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ABSTRACT

Low Dimensional Formal Fibers
in Characteristic p>0

By:

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The goal of this dissertation is to answer a question originally asked by Matsumura in [3] : is it possible for a local ring to have formal fibers with dimension 1 through the dimension of the ring minus 3? Rotthaus (in [6]) answered this question positively for rings of characteristic 0; in the following I again answer it in the affirmative for rings of characteristic $p > 0$. I also prove that the examples are excellent rings.

The proofs rely on Weierstrass Preparation, differential algebra in characteristic $p > 0$, and the existence of hypertranscendental elements. To prove that the examples have low dimensional formal fibers the crucial result was to show that $\mathrm{trdeg}_{R_S}\hat{R}_S = \infty$. For this, I discovered a new proof quite different from that of Rotthaus which takes advantage of certain Weierstrass automorphisms, under which R_s is particularly well behaved. To show the examples are excellent, it was enough to show that $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_S)$ is a separable extension. I computed explicitly a p-basis of a general power series ring and then used the basis, along with the actual construction of R_s, to prove this.

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Introduction

In this paper we obtain the results of Rotthaus' *On Rings with Low Dimensional Formal Fibers* [6] for rings of positive characteristic $p>0$. We show that there exist suitably nice ("excellent") rings of characteristic p which have formal fibers of dimension 1 through the dimension of the ring minus three. Together with what was known previously, this means that excellent rings of any characteristic may have formal fibers of dimension 0 through the dimension of the ring minus one. Our proof relies heavily on "Weierstrass automorphisms", which motivated Rotthaus' original construction in characteristic 0.

Any homomorphism of commutative rings $f : R \to S$ induces a mapping ${}^{a}f$: spec $S \to$ spec R which takes a prime ideal of S to its preimage under f, which will be a prime ideal of R. Given an ideal $p \in$ spec R, one can examine its preimage under the map ${}^{a}f$ i.e. the <u>fiber</u> of $p \in$ spec R under ${}^{a}f$. It can easily be shown that this fiber can be identified with the spectrum of the ring $S \otimes_R k(p)$, where $k(p) = R_p / pR_p$ (see [2], p. 47). In the special case where the above map f is the natural inclusion $i : R \to \hat{R}$, where R is a local ring and $\hat{R}$ its completion with respect to its maximal ideal, the rings $\hat{R} \otimes_R k(p)$ are called the formal fibers of R. When R is a domain, the fiber of the prime ideal (0), i.e. $\hat{R} \otimes_R Q(R)$, is called the generic formal fiber.

One of the main goals of commutative algebra has been to understand under what conditions certain properties of rings will pass between the ring and its completion. Grothendieck formulated the definition of an excellent ring with this in mind: his excellent rings preserved "Serre's conditions", which could be used to express many interesting properties. Formal fibers play a central role in Grothendieck's definition. In order for a ring to be excellent the ring must have geometrically regular formal fibers. This definition paved the way for formal fibers to become important objects of study for the commutative algebraist. More recently, formal fibers have been used to create a new class of examples of rings via birational extensions (see Rotthaus, Heinzer, and Sally [1]). Here maximal ideals in the generic formal fiber are an intrinsic part of the construction of certain Noetherian rings. The occurrence of formal fibers in their research and in Grothendieck's definition of excellence justify the further study of these rings and their properties.

Yet with all that said, it is surprising how little is really known about formal fibers. It was not until 1987 that Matsumura first launched the study of the dimension of formal fibers, surely one of the most fundamental questions that can be asked about a ring. In his paper, *On the Dimension of Formal Fibres of a Local Ring* [3], he shows that the maximum value of the dimension of a ring's formal fibers (which he denotes by $\alpha(R)$ and shows is equal to dim $\hat{R} \otimes_R Q(R)$ for a local noetherian domain R) can be either 0, the dimension of the ring minus one, or the dimension of the ring minus two. He then posed the question whether or not $\alpha(R)$ could ever be any of the intermediate values, one to the dimension of the ring minus three.

In 1990, Christel Rotthaus was able to answer this question in the affirmative in her paper *On Rings with Low Dimensional Formal Fibers* [6]. She produced examples of excellent rings in characteristic 0 which had all the above intermediate values as the dimension of their generic formal fiber, Matsumura's α. She began by defining a ring R_s which is the basic building block of her examples. The construction of R_s was at least partly motivated by the so called Weierstrass automorphisms: because her ring R_s behaves nicely with respect to these automorphisms, she is easily able to prove that R_s is Noetherian and even excellent. By adding power series variables to the ring R_s, she is able to obtain the desired values for α.

Perhaps the centerpiece of Rotthaus' argument is the fact that $\mathrm{trdeg}_{R_S}\hat{R}_S = \infty$. In order to prove this she uses differential methods: she proves that a hypertranscendental element of k[[T]] (which is contained in $\hat{R}_S$) will still be hypertranscendental over R_S. She relies on, of course, the well known fact that hypertranscendentals exist in the formal power series ring k[[T]] when k is a field of characteristic 0. It has now been recently shown by Shikishima-Tsuji and Katsura that such hypertranscendental elements also exist in characteristic p. In their paper *Hypertranscendental Elements of a Formal Power Series Ring of Positive Characteristic* [7] they use Hasse-Schmidt derivations to find sufficient conditions for a power series to be hypertranscendental and then produce several examples. It was the results of this article that first led us to believe that a generalization of Rotthaus' example to characteristic p might be possible.

Two major obstacles stood in the way of proving that our example in characteristic p satisfied the desired properties. The first arose from fundamental differences between the behavior of derivations in characteristic 0 and in characteristic p: even though we knew of the existence of hypertranscendentals in either characteristic, Rotthaus' proof that $\mathrm{trdeg}_{R_S}\hat{R}_S = \infty$ no longer went through. Binomial coefficients, which in characteristic 0 were certainly adequate nonzero elements, may suddenly vanish modulo p. In addition, Hasse-Schmidt derivations do not behave exactly like normal derivations: a Hasse-Schmidt derivation is really a family of mappings $\{d_n : n \in N\}$, where d_n replaces the composition δ^n of an ordinary derivation δ. These facts combined to make one of Rotthaus' lemmas (which happened to be used no less than three times in her main argument) completely false in characteristic p. The task was then to try to reformulate her lemma in such a way that it was true in characteristic p, and was still strong enough to show that $\mathrm{trdeg}_{R_S}\hat{R}_S = \infty$. What we ended up with was a lemma that mimicked the proof of Rotthaus' lemma more than the statement, and a completely new argument for $\mathrm{trdeg}_{R_S}\hat{R}_S = \infty$, which relies heavily on Weierstrass automorphisms. The new proof also has the advantage of working well in either characteristic 0 or p.

Once we knew that $\mathrm{trdeg}_{R_S}\hat{R}_S = \infty$, it followed easily that we could construct rings with low dimensional formal fibers, just as Rotthaus did. What didn't follow was that the rings we had constructed were still excellent. This was the second major obstacle. For this result, it suffices to prove that the extension $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_s)$ is a separable extension. For the proof of this we utilized the notion of a p-basis, and the natural p-basis that exists for a power series ring.

Philosophically speaking, there was no reason why rings with low dimensional formal fibers shouldn't exist in characteristic p just as they do in characteristic 0. This dissertation confirms this by showing that the difficulties of generalizing Rotthaus' work to characteristic p can be overcome. We now know that excellent rings exist, in either characteristic, with generic formal fibers of any dimension from 0 to the dimension of the ring minus one. Because the new proof works well in both characteristic p and 0, I hope to have also shown how Weierstrass automorphisms, which are an intrinsic part of the structure of these rings, can be a powerful tool in their study.

1. Definition of R_s and $A_{n,t}$

Let k be an infinite field of characteristic p>0 such that $k^p = k$, with $x_1, \ldots, x_s$ and T variables over k.

Let $M \subseteq k$ be an infinite additive subgroup of k.

(1.1) **Definition:** Let s be an integer, s>0. For $(n) = (n_1, \ldots, n_s) \in M^s$, define :

$$A_{(n)} = k[[x_1 + n_1 T, \ldots, x_s + n_s T]] \subseteq k[[x_1, \ldots, x_s, T]]$$

$$R_0 = k[\, A_{(n)} : (n) \in M^s \,] \subseteq k[[x_1, \ldots, x_s, T]]$$

$$R_s = Q(R_0) \cap k[[x_1, \ldots, x_s, T]] \text{ (Here Q denotes the quotient field).}$$

R_s is the fundamental building block of our construction. The definition here differs from that of Rotthaus' only in the set M, which replaces the role of the integers in characteristic 0 (see [6]). We have also added the condition $k^p = k$, which will aid in the proof that R_s is an excellent ring (see section 5).

We now introduce the definition of $A_{n,t}$, which, though the ultimate goal of the paper, we will hardly encounter until section 8.

(1.2) **Definition :** Let n,t$\in$ N be integers with $n > 2$ and $1 \leq t \leq n-3$. Let $s = n - 2 - t$ ($s \geq 1$) and $v_1, \ldots, v_{t+1}$ be variables over R_s. Define:

$$A_{n,t} = R_s[[v_1, \ldots, v_{t+1}]].$$

The following definition, first formulated by Matsumura in [3], was intended to numerically measure the 'difference between a local Noetherian ring and its completion:

(1.3) **Definition**: For R a local Noetherian ring, define $\alpha(R)$ to be the supremum of the dimension of the formal fibers of R:

$$\alpha(R) = \sup\{\dim \hat{R} \otimes_R k(p) \ : p \in \mathrm{spec}(R)\} \text{ where } k(p) = {R_p}/{pR_p}.$$

(1.4) **Note:** Matsumura shows in [3] that if R is also a domain, then $\alpha(R) \;=\; \dim \hat{R} \otimes_R Q(R)$, the dimension of the generic formal fiber of R.

The goal of this dissertation is to show that $A_{n,t}$ is an excellent ring of dimension n, and that $\alpha(A_{n,t}) = t$.

2. Properties of Power Series Rings and Weierstrass Preparation

In this section, we review the fundamental properties of power series rings necessary for the main part of the proof. I also include the results of Weierstrass Preparation, which motivated the construction of R_s and play critical roles in most of the proofs that follow.

(2.1) **Definition:** We can write an element P of the power series ring $k[[x_1, \ldots, x_s, T]]$ as $\sum_{i=0}^{\infty} P_i$ where the P_i are homogeneous polynomials in $x_1, \ldots, x_s$ and T of degree i.

With this in mind, make the following definitions:

$$o(P) = \text{"the order of P"} = \min \{ i \in \mathbf{N} : P_i \neq 0 \}$$

(the order of 0 is defined to be ∞)

$$\text{"the initial form of P"} = P_{o(P)} .$$

(2.2) **Proposition:** The power series ring $k[[x_1, \ldots, x_s, T]]$ is local with maximal ideal generated by $\{x_1, \ldots, x_s, T\}$, and the dimension of $k[[x_1, \ldots, x_s, T]] = s + 1$.

(2.3) **Proposition:** Let $P^1, \ldots, P^{s+1}$ be s + 1 power series contained in the maximal ideal $(x_1, \ldots, x_s, T)$ of $k[[x_1, \ldots, x_s, T]]$. Assume that the degree 1 forms of the $P^1, \ldots, P^{s+1}$ are linearly independent over k and that the determinant of the coefficients in these degree one forms is not zero. Then the map :

$$k[[x_1, \ldots, x_s, T]] \rightarrow k[[x_1, \ldots, x_s, T]]$$
$$x_i \mapsto P^i \qquad i = 1 \text{ to } s$$
$$T \mapsto P^{s+1}$$

is an automorphism of the ring $k[[x_1, \ldots, x_s, T]]$.

Proof: See Corollary 2, p. 137; Zariski and Samuel [8].

(2.4) **Definition:** An element P of the power series ring $k[[x_1, \ldots, x_s, T]]$ is said to be regular in T if a term cT^n occurs in P, where $0 \neq c \in k$ and $n > 0$.

(2.5) **Theorem:** Let $P \in k[[x_1, \ldots, x_s, T]]$ be a power series which is regular in T, and not a unit. Let r be the smallest r such that cT^r occurs in P, with $0 \neq c \in k$. Then there exist power series $\varepsilon \in k[[x_1, \ldots, x_s, T]]$, which is a unit, and $a_i \in k[[x_1, \ldots, x_s]]$ for i = 0 to r - 1 such that :

$$P = \varepsilon \cdot \left(T^r + a_{r-1}T^{r-1} + \ldots + a_0\right).$$

Proof: See Theorem 5 (Weierstrass Preparation Theorem), p. 139; Zariski and Samuel [8]).

