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ABSTRACT

MINMAX MODELING AND CONTROL APPROACH TO UNCERTAIN SYSTEMS

By

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This thesis is concerned with the determination of a controller for a linear time-invariant parameter uncertain system with corresponding performance measure. The uncertain system is described in state space and the system matrices, which contain constant but uncertain parameters, are given in companion form. The true parameter values are not known, but are assumed to lie within a given rectangular set.

A two step, minmax modeling and control procedure is proposed as a new and improved method of obtaining a controller for the system in the presence of parameter uncertainty. In the first step, an optimal model for the uncertain system is derived. The modeling problem is viewed as a two-person game of design against nature, and the game is played with the designer minimizing the maximum value of game cost. The model obtained from step one is optimal in the minmax sense and represents a guaranteed cost model for the uncertain system. The game cost is taken as the norm of the difference between system and model matrices.

The concepts of controllability and stability are applied to the uncertain system and optimal minmax model. An upper bound on the error between uncertain system and minmax model trajectories is formulated as a function of the game cost, for the case where the uncertain

system is asymptotically stable for all parameter values.

In the second step of the proposed procedure, a controller for the uncertain system is sought, based on the optimal minmax model state equation and a model performance index analogous in form to the given system performance measure. It is claimed that the minmax modeling and control procedure provides a solution to the problem of controlling the uncertain system, whenever a solution to the control problem in step two exists. The application of the minmax modeling and control approach is illustrated by example problems. Comparisons are made with alternate techniques from the literature.

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CHAPTER I

INTRODUCTION

In applications of optimal control theory, an engineer is often confronted with the problem of determining a controller for a system whose characteristics are not known exactly. It is assumed that the structure of the system is accurately represented by a vector differential equation which depends upon a finite number of parameters. It is the uncertainty in these parameters that defines the system uncertainty.

For complex physical systems, this uncertainty may be due to numerical limitations of the identification procedure, or the system parameters may vary slightly with environmental conditions. Alternately, it may be required to determine a fixed controller for an ensemble of systems which differ because of nonzero component tolerances.

The control problem is to determine the system inputs which optimize a given performance criterion, and satisfy all necessary system and control constraints. The performance criterion is generally chosen subjectively and represents the control cost. However, controller complexity imposes a cost which is difficult to represent in a performance index. Therefore, a useful control policy should also be simple to determine and implement, while maintaining satisfactory performance.

Several approaches have been used to determine the optimal controller for an uncertain system, and three are mentioned here. The approaches differ in their assumptions about the parameter uncertainty, and in their definition of an optimal control policy.

1.1 Stochastic Approach

The first approach is that of stochastic optimal control, where an *a priori* probability is assumed for the uncertain parameters. A controller is sought which minimizes the expected value of a given performance index [A-1]. If the statistical properties of the uncertain parameters are not known *a priori*, but can be estimated during the dynamic process, the concept of adaptive stochastic control is used ([C-1], [C-2], [S-4]). Adaptive control implies further identification of parameters as well as the ability to modify the control as the parameter estimates change [S-3].

1.2 Minmax Cost Approach

The minmax cost approach has received some attention in control literature. The true parameter values are not known, but are assumed to belong to a given compact set. When the performance criterion is written as a cost functional, the determination of the minmax cost controller is viewed as a two person game [B-1]. The first player is the designer whose objective is to determine the control law. His adversary, referred to as nature, chooses the system parameters. It is assumed that nature is perverse and actively seeks to maximize the cost which the designer is attempting to minimize. The design objective is to choose a control which minimizes the maximum value of the cost over all possible parameter variations. The optimal control derived from the minmax strategy yields the smallest guaranteed upper bound on the performance functional [B-1]. It is also referred to as the worst case design [W-1].

Dorato and Kestenbaum [D-3] consider several minmax cost control problems where a saddle point solution exists. Schmitendorf [S-2] develops sufficient conditions and a technique for determining a minmax cost controller. His method is applicable to problems in which a saddle point solution does not exist. Blum [B-1] states a minmax theorem which is applicable to the minmax feedback control problem. By introducing mixed strategies over the uncertainty set, a saddle point problem is created which is equivalent to the original problem. Barmish [B-2] determines a guaranteed performance controller for linear systems when the initial state is uncertain. Witenhausen [W-1] considers minmax cost control of sampled linear systems.

By introducing a Lyapunov-like function, Chang and Peng [C-3] have developed a method for determining a simple guaranteed cost controller. Their method, along with the work done by Sworder [S-4], is also applicable to the minmax adaptive control problem.

1.3 Minmax Sensitivity Approach

Dorato and Kestenbaum [D-3] have commented that the minmax cost controller is overly pessimistic and is too concerned with the worst that can happen. Furthermore, controls based on this design methodology induce conservative system performance. To meet these objections, the minmax sensitivity approach was developed.

In this approach, a controller is sought which minmaximizes not the performance index (or cost), but the sensitivity of that cost [R-1], [S-3]. Analogous to the minmax cost approach, the design procedure is posed in a game-theoretic setting. That is, the performance index is written as a cost functional and the true parameter values are unknown but assumed to belong to a given compact set. Contrary to many perturbation techniques (see for example [C-4], [C-5]), the minmax sensitivity design procedure is concerned with the large parameter deviation case [S-3].

Rohrer and Sobral [R-1] define a relative sensitivity functional which is dependent on both the unknown system parameters and the controller. The control function is chosen which minimizes the relative sensitivity when the system parameters are at their worst possible values. Implicit in their argument is that the controller obtained be exactly optimal at some point within the parameter uncertainty set.

Salmon [S-1] expands the concept of relative sensitivity and determines a minmax sensitivity controller based on this generalization. He assumes that the controller structure is known beforehand and thus, he has complete freedom in specifying the control parameters. This additional degree of freedom allows Salmon to design a controller which may be suboptimal at every point within the parameter uncertainty set. For the case where the uncertain system parameters are constants, he develops an algebraic minmax algorithm which aids in the search for the desired controller parameters.

Werner and Cruz [W-2] and Kokotovic, et.al. [K-3] have considered the optimally adaptive sensitivity control problem, but these techniques do not use a minmax criterion.

1.4 Comparison of the Basic Approaches

The stochastic approach requires the knowledge of an a priori probability distribution for the uncertain parameters. Such information is often not available to a designer. Computation of an optimal control is generally very complex, with the end result being that only the expected value of the performance index is minimized. Furthermore, the optimal controller is typically a random device, or a deterministic device selected at random [W-1].

Minmax cost control yields the smallest upper bound on the cost functional: there is some value in knowing what the worst case system performance is. But this approach is too pessimistic and results in conservative system performance even when the system parameters are perturbed from their worst case values. In general, game-theoretic saddle points fail to exist, thereby complicating the computation of a minmax cost control. Mixed strategies can create a saddle point condition, but only at the expense of introducing probabilities over the uncertainty set.

Controllers based on the minmax sensitivity approach yield good performance over most of the parameter uncertainty set. This methodology eliminates much of the pessimism found in minmax cost controller design. But this improvement is achieved by increasing the complexity of the required computations. Analytic methods are rare even in the simplest cases [R-1], and the designer is forced to employ graphical approximation procedures, or to develop iterative computer search techniques. These can be both time consuming and expensive. Numerical stability and convergence problems further complicate this approach.

Adaptive stochastic, guaranteed, and sensitivity controllers give good system performance. But the difficulty and expense of realizing these control policies prohibits their usefulness.

1.5 Minmax Modeling and Control (MMAC) Approach

The purpose of this dissertation is to present a different and improved approach to the determination of a controller for an uncertain system. A two step procedure, hereafter referred to as the minmax modeling and control (MMAC) approach, is proposed. The first step is to determine an optimal model for the uncertain system by using a minmax criterion. In the second step, the controller for the uncertain system is designed, based on the optimal model parameters and a pre-specified cost functional. It is important to note that the placement of the minmax criterion into the modeling phase followed in sequence by the determination of an optimal controller is significantly distinct from the aforementioned approaches, which apply the minmax criterion directly to controller determination.

Attention is restricted to systems described by single input, multi-output, linear differential equations of n^{th} order, with constant but uncertain coefficients. The initial conditions are known exactly. The true coefficient (or parameter) values are not known, but are assumed to belong to a rectangular set in Euclidean m-space, where m is the number of uncertain parameters. There are at most n + 1 unknown coefficients. No probability measure is assigned to the rectangle. Furthermore, no input-output data pairs are available from the uncertain system.

It is assumed that the bounded set description of the parameter uncertainties is the end result of a system identification phase. That is, further refinement of set estimates is neither cost effective nor desired at this time.

1.6 MMAC - Related Works

Perkins, et. al. [P-1] have considered a two step procedure in the feedback design of linear time-invariant parameter uncertain systems, described by Laplace transfer functions. Their first step is to specify a desired overall system transfer function. In the second step, a feedback structure is chosen which optimizes a scalar sensitivity index, for a specific system input. Perkins, et. al. comment that many meaningful problems can be attacked using a two step procedure. However, they do not consider the optimal control problem.

Bandler and Srimivasan [B-3] have developed computer algorithms which aid in determining minimax models for linear time-invariant systems. This is not minmax in the game-theoretic sense. Rather, their criterion is the determination of model parameters which minimize a Chebychev norm [D-5], [K-2] of the difference between the model trajectory and a known system trajectory, for a specific input. The Chebychev measure is taken over the independent variable, which is usually time.

Genesio and Pome [G-2] have presented a minmax modeling and control approach to the problem of controlling a plant which is described only by input-output data. The plant is supposed linear and time-invariant, and is represented by state equations with constant but unknown matrices. The performance measure is restricted to be quadratic in form, with no terminal error cost. A linear, stationary, reduced order state space model is sought which guarantees the minimum deviation between the system performance and the model performance. The model performance measure is also quadratic in form. The optimal model is determined by minmaximizing a suitable function of the performance deviation. The solution of the modeling problem involves the use of the plant input-output data, which is further assumed to be uncorrupted by noise. A numerical procedure is necessary to obtain the optimal model parameters [G-2]. Once the optimal model is obtained, a minmax control policy for the plant is found by solving an appropriate matrix Riccati equation.

The approach formulated by Genesio and Pome is similar in outline to the MMAC approach presented here. However. two comments are in order. First, Genesio and Pome's method assumes that the performance index is quadratic in form, which is not the case with the MMAC approach. Second, and most important, the input-output data pairs play an integral role in obtaining a solution to the modeling problem. This is not true in the MMAC approach - data pairs are neither necessary nor required. Thus, the two approaches share only a common name and purpose.

1.7 MMAC - Step One

The MMAC procedure presented in this thesis is an analytic approach that differs considerably from previous works. As a starting point, the uncertain system is given a state space representation, where the sytem matrix is written as a companion matrix. For single input, n^{th} order linear differential equations, this is always possible [B-4], [D-4]. The system matrix and corresponding input matrix become functions of the uncertain parameters.

The determination of the optimal model in step one is viewed as a two person game. The first player is the designer who must choose a stationary model matrix (expressed in companion form) and a corresponding input matrix. His opponent, referred to as nature, chooses the system matrix and input matrix. The norm of the difference between system and model matrices is taken as the cost functional. The value of the cost index is viewed as the designer's loss and nature's gain, resulting from a choice of a candidate model. Here, nature takes the role of an intelligent adversary, who attempts to maximize the cost which the designer is attempting to minimize. The design strategy is to choose the model which minimizes the maximum value of the cost for all possible system and input matrices.

The matrix norm induces a scalar algebraic cost equation of the uncertain system and model parameters. Therefore, the selection of the optimal model and corresponding worst case system matrices is accomplished by solving a parameter optimization problem, using a minmax criterion.

For linear time-invariant systems as described above, completion of step one results in the determination of a unique model, optimal in the minmax sense. The model parameters are unique, completely known, and lie in the interior of the rectangular uncertainty set. Thus, step one of the MMAC method removes the uncertainty from the problem by specifying a fixed and known model, from which a controller can be designed (in step two).

The minmax criterion by which the optimal model is chosen yields the smallest guaranteed upper bound on the cost. It is therefore appropriate to describe the minmax model as a guaranteed cost model. Here the cost, taken as a matrix norm of differences, represents the mismatch between the system and model matrices.

An upper bound on the norm of the error between system and model trajectories can be determined as a product of a constant times the cost functional specified in step one of the MMAC approach. All states are considered observable. Thus, as the modeling error is made arbitrarily small, the system and model trajectories are forced to coincide.

1.8 MMAC - Step Two

In step two of the MMAC approach, the controller for the uncertain system is designed, based on the minmax model and a given performance index. Since the minmax model is completely known, step two can be stated as a deterministic optimal control problem. This is easier to solve than the corresponding minmax cost control or sensitivity problems.

The control law derived from the MMAC approach no longer achieves the best guaranteed performance. This is the main disadvantage of the approach. However, the MMAC controller exhibits more satisfactory system performance than a guaranteed cost controller when the system parameters are perturbed from their worst case values. Also, the minmax modeling and control policy is designed optimally for parameter values which lie in the interior of the uncertainty set.

1.9 Outline of the Dissertation

The outline of the dissertation is as follows. Chapter II provides pertinent definitions and results from the mathematical theory of games. Only those elements of game theory which are necessary in understanding the problem of modeling and controlling an uncertain system are presented.

In Chapter III, the problem of controlling an uncertain system is formulated. The two-step, MMAC procedure is proposed as a solution. Formal statement and proof of the optimal modeling problem (step one) is given. Several examples which demonstrate step one of the MMAC approach are presented. Comments regarding controllability and stability are made. An upper bound on the error between uncertain system and optimal model trajectories is derived.

In Chapter IV, the application of the second step of the MMAC approach is illustrated by continuing the examples presented in the previous chapter. Comparisons are made with several techniques from the literature.

Chapter V contains a summary of the results in this thesis and recommendations for further research.

CHAPTER II

MATHEMATICAL BACKGROUND - GAME THEORY

The mathematical theory of games is concerned with optimization problems involving two or more players with conflicting interests. A basic feature of game theory is that the final outcome depends primarily on the combination of strategies selected by the adversaries. Therefore, particular emphasis is placed on the decision-making processes of the players.

The discussion in this chapter is limited to the topic of two-person, zero-sum continuous games. For a more general treatment of the theory of games, see Karlin [K-1], McKinsey [M-1], or Owen [O-1].

2.1 Two-person, Zero-sum Continuous Games

As the name implies, a two-person game involves only two adversaries, a player I and a player II. Zero-sum indicates that one player wins whatever the other player loses, so that the sum of the net winnings is zero. In a continuous game, both players have a continuum of possible strategies from which to choose. For each pair of strategies, there is a corresponding payoff or cost. The payoff function represents player I's loss and player II's gain. Therefore, player I attempts to minimize the payoff, while II attempts to maximize it. This is a perfect information game in the sense that each player knows the strategies available to himself, the ones available to his opponent, and the corresponding cost.

The development of rational criteria for selecting a strategy is a primary objective of game theory. This is accomplished by assuming that both players are rational, and that each will actively attempt to do as well as possible, relative to the opposition. This is in contrast to statistical decision theory (see for example, [H-1] Chapter 4), where it is assumed that a decision-maker is playing a game with a passive opponent who chooses his strategies in a random fashion.

In general, a game is characterized by:

i) the strategies for player I

ii) the strategies for player II

iii) the payoff or cost function.

The following definitions clarify these concepts.

2.2 Games with Pure Strategies

Definition 2.2.1: A pure strategy for player I is any element $u \in U$, where U is compact and represents the set of all choices for I. Similarly, a pure strategy for player II is any element $v \in V$, where V is compact and represents the set of all choices for II.

Definition 2.2.2: The payoff (or cost) is a real-valued, continuous function $L(u,v)$, defined on the Cartesian product space $U \times V$.

Remark 2.2.3: To facilitate the presentation, u and v will hereafter be regarded as scalar variables, $L(u,v)$ a continuous function of two variables, and U and V closed real line intervals.

Example 2.2.4: A typical cost function might be

$$L(u,v) = 2u^2 - v^2$$

where

$$U = [0,1], \; V = [2,3].$$

A pure strategy for player I would be any u such that $0 \leq u \leq 1$. Player II's pure strategies would consist of any v such that $2 \leq v \leq 3$.

As previously mentioned, Player I attempts to choose a pure strategy such that the cost is minimized, while II attempts to choose a pure strategy that maximizes the cost. The following theorem shows that the order in which maximization and minimization are performed is important. Thus, it makes a difference which opponent plays first.

<u>Theorem 2.2.5</u> ([M-1]): Let L(u,v) denote a real-valued, continuous function defined whenever $u \in U$, $v \in V$, where U and V are compact sets.

Suppose that

$$\max_{v \in V} \min_{u \in U} L(u,v)$$

and

$$\min_{u \in U} \max_{v \in V} L(u,v) \qquad \text{both exist.}$$

Then

$$\max_{v \in V} \min_{u \in U} L(u,v) \leq \min_{u \in U} \max_{v \in V} L(u,v).$$

Proof of Theorem 2.2.5 follows immediately from the definition of minima and maxima for a continuous function. Details are given in McKinsey [M-1] and Karlin [K-1].

Next, the concept of value is defined.

<u>Definition 2.2.6</u> ([K-1]): A real number ℓ_1 is called the <u>upper value</u> of the cost iff

$$\min_{u\varepsilon U} \max_{v\varepsilon V} L(u,v) = \ell_1.$$

Similarly, a real number ℓ_2 is called the <u>lower value</u> of the cost iff

$$\max_{v\varepsilon V} \min_{u\varepsilon U} L(u,v) = \ell_2.$$

For the case where $L(u,v)$ is a real-valued, continuous function defined for every $u \varepsilon U$, $v \varepsilon V$ where U and V are compact sets, ℓ_1 and ℓ_2 always exist [O-1]. Furthermore, Theorem 2.2.5 implies that $\ell_2 \leq \ell_1$.

<u>Example 2.2.7</u>: Let

$$L(u,v) = (u-v)^2; \; 0 \leq u \leq 1, \; 0 \leq v \leq 1.$$

It is easy to see that

$$\min_{u\varepsilon U} \max_{v\varepsilon V} (u-v)^2 = 1/4 = \ell_1$$

(u plays first)

and

$$\max_{v\varepsilon V} \min_{u\varepsilon U} (u-v)^2 = 0 = \ell_2.$$

(v plays first)

Note that

$$\ell_2 = 0 \leq 1/4 = \ell_1.$$

It is possible to state a necessary and sufficient condition for the equality of the upper and lower values, ℓ_1 and ℓ_2. First, the concept of a game-theoretic saddle point needs to be defined.

<u>Definition 2.2.8</u> ([M-1]): Suppose that $L(u,v)$ is a real-valued, continuous function defined whenever $u \varepsilon U$, $v \varepsilon V$, where U and V are compact sets; then a point (u_o,v_o), $u_o \varepsilon U$, $v_o \varepsilon V$ is called a <u>game-theoretic saddle point</u> of $L(u,v)$ if the following conditions are

satisfied:

i) $L(u_o,v) \leq L(u_o,v_o) \qquad \forall\, v \in V$

ii) $L(u_o,v_o) \leq L(u,v_o) \qquad \forall\, u \in U.$

Example 2.2.9: Consider

$$L(u,v) = u^2 - v^2; -1 \leq u \leq 1, \quad -1 \leq v \leq 1.$$

The function $L(u,v)$ has a game-theoretic saddle point at (0,0) since for all $u \in [-1,1]$ and $v \in [-1,1]$,

$$0^2 - v^2 \leq 0^2 - 0^2 \leq u^2 - 0^2.$$

Remark 2.2.10: Note that the definition for a game-theoretic saddle point is not equivalent to the usual conditions for a calculus saddle point, which are ([0-2], [K-4], or [B-5] Chapter 9):

i) $\frac{\partial L}{\partial u} = \frac{\partial L}{\partial v} = 0$

ii) $\left(\frac{\partial^2 L}{\partial u^2}\right)\left(\frac{\partial^2 L}{\partial v^2}\right) - \left(\frac{\partial^2 L}{\partial u \partial v}\right)^2 \leq 0.$

The next example illustrates this point.

