NUMERICAL METHODS OF BIFURCATION PROBLEMS VIA SINGULAR VALUE DECOMPOSITIONS AND HOMOTOPY METHODS

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ABSTRACT

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A relation between bifurcation theory and the singular value decomposition, homotopy methods in numerical analysis is studied. Given a nonlinear equation, we give a local analysis in a neighborhood of a solution via the Liapunov-Schmidt method and the singular value decomposition. This analysis is applicable to regular, turning or bifurcation points. In the case of a bifurcation point, homotopy methods are used for solving the bifurcation equation. A numerical method for global bifurcation problems based on the above analysis is presented.

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CHAPTER 1

INTRODUCTION

Let F: $X \times R^s \rightarrow Z$ be a Fredholm operator, where X, Z are Banach spaces and R^s is the usual s dimensional space in which the parameter vector λ sets. The Liapunov - Schmidt method can be used to reduce $F(x,\lambda) = 0$ to a finite dimensional system $f(y,\lambda) = 0$ with f: $R^m \times R^s \rightarrow R^n$. The bifurcation behavior of the solution set of f determines the bifurcation behavior of the solution set of F. In the boundary value problem of ODE or PDE, a finite element method or a finite difference method also can be used to approximate the problem by a finite dimensional system. If the original differential equation contains parameters, then the finite dimensional system which is of the same form as f obtained from the Liapunov- Schmidt method also contains parameters. Furthermore, many problems themselves are finite dimensional problems. Therefore studying the behavior of the finite dimensional system with parameters is meaningful.

In this paper we present a new numerical methods for the bifurcation problem $f(x,\lambda) = 0$ with f : $R^m \times R \rightarrow R^m$. The methods include using reliable ways to distinguish the bifurcation and non-bifurcation, to factor out the bifurcation equation, and to solve the problem in either cases.

Bifurcation occurs at the points of the solution set when the Jacobian matrix of f at these points is rank deficient and a point of the solution set is near a bifurcation point when the Jacobian matrix at that point is nearly rank deficient. We relate this Jacobian matrix with the *singular value decomposition(SVD).* Because an effective way to detect and treat rank deficient problems in numerical analysis is to compute the singular value decomposition. see [6]. If the singular value decomposition of the Jacobian matrix at some point is performed, we certainly want to use the informations from it as much as we can. In this thesis, we connect it with the Liapunov- Schmidt method in the bifurcation theory. The major results which relate this are:

Theorem 3.2.2 The Liapunov-Schmidt method via SVD;

Theorem 4.1.1 A numerical bifurcation equation via SVD;

Theorem 6.1.1 A matrix result via SVD for the Newton's iterates;

Algorithm 7.2.1 A numerical method of global bifurcations via SVD.

After a numerical bifurcation equation, which is a system of special polynomials, is obtained via SVD, we need a reliable method to solve it. The development in 1976 by Chow, Mallet-Paret and Yorke [2] is an advance in the homotopy methods. The methods are called the *probability one homotopy methods* which provide practical ways for solving nonlinear equations [10]. For the general case of a system of polynomials, the problem of finding all zeros has already been solved in [3]. The numerical bifurcation equation which appears in this paper is a system of special polynomials. Only some of the solutions are to be solved. The others can be obtained by symmetry. We develop a special homotopy equation to find these required solutions by using some techniques in [2] [3]. We obtain a method which only requires half of the usual number of computations. The result is obtained in:

Theorem 5.2.6 A probability one homotopy method with symmetry for solving the numerical bifurcation equation.

In most chapters, we assume f satisfies the following two conditions:

(a) f is C^k, $k \geq 2$;

(b) there are only finite smooth curves passing each bifurcation point of the solution set of $f(x,\lambda) = 0$.

CHAPTER 2

THE SINGULAR VALUE DECOMPOSITION

§ 2.1. Definitions and Theorems

Singular value decomposition (SVD) is one of the most important tools in matrix computations. It is a reliable method for detecting and treating rank deficient problems. In this section we briefly describe the results of SVD which are needed in this thesis. We will present them by restricting to real matrices. For proofs or details, see, for example, Golub and Van Loan [6], Dongarra et al [4]. Similar results for complex matrices also can be found in Stoer and Burlisch [9]. But for the purpose of our research, we only need the case for real matrices which is presented in the following definitions and theorems.

Let A be a real m by n matrix. It is known that there exist an m by m orthogonal matrix U and an n by n orthogonal matrix V such that

$$A = U \Sigma V^T, \tag{1.1}$$

where

$$\Sigma = \begin{bmatrix} D_r & O \\ O & O \end{bmatrix} \tag{1.2}$$

is an m by n matrix, and D_r is a r by r diagonal matrix. Furthermore $D_r = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. V^T is the transpose of V.

Definition 1.1. (1.1) is called the *singular value decomposition* of the matrix A. Let

p = min(m,n), and define $\sigma_{r+1}=\sigma_{r+2}=...=\sigma_p = 0$. Then $\sigma_1, \sigma_2, \cdots, \sigma_p$ are called *the singular values* of A.

Theorem 1.2.

(a) The number of nonzero singular values is equal to the rank of A ;

(b) $\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2$ are the positive eigenvalues of A^TA and AA^T ;

(c) the columns of V are corresponding eigenvectors of A^TA ;

(d) the columns of U are corresponding eigenvectors of AA^T .

Definition 1.3. The columns of V are called the *right singular vectors* of A, while the columns of U are called *the left singular vectors* of A.

Definition 1.4. The *pseudo-inverse* A^+ is defined to be the matrix

$$A^+ = V \Sigma^+ U^T, \tag{1.3}$$

where

$$\Sigma^+ = \begin{bmatrix} D_r^{-1} & O \\ O & O \end{bmatrix} \tag{1.4}$$

with $D_r^{-1} = \mathrm{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_r})$.

It's easy to verify the properties of the pseudo-inverse:

Theorem 1.5.

(a) $AA^+A = A$;

(b) $A^+AA^+ = A^+$;

(c) $AA^+ = U \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} U^T$;

(d) $A^+A = V \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} V^T$;

(e) $(AA^+)^2 = AA^+$;

(f) $(A^+A)^2 = A^+A$.

Regarding A as a linear transformation from R^n to R^m under certain bases, denoting U = [$u_1, u_2, \cdots, u_m$] and V = [$v_1, v_2, \cdots, v_n$], the following theorem is obtained:

Theorem 1.6.

(a) Null(A) = Span{ $v_{r+1}, v_{r+2}, \cdots, v_n$}; and

(b) Range (A) = Span { $u_1, u_2, \cdots, u_r$}.

§ 2.2. Computational Methods

Theoretically Theorem 1.2 already gives a way to compute SVD of A. i.e., A^TA and AA^T can be used to obtain SVD of A. However it is not a satisfactory way for the computational purpose due to that the round off errors often destroy pertinent information. For example, let $A = \begin{bmatrix} 1 & 0 \\ a & 0 \\ 0 & a \end{bmatrix}$,where a satisfies $\varepsilon_0 < a < \sqrt{\varepsilon_0}$ and ε_0 is the machine precision. Then theoretically $A^TA = \begin{bmatrix} 1+a^2 & 0 \\ 0 & a^2 \end{bmatrix}$ gives the singular values of A with $\sigma_1 = \sqrt{1+a^2}$ and $\sigma_2 = a$ since obviously the eigenvalues of A^TA are $1+a^2$ and a^2. But the computational results due to the round off errors yield $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ gives the singular values of A with $\sigma_1 = 1$ and $\sigma_2 = 0$. The second singular value is qualitatively incorrect.

The basic computational method is the *Golub-Reisch SVD algorithm* [6] in 1970 which contains Householder bidiagonalization, the decoupling calculation and the Golub-Kahan SVD step of a bidiagonal square matrix having no zeros on its diagonal and superdiagonal. We will briefly illustrate these methods. For more details, see [4] [5] [6] [7] .

Let A be an m by n matrix. In this section, we assume m ≥ n, otherwise consider A^T. Two orthogonal matrices are involved in this algorithm, they are:

Definition 2.1. A *Householder matrix* U is a matrix of the form U = I - $2uu^T/u^Tu$, where u is a column vector.