(2.6) **Proposition:** Let $P \in k[[x_1, \ldots, x_s, T]]$ be a power series which is regular in T and not a unit. By (2.5) we write:

$$P = \varepsilon \cdot \left(T^r + a_{r-1}T^{r-1} + \ldots + a_0\right).$$

Let $Q = T^r + a_{r-1}T^{r-1} + \ldots + a_0 \in k[[x_1, \ldots, x_s]]\,[T]$. Then:

$$k[[x_1, \ldots, x_s, T]]\,/\,(P) \;\; \cong \;\; k[[x_1, \ldots, x_s]]\,[T]\,/\,(Q).$$

Here (P) and (Q) denote the principal ideals generated in the rings $k[[x_1, \ldots, x_s, T]]$ and $k[[x_1, \ldots, x_s]]\,[T]$ respectively. Also note that the isomorphism is that induced by the natural inclusion $k[[x_1, \ldots, x_s]]\,[T] \subseteq k[[x_1, \ldots, x_s, T]]$.

Proof: See Corollary 2, p. 146; Zariski and Samuel [8].

The previous results are the basic results of Weierstrass. Now we show how these apply to the rings at hand : R_s in (2.9), (2.10) and (2.11), $A_{n,t}$ in (2.12). We begin with the following:

(2.7) **Definition:** For $(n) = (n_1, \ldots, n_s) \in M^s$ set :

$$\begin{aligned} \sigma_{(n)} \; : \; k[[x_1, \ldots, x_s, T]] &\to k[[x_1, \ldots, x_s, T]] \\ x_i &\mapsto x_i + n_i T \quad \text{for } i = 1 \text{ to } s \\ T &\mapsto T \end{aligned}$$

(2.8) **Note:** (i) It follows from (2.3) that such a map is an automorphism of $k[[x_1, \ldots, x_s, T]]$.

(ii) $\sigma_{(n)} \circ \sigma_{(m)} = \sigma_{(n)+(m)}$

(iii) $\sigma^{-1}_{(n)} = \sigma_{-(n)}$

(iv) $\sigma_{(n)}\big|_{k[[T]]} = id_{k[[T]]}$

These $\sigma_{(n)}$ are examples of "Weierstrass automorphisms", and play a central role in almost all of the results of this paper. One can motivate the definition of R_s via the $\sigma_{(n)}$'s as follows :

Let $S = k[[x_1, \ldots, x_s]]$.
Then $\sigma_{(n)}(S) = k[[x_1 + n_1T, \ldots, x_s + n_sT]] = A_{(n)}$.

Now $R_0 = k[\sigma_{(n)}(S) : (n) \in M^s]$, which is closed under $\{\sigma_{(n)} : (n) \in M^s\}$. Now R_s differs from R_0 only in that inverses from $k[[x_1, \ldots, x_s, T]]$ have been added, so that R_s is also closed under all $\sigma_{(n)}$. Thus R_s can be viewed as the "closure" of $k[[x_1, \ldots, x_s]]$ in $k[[x_1, \ldots, x_s, T]]$ under inverses and $\sigma_{(n)}$, $(n) \in M^s$.

The $\sigma_{(n)}$ will, in a crucial place later in the paper, allow us to assume that a specific $A_{(n)}$ is actually $A_{(0, \ldots, 0)} = k[[x_1, \ldots, x_s]]$, a critical simplification.

(2.9) **Proposition:** Let $0 \neq P \in k[[x_1, \ldots, x_s, T]]$ which is not a unit (i.e. $o(P) > 0$). Then there exists an $(n) \in M^s$ such that $\sigma_{(n)}(P) \in k[[x_1, \ldots, x_s, T]]$ is regular in T.

Proof: Let P_r be the initial form of P ($r = o(P)$). Because P_r is homogeneous, $P_r(x_1, \ldots, x_s, 1)$ is a nonzero polynomial in the variables $x_1, \ldots, x_s$. Since M is infinite, there exists an $(n) = (n_1, \ldots, n_s) \in M^s$ such that $P_r(n_1, \ldots, n_s, 1) \neq 0$. Now $\sigma_{(n)}(P_r) = P_r(x_1 + n_1 T, \ldots, x_s + n_s T, T)$ which has the term $P_r(n_1, \ldots, n_s, 1)\, T^r$. Since $0 \neq P_r(n_1, \ldots, n_s, 1) \in k$, we have $\sigma_{(n)}(P_r)$, and hence $\sigma_{(n)}(P)$, is regular in T.

(2.10) **Corollary:** Let $0 \neq P \in R_s$, which is not a unit. Then we can write:

$$P = e \cdot \left(T^r + b_{r-1}T^{r-1} + \ldots + b_0\right)$$

where e is a unit in R_s and $b_0, \ldots, b_{r-1} \in A_{(m)}$ for some suitable $(m) \in M^s$.

Proof: We apply a $\sigma_{(n)}^{-1}$ to the results of (2.5) as follows:

By (2.9), choose $(n) = (n_1, \ldots, n_s) \in M^s$ such that $\sigma_{(n)}(P)$ is regular in T. By (2.6) we can write:

$$(*) \qquad \sigma_{(n)}(P) = \varepsilon \cdot \left(T^r + a_{r-1}T^{r-1} + \ldots + a_0\right)$$

where $\varepsilon \in k[[x_1, \ldots, x_s, T]]$ is a unit and $a_i \in k[[x_1, \ldots, x_s]]$ for i = 0 to r-1. Since $k[[x_1, \ldots, x_s]] = A_{(0,\ldots,0)} \subseteq R_0 \subseteq R_s$, ε is actually a unit in R_s. We apply $\sigma_{(n)}^{-1}$ to (*) :

$$P = \sigma_{(n)}^{-1}(\varepsilon)(T^r + \sigma_{(n)}^{-1}(a_{r-1})T^{r-1} + \ldots + \sigma_{(n)}^{-1}(a_1)T + \sigma_{(n)}^{-1}(a_0))$$

Now let $\sigma_{(n)}^{-1}(\varepsilon) = e$, which will still be a unit in $k[[x_1, \ldots, x_s, T]]$. Also let $b_i = \sigma_{(n)}^{-1}(a_i)$ for $i = 0$ to $r-1$.

Now the $b_i \in k[[x_1 - n_1T, \ldots, x_s - n_sT]] = A_{-(n)}$.

Hence: $P = e \cdot \left(T^r + b_{r-1}T^{r-1} + \ldots + b_0\right)$ as desired. Note that since the $b_i \in A_{-(n)} \subseteq R_0 \subseteq R_s$, e is actually a unit in R_s.

(2.11) **Proposition:** Let $0 \neq P \in k[[x_1, \ldots, x_s, T]]$ which is not a unit. By (2.10) we write :

$$P = e \cdot \left(T^r + b_{r-1}T^{r-1} + \ldots + b_0\right)$$

where e is a unit in $k[[x_1, \ldots, x_s, T]]$ and $b_0, \ldots, b_{r-1} \in A_{(n)}$.
We let $Q = T^r + b_{r-1}T^{r-1} + \ldots + b_0 \in A_{(n)}[T]$.
Then:

$$A_{(n)}[T] / (Q) \cong k[[x_1, \ldots, x_s, T]] / (P),$$

where (Q) and (P) denote the principal ideals generated by Q and P in $A_{(n)}$ and $k[[x_1, \ldots, x_s, T]]$ respectively.

Proof: The set up is as follows from (2.9) and (2.10) :

$\sigma_{(n)}$ (P) is regular and $\sigma_{(n)}(P) = \varepsilon \cdot \left(T^r + a_{r-1}T^{r-1} + \ldots + a_0\right)$ where $a_i \in k[[x_1, \ldots, x_s]]$ for $i = 0$ to $r - 1$, $e = \sigma_{(n)}^{-1}(\varepsilon)$ and $Q = \sigma_{(n)}^{-1}($ $T^r + a_{r-1}T^{r-1} + \ldots + a_0)$.

Now by (2.6):

$$k[[x_1, \ldots, x_s]]\,[T] / (\sigma_{(n)}(Q)) \;\cong\; k[[x_1, \ldots, x_s, T]] / (\sigma_{(n)}(P)).$$

First note that we have the following commutative diagram :

$$\begin{array}{ccc} k[[x_1, \ldots, x_s]]\,[T] & \xrightarrow{\subseteq} & k[[x_1, \ldots, x_s, T]] \\ \cong\downarrow\sigma_{(n)}^{-1} & & \cong\downarrow\sigma_{(n)}^{-1} \\ k[[x_1 - n_1T, \ldots, x_s - n_sT]]\,[T] & \xrightarrow{\subseteq} & k[[x_1, \ldots, x_s, T]] \end{array}$$

Now since $\sigma_{(n)}^{-1}((\sigma_{(n)}(P))) = (P)$ and $\sigma_{(n)}^{-1}((\sigma_{(n)}(Q))) = (Q)$, it is clear that :

$$A_{-(n)}[T] / (Q) \;\cong\; k[[x_1, \ldots, x_s, T]] / (P)$$

and that this map is induced by the natural inclusion of $A_{-(n)}[T]$ into $k[[x_1, \ldots, x_s, T]]$.

Now we apply Weierstrass Preparation to the larger ring $A_{n,t}$.

(2.12) **Proposition:** Let $0 \neq P \in A_{n,t} \backslash (v_1, \ldots, v_{t+1})$. Then we can express P as :

$$P = e \cdot (T^r + b_{r-1}T^{r-1} + \ldots + b_0)$$

where e is a unit in $\hat{A}_{n,t} = k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}]]$, and the $b_i \in A_{(m)}[[v_1, \ldots, v_{t+1}]]$ for some $(m) \in M^s$.

Proof : This fact is essentially the same as (2.9) and (2.10), but we must make adjustments for the extra variables $v_1, \ldots, v_{t+1}$. We extend each map $\sigma_{(n)}$, as follows:

$$\sigma^e_{(n)} : k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}]] \to k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}]],$$
$$x_i \mapsto x_i + n_i T \quad \text{for } i = 1 \text{ to } s,$$
$$T \mapsto T,$$
$$v_i \mapsto v_i \quad \text{for } i = 1 \text{ to } t+1.$$

Hence $\sigma^e_{(n)}\Big|_{k[[x_1, \ldots, x_s, T]]} = \sigma_{(n)}$.

Note that $\sigma^e_{(n)}$ is an automorphism of $k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}]]$ by (2.3), with inverse $\sigma^e_{-(n)}$. First we show that $\sigma^e_{(n)}(P)$ is regular for some $(n) \in M^s$.

We first write P as $\sum_{i=0}^{\infty} P_i$ where the P_i are homogeneous polynomials in $x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}$ of degree i. Choose r such that P_r contains a monomial of the form $T^{r_0} x_1^{r_1} \cdot \ldots \cdot x_s^{r_s}$. The existence of such an r is guaranteed by $P \notin (v_1, \ldots, v_{t+1})$.

Now $P_r(x_1, \ldots, x_s, T, 0, \ldots, 0)$ is a nonzero polynomial in $x_1, \ldots, x_s, T$, which is also homogeneous. Consequently, $P_r(x_1, \ldots, x_s, 1, 0, \ldots, 0)$ is a nonzero polynomial in $x_1, \ldots, x_s$. Since M is infinite, there exists an $(n) = (n_1, \ldots, n_s) \in M^s$ such that $P_r(n_1, \ldots, n_s, 1, 0, \ldots, 0) \neq 0$. Now $\sigma^e_{(n)}(P_r) = P_r(x_1 + n_1 T, \ldots, x_s + n_s T, T, v_1, \ldots, v_{t+1})$ which has the term $P_r(n_1, \ldots, n_s, 1, 0, \ldots, 0) T^r$. Since $P_r(n_1, \ldots, n_s, 1, 0, \ldots, 0) \neq 0$, $\sigma^e_{(n)}(P_r)$ and hence $\sigma^e_{(n)}(P)$ is regular in T.