Example 2.2.11: Let

$$L(u,v) = u^2 + v^2; \; 0 \leq u \leq 1, \; 0 \leq v \leq 1.$$

$L(u,v)$ has a game-theoretic saddle point at (0,1) since for all $u \in [0,1]$ and $v \in [0,1]$,

$$0^2 + v^2 \leq 0^2 + 1^2 \leq u^2 + 1^2.$$

However

$$\left(\frac{\partial^2 L}{\partial u^2}\right)\left(\frac{\partial^2 L}{\partial v^2}\right) - \left(\frac{\partial^2 L}{\partial u \partial v}\right)^2 = (2)\,(2) - 0^2 \not\leq 0,$$

for any $u \in [0,1]$, $v \in [0,1]$. Therefore no calculus saddle point exists for $L(u,v) = u^2 + v^2$.

Remark 2.2.12: From Definitions 2.2.6 and 2.2.8, it can immediately be verified that:

i) $L(u_o,v) \leq \ell_2 \qquad \forall\ v \in V$

ii) $\ell_1 \leq L(u,v_o) \qquad \forall\ u \in U.$

The following simple criterion is often useful in determining when ℓ_1 and ℓ_2 are equal. Moreover, it demonstrates the connection between the existence of a game-theoretic saddle point and the equality of the upper and lower values.

Theoreum 2.2.13: ([M-1]): Let $L(u,v)$ be a real-valued, continuous function defined whenever $u \in U$, $v \in V$, where U and V are compact sets. Suppose that

$$\max_{v\varepsilon V}\min_{u\varepsilon U} L(u,v) = \ell_2$$

and

$$\min_{u\varepsilon U}\max_{v\varepsilon V} L(u,v) = \ell_1 \qquad \text{both exist.}$$

Then a necessary and sufficient condition for $\ell_1 = \ell_2$ is that $L(u,v)$ posses a game-theoretic saddle point. If (u_o,v_o) is any game-theoretic saddle point of $L(u,v)$, then $L(u_o,v_o) = \ell_1 = \ell_2$.

Standard proofs are provided by McKinsey [M-1], Karlin [K-1], or Owen [O-1].

The existence of a game-theoretic saddle point provides a necessary and sufficient condition for the equality of the upper and lower values of the cost. Theorem 2.2.13 further implies that the order of minimization and maximization may be interchanged: it makes no difference which opponent plays first. This consequence becomes useful when it is easier to solve a maxmin problem than to obtain the solution directly from the corresponding minmax problem (or vice-versa). Unfortunately, game-theoretic saddle point solutions involving pure strategies seldom exist.

2.3 Games with Mixed Strategies

Consider a game in which

$$\min_{u \varepsilon U} \max_{v \varepsilon V} L(u,v) > \max_{v \varepsilon V} \min_{u \varepsilon U} L(u,v)$$

where u,v, U,V, and L(u,v) have been appropriately defined. By Theorem 2.2.13, the inequality of the upper and lower cost values implies that a game-theoretic saddle point solution involving pure strategies does not exist. Furthermore, the order of minimization and maximization cannot be interchanged. This poses a dilemna for game-theoreticians who seek solutions to problems in which the order of play does not lead to an advantage.

They resolve this difficulty by having each player assign a probability distribution over his set of pure strategies. This randomization defines a mixed strategy. The problem then becomes one of determining a mixed strategy pair which optimizes the expected value of cost, where the expectation is taken over the probability mixes for both players. It is a consequence of the minmax principle of Von Neuman and Morgenstern [N-1] that the difference between minmax and maxmin can be equalized on an expected value basis.

Unfortunately, the solution to the revised problem yields a mixed strategy pair (i.e., a pair of optimal probability distributions). The solution to the original problem (of optimizing a deterministic payoff function) yields a pure strategy pair. These answers are not equivalent.

The following remarks formalize these ideas.

Definition 2.3.1 ([0-1]): A mixed strategy for player I is a probability distribution F(u) defined over the set U of all pure strategies. Similarly, a mixed strategy for player II is a probability distribution G(v) defined over the set V of all pure strategies.

Definition 2.3.2 ([0-1], [M-1]): Let the payoff function L(u,v) be given. Then for each pair of mixed strategies (F(u), G(v)), the expected payoff E(F,G) is defined as the Stieltjes integral:

$$E(F,G) \triangleq \int_V \int_U L(u,v)\, dF(u)\, dG(v).$$

The revised game is viewed as follows: player I attempts to choose a probability distribution F(u) which minimizes the expected payoff. Player II tries to choose a probability distribution G(v) which maximizes the expected cost. To see that the order of play is no longer important in this revised game, consider the following theorem:

Theorem 2.3.3 ([M-1]): If L(u,v) is a real-valued, continuous funtion defined on the Cartesian product space U x V with U and V compact, then the quantities

$$\min_{F \varepsilon D_u} \max_{G \varepsilon D_v} E(F,G)$$

and

$$\max_{G \varepsilon D_v} \min_{F \varepsilon D_u} E(F,G)$$

exist and are equal.

D_u and D_v are given as the sets of all possible probability distributions for U and V respectively.

Proof of this theorem is provided in McKinsey [M-1].

Theorem 2.3.3 states that the introduction of mixed strategies creates a game-theoretic saddle point solution to the problem of optimizing the expected payoff. Therefore, the interchange of the order of minimization and maximization is permitted, since it makes no difference which opponent plays first. Note however, that the optimization is performed over the mixed strategies F and G. Thus, the solution to this modified problem is a pair of probability distributions (F^*,G^*) which optimize the expected payoff $E(F,G)$. This revised problem is not the same as the original problem of determining a pure strategy pair (u^*,v^*) which optimizes a given payoff function $L(u,v)$.

2.4 Solution of Minmax Games

Consider a minmax game in which a pure strategy solution is sought. That is, given an appropriate payoff function $L(u,v)$, it is desired to find a pure strategy pair (u^*,v^*) such that

$$\min_{u\varepsilon U} \max_{v\varepsilon V} L(u,v) = L(u^*,v^*),$$

where U and V are given compact sets.

Unfortunately, there is no general approach to obtaining a pure strategy solution to a game-theoretic minmax problem#. However,

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\# The same statement is true concerning the solution of a game-theoretic maxmin problem.

there are several basic methods which may lead to a solution. These are [S-1]:

1) To locate a game-theoretic saddle point (if it exists), and show that it represents a global solution,

2) To analytically solve the maximization step for a fixed $u \in U$, and then to minimize,

3) To develop an iterative procedure and search for a solution,

4) To introduce mixed strategies over V and create a game-theoretic saddle point solution.

In method (1), the existence of a game-theoretic saddle point involving pure strategies implies that the order in which maximization and minimization are performed is not important. It is therefore possible to apply the necessary and sufficient conditions for locating extremum, given in the calculus, to optimize the cost. The major difficulty with this method is that game-theoretic saddle points involving pure strategies seldom exist.

Method (2) ignores the existence of a game-theoretic saddle point. For a fixed $u \in U$, an analytic solution to the maximization step is sought such that

$$\phi(u) = \max_{v \in V} L(u,v).$$

$\phi(u)$ is then substituted into the payoff function. The next step involves minimizing $L(u,\phi(u))$ with respect to u, and the desired solution pair is given by

$$(u^*,v^*) = (u^*,\phi(u^*)).$$

Danskin [D-1] has shown that the principal difficulty with this method is that $\phi(u)$ is, in general, a non-differentiable function even when $L(u,v)$ is quite smooth. Nevertheless, method (2) provides an analytic approach to solving minmax problems in the absence of game-theoretic saddle points.

Computer programs typically facilitate the implementation of method (3). Salmon [S-1] and Demjanov [D-2] have developed search techniques which allow for the non-differentiability that generally arises in the absence of game-theoretic saddle points. A typical search algorithm decomposes the original minmax problem into a series of simpler optimizations. The simpler problems are not trivial, and the algorithm fails when the minimizations cannot be solved. However, method (3) is capable of solving many practical minmax problems.

Method (4) presents a different approach to obtaining at least a partial solution to a minmax problem. Two additional assumptions are implicit with this method. These are:

i) That it is easier to solve the corresponding maxmin problem (i.e., to minmize first),

ii) That interest is centered on obtaining <u>explicitly</u> the optimal pure strategy u^*, but not necessarily v^*.

Often the form of the cost function $L(u,v)$ is such that there exists an advantage to minimizing first. This is particularly true in minmax cost control problems (cf. [B-1]). Under such circumstances, the first assumption is valid.

The second assumption is realistic when considering most games of design against nature. Here, the designer is player I whose goal is to minimize the payoff. His opponent, called nature, actively seeks to maximize the cost. The designer is interested in obtaining an explicit pure strategy $u^* \varepsilon U$ which minimizes the maximum value of cost over all of his opponent's choices. The strategy $u^* \varepsilon U$ and the optimal value of cost $L(u^*,v^*)$ are desired, but not necessarily the corresponding strategy $v^* \varepsilon V$.

The following steps outline the theoretical basis of method (4) as presented by Blum [B-1]. This procedure is typical of the method (cf., Schmitendorf [S-2]).

a) An explicit $u^* \varepsilon U$ is sought which optimizes

$$\min_{u \varepsilon U} \max_{v \varepsilon V} L(u,v).$$

This is a standard minmax problem, except that only the optimal $u^* \varepsilon U$ is desired explicitly.

b) Mixed strategies $G(v)$ are introduced over player II's set of pure strategies V.

c) A lemma is proven which shows that there exists a pure strategy $u^* \varepsilon U$ such that

$$\min_{u \varepsilon U} \max_{v \varepsilon V} L(u,v) = \min_{u \varepsilon U} \max_{G \varepsilon D_V} E(u,G).$$

Thus, the u^* that optimizes the payoff $L(u,v)$ also optimizes the expected payoff $E(u,G)$#. D_V is the set of all possible probability distributions for V.

d) Finally, the proof of a minmax theorem permits the interchange of the order of maximization and minimization in the modified problem. That is, there exists a pure strategy $u^* \varepsilon U$ such that

By a slight abuse of notation, the cost criterion with mixed strategies over V is denoted by $E(u,G) = \int_V L(u,v)\, dG(v)$.

$$\min_{u \varepsilon U} \max_{v \varepsilon V} L(u,v) = \max_{G \varepsilon D_V} \min_{u \varepsilon U} E(u,G).$$

The solution to the modified problem is given by the pure strategy $u^* \varepsilon U$ and the optimal probability distribution $G^*(v)$. Hence, v^* is not stated explicity. Frequently, the solution to the maximization step is obtained over a finite subset of V. In this case, v^* (which is often not unique) may be readily identified.

Schmitendorf [S-2], Blum [B-1] and others have developed techniques which implement the theory of method (4). Their applications are in the area of optimal control theory.

This completes the presentation of background material in the mathematical theory of games. However, these concepts will be used in the chapters that follow, when the problem of modeling and controlling an uncertain system is considered.

CHAPTER III

MMAC APPROACH - THE OPTIMAL MODELING PART OF THE PROBLEM

In this chapter, the problem of controlling a linear time-invariant parameter uncertain system is formulated. The two step MMAC procedure is proposed as a solution. A precise statement and proof of step one of the MMAC approach, the optimal modeling problem, is given. Several examples which demonstrate step one of the MMAC approach are presented. Comments regarding controllability and stability are made. Chapter III concludes with a derivation of an upper bound on the error between uncertain system and optimal model trajectories.

3.1 Preliminary Remarks

Consider the class S of linear time-invariant parameter uncertain systems (to be controlled)

$$S: \quad \dot{x}(t) = A\,x(t) + B\,u(t), \quad x(t_o) = x_o, \quad t \in [t_o, t_f] \tag{3.1.1}$$

where $x(t) \in R^n$ is the system state,

$u(t) \in R^1$ is the control,

A is the n x n system matrix written in companion form ([B-4], [D-4]) and parameterized by the first n-entries of the uncertainty vector q,

$$A = A(q) = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & & & & & & & . \\ . & & & & & & & . \\ . & & & & & & & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 \\ q_1 & q_2 & q_3 & q_4 & . & . & . & q_n \end{bmatrix} ,$$

B is the $n \times 1$ input matrix parameterized by the $(n+1)$th entry of the uncertainty vector q,

$$B = B(q) = [0 \quad 0 \quad 0 \quad 0 \quad . \quad . \quad . \quad 0 \quad q_{n+1} \;]^T,$$

$q \in Q \subset R^{n+1}$ is the $(n+1)$-vector of time-invariant uncertain parameters, and the set Q is a given rectangle in R^{n+1},

$x(t_o) = x_o$ are known initial conditions,

and the time interval $[t_o, t_f]$ is prescribed.

If q were known exactly then the usual statement of the optimal control problem could be given. Here, however, the actual value of q is not known but is assumed to lie in the compact rectangle Q in R^{n+1}.

A control $u(\cdot)$ will be called admissible if it is piecewise continuous and $u(t) \in U$ for every $t \in [t_o, t_f]$, where $U \subset R^1$ is a given set. The set of admissible controls will be denoted by M. Note that for every $u(\cdot) \in M$ and $q \in Q$, there always exists a solution of (3.1.1) on $[t_o, t_f]$, where $x(\cdot)$ is the solution or system trajectory which corresponds to the input $u(\cdot) \in M$ [A-2].

The control cost depends on the choice of input $u(\cdot)$ and the parameter vector q:

$$J(u(\cdot), q) = h(x(t_f)) + \int_{t_o}^{t_f} g(x(t), u(t))\, dt \qquad (3.1.2)$$

where $h(x(t_f)) \geq 0$ represents the terminal cost and the integral from t_o to t_f represents accumulated cost along the path.

Until q is known exactly, the minimization of $J(u(\cdot),q)$ with respect to $u(\cdot)$ cannot be carried out. Using the MMAC approach, the procedure for deriving a controller begins with a determination of an optimal model for S, with a corresponding fixed parameter vector $p^* \varepsilon Q$. The optimal modeling criterion is taken as minmax. The uncertainty is removed from the problem once the unknown parameters have been assigned the stationary values p^*. The remaining control problem is to determine the optimal control $u^*(\cdot)$ based on the minmax model, i.e., find an admissible $u^*(\cdot) \varepsilon M$ which minimizes

$$\hat{J}(u(\cdot)) = h(\hat{x}(t_f)) + \int_{t_o}^{t_f} g(\hat{x}(t), u(t))\, dt \tag{3.1.3}$$

where $\hat{x}(t) \varepsilon R^n$ is the model state which corresponds to the control input $u(\cdot)$. That is, $u(\cdot)$ drives the state equation

$$M: \quad \dot{\hat{x}}(t) = \hat{A}\,\hat{x}(t) + \hat{B}\,u(t), \quad \hat{x}(t_o) = \hat{x}_o = x_o \tag{3.1.4}$$

to generate a trajectory $\hat{x}(\cdot)$. Note that the functionals $h(\cdot)$ and $g(\cdot)$ are the same as in (3.1.2) except that $\hat{x}(t)$ replaces $x(t)$.

Existence of the optimal control $u^*(\cdot)$, of course, depends upon the functionals $h(\cdot)$ and $g(\cdot)$, the time interval $[t_o,t_f]$, the class of systems S, the initial conditions $x(t_o) = x_o$, the given sets U and Q, and the function space M. The question of existence of an optimal control is a very difficult one to answer [A-2], [K-5]; nevertheless, Lee and Markus [L-1] have proven existence theorems for several linear and nonlinear processes with various cost functionals. In the specific examples considered in this thesis, the existence of optimal solutions is guaranteed.

Remark 3.1.1: The problem statements which result from employing the minmax cost or minmax sensitivity procedures are fundamentally different but closely related to the MMAC approach taken here. In the minmax cost approach, the problem is to find an admissible control $u^*(\cdot)$ which satisfies

$$\max_{q\varepsilon Q} J(u^*(\cdot),q) \leq \max_{q\varepsilon Q} J(u(\cdot),q) \tag{3.1.5}$$

for all $u(\cdot)\ \varepsilon\ M$. The performance functional $J(u(\cdot),q)$ is given by equation (3.1.2). Thus, the worst case parameter vector $q^*\ \varepsilon\ Q$ and the corresponding minmax controller $u^*(\cdot)\ \varepsilon\ M$ are found by minmaximizing the cost functional $J(u(\cdot),q)$

$$\begin{aligned} J(u^*(\cdot),q^*) &= \min_{u(\cdot)\varepsilon M}\ \max_{q\varepsilon Q} J(u(\cdot),q) \qquad (3.1.6)\\ &= \min_{u(\cdot)\varepsilon M}\ \max_{q\varepsilon Q} \left[h(x(t_f)) + \int_{t_o}^{t_f} g(x(t),u(t))\, dt\right] \end{aligned}$$

with $x(t)\ \varepsilon\ R^n$ given as the solution of equation (3.1.1).

In the minmax sensitivity approach the problem is to find an admissible $u^*(\cdot)$ which satisfies

$$\max_{q\varepsilon Q} S(u^*(\cdot),q) \leq \max_{q\varepsilon Q} S(u(\cdot),q) \tag{3.1.7}$$

for every $u(\cdot)\ \varepsilon\ M$. The sensitivity functional $S(u(\cdot),q)$ is typically written as ([R-1], [S-3])

$$S(u(\cdot),q) = \frac{J(u(\cdot),q) - J(u^o(\cdot),q)}{J(u^o(\cdot),q)} \tag{3.1.8}$$

indicating a relative index, or

$$S(u(\cdot),q) = J(u(\cdot),q) - J(u^o(\cdot),q) \tag{3.1.9}$$

which is the expression for the absolute sensitivity. The functional

$J(u(\cdot),q)$ is specified in equation (3.1.2), and $u^o(\cdot)$ is the optimal control corresponding to a fixed choice of $q \varepsilon Q$ [R-1]

$$J(u^o(\cdot),q) = \min_{u(\cdot)\varepsilon M} J(u(\cdot),q). \tag{3.1.10}$$

The basic philosophy is to choose the control which makes $J(u(\cdot),q)$ stay as close as possible to the optimal value $J(u^o(\cdot),q)$ for all values of $q \varepsilon Q$. Analogous to the minmax cost approach, the parameter vector $q^* \varepsilon Q$ and the corresponding minmax sensitivity controller $u^*(\cdot) \varepsilon M$ are found by minmaximizing the sensitivity functional $S(u(\cdot),q)$, and

$$S(u^*(\cdot),q^*) = \min_{u(\cdot)\varepsilon M} \max_{q\varepsilon Q} S(u(\cdot),q), \tag{3.1.11}$$

where all quantities in the above expression have been previously defined.