Definition 2.2. A *Givens rotation matrix* U is a matrix of the form

$$U = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \cos\theta & & & \sin\theta & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & -\sin\theta & & & \cos\theta & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix} .$$

Then the matrix A can be transformed into a bidiagonal matrix by:

Theorem 2.3 (Householder Bidiagonalization). There exist products of Householder matrices $U_B = U_1 U_2 \cdots U_n$ and $V_B = V_1 V_2 \cdots V_{n-2}$ such that

$$U_B^T A V_B = \begin{bmatrix} B \\ \cdots\cdots \\ O \end{bmatrix}, \tag{2.1}$$

where

$$B = \begin{bmatrix} d_1 f_2 & & & \\ & d_2 f_3 & & \\ & & d_3 \ldots & \\ & & & \cdots f_n \\ & & & d_n \end{bmatrix}.$$

An example of 5×4 matrix is illustrated as following:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \xrightarrow{U_1 \times} \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \xrightarrow{\times V_1} \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \xrightarrow{U_2 \times} \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \xrightarrow{\times V_2} \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \xrightarrow{U_3 \times} \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} \xrightarrow{U_4 \times} \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As soon as SVD of B is obtained, SVD of A can be immediately obtained since

$$A = U_B \begin{bmatrix} B \\ \cdots \\ O \end{bmatrix} V_B^T = U_B \begin{bmatrix} U_C \Sigma_C V_C^T \\ \cdots\cdots \\ O \end{bmatrix} V_B^T = (U_B \begin{bmatrix} U_C & O \\ O & I \end{bmatrix}) \begin{bmatrix} \Sigma_C \\ \cdots\cdots \\ O \end{bmatrix} (V_B V_C)^T.$$

If B has a zero on its superdiagonal, B can be immediately decoupled into two smaller upper bidiagonal square matrices. If B has a zero on its diagonal, B also can be decoupled into two smaller upper bidiagonal square matrices by multiplying by a Givens matrix. Therefore using the decoupling calculations, upper bidiagonal square submatrices with no zero on their diagonal and superbidiagonal always can be obtained except for these submatrices on the diagonal of B with size 1.

An example of a 5×5 upper bidiagonal matrix with (3,3)-th entry zero is illustrated as following($+, O$ denote the entries which are changed in that step):

$$
\begin{bmatrix} * & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}
\xrightarrow{U_1 \times}
\begin{bmatrix} * & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & O & + \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}
\xrightarrow{U_2 \times}
\begin{bmatrix} * & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}
\xrightarrow{\times V_1}
\begin{bmatrix} * & * & + & 0 & 0 \\ 0 & * & O & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}
\xrightarrow{\times V_2}
\begin{bmatrix} * & * & O & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}.
$$

Now we discuss a method to diagonalize the above mentioned upperbidiagonal submatrices . Without losting generality we just assume B with no zero on its diagonal or superdiagonal since the decoupling calculations always can be used to change to smaller matrices.

Choosing Givens rotations $S_{i,i+1}$, $T_{i,i+1}$, i=1,2,...,n-1, leaving $T_{1,2}$ open, B is transformed as following:

$$
B \xrightarrow{\times T_{1,2}}
\begin{bmatrix} * & * & & & & \\ + & * & * & & & \\ & & * & * & & \\ & & & * & \cdots & \\ & & & & \cdots & * \\ & & & & & * \end{bmatrix}
\xrightarrow{S^T_{1,2} \times}
\begin{bmatrix} * & * & + & & & \\ O & * & * & & & \\ & & * & * & & \\ & & & * & \cdots & \\ & & & & \cdots & * \\ & & & & & * \end{bmatrix}
\xrightarrow{\times T_{2,3}}
\begin{bmatrix} * & * & O & & & \\ & * & * & & & \\ & + & * & * & & \\ & & & * & \cdots & \\ & & & & \cdots & * \\ & & & & & * \end{bmatrix}
$$

$$
\xrightarrow{S^T_{2,3} \times}
\begin{bmatrix} * & * & & & & \\ & * & * & + & & \\ & O & * & * & & \\ & & & * & \cdots & \\ & & & & \cdots & * \\ & & & & & * \end{bmatrix}
\rightarrow \cdots \xrightarrow{S^T_{n-1,n} \times}
\begin{bmatrix} * & * & & & \\ & * & * & & \\ & & * & \cdots & \\ & & & \cdots & \\ & & & * & * \\ & & & O & * \end{bmatrix}
$$

$$
\tilde{B} = (S_{1,2} S_{2,3} \cdots S_{n-1,n})^T B (T_{1,2} T_{2,3} \cdots T_{n-1,n}) = S^T B T \tag{2.2}
$$

which is again an upper bidiagonal matrix.

Now $T_{1,2}$ is chosen as following:

$$T_{1,2} = \begin{bmatrix} \cos\phi_{12} & \sin\phi_{12} & & & \\ -sin\phi_{12} & \cos\phi_{12} & & & \\ & & 1 & & \\ & & & \cdots & \\ & & & & 1 \end{bmatrix} \tag{2.3}$$

such that

$$\begin{bmatrix} \cos\phi_{1,2} & -sin\phi_{1,2} \\ \sin\phi_{1,2} & \cos\phi_{1,2} \end{bmatrix} \begin{bmatrix} d_1^2-\mu \\ d_1 f_2 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix} , \tag{2.4}$$

where μ is the eigenvalue of

$$\begin{bmatrix} d_{n-1}^2+f_{n-1}^2 & d_{n-1}f_n \\ d_{n-1}f_n & d_n^2+f_n^2 \end{bmatrix} \tag{2.5}$$

which is closer to $d_n^2 + f_n^2$.

The algorithm for obtaining $\overline{B}$ from B by (2.2) - (2.5) is just the *Golub-Kahan SVD step.*

Let $B^{(1)}$ = B. Recursively $B^{(i+1)}$ can be obtained by using $B^{(i)}$ instead of B in the above step. If for some $B^{(i)}$, there are zeros on its diagonal or superdiagonal, smaller matrices will be considered. It is known that $f_n^{(i)} \rightarrow 0$ and $d_n^{(i)} \rightarrow \sigma_n$ at least quadratically when i $\rightarrow \infty$. Disregarding exception, the convergent rate is even cubic.

Hence the singular value decomposition of B always can be obtained by iterations.

The subroutine SSVD in Linpack [4], SVD and MINIF in Eispack [5] are all based on the Golub-Reisch SVD algorithm.

A different way of computing the bidiagonalization in the *Lawson-Harson algorithm* in 1974 is to upper triangularize the matrix A first, which is faster when m $\gg$ n. The subroutine LSVDF in IMSL Library [7] is based on the Lawson-Harson SVD algorithm.

CHAPTER 3

THE SINGULAR VALUE DECOMPOSITION AND THE LIAPUNOV-SCHMIDT METHOD

§ 3.1. The Liapunov-Schmidt Method

Many problems in analysis and applied mathematics can be reduced to the determination of the zeros of a function in a Banach space. A bifurcation occurs when a multiple zero exists. A technique which is called the *Liapunov-Schmidt Method* (LSM) can be used to simplify bifurcation problems. In this section we want to establish the connection between the singular value decomposition and the Liapunov-Schmidt method. Therefore a reliable numerical method can be used to solve the bifurcation problems.

We first state the following theorem which gives the Liapunov-Schmidt method.

Theorem 1.1 (Chow-Hale [1]). Suppose X, Z are Banach spaces, A: $X \to Z$ is a continuous linear operator, N: $X \to Z$ is a continuous nonlinear operator and I is the identity operator in X. Let W and E be continuous projections in X and Z respectively. Suppose

$$Null(A) = X_W \quad , \qquad Range(A) = Z_E \quad ,$$

where X_w is the range of W in X and Z_E is the range of E in Z. Then there is a bounded linear operator K: $Z_E \rightarrow X_{I-W}$, called the *right inverse* of A , such that AK = I on Z_E , KA = I - W on X. Moreover, the equation

$$Ax - Nx = 0 \tag{1.1}$$

is equivalent to the following equations

$$z - KEN(y+z) = 0, \tag{1.2}$$

$$(I-E)N(y+z) = 0, \tag{1.3}$$

where

$$x = y+z, \quad y \in X_W, \quad z \in X_{I-W}.$$

For the proof of this theorem, see [1].

§ 3.2. A Connection of SVD and LSM

Consider a C^1 function f: $R^m \times R^s \rightarrow R^m$. Such a function can be viewed as a function from R^m to itself with s parameters. In our discussion, we treat $R^m \times R^s$ as R^{m+s}.

The first derivative of f at some point $x_0 \in R^{m+s}$ can be represented as a real m by m+s Jocabian matrix, which is denoted by Df(x_0). Assume the rank of this matrix is r, $0 \le r \le m$. The singular value decomposition of Df(x_0) is denoted by

$$Df(x_0) = U_0 \Sigma_0 V_0^T \quad ,$$

The following Lemma is obvious.

Lemma 2.1.

(a) $U_0 \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} U_0^T$ is the projection from R^n to {Range[Df(x_0)]};

(b) $V_0 \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} V_0^T$ is the projection from R^{m+s} to { Null[Df(x_0)] $\}^{\perp}$;

(c) [Df(x_0) $]^+ = V_0 \Sigma_0^+ U_0^T$ is the right inverse of Df(x_0).