Since $\sigma^e_{(n)}(P)$ is regular in T, by (2.5) we can write $\sigma^e_{(n)}(P)$ as:

$$\sigma^e_{(n)}(P) = \varepsilon \cdot \left(T^r + a_{r-1} T^{r-1} + \ldots + a_0\right)$$

where ε is a unit in $k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}]]$ and the $a_i \in k[[x_1, \ldots, x_s, v_1, \ldots, v_{t+1}]]$.

As in (2.10), we apply $\left(\sigma^e_{(n)}\right)^{-1}$ to obtain:

$$P = e \cdot \left(T^r + b_{r-1}T^{r-1} + \ldots + b_0\right)$$

where e is a unit in $k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}]]$ and the $b_i \in A_{-(n)}[[v_1, \ldots, v_{t+1}]]$, as desired.

3. The Elementary Properties of R_S

In this section, elementary facts are presented which will enable us to prove that R_s is excellent in sections 4 and 5. All the results are immediate consequences of the basic facts about power series rings and Weierstrass Preparation.

(3.1) **Proposition:** R_s is a local ring of dimension $s + 1$, with maximal ideal $(x_1, \ldots, x_s, T)$, and completion $\hat{R}_s = k[[x_1, \ldots, x_s, T]]$.

Proof: An element of the power series ring $k[[x_1, \ldots, x_s, T]]$ is invertible if and only if it has a nonzero term in degree zero. Since $R_s = Q(R_0) \cap k[[x_1, \ldots, x_s, T]]$, the same is true of R_s. Hence R_s is a local ring with maximal ideal generated by $x_1, \ldots, x_s$, and T. Clearly $\hat{R}_s = k[[x_1, \ldots, x_s, T]]$ which implies that $\dim R_s = s + 1$.

(3.2) **Proposition:** R_s is a Noetherian ring.

Proof: Let $I \subseteq R_s$ be a proper ideal. It must be shown that I is finitely generated. Pick $0 \neq P \in I$. By (2.10), express P as:

$$P = e \cdot \left(T^r + b_{r-1}T^{r-1} + \ldots + b_0\right)$$

where e is a unit in $k[[x_1, \ldots, x_s, T]]$ and $b_0, \ldots, b_{r-1} \in A_{(n)}$ for some suitable $(n) \in M^s$.

Let $Q = T^r + b_{r-1}T^{r-1} + \ldots + b_0 \in A_{(n)}[T]$. Consider the following commutative diagram of maps:

$$\begin{array}{ccc} R_s/(P) & \overset{=}{\longleftrightarrow} & R_s/(P) \\ a\uparrow & & b\downarrow \\ A_{(n)}[T]/(Q) & \xrightarrow{c\,=\,\text{nat.}} & k[[x_1, \ldots, x_s, T]]/(P) \end{array}$$

Claim: The map b is an isomorphism.

Proof (of claim): (i) Let $\bar{r} \in R_s/(P)$, where $r \in R_s$, such that $b(\bar{r}) = 0$. Then $r = \ell P$ where $\ell \in k[[x_1, \ldots, x_s, T]]$. Then $\ell = r/P \in Q(R_s)$. Since $R_s \subseteq Q(R_0)$, $Q(R_s) \subseteq Q(R_0)$ or $Q(R_s) \cap k[[x_1, \ldots, x_s, T]] \subseteq Q(R_0) \cap k[[x_1, \ldots, x_s, T]] = R_s$. Hence $\ell \in R_s$, which shows that $\bar{r} = 0$ in $R_s/(P)$, and that b is injective.

(ii) By (2.11), the map c is an isomorphism. Now by the commutativity of the diagram, $b \circ a$ is surjective, hence b is surjective.

But $k[[x_1, \ldots, x_s, T]]/(P)$ is a Noetherian ring, so $R_s/(P)$ must also be Noetherian. Hence the ideal $\bar{I} \subseteq R_s/(P)$ is finitely generated by say $\bar{P}_1, \ldots, \bar{P}_r$ where $P_1, \ldots, P_r \in R_s$. Now clearly I is generated in R_s by $P, P_1, \ldots, P_r$, which is all that needed to be shown.

(3.3) **Proposition:** For any proper ideal $I \subseteq R_s$, the quotient ring R_s/I is complete.

Proof: From the proof of (3.2), we have that for some $0 \neq P \in I$:

$$R_s/(P) \cong k[[x_1, \ldots, x_s, T]] \,/\, (P).$$

Hence:

$$R_s/I \cong {}^{R_s/(p)}\!\big/_{\bar{I}} \cong {}^{k[[x_1, \ldots, x_s, T]]/(P)}\!\big/_{\bar{I}} \cong k[[x_1, \ldots, x_s, T]]/(I)$$

and the latter is a complete ring.

(3.4) **Proposition:** $\alpha(R_s) = 0$.

Proof: It is enough to show that for any prime ideal $(0) \neq q \in$ spec ($k[[x_1, \ldots, x_s, T]]$), $q \cap R_s \neq (0)$. Let $0 \neq x \in q$. Since $x \in k[[x_1, \ldots, x_s, T]]$, by (2.10) express $x = eQ$ where e is a unit in $k[[x_1, \ldots, x_s, T]]$ and $Q \in A_{(n)}[T]$ for some suitable $(n) \in M^s$. Since $eQ \in q$, a prime ideal, and e is a unit, Q must be in q. Hence $0 \neq Q \in q \cap R_s$, which is all that needed to be shown.

4. The Excellence of R_S and the Separability of $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_s)$

In this section, we reduce the problem of excellence for R_s to the separability of the extension $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_S)$, using the elementary properties from section 3.

Let R be a commutative ring.

(4.1) **Proposition:** R is reduced if and only if R_q is reduced for all $q \in$ spec R.

Proof: "$\Rightarrow$": Let $r/s \in R_q$ such that $(r/s)^n = 0$ in R_q.
Then $tr^n = 0 = t^n r^n = (tr)^n$ for some $t \in R \backslash q$. R reduced implies that $tr = 0$ and hence $r/s = 0 \in R_q$.

"$\Leftarrow$": Suppose $x^n = 0$, $x \in R \backslash (0)$. Then ann (x) is properly contained in R, so there exists an $m \in$ Max (R) such that ann $(x) \subseteq m$.
By hypothesis R_m is reduced. Since $(x/1)^n = 0$ in R_m, $x/1 = 0$ or there exists a $t \in R \backslash m$ such that $tx = 0$. But since ann $(x) \subseteq m$ this is a contradiction.

(4.2) **Proposition:** Assume R is Noetherian, dim R = 0. Then R is regular if and only if R is reduced.

Proof: R is regular if and only if R_q is regular for all $q \in$ spec R.

Dim R = 0 implies that dim $R_q = 0$, so that R_q regular means that qR_q is generated by zero elements, which can happen if and only if R_q is reduced. Applying (4.1), the result follows.

(4.3) **Proposition:** (R, m) a local Noetherian ring, $\alpha(R) = 0$. Then:

R has geometrically regular formal fibers if and only if R has geometrically reduced formal fibers.

Proof: Let $q \in$ spec R , L finite over $k(q) = R_q/qR_q$.
Since $\alpha(R) = 0$, dim $\hat{R} \otimes_R k(p) = 0$.

Since the extension $\hat{R} \otimes_R k(p) \xrightarrow{\subseteq} \hat{R} \otimes_R L$ is finite and flat, dim $\hat{R} \otimes_R L = 0$ also.

Now by (4.2), $\hat{R} \otimes_R L$ is regular if and only if $\hat{R} \otimes_R L$ is reduced.

(4.4) **Proposition:** (R, m) local Noetherian domain, $\alpha(R) = 0$, and for all $0 \neq q \in$ spec R the ring R/q is complete.

If L is a finite extension field of k(q), where $0 \neq q \in$ spec R then $\hat{R} \otimes_R L = L$. In particular, $\hat{R} \otimes_R L$ is reduced.

Proof: $\hat{R} \otimes_R L = \hat{R} \otimes_R k(q) \otimes_{k(q)} L = \hat{R} \otimes_R R/q \otimes_R R_q \otimes_{k(q)} L$

$$= (R/q)^{\wedge} \underset{R}{\otimes} R_q \underset{k(q)}{\otimes} L = (R/q \underset{R}{\otimes} R_q) \underset{k(q)}{\otimes} L$$

$$= k(q) \underset{k(q)}{\otimes} L = L.$$

(4.5) Proposition: (R, m) local Noetherian domain with R^ also a domain. Assume also that $Q(R) \xrightarrow{\subseteq} Q(\hat{R})$ is separable. If L is a finite extension field of Q(R), then $\hat{R} \underset{R}{\otimes} L$ is reduced.

Proof: $\hat{R}$ a domain implies that $\hat{R} \xrightarrow{\subseteq} Q(\hat{R})$.
Q(R) a flat R-module implies that $\hat{R} \underset{R}{\otimes} Q(R) \xrightarrow{\subseteq} Q(\hat{R}) \underset{R}{\otimes} Q(R) = Q(\hat{R})$.

Since Q(R) is a field, L is flat as a Q(R) module :

$$\begin{array}{ccc} \hat{R} \underset{R}{\otimes} Q(R) \underset{Q(R)}{\otimes} L & \xrightarrow{\subseteq} & Q(\hat{R}) \underset{Q(R)}{\otimes} L \\ = \updownarrow & & = \updownarrow \\ \hat{R} \underset{R}{\otimes} L & \xrightarrow{\subseteq} & Q(\hat{R}) \underset{R}{\otimes} L \end{array}$$

But $Q(\hat{R}) \underset{R}{\otimes} L = Q(\hat{R}) \underset{Q(R)}{\otimes} L$ which, by the definition of separable, is reduced. By the above inclusion then $\hat{R} \underset{R}{\otimes} L$ is reduced.

(4.6) Corollary: (R, m) local Noetherian domain, $\alpha(R) = 0$ and $\hat{R}$ also a domain. Assume that R is universally catenary, that R/q is complete for all $0 \neq q \in \operatorname{spec} R$, and finally that $Q(R) \xrightarrow{\subseteq} Q(\hat{R})$ is separable. Then : R is excellent.

Proof: It is enough to show that the formal fibers of R are geometrically regular (see Matsumura [2] for this result, and also the definition of excellence). By (4.3), it is enough to show that the formal fibers of R are geometrically reduced. This is shown for $q \neq 0$ by (4.4), and for $q = 0$ by (4.5).

(4.7) **Corollary:** If the extension $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_s)$ is separable, then R_s is an excellent ring.

Proof: In view of (4.6), the result follows from (3.1), (3.2), (3.3), and (3.4). Note that R_s is universally catenary because R_s is regular (which also follows from (3.1) and (3.2)).

5. The Separability of $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_s)$

In section 4, we saw that R_s is excellent if $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_s)$ is a separable extension of fields. In characteristic 0 this meant that R_s is clearly excellent. In characteristic p, it meant that a considerable amount of work still remained. In this section, I include not only the proof of separability in characteristic p, but also a result that we obtained en route to the proof. The proposition didn't prove useful in the end, but is interesting in its own right.

Our initial attack on the problem of separability was based on the idea that $Q(R_s) = Q(R_0)$, and that we could obtain $Q(R_s)$ as the direct limit of a family of finite products of fields (namely the composita of finitely many $Q(A_{(n)})$). It was known that $Q(A_{(n)}) \xrightarrow{\subseteq} Q(\hat{R}_s)$ is separable, and that the direct limit of fields which form separable extensions when extended to $Q(\hat{R}_s)$ would also form a separable extension when extended to $Q(\hat{R}_s)$. This meant that all we needed to show was that a finite product of subfields, each of which extend to a larger field in a separable fashion, will also extend to the larger field in a separable fashion. In what follows I present our results in this direction, before preceding in (5.6) to the actual argument that $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_s)$ is a separable extension.

Suppose $K \xrightarrow{\subseteq} E$ and $L \xrightarrow{\subseteq} E$ are both separable extensions of characteristic p. We are interested in what conditions are sufficient to conclude that $KL \xrightarrow{\subseteq} E$ is also a separable extension. This is always the case if, say, $K \xrightarrow{\subseteq} E$ is algebraic, but fails in general , as the following example shows.

(5.1) **Example:** Let K be a field of characteristic p, x and y variables over K. Let $E = K(x, y, (xy)^{1/p})$. The extensions $K(x) \xrightarrow{\subseteq} E$ and $K(y) \xrightarrow{\subseteq} E$ are both separable extensions while $K(x)K(y) = K(x, y) \xrightarrow{\subseteq} E$ is <u>not</u> a separable extension.