In the MMAC approach, the problem of determining a controller for an uncertain system is essentially decomposed into two parts:

1) determine a minmax model for the uncertain system,
2) find an optimal controller based on the minmax model (equation (3.1.4)) and a specified performance index (equation (3.1.3)).

A precise development for the selection of the optimal model with stationary parameters $p^* \varepsilon Q$ is given in the next section.

3.2 Optimal Modeling Problem Formulation - MMAC Step One

Consider the class S of linear time-invariant parameter uncertain systems (3.1.1) written in companion form and parameterized by the uncertainty vector $q \varepsilon Q$. It is desired to find a model M (3.1.4) with stationary model and input matrices $\hat{A}$, $\hat{B}$ also written in companion form, which provides an optimal representation of S. The following comments are in order.

<u>Remark 3.2.1</u>: The model M takes the following form

$$M: \dot{\hat{x}}(t) = \hat{A}\,\hat{x}(t) + \hat{B}u(t), \quad \hat{x}(t_o) = \hat{x}_o = x_o \qquad (3.2.1)$$

where $\hat{x}(t) \in R^n$ is the model state,

$u(t) \in R^1$ is the control,

$\hat{A}$ is the $n \times n$ model matrix written in companion form and parameterized by the first n-entries of an uncertainty vector p,

$$\hat{A} = \hat{A}(p) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ p_1 & p_2 & p_3 & p_4 & \cdot & \cdot & \cdot & p_n \end{bmatrix},$$

$\hat{B}$ is the $n \times 1$ input matrix parameterized by the (n+1)th entry of the uncertainty vector p,

$$\hat{B} = \hat{B}(p) = [0 \;\; 0 \;\; 0 \;\; 0 \;\; . \;\; . \;\; . \;\; 0 \;\; p_{n+1}]^T,$$

and $p \in Q \subset R^{n+1}$ is the (n+1)-vector of time-invariant uncertain parameters (to be determined), and the set Q is the identical rectangle contained in R^{n+1} which is given in (3.1.1).

The next definition describes what is meant by a rectangular set in R^m.

<u>Definition 3.2.2</u>: A <u>rectangle in R^m</u> is a set W of the form

$$W = \{ (w_1, w_2, \ldots, w_m): a_i \leq w_i \leq b_i, \quad i = 1,2,\ldots,m\}.$$

All a_i and b_i are finite numbers.

<u>Remark 3.2.3</u>: Alternately, if W is a rectangle in R^m, then

$$W = I_1 \times I_2 \times \ldots \times I_m, \; I_i = [a_i, b_i], \quad i = 1,2,\ldots,m,$$

where x indicates the Cartesian product [P-2].

<u>Remark 3.2.4</u>: From Definition 3.2.2 and Remark 3.2.3, it is clear that the uncertainty vectors $q = [q_1 \, q_2 \ldots q_n \, q_{n+1}]^T$ and $p = [p_1 \, p_2 \ldots p_n \, p_{n+1}]^T$ can be described by

$$q \in Q \implies a_i \leq q_i \leq b_i, \; i = 1,2,\ldots,n+1 \tag{3.2.2}$$

$$p \in Q \implies a_i \leq p_i \leq b_i, \; i = 1,2,\ldots,n+1 \tag{3.2.3}$$

where Q is given by the Cartesian product

$$Q = I_1 \times I_2 \times \ldots \times I_n \times I_{n+1}, \; I_i = [a_i, b_i], \quad i = 1,2,\ldots,n+1. \tag{3.2.4}$$

The determination of the optimal model in step one of the MMAC approach is now viewed as a two person game with pure strategies $p \in Q$, $q \in Q$ [cf. Section 2.2]. The p-player is the designer who must choose a model matrix $\hat{A}$ and a corresponding input matrix $\hat{B}$. The q-player, also referred to as nature, chooses the system matrix A and the input matrix B. Heuristically, if the designer can select a model matrix $\hat{A}$ which is "close" to A and a matrix $\hat{B}$ which is "close" to B, then the model and system trajectories $\hat{x}(\cdot)$ and $x(\cdot)$ should also be "close" for every input $u(\cdot)$. Alternately, nature desires to choose her matrices in such a way that the correspondence between the model and system is as poor as possible.

The natural selection for the measure of game cost is the norm of the difference between system and model matrices. The value of the cost index is given as the designer's loss and nature's gain, resulting from a choice of a candidate model. Thus, the designer's strategy is to select the model which minimizes the maximum value of cost for all possible system and input matrices.

The matrix norm induces a scalar algebraic cost equation of the uncertain system and model parameters (q and p). Therefore, the determination of the optimal model and corresponding worst case system matrices is accomplished by solving a parameter optimization problem, using a minmax criterion.

The following definitions formalize these ideas.

Definition 3.2.5: A model M (3.2.1) for the system S (3.1.1) is optimal (in the minmax sense) when the n+1 model parameters p_i are given as the solution of the following criterion:

i) $$\min_{p \varepsilon Q} \max_{q \varepsilon Q} \|A - \hat{A}\| \tag{3.2.5}$$

ii) $$\min_{p \varepsilon Q} \max_{q \varepsilon Q} \|B - \hat{B}\| \,. \tag{3.2.6}$$

The choice of matrix norm $\|\cdot\|$ in equations (3.2.5) and (3.2.6) is taken to be the square root of the maximum eigenvalue of F^TF, where F is any arbitary $m \times n$ matrix [B-6], [D-4], [K-6]. That is,

$$\|F\| \triangleq \sqrt{\lambda_{max}} \tag{3.2.7}$$

where λ_{max} is the maximum eigenvalue of F^TF.

The optimal modeling problem can now be stated:

Optimal Modeling Problem: Given an uncertain system (3.1.1) with uncertainty vector $q \varepsilon Q$, determine an optimal minmax model M (3.2.1) with corresponding parameter vector $p^* \varepsilon Q$ such that

$$\max_{q \varepsilon Q} ||A(q) - \hat{A}(p^*)|| \leq \max_{q \varepsilon Q} ||A(q) - \hat{A}(p)|| \qquad (3.2.8)$$

and

$$\max_{q \varepsilon Q} ||B(q) - \hat{B}(p^*)|| \leq \max_{q \varepsilon Q} ||B(q) - \hat{B}(p)|| \qquad (3.2.9)$$

for all $p \varepsilon Q$, where the induced matrix norm $||\cdot||$ is defined by (3.2.7).

3.3 Solution of the Optimal Modeling Problem

In order to solve the optimal modeling problem, it is necessary to determine the eigenvalues of $(A-\hat{A})^T (A-\hat{A})$ and $(B-\hat{B})^T (B-\hat{B})$, where

$$A - \hat{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ (q_1-p_1) & (q_2-p_2) & \cdots & (q_n-p_n) \end{bmatrix}, \qquad (3.3.1)$$

and

$$B - \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ (q_{n+1}-p_{n+1}) \end{bmatrix}. \qquad (3.3.2)$$

The following lemma will aid in this development.

Lemma 3.3.1 ([B-4], p. 140): Let F be an arbitrary real $m \times n$ matrix, so that F^TF and FF^T are $n \times n$ and $m \times m$ matrices respectively. Then λ is a nonzero eigenvalue of F^TF iff it is a nonzero eigenvalue of FF^T.

Proof: Assume $\lambda \neq 0$ is an eigenvalue of F^TF. Then by definition ([D-4], p. 151), there exists some nonzero vector x (called the eigenvector of F^TF associated with the eigenvalue λ) such that

$$F^TF\ x = \lambda\ x. \tag{3.3.3}$$

Multiplying by F gives

$$FF^T(F\ x) = \lambda\ (F\ x) \tag{3.3.4}$$

or

$$FF^T z = \lambda\ z, \text{ where } z = F\ x. \tag{3.3.5}$$

Thus, λ and z are an eigenvalue and eigenvector pair of FF^T provided that $z \neq 0$. But since $\lambda x \neq 0$, $z = Fx \neq 0$; otherwise $F^TFx = 0$, which would violate the assumption. Therefore, $\lambda \neq 0$ and $x \neq 0$ are an eigenvalue and eigenvector for F^TF implies that λ and z are an eigenvalue and eigenvector pair for FF^T.

Now assume that

$$FF^Tz = \lambda\ z, \quad \lambda \neq 0, \quad z \neq 0. \tag{3.3.6}$$

Multiplying by F^T gives

$$F^TF\ (F^Tz) = \lambda\ (F^Tz) \tag{3.3.7}$$

or

$$F^TF\ y = \lambda\ y, \text{ where } y = F^Tz\ . \tag{3.3.8}$$

Since $\lambda z \neq 0$, then $y = F^Tz \neq 0$; otherwise $FF^Tz = 0$, which would violate the assumption. Therefore, $\lambda \neq 0$ and $z \neq 0$ are an eigenvalue and eigenvector for FF^T implies that λ and y are an eigenvalue and eigenvector pair for F^TF. This proves the lemma.

Q.E.D.

An immediate consequence of Lemma 3.3.1 is that $(A-\hat{A})\ (A-\hat{A})^T$ has the same nonzero eigenvalues as $(A-\hat{A})^T\ (A-\hat{A})$. Forming the product $(A-\hat{A})\ (A-\hat{A})^T$ yields

$$(A-\hat{A})(A-\hat{A})^T = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdot & \sum_{i=1}^{n} (q_i-p_i)^2 \end{bmatrix}, \qquad (3.3.9)$$

where the only nonzero coefficient is

$$[(A-\hat{A})\ (A-\hat{A})^T]_{(n,n)} = \sum_{i=1}^{n} (q_i-p_i)^2. \qquad (3.3.10)$$

Taking the product $(B-\hat{B})^T\ (B-\hat{B})$ yields the scalar quantity

$$(B-\hat{B})^T\ (B-\hat{B}) = (q_{n+1} - p_{n+1})^2. \qquad (3.3.11)$$

The eigenvalues of $(A-\hat{A})\ (A-\hat{A})^T$ are found by solving the characteristic equation [B-4]

$$c(\lambda) = \det \{\lambda I - (A-\hat{A})\ (A-\hat{A})^T\} = 0, \qquad (3.3.12)$$

where I is the identity matrix, and det $\{\cdot\}$ denotes the determinant.

From (3.3.9) and (3.3.12), it is easy to see that

$$c(\lambda) = \lambda^{n-1}(\lambda - \sum_{i=1}^{n} (q_i-p_i)^2) = 0, \qquad (3.3.13)$$

and therefore,

$$\lambda_j = 0 \qquad j = 1,2,\ldots,n-1 \tag{3.3.14}$$

$$\lambda_n = \sum_{i=1}^{n} (q_i - p_i)^2. \tag{3.3.15}$$

By Lemma 3.3.1, the maximum eigenvalue of $(A-\hat{A})^T (A-\hat{A})$ is given by (3.3.15),

or

$$\lambda_{max} = \sum_{i=1}^{n} (q_i - p_i)^2. \tag{3.3.16}$$

Finally, the norm of $(A-\hat{A})$ can be written as

$$\|A-\hat{A}\| \triangleq \sqrt{\lambda_{max}} = \sqrt{\sum_{i=1}^{n} (q_i - p_i)^2}\ . \tag{3.3.17}$$

Since $(B-\hat{B})^T(B-\hat{B})$ is a scalar quantity (3.3.11), a trivial calculation yields

$$\| B-\hat{B} \| = \sqrt{(q_{n+1} - p_{n+1})^2}. \tag{3.3.18}$$

Remark 3.3.2: The p* and q* which minmaximize $\|A-\hat{A}\|$ (3.2.5) and $\| B-\hat{B}\|$ (3.2.6) can be obtained by optimizing $\| A-\hat{A}\|^2$ and $\| B-\hat{B}\|^2$ [B-4]. Therefore, using the expressions for $\| A-\hat{A}\|$ and $\| B - \hat{B}\|$ given by (3.3.17) and (3.3.18), and recalling (3.2.2) thru (3.2.4), the solution of the optimal modeling problem involves determining values for the vector $p^* = [p_1{}^* \, p_2{}^* \ldots \, p_n{}^* \; p_{n+1}{}^*]^T$ such that

$$\max_{\substack{q_i \varepsilon [a_i, b_i] \\ i=1,2,\ldots n}} \sum_{i=1}^{n} (q_i - p_i{}^*)^2 \leq \max_{\substack{q_i \varepsilon [a_i, b_i] \\ i=1,2,\ldots,n}} \sum_{i=1}^{n} (q_i - p_i)^2 \tag{3.3.19}$$

and

$$\max_{q_{n+1} \varepsilon [a_{n+1}, b_{n+1}]} (q_{n+1} - p_{n+1}^*)^2 \leq \max_{q_{n+1} \varepsilon [a_{n+1}, b_{n+1}]} (q_{n+1} - p_{n+1})^2 \tag{3.3.20}$$

for all $p_i \varepsilon [a_i, b_i]$, $i = 1,2, \ldots, n, n+1$.

For notational convenience, define the functions

$$L_i(p_i,q_i) \triangleq (q_i-p_i)^2 \ , \quad i = 1,2,\ldots, n, n+1 \tag{3.3.21}$$

and let

$$\begin{aligned} L(p,q) &= L(p_1,p_2, \ldots, p_n, p_{n+1}, q_1, q_2, \ldots, q_n, q_{n+1}) \\ &\triangleq \sum_{i=1}^{n+1} L_i(p_i,q_i) = \sum_{i=1}^{n+1} (q_i - p_i)^2. \end{aligned} \tag{3.3.22}$$

<u>Remark 3.3.3</u>: Note that

$$L(p,q) = \|A-\hat{A}\|^2 + \|B-\hat{B}\|^2 \tag{3.3.23}$$

since

$$L(p,q) = \sum_{i=1}^{n+1} (q_i-p_i)^2 = \underbrace{\sum_{i=1}^{n} (q_i-p_i)^2}_{\|A-\hat{A}\|^2} + \underbrace{(q_{n+1}-p_{n+1})^2}_{\|B-\hat{B}\|^2} \tag{3.3.24}$$

It is obvious that $\|A-\hat{A}\|^2$ and $\|B-\hat{B}\|^2$ can be written as

$$\begin{aligned} \|A-\hat{A}\|^2 &= L(p_1,p_2, \ldots, p_n, 0, q_1, q_2, \ldots, q_n, 0) \\ &= \sum_{i=1}^{n} (q_i-p_i)^2 + 0 \ , \end{aligned} \tag{3.3.25}$$

and

$$\begin{aligned} \|B-\hat{B}\|^2 &= L(0,0, \ldots, 0, p_{n+1}, 0, 0, \ldots, 0, q_{n+1}) \\ &= \underbrace{0 + 0 + \ldots + 0}_{n \text{ terms}} + (q_{n+1} - p_{n+1})^2 \ . \end{aligned} \tag{3.3.26}$$

From Remark's 3.3.2 and 3.3.3, it is clear that the optimal modeling problem defined in Section 3.2 can be solved by minmaximizing $L(p,q)$ given in (3.3.22). Consider the following lemma.

Lemma 3.3.4:

$$\min_{p\varepsilon Q}\max_{q\varepsilon Q} L(p,q) = \sum_{i=1}^{n+1} \min_{p_i\varepsilon I_i}\max_{q_i\varepsilon I_i} L_i(p_i,q_i) \tag{3.3.27}$$

where $p = [p_1\,p_2 \cdots p_n\,p_{n+1}]^T$, $q = [q_1\,q_2 \cdots q_n\,q_{n+1}]^T$,

$Q = I_1 \times I_2 \times \cdots \times I_n \times I_{n+1}$, $I_i = [a_i,b_i]$, $i = 1,2,\ldots,n,n+1$,

and $L_i(\cdot)$ and $L(\cdot)$ are defined in (3.3.21) and (3.3.22).

Proof: Using (3.2.2), (3.2.3), (3.2.4) and (3.3.22),

$$\min_{p\varepsilon Q}\max_{q\varepsilon Q} L(p,q) = \min_{\substack{p_j\varepsilon I_j\\ j=1,2,\ldots,n+1}}\ \max_{\substack{q_j\varepsilon I_j\\ j=1,2,\ldots,n+1}} L(p_1,p_2,\ldots,p_{n+1},q_1,q_2\ldots,q_{n+1})$$

$$= \min_{\substack{p_j\varepsilon I_j\\ j=1,2,\ldots,n+1}}\ \max_{\substack{q_j\varepsilon I_j\\ j=1,2,\ldots,n+1}} \sum_{i=1}^{n+1}(q_i-p_i)^2. \tag{3.3.28}$$

Expanding the sum in (3.3.28) gives

$$\min_{p\varepsilon Q}\max_{q\varepsilon Q} L(p,q) = \min_{\substack{p_j\varepsilon I_j\\ j=1,2,\ldots,n+1}}\ \max_{\substack{q_j\varepsilon I_j\\ j=1,2,\ldots,n+1}} \{(q_1-p_1)^2 + (q_2-p_2)^2 + \cdots + (q_{n+1}-p_{n+1})^2\}\ . \tag{3.3.29}$$

Minmaximizing each term in (3.3.29) yields

$$\min_{p\varepsilon Q}\max_{q\varepsilon Q} L(p,q) = \min_{\substack{p_j\varepsilon I_j \\ j=1,2,\ldots,n+1}} \max_{\substack{q_j\varepsilon I_j \\ j=1,2,\ldots,n+1}} (q_1-p_1)^2 + \min_{\substack{p_j\varepsilon I_j \\ j=1,2,\ldots,n+1}} \max_{\substack{q_j\varepsilon I_j \\ j=1,2,\ldots,n+1}} (q_2-p_2)^2 + \ldots + \min_{\substack{p_j\varepsilon I_j \\ j=1,2,\ldots,n+1}} \max_{\substack{q_i\varepsilon I_j \\ j=1,2,\ldots,n+1}} (q_{n+1}-p_{n+1})^2. \tag{3.3.30}$$

But for each term in (3.3.30),

$$\min_{\substack{p_j\varepsilon I_j \\ j=1,2,\ldots,n+1}} \max_{\substack{q_j\varepsilon I_j \\ j=1,2,\ldots,n+1}} (q_k-p_k)^2 = \min_{p_k\varepsilon I_k} \max_{q_k\varepsilon I_k} (q_k-p_k)^2, \tag{3.3.31}$$

where k is arbitrary.

Therefore,

$$\min_{p\varepsilon Q}\max_{q\varepsilon Q} L(p,q) = \min_{p_1\varepsilon I_1}\max_{q_1\varepsilon I_1} (q_1-p_1)^2 + \min_{p_2\varepsilon I_2}\max_{q_2\varepsilon I_2} (q_2-p_2)^2 + \ldots + \min_{p_{n+1}\varepsilon I_{n+1}}\max_{q_{n+1}\varepsilon I_{n+1}} (q_{n+1}-p_{n+1})^2. \tag{3.3.32}$$

Finally, collecting terms and using (3.3.21)

$$\min_{p\varepsilon Q}\max_{q\varepsilon Q} L(p,q) = \sum_{i=1}^{n+1} \min_{p_i\varepsilon I_i}\max_{q_i\varepsilon I_i} (q_i-p_i)^2$$

$$= \sum_{i=1}^{n+1} \min_{p_i\varepsilon I_i}\max_{q_i\varepsilon I_i} L_i(p_i,q_i) \tag{3.3.33}$$

which is the desired result. Q.E.D.