Proof: Identify Df(x_0) as A in Theorem 2.1.5, then Theorem 2.1.5 /(c),(e) give (a); Theorem 2.1.5/(d),(f) give (b). (Df(x_0) $)^+$ is just the pseudo-inverse of Df(x_0), therefore Theorem 2.1.5/(c),(d) give

$$Df\,(x_0)[Df\,(x_0)]^+ = U_0 \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} U_0^T$$

$$[Df\,(x_0)]^+ Df\,(x_0) = V_0 \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} V_0^T,$$

then (a),(b) and the definition implies (c). □

Theorem 2.2. Let f: $R^{m+s} \rightarrow R^m$ be C^1, $f\,(x_0)$=0. Let $Df\,(x_0)=U_0 \Sigma_0 V_0^T$ has rank r with $0 \le r \le m$. Then there is a neighborhood η of x_0 such that f(x)=0 , x $\in$ η if and only if

$$z-z_0=V_0 \Sigma_0^+ U_0^T \left[Df\,(x_0)(x-x_0)-f\,(x)\right] \quad \text{and} \tag{2.1.a}$$

$$\begin{bmatrix} \sum_{i=1}^{m} u_{0,i,r+1} f_i(x) \\ \cdots \\ \sum_{i=1}^{m} u_{0,i,m} f_i(x) \end{bmatrix} = 0\ , \tag{2.1.b}$$

where $z - z_0$ is the projector of $x - x_0$ in R^{m+s} to $\{ \text{Null} [Df(x_0)] \}^{\perp}$ which is the orthogonal complementary subspace of Null[Df(x_0)] in R^{m+s} and

$$f(x) = \begin{bmatrix} f_1(x) \\ \cdots \\ f_m(x) \end{bmatrix}.$$

In the case r=m, the second equation disappears.

Proof: Consider f(x)=0, which is same as

$$Df(x_0)(x - x_0) - [Df(x_0)(x - x_0) - f(x)] = 0 \quad . \tag{2.2}$$

Regarding Df(x_0) as the operator A, Df(x_0)(x- x_0) -f(x) as the continuous operator N(x), x- x_0 as x in Theorem 1.1, the right hand side of (1.1) in the theorem gives that (2.2) is equivalent to:

$$\begin{cases} (z - z_0) - KE[Df(x_0)(x - x_0) - f(x)] = 0 \\ (I - E)[Df(x_0)(x - x_0) - f(x)] = 0 \quad , \end{cases} \tag{2.3}$$

where (z- z_0) is the projector of $x - x_0$ in R^{m+s} to $\{ \text{Null}[Df(x_0)] \}^{\perp}$.

From Lemma 2.1/(a), we have:

$$(I - E)[Df(x_0)(x - x_0) - f(x)]$$

$$= U_0 \begin{bmatrix} O & O \\ O & I_{m-r} \end{bmatrix} U_0^T [U_0 \Sigma_0 V_0^T (x - x_0) - f(x)]$$

$$= U_0 \begin{bmatrix} O & O \\ O & I_{m-r} \end{bmatrix} U_0^T [-f(x)]$$

$$= U_0 \begin{bmatrix} O & O \\ O & I_{m-r} \end{bmatrix} \begin{bmatrix} -(u_{0,1,1}f_1(x)+u_{0,2,1}f_2(x)+\ldots+u_{0,m,1}f_m(x)) \\ \cdots \\ -(u_{0,1,m}f_1(x)+u_{0,2,m}f_2(x)+\ldots+u_{0,m,m}f\,(x)) \end{bmatrix}$$

$$= U_0 \begin{bmatrix} O \\ \cdots \\ O \\ -\sum_{i=1}^{m} u_{0,i,r+1}f_i(x) \\ \cdots \\ -\sum_{i=1}^{m} u_{0,i,m}f_i(x) \end{bmatrix} .$$

From Lemma 2.1/(a),(c), we also have:

$$KE = (V_0\Sigma_0^+U_0^T)(U_0 \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} U_0^T) = V_0\Sigma_0^+U_0^T \quad .$$

Combining (2.3) and the above two equalities, the conclusion of the above theorem connecting SVD and LSM is followed. □

In Chapter 4 and Chapter 6, we will restrict to the case s = 1. We are going to change (2.1) further in different situations according the rank r. In the deficient rank case, we obtain a bifurcation equation via the singular value decomposition in Chapter 4, and in the full rank case, we obtain a matrix result via the singular value decomposition in Chapter 6.

CHAPTER 4

A NUMERICAL BIFURCATION EQUATION

§ 4.1. Theorems

SVD can be used for detecting bifurcations because that it is a way for detecting rank deficiency, and a bifurcation occurs at the point where the rank of the Jacobian matrix of the derivative of the map is deficient. Now we are going to derive a bifurcation equation by using the Liapunov-Schmidt method via SVD which is a system of polynomials whose coefficients are expressed in terms of the entries of the orthonormal matrices in SVD.

In Chapter 4, 6, 7, we use s=1, i.e. f: $R^{m+1} \to R^m$ is a function containing one parameter. In other words, we deal with one-parameter problems or equivalent one-parameter problems (if several parameters are involved, we give more conditions to change them to one-parameter problems). Furthermore we assume that there are only finite smooth curves passing each bifurcation point of the solution set of f(x)=0, and f $\in C^k$, k $\geq$ 2 for some integer k which is discussed later in this chapter such that (1.1) holds.

In this chapter, we assume f(x_0)=0 for some $x_0=(x_{0,1}, x_{0,2}, \cdots, x_{0,m+1})^T$ in R^{m+1} . Df(x_0)=$U_0 \Sigma_0 V_0^T$ with rank r, $0 \leq r \leq m-1$, where

$$V_0 = \begin{bmatrix} v_{0,1,1,} & \cdots & v_{0,1,m+1} \\ \cdots & & \cdots \\ v_{0,m+1,1} & \cdots & v_{0,m+1,m+1} \end{bmatrix}.$$

Consider the Taylor series of $\begin{bmatrix} \sum_{i=1}^{m} u_{0,i,r+1} f_i(x) \\ \cdots \\ \sum_{i=1}^{m} u_{0,i,m} f_i(x) \end{bmatrix}$ about x=x_0. Obviously the constant term is zero, Also its linear term is zero, since

$$D(\begin{bmatrix} \sum_{i=1}^{m} u_{0,i,r+1} f_i(x) \\ \cdots \\ \sum_{i=1}^{m} u_{0,i,m} f_i(x) \end{bmatrix}(x_0)) = D(U_0 \begin{bmatrix} O & O \\ O & I_{m-r} \end{bmatrix} U_0^T f)(x_0)$$

$$= U_0 \begin{bmatrix} O & O \\ O & I_{m-r} \end{bmatrix} U_0^T (Df\,(x_0)) = (U_0 \begin{bmatrix} O & O \\ O & I_{m-r} \end{bmatrix} U_0^T)(U_0 \Sigma_0 V_0^T)$$

$$= O \ .$$

Denote x=$(x_1, x_2, \cdots, x_{m+1})^T$. We assume that f has enough smoothness such that Taylor's Theorem holds:

$$\begin{bmatrix} \sum_{i=1}^{m} u_{0,i,r+1} f_i(x) \\ \cdots \\ \sum_{i=1}^{m} u_{0,i,m} f_i(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{d_{r+1}!}[(x_1 - x_{0,1})\frac{\partial}{\partial x_1} + \cdots + (x_{m+1} - x_{0,m+1})\frac{\partial}{\partial x_{m+1}}]^{d_{r+1}} (\sum_{i=1}^{m} u_{0,i,r+1} f_i(x_0)) \\ \cdots \\ \frac{1}{d_m!}[(x_1 - x_{0,1})\frac{\partial}{\partial x_1} + \cdots + (x_{m+1} - x_{0,m+1})\frac{\partial}{\partial x_{m+1}}]^{d_m} (\sum_{i=1}^{m} u_{0,i,m} f_i(x_0)) \end{bmatrix} +$$

$$\begin{bmatrix} \frac{1}{(d_{r+1}+1)!}[(x_1 - x_{0,1})\frac{\partial}{\partial x_1} + \cdots + (x_{m+1} - x_{0,m+1})\frac{\partial}{\partial x_{m+1}}]^{d_{r+1}+1} (\sum_{i=1}^{m} u_{0,i,r+1} f_i(x_0 + \theta(x - x_0)) \\ \cdots \\ \frac{1}{(d_m+1)!}[(x_1 - x_{0,1})\frac{\partial}{\partial x_1} + \cdots + (x_{m+1} - x_{0,m+1})\frac{\partial}{\partial x_{m+1}}]^{d_m+1} (\sum_{i=1}^{m} u_{0,i,m} f_i(x_0 + \theta(x - x_0)) \end{bmatrix},$$

(1.1)

where $0 < \theta < 1$, $d_j \geq 2$, j=r+1, r+2, ... , m are positive integers and the second term of the right hand is continuous.

Actually if there exists a positive integer k for the smoothness such that:

$$k \geq \max_{r+1 \leq p \leq m} (d_p + 1), \tag{1.2}$$

then $f \in C^k$ will have (1.1). Note that $r \leq m-1$, $k \geq 3$.