Proof: $E = K(x, y, (xy)^{1/p}) = K(x, (xy)^{1/p})$, and the element $(xy)^{1/p}$ is transcendental over K(x). Since purely transcendental extensions are separable, $K(x) \xrightarrow{\subseteq} E$ (and by symmetry $K(y) \xrightarrow{\subseteq} E$) is a separable extension.

On the other hand, the irreducible polynomial of $(xy)^{1/p}$ is:

$$V^p - xy$$

which is not separable. Hence the extension $K(x, y) \xrightarrow{\subseteq} E$ is not separable.

The following Proposition gives sufficient conditions on the extensions $K \xrightarrow{\subseteq} E$ and $L \xrightarrow{\subseteq} E$ for $KL \xrightarrow{\subseteq} E$ to be separable. It was the strongest result we found in this direction, though it did not help us to show that $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_s)$ is separable. First we make the following definitions:

(5.2) **Definition:** Let $K \subseteq L$ be an extension of fields of characteristic p. Let $B = \{b_1, \ldots, b_s\} \subseteq L$.

i) Define $\Gamma_B = \{ b_1^{a_1} \cdot \ldots \cdot b_n^{a_n} : 0 \le a_i \le p-1,\ i = 0,\ldots,n \}$

ii) B is <u>p-independent for $K \subseteq L$</u> if Γ_B is linearly independent over $L^p(K)$.

iii) B is a <u>p-basis of L over K</u> if B is p-independent for $K \subseteq L$ and $L^p(K,B) = L$.

(5.3) **Note:** B is a p-basis of L over K $\Leftrightarrow$ B is a differential basis of L over K (i.e. $\{db : b \in B\}$ is a basis of the L-vector space $\Omega_{L/K}$, where $\Omega_{L/K}$ is the module of differentials of L over K, and d the universal derivation). See [2], p.202.

(5.4) **Proposition:** Let K, $L \subseteq E$ be fields of characteristic p. Let $F \subseteq K^{p^{-1}}$ be a subset such that :

$$(E(F))^p (K \cap L) = E^p(K).$$

If $K \xrightarrow{\subseteq} E$ and $L \xrightarrow{\subseteq} E(F)$ are separable extensions, then $KL \xrightarrow{\subseteq} E$ is separable.

Proof: $K \xrightarrow{\subseteq} E$ separable implies that $K \xrightarrow{\subseteq} KL$ is separable, or that we have the following sequence:

$$0 \to \Omega_K \underset{K}{\otimes} KL \to \Omega_{KL} \to \Omega_{KL/K} \to 0$$

Tensoring this sequence over KL with E, and comparing it to the sequence above with E replacing KL, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \Omega_K \otimes_K E & \to & \Omega_{KL} \otimes_{KL} E & \to & \Omega_{KL/K} \otimes_{KL} E & \to 0 \\ & = \downarrow & & f \downarrow & & g \downarrow & \\ 0 \to & \Omega_K \otimes_K E & \longrightarrow & \Omega_E & \longrightarrow & \Omega_{E/K} & \to 0 \end{array}$$

Now $KL \xrightarrow{\subseteq} E$ is separable if and only if f is injective. By the above diagram, it is enough to show that g is injective. For this, it must be shown that there exists a p-basis of KL over K which remains p-independent for $K \subseteq E$.

Now $\Omega_{KL/K}$ is generated as a KL-vector space by $\{db : b \in KL\}$, where $d: KL \to \Omega_{KL/K}$ is the universal derivation. Writing $b = \Sigma x_i y_i$ where $x_i \in K$ and $y_i \in L$, we have:

$$d(b) = d\left(\sum x_i y_i\right) = \sum x_i d(y_i).$$

Thus $\Omega_{KL/K}$ is also generated as a KL-vector space by the smaller set $\{dy : y \in L\}$.

Now choose a p-basis $B \subseteq L$ of KL over K. This implies that Γ_B is linearly independent over $(KL)^p(K) = L^p(K)$. Thus Γ_B is also linearly independent over $L^p(K \cap L)$. Since $B \subseteq L$, B is p-independent for $K \cap L \subseteq L$. Because $L \xrightarrow{\subseteq} E(F)$ is separable, the map :

$$\Omega_{L/K\cap L} \otimes_L E(F) \longrightarrow \Omega_{E(F)/K\cap L}$$

is injective (see [2], Theorem 26.6, p.202-3). This means that any set which is p-independent for $K \cap L \subseteq L$ is also p-independent for $K \cap L \subseteq E(F)$. Hence B is also p-independent for $K \cap L \subseteq E(F)$.

But this means that Γ_B is linearly independent over $(E^p(F))(K \cap L) = E^p(K)$, by hypothesis. Thus B is p-independent for $K \subseteq E$, which is all that had to be shown.

(5.5) **Corollary:** K, L$\subseteq$E fields of characteristic p. If $K \xrightarrow{\subseteq} E$ is separable and $L \xrightarrow{\subseteq} E(K^{p^{-1}})$ is separable, then $KL \xrightarrow{\subseteq} E$ is also separable.

Proof: Let $F=K^{p^{-1}}$ in (5.4). This result also follows directly from MacLane's criterion for separability (Theorem 26.4, p.201 [2]).

Useful as the above Corollary appears, neither it nor (5.4) were sufficient to show that a compositum of finitely many $Q(A_{(n)})$ extended separably to $k((x_1, \ldots, x_s, T))$. Instead, we computed a differential (p-) basis of a power series ring, and relied on the specifics of our example to prove the rest. The result is surprisingly straightforward, and exactly what we needed.

(5.6) **Proposition:** Let K be a field of characteristic p, $K = K^p$, $y_1,\ldots,y_r$ variables over K. Then the set $\{y_1,\ldots,y_r\}$ is a p-basis of the extension π (the prime field) $\subseteq K((y_1,\ldots,y_r))$.

Proof: Let $B = \{y_1,\ldots,y_r\}$. All that needs to be shown is :

i) $[K((y_1,\ldots,y_r))]^p(B) = K((y_1,\ldots,y_r))$,

ii) Γ_B is linearly independent over $[K((y_1,\ldots,y_r))]^p$.

For i), let $\alpha=\sum_{(i)} a_{(i)} y_1^{i_1}\cdot\ldots\cdot y_r^{i_r} \in K[[y_1,\ldots y_r]]$ where $(i) = (i_1,\ldots,i_r) \in \mathbf{N}^r$.

For each (i), express $i_t = j_t + m_t p$ where $0 \le j_t \le p-1$ for all $t = 1$ to r.

Thus: $$a_{(i)}y_1^{i_1}\cdot\ldots\cdot y_r^{i_r} = y_1^{j_1}\cdot\ldots\cdot y_r^{j_r}\cdot a_{(i)}y_1^{pm_1}\cdot\ldots\cdot y_r^{pm_r}.$$

Let $M_{(j)} = y_1^{j_1}\cdot\ldots\cdot y_r^{j_r} \in \Gamma_B$, so that now:

$$\alpha = \sum_{M_{(j)}\in\Gamma_B} M_{(j)}A_{(j)} \text{ where } A_{(j)} \in K[[y_1^p,\ldots y_r^p]].$$

Since $K = K^p$, this shows that:

$$K[[y_1,\ldots,y_r]] = (K[[y_1,\ldots,y_r]])^p[y_1,\ldots,y_r]$$

and clearly i) follows.

For ii), it is enough to show linear independence over $K[[y_1^p,\ldots y_r^p]]$, or that the above representation of α is unique. Thus we show that :

$$\text{if } 0 = \sum_{M_{(j)}\in\Gamma_B} M_{(j)}A_{(j)} \text{ then } A_{(j)} = 0 \text{ for all } (j).$$

But since the coefficient of $y_1^{j_1+m_1p}\cdot\ldots\cdot y_r^{j_r+m_rp}$ in $\sum M_{(j)}A_{(j)}$ is just the coefficient of $y_1^{pm_1}\cdot\ldots\cdot y_r^{pm_r}$ in $A_{(j_1,\ldots,j_r)}$, it is clear that if any $A_{(j)} \neq 0$, then also $\sum M_{(j)}A_{(j)} \neq 0$, which is what needs to be shown.

(5.7) Note: (5.6) implies that the dimension of $\Omega_{K((y_1,\ldots,y_r))/\pi}$ as a $K((y_1,\ldots,y_r))$-vector space is r. Compare this to characteristic 0: If L is a field of characteristic 0, then the dimension of the vector space $\Omega_{L((y_1,\ldots,y_r))/\pi}$ is equal to infinity (in characteristic 0, the notions of transcendence basis and differential basis coincide). In this sense, the characteristic p case is surprisingly "simpler" than characteristic 0.

Now we apply (5.6) in successive corollaries to show R_s is an excellent ring:

(5.8) **Corollary:** $\{x_1 + n_1T,\ldots,x_s + n_sT\}$ is a differential basis of the field extension $\pi \subseteq Q(A_{(n)})$.

Proof: Immediate from:

$$Q(A_{(n)})=k((x_1 + n_1T,\ldots,x_s + n_sT))\cong k((y_1,\ldots,y_s)),$$

where $y_i = x_i + n_iT$.

(5.9) **Corollary:** If $L = k(A_{(n)} : (n)\in Q)$ where $Q \subseteq M^s$ and $|Q| \geq 2$, then $\{x_1,\ldots,x_s,T\}$ is a differential basis of $\pi \subseteq L$.

Proof: By (5.8), $Q(A_{(n)}) = Q(A_{(n)})^p(x_1 + n_1T,\ldots,x_s + n_sT)$, and it is also true that $L^p = k(Q(A_{(n)})^p \ : (n) \in Q)$.

Now:

$$\begin{aligned} L^p(x_1 + n_1T,\ldots,x_s + n_sT \ : \ (n) \in Q) &= \\ &= k(Q(A_{(n)})^p \ : \ (n) \in Q)(x_1 + n_1T,\ldots,x_s + n_sT \ : \ (n) \in Q), \\ &= k(Q(A_{(n)}) \ : \ (n) \in Q) = L. \end{aligned}$$

Since $|Q| \geq 2$, there exist

$(n) = (n_1, \ldots, n_s) \in Q$ and $(m) = (m_1, \ldots, m_s) \in Q$ with some $m_i \neq n_i$.

Now: $$T = \frac{(x_i + m_i T) - (x_i + n_i T)}{m_i - n_i} \in k(x_i + m_i T, x_i + n_i T).$$

Hence $T \in L^p(x_1 + n_1 T, \ldots, x_s + n_s T \;:\; (n) \in Q)$.

But then: $L^p(x_1 + n_1 T, \ldots, x_s + n_s T \;:\; (n) \in Q) = L^p(x_1, \ldots, x_s, T)$

and thus $L^p(x_1, \ldots, x_s, T) = L$ from above.

By (5.6), $\Gamma_{\{x_1, \ldots, x_s, T\}}$ is linearly independent over $k((x_1^p, \ldots, x_s^p, T)) \supseteq L^p$. Thus $\Gamma_{\{x_1, \ldots, x_s, T\}}$ is linearly independent over L^p, which, combined with the above, shows that $\{x_1, \ldots, x_s, T\}$ is the desired differential basis.

(5.10) **Corollary:** If L is as in (5.9), then $L \xrightarrow{\subseteq} k((x_1, \ldots, x_s, T))$ is separable.

Proof: A differential basis of $\pi \subseteq L$, $\{x_1, \ldots, x_s, T\}$, is also a differential basis of $\pi \subseteq k((x_1, \ldots, x_s, T))$, which means that the map:

$$\Omega_L \underset{L}{\otimes} k((x_1, \ldots, x_s, T)) \to \Omega_{k((x_1, \ldots, x_s, T))}$$

is injective. This shows that the extension is separable.

(5.11) **Note:** It is also true that $Q(A_{(n)}) \xrightarrow{\subseteq} k((x_1,\ldots,x_s,T))$ is a separable extension.

(5.12) **Corollary:** $Q(R_s) \xrightarrow{\subseteq} Q(\hat{R}_s)$ is separable.

Proof: $Q(R_s)$ satisfies the conditions for L in (5.9) and (5.10), and $Q(\hat{R}_s) = k((x_1,\ldots,x_s,T))$.

(5.13) **Corollary:** R_s is an excellent ring.