An immediate consequence of Lemma 3.3.4 is that the minmaximizing solution of $L(p,q)$, a function of $2(n+1)$ variables, can be obtained by minmaximizing the $L_i(p_i,q_i)$, where the constraints are $a_i \le p_i \le b_i$ and $a_i \le q_i \le b_i$, for $i = 1,2,\ldots,n+1$. Thus, the original minmax problem has been transformed into n+1 simpler problems. Since the functions $L_i(p_i,q_i)$ are identical in form for every i, it will suffice to study a single function $L_k(p_k,q_k)$ for an arbitrary integer k, $1 \le k \le n+1$.

In view of the preceding discussion, consider the following subproblem:

Subproblem: Determine the optimal $p_k{}^*$ and $q_k{}^*$ such that

$$L_k(p_k{}^*,q_k{}^*) = \min_{p_k \varepsilon I_k} \max_{q_k \varepsilon I_k} L_k(p_k,q_k) \quad , \qquad (3.3.34)$$

where $L_k(p_k,q_k) = (q_k-p_k)^2$, $I_k = [a_k,b_k]$, and k is an arbitrary integer $1 \le k \le n+1$.

The subproblem is solved as follows.

Lemma 3.3.5: The $q_k{}^*$ which maximizes $L_k(p_k,q_k)$ does not lie in the open interval (a_k,b_k).

Proof: Let p_k be arbitrary and $p_k \varepsilon (a_k,b_k)$.

From the calculus, necessary conditions for a $q_k{}^* \varepsilon (a_k,b_k)$ to maximize $L_k(\cdot)$ are [0-2]:

$$\text{i)} \quad \frac{\partial L_k(\cdot)}{\partial q_k} = 0 \qquad (3.3.35a)$$

$$\text{ii)} \quad \frac{\partial^2 L_k(\cdot)}{\partial q_k{}^2} \le 0 \quad . \qquad (3.3.35b)$$

Performing (3.3.35a) yields

$$\frac{\partial L_k(\cdot)}{\partial q_k} = 2\ (q_k - p_k) = 0 \Rightarrow q_k{}^* = p_k \tag{3.3.36}$$

as a candidate for a maximum.

Checking (3.3.35b) at $q_k^* = p_k$ gives

$$\frac{\partial^2 L_k(\cdot)}{\partial q_k{}^2} = 2 \not\leq 0 \ . \tag{3.3.37}$$

Condition (ii) fails at $q_k{}^* = p_k$. Therefore, there is no $q_k{}^* \in (a_k, b_k)$ which maximizes $L_k(p_k, q_k)$.

Q.E.D.

From Lemma 3.3.5, the $q_k{}^*$ which maximizes $L_k(\cdot)$ must lie on the boundary of I_k. Therefore, there are two possibilities:

$$q_k{}^* = a_k \tag{3.3.38}$$

or

$$q_k{}^* = b_k \ . \tag{3.3.39}$$

Evaluating $L_k(p_k, q_k)$ at $q_k{}^* = a_k$ and $q_k{}^* = b_k$ yields

$$L_k(p_k, a_k) = (a_k - p_k)^2 = a_k{}^2 - 2a_k p_k + p_k{}^2 \tag{3.3.40}$$

and

$$L_k(p_k, b_k) = (b_k - p_k)^2 = b_k{}^2 - 2b_k p_k + p_k{}^2 \ . \tag{3.3.41}$$

Writing $b_k = (b_k - a_k) + a_k$ and substituting this expression into the right hand side of (3.3.41) gives

$$L_k(p_k, b_k) = L_k(p_k, a_k) + 2\ (a_k - b_k)\ [p_k - \frac{(a_k + b_k)}{2}]. \tag{3.3.42}$$

Now

$$L_k(p_k,b_k) > L_k(p_k,a_k) \text{ when } (a_k-b_k)\ [p_k - \frac{(a_k+b_k)}{2}] > 0 \quad (3.3.43)$$

and

$$L_k(p_k,b_k) < L_k(p_k,a_k) \text{ when } (a_k-b_k)\ [p_k - \frac{(a_k+b_k)}{2}] < 0. \quad (3.3.44)$$

Therefore, the maximizing q_k^* is given by

$$q_k^* = \begin{cases} b_k & \text{when } (a_k-b_k)\ [p_k - \frac{(a_k+b_k)}{2}] > 0 \\ a_k & \text{when } (a_k-b_k)\ [p_k - \frac{(a_k+b_k)}{2}] < 0 \\ \text{either } a_k \text{ or } b_k & \text{when } (a_k-b_k)\ [p_k - \frac{(a_k+b_k)}{2}] = 0 \end{cases} \quad (3.3.45)$$

Using a slightly modified signum notation (cf. [0-2] p. 36)

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \pm 1 & x = 0 \end{cases}, \quad (3.3.46)$$

(3.3.45) can be written as

$$q_k^* = \frac{a_k+b_k}{2} - \frac{(a_k-b_k)}{2} \operatorname{sgn} \{(a_k-b_k)\ [p_k - \frac{(a_k+b_k)}{2}]\}. \quad (3.3.47)$$

Equations (3.3.40) and (3.3.41) can be combined and rewritten

$$L_k(p_k,q_k^*) = (q_k^*-p_k)^2 = [(\frac{a_k+b_k}{2} - p_k) - \frac{(a_k-b_k)}{2} \operatorname{sgn} \{(a_k-b_k)\ [p_k - \frac{(a_k+b_k)}{2}]\}]^2 \quad (3.3.48)$$

or

$$L_k(p_k,q_k^*) = (p_k - \frac{a_k+b_k}{2})^2 + |a_k-b_k|\ |p_k - \frac{(a_k+b_k)}{2}| + \frac{(a_k-b_k)^2}{4} \tag{3.3.49}$$

since x sgn x = |x| [0-2], where |·| denotes the absolute value.

Clearly the p_k^* which minimizes (3.3.49) occurs when

$$p_k^* = \frac{a_k+b_k}{2}\ , \tag{3.3.50}$$

and

$$\min_{p_k \varepsilon I_k}\ \max_{q_k \varepsilon I_k} L_k(p_k,q_k) = \frac{(a_k-b_k)^2}{4}\ . \tag{3.3.51}$$

Thus, the solution of the subproblem (3.3.34) for every k, $1 \leq k \leq n + 1$, is

$$p_k^* = \frac{a_k+b_k}{2} \tag{3.3.52}$$

$$q_k^* = \text{either } a_k \text{ or } b_k \tag{3.3.53}$$

$$L_k(p_k^*,q_k^*) = \frac{(a_k-b_k)^2}{4}\ . \tag{3.3.54}$$

From Lemma 3.3.4, $L(p^*,q^*)$, p^*, and q^* are given as

$$L(p^*,q^*) = \sum_{i=1}^{n+1} L_i(p_i^*,q_i^*) = \frac{1}{4}\sum_{i=1}^{n+1} (a_i-b_i)^2 \tag{3.3.55}$$

$$p^* = [p_1^* p_2^* \ldots p_n^* p_{n+1}^*]^T,\ p_i^* = \frac{a_i+b_i}{2}\ ,\ i = 1,2,\ldots,n+1 \tag{3.3.56}$$

and

$$q^* = [q_1^* q_2^* \ldots q_n^* q_{n+1}^*]^T,\ q_i = \text{either } a_i \text{ or } b_i,\ i = 1,2,\ldots,n+1. \tag{3.3.57}$$

From Remark 3.3.3,

$$L(p^*,q^*) = \| A(q^*) - \hat{A}(p^*) \|^2 + \| B(q^*) - \hat{B}(p^*) \|^2, \tag{3.3.58}$$

where

$$\|A(q^*) - \hat{A}(p^*)\|^2 = L(p_1^*,p_2^*\ldots,p_n^*,0,q_1^*,q_2^*,\ldots,q_n^*,0)$$

$$= \sum_{i=1}^{n} (q_i^*-p_i^*)^2 \tag{3.3.59}$$

and

$$\| B(q^*) - \hat{B}(p^*) \|^2 = L(0,0,\ldots,0,p_{n+1}^*,0,0,\ldots,0,q_{n+1}^*)$$

$$= (q_{n+1}^*-p_{n+1}^*)^2 \quad . \tag{3.3.60}$$

Finally, Remark 3.3.2 equivalences the solution of (3.3.19) and (3.3.20) (given by (3.3.56), (3.3.57), (3.3.59) and (3.3.60)) to (3.2.8) and (3.2.9). Therefore, the solution of the Optimal Modeling Problem using step one of the MMAC approach is given as

$$p^* = [p_1^* p_2^* \cdots p_{n+1}^*]^T, \; p_i^* = \frac{a_i+b_i}{2}, \; i = 1,2,\ldots,n+1 \tag{3.3.61}$$

$$\hat{A}(p^*) = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 \\ p_1^* & p_2^* & p_3^* & p_4^* & . & . & . & p_{n+1}^* \end{bmatrix} \tag{3.3.62}$$

$$\hat{B}(p^*) = [0\ \ 0\ \ 0\ \ 0\ \ .\ \ .\ \ .\ \ 0\ \ p_{n+1}{}^*]^T \tag{3.3.63}$$

$$q^* = [q_1{}^* q_2{}^* \ldots q_{n+1}{}^*]^T,\ q_i{}^* = \text{either } a_i \text{ or } b_i,\ i = 1,2,\ldots,n+1 \tag{3.3.64}$$

$$A(q^*) = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 \\ q_1{}^* & q_2{}^* & q_3{}^* & q_4{}^* & . & . & . & q_n{}^* \end{bmatrix} \tag{3.3.65}$$

$$B(q^*) = [0\ \ 0\ \ 0\ \ 0\ \ .\ \ .\ \ .\ \ 0\ \ q_{n+1}{}^*]^T \tag{3.3.66}$$

$$\| A(q^*) - \hat{A}(p^*) \| = \frac{1}{2} \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2} \tag{3.3.67}$$

and

$$\| B(q^*) - \hat{B}(p^*) \| = \frac{1}{2} \sqrt{(a_{n+1} - b_{n+1})^2}\ . \tag{3.3.68}$$

Remark 3.3.6: When the uncertainty set Q is given as a rectangle in R^{n+1} (cf. Definition 3.2.2 and Remark 3.2.3), the solution of the Optimal Modeling Problem yields an optimal model parameter vector p* of values $p_i{}^*$, i = 1,2,...,n+1 which are uniquely specified in (3.3.61). Not only does p* lie in the interior of the rectangular uncertainty set - it can be considered as defining the center point of Q. In a different yet related problem Schweppe [S-5] also chooses the center point as a "best estimate" when he considers the estimation of parameters for a static linear system described by an unknown but bounded model (cf. [S-5], Chapter 5). Since there is a strong correspondence between the vector p^* and the matrices $\hat{A}(p^*)$ and $\hat{B}(p^*)$, the uniqueness of $p^* \varepsilon Q$ implies that the model (3.2.1) is also unique.

Remark 3.3.7: Note that the maximizing $q^* \varepsilon Q$ given by (3.3.64) is not unique. However, each q_k^*, $k = 1,2,\ldots,n+1$ lies on the boundary of the uncertainty set Q. It is also interesting to note that the q_k^*, written as a function of p_k in (3.3.47) with k arbitrary, is not differentiable at $p_k = p_k^*$ (3.3.52). This phenomena was discussed in Section 2.4.

Remark 3.3.8: The optimal minmax model (3.2.1) with matrices $\hat{A}(p^*)$ and $\hat{B}(p^*)$ given by (3.3.62) and (3.3.63), yields the smallest guaranteed upper bound on the cost (3.2.8) and (3.2.9). It is therefore appropriate to describe the optimal minmax model as a guaranteed cost model. That is, if the true value of the uncertain system parameter vector $q \varepsilon Q$ is given as $q_{true} \neq q^*$, with $q_{true} \varepsilon Q$, then the actual cost will always be less than or equal to that specified in (3.3.67) and (3.3.68).

Remark 3.3.9: If $1 \leq r < n+1$ of the system parameters are known exactly, the determination of the optimal minmax model can proceed without difficulty if the uncertainty intervals, $I_k = [a_k, b_k]$, corresponding to the r known parameters are written as point sets.

Remark 3.3.10: After the optimal model has been found using step one of the MMAC approach, the remaining control problem is to determine an admissible control function for the uncertain system (3.1.1) based on the optimal minmax model (3.2.1), (3.3.62), and (3.3.63) and the specified performance index (3.1.3). That is, find an admissible $u^*(\cdot) \in M$ which minimizes

$$\hat{J}(u(\cdot)) = h(\hat{x}(t_f)) + \int_{t_o}^{t_f} g(\hat{x}(t), u(t))\, dt \qquad (3.3.69)$$

subject to the constraints

$$\dot{\hat{x}}(t) = \hat{A}(p^*)\, \hat{x}(t) + \hat{B}(p^*)\, u(t), \quad \hat{x}(t_o) = x_o. \qquad (3.3.70)$$

The solution of (3.3.69) and (3.3.70) is step two of the MMAC approach. Since (3.3.70) involves quantities that are completely known, step two requires the solution of a deterministic optimal control problem. The control portion of the MMAC approach will be considered in Chapter IV.

In the next section, three examples which illustrate the selection of the optimal model (3.2.1) using step one of the MMAC approach are presented.

3.4 Optimal Modeling Examples

The following examples illustrate the technique developed in Section 3.3 for optimal model selection.

Example 3.4.1: Let the uncertain system S (to be controlled) have the following representation:

$$S: \quad \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad (3.4.1)$$

$$x(0) = [.5\ 0]^T, \quad t \in [0,\pi], \quad \alpha \in [1,4], \quad |u(t)| \leq 1$$

with control cost

$$J(u(\cdot),\alpha) = -x_1(\pi) \tag{3.4.2}$$

where $x(t) = [x_1(t) \quad x_2(t)]^T$.

Using the notation developed in Section 3.1 thru 3.3, S can be rewritten as

$$S: \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ q_1 & q_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ q_3 \end{bmatrix} u(t) \tag{3.4.3}$$

$$x(0) = [.5\ 0]^T, \ t \in [0,\pi], \ |u(t)| \leq 1$$

$$q_1 \in [-4,-1], \ q_2 \in [0,0], \ q_3 \in [1,1]$$

and the control cost $J(u(\cdot),\alpha)$ is the same as (3.4.2).

The determination of the optimal model M of the form (3.2.1) is trivial and is accomplished by selecting (cf. (3.3.61))

$$p_1{}^* = \frac{-4-1}{2} = -2.5, \ p_2{}^* = \frac{0+0}{2} = 0, \ p_3{}^* = \frac{1+1}{2} = 1. \tag{3.4.4}$$

The optimal model is given as

$$M: \dot{\hat{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2.5 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{3.4.5}$$

$$\hat{x}(0) = x(0) = [.5\ 0]^T, \ t \in [0,\pi], \ |u(t)| \leq 1$$

and the control cost for the model $\hat{J}(u(\cdot))$ is

$$\hat{J}(u(\cdot)) = -\hat{x}_1(\pi) \tag{3.4.6}$$

where $\hat{x}(t) = [\hat{x}_1(t)\ \hat{x}_2(t)]^T$.

As can be seen, the determination of the optimal model simply requires the computation of the center of each uncertainty interval $[a_i, b_i]$. Note that the dimension of the state equations remains unchanged. It is therefore possible to merely write down the optimal model and control cost for any uncertain system described by (3.1.1). The next two examples utilize this abbreviated approach.

Example 3.4.2: Let the uncertain system be

$$S:\ \dot{x}(t) = \alpha\, x(t) + u(t) \tag{3.4.7}$$

$$x(0) = 2, \quad t \in [0,\infty), \quad \alpha \in [-2.5,-0.5],$$

$u(t)$ is unconstrained, and the control cost is

$$J(u(\cdot),\alpha) = \frac{1}{2}\int_0^\infty (x^2(t) + u^2(t))\, dt, \tag{3.4.8}$$

where $x(t)$ is a scalar quantity.

The minmax model M is given by

$$M:\ \dot{\hat{x}}(t) = -1.5\, \hat{x}(t) + u(t) \tag{3.4.9}$$

$$\hat{x}(0) = 2, \ t \in [0,\infty),\ u(t) \text{ is unconstrained,}$$

and

$$\hat{J}(u(\cdot)) = \frac{1}{2}\int_0^\infty (\hat{x}^2(t) + u^2(t))\, dt, \tag{3.4.10}$$

where $\hat{x}(t)$ is a scalar quantity.

Example 3.4.3: Let the uncertain system be

$$S:\ \dot{x}(t) = -2x(t) + \beta\, u(t) \tag{3.4.11}$$

$$x(0) = 5,\ t \in [0,\infty),\ \beta \in [1,5],\ u(t) \text{ is unconstrained,}$$

and the control cost is

$$J(u(\cdot),\beta) \;=\; \frac{1}{2}\int_0^\infty (x^2(t) + u^2(t))\,dt \qquad (3.4.12)$$

where x(t) is a scalar quantity.

The optimal model M is

$$M:\ \dot{\hat{x}}(t) \;=\; -2\hat{x}(t) + 3u(t) \qquad (3.4.12)$$

$\hat{x}(0) = 5$, $t \in [0,\infty)$, u(t) is unconstrained,

$$\hat{J}(u(\cdot)) \;=\; \frac{1}{2}\int_0^\infty (\hat{x}^2(t) + u^2(t))\,dt \qquad (3.4.13)$$

where $\hat{x}(t)$ is a scalar quantity.

These three examples will be considered again in Chapter IV, when the problem of determining the MMAC controller is presented.

3.5 Controllability

In control theory, a basic question is if it is possible to transfer any initial state to any desired state (often taken as the origin) in a finite length of time by applying an appropriate control input. Kalman [K-7] introduced the concept of controllability and gave an answer to this basic question. This concept can be applied to the optimal minmax model (3.2.1) and to the uncertain system (3.1.1).

First, consider the deterministic linear system

$$\dot{x}(t) \;=\; F\,x(t) + G\,u(t),\ x(t_o) \;=\; x_o \qquad (3.5.1)$$

where $x(t) \in R^n$ is the state,

$u(t) \in R^m$ is the control,

F is a constant n x n matrix,

and G is a constant n x m matrix.

Definition 3.5.1 ([K-5], p.21): If there is a finite time $t_1 \geq t_o$ and a bounded measurable control u(t), $t \in [t_o, t_1]$, which transfers the state x_o to the origin at time t_1, the state x_o is said to be controllable at time t_o. If all values of x_o are controllable for all t_o, the system is completely controllable, or simply, controllable.

Relative to the optimal control problem, the significance of complete controllability can be easily grasped. It would be meaningless to search for an optimal control if for a given initial state, no bounded input exists which can drive the system to the zero state in finite time. It should be noted however, that controllability does not guarantee the existence of a solution to every optimal control problem (cf. [B-4], p. 346).

Remark 3.5.2: Kalman [K-7] has shown that a linear time-invariant system (3.5.1) is controllable iff the n x mn matrix

$$E \triangleq [G \mid FG \mid F^2G \mid \ . \ . \ . \mid F^{n-1}G] \qquad (3.5.2)$$

has rank n. If there is only one control input (m = 1), a necessary and sufficient condition for controllability is that the n x n matrix E be nonsingular ([K-5], p.21).

It can be shown that the optimal minmax model (3.2.1) with fixed parameter vector $p^* \in Q$ given by (3.3.61) is completely controllable provided p_{n+1}^* is restricted from assuming a zero value. The next theorem formalizes this statement.