Now we have the following theorem :

Theorem 1.1. The unit tangent vector ξ at x_0 along one of the branches satisfies:

$$(a) \quad \xi = y_{r+1} \begin{bmatrix} v_{0,1,r+1} \\ \cdots \\ v_{0,m+1,r+1} \end{bmatrix} + y_{r+2} \begin{bmatrix} v_{0,1,r+2} \\ \cdots \\ v_{0,m+1,r+2} \end{bmatrix} + \cdots + y_{m+1} \begin{bmatrix} v_{0,1,m+1} \\ \cdots \\ v_{0,m+1,m+1} \end{bmatrix};$$

$$(b) \quad [(\sum_{j=r+1}^{m+1} y_j v_{0,1,j}) \frac{\partial}{\partial x_1} + (\sum_{j=r+1}^{m+1} y_j v_{0,2,j}) \frac{\partial}{\partial x_2} + \cdots + (\sum_{j=r+1}^{m+1} y_j v_{0,m+1,j}) \frac{\partial}{\partial x_{m+1}}]^{d_p} (\sum_{i=1}^{m} u_{0,i,p} f_i)(x_0) = 0$$

for p = r+1, r+2, ... , m, where d_p is from (1.1);

$$(c) \quad y_{r+1}^2 + y_{r+2}^2 + \cdots + y_{m+1}^2 = 1.$$

Here y_i, i = r+1, ... , m+1, are real numbers.

In the generic case, $d_{r+1} = d_{r+2} = \cdots = d_m = 2$, we have the following corollary:

Corollary 1.2 (The Quadratic Bifurcation Equation).

In Theorem 1.1, if $d_{r+1} = d_{r+2} = \cdots = d_m = 2$, then the conclusion (b) has the following matrix form:

$$(y_{r+1}, \cdots, y_{m+1}) \begin{bmatrix} v_{0,1,r+1} & \cdots & v_{0,m+1,r+1} \\ \cdots & & \\ v_{0,1,m+1} & \cdots & v_{0,m+1,m+1} \end{bmatrix} Q_p \begin{bmatrix} v_{0,1,r+1} & \cdots & v_{0,1,m+1} \\ \cdots & & \\ v_{0,m+1,r+1} & \cdots & v_{0,m+1,m+1} \end{bmatrix} \begin{bmatrix} y_{r+1} \\ \cdots \\ y_{m+1} \end{bmatrix} = 0$$

for p = r+1, r+2, ... , m, and

$$Q_p = \begin{bmatrix} \frac{\partial^2}{\partial x_1 \partial x_1}(\sum_{i=1}^m u_{0,i,p} f_i(x_0)), \ldots, \frac{\partial^2}{\partial x_1 \partial x_{m+1}}(\sum_{i=1}^m u_{0,i,p} f_i(x_0)) \\ \cdots \\ \frac{\partial^2}{\partial x_{m+1} \partial x_1}(\sum_{i=1}^m u_{0,i,p} f_i(x_0)), \ldots, \frac{\partial^2}{\partial x_{m+1} \partial x_{m+1}}(\sum_{i=1}^m u_{0,i,p} f_i(x_0)) \end{bmatrix} .$$

This corollary gives a different way to derive the quadratic bifurcation equation with [8] in the generic case.

§ 4.2. Proofs

Proof of Theorem 1.1: We first prove Theorem 1.1 by starting with (3.2.1), f(x)=0 is equivalent to the equations:

$$z - z_0 = V_0 \Sigma_0^{+} U_0^T \left[Df(x_0)(x - x_0) - f(x) \right]$$

and

$$\begin{bmatrix} \sum_{i=1}^{m} u_{0,i,r+1} f_i(x) \\ \cdots \\ \sum_{i=1}^{m} u_{0,i,m} f_i(x) \end{bmatrix} = 0.$$

Let $\| x - x_0 \|$ be small. Dividing both sides of the first equation of the right hand side of the above expression by $\| x - x_0 \|$, note that

$$z - z_0 = V_0 \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} V_0^T (x - x_0),$$

we get

$$V_0 \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} V_0^T \left(\frac{x - x_0}{\| x - x_0 \|} \right) = V_0 \Sigma_0^{+} U_0^T \left[\frac{Df(x_0)(x - x_0) - f(x)}{\| x - x_0 \|} \right]. \qquad (2.1)$$

Let x set on the branch of the solution set of f(x) = 0 near x_0 and tend to x_0. Then $\frac{x - x_0}{\| x - x_0 \|} \to \xi$, the unit tangent vector along that branch by the assumed smoothness condition. Note that $Df(x_0)(x - x_0) - f(x) = o(\| x - x_0 \|)$, therefore the following equation is obtained from (2.1):

$$V_0 \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} V_0^T \xi = 0,$$

i.e.,

$$\xi = V_0 \begin{bmatrix} 0 \\ \cdots \\ 0 \\ y_{r+1} \\ \cdots \\ y_{m+1} \end{bmatrix}$$

$$= y_{r+1} \begin{bmatrix} v_{0,1,r+1} \\ \cdots \\ v_{0,m+1,r+1} \end{bmatrix} + y_{r+2} \begin{bmatrix} v_{0,1,r+2} \\ \cdots \\ v_{0,m+1,r+2} \end{bmatrix} + \cdots + y_{m+1} \begin{bmatrix} v_{0,1,m+1} \\ \cdots \\ v_{0,m+1,m+1} \end{bmatrix} \tag{2.2}$$

that is exactly (a) in Theorem 1.1.

The second equation $\begin{bmatrix} \sum_{i=1}^{m} u_{0,i,r+1} f_i(x) \\ \cdots \\ \sum_{i=1}^{m} u_{0,i,m} f_i(x) \end{bmatrix} = 0$ in (3.2.1) can be replaced due to (1.1) and the smoothness condition (1.2) of f by

$$\begin{bmatrix} [(x_1 - x_{0,1})\frac{\partial}{\partial x_1} + \cdots + (x_{m+1} - x_{0,m+1})\frac{\partial}{\partial x_{m+1}}]^{d_r+1} (\sum_{i=1}^{m} u_{0,i,r+1} f_i(x_0)) + \frac{d_{r+1}!}{(d_{r+1}+1)!}[o(\,||x - x_0||^{d_{r+1}}] \\ \cdots \\ [(x_1 - x_{0,1})\frac{\partial}{\partial x_1} + \cdots + (x_{m+1} - x_{0,m+1})\frac{\partial}{\partial x_{m+1}}]^{d_m} (\sum_{i=1}^{m} u_{0,i,m} f_i(x_0)) + \frac{d_m!}{(d_m+1)!}[o(\,||x - x_0||^{d_m}] \end{bmatrix} = 0 \;. \tag{2.3}$$

Divide both sides of the first component by $||x - x_0||^{d_{r+1}}$, the second component by $||x - x_0||^{d_{r+2}}$, ... , the last component by $||x - x_0||^{d_{m+1}}$, then let $||x - x_0|| \to 0$, hence the second term in each component goes to zero. For the first terms in all components, the orders of the derivatives are just the same as the orders of the powers of $\| x - x_0 \|$, therefore we distribute through to each of $x_1 - x_{0,1}$, $x_2 - x_{0,2}$,..., $x_{m+1} - x_{0,m+1}$ a denominator $\| x - x_0 \|$. By all the hypotheses of f assumed, the limit can be performed in all above mentioned quotients, and the limits of these quotients are just the components of the unit tangent vector, by (a), they are $\sum_{j=r+1}^{m+1} y_j v_{0,1,j}$, $\sum_{j=r+1}^{m+1} y_j v_{0,2,j}$, ..., $\sum_{j=r+1}^{m+1} y_j v_{0,m+1,j}$. The limit of the second term of each component is zero. Therefore the limit of (2.3) gives (b), i.e.,

$$\begin{bmatrix} [(\sum_{j=r+1}^{m+1} y_j v_{0,1,j})\frac{\partial}{\partial x_1} + \cdots + (\sum_{j=r+1}^{m+1} y_j v_{0,m+1,j})\frac{\partial}{\partial x_{m+1}}]^{d_{r+1}}(\sum_{i=1}^{m} u_{0,i,r+1} f_i)(x_0) \\ \cdots \\ [(\sum_{j=r+1}^{m+1} y_j v_{0,1,j})\frac{\partial}{\partial x_1} + \cdots + (\sum_{j=r+1}^{m+1} y_j v_{0,m+1,j})\frac{\partial}{\partial x_{m+1}}]^{d_m}(\sum_{i=1}^{m} u_{0,i,m} f_i)(x_0) \end{bmatrix} = 0 \quad (2.4)$$

