and $|\alpha| = p$, it follows that $\beta \notin \mathrm{Inn}\, K$. Now

$$g^{-1}kg = k \Leftrightarrow \beta^{-1}x^{-1}kx\beta = k$$

$$\Leftrightarrow (k^{\varphi_x})^\beta = k \quad \text{where } \varphi_x \in \mathrm{Inn}\, K$$

$$\Leftrightarrow (k^\beta)^{\varphi_x} = k \quad \text{since } \mathrm{Inn}\, K \leq A_K \text{ and } A_K \text{ is abelian}$$

$$\Leftrightarrow k^\beta = k^{(\varphi_x)^{-1}} \Leftrightarrow \beta \in \mathrm{Inn}\, K.$$

Thus $C_G(K) = Z(K)$.

(ii) Since K is a p-group, $\Phi(K) = K' \cdot \mho^1(K)$. (See [5], p.272). Thus $K' \leq \Phi(K) \leq Z(K)$. By 3.13, $[K \rtimes A_K, K] \leq Z(K)$. Since $[G,K] = [K \rtimes \langle\alpha\rangle, K] \leq [K \rtimes A_K, K]$, we have $[G,K] \leq Z(K)$.

(iii) Result (iii) follows from the fact that $\Phi(K) \leq Z(K)$.

By result 1.8, abelian stability groups may be found by considering $\mathrm{Stab}(G \geq H \geq 1)$ where H is an abelian subgroup of the group G.

The following results produce closed abelian stability groups.

Theorem 3.16. Let G be a non-abelian p-group. Let $1 \neq H \leq Z(G)$ such that $\exp H = p$. Then $B = \mathrm{Stab}(G \geq H \geq 1)$ is closed.

Proof: By 1.5 (i), $[G,B] \leq H$ and $[G,B,B] = 1$. Let $1 \neq x \in H$ and M_x be a maximal subgroup of G containing $Z(G)$. Then $M_x \trianglelefteq G$ of index p and $|x| = p$. Hence there exists $f \in \mathrm{Hom}(G,\langle x\rangle)$ with $M_x \leq \ker f$. Thus $f \in \mathrm{Hom}(G/H,H)$. Let $\alpha_f \in B$ be the corresponding automorphism of G guaranteed by 1.7. Then $\langle x\rangle = [G,\alpha_f] \leq [G,B]$. Hence $H \leq [G,B]$. Therefore $H = [G,B]$ and $\bar{B} = \mathrm{Stab}(G \geq [G,B] \geq 1) = \mathrm{Stab}(G \geq H \geq 1) = B$.

Corollary 3.17. Let G be a non-abelian p-group. Let $1 \neq H \leq Z(G)$ and $\exp Z(G) = p$. Then $B = \mathrm{Stab}(G \geq H \geq 1)$ is closed.

Proof: Since $H \leq Z(G)$ and $\exp(Z(G)) = p$, H has exponent p. Thus 3.10 applies to give the desired conclusion.

Corollary 3.18. Let G be a non-abelian p-group such that $\Phi(G) \leq Z(G)$. Let $H \leq G'$, the derived group. Then $B = \mathrm{Stab}(G \geq H \geq 1)$ is closed.

We first prove the following lemma.

Lemma 3.19. Let G be a p-group such that $G' \leq Z(G)$. Then $\exp G' = \exp(G/Z(G))$.

Proof: Since $G' \leq Z(G)$, $[x,y]^j = [x,y^j]$ for $j \geq 0$ and $x,y \in G$.

If $t = \exp(G/Z(G))$ then $y^t \in Z(G)$ for every $y \in G$. Thus $[x,y^t] = 1$ for every $x \in G$ and hence $[x,y]^t = 1$ for every $x,y \in G$. Thus $\exp G' \leq \exp(G/Z(G))$.

Now suppose $t = \exp G'$ and $x,y \in G$. Then $[x,y]^t = 1$ $\Rightarrow [x,y^t] = 1 \Rightarrow y^t \in Z(G) \Rightarrow \exp(G/Z(G)) \leq \exp G'$.

Hence $\exp G' = \exp(G/Z(G))$.

Proof of 3.18. Since $\Phi(G) \leq Z(G)$, $\exp(G/Z(G)) = p$. Thus by 3.19, $\exp G' = p$. Also, since G is a p-group,

$\Phi(G) = G'\mho^1(G)$. (See [5], p.272). Thus $G' \leq Z(G)$. Since $H \leq G'$, it follows that $\exp H = p$ and $H \leq Z(G)$. Therefore by 3.16, B is closed.

<u>Theorem 3.20</u>. Let G be a purely non-abelian p-group then the group A_c of central automorphisms is closed.

<u>Proof</u>: By results 2.9 and 2.10, A_c is a normal p-subgroup of Aut G. Hence $O_p(\text{Aut } G)$ is a stability group containing A_c and we may consider $\overline{A_c}$. Since $\gamma GA_c \leq Z(G)$, we have

$$\overline{A_c} = \text{Stab}(G \geq \gamma GA_c \geq \gamma GA_c^2 \geq \cdots \geq 1) \leq A_c.$$

Thus $A_c = \overline{A_c}$.

<u>Definition 3.21</u>. A p-group G is called special if either G is elementary abelian or G is of class 2 and $G' = Z(G) = \Phi(G)$ is elementary abelian.

<u>Example 3.22</u>. Let K be a special non-abelian p-group such that $|Z(K)| > p$ then

(i) K is purely non-abelian.

(ii) $A_c = \text{Stab}(K > Z(K) > 1)$ is a closed stability group which is elementary abelian and contains an outer automorphism α.

(iii) Let $\alpha \in A_c$ be an outer automorphism. If $G = K \rtimes \langle\alpha\rangle$, then A_c can be extended to a normal subgroup of Aut G which centralizes G/K and is closed as a stability group in Aut G.

<u>Proof</u>: (i) Suppose K is not purely non-abelian. Then $K = A \times B$, the direct product of A and B, where A is abelian. It follows that $A \leq Z(K)$. Also, $A \not\leq K'$ since $K' \leq B$ and $A \cap B = 1$. But this is a contradiction since K is special and therefore $K' = Z(K)$. Thus K is purely non-abelian.

(ii) By 2.10, $A_c = \mathrm{Stab}(K \geq Z(K) \geq 1)$ since $K' = Z(K)$. Moreover part (i) and 3.20 imply that A_c is closed. Since $\exp Z(K) = p$, it follows from 3.14 that A_c is elementary abelian. Finally, using Result 1.23, $A_c > \mathrm{Inn}\, K$ since $\Phi(K) = Z(K)$ and $Z(K)$ is not cyclic.

(iii) Since $\alpha \in A_c$, A_c is abelian and the automorphism α induces on K, namely $\alpha^{-1}k\alpha = k^{\alpha}$ for every $k \in K$, is just α, Remark 3.11 states that A_c can be extended to a subgroup A of G which centralizes G/K. Moreover, by Corollary 3.8, A is closed.

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