Theorem 3.5.3: The linear time-invariant minmax model described by (3.2.1), (3.3.61), (3.3.62) and (3.3.63) is completely controllable iff $p_{n+1}{}^* \neq 0$.

Proof: Let $p^* \in Q$ be fixed according to (3.3.61). It then follows that the minmax model has a completely deterministic representation. From Remark 3.5.2, a necessary and sufficient condition for a linear time-invariant system with a single input to be controllable is that the n x n matrix E defined in (3.5.2) be nonsingular. It is a well known result in the theory of matrices that a square matrix E is nonsingular iff the determinant of E does not equal zero [G-1]. Forming the partitioned matrix E and taking the determinant yields

$$|\det\{E\}| = |(p^*_{n+1})^n|. \tag{3.5.3}$$

Now if $p^*_{n+1} \neq 0$, then E is nonsingular. Consequently the optimal minmax model is controllable. Conversely, if the minmax model is controllable, then $\det\{E\} \neq 0$, which implies that $p^*_{n+1} \neq 0$. This proves the theorem. Q.E.D.

From (3.3.61), it is easy to see that p^*_{n+1} will never equal zero provided that the $(n+1)^{th}$ interval of the rectangular uncertainty set Q is not symmetric with respect to the origin. Therefore, the linear time-invariant minmax model is completely controllable iff

$$a_{n+1} \neq -b_{n+1}, \tag{3.5.4}$$

where $Q = I_1 \times I_2 \times \dots \times I_n \times I_{n+1}$,

$I_i = [a_i, b_i], \quad i = 1,2,\dots,n,n+1.$

Next, consider the following definition (cf. [B-1]).

Definition 3.5.4: The linear time-invariant parameter uncertain system (3.1.1) with uncertainty vector $q \in Q$ is completely controllable iff it is completely controllable for each $q \in Q$.

Checking the controllability of an uncertain system is a formidable task since by definition, the system must be examined for each fixed $q \in Q$. It is doubtful that the rank condition (3.5.2) represents an appropriate test for controllability of the uncertain system (3.1.1). An integral test may provide a more reasonable approach. No attempt to verify the controllability property of the uncertain system (3.1.1) is made in this thesis.

However, it is obvious that if the problem of controlling the linear time-invariant parameter uncertain system (3.1.1) is to be meaningful, it is necessary that I_{n+1} be restricted from containing the zero element. That is,

$$0 \notin I_{n+1} = [a_{n+1}, b_{n+1}]. \tag{3.5.5}$$

For if $q_{n+1} = 0$, then the control input has absolutely no influence on the state. This condition (3.5.5) is reasonable and will be assumed in all subsequent discussions.

3.6 Stability

Stability is an important concept to be considered in the design of a controller for a system. In this section, the stability properties for deterministic, linear time-invariant systems are applied to the uncertain system described by (3.1.1). For a more extensive treatment of stability theory, see [K-6], [O-3] and [B-4].

Consider the deterministic system

$$\dot{x}(t) = F\,x(t) + G\,u(t), \quad x(t_o) = x_o \tag{3.6.1}$$

where $x(t) \varepsilon R^n$ is the state,

$u(t) \varepsilon R^m$ is the control,

F is a constant $n \times n$ matrix,

and G is a constant $n \times m$ matrix.

If $u(t) \equiv 0$ for all t, the system in (3.6.1) is said to be free (or unforced) [K-6]:

$$\dot{x}(t) = F\, x(t). \tag{3.6.2}$$

<u>Definition 3.6.1</u>: ([O-3], p. 438): For the free system (3.6.2), x_e is called an <u>equilibrium state</u> iff

$$F\, x_e = 0 \tag{3.6.3}$$

for all t.

Note that if F is nonsingular, then the origin is the unique equilibrium state of (3.6.2). If F is singular, then there exists an infinite number of equilibrium states [B-4].

Next, the concepts of stability, asymptotic stability, and instability (in the sense of Lyapunov) are defined.

<u>Definition 3.6.2</u> ([O-3], pp 438-439): An equilibrium state x_e of the free dynamic system (3.6.2) is <u>stable</u> if for every real number $\varepsilon > 0$ there exists a real number $\delta(\varepsilon, t_o) > 0$ such that

$$\| x_o - x_e \| \leq \delta \text{ implies } \| x(t) - x_e \| \leq \varepsilon \text{ for all } t \geq t_o.$$

<u>Definition 3.6.3</u> ([O-3], pp 439-440): An equilibrium state x_e of the free dynamic system (3.6.2) is <u>asymptotically stable</u> if

(i) it is stable,

(ii) every solution starting at a state x_o sufficiently near x_e

converges to x_e as t increases indefinitely. Namely, given two real numbers $\delta > 0$ and $\mu > 0$, there are real numbers $\varepsilon > 0$ and $T(\mu,\delta,t)$ such that

$$\| x_0 - x_e \| \leq \delta \text{ implies } \| x(t) - x_e \| \leq \varepsilon \text{ for all } t \geq t_0 \text{ and}$$

$$\| x(t) - x_e \| \leq \mu \quad \text{for} \quad t \geq t_0 + T(\mu,\delta,t_0).$$

Definition 3.6.4 ([O-3], p. 441): An equilibrium state x_e of the free dynamic system (3.6.2) is unstable if it is neither stable nor asymptotically stable.

Definitions 3.6.2 and 3.6.3 represent the precise and formal definitions for stability in the sense of Lyapunov and asymptotic stability in the sense of Lyapunov, respectively. Note that stability and asymptotic stability are defined with respect to the equilibrium state x_e, which may be nontrivial if F is singular. Nevertheless, the stability characteristics of all equilibrium states for (3.6.2) are the same [K-8] and are related to the eigenvalues of the coefficient matrix F as stated in the following criterion:

Stability Criterion ([A-2], p. 149 and [B-7], p. 241): Let the eigenvalues of the matrix F in (3.6.2) be denoted by

$$\alpha_i + j\beta_i \qquad i = 1,2,\ldots,n, \tag{3.6.4}$$

where the α's and the β's are real numbers and $j = \sqrt{-1}$. The system (3.6.2) is stable iff 1) $\alpha_i \leq 0$ for all i, 2) if $\alpha_k + j\beta_k$ is a multiple root of the characteristic polynomial of F, then $\alpha_k < 0$. The system (3.6.2) is asymptotically stable iff $\alpha_i < 0$ for all i.

Once the characteristic polynomial of F is calculated, there are a variety of methods which implement the stability criterion stated above. Two commonly used approaches are Routh's method and Hurwitz's method. These procedures will not be detailed here, but may be found in Ogata ([O-3], Chapter 8).

Now, consider the linear time-invariant parameter uncertain system described in (3.1.1) with uncertainty vector $q \in Q$, where Q is compact. When $u(t) \equiv 0$ for all t

$$\dot{x}(t) = A(q)\, x(t) \tag{3.6.5}$$

is said to be unforced. From the previous discussion, the following definition is appropriate:

Definition 3.6.5: The free, linear time-invariant parameter uncertain system (3.6.5), with q, A(q), and Q described in (3.1.1), is stable (asymptotically stable) iff it is stable (asymptotically stable) for each $q \in Q$.

Remark 3.6.6: Suppose it is desired to find if the uncertain system (3.6.5) is asymptotically stable. The characteristic equation for (3.6.5) is readily determined as

$$c(\lambda) = \lambda^n - q_1\lambda^{n-1} - q_2\lambda^{n-2} - \dots - q_{n-1}\lambda - q_n = 0, \tag{3.6.6}$$

where the coefficients q_i are constant but arbitrary. Using Routh's implementation of the Stability Criterion ([O-3], Chapter 8), conditions which the q_i must satisfy in order that all the eigenvalues in (3.6.5) have negative real parts are given. These conditions are written as algebraic inequalities in the parameters q_i. It is then a simple matter to check if the constraints $q_i \in [a_i, b_i]$ are compatible with the derived conditions for asymptotic stability (see [O-3], Example 8-1, p. 445).

Other methods of investigating the stability properties of linear time-invariant parameter uncertain systems are available (see for example, [H-2]). However, Routh's or Hurwitz's methods ([O-3] Chapter 8 or [D-6] Chapter 5) are appealing since the characteristic equation for the uncertain system (3.6.5) is easily found.

Finally, it is obvious that if the free portion of the uncertain system (3.1.1) is stable for every $q \in Q$, then the unforced part of the optimal minmax model, described by (3.2.1), (3.3.61) and (3.3.62), is also stable. The more interesting case is when the free part of the uncertain system (3.1.1) is not stable for every $q \in Q$. In this case, it is entirely possible to determine a minmax model with parameters $p^* \in Q$ whose free portion is stable, while for some $q \in Q$, $q \neq p^*$, the unforced uncertain system is unstable (or vice-versa).

A difference between model and system stability properties may prove to be disastrous when implementing step two of the MMAC approach, especially over the infinite time interval $t \in [0,\infty)$. This presents an area of further study which is not pursued in this thesis. In the specific examples considered in this thesis, no complications arise.

3.7 Trajectory Error Bound

In this section an upper bound on the norm of the error between system and model trajectories is determined as a function of the matrix error norms (3.3.67) and (3.3.68). This adds substance to the claim made in Section 3.2: that if the model matrix $\hat{A}$ is chosen "close" to

A and if the matrix $\hat{B}$ is chosen "close" to B then the model and system trajectories $\hat{x}(\cdot)$ and $x(\cdot)$ should also be "close". The norm of differences is taken as the measure of closeness.

Use will be made of the matrix norm $\|\cdot\|$ defined by [K-6]:

$$\| F \| \triangleq \sqrt{\lambda_{max}} \tag{3.7.1}$$

where λ_{max} is the maximum eigenvalue of $F^T F$, and of the norm inequalities [K-6]

$$\| F + G \| \leq \| F \| + \| G \| \tag{3.7.2}$$

$$\| F\,G \| \leq \| F \| \cdot \| G \| \,. \tag{3.7.3}$$

The vector norm $\|\cdot\|_2$ defined on $L_2([t_o,t_f],R^r)$ as [A-2]:

$$\| z(t) \|_2 \triangleq \{\int_{t_o}^{t_f} z^T(t)\; z(t)\; dt \}^{\frac{1}{2}},\; z(t) \in R^r \tag{3.7.4}$$

and the inequalities ([K-6] and [B-6] p. 184)

$$\| y(t) + z(t) \|_2 \leq \| y(t) \|_2 + \| z(t) \|_2 \tag{3.7.5}$$

$$\| y(t)\cdot z(t) \|_2 \leq \| y(t) \|_2 \cdot \| z(t) \|_2 \tag{3.7.6}$$

$$\| \int_{t_o}^{t_1} z(t)\; dt \|_2 \leq \int_{t_o}^{t_1} \| z(t) \|_2\; dt,\; t_1 \geq t_o \tag{3.7.7}$$

will also be useful.

Before proceeding with the derivation of the trajectory error bound, some preliminary results are required. Consider the linear time-invariant system

$$\dot{z}(t) = F\, z(t) + G\, u(t) \tag{3.7.8}$$

where

$$z(t_o) = z_o \in \Sigma_o, \ \Sigma_o \text{ is a bounded subset of } R^n, \tag{3.7.9}$$

$$t \in [t_o, t_f], \tag{3.7.10}$$

$$u(t) \in U \text{ for every } t \in [t_o, t_f], \ U \text{ is a bounded subset of } R^1, \tag{3.7.11}$$

and $z(t)$ is the n x 1 state vector,

$u(t)$ is the scalar control,

F is an n x n matrix,

G is an n x 1 matrix.

Let the origin by an asymptotically stable equilibrium point (cf. Section 3.6, Definition 3.6.3) for the unforced system

$$\dot{z}(t) = F\, z(t) \ . \tag{3.7.12}$$

Kalman [K-6] and others [B-4], [D-7] have shown that the origin is an asymptotically stable equilibrium point of (3.7.12) if and only if there exist finite positive constants M_1 and k_1 such that the transition matrix $\Phi(t,\tau)$, defined by the infinite series [B-4]

$$\Phi(t,\tau) = e^{F(t-\tau)} \triangleq I + F(t-\tau) + \frac{1}{2!}F^2(t-\tau)^2 + \frac{1}{3!}F^3(t-\tau)^3 + \cdots,$$

$$t \geq \tau \geq t_o \tag{3.7.13}$$

is bounded in norm by

$$\| \Phi(t,\tau) \| = \| e^{F(t-\tau)} \| \leq M_1 e^{-k_1(t-\tau)}, \ t \geq \tau \geq t_o. \tag{3.7.14}$$

By using (3.7.14) and observing that (3.7.9) and (3.7.11) imply that there exist finite positive constants M_2 and M_3 such that:

$$\| z_o \| \leq M_2 < \infty \tag{3.7.15}$$

and

$$\| u(t) \|_2 \leq M_3 < \infty \text{ for all } t \geq t_o, \tag{3.7.16}$$

it is possible to bound the norm of the solution to (3.7.8) by a finite constant.

The derivation proceeds as follows.

From variation of parameters, the solution of (3.7.8) is [B-4]

$$z(t) = e^{F(t-t_o)} z_o + \int_{t_o}^{t} e^{F(t-\tau)} G\, u(\tau)\, d\tau . \tag{3.7.17}$$

Taking the norm of z(t) and generating the string of inequalities

$$\| z(t) \|_2 = \| e^{F(t-t_o)} z_o + \int_{t_o}^{t} e^{F(t-\tau)} G\, u(\tau)\, d\tau \|_2 \tag{3.7.18}$$

$$\leq \| e^{F(t-t_o)} z_o \|_2 + \| \int_{t_o}^{t} e^{F(t-\tau)} G\, u(\tau)\, d\tau \|_2 \text{ by (3.7.5)} \tag{3.7.19}$$

$$\leq \| e^{F(t-t_o)} \| \cdot \| z_o \| + \int_{t_o}^{t} \| e^{F(t-\tau)} \| \cdot \| G \| \cdot \| u(\tau) \|_2 d\tau \tag{3.7.20}$$

by (3.7.3), (3.7.6), (3.7.7)

$$\leq M_1\, e^{-k_1(t-t_o)} \cdot M_2 + \int_{t_o}^{t} M_1\, e^{-k_1(t-\tau)} \cdot M_4\, M_3\, d\tau \tag{3.7.21}$$

by (3.7.14), (3.7.15), (3.7.16) and by observing that

$$0 \leq \| G \| < M_4 < \infty \tag{3.7.22}$$

is true for finite dimensional linear transformations [B-6].

Integrating (3.7.21) and continuing the string of inequalities yields

$$\| z(t) \|_2 \leq M_1M_2 \, e^{-k_1(t-t_o)} + \frac{M_1M_3M_4}{k_1} \cdot (1 - e^{-k_1(t-t_o)}) \tag{3.7.23}$$

$$= (M_1M_2 - \frac{M_1M_3M_4}{k_1}) \, e^{-k_1(t-t_o)} + \frac{M_1M_3M_4}{k_1} \tag{3.7.24}$$

$$\leq |M_1M_2 - \frac{M_1M_3M_4}{k_1}| \; |e^{-k_1(t-t_o)}| + |\frac{M_1M_3M_4}{k_1}| \tag{3.7.25}$$

by well known properties of the absolute value [O-2].

Since $|e^{-k_1(t-t_o)}| \leq 1$ for $t \geq t_o$, $\| z(t) \|_2$ can be bounded by

$$\| z(t) \|_2 \leq N_1 < \infty , \tag{3.7.26}$$

where

$$N_1 = |M_1M_2 - \frac{M_1M_3M_4}{k_1}| + |\frac{M_1M_3M_4}{k_1}| > 0. \tag{3.7.27}$$

It is now possible to derive an upper bound for the norm of the difference between the model and uncertain system trajectories.

Consider the uncertain system (3.1.1) with an arbitrary choice of $q \in Q$. In order to make use of the previous results, it is convenient to assume that the origin is an asymptotically stable equilibrium point for the unforced part of (3.1.1) and for all $q \in Q$ (cf. Section 3.6, Definition 3.6.5). Also let the input u(t) be bounded in norm by a finite and positive constant M_5. Furthermore, let the optimal minmax model be given by (3.2.1), (3.3.61), (3.3.62) and (3.3.63). The uncertain system and optimal model state equations are repeated here for convenience:

$$S:\quad \dot{x}(t) = A\,x(t) + B\,u(t),\ x(t_o) = x_o,\ t \in [t_o,t_f] \tag{3.7.28}$$

$u(t) \in U$, $U \subset R^1$ is a bounded set.

$$M:\quad \dot{\hat{x}}(t) = \hat{A}^*\,\hat{x}(t) + \hat{B}^*\,u(t),\quad \hat{x}(t_o) = \hat{x}_o = x_o,\quad t \in [t_o,t_f], \tag{3.7.29}$$

where the star (*) indicates that the matrices $\hat{A}$ and $\hat{B}$ are evaluated at the optimal p^*. That is,

$$\hat{A}^* = \hat{A}(p^*) \tag{3.7.30}$$

$$\hat{B}^* = \hat{B}(p^*), \tag{3.7.31}$$

where p^* is given by (3.3.61).

Forming the difference between system and model state equations gives

$$\dot{x}(t) - \dot{\hat{x}}(t) = A\,x(t) - \hat{A}^*\,\hat{x}(t) + (B-\hat{B}^*)\,u(t). \tag{3.7.32}$$

Adding $A\,\hat{x}(t) - A\,\hat{x}(t)$ to the right hand side of (3.7.32) and collecting terms yields

$$\dot{x}(t) - \dot{\hat{x}}(t) = A\,(x(t) - \hat{x}(t)) + (A-\hat{A}^*)\,\hat{x}(t) + (B-\hat{B}^*)\,u(t). \tag{3.7.33}$$

Using variation of parameters, the solution of (3.7.33) is

$$x(t) - \hat{x}(t) = e^{A(t-t_o)}\,(x_o-\hat{x}_o) + \int_{t_o}^{t} e^{A(t-\tau)}\,(A-\hat{A}^*)\,\hat{x}(\tau)\,d\tau + \int_{t_o}^{t} e^{A(t-\tau)}\,(B-\hat{B}^*)\,u(\tau)\,d\tau. \tag{3.7.34}$$

Since $\hat{x}_o = x_o$, (3.7.34) reduces to

$$x(t) - \hat{x}(t) = \int_{t_o}^{t} e^{A(t-\tau)} (A-\hat{A}^*)\hat{x}(\tau)\, d\tau + \int_{t_o}^{t} e^{A(t-\tau)} (B-\hat{B}^*)\, u(\tau)\, d\tau. \tag{3.7.35}$$

Taking the norm of $x(t) - \hat{x}(t)$ and generating the string of inequalities

$$\| x(t) - \hat{x}(t) \|_2 = \| \int_{t_o}^{t} e^{A(t-\tau)} (A-\hat{A}^*)\, \hat{x}(\tau)\, d\tau + \int_{t_o}^{t} e^{A(t-\tau)} (B-\hat{B}^*)\, u(\tau)\, d\tau \|_2 \tag{3.7.36}$$

$$\leq \| \int_{t_o}^{t} e^{A(t-\tau)} (A-\hat{A}^*)\, \hat{x}(\tau)\, d\tau \|_2 + \| \int_{t_o}^{t} e^{A(t-\tau)} (B-\hat{B}^*)\, u(\tau)\, d\tau \|_2 \tag{3.7.37}$$

by (3.7.5)

$$\leq \int_{t_o}^{t} \| e^{A(t-\tau)} \| \cdot \| A-\hat{A}^* \| \cdot \| \hat{x}(\tau) \|_2\, d\tau + \int_{t_o}^{t} \| e^{A(t-\tau)} \| \cdot \| B-\hat{B}^* \| \cdot \| u(\tau) \|_2\, d\tau \tag{3.7.38}$$

by (3.7.3), (3.7.6), (3.7.7)

$$\leq \| A-\hat{A}^* \| \int_{t_o}^{t} M_6\, e^{-k_2(t-\tau)} \cdot N_2\, d\tau + \| B-\hat{B}^* \| \int_{t_o}^{t} M_6\, e^{-k_2(t-\tau)} \cdot M_5\, d\tau, \tag{3.7.39}$$

where

$$\| e^{A(t-\tau)} \| \leq M_6\, e^{-k_2(t-\tau)}, \quad t \geq \tau \geq t_o \tag{3.7.40}$$

$$\| \hat{x}(\tau) \|_2 \leq N_2 < \infty, \quad \tau \geq t_o \tag{3.7.41}$$

$$\| u(\tau) \|_2 \leq M_5 < \infty, \quad \tau \geq t_o \tag{3.7.42}$$

follow from (3.7.14), (3.7.26), and the assumptions of asymptotic stability of the free uncertain system and the boundness of $u(t)$, $t \geq t_o$. Note that M_5, M_6, N_2 and k_2 are all positive finite numbers.