(c) of Theorem 1.1 is obvious. Hence the theorem is proved. $\square$

Proof of Corollary 1.2: Corollary 1.2 can be immediately followed since the second derivative can be written in the matrix form:

$$[(\sum_{j=r+1}^{m+1} y_j v_{0,1,j})\frac{\partial}{\partial x_1} + \cdots + (\sum_{j=r+1}^{m+1} y_j v_{0,m+1,j})\frac{\partial}{\partial x_{m+1}}]^2(g(x_0))$$

$$= \sum_{p,q=1}^{m+1} (\sum_{j=r+1}^{m+1} y_j v_{0,p,j})(\sum_{j=r+1}^{m+1} y_j v_{0,q,j})\frac{\partial^2 g(x_0)}{\partial x_p \partial x_q}$$

$$= [(\sum_{j=r+1}^{m+1} y_j v_{0,1,j}), \cdots, (\sum_{j=r+1}^{m+1} y_j v_{0,m+1,j})]$$

$$\begin{bmatrix} \frac{\partial^2 g(x_0)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 g(x_0)}{\partial x_1 \partial x_{m+1}} \\ \cdots & & \cdots \\ \frac{\partial^2 g(x_0)}{\partial x_{m+1} \partial x_1} & \cdots & \frac{\partial^2 g(x_0)}{\partial x_{m+1} \partial x_{m+1}} \end{bmatrix} \begin{bmatrix} \sum_{j=r+1}^{m+1} y_j v_{0,1,j} \\ \cdots \\ \sum_{j=r+1}^{m+1} y_j v_{0,m+1,j} \end{bmatrix}$$

$$= (y_{r+1}, \ldots, y_{m+1}) \begin{bmatrix} v_{0,1,r+1} & \cdots & v_{0,m+1,r+1} \\ \cdots & & \\ v_{0,1,m+1} & \cdots & v_{0,m+1,m+1} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial^2 g(x_0)}{\partial x_1 \partial x_1} & \cdots & \dfrac{\partial^2 g(x_0)}{\partial x_1 \partial x_{m+1}} \\ \cdots & & \cdots \\ \dfrac{\partial^2 g(x_0)}{\partial x_{m+1} \partial x_1} & \cdots & \dfrac{\partial^2 g(x_0)}{\partial x_{m+1} \partial x_{m+1}} \end{bmatrix} \begin{bmatrix} v_{0,1,r+1} & \cdots & v_{0,1,m+1} \\ \cdots & & \\ v_{0,m+1,r+1} & \cdots & v_{0,m+1,m+1} \end{bmatrix} \begin{bmatrix} y_{r+1} \\ \cdots \\ y_{m+1} \end{bmatrix} .$$

$\square$

The bifurcation equation in Theorem 1.1 and Corollary 1.2 are systems of polynomials. They can be solved by the probability one homotopy methods which can guarantee finding all the roots. We will describe them in the next chapter.

CHAPTER 5

A PROBABILITY ONE HOMOTOPY METHOD

§ 5.1. Introduction

The bifurcation equation reduced from bifurcation problems via SVD in Chapter 4 is a system of special polynomial equations. There are m+1-r equations and variables. For sake of convenience, in this chapter, we use n instead of m+1-r. Denote ($y_{r+1}, y_{r+2}, \cdots, y_{m+1}$) by Z = ($z_1, z_2, \cdots, z_n$) and n polynomials by $P_1(Z), P_2(Z), \cdots, P_n(Z)$ which can be regarded as components of a polynomial vector P(Z).

Homotopy methods can be used to solve nonlinear equations, i.e., if we want to solve P(Z) = 0 , we first solve a simple equation Q(Z) = 0, and then set a homotopy function H(Z,t) = (1-t) Q(Z) + t P(Z). Solve H(Z,t) = 0 by following the homotopy curves (solution set of H(Z,t) = 0) from t=0 to t=1, hence the zeros of Q(Z) lead to the zeros of P(Z). The nonsingular Jacobian matrices of H(Z,t) are important for tracing the homotopy curves.

The development in 1976 by Chow et al can finally avoid singular Jacobian matrices by constructions called the *probability one homotopy methods* . In the methods, for almost all the choices of the homotopy parameters, the methods are globally convergent. This is an advance over earlier homotopies, since the philosophy and the resulting software are fundamentally new [10].

The numerical bifurcation equation is a system of special polynomial equations.

Observe that if Z = ($z_1, z_2, \ldots, z_n$) is a solution of this system, then -Z = ($-z_1, -z_2, \ldots, -z_n$) is also a solution of it. Therefore we wish to construct a symmetric homotopy function such that only half of the solutions need to be computed. In the next section, we will use the fundamental idea of "probability one" and the techniques in [2][3] to construct such a function.

§ 5.2. A Probability One Homotopy Method for the Bifurcation Equation

The system of polynomials appear in the bifurcation equation above is a system of homogeneous polynomials with degree $d_i \geq 2$ for each polynomial $P_i(Z)$, i = 1, 2, ..., n-1, except the last one $P_n(Z) = z_1^2 + z_2^2 + \ldots + z_n^2 - 1$. Without losing generality, we can assume that the first s polynomials are of odd degree, i.e., $d_i \geq 3$, for i = 1, 2, ..., s, and the rest of the polynomials are of even degree, $0 \leq s \leq n-1$. Note first that there is at least one even degree polynomial (the last one), and secondly that Z = 0 is not a solution of the above system, although it satisfies all the polynomial equations except the last one. Thirdly DP(0) = 0, since P(Z) has no linear term.

We constrict the following symmetric homotopy function:

$$H(Z,t) = \begin{bmatrix} H_1(Z,t) \\ \cdots \\ H_n(Z,t) \end{bmatrix}$$

$$= (1-t) \begin{bmatrix} z_1^{d_1} - b_1 z_1 \\ \cdots \\ z_s^{d_s} - b_s z_s \\ z_{s+1}^{d_{s+1}} - b_{s+1} \\ \cdots \\ z_{n-1}^{d_{n-1}} - b_{n-1} \\ z_n^2 - b_n \end{bmatrix} + t \begin{bmatrix} P_1(Z) \\ \cdots \\ P_n(Z) \end{bmatrix} + t(1-t) \begin{bmatrix} \sum_{j=1}^{n} a_{1,j} z_j^{d_1} \\ \cdots \\ \sum_{j=1}^{n} a_{n-1,j} z_j^{d_{n-1}} \\ \sum_{j=1}^{n} a_{n,j} z_j^2 \end{bmatrix}, \tag{2.1}$$

where $Z = (z_1, z_2, \cdots, z_n)^T \in \mathbf{C}^n$ and $H(Z,t) = (H_1(Z,t), H_2(Z,t), \cdots, H_n(Z,t))^T \in \mathbf{C}^n$. The parameters are chosen in random by $a = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{nn})^T \in \mathbf{C}^{n^2}$ and $b = (b_1, \ldots, b_n)^T \in \mathbf{C}^n$.

The difference of this homotopy function and the known homotopy function is that $b_1 z_1, \ldots, b_s z_s$ are used instead of $b_1, \ldots, b_s$ in the first s components. Therefore the transversality condition should be checked. We have:

Lemma 2.1. Let $W = \{ (a,b) \in \mathbf{C}^{n^2} \times \mathbf{C}^n \mid b_1, \cdots, b_n \neq 0 \}$, and $A = \{ (Z,t) \in \mathbf{C}^n \times R \}$. Then the submatrix of the Jacobi of (2.1) $\dfrac{\partial(H_1, \cdots, H_n)}{\partial(Z,\, a,\, b)}$ has full rank on $A \times W$ (i.e., rank is n if one regards the matrix as a $n \times [n + n^2 + n]$ complex matrix, or the rank is 2n if one regards the matrix as a $2n \times [\, 2n + (2n)^2 + 2n \,]$ real matrix).

Proof: Case 1: $t = 0$.

Consider a submatrix :

$$\frac{\partial H}{\partial(z,b)} = \begin{bmatrix} d_1 z_1^{d_1-1} - b_1 & & & & & & -z_1 & & & & \\ & \cdots & & & & & & \cdots & & & \\ & & d_s z_s^{d_s-1} - b_s & & & & & & -z_s & & \\ & & & d_{s+1} z_{s+1}^{d_{s+1}-1} & & & & & & -1 & \\ & & & & \cdots & & & & & & \cdots \\ & & & & & d_{n-1} z_{n-1}^{d_{n-1}-1} & & & & & \cdots \\ & & & & & & 2z_n & & & & -1 \end{bmatrix} \tag{2.2}$$

which has full rank since one of $d_j z_j^{d_j-1} - b_j$, $-z_j$ is nonzero, for $1 \le j \le s$. Therefore the assertion is true in this case.

Case 2: $t \in (0,1)$ and $z_1, \cdots, z_n$ are not all zero.