Proof: This follows from (4.7).

6. Derivations in Characteristic p and an Important Lemma

To show $\operatorname{trdeg}_{R_S}\hat{R}_S = \infty$, we rely heavily on the theory of derivations. In this section the basic information we will need is presented. In particular, we define Hasse-Schmidt derivations, which are a generalization of the iteration of ordinary derivations for rings of characteristic p. After reviewing the elementary properties of these derivations, we move on to prove the major stepping stone in the proof that $\operatorname{trdeg}_{R_S}\hat{R}_S = \infty$, a lemma generalized from the work of Rotthaus and the most technical result of this paper. In section 7, we will finish off the proof that $\operatorname{trdeg}_{R_S}\hat{R}_S = \infty$.

Let R be a ring of characteristic p.

(6.1) **Definition:** $\{d_n : n \in N\}$, a sequence of maps $d_n : R \to R$, is a <u>derivation</u> of R if :

(D1): $d_0 = id_R$

(D2): $d_n(x+y) = d_n(x) + d_n(y)$

(D3): $d_n(xy) = \sum_{i=0}^{n} d_i(x) d_{n-i}(y)$

(D4): $d_n \circ d_m(x) = \binom{m+n}{m} d_{m+n}(x)$

for all $x,y \in R$ and $m,n \in N$.

Note: d_1 is an ordinary derivation; the higher "nth derivative" differ from the composition d_1^n by a constant, arising from the binomial coefficient which occurs in (D4).

(6.2) **Example:** Let R[[T]] be the formal power series ring in one variable T. Set:

$$d_n(\sum_{i=0}^{\infty} r_i T^i) = \sum_{i=n}^{\infty} \binom{i}{n} r_i T^{i-n} \text{ for } n \in N.$$

Then $\{d_n : n \in N\}$ is a derivation of R[[T]].

Note that the binomial coefficients appearing in this definition and in (6.1) above are understood as integers modulo p. The fact that many of these vanish (=0) is what makes the characteristic p case so much more complicated than the characteristic 0 case. The following proposition proved exceedingly useful in determining when these binomial coefficients do not vanish.

(6.3) **Proposition:** Let $m,n \in N$. Let :

$$m = \sum_{i=0}^{e} m_i p^i \quad \text{and} \quad n = \sum_{i=0}^{e} n_i p^i$$

be the p-adic expressions of m and n $(0 \le m_i, n_i \le p-1$ for all $i)$.

Then:

$$\binom{m}{n} \equiv \binom{m_e}{n_e} \cdot \binom{m_{e-1}}{n_{e-1}} \cdot \ldots \cdot \binom{m_0}{n_0} \text{ mod } p.$$

Thus $\binom{m}{n} \not\equiv 0 \text{ mod } p \Leftrightarrow m_i \ge n_i$ for $i = 0$ to e.

Proof: See Shikishima-Tsuji and Katsura [7], Lemma 1, p.94.

For (6.4) and (6.5), assume that $\{d_n : n \in N\}$ is a derivation on a ring R of characteristic p. Let $x \in R$.

(6.4) **Proposition:** $d_n(x^i) = i \cdot x^{i-1} \cdot d_n x + a_i$ where $a_i \in Z/pZ[d_0 x, \ldots, d_{n-1}x]$.

(6.5) **Proposition:** i) $d_n x^{p^e} = 0$ if $n \not\equiv 0 \bmod p^e$.
ii) $d_{np^e} x^{p^e} = (d_n x)^{p^e}$.

Proof: For proofs of (6.4) and (6.5) see Okugawa, Proposition 2 and its corollary, p.12.

In Rotthaus, the following useful lemma appears:

(6.6) **Lemma:** Suppose $K \subseteq L$ is an extension of fields of characteristic 0. Let δ be a derivation of L with $\delta(K) \subseteq K$. Then for any element $\alpha \in L$, the following conditions are equivalent:

i) the set $\{\delta^n(\alpha) : n \in N\}$ is algebraically independent over K.

ii) $trdeg_K K(\delta^n(\alpha) : n \in N)$ is infinite.

In particular, if (i) fails then $K(\delta^n(\alpha) : n \in N) = K(\alpha, \delta\alpha, \ldots, \delta^m\alpha)$ for some $m \in N$.

Proof: See Rotthaus' paper for the details. The idea is to suppose that i) fails, then there exists an algebraic relation among the first, say, $m+1$ derivatives of α: $\alpha, \delta\alpha, \ldots, \delta^m\alpha$. We apply δ to this equation to obtain a new equation, which now involves $\delta^{m+1}(\alpha)$ also. Continuing in this manner one produces enough relations to conclude that $\mathrm{trdeg}_K K(\delta^n(\alpha) : n \in N) < \infty$.

The above lemma is used in Rotthaus' proof that $\mathrm{trdeg}_{R_S} \hat{R}_S = \infty$ no less than three times - in seemingly critical places. It was clear immediately that we needed a similar statement for char p, and yet here the lemma is patently false, as the following example shows.

(6.7) **Example:** Let $K \subseteq L$ be fields of characteristic p. Let $\{d_n : n \in N\}$ be a derivation of L, under which K is closed ($d_n(K) \subseteq K$ for all $n \in N$). Let $\alpha \in L$ be such that $\{d_n\alpha : n \in N\}$ is algebraically independent over K (these do exist in certain cases, see discussion (7.3)). Let $a = d_1\alpha$. Then :

i) $\{d_n a : n \in N\}$ is algebraically dependent over K

and

ii) $\mathrm{trdeg}_K K(d_n a : n \in N) = \infty$.

Proof: $d_n a = d_n(d_1\alpha) = \binom{n+1}{n} d_n\alpha$ by (D4).

If m>0, then $\binom{p^m+1}{p^m} \equiv 1 \bmod p$ and $\binom{p^m-1+1}{p^m-1} \equiv 0 \bmod p$ by (6.3) so that $d_{p^m}a = d_{p^m+1}\alpha$ and $d_{p^m-1}a = 0$ for all m>0. These facts imply ii) and i) respectively.

In view of (6.7), the question is how to restate Rotthaus' lemma (6.6) so that it will be true for char p, and still be strong enough to show that $\operatorname{trdeg}_{R_S}\hat{R}_S = \infty$. We follow the exact same method of proof as Rotthaus - applying successive derivations to an algebraic relation - but vanishing coefficients will make the restatement of the lemma and the argument quite technical, and at least slightly tricky. The following notion, interesting in its own right, proves to be useful to our understanding of the revised lemma.

(6.8) **Definition:** Let $K \subseteq L$ be fields of characteristic p (resp. 0), $\{d_n : n \in N\}$ (resp. δ) a derivation on L under which K is closed. Let $\alpha \in L$. Set :

$$v_K(\alpha) = \lim_{r\to\infty} \frac{1}{r+1}\left(\operatorname{trdeg}_K K(d_0\alpha, \ldots d_r\alpha)\right)$$

$$\left(\text{resp.} \quad = \lim_{r\to\infty} \frac{1}{r+1}\left(\operatorname{trdeg} K(\alpha, \delta\alpha, \ldots, \delta^r\alpha)\right)\right).$$

(6.9) **Note:** (i) It follows from (6.6) that, for the analogous definition in characteristic 0, $v_K(\alpha) = 0$ or $v_K(\alpha) = 1$.

(ii) Shikishima-Tsuji and Katsura, in Theorem 3 of [7], show that for the extension of fields $K \subseteq K((X))$ in characteristic p, for every real number $0 \leq r \leq 1$, there exists an $\alpha \in K[[X]]$ such that $v_K(\alpha) = r$. Here we see how much more complicated the characteristic p case is in an explicit manner.

(iii) unfortunately, we do not know that the limit actually exists for any element (in characteristic p), and allowances for this must be made in the following proposition.

The next result is the strongest reformulation of Rotthaus' lemma that we could find; then we deduce a weaker corollary which turns out to be sufficient for the result of section 7, $\mathrm{trdeg}_{R_S}\hat{R}_S = \infty$.

Let $L_1 \subseteq L_2 \subseteq E$ be fields of characteristic p, $\{d_n : n \in N\}$ a derivation of E under which both L_1 and L_2 are closed. Let $\alpha \in E$.

(6.10) **Lemma:** If $\{d_n\alpha : n \in N\}$ is algebraically independent over L_1 and algebraically dependent over L_2, then there exists a $\lambda \in L_2$ such that either $v_{L_1}(\lambda)$ does not exist, or $v_{L_1}(\lambda)>0$.

Proof: Since A= $\{d_n\alpha : n \in N\}$ is algebraically dependent over L_2, there exist $w_0, \ldots, w_r \in L_2$ such that A is algebraically dependent over $L_1(w_0, \ldots, w_r)$.

Let $\Gamma_i = \{d_n w_i : n \in N\}$ for i=0, . . . , r.

Clearly A is algebraically dependent over $L_1(\Gamma_0, \ldots, \Gamma_r)$.

By reducing r if necessary, it can be assumed that:

i) A is algebraically independent over $L_1(\Gamma_0, \ldots, \Gamma_{r-1})$,

ii) A is algebraically dependent over $L_1(\Gamma_0, \ldots, \Gamma_r)$.

Writing $L = L_1(\Gamma_0, \ldots, \Gamma_{r-1})$, $w = w_r$ and $\Gamma = \Gamma_r$, these become:

i) A is algebraically independent over L,

ii) A is algebraically dependent over L[Γ],

where L is closed under $\{d_n : n \in N\}$.

Thus there exists an $m \in N$ such that $\{d_0\alpha, \ldots, d_m\alpha\}$ is algebraically dependent over L[Γ].

Choose $0 \neq f(v_0, \ldots, v_m) \in L[\Gamma][v_0, \ldots, v_m]$ (v_i are variables) of minimal total degree in the v_i's such that $0 = f(d_0\alpha, \ldots, d_m\alpha)$.

As in the proof of Rotthaus' lemma, we would like to apply a derivation to the equation $0 = f(d_0\alpha, \ldots, d_m\alpha)$ and obtain a new relation, but in the characteristic p case this is much more difficult. How the above equation will behave under different d_n is determined by numerical aspects of the polynomial f, defined below. These will help us decide which d_n we should apply, and what to expect afterwards.

Define:

$$e = \max\{ e : f(v_0, \ldots, v_m) \in L[\Gamma][v_0^{p^e}, \ldots, v_m^{p^e}]\}$$

$$j = \min\{ j : f(v_0, \ldots, v_m) \in L[\Gamma][v_0^{p^e}, \ldots, v_j^{p^e}, v_{j+1}^{p^{e+1}}, \ldots, v_m^{p^{e+1}}]\}$$

$$c = \left\lfloor \frac{mp - j}{p - 1} \right\rfloor = \text{the greatest integer less than or equal to } \frac{mp - j}{p - 1}.$$

Note that $j \leq m$ so $\frac{mp - j}{p - 1} \geq \frac{mp - m}{p - 1} = m$. This implies that $c \geq m$.

Let μ be such that $f(v_0, \ldots, v_m) \in L[d_0 w, \ldots, d_\mu w][v_0, \ldots, v_m]$.

The significance of e, j, and μ is that now $f(v_0, \ldots, v_m)$ can be expressed as :

$$f(v_0, \ldots, v_m) = \sum_{i=0}^{\eta} F_i(v_0, \ldots, v_m) v_j^{ip^e} \quad \text{where}$$

$$F_i(v_0, \ldots, v_m) \in L[d_0 w, \ldots, d_\mu w][v_0^{p^e}, \ldots, v_{j-1}^{p^e}, v_{j+1}^{p^{e+1}}, \ldots, v_m^{p^{e+1}}]$$

and there exists an $i \not\equiv 0 \bmod p$ such that $F_i(v_0, \ldots, v_m) \neq 0$.

One should think of v_j as corresponding to the "highest derivative" which occurs in f, even though derivatives which are higher (up to m) certainly may exist. This is because the higher derivatives all occur raised to a multiple of a strictly greater power of p (at least e+1). When we apply higher and higher derivatives to the equation $0 = f(v_0, \ldots, v_m)$, according to (6.5) the terms derived from $d_j\alpha$ **eventually** will be the highest derivatives in our new relation. For this reason, $d_j\alpha$ is isolated and kept track of specifically, while the other derivatives are of only secondary importance.