Integrating (3.7.39) and continuing the string of inequalities yields

$$\| x(t) - \hat{x}(t) \|_2 \leq \| A-\hat{A}^* \| \cdot f(t) + \| B-\hat{B}^* \| \cdot g(t) \tag{3.7.43}$$

where $f(\cdot)$ and $g(\cdot)$ are positive-valued monotonically-increasing functions of t given by

$$f(t) = \frac{M_6 N_2}{k_2} (1-e^{-k_2(t-t_o)}), \; t \geq t_o \tag{3.7.44}$$

$$g(t) = \frac{M_5 M_6}{k_2} (1-e^{-k_2(t-t_o)}), \; t \geq t_o \; . \tag{3.7.45}$$

Since $f(\cdot)$ and $g(\cdot)$ are positive-valued, monotonically-increasing functions, the maximum values for each are finite and can be written as

$$c_1 = \max_{t \varepsilon [t_o, t_f]} f(t) = f(t_f) < \infty \tag{3.7.46}$$

$$c_2 = \max_{t \varepsilon [t_o, t_f]} g(t) = g(t_f) < \infty \; . \tag{3.7.47}$$

Combining these results yields

$$\| x(t) - \hat{x}(t) \|_2 \leq c_1 \| A-\hat{A}^* \| + c_2 \| B-\hat{B}^* \| \; . \tag{3.7.48}$$

Evaluating $\| A-\hat{A}^* \|$ and $\| B-\hat{B}^* \|$ at $q = q^*$ gives the upper bound

$$\| x(t) - \hat{x}(t) \|_2 \leq c_1 \| A-\hat{A}^* \| + c_2 \| B-\hat{B}^* \|$$

$$\leq c_1 \| A^*-\hat{A}^* \| + c_2 \| B^*-\hat{B}^* \| \qquad (3.7.49)$$

where $A^* = A(q^*)$, $B^* = B(q^*)$, and q^* is given by (3.3.64). This is the desired result.

CHAPTER IV

MMAC APPROACH - THE OPTIMAL CONTROL PART OF THE PROBLEM

In the previous chapter, the problem of controlling a linear time-invariant parameter uncertain system was formulated. The two step, MMAC procedure was proposed as a solution methodology, and the optimal minmax model was derived from step one.

The problem that remains is to determine an admissible controller for the uncertain system based on the optimal minmax model and a specified performance index. This is the second step of the MMAC approach and it is considered in this chapter.

The application of step two requires the solution of example problems and several are presented. Comparisons are made with various techniques from the literature.

4.1 MMAC Approach - A Brief Review

The problem of controlling the linear time-invariant parameter uncertain system (3.1.1) is stated in Section 3.1. Since the performance index $J(u(\cdot),q)$ (3.1.2) is functionally dependent on the uncertainty vector q, the minimization of $J(u(\cdot),q)$ with respect to $u(\cdot)$ cannot be carried out unless q is fixed and known. Thus, the usual statement of the optimal control problem cannot be given.

In order to obtain a solution to the problem of controlling the uncertain system, the two step MMAC approach is proposed. In the first step, an optimal minmax model for the uncertain system is derived (Sections 3.2 and 3.3). The determination of the optimal model is viewed as a two-person game, where the designer selects the model matrices $(\hat{A},\hat{B})$ and his opponent, called nature, chooses the system matrices (A,B). The game cost is taken as the norm of the difference between system and model matrices, and the game is played with the designer minimizing the maximum value of cost.

The completion of step one results in the determination of a unique model, optimal in the minmax sense (cf. Chapter III, Definition 3.2.5). The model parameters are unique, completely known, and lie in the interior of the rectangular uncertainty set. For convenience, the optimal modeling solution is repeated here:

$$\dot{\hat{x}}(t) = \hat{A}(p^*)\,\hat{x}(t) + \hat{B}(p^*)\,u(t), \quad \hat{x}(t_o) = \hat{x}_o = x_o, \quad t \in [t_o, t_f] \tag{4.1.1}$$

$$\hat{A}(p^*) = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 \\ p_1^* & p_2^* & p_3^* & p_4^* & . & . & . & p_n^* \end{bmatrix} \tag{4.1.2}$$

$$\hat{B}(p^*) = [\,0 \;\; 0 \;\; 0 \;\; 0 \;\; . \;\; . \;\; . \;\; p_{n+1}^*\,]^T \tag{4.1.3}$$

$$p^* = [p_1^* \; p_2^* \; . \; . \; . \; p_{n+1}^*]^T, \; p_i^* = \frac{a_i+b_i}{2}, \; i = 1,2,..,n+1 \quad (4.1.4)$$

where $\hat{x}(t) \in R^n$ is the state,

$u(t) \in R^1$ is the control,

and the a_i's and b_i's describe the rectangular uncertainty set (cf. Section 3.2, equation (3.2.4)).

It is possible to write a performance index for the optimal model which is analogous in form to (3.1.2). This index is given by (3.1.3) and repeated here for convenience

$$\hat{J}(u(\cdot)) = h(\hat{x}(t_f)) + \int_{t_o}^{t_f} g(\hat{x}(t), u(t))\, dt, \quad (4.1.5)$$

where $h(\hat{x}(t_f))$ represents the terminal cost and the integral from t_o to t_f represents accumulated cost along the path.

In summary, step one of the MMAC approach effectively removes the uncertainty from the problem of controlling (3.1.1) by specifying a fixed and known model, from which a controller can be designed (in step two). This completes the review of the MMAC approach.

4.2 Control Problem Formulation - MMAC Step Two

After the minmax model has been found using step one of the MMAC approach, the remaining problem is to determine an admissible control function for the uncertain system (3.1.1) based on the optimal minmax model (4.1.1), (4.1.2), (4.1.3), (4.1.4) and the performance index (4.1.5). This control problem is stated as follows:

Control Problem: Find an admissible $u^*(\cdot) \in M$ which minimizes

$$\hat{J}(u(\cdot)) = h(\hat{x}(t_f)) + \int_{t_o}^{t_f} g(\hat{x}(t), u(t))\, dt \quad (4.2.1)$$

subject to the constraints

$$\dot{\hat{x}}(t) = \hat{A}(p^*)\,\hat{x}(t) + \hat{B}(p^*)\,u(t),\quad \hat{x}(t_o) = \hat{x}_o = x_o, \qquad (4.2.2)$$

where $\hat{A}(p^*)$ and $\hat{B}(p^*)$ are given by (4.1.2) and (4.1.3) and x_o is given by (3.1.1).

Definition 4.2.1: If a $u^*(\cdot) \in M$ exists which solves the Control Problem, then let it be called the MMAC controller for the uncertain system given by (3.1.1).

The solution of the Control Problem is step two of the MMAC approach. Since the minmax model state equation and initial condition vector (4.2.2) involve parameters that are completely specified, step two requires the solution of a deterministic optimal control problem. Hence, the necessary conditions for optimality developed by Pontryagin [P-3], and the interpretations and refinements given by Athans and Falb [A-2], Kirk [K-5], Lee and Markus [L-1] and others, may aid in deriving the MMAC controller.

Of course, there is no guarantee that a $u^*(\cdot) \in M$ exists which solves (4.2.1) and (4.2.2) (cf. Section 3.1). Nevertheless, since existence theorems are in rather short supply, it is claimed that the two step MMAC approach provides a solution to the problem of controlling an uncertain system, whenever a $u^*(\cdot) \in M$ exists.

To demonstrate the MMAC approach, specific choice of parameters for the uncertain system (3.1.1) (to be controlled) and performance index (3.1.2) must be made. In order to emphasize step two of the approach, the three examples presented in Section 3.4 are solved for the MMAC controller#. As a comparison, the examples are reworked using various

#The solutions of step one of the MMAC approach are presented in Section 3.4.

techniques from the literature.

Since these are specific examples, no general conclusions should be drawn from the comparisons. That is, it may be possible to construct other more pathological problems which exhibit drastically different results. The basic intent is to exhibit the MMAC approach as a method which works well in some cases.

4.3 Notation

In the examples which follow, a consistent notation is needed in order to differentiate between the various performance indices and optimal controllers. Therefore, the MMAC performance index and optimal MMAC controller are denoted by:

$$J_1(u_1(\cdot)) \quad \text{and} \quad u_1^*(\cdot) \in M,$$

the minmax cost performance index and optimal minmax cost controller are denoted by:

$$J_2(u_2(\cdot),q) \quad \text{and} \quad u_2^*(\cdot) \in M,$$

the optimal performance index and corresponding optimal controller for each $q \in Q$ are given by:

$$J_3(u_3(\cdot,q),q) \text{ and } u_3^*(\cdot,q) \in M,$$

and the minmax sensitivity performance index and optimal minmax sensitivity controller (using the method in [R-1]) are denoted by:

$$J_4(u_4(\cdot),q) \quad \text{and} \quad u_4^*(\cdot) \in M,$$

where $u_4^*(\cdot)$ minmaximizes the relative sensitivity

$$S(u_4(\cdot),q).$$

4.4 Example I

Consider the determination of a controller for the uncertain first order stationary linear system described by

$$\dot{x}(t) = q_1 \, x(t) + u(t) \tag{4.4.1}$$

with

$$q_1 \in [-2.5, -0.5], \tag{4.4.2}$$

time interval

$$t \in [0,\infty), \tag{4.4.3}$$

initial condition

$$x(0) = 2, \tag{4.4.4}$$

and the quadratic cost functional

$$J(u(\cdot), q_1) = \frac{1}{2} \int_0^\infty \{P\, x^2(t) + R\, u^2(t)\}\, dt \tag{4.4.5}$$

where P = 1 and R = 1.

Four controllers are derived for comparison:

1) MMAC controller $u_1^*(\cdot)$
2) minmax cost controller $u_2^*(\cdot)$
3) optimal controller $u_3^*(\cdot, q_1)$
4) minmax sensitivity controller $u_4^*(\cdot)$.

<u>MMAC Approach</u>

Using the optimal minmax model given in Example 3.4.2, the problem is to determine an admissible $u_1^*(\cdot) \in M$ that minimizes

$$J_1(u_1(\cdot)) = \frac{1}{2} \int_0^\infty \{P\, \hat{x}^2(t) + R\, u_1^2(t)\}\, dt \tag{4.4.6}$$

subject to the constraints

$$\dot{\hat{x}}(t) = -1.5\, \hat{x}(t) + u_1(t), \tag{4.4.7}$$

$$t \in [0,\infty), \tag{4.4.8}$$

$$\hat{x}(0) = 2, \tag{4.4.9}$$

where $u_1(t)$ is unconstrained and $P = 1$, $R = 1$.

This is a classical linear regulator problem with quadratic performance measure. From deterministic optimal control theory, a solution exists and is given by [K-5]:

$$u_1^*(t) = -R^{-1}\hat{b}\, k\, \hat{x}(t), \tag{4.4.10}$$

where k satisfies the scalar Riccati equation [K-5]

$$-2k\,\hat{a} - P + R^{-1}\hat{b}^2 k^2 = 0 \tag{4.4.11}$$

with $\hat{a} = -1.5$ and $\hat{b} = 1$.

Solving for k and substituting into (4.4.10) yields

$$u_1^*(t) = c_1\, \hat{x}(t) \tag{4.4.12}$$

where

$$c_1 \simeq -0.303. \tag{4.4.13}$$

Evaluating $J_1(\cdot)$ at $u_1^*(\cdot)$ gives

$$J_1(u_1^*(\cdot)) \simeq 0.6056. \tag{4.4.14}$$

Minmax Cost Approach

In the minmax cost approach, the problem is to find a $u_2^*(\cdot) \in M$ which minimizes

$$\max_{q_1 \in [-2.5,-0.5]} J_2(u_2(\cdot),q_1) = \max_{q_1 \in [-2.5,-0.5]} \frac{1}{2}\int_0^\infty \{P\, x^2(t) + R\, u_2^2(t)\}\, dt, \tag{4.4.15}$$

subject to the constraints given by (4.4.1), (4.4.2), (4.4.3), and (4.4.4). Also $P = 1$, $R = 1$.

As in the previous calculation, the optimal control $u_2^*(t)$ can be realized by linear feedback of the state variable

$$u_2{}^*(t) \;=\; c_1\, x(t). \tag{4.4.16}$$

By solving a scalar Riccati equation, it is found that c_1 must lie in the interval

$$.5 - \sqrt{1.25} \leq c_1 \leq 2.5 - \sqrt{7.25}. \tag{4.4.17}$$

Following the procedure given in [R-1], the performance index $J_2(\cdot)$ can be written as a function of the unknowns c_1 and q_1.

A simple calculation yields

$$J_2(c_1,q_1) \;=\; -\frac{(1 + c_1{}^2)}{q_1 + c_1}\;. \tag{4.4.18}$$

It is easy to see that the problem of determining a $u_2{}^*(\cdot) \,\varepsilon\, M$ has been transformed into a parameter optimization problem. That is, find the optimal $c_1{}^*$ which minimizes

$$\max_{q_1 \varepsilon [-2.5,-0.5]} \; -\frac{(1 + c_1{}^2)}{q_1 + c_1}\;, \tag{4.4.19}$$

where

$$c_1{}^* \;\varepsilon\; [.5 - \sqrt{1.25},\; 2.5 - \sqrt{7.5}]. \tag{4.4.20}$$

Solving this equivalent problem gives

$$c_1{}^* \;\simeq\; -0.618, \tag{4.4.21}$$

and therefore

$$u_2{}^*(t) \;=\; -0.618\, x(t). \tag{4.4.22}$$

Also the worst case value of q_1 is

$$q_1{}^* \;=\; -0.5 \tag{4.4.23}$$

and the corresponding cost is

$$J_2(u_2^*(\cdot),q_1^*) \quad \simeq \quad 1.2361. \qquad (4.4.24)$$

Using the minmax cost approach, (4.4.24) is the guaranteed upper bound on the cost. That is for any $q_1 \in [-2.5, -0.5]$,

$$J_2(u_2^*(\cdot),q_1) \quad \leq \quad J_2(u_2^*(\cdot),q_1^*) \quad \simeq \quad 1.2361. \qquad (4.4.25)$$

Optimal Control Approach

Consider the uncertain system described by (4.4.1) thru (4.4.4), and the performance index (4.4.5). Since q_1 is not known exactly, the usual statement of the optimal control problem cannot be given. Therefore, the determination of a controller requires the solution of a modified problem, and the previous two methods typify such approaches.

However, for comparison purposes only, it is possible to determine the optimal controller $u_3^*(\cdot\ q_1)$ for each fixed $q_1 \in [-2.5, -0.5]$. From deterministic optimal control theory, $u_3^*(t,q_1)$ is given by [K-5]

$$u_3^*(t,q_1) \quad = \quad -R^{-1} b\ k\ x(t), \qquad (4.4.26)$$

where k satisfies

$$-2k\ a - P + R^{-1}\ b^2\ k^2 \quad = \quad 0 \qquad (4.4.27)$$

and $a = q_1$, $b=1$ for $q_1 \in [-2.5, -0.5]$.

Solving for k and substituting into (4.4.26) gives

$$u_3^*(t,q_1) \quad = \quad -(q_1 + \sqrt{q_1^2+1}\ )\ x(t). \qquad (4.4.28)$$

For each $q_1 \in [-2.5,-0.5]$, the optimal cost can be written as

$$J_3(u_3^*(\cdot,q_1),q_1) = \frac{1 + (q_1 + \sqrt{q_1^2 + 1}\,)^2}{\sqrt{q_1^2 + 1}} , \qquad (4.4.29)$$

Minmax Sensitivity Approach

In the minmax sensitivity approach as reported by Rohrer and Sobral [R-1], the problem is to find a $u_4^*(\cdot)$ ε M which minimizes

$$\max_{q_1 \varepsilon [-2.5,-0.5]} S(u_4(\cdot),q_1) = \max_{q_1 \varepsilon [-2.5,-0.5]} \left\{\frac{J_4(u_4(\cdot),q_1) - J_4(u_4^o(\cdot),q_1)}{J_4(u_4^o(\cdot),q_1)}\right\} \qquad (4.4.30)$$

subject to the constraints given by (4.4.1), (4.4.2), (4.4.3) and (4.4.4).

$J_4(u_4(\cdot),q_1)$ is given as

$$J_4(u_4(\cdot),q_1) = \frac{1}{2}\int_0^{\infty} \{x^2(t) + u_4^2(t)\}\, dt \qquad (4.4.31)$$

and

$$J_4(u_4^o(\cdot),q_1) = \min_{u_4(\cdot)\varepsilon M} J_4(u_4(\cdot),q_1) \qquad (4.4.32)$$

for each fixed q_1 ε $[-2.5, -0.5]$.

As before, the optimal control $u_4^*(t)$ can be realized by linear feedback of the state. By solving the Riccati equation, it is found that

$$u_4^*(t) = c_1\, x(t) \qquad (4.4.33)$$

where

$$.5 - \sqrt{1.25} \le c_1 \le 2.5 - \sqrt{7.25} . \qquad (4.4.34)$$

Identical to the minmax cost approach, $J_4(\cdot)$ can be written as a function of c_1 and q_1

$$J_4(c_1,q_1) = \frac{-(1 + c_1^{\,2})}{q_1 + c_1} . \tag{4.4.35}$$

From (4.4.29), it is obvious that $J_4(u_4^{\,o}(\cdot),q_1)$ is given by

$$J_4(u_4^{\,o}(\cdot),q_1) = \frac{1 + (q_1 + \sqrt{q_1^{\,2}+1}\,)^2}{\sqrt{q_1^{\,2} + 1}} . \tag{4.4.36}$$

Combining (4.4.35) and (4.4.36) yields

$$S(c_1,q_1) = \frac{\dfrac{-(1+c_1^{\,2})}{q_1+c_1} - \dfrac{1 + (q_1 + \sqrt{q_1^{\,2} + 1}\,)^2}{\sqrt{q_1^{\,2}+1}}}{\dfrac{1 + (q_1 + \sqrt{q_1^{\,2}+1}\,)^2}{\sqrt{q_1^{\,2} + 1}}} , \tag{4.4.37}$$

where the relative sensitivity $S(\cdot)$ has been written as a function of c_1 and q_1.