Consider a submatrix:

$$\frac{\partial H}{\partial a} = \begin{bmatrix} t(1-t)z_1^{d_1},\ldots,t(1-t)z_n^{d_n} & & & & \\ & t(1-t)z_1^{d_2},\ldots,t(1-t)z_n^{d_2} & & & \\ & & \cdots\cdots & & \\ & & & t(1-t)z_1^{d_{n-1}},\ldots,t(1-t)z_n^{d_{n-1}} & \\ & & & & t(1-t)z_1^2,\ldots,t(1-t)z_n^2 \end{bmatrix}$$

(2.3)

which has rank n as soon as one of $z_1, \cdots, z_n$ is nonzero.

Case 3 : $t \in (0,1)$ and $z_1 = \cdots = z_n = 0$.

In this case $\dfrac{\partial P}{\partial Z} = 0$ and also the partial derivatives to Z of the terms in (2.1), which have degrees of Z greater than one, are zero.

$$\frac{\partial H}{\partial (Z,b)} = \begin{bmatrix} -(1-t)b_1 & & & 0 & & \\ & \cdots & & & \cdots & \\ & & -(1-t)b_n & & & 0 \\ 0 & & & -(1-t) & & \\ & \cdots & & & \cdots & \\ & & 0 & & & -(1-t) \end{bmatrix}$$

(2.4)

which gives rank n. □

Now define the homogeneous part of (2.1) by:

$$\tilde{H}(Z,t) = \begin{bmatrix} \tilde{H}_1(Z,t) \\ \cdots \\ \tilde{H}_n(Z,t) \end{bmatrix}$$

$$= (1-t)\begin{bmatrix} z_1^{d_1} \\ \cdots \\ z_{n-1}^{d_{n-1}} \\ z_n^2 \end{bmatrix} + t\begin{bmatrix} P_1(Z) \\ \cdots \\ P_{n-1}(Z) \\ z_1^2 + \cdots + z_n^2 \end{bmatrix} + t(1-t)\begin{bmatrix} \sum_{j=1}^n a_{1,j} z_j^{d_1} \\ \cdots \\ \sum_{j=1}^n a_{n-1,j} z_j^{d_{n-1}} \\ \sum_{j=1}^n a_{n,j} z_j^2 \end{bmatrix}. \qquad (2.5)$$

Since $\frac{\partial \tilde{H}}{\partial a}$ is just as case 2 of (2.3), we have:

Lemma 2.2. The Jacobi of (2.5) has full rank w.r.t. $(z,t) \times (a,b)$ on $\{ (\mathbf{C}^n - \{0\}) \times (0,1] \times W$.

Now we need a *transversality theorem* (see [3]).

Definition 2.3. Let F be a smooth map : open set $A \subseteq R^d \rightarrow R^p$, then a point $y \in R^p$ is called a *regular value* of F on $S \subseteq A$ provided that Range $\{ DF(x) \} = R^p$ for

all $x \in S \cap F^{-1}(y)$. Those x's are called *regular points.*

Theorem 2.4 (Transversality Theorem). Let $A \subseteq R^d$ and $W \subseteq R^q$ be open sets, and F: $A \times W \rightarrow R^p$ be C^r smooth with $r > \max\{0, d-p\}$. Suppose for some set $S \subseteq A$ that $y \in R^p$ is a regular value of F on $S \times W$. Then for almost every $w \in W$ (in the sense of either Baire category or Lebesgue measure), y is a regular value of $F(\bullet, w)$ on S.

Lemma 2.1 and Lemma 2.2 give the full rank of the Jacobians of H and $\tilde{H}$ (regarding them as real matrices by $\mathbf{C} = R^2$), therefore they give two onto linear transformations. This implies that $0 \in \mathbf{C}^n$ is a regular value of H on $[\mathbf{C}^n \times [0,1)] \times W$ and of $\tilde{H}$ on $[(\mathbf{C}^n - \{0\}) \times (0,1)] \times W$. Also direct computation gives that 0 is a regular value of $\tilde{H}$ on $[(\mathbf{C}^n - \{0\}) \times 0] \times W$. Hence the transversality theorem gives:

Lemma 2.5. For almost every (a,b) $\in \mathbf{C}^{n^2} \times \mathbf{C}^n$, $0 \in \mathbf{C}^n$ is a regular value both of H(• , • ,a,b) on $\mathbf{C}^n \times [0,1)$ and of $\tilde{H}(\bullet,\bullet,a)$ on $(\mathbf{C}^n - \{0\}) \times [0,1)$.

As soon as Lemma 2.5 is established, the rest of work is same as [3] . i.e., the first part implies the homotopy curves are one-dimensional manifolds, also $\frac{dt}{ds} > 0$ where s is the arclength; the second part guarantees the curves not going to the infinity before t $\to$ 1. The degree theory argument guarantees all the solutions of P are the end points of these curves. Therefore we have:

Theorem 2.6. For almost every (a,b) $\in \mathbf{C}^{n^2} \times \mathbf{C}^n$, the solution set of H(Z,t) = 0 forms $d = d_1 \times \cdots \times d_{n-1} \times 2$ one-dimensional homotopy curves beginning with d distinct roots of H(Z,0) = 0 which are easily obtained and leading to all solutions of H(Z,1) = P(Z) = 0 with each curve reaching one zero of P(Z) or approaching the infinite (this occurs if the number of zeros of P(Z) is less than d including multiplicity) when t $\to$ 1.

Observe that if Z is a solution of (2.1), so is -Z. Thus only half of the curves are needed to be followed. We can pick the beginning points by choosing z_1 coordinate zero or one of (d_1 -1)-th roots of b_1, ... , z_s coordinate zero or one of (d_s -1)-th roots of b_s, z_{s+1} coordinate one of d_{s+1}-th roots of b_{s+1}, ... , z_{n-1} coordinate one of d_{n-1} -th roots of b_{n-1}, but choosing z_n coordinate only the positive square root of b_n (or only the negative square root of b_n). As soon as half of the solutions of P(Z)=0 are obtained, the another half are just negative of them.

The solutions of the numerical bifurcation should be real numbers, and the homotopy method gives the complex numbers, hence we only pick those solutions of the homotopy function with imaginary part zero theoretically and near zero numerically.

CHAPTER 6

THE SINGULAR VALUE DECOMPOSITION AND REGULAR POINTS

§ 6.1. A Matrix Result

Let f: $R^{m+1} \to R^m$, $f(x_0) = 0$, $f \in C^2$. Suppose x_0 is a regular point of the solution set. i.e., $Df(x_0)$ has full rank. The question is how to find some points near x_0 in the solution set, on a one-dimensional smooth manifold. Let ξ be the unit tangent vector along this one-dimensional smooth manifold. Using a limit procedure as in Chapter 4, we have:

$$f(x) = 0 \implies \xi^{\perp} = 0$$

$$i.e., \ \xi = \begin{bmatrix} v_{1,m+1} \\ \cdots \\ v_{m+1,m+1} \end{bmatrix}. \tag{1.1}$$

Let f(x) = 0, $Df(x) = U\Sigma V^T$ with rank m.

$$V = \begin{bmatrix} v_{1,1} & \cdots & v_{1,m+1} \\ \cdots & & \\ v_{m+1,1} & \cdots & v_{m+1,m+1} \end{bmatrix}, \qquad Df(x) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_{m+1}} \\ \cdots & & \cdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_{m+1}} \end{bmatrix}.$$

Let $|v_{k,m+1}| = \max_{1 \le i \le m+1} |v_{i,m+1}| \neq 0$. Denote $\tilde{Df}(x)$ as the matrix from Df(x) by deleting one column :

$$\tilde{Df}(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{k-1}} & \frac{\partial f_1}{\partial x_{k+1}} & \cdots & \frac{\partial f_1}{\partial x_{m+1}} \\ \cdots & & \cdots & \cdots & & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_{k-1}} & \frac{\partial f_m}{\partial x_{k+1}} & \cdots & \frac{\partial f_m}{\partial x_{m+1}} \end{bmatrix} \tag{1.2}$$

and we have:

Theorem 1.1. Among all m × m submatrices of Df(x) , $\tilde{Df}$ (x) is the only submatrix which is always nonsingular.

Proof: First we prove that it is nonsingular. From f(x) = 0, we have by (1.1)

$$O = Df(x) \begin{bmatrix} v_{1,m+1} \\ \cdots \\ v_{m+1,m+1} \end{bmatrix} = \tilde{Df}(x) \begin{bmatrix} v_{1,m+1} \\ \cdots \\ v_{k-1,m+1} \\ v_{k+1,m+1} \\ \cdots \\ v_{m+1,m+1} \end{bmatrix} + V_{k,m+1} \begin{bmatrix} \frac{\partial f_1}{\partial x_k} \\ \cdots \\ \frac{\partial f_m}{\partial x_k} \end{bmatrix}$$

$$\tilde{Df}(x) \begin{bmatrix} v_{1,m+1} \\ \cdots \\ v_{k-1,m+1} \\ v_{k+1,m+1} \\ \cdots \\ v_{m+1,m+1} \end{bmatrix} = \begin{bmatrix} -v_{k,m+1} \frac{\partial f_1}{\partial x_k} \\ \cdots \\ -v_{k,m+1} \frac{\partial f_m}{\partial x_k} \end{bmatrix} \tag{1.3}$$

$v_{k,m+1} \neq 0$ means as soon as $v_{k,m+1}$ is known, the rest of the components of $(v_{1,m+1}, \ldots, v_{m+1,m+1})^T$ are known since the null space is one-dimensional. That is equivalent to say (1.3) is uniquely solved, i.e., $\tilde{Df}$ (x) is nonsingular.