Certain derivatives of α will never be able to be expressed as derivatives of $d_j\alpha$ (the $d_n\alpha$ for which $\binom{n}{j} \equiv 0 \bmod p$, see (D4)), so we essentially add these to the base field L in the claim that follows. We also add the first c derivatives, in view of the word **eventually** in the previous paragraph. The point of the following claim is, more or less, to express how many derivatives of w have to be added to obtain a new derivative of α. By comparing w and α in this way, we'll subsequently be able to argue that certain derivatives of w are algebraically independent over L, and finally that $v_{L_1}(w)>0$ (or doesn't exist), which is the goal of the lemma.

Let $B = \{d_n\alpha : \binom{n}{j} \equiv 0 \bmod p\}$

$C = \{d_0\alpha, \ldots, d_c\alpha\}$

$W_i = \{d_n w : n \leq i\}$.

Claim: For all $t \in N$, $d_t\alpha$ is algebraically dependent over $L[B, C, W_{\mu+p^e(t-j)}]$.

Proof (of claim): Suppose the claim is false, then there exists a minimal t such that $d_t\alpha$ is algebraically independent over the above ring. Clearly $d_t\alpha \notin B$ and $d_t\alpha \notin C$, so we can assume that $t>c$ and $\binom{t}{j} \not\equiv 0 \bmod p$.

We want to show that $d_t\alpha$ is algebraic over the above ring. To do this we apply $d_{p^e(t-j)}$ to the equation:

$$0 = f(d_0\alpha, \ldots, d_m\alpha) = \sum_{i=0}^{\eta} f_i \cdot (d_j\alpha)^{ip^e} \quad \text{where}$$

$$f_i = F_i(d_0\alpha, \ldots, d_m\alpha) \in L[W_\mu][(d_0\alpha)^{p^e}, \ldots (d_{j-1}\alpha)^{p^e}, (d_{j+1}\alpha)^{p^{e+1}}, \ldots, (d_m\alpha)^{p^{e+1}}]$$

For greater ease in notation, let $D_{p^e} = \{(d_0\alpha)^{p^e}, \ldots, (d_{j-1}\alpha)^{p^e}\}$

$D_{p^{e+1}} = \{(d_{j+1}\alpha)^{p^{e+1}}, \ldots, (d_m\alpha)^{p^{e+1}}\}$.

Now $f_i \in L[W_\mu, D_{p^e}, D_{p^{e+1}}]$.

Thus: $0 = d_{p^e(t-j)}\left(\sum_{i=0}^{\eta} f_i \cdot (d_j\alpha)^{ip^e}\right).$

By (D3): $$0 = \sum_{\ell=0}^{p^e(t-j)} \sum_{i=0}^{\eta} (d_{p^e(t-j)-\ell} f_i) \cdot (d_\ell (d_j\alpha)^{ip^e}).$$

If i>0, $\ell \not\equiv 0 \bmod p^e$ then by (6.5) $d_\ell (d_j\alpha)^{ip^e} = 0$, and if $i = 0$, $\ell > 0$, then still $d_\ell (d_j\alpha)^{ip^e} = 0$. Using these facts we obtain a new equation:

$$0 = \sum_{\ell=0}^{t-j} \sum_{i=0}^{\eta} (d_{p^e(t-j-\ell)} f_i) \cdot (d_{\ell p^e} (d_j\alpha)^{ip^e}).$$

Now apply the formula from the second part of (6.5) to obtain:

$$0 = \sum_{\ell=0}^{t-j} \sum_{i=0}^{\eta} (d_{p^e(t-j-\ell)} f_i) \cdot (d_\ell (d_j\alpha)^i)^{p^e}.$$

We separate the terms for which $\ell = t - j$:

(*) $$0 = \sum_{\ell=0}^{t-j-1} \sum_{i=0}^{\eta} (d_{p^e(t-j-\ell)} f_i) \cdot (d_\ell (d_j\alpha)^i)^{p^e} + \sum_{i=0}^{\eta} f_i \cdot (d_{t-j}(d_j\alpha)^i)^{p^e}.$$

We examine each of the factors in the above equation in turn:

i) What is $d_{p^e(t-j-\ell)} f_i$? Or even better, what ring is it in ?

We know $f_i \in L[W_\mu, D_{p^e}, D_{p^{e+1}}]$. We compute what happens to W_μ, D_{p^e}, and $D_{p^{e+1}}$ in turn:

a) $d_{p^e(t-j-\ell)}(L[W_\mu]) \subseteq L[d_\delta(d_k w) : \delta = 0 \text{ to } p^e(t-j),\ k = 0 \text{ to } \mu]$

$$\subseteq L[W_{\mu+p^e(t-j)}] \text{ for } \ell = 0 \text{ to } t-j-1.$$

b) Similarly,

$$d_{p^e(t-j-\ell)}(L[D_{p^e}]) \subseteq L[d_\delta(d_k\alpha)^{p^e} : k = 0 \text{ to } j-1,\ \delta = 0 \text{ to } p^e(t-j)]$$

by (6.5) $\subseteq L[(d_k\alpha)^{p^e} : k = 0 \text{ to } j-1+t-j]$

$$\subseteq L[d_0\alpha, \ldots, d_{t-1}\alpha].$$

c) Finally,

$$d_{p^e(t-j-\ell)}(L[D_{p^{e+1}}]) \subseteq L[d_\delta(d_k\alpha)^{p^{e+1}} : k = j+1, \ldots, m,\ \delta = 0, \ldots, p^e(t-j)]$$

by (6.5) $\subseteq L[(d_k\alpha)^{p^{e+1}} : k = j+1, \ldots, m + \left\lfloor \frac{p^e(t-j)}{p^{e+1}} \right\rfloor].$

Note that $t>c$ implies that $t>\frac{mp-j}{p-1}$, or that $tp - t > mp - j$ and

$t > m + \frac{t-j}{p} \geq m + \left\lfloor \frac{t-j}{p} \right\rfloor$. Hence $m + \left\lfloor \frac{p^e(t-j)}{p^{e+1}} \right\rfloor \leq t-1$.

This, with the above, shows that :

$$d_{p^e(t-j-\ell)}(L[D_{p^{e+1}}]) \subseteq L[d_0\alpha, \ldots, d_{t-1}\alpha].$$

Combining a), b) and c), we conclude that :

$d_{p^e(t-j-\ell)}f_i \in L[W_{\mu+p^e(t-j)}][d_0\alpha, \ldots, d_{t-1}\alpha]$. This ends i).

ii) As for $(d_\ell(d_j\alpha)^i)^{p^e}$, the second factor in (*), we first note that $\ell \leq t-j-1$. By (6.4) and (D4),

$$d_\ell(d_j\alpha)^i = i\cdot(d_j\alpha)^{i-1}\cdot\binom{\ell+j}{\ell}\cdot d_{\ell+j}\alpha + a_i \quad \text{where } a_i \in \mathbb{Z}/p\mathbb{Z}[d_k\alpha \ : k=j,\ldots,\ell+j-1]$$

But $\ell+j \leq t-1$, so clearly: $(d_\ell(d_j\alpha)^i)^{p^e} \in L[d_0\alpha, \ldots, d_{t-1}\alpha]$.

iii) Finally, $d_{t-j}(d_j\alpha)^i = i\cdot(d_j\alpha)^{i-1}\cdot\binom{t}{j}\cdot d_t\alpha + a_i$ where $a_i \in L[d_j\alpha, \ldots, d_{t-1}\alpha]$

by (6.4).

Thus we rewrite (*) as:

$$0 = \sum_{\ell=0}^{t-j-1}\sum_{i=0}^{\eta} d_{p^e(t-j-\ell)}f_i\cdot(d_\ell(d_j\alpha)^i)^{p^e} + \sum_{i=0}^{\eta} f_i\cdot a_i^{p^e} + \left(\sum_{i=1}^{\eta} f_i\cdot i\cdot\binom{t}{j}\cdot(d_j\alpha)^{(i-1)p^e}\right)\cdot(d_t\alpha)^{p^e}$$

Letting $c_0 = \sum_{\ell=0}^{t-j-1}\sum_{i=0}^{\eta} d_{p^e(t-j-\ell)} f_i \cdot (d_\ell (d_j\alpha)^i)^{p^e} + \sum_{i=0}^{\eta} f_i \cdot a_i^{p^e}$

$$c_1 = \left(\sum_{i=1}^{\eta} f_i \cdot i \cdot \binom{t}{j} \cdot (d_j\alpha)^{(i-1)p^e}\right),$$

we get that $0 = c_0 + c_1 \cdot (d_t\alpha)^{p^e}$ where by i), ii), and iii), c_0, $c_1 \in L[W_{\mu+p^e(t-j)}][d_0\alpha, \ldots, d_{t-1}\alpha]$.

We now show that in fact $c_1 \neq 0$:

First note that the polynomial in $v_0, \ldots, v_m$:

$$\sum_{i=1}^{\eta} F_i(v_0, \ldots, v_m) \cdot i \cdot \binom{t}{j} \cdot v_j^{(i-1)p^e} \in L[d_0 w, \ldots, d_\mu w][v_0, \ldots, v_m]$$

is not equal to 0 because :

i) $\binom{t}{j} \not\equiv 0 \mod p$

and ii) there exists an $i \not\equiv 0 \mod p$ such that $F_i(v_0, \ldots, v_m) \neq 0$.

If $F_i(v_0, \ldots, v_m) \neq 0$ and $i>0$, then:

$$\text{Tot.Deg.}\left(F_i(v_0, \ldots, v_m) v_j^{ip^e}\right) \geq p^e + \text{Tot.Deg.}\left(F_i(v_0, \ldots, v_m) \cdot i \cdot \binom{t}{j} \cdot v_j^{(i-1)p^e}\right)$$

It follows that :

$$\text{Tot.Deg.}\left(\sum_{i=0}^{\eta} F_i(v_0, \ldots, v_m) v_j^{ip^e}\right) > \text{Tot.Deg.}\left(\sum_{i=1}^{\eta} F_i(v_0, \ldots, v_m)\cdot i \cdot \binom{t}{j} \cdot v_j^{(i-1)p^e}\right).$$

So that, by choice of f :

$$0 \neq c_1 = \sum_{i=1}^{\eta} F_i(d_0\alpha, \ldots, d_m\alpha)\cdot i \cdot \binom{t}{j} \cdot (d_j\alpha)^{(i-1)p^e}.$$

Hence $0 = c_0 + c_1\cdot(d_t\alpha)^{p^e}$ where $c_0, c_1 \in L[W_{\mu+p^e(t-j)}][d_0\alpha, \ldots, d_{t-1}\alpha]$ and $c_1 \neq 0$. Thus $d_t\alpha$ is algebraically dependent over $L[W_{\mu+p^e(t-j)}][d_0\alpha, \ldots, d_{t-1}\alpha]$.

Let $u < t$ be an integer. By our choice of t, we can assume that $d_u\alpha$ is algebraically dependent over $L[B, C, W_{\mu+p^e(u-j)}]$. Hence $d_u\alpha$ is also algebraically dependent over the larger ring $L[B, C, W_{\mu+p^e(t-j)}]$. This shows that $L[B, C, W_{\mu+p^e(t-j)}][d_0\alpha, \ldots, d_{t-1}\alpha]$ is an algebraic extension of $L[B, C, W_{\mu+p^e(t-j)}]$. Now we have shown $d_t\alpha$ is algebraically dependent over $L[W_{\mu+p^e(t-j)}][d_0\alpha, \ldots, d_{t-1}\alpha]$ and hence over $L[B, C, W_{\mu+p^e(t-j)}][d_0\alpha, \ldots, d_{t-1}\alpha]$. Hence $d_t\alpha$ is algebraically dependent over $L[B, C, W_{\mu+p^e(t-j)}]$, which is a contradiction to the choice of t, and ends the proof of the claim.

Because $A = \{d_n\alpha : n \in N\}$ is algebraically independent over L, $A \setminus (B \cup C) = \{d_n\alpha : \binom{n}{j} \not\equiv 0 \bmod p,\ n > c\}$ is algebraically independent over L[B, C]. We would like to describe how 'big' this algebraically independent set $A\setminus(B\cup C)$ is, or more precisely, since $|A\setminus(B\cup C)| = \infty$, how frequently can we expect to find a member of $A\setminus(B\cup C)$.