The value of c_1 which minimizes

$$\max_{q_1 \varepsilon [-2.5,-0.5]} S(c_1,q_1)$$

is given by

$$c_1^{\,*} \simeq 0.405. \tag{4.4.38}$$

Therefore,

$$u_4^{\,*}(t) = -0.405\, x(t). \tag{4.4.39}$$

Comparison

In order to compare the performance of the uncertain system (4.4.1) to the different control laws, each of the four derived controllers is applied as the input and q_1 is allowed to vary from -2.5 to -0.5. Plots are made of the system performance (4.4.5)

$$J(u_i^*(.),q_1) = \frac{1}{2}\int_0^\infty (x^2(t) + u_i^2(t))\, dt \qquad (4.4.40)$$

versus q_1, where i = 1,2,3,4.

Figure 4.4.1 shows the performance of the MMAC controller $u_1^*(\cdot)$, the minmax cost controller $u_2^*(\cdot)$ and the optimal controller $u_3^*(\cdot,q_1)$ versus q_1. Figure 4.4.2 compares the performance of the MMAC controller $u_1^*(\cdot)$ and the minmax sensitivity controller $u_4^*(\cdot)$ versus q_1. Table 4.4.1 tabulates $J(u_i^*(\cdot), q_1)$ versus q_1 for discrete values of $q_1 \in [-2.5,-0.5]$.

As can be seen, the MMAC controller $u_1^*(\cdot)$ competes very well with the minmax cost and sensitivity controllers (the optimal controller $u_3^*(\cdot,q_1)$ is displayed as a reference). The MMAC controller deviates furthest from optimal when q_1 is in the range $-0.75 \leq q_1 \leq -0.50$. However, the MMAC controller outperforms the minmax cost controller $u_2^*(\cdot)$ for $-2.5 \leq q_1 \leq -1.0$, which is more than half of the uncertainty interval [-2.5, -0.5]. Also $u_1^*(\cdot)$ exhibits a better performance than the minmax sensitivity controller $u_4^*(\cdot)$ when $-2.5 \leq q \leq -1.5$.

Example I demonstrates that the MMAC approach may be useful in determining a controller for a linear system with uncertainty in the system matrix, and whose performance index is quadratic in form.

4.5 Example II

In this example, reported by Schmitendorf [S-2], the problem is to determine a controller for the uncertain system described by

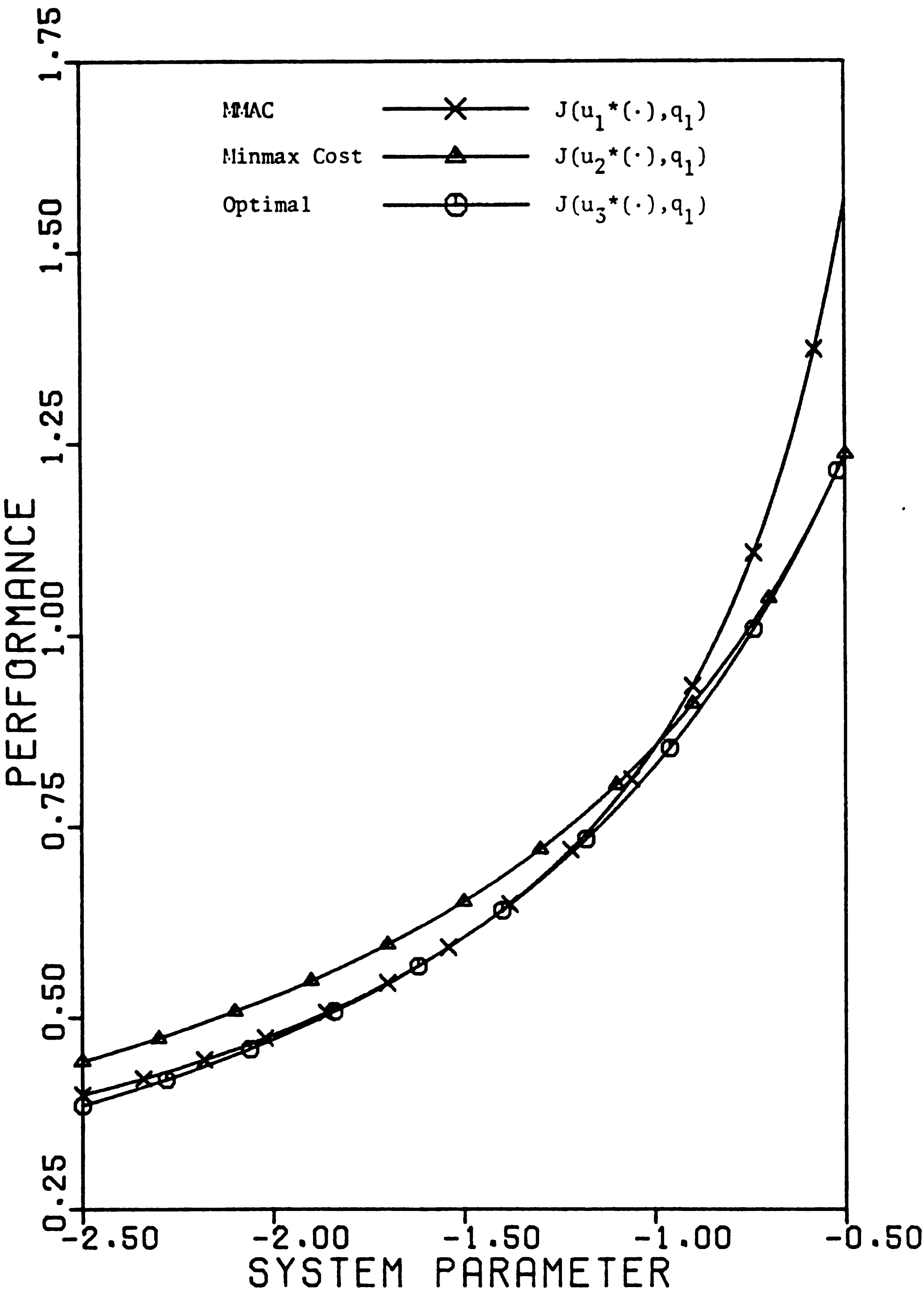

Figure 4.4.1 $J(u_i^*(\cdot),q_1)$ i = 1,2,3 versus q_1 For Example I.

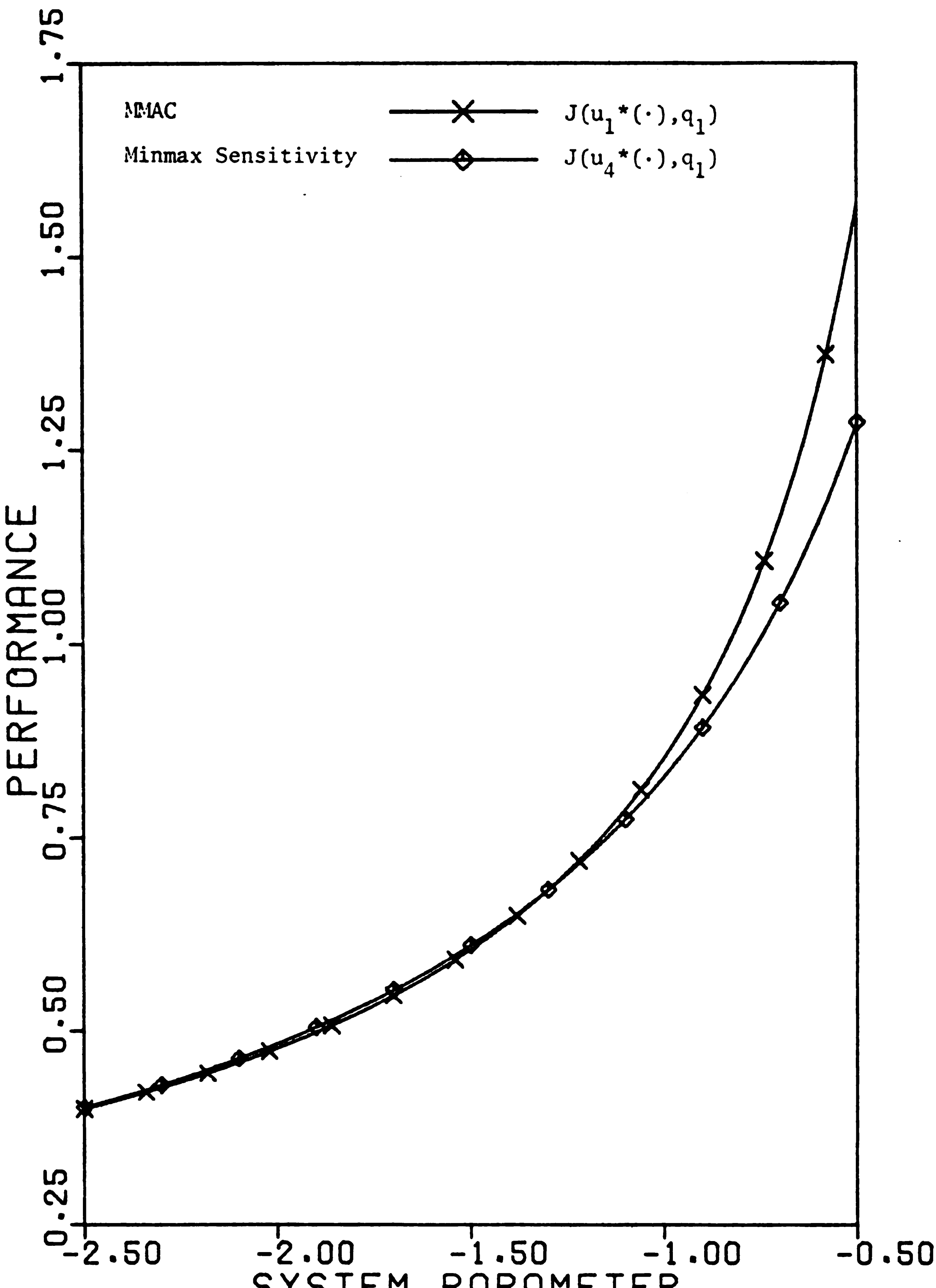

Figure 4.4.2 $J(u_i*(\cdot),q_1)$ i = 1,4 versus q_1

For Example I.

TABLE 4.4.1 $J(u_i^*(\cdot),q_1)$ i = 1,2,3,4 versus q_1

for Example I.

q_1	MMAC $J(u_1^*(\cdot),q_1)$	Minmax Cost $J(u_2^*(\cdot),q_1)$	Optimal $J(u_3^*(\cdot),q_1)$	Minmax Sensitivity $J(u_4^*(\cdot),q_1)$
-2.50	0.3993	0.4432	0.3852	0.4007
-2.25	0.4345	0.4819	0.4244	0.4385
-2.00	0.4779	0.5279	0.4721	0.4840
-1.75	0.5331	0.5836	0.5311	0.5402
-1.50	0.6056	0.6525	0.6056	0.6111
-1.25	0.7055	0.7398	0.7016	0.7034
-1.00	0.8529	0.8541	0.8284	0.8285
-0.75	1.0945	1.0102	1.0000	1.0078
-0.50	1.5691	1.2361	1.2361	1.2861

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ q_1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(t) = [x_1(t) \; x_2(t)]^T, \qquad (4.5.1)$$

with

$$q_1 \in [-4,-1], \qquad (4.5.2)$$

time interval

$$t \in [0,\pi], \qquad (4.5.3)$$

initial conditions

$$x(0) = [.5 \; 0]^T, \qquad (4.5.4)$$

controller constraint

$$|u(t)| \leq 1 \text{ for } t \in [0,\pi], \qquad (4.5.5)$$

and the non-quadratic performance measure

$$J(u(\cdot),q_1) = -x_1(\pi). \qquad (4.5.6)$$

Three controllers are derived for comparison:

1) MMAC controller $u_1^*(\cdot)$
2) Minmax cost controller $u_2^*(\cdot)$ (from [S-2])
3) Optimal controller $u_3^*(\cdot,q_1)$.

MMAC Approach

Using the optimal minmax model given in Example 3.4.1 the control problem is to determine a $u_1^*(\cdot)$ which satisfies (4.5.5) and minimizes

$$J_1(u_1(\cdot)) = -\hat{x}_1(\pi). \qquad (4.5.7)$$

The minmax model state equation is

$$\dot{\hat{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2.5 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t) \qquad (4.5.8)$$

where the initial conditions and time interval are the same as (4.5.4) and (4.5.3).

Writing the Hamiltonian [K-5]

$$H(\hat{x}(t), u_1(t), \lambda(t)) = \lambda_1(t)\, \hat{x}_2(t) - 2.5\, \lambda_2(t)\, \hat{x}_1(t) + \lambda_2(t)\, u_1(t), \qquad (4.5.9)$$

where $\lambda_1(t)$ and $\lambda_2(t)$ denote the costate, necessary conditions for optimality are [K-5]:

$$\dot{\lambda}_1(t) = 2.5\, \lambda_2(t), \quad \lambda_1(\pi) = -1 \qquad (4.5.10)$$

$$\dot{\lambda}_2(t) = -\lambda_1(t), \quad \lambda_2(\pi) = 0 \qquad (4.5.11)$$

and (4.5.8) with initial conditions (4.5.4).

Since $H(\cdot)$ is linear in the control, the minimum principle implies that [K-5], [B-5]

$$u_1^*(t) = \begin{cases} -1 & \lambda_2^*(t) > 0 \\ +1 & \lambda_2^*(t) < 0 \\ \text{undetermined} & \lambda_2^*(t) = 0 \end{cases} \qquad (4.5.12)$$

After straightforward but tedious calculation $\lambda_2^*(t)$ is given by:

$$\lambda_2^*(t) = c_1^* \cos(\sqrt{2.5}\, t) + c_2^* \sin(\sqrt{2.5}\, t), \quad 0 \le t \le \pi \qquad (4.5.13)$$

where

$$c_1^* = \frac{-\sin(\sqrt{2.5}\, \pi)}{\sqrt{2.5}} \simeq .6120, \qquad (4.5.14)$$

and

$$c_2^* \;=\; \frac{\cos(\sqrt{2.5}\,\pi)}{\sqrt{2.5}} \;\simeq\; .1595 \quad . \tag{4.5.15}$$

The switching time t_1^* is found by equating (4.5.13) to zero:

$$t_1^* \;=\; \frac{(\sqrt{2.5}\,-\,1)\pi}{\sqrt{2.5}} \;\simeq\; 1.1547 \cdot \tag{4.5.16}$$

Combining (4.5.12) thru (4.5.16) yields

$$u_1^*(t) = \begin{cases} -1 & 0 \leq t < t_1^* \\ +1 & t_1^* < t \leq \pi \end{cases} . \tag{4.5.17}$$

Evaluating $J_1(u_1(\cdot))$ at $u_1^*(.)$ gives

$$J_1(u_1^*(\cdot)) \;=\; -\hat{x}_1^*(\pi) \;\simeq\; -1.4269. \tag{4.5.18}$$

Minmax Cost Approach

In the minmax cost approach, the problem is to find an admissible $u_2^*(\cdot)$ which satisfies (4.5.5) and minimizes

$$\max_{q_1\varepsilon[-4,-1]} J_2(u_2(.),q_1) \;=\; \max_{q_1\varepsilon[-4,-1]} -x_1(\pi), \tag{4.5.19}$$

subject to the constraints

$$\dot{x}(t) \;=\; \begin{bmatrix} 0 & 1 \\ q_1 & 0 \end{bmatrix} x(t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2(t), \tag{4.5.20}$$

$$q_1 \;\varepsilon\; [-4,\,-1], \quad |u(t)| \;\leq\; 1, \tag{4.5.21}$$

where the initial conditions and time interval are the same as (4.5.4) and (4.5.3).

Schmitendorf solves this example in [S-2] by following the algorithm which he develops in the same report. A Hamiltonian function similar in form to (4.5.9) is written, except that there are twice as many state and costate equations. By invoking the minimum principle, the minimizing $u_2^*(\cdot)$ again has the form of a bang-bang controller.

From [S-2], $u_2^*(t)$ is given by

$$u_2^*(t) = \begin{cases} -1 & 0 \leq t < t_2^* \\ +1 & t_2^* < t \leq \pi \end{cases} \tag{4.5.22}$$

where

$$\cos t_2^* = \sqrt{3} - 1. \tag{4.5.23}$$

The worst case values of q_1 are

$$q_1^* = -1 \quad \text{or} \quad -4, \tag{4.5.24}$$

and the corresponding cost is

$$J_2(u_2^*(\cdot),q_1^*) = -x_1^*(\pi) \simeq -.9641 \ . \tag{4.5.25}$$

Optimal Control Approach

Consider the uncertain system described in Example II by (4.5.1) thru (4.5.6). As in Example I, it is possible to compute the optimal controller $u_3^*(\cdot,q_1)$ for each fixed value of $q_1 \in [-4,-1]$.

Writing the Hamiltonian function and proceeding as in equations (4.5.9) thru (4.5.12), it becomes obvious that for each $q_1 \in [-4,-1]$:

$$u_3^*(t,q_1) = \begin{cases} -1 & \lambda_2^*(t,q_1) > 0 \\ +1 & \lambda_2^*(t,q_1) < 0 \\ \text{undetermined} & \lambda_2^*(t,q_1) = 0 \end{cases} \tag{4.5.26}$$

where

$$\lambda_2^*(t,q_1) = c_1^*(q_1)\cdot \cos(\sqrt{-q_1}\,t) + c_2^*(q_1)\cdot \sin(\sqrt{-q_1}\,t), \tag{4.5.27}$$

$$c_1^*(q_1) = \frac{-\sin(\sqrt{-q_1}\,\pi)}{\sqrt{-q_1}} \tag{4.5.28}$$

and

$$c_2^*(q_1) = \frac{\cos(\sqrt{-q_1}\,\pi)}{\sqrt{-q_1}} . \tag{4.5.29}$$

The switching time $t_3^*(q_1)$ is found by equating (4.5.27) to zero:

$$t_3^*(q_1) = \frac{(\sqrt{-q_1}-1)\cdot\pi}{\sqrt{-q_1}} . \tag{4.5.30}$$

Therefore,

$$u_3^*(t,q_1) = \begin{cases} -1 & 0 \le t < t_3^*(q_1) \\ +1 & t_3^*(q_1) < t \le \pi, \end{cases}$$

where $t_3^*(q_1)$ is given by (4.5.30).

After much tedious work, the optimal cost for each $q_1 \in [-4,-1]$ can be written as:

$$J(u_3^*(\cdot,q_1),q_1) = -x_1^*(\pi) = -d_2(q_1)\cdot\cos(\sqrt{-q_1}\cdot\pi) - d_3(q_1)\cdot\sin(\sqrt{-q_1}\cdot\pi) + \frac{1}{q_1} \tag{4.5.31}$$

where

$$d_2(q_1) = \cos(\sqrt{-q_1}\cdot t_3^*(q_1))\cdot [d_1(q_1)\cdot \cos(\sqrt{-q_1}\cdot t_3^*(q_1)) + \frac{2}{q_1}]$$
$$+ d_1(q_1)\cdot \sin^2(\sqrt{-q_1}\cdot t_3^*(q_1)), \qquad (4.5.32)$$

$$d_3(q_1) = [d_1(q_1)\cdot\cos(\sqrt{-q_1}\cdot t_3^*(q_1)) + \frac{2}{q_1}]\cdot \sin(\sqrt{-q_1}\cdot t_3^*(q_1))$$
$$- d_1(q_1)\cdot \sin(\sqrt{-q_1}\cdot t_3^*(q_1))\cdot \cos(\sqrt{-q_1}\cdot t_3^*(q_1)),$$
$$(4.5.33)$$

$$d_1(q_1) = \frac{1}{2} - \frac{1}{q_1}, \qquad (4.5.34)$$

and $t_3^*(q_1)$ is given by (4.5.30).