To prove that it is the only m × m submatrix which is always nonsingular, we pick the case when $\xi = \begin{bmatrix} v_{1,m+1} \\ \cdots \\ v_{m+1,m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$ --- k-th . Therefore

$$Df(x) = U \, \Sigma \, V^T$$

$$= U \begin{bmatrix} \sigma_1 & & & 0 \\ & \cdots & & \cdots \\ & & \sigma_m & 0 \end{bmatrix} \begin{bmatrix} v_{1,1} & \cdots & v_{1,m} & 0 \\ \cdots & & \cdots & \cdots \\ v_{k-1,1} & & v_{k-1,m} & 0 \\ 0 & \cdots & 0 & 1 \\ v_{k+1,1} & & v_{k+1,m} & 0 \\ \cdots & & \cdots & \cdots \\ v_{m+1,1} & \cdots & v_{m+1,m} & 0 \end{bmatrix}^T$$

$$= U \begin{bmatrix} \sigma_1 & & & 0 \\ & \cdots & & \cdots \\ & & \sigma_m & 0 \end{bmatrix} \begin{bmatrix} v_{1,1} & \cdots & v_{k-1,1} & 0 & v_{k+1,1} & \cdots & v_{m+1,1} \\ \cdots & & \cdots & \cdots & \cdots & & \cdots \\ v_{1,m} & \cdots & v_{k-1,m} & 0 & v_{k+1,m} & \cdots & v_{m+1,m} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

Thus any m × m submatrix deleting the i-th column of Df(x) with i ≠ k must include the column

$$U \begin{bmatrix} \sigma_1 & & & 0 \\ & \cdots & & \cdots \\ & & \sigma_m & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}$$

which is the zero column. Thus the theorem is proved. □

By this theorem, among all m+1 coordinate directions, this is the best direction for keeping the derivative invertible.

§ 6.2. A Newton's Method via SVD

We may use the following method to find some point near x_0 on the solution set of f.

Method 2.1. Compute SVD of f at x_0. If k is the coordinate index such that

$|v_{k,m+1}| = \max\limits_{1 \le i \le m+1} |v_{i,m+1}|$, then

set predictor:

$$x^{(0)} = x_0 = h \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ ---k-th \\ \\ \\ \\ \end{matrix} \qquad (2.1)$$

for some small real number h>0 .

Obtain $x^* = \lim\limits_{\nu \to \infty} x^{(\nu)}$ by keeping the k-th coordinate fixed $x_k^{(\nu)} = x_k^{(0)}$, and changing the other coordinates from

$$\tilde{D}f(x^{(\nu)}) \begin{bmatrix} x_1^{(\nu+1)} - x_1^{(\nu)} \\ \cdots \\ x_{k-1}^{(\nu+1)} - x_{k-1}^{(\nu)} \\ x_{k+1}^{(\nu+1)} - x_{k+1}^{(\nu)} \\ \cdots \\ x_{m+1}^{(\nu+1)} - x_{m+1}^{(\nu)} \end{bmatrix} = -f(x^{(\nu)}) \qquad (2.2)$$

for $\nu = 0, 1, 2, \ldots$.

If $f(x_0) = 0$, in the neighborhood of x_0, f(x) is small. And $\tilde{D}f(x_0)$ is non-singular. $f \in C^2$ implies $\tilde{D}f$ is Lipschitz in that subspace. Therefore the following theorem can guarantee the above convergence quadratically. After a few iterates, x^* can be approximated by high accuracy.

Theorem 2.2 (Newton-Kantorovich [9]). Given $g : \Omega \subset R^m \to R^m$ and the convex set $\Omega_1 \subset \Omega$, let g be continuously differentiable on Ω_1 and satisfy the conditions:

(a) $\| Dg(y) - Dg(z) \| \leq \gamma \| y - z \|$ for all y, z in Ω_1;

(b) $\| Dg(y^{(0)})^{-1} g(y^{(0)}) \| \leq \alpha$;

(c) $\| Dg(y^{(0)})^{-1} \| \leq \beta$, for some $y^{(0)} \in \Omega_1$.

Consider the quantities

$$h := \alpha\beta\gamma$$

$$\gamma_1 = \frac{1 - \sqrt{1-2h}}{h}\alpha.$$

$$\gamma_2 = \frac{1 + \sqrt{1-2h}}{h}\alpha.$$

If $h \leq 1/2$ and the closed ball $\overline{S_{\gamma_1}(y^{(0)})} \subset \Omega_1$, then the sequence $\{ y^{(\nu)} \}$ defined by $y^{(\nu+1)} = y^{(\nu)} - Dg(y^{(\nu)})^{-1} g(y^{(\nu)})$ for $\nu = 0, 1, \ldots$ remains in $S_{\gamma_1}(y^{(0)})$ and converges to the unique zero of $g(y)$ in $\Omega_1 \cap S_{\gamma_2}(y^{(0)})$.

CHAPTER 7

A NUMERICAL METHOD OF GLOBAL BIFURCATIONS

§ 7.1. Rank Detecting, Curve Following and Curve Switching Via SVD

Let f: $R^m \times R \rightarrow R^m$, and suppose f is smooth enough so that the Taylor Theorem in Chapter 4 holds. Df(x)=U Σ V^t , where

$$\Sigma = \begin{bmatrix} D_r & O \\ O & O \end{bmatrix} = \begin{bmatrix} \sigma_1 & & & & & 0 \\ & \cdots & & & & \\ & & \sigma_r & & & \cdots \\ & & & \cdots & & \\ & & & & \sigma_m & 0 \end{bmatrix}$$

by setting $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_m = 0$.

We also assume that there are only finite smooth curves passing each bifurcation point of the solution set of f(x)=0. In this chapter, we assume that the solution set is connected, otherwise we just consider a component of the solution set.

We present in this section several numerical methods by using the singular value decomposition which are going to be used in the next section for an algorithm.

Theoretically $\sigma_m = 0$ is a criterion for a bifurcation point. Numerically we can choose a very small number ε which is machine dependent, hence

$$\sigma_m < \varepsilon \tag{1.1}$$

is a criterion for the bifurcation point. Since SVD is a reliable way for detecting rank deficiency and near deficiency, the above method should be accurate to decide the bifurcation point.

If a point x is detected as a regular point, according to the tangent vector which is obtained by SVD, i.e., the m+1-th right singular vector

$(v_{1,m+1}, v_{2,m+1}, \cdots, v_{m+1,m+1})^T$, we can find x_k the largest coordinate changing the solution. Consequently Newton's method, i.e., Method 6.2.2., can be used. Namely choose a suitable positive number h which is dependent on σ_m monotonically, denoted it by :

$$h = g(\sigma_m) \tag{1.2}$$

such that

$$x^{(0)} = x + h \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \text{---} kth; \tag{1.3}$$

and

$$\tilde{Df}(x^{\nu}) \begin{bmatrix} x_1^{(\nu+1)} - x_1^{(\nu)} \\ \cdots \\ x_{k-1}^{(\nu+1)} - x_{k-1}^{(\nu)} \\ x_{k+1}^{(\nu+1)} - x_{k+1}^{(\nu)} \\ \cdots \\ x_{m+1}^{(\nu+1)} - x_{m+1}^{(\nu)} \end{bmatrix} = -f(x^{(\nu)}) \tag{1.4}$$

for $\nu = 0,1,2, \ldots$, where

$$\tilde{Df}(x^{v}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{k-1}} & \frac{\partial f_1}{\partial x_{k+1}} & \cdots & \frac{\partial f_1}{\partial x_{m+1}} \\ \cdots & & \cdots & \cdots & & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_{k-1}} & \frac{\partial f_m}{\partial x_{k+1}} & \cdots & \frac{\partial f_m}{\partial x_{m+1}} \end{bmatrix}.$$

After a few iterates, we get a point $x^{(v_0)}$ for some v_0 which is a numerical solution of $f(x) = 0$ different from x. Note that (1.2) also can be used to avoid missing bifurcation points since by our methods the step size h varies and the h is smaller when x is close to the bifurcation point, especially h is smaller when x is near the bifurcation point.