First choose e' such that $p^{e'} > j$. By the computational result of Shikishima-Tsuji and Katsura (6.3), we know that:

$$\binom{np^{e'}+j}{j} \equiv 1 \mod p \text{ for all n.}$$

This shows that once past c, at least one out of every $p^{e'}$ derivatives of α will be in $A\setminus(B\cup C)$.

On the other hand, by the claim and the above comment for $u < t$, for all $c < u \leq t$ $d_u\alpha$ is algebraically dependent over $L[B, C, W_{\mu+p^e(t-j)}]$, or:

$$\text{trdeg}_{L(B, C, W_{\mu+p^e(t-j)})} L(B, C, W_{\mu+p^e(t-j)})(d_0\alpha, \ldots, d_t\alpha) = 0.$$

We now have:

$$\text{trdeg} \geq \left\lfloor \frac{t-c}{p^{e'}} \right\rfloor \left(\begin{array}{c} L(B, C, W_{\mu+p^e(t-j)})(d_0\alpha, \ldots, d_t\alpha) \\ \uparrow \\ L(B, C, W_{\mu+p^e(t-j)}) \\ \uparrow \\ L(B, C) \end{array} \right) \text{trdeg} = 0$$

So that we must also have $\text{trdeg}_{L(B, C)} L(B, C, W_{\mu+p^e(t-j)}) \geq \left\lfloor \dfrac{t-c}{p^{e'}} \right\rfloor$.

Since $L_1 \subseteq L \subseteq L(B, C)$, this implies that :

$$\mathrm{trdeg}_{L_1} L_1[W_{\mu+p^e(t-j)}] > \frac{t-c}{p^{e'}} - 1 .$$

We let $\vartheta_\ell = \dfrac{\mathrm{trdeg}_{L_1} L_1(W_\ell)}{\ell+1}$, the term which appears in the limit of definition (6.8).

The above has shown that:

$$\vartheta_{\mu+p^e(t-j)} > \frac{\frac{t-c}{p^{e'}} - 1}{\mu + p^e(t-j)+1} = \frac{t-c-p^{e'}}{p^{e+e'}t - p^{e+e'}j + p^{e'}\mu + p^{e'}}$$

$$= \frac{t + \text{ (a constant)}}{p^{e+e'}t + \text{ (a constant)}} .$$

For large t, this latter quotient approaches $1/(p^{e+e'})$, so that :

$$\lim_{t\to\infty} \vartheta_{\mu+p^e(t-j)} \geq \frac{1}{p^{e+e'}} > 0, \text{ if the limit actually exists.}$$

But this implies that the limit $v_{L_1}(w) \neq 0$, if it exists. Since $w \in L_2$, setting $w = \lambda$ proves the lemma.

Were we to attempt to reformulate this lemma (6.10) for characteristic 0, what would we end up with? We first make a definition, and then prove a similar lemma for characteristic 0. The fact that the proof will rely exclusively on Rotthaus' lemma (6.6) shows that (6.10) is an appropriate analogy to (6.6).

(6.11) **Definition :** Let $R \subseteq S$ be rings of characteristic p (resp. char 0). Let $\{d_n : n \in N\}$ (resp. δ) be a derivation of S under which R is closed. An element $\alpha \in S$ is hypertranscendental over R if $\{d_n\alpha : n \in N\}$ (resp. $\{\delta^n\alpha : n \in N\}$) is algebraically independent over R.

Let $L_1 \subseteq L_2 \subseteq E$ be fields of characteristic 0, $\delta : E \rightarrow E$ be a derivation of E such that L_1 and L_2 are closed under δ. Let $\alpha \in E$.

(6.12) **Lemma :** If α is hypertranscendental over L_1 and not hypertranscendental over L_2 , then some element of L_2 is hypertranscendental over L_1.

Proof : As in characteristic p (see beginning of proof of (6.10)) , the situation can be reduced to the following:

$L_1 \subseteq L$ is a field, closed under δ, and $w \in L_2$ such that :

i) α is hypertranscendental over L,

ii) α is not hypertranscendental over $L(\delta^n w : n \in N)$.

By Rotthaus' lemma (6.6), $\operatorname{trdeg}_{L(\delta^n w : n \in N)} L(\delta^n w, \delta^n\alpha : n \in N)$ is finite. Since $\operatorname{trdeg}_L L(\delta^n w, \delta^n\alpha : n \in N) = \infty$, it must be true that $\operatorname{trdeg}_L L(\delta^n w : n \in N) = \infty$. But

, by lemma (6.6) once more, this implies that $w \in L_2$ is hypertranscendental over L, and hence over L_1.

Thus we obtain the following corollary:

(6.13) **Corollary :** Let $L_1 \subseteq L_2 \subseteq E$ be fields of characteristic p (or 0), $\{d_n : n \in N\}$ (or δ) a derivation of E under which both L_1 and L_2 are closed. If $\alpha \in E$ is hypertranscendental over L_1 and not hypertranscendental over L_2, then $\mathrm{trdeg}_{L_1} L_2 = \infty$.

Proof : In characteristic 0 this follows immediately from (6.12). In characteristic p, (6.10) guarantees a $\lambda \in L_2$ such that $v_{L_1}(\lambda)$ either doesn't exist or is greater than zero. Either case implies $\mathrm{trdeg}_{L_1} L_1(d_n\lambda : n \in N) = \infty$, which in turn implies that $\mathrm{trdeg}_{L_1} L_2 = \infty$.

7. $\mathbf{Trdeg_{R_s}\,\hat{R}_s = \infty}$

In this section we present the proof that $\operatorname{trdeg}_{R_s}\hat{R}_s = \infty$, which will be essential in the next section when we prove that the ring $A_{n,t}$, constructed from R_s, has low dimensional formal fibers. The proof here depends on the corollary (6.13) from the previous section and the Weierstrass automorphisms introduced earlier (2.7). The proof is quite different than that of Rotthaus for characteristic 0. The advantage is that the new proof works in any characteristic, as we shall see in (7.8).

First we state a result of Shikishima-Tsuji and Katsura (Theorem 1, p.9 , [7]):

(7.1) **Notation:** Let $\{d_n : n \in N\}$ represent the Hasse-Schmidt derivation with respect to the variable T on the ring $k[[x_1, \ldots, x_s, T]]$ (or on $k[[T]]$) introduced as example (6.2).

(7.2) **Theorem :** There exists an $\alpha \in k[[x_1, \ldots, x_s, T]]$ which is hypertranscendental over k(T).

Note: This is a crucial result. Shikishima-Tsuji and Katsura's proof does not require that $k^p = k$, or that k be infinite.

Before stating and proving the next theorem, we prove the following lemma, which shows that our rings $A_{(n)} = k[[x_1 + n_1T, \ldots, x_s + n_sT]]$ are closed under the derivation $\{d_n : n \in N\}$. We will need this to show that certain fields are closed under $\{d_n : n \in N\}$, which will enable us to apply the corollary (6.13).

(7.3) **Lemma :** Let $(n) = (n_1, \ldots, n_s) \in M^s$, $r>0$. Then $d_r(A_{(n)}) \subseteq A_{(n)}$.

Proof : First note that $d_r : k[[x_1, \ldots, x_s, T]] \longrightarrow k[[x_1, \ldots, x_s, T]]$ takes terms of total degree n to terms of total degree n - r. Hence $d_r(m^{n+r}) \subseteq m^n$ (where $m = (x_1, \ldots, x_s, T)$, the maximal ideal of $k[[x_1, \ldots, x_s, T]]$). Thus d_r is continuous with respect to the topology on $k[[x_1, \ldots, x_s, T]]$ induced by m.

Define a map $\phi : k[[y_1, \ldots, y_s]] \longrightarrow k[[x_1, \ldots, x_s, T]]$

$$y_i \mapsto x_i + n_i T \quad \text{for } i = 1 \text{ to } s.$$

Then by (2.3) we have that :

$$k[[y_1, \ldots, y_s]] \underset{\phi}{\cong} A_{(n)} = k[[x_1 + n_1 T, \ldots, x_s + n_s T]] \subseteq k[[x_1, \ldots, x_s, T]].$$

Let $a \in A_{(n)}$. We write a as $a = \sum_{i=0}^{\infty} a_i$, where each a_i is a homogeneous polynomial of degree i (in $x_1, \ldots, x_s, T$).

Claim: $d_r(a_i) \in A_{(n)}$ is a homogeneous polynomial of degree i - r (if $i < r$ then $d_r(a_i) = 0$) .

Proof (of claim): Let $i_1 + \ldots + i_s = i$. We compute d_r on a monomial:

$$d_r((x_1 + n_1 T)^{i_1} \cdot \ldots \cdot (x_s + n_s T)^{i_s}) = \sum_{r_1 + \ldots + r_s = r} d_{r_1}(x_1 + n_1 T)^{i_1} \cdot \ldots \cdot d_{r_s}(x_s + n_s T)^{i_s}$$

by the product rule. Similarly :

$$d_{r_j}(x_j + n_jT)^{i_j} = \sum_{\ell_1+\ldots+\ell_{i_j}=r_j} d_{\ell_1}(x_j + n_jT)\cdot \ldots \cdot d_{\ell_{i_j}}(x_j + n_jT)$$

and $$d_{\ell_m}(x_j + n_jT) = \begin{cases} 0 \text{ if } \ell_m > 1 \\ n_j \text{ if } \ell_m = 1 \\ x_j + n_jT \in A_{(n)} \text{ if } \ell_m = 0 \end{cases}$$

by the definition (6.2) of our derivation. Hence:

$$d_{r_j}(x_j + n_jT)^{i_j} = \binom{i_j}{r_j}\cdot n_j^{r_j}\cdot(x_j + n_jT)^{i_j - r_j} \in A_{(n)}.$$

Now: $d_r((x_1 + n_1T)^{i_1}\cdot \ldots \cdot(x_s + n_sT)^{i_s}) =$

$$\sum_{r_1+\ldots+r_s=r} \binom{i_1}{r_1}\cdot n_1^{r_1}\cdot(x_1 + n_1T)^{i_1-r_1}\cdot \ldots \cdot \binom{i_s}{r_s}\cdot n_s^{r_s}\cdot(x_s + n_sT)^{i_s-r_s},$$

which is homogeneous of degree i - r. Note that for a term in the above sum to be nonzero, i_j must be greater than or equal to r_j for all j. Hence $d_r(a_i) = 0$ if $i < r$. This proves the claim.

Since $d_r(a_i) \in A_{(n)}$, we can consider the elements $\phi^{-1}(d_r(a_i))$, which will also be homogeneous polynomials (in the y_i 's) of degree i - r. Since $k[[y_1, \ldots, y_s]]$ is complete, $\lim_{n\to\infty} \sum_{i=r}^{n} \phi^{-1}(d_r(a_i)) = \sum_{i=r}^{\infty} \phi^{-1}(d_r(a_i))$ is in $k[[y_1, \ldots, y_s]]$.

$$\text{Now } \phi\left(\sum_{i=r}^{\infty} \phi^{-1}(d_r(a_i))\right) = \sum_{i=r}^{\infty} \phi\left(\phi^{-1}(d_r(a_i))\right) = \sum_{i=r}^{\infty} d_r(a_i) = d_r\left(\sum_{i=0}^{\infty} a_i\right) = d_r(a)$$

since d_r is continuous. Hence $d_r(a) \in \operatorname{Im}\phi = A_{(n)}$ which proves the lemma.

Now for our theorem:

(7.4) **Theorem :** Let $\alpha \in k[[T]]$ which is hypertranscendental over $k(T)$. Then α is also hypertranscendental over R_s. In particular, $\mathrm{trdeg}_{R_s}\hat{R}_s = \infty$.

Proof: Let $A = \{d_n\alpha : n \in N\}$. Assume α is not hypertranscendental over R_s, i.e. A is algebraically dependent over R_s.

Since $R_s \subseteq Q(R_0)$, this means that A is also algebraically dependent over $Q(R_0) = k($ $A_{(n)} : (n) \in M^s$). Since only finitely many $A_{(n)}$ can be involved in an algebraic relation, A must be algebraically dependent over:

$$k(A_{(n_1)}, \ldots, A_{(n_r)}) \text{ for some } (n_1), \ldots, (n_r) \in M^s .$$

By reducing r if necessary, it can be assumed that :

i) A is algebraically independent over $k(A_{(n_1)}, \ldots, A_{(n_{r-1})})$,

ii) A is algebraically dependent over $k(A_{(n_1)}, \ldots, A_{(n_r)})$.