Comparison

Comparison is made by applying each of the three derived controllers as the input to the uncertain system (4.5.1), while q_1 varies from -4 to -1. Figure 4.5.1 compares the performance of the MMAC controller $u_1^*(\cdot)$ with the minmax cost and optimal controllers $u_2^*(\cdot)$ and $u_3^*(\cdot,q_1)$ for $-4 \leq q_1 \leq -1$. Table 4.5.1 gives the value of $J(u_i^*(\cdot),q_1)$ versus q_1 for selected $q_1 \in [-4,-1]$.

Note that the system performance (4.5.6) $J(u_i^*(\cdot),q_i)$ is given as

$$J(u_i^*(\cdot),q_1) = -x_1(\pi) \qquad (4.5.35)$$

for $i = 1,2,3$.

As can be seen, the MMAC controller $u_1^*(\cdot)$ outperforms the minmax cost controller $u_2^*(\cdot)$ for $-4 \leq q_1 \leq -2.25$, which is over half of the uncertainty interval [-4,-1]. In this same region, the performance with

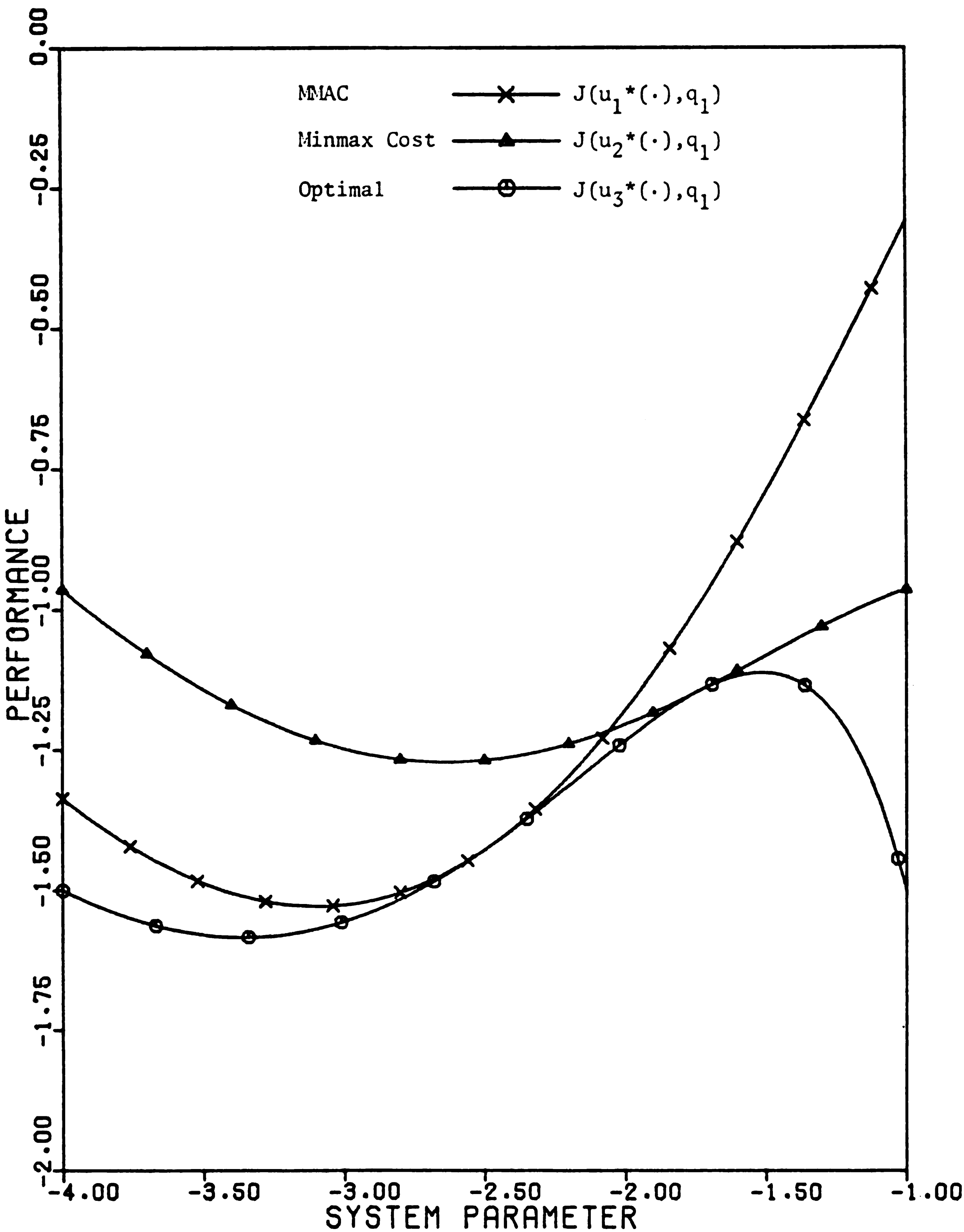

Figure 4.5.1 $J(u_i^*(\cdot),q_1)$ $i = 1,2,3$ versus q_1 For Example II.

TABLE 4.5.1 $J(u_i^*(\cdot),q_1)$ $i = 1,2,3$, versus q_1 for Example II.

q_1	MMAC $J(u_1^*(\cdot),q_1)$	Minmax Cost $J(u_2^*(\cdot),q_1)$	Optimal $J(u_3^*(\cdot),q_1)$
-4.00	-1.3366	-0.9641	-1.5000
-3.75	-1.4240	-1.0608	-1.5515
-3.50	-1.4870	-1.1419	-1.5790
-3.25	-1.5219	-1.2048	-1.5806
-3.00	-1.5254	-1.2477	-1.5551
-2.75	-1.4944	-1.2690	-1.5030
-2.50	-1.4269	-1.2683	-1.4269
-2.25	-1.3218	-1.2462	-1.3333
-2.00	-1.1792	-1.2048	-1.2337
-1.75	-1.0012	-1.1481	-1.1484
-1.50	-0.7917	-1.0824	-1.1123
-1.25	-0.5575	-1.0170	-1.1884
-1.00	-0.3084	-0.9641	-1.5000

the MMAC controller is very close to optimal. However, for $-2 \leq q_1 \leq -1$, the minmax cost controller $u_2^*(\cdot)$ is far superior to $u_1^*(\cdot)$.

Example II demonstrates that the MMAC approach may be useful in determining controls for uncertain systems with non-quadratic cost criteria.

4.6 Example III

Consider the determination of a controller for the uncertain first order linear system described by

$$\dot{x}(t) = -2x(t) + q_1 u(t) \tag{4.6.1}$$

with

$$q_1 \varepsilon [1,5], \tag{4.6.2}$$

time interval

$$t \varepsilon [0,\infty), \tag{4.6.3}$$

initial condition

$$x(0) = 5, \tag{4.6.4}$$

and the quadratic cost functional

$$J(u(\cdot),q_1) = \frac{1}{2}\int_0^\infty \{x^2(t) + u^2(t)\}\, dt \tag{4.6.5}$$

with $u(t)$ unconstrained for $0 \leq t < \infty$.

In this problem, the parameter uncertainty is in the input matrix (i.e., $B(q) = q_1$). Since none of the authors cited have considered such an example, only two controllers are derived for comparison:

1) MMAC controller $u_1^*(\cdot)$

2) Optimal controller $u_3^*(\cdot,q_1)$.

MMAC Approach

The optimal minmax model found in Example 3.4.3 is

$$\dot{\hat{x}}(t) = -2\,\hat{x}(t) + 3u_1(t),\ t \in [0,\infty),\ \hat{x}(0) = 5 \qquad (4.6.6)$$

and the corresponding model performance index is

$$J_1(u_1(\cdot)) = \frac{1}{2}\int_0^\infty \{\hat{x}^2(t) + u_1^2(t)\}\,dt. \qquad (4.6.7)$$

The control problem is to determine a $u_1^*(\cdot)$, where $u_1^*(t)$ is unconstrained for $0 \leq t < \infty$, that minimizes (4.6.7) subject to the constraints given in (4.6.6). This is a scalar linear regulator problem with quadratic cost index. A solution exists and is given by [K-5]

$$u_1^*(t) = -R^{-1}\,\hat{b}\,k\,\hat{x}(t) = -\,3k\,\hat{x}(t), \qquad (4.6.8)$$

where k satisfies the scalar Riccati equation [K-5]

$$4k - 1 + 9k^2 = 0 \qquad (4.6.9)$$

Solving for k and substituting into (4.6.8) yields

$$u_1^*(t) = \frac{(2-\sqrt{13})}{3}\,\hat{x}(t) \simeq -0.5352\,\hat{x}(t). \qquad (4.6.10)$$

Evaluating $J_1(\cdot)$ at $u_1^*(\cdot)$ gives

$$J_1(u_1^*(\cdot)) \simeq 2.2299\ . \qquad (4.6.11)$$

Optimal Control Approach

As in the previous two examples, it is possible to compute the optimal controller $u_3^*(\cdot,q_1)$ for each $q_1 \in [1,5]$. From deterministic control theory, $u_3^*(t,q_1)$ is given by [K-5]

$$u_3^*(t,q_1) = -R^{-1}\,b\,k\,x(t) = -q_1 k\,x(t), \qquad (4.6.12)$$

where k satisfies

$$4k - 1 + q_1^2 k^2 = 0. \qquad (4.6.13)$$

Solving for k and substituting into (4.6.12) gives

$$u_3^*(t,q_1) = \frac{(2 - \sqrt{4+q_1^2})}{q_1} \cdot x(t) \quad . \qquad (4.6.14)$$

For each $q_1 \varepsilon [1,5]$, the optimal cost can be written as

$$J_3(u_3^*(\cdot,q_1),q_1) = \frac{[1 + (c_1^*(q_1))^2] \cdot 25}{4 \cdot [2 - q_1 \cdot c_1^*(q_1)]} , \qquad (4.6.15)$$

where

$$c_1^*(q_1) = \frac{2 - \sqrt{4+q_1^2}}{q_1} \qquad (4.6.16)$$

Comparison

Comparison is made by applying $u_1^*(\cdot)$ and $u_3^*(\cdot,q_1)$ as the input to the uncertain system (4.6.1), while q_1 varies from 1 to 5. Figure 4.6.1 compares the performance of the MMAC controller $u_1^*(\cdot)$ with the optimal controller $u_3^*(\cdot,q_1)$ for $1 \leq q_1 \leq 5$. Table 4.6.1 lists values of $J(u_i^*(\cdot),q_1)$ versus q_1 for discrete $q_1 \varepsilon [1,5]$.

Note that the system performance (4.6.5) is given as

$$J(u_i^*(\cdot),q_1) = \frac{1}{2} \int_0^\infty \{x^2(t) + u_i^2(t)\} \, dt , \qquad (4.6.17)$$

for i = 1 and 3.

The MMAC controller gives near optimal performance for $2 \leq q_1 \leq 4$. For $1 \leq q_1 < 2$ and $4 < q_1 \leq 5$, the MMAC performance deviates less than 4.5% from optimal.

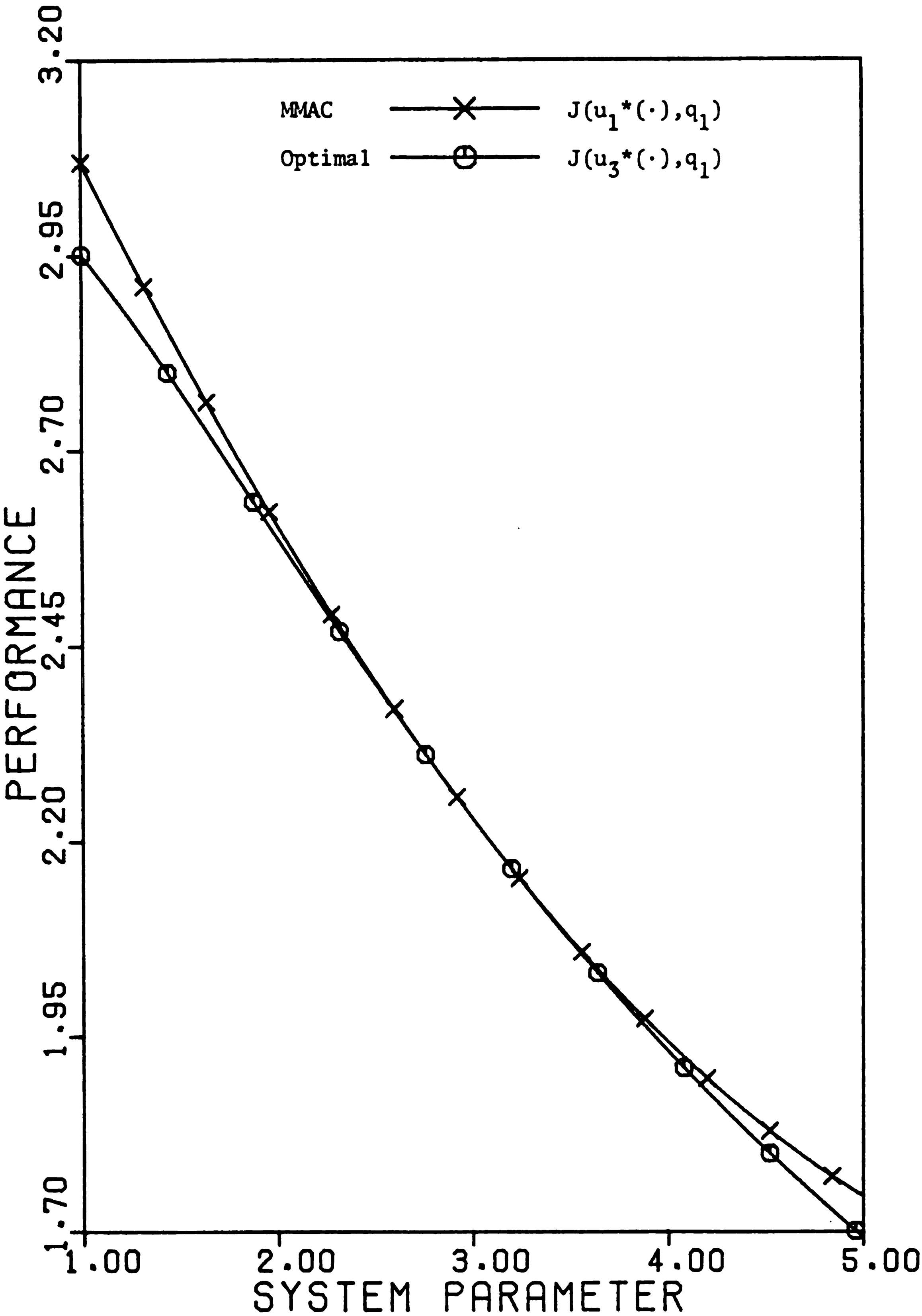

Figure 4.6.1 $J(u_i*(\cdot),q_1)$ $i = 1,3$ versus q_1 for Example III.

TABLE 4.6.1 $J(u_i^*(\cdot),q_1)$ $i = 1,3$ versus q_1 for Example III.

q_1	MMAC $J(u_1^*(\cdot),q_1)$	Optimal $J(u_3^*(\cdot),q_1)$
1.0	3.0691	2.9508
1.5	2.8261	2.7778
2.0	2.6052	2.5888
2.5	2.4065	2.4031
3.0	2.2299	2.2299
3.5	2.0755	2.0726
4.0	1.9432	1.9314
4.5	1.8331	1.8052
5.0	1.7451	1.6926

Example III demonstrates that the MMAC approach may be useful in determining controls for systems with uncertainty in the input matrix.

CHAPTER V

SUMMARY AND RECOMMENDATIONS

5.1 Summary

This dissertation is concerned with the determination of a controller for a linear time-invariant parameter uncertain system. Due to the uncertainty in the system state equation the usual statement of the optimal control problem cannot be given. To circumvent this difficulty, a two-step, minmax modeling and control (MMAC) approach is proposed as a new method for selecting a controller.

In the first step, an optimal minmax model for the uncertain system is derived. The determination of the optimal model is viewed as a two-person game of design against nature. The game cost is taken as the norm of the difference between system and model matrices, and the game is played with the designer minimizing the maximum value of cost. When the true parameter values are known to lie within a bounded rectangular set, it is shown that the optimal model exists and is unique. The optimizing model parameters define the center point of the uncertainty set.

The minmax criterion by which the optimal model is chosen yields the smallest guaranteed upper bound on the cost. It is therefore appropriate to describe the minmax model as a guaranteed cost model. The mismatch between system and model matrices is shown to induce an upper bound on the error between uncertain system and optimal model trajectories, for the case where the origin is an asymptotically stable equilibrium point for the uncertain system with arbitrary $q \in Q$. It is also shown that the optimal minmax model is completely controllable provided that the model parameter in the input matrix is restricted from assuming a zero value.

In the second step of the MMAC approach, a controller for the uncertain system is sought, based on the minmax model state equation and a model performance index analogous in form to the given system performance measure. This controller is denoted as the MMAC controller for the uncertain system. It is claimed that the two step MMAC approach provides a solution to the problem of controlling the uncertain system, whenever an admissible MMAC controller exists.

The application of the MMAC approach requires the solution of example problems and several are presented. Comparisons are made with various techniques from the literature. Although these examples are specific, they show that the MMAC approach competes well with the opposing techniques.

In summary, this dissertation proposes a new and improved approach to the determination of a controller for an uncertain system. The MMAC approach may be useful in deriving controls for linear time-invariant systems with uncertain parameters in both system and input matrices, and whose performance measure may be either quadratic or non-quadratic.

5.2 Recommendations for Further Research

There are a number of topics for further research based on this work, for example:

(1) The completely analytic solution of the optimal modeling problem relied heavily on the geometry of the uncertainty set, i.e., Q was a known and bounded rectangle. If this restriction was relaxed and Q was simply given as a compact, convex set, then the optimal model parameters and the corresponding worst case system parameters might be attainable by employing a nonlinear programming algorithm.

(2) The problem of controlling a linear time-invariant parameter uncertain system where the system and input matrices, A(q) and B(q), are not restricted to companion form should be investigated. This may require the definition of a more suitable matrix norm for computation.

(3) As noted in Section 3.6, there may be some $q \in Q$ for which the stability properties of the uncertain system and minmax model are different. This difference may prove to be disastrous when implementing step two of the MMAC approach, especially over an infinite time interval, and should be studied.

(4) An interesting extension to this work would be the design of an MMAC controller based on a reduced order minmax model, determined from observations of the uncertain system states.

(5) By allowing the uncertain system matrices to be time varying, the MMAC approach might be extended to a broader class of problems. In this case, the modeling problem would be to determine stationary model matrices which minimize the maximum value of game cost. The game cost might be written as a Chebychev norm of matrix differences.

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