Using Newton's method via SVD, we can follow the branch to the bifurcation point x. When h is very small, we can detect the bifurcation point by detecting the last singular values along the tangent direction. If a bifurcation point x is detected, using Theorem 4.1.1 and 5.2.6, all the unit tangential directions ξ can be determined. Deleting one direction which is opposite the direction while the bifurcation point is obtained, we have points on the other branchs near the bifurcation point numerically:

$$x := x + \delta\xi, \tag{1.5}$$

where δ is a small positive number.

Now connecting all methods above, we have a numerical method of global bifurcations which is described in the next section.

§ 7.2. A Numerical Method of Global Bifurcations via SVD

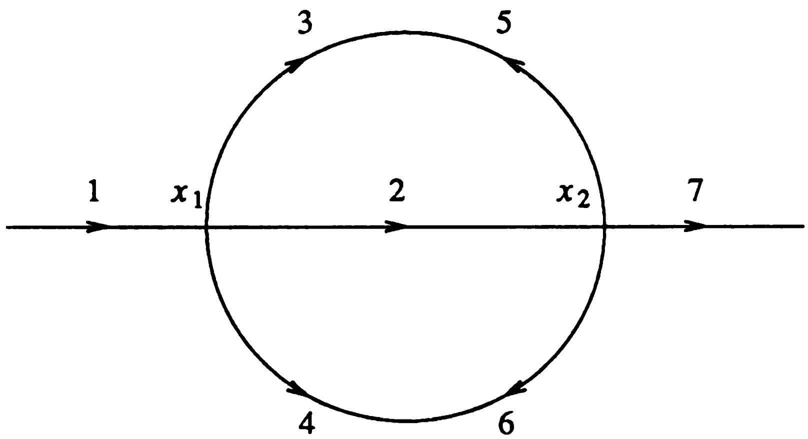
We consider an open bounded region. So if the solution set is bounded in this region, then the algorithm which we will describe below gives all branchs numerically.

If the solution set is unbounded, or the set is too large comparing the open region we set, then the algorithm gives the subset of the solution inside the region numerically. A suitable chosen region will give the major character of the solution set.

The method begins at one point of the solution set in the region. Usually if the problem has a trivial solution, we will take it with some value of the parameter as a starting point which is a regular point. Then trace the solution curve to get the global behavior.

Algorithm 2.1. Given f(x)=0 with f: $R^{m+1} \rightarrow R^m$ and the hypotheses in the beginning of this chapter, the following algorithm gives the solution set numerically.

Procedure Solution_Via_SVD
begin
define a bounded open region B;
initialize x=$(x_1, \ldots, x_{m+1})^T$ (* in B and fails to satisfy (1.1) $\sigma_m < \varepsilon$ *)
and two directions (* based on the last right singular vector of Df *);
L=2 (* L is the label of branchs to be followed *);
while L>0 **do**
L:=L-1;
while $\sigma_m \geq \varepsilon$ in (1.1) and x in B **do**
generate x_1 (* based on (1.2),(1.3),(1.4) *), set x:= x_1 ;
end while
if x in B **then**
case (to judge if the bifurcation point x has been met before) **begin**
x is a new bifurcation point:
find points with directions on N branchs to be followed
(* based on (1.5) *), set L:=L+N;
x is a previously found bifurcation point:
then the branch to this point x determines a branch with opposite
direction emanating from x to the starting point. Delete this
branch that emanates from x from the list of branches to be followed
when x is used as a starting point; Set L: = L-1; (* See Figure 1 *)
end case
end if
end while
end procedure ;



For example if path 3 starting at x_1 leads to x_2, then the path 5 leading from x_2 to x_1 is deleted when x_2 is used as a starting point

Figure 1. An illustration of the algorithm.

§ 7.3. An Example

We give an example to show the above algorithm. This example is obtained from the bifurcation problem in Banach space of codimension 4, reducing into the null space by the Liapunov-Schmidt method which has been theoretically done in [1] .

Let f: $R^2 \times R^4 \rightarrow R^2$ by $w \times (\mu,\nu,\alpha_1,\alpha_2) \rightarrow 0$ with $C(w) + \alpha_1 L_1 w + \alpha_2 L_2 w = 0$, where

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad C(w) = \begin{bmatrix} w_1^3 + \mu w_1 w_2^2 \\ \nu w_1^2 w_2 + w_2^3 \end{bmatrix}, \quad L_1 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } L_2 = -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By selecting 3 parameters, we get an equivalent one-parameter problem , and so the Algorithm 2.1 can be used to get the global bifurcation numerically. Figure 2 is the computer graph of this example. The value on the horizontal axis represents the value of the parameter α_1, and the value on the vertical axis represents the sum of the values of w_1 and w_2 . The following table gives five different choices of μ , ν, α_2 (for the reason of such a choice also see [1]). For all such choices beginning with(w_1 , w_2 , α_1) = (0.0, 0.0, -1.5), five different diagrams are obtained.

case	(a)	(b)	(c)	(d)	(e)
μ	2.0	0.75	0.5	0.5	1.5
ν	2.0	2.0	1.5	0.5	0.5
α_2	1.0	1.0	1.0	1.0	1.0

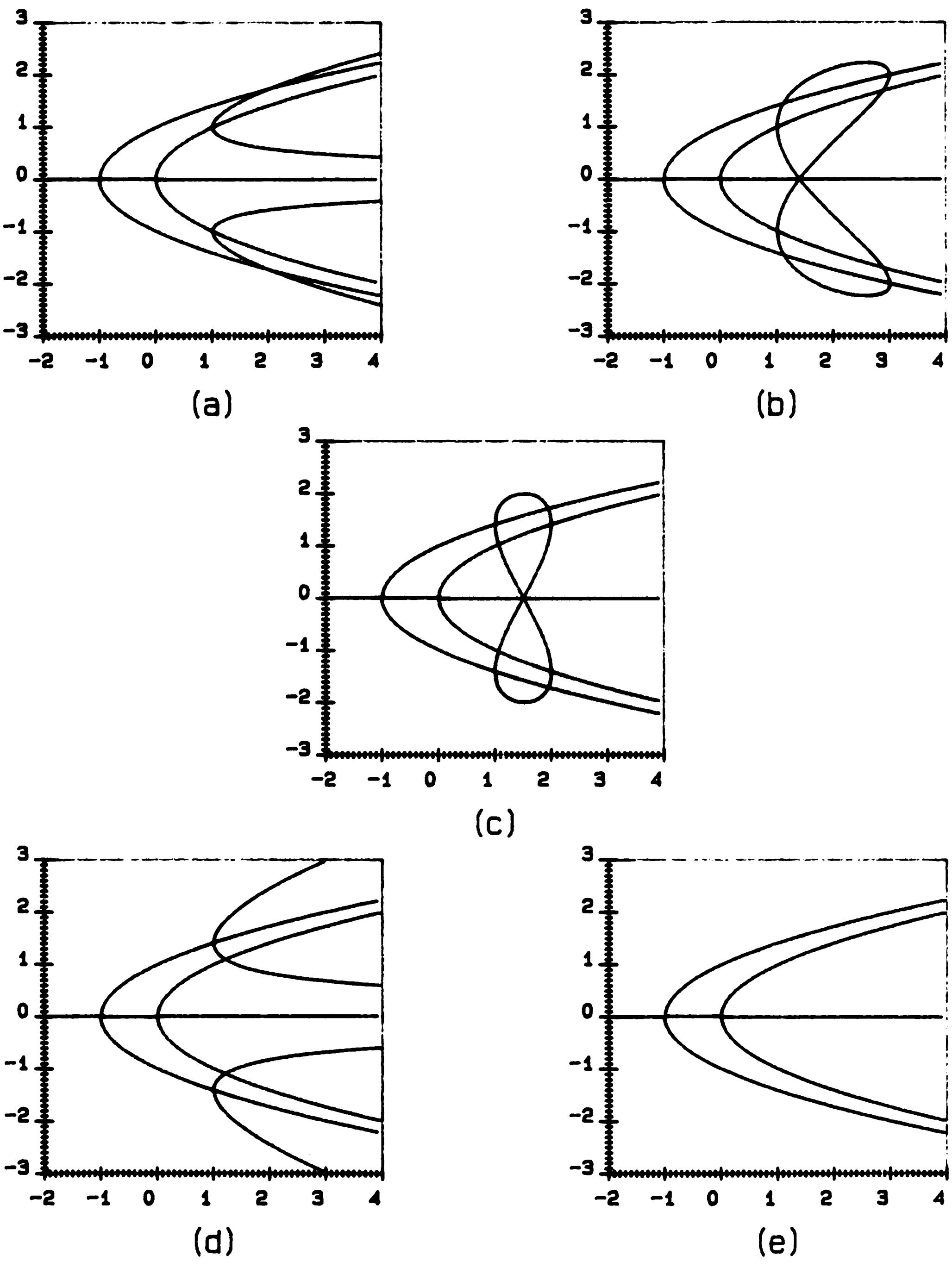


Figure 2. Five computer graphs for the example.

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