Again, an algebraic relation can only involve finitely many elements $\tau_1, \ldots, \tau_t \in A_{(n_r)}$. Letting $L = k(A_{(n_1)}, \ldots, A_{(n_{r-1})})$, we get:

i) A is algebraically independent over L,

ii) A is algebraically dependent over $L(\tau_1, \ldots, \tau_t)$.

We would like to argue that this is a contradiction to (6.13), but the field $L(\tau_1, \ldots, \tau_t)$ is not necessarily closed under our derivation. To arrange for this, we could add to the field all the derivatives of $\tau_1, \ldots, \tau_t$, but then we could potentially have infinite transcendence degree between the two fields, so no contradiction. Instead, we apply a Weierstrass automorphism to adjust the situation.

Apply the automorphism $\sigma_{-(n_r)}$ to i) and ii) above:

$$\begin{aligned}\sigma_{-(n_r)}\left(A_{(n_i)}\right) &= \sigma_{-(n_r)} \circ \sigma_{(n_i)}\left(k[[x_1, \ldots, x_s]]\right) \\ &= \sigma_{(n_i)-(n_r)}\left(k[[x_1, \ldots, x_s]]\right) = A_{(m_i)},\end{aligned}$$

where $(m_i) = (n_i) - (n_r)$ for $i = 1$ to $r-1$. For the $\tau_1, \ldots, \tau_t$, we note that:

$$\sigma_{-(n_r)}(A_{(n_r)}) = \sigma_{-(n_r)} \circ \sigma_{(n_r)}(k[[x_1, \ldots, x_s]]) = k[[x_1, \ldots, x_s]] .$$

Thus $\sigma_{-(n_r)}(\tau_1), \ldots, \sigma_{-(n_r)}(\tau_t) \in k[[x_1, \ldots, x_s]]$. The set A remains fixed under $\sigma_{-(n_r)}$, as noted in (2.7)iv. Thus we get :

i) A is algebraically independent over $L_1 = k(A_{(m_1)}, \ldots, A_{(m_{r-1})})$,

ii) A is algebraically dependent over $L_2 = L_1(\sigma_{-(n_r)}(\tau_1), \ldots, \sigma_{-(n_r)}(\tau_t))$.

Now the field L_1 is closed under $\{d_n : n \in N\}$ by (7.3). L_2 is also closed under $\{d_n : n \in N\}$ because $\sigma_{-(n_r)}(\tau_1), \ldots, \sigma_{-(n_r)}(\tau_t) \in k[[x_1, \ldots, x_s]]$ and:

$$d_n\big|_{k[[x_1, \ldots, x_s]]} = 0 \text{ if } n>0.$$

But now α is hypertranscendental over L_1, not hypertranscendental over L_2, and $\mathrm{trdeg}_{L_1} L_2 \leq t < \infty$. This contradicts corollary (6.13), and thus proves the theorem.

The proof above works because we isolate a particular $A_{(n)}$, and then apply the appropriate σ to assume that $(n) = (0, \ldots, 0)$. Thus $A_{(n)} = k[[x_1, \ldots, x_s]]$, and this ring vanishes under our derivation, which differentiates with respect to T. But these remarks are equally true in characteristic 0. The only changes that have to be made are purely formal, and outlined below:

(7.5) **Definition:** Let k' be an infinite field of characteristic 0, and $M' \subseteq k'$ be an infinite additive subgroup of k. For $(n) = (n_1, \ldots, n_s) \in M'^s$, define :

$$A'_{(n)} = k'[[x_1 + n_1 T, \ldots, x_s + n_s T]] \subseteq k'[[x_1, \ldots, x_s, T]]$$

$$R'_0 = k'[\, A'_{(n)} : (n) \in M'^s \,] \subseteq k'[[x_1, \ldots, x_s, T]]$$

$$R'_s = Q(R'_0) \cap k'[[x_1, \ldots, x_s, T]] .$$

(7.6) **Note :** In Rotthaus' definition, M' = **Z**, but all the proofs go through with this slight generalization.

(7.7) **Notation:** Let δ be the derivation on $k'[[x_1, \ldots, x_s, T]]$ which differentiates with respect to T.

(7.8) **Theorem :** Let $\alpha \in k'[[T]]$ which is hypertranscendental over $k'(T)$. Then α is also hypertranscendental over R'_s. In particular, $\mathrm{trdeg}_{R'_s}\hat{R}'_s = \infty$.

Proof : We repeat the proof of (7.4), with the substitution of δ^n in the place of d_n. Note that lemma (6.12) was proved for any characteristic. Lemma (7.3) will also hold in characteristic 0 : one just has to show that $\delta(A'_{(n)}) \subseteq A'_{(n)}$, which is a quick check.

8. Low Dimensional Formal Fibers

In the previous sections we have introduced a ring R_s, which has various properties, as we have discussed and proved. But of course, $\alpha(R_s) = 0$ (3.4), so it is not an example of the rings in which we are interested, those with α between 0 and the dimension of the ring minus 2. In this final section of the paper we show that the rings $A_{n,t}$ constructed from R_s (see (1.2)) are excellent and have low dimensional formal fibers.

(8.1) **Theorem :** $A_{n,t}$ is an excellent Noetherian local domain of dimension n such that $\alpha(A_{n,t}) = t$, i.e. the generic formal fiber of $A_{n,t}$ has dimension t.

Proof : It is clear that $A_{n,t}$ is a local domain. It is Noetherian since R_s is Noetherian (1.3) and has dimension n because :

$$\hat{A}_{n,t} = k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}]]$$

and $s + t + 2 = n$. The ring $A_{n,t}$ is excellent because R_s is excellent (this follows from Rotthaus [5]).

It remains to see that $\dim \hat{A}_{n,t} \otimes_{A_{n,t}} Q(A_{n,t}) = t$.

"$\leq$" : Let $P \in \hat{A}_{n,t} \setminus (v_1, \ldots, v_{t+1})$. By (2.12) we can express P as:

$$P = e\cdot\left(T^m + b_{m-1}T^{m-1} + \ldots + b_0\right)$$

where e is a unit in $\hat{A}_{n,t}$ and $b_i \in A_{(n)}[[v_1, \ldots v_{t+1}]]$ for all $i = 0$ to $m-1$.

Thus $(P) \cap A_{n,t} \neq (0)$, so any prime ideal $q \subsetneq \hat{A}_{n,t}$ with $q \cap A_{n,t} = (0)$ must be properly contained in $(v_1, \ldots, v_{t+1})$. Since ht $(v_1, \ldots, v_{t+1}) = t + 1$, this shows that dim $\hat{A}_{n,t} \otimes_{A_{n,t}} Q(A_{n,t}) \leq t$.

"$\geq$" : Let $w_1, \ldots, w_t \in k[[x_1, \ldots, x_s, T]]$ which are algebraically independent over R_s (here we rely on the main result of section 7, $\mathrm{trdeg}_{R_s}\hat{R}_s = \infty$). Define a homomorphism:

$$\phi : \hat{A}_{n,t} = k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}]] \longrightarrow \hat{R}_s[[v_{t+1}]]$$

$$x_i \mapsto x_i \quad \text{for } i = 1 \text{ to } s$$

$$T \mapsto T$$

$$v_j \mapsto w_j v_{t+1} \quad \text{for } j = 1, \ldots, t$$

$$v_{t+1} \mapsto v_{t+1}$$

Clearly ϕ is surjective, and $Q = \ker \phi$ must be a prime ideal of height t.

It suffices to prove the following claim:

Claim: $Q \cap A_{n,t} = (0)$.

Proof (of claim): Let $F \in A_{n,t}$, $F \neq 0$.

Then $F = \sum_{\upsilon=0}^{\infty} f_\upsilon(v_1, \ldots, v_{t+1})$ where the f_υ are homogeneous polynomials in the variables $v_1, \ldots, v_{t+1}$ with coefficients in R_s.

Write $f_\upsilon(v_1, \ldots, v_{t+1}) = \sum_{\mu_1+\ldots+\mu_{t+1}=\upsilon} b_{(\mu)} \cdot v_1^{\mu_1} \cdot \ldots \cdot v_{t+1}^{\mu_t}$ where $b_{(\mu)} \in R_s$.

Now $\phi(F) = \sum_{\upsilon=0}^{\infty} f_\upsilon(w_1 v_{t+1}, \ldots, w_t v_{t+1}, v_{t+1}) =$

$$= \sum_{\upsilon=0}^{\infty} v_{t+1}^{\upsilon} \cdot \left(\sum_{\mu_1+\ldots+\mu_t=\upsilon} b_{(\mu)} w_1^{\mu_1} \cdot \ldots \cdot w_t^{\mu_t} \right).$$

If $F \neq 0$, then some $f_\upsilon(v_1, \ldots, v_{t+1}) \neq 0$. But then, since $\{w_1, \ldots, w_t\}$ is algebraically independent over R_s:

$$\sum_{\mu_1+\ldots+\mu_t=\upsilon} b_{(\mu)} w_1^{\mu_1} \cdot \ldots \cdot w_t^{\mu_t} \neq 0 \text{ for that } \upsilon.$$

Hence $\phi(F) \neq 0$, or $F \notin Q$, which proves the claim and the theorem.

The following corollary, which shows that maximal ideals in a formal fiber may have height strictly less than the dimension of the formal fiber, is proved in Rotthaus [6] for characteristic 0. For the sake of completeness, we also prove it here for characteristic p.

(8.2) **Corollary:** Let w be an additional variable. Then the generic formal fiber of $B = A_{n,t}[w]_{(m_{A_{n,t}}, w)}$ contains a maximal ideal of height $\leq t+1 < n-1$ while the generic formal fiber has dimension $\geq n-1$. (Here $m_{A_{n,t}}$ denotes the maximal ideal of $A_{n,t}$.)

Proof: $B \subsetneq \hat{A}_{n,t}[w]_{(m_{A_{n,t}}, w)} \subsetneq k[[x_1, \ldots, x_s, T, v_1, \ldots, v_{t+1}, w]] = \hat{B}$.

By Example 2 of Matsumura, the generic formal fiber of $\hat{A}_{n,t}[w]_{(m_{A_{n,t}}, w)}$ has dimension n - 1. Since $\hat{B} = \left(\hat{A}_{n,t}[w]_{(m_{A_{n,t}}, w)}\right)^{\wedge}$, we must also have: $\dim \hat{B} \underset{B}{\otimes} Q(B) \geq n - 1$.

Let $b \in \hat{A}_{n,t}$ be a non-invertible element which is algebraically independent over $A_{n,t}$. Take a maximal prime ideal $Q \subseteq \hat{B}$ such that $w - b \in Q$ and $Q \cap B = (0)$. Then $Q = (w - b, q)$, where $q = Q \cap \hat{A}_{n,t}$ is a prime ideal in $\hat{A}_{n,t}$ with $b \cap A_{n,t} = (0)$. By (8.1), ht $q \leq t$, thus ht $Q \leq t + 1$.

9. Summary

We have seen that excellent rings of characteristic p can have formal fibers of arbitrary dimension. On the way we have combined facts from Weierstrass Preparation and differential algebra. Hypertranscendental elements played a key role in proving $\mathrm{trdeg}_{R_s}\hat{R}_s = \infty$ in both characteristic 0 and p>0. And the intervention of a single Weierstrass automorphism, $\sigma_{-(n_r)}$, in the midst of Theorem (7.2), salvaged a proof which otherwise would have been very technical and messy, if possible at all. We are left with a thorough and complete answer (when taken in conjunction with the work of Rotthaus [6]) to Matsumura's original inquiry into the dimension of formal fibers.

Of the many questions that still remain about formal fibers, it might be appropriate to ask about the dimension of the formal fibers of a non-excellent ring. To my knowledge, no examples of low dimensional formal fibers exist for this case. Another major unanswered question involves the definition (6.8). As mentioned, we have been forced to allow for the fact that such a limit may not exist. Surely it should be possible to answer the existence question in a more definitive way. For this I believe that we will have to acquire a better understanding of how derivatives can be applied to algebraic relations, of which Lemma (6.10) is just the beginning. I also believe that the basic method of (7.2), where we are able to assume that $A_{(n)}$ is indeed $A_{(0,\ldots,0)}$, may be of use in furthering the study of Rotthaus' extraordinary ring R_S.

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List of References

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