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INVOLUTIONS OF 3-MANIFOLDS WITH A 2-DIMENSIONAL FIXED POINT SET COMPONENT

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ABSTRACT

INVOLUTIONS OF 3-MANIFOLDS WITH A
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By

Donald K. Showers

Let T_1 and T_2 be involutions of the 3-manifolds M_1 and M_2 respectively and assume T_1 and T_2 have 2-dimensional fixed point set components F_1 and F_2. Taking the connected sum of F_1 and F_2 in the connected sum of M_1 and M_2 gives a manifold $M_1 \# M_2$ with an induced involution $T_1 \# T_2$ and a fixed point set component $F_1 \# F_2$. The question studied in Chapter I of this thesis is the converse of this construction.

It is found that under certain conditions it is possible to detect that a manifold M with involution T can be constructed as a connected sum of two other manifolds with involution by finding a non-zero kernel of the inclusion map of a 2-dimensional fixed point set component of T in homotopy. Thus the inclusion of 2-dimensional fixed point set components into an irreducible manifold is a monomorphism in homotopy. This allows for classification of these involutions of $S^1 \times S^1 \times S^1$ and of an investigation of $S^1 \times K$ where K is the "Klein bottle".

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INTRODUCTION

The techniques involved in dividing mathematical objects into basic components are of fundamental interest in mathematics. In this thesis the objects are piecewise linear 3-manifolds with an involution having a 2-dimensional fixed point set component. The result of a general investigation indicate that under suitable conditions one can detect whether an involution can be considered as induced from the connected sum of two involutions by the kernel of the inclusion of a 2-dimensional component of the fixed point set in homotopy. The results are applied to $S^1 \times S^1 \times S^1$ and to $S^1 \times K$ where K is the "Klein bottle". In addition, uniqueness questions are answered.

CHAPTER I

BASIC DEFINITIONS AND FIRST THEOREM

In this work all topological manifolds are assumed to have a piecewise linear structure and all continuous functions are assumed to be piecewise linear functions unless stated differently.

Let M^n denote a closed manifold of dimension n. An involution on M^n is a continuous function T not equal to the identity function which maps M^n to M^n such that $T \circ T$ is the identity function on M^n. The set $F(T)$ will denote the set of fixed points of T, $F(T)$ is given the subspace topology which makes the components of $F(T)$ manifolds without boundary. A n-sphere is said to be the trivial n-manifold since $M^n = M^n \# S^n$. A manifold M^n is said to be prime if it cannot be written as a connected sum of two non-trivial manifolds. J. Milnor has shown [3] that if $n = 3$ every closed orientable manifold is the connected sum of prime manifolds and every such decomposition is unique up to a permutation of factors. Among prime 3-manifolds $S^1 \times S^2$ is unique in that it contains an embedded 2-sphere which does not bound a 3-cell, the property that every embedded 2-sphere bounded a 3-cell is called irreducible.

Let M_1 and M_2 be two 3-manifolds on which involutions T_1 and T_2 have 2-dimensional fixed point set components F_1 and F_2 respectively. By removing "small" invariant 3-cells c_1 and c_2 such that $\partial c_i \cap F_i \simeq S^1$ $i = 1,2$ and noting that $T_1 | \partial c_1$ is isotopic to $T_2 | \partial c_2$ through any orientation reversing identification of ∂c_1 with ∂c_2 it is possible to induce an involution on the connected sum of M_1 and M_2 which has a fixed point set component $F_1 \# F_2$. This construction of an involution is denoted by $T_1 \#_{F_1 \# F_2} T_2$ since it depends on the choice of F_1 and F_2. Note

$$F(T_1 \#_{F_1 \# F_2} T_2) = (F(T_1)-F_1) \cup (F(T_2)-F_2) \cup (F_1 \# F_2).$$

The purpose of this chapter is to show that under reasonable conditions involutions on orientable manifolds induced by the above constructions, can be detected by a non-trivial kernel of $i_* \pi_1(F_i) \to \pi_1(M^3)$ where F_i is any 2-dimensional component of $F(T)$.

<u>Theorem 1</u>. Let M be an orientable closed 3-manifold with T an involution on M. Suppose $S^1 \times S^2$ is not a connected sum summand of M and assume further that there is a 2-dimensional component $F_c \neq$ projective 2-space

such that $i_*\pi_1(F_c) \to \pi_1(M)$ is not a monomorphism, then M is a connected sum of non-trivial manifolds and T is induced.

Proof: The proof is divided into several cases:

Case A: F_c is two sided;

Case 1: F_c is two sided and separates M.

In this case F_c separates M into two components M_1 and M_2 with $\partial M_1 \simeq F_c \simeq \partial M_2$. By the VanKampen theorem, $i_*\pi_1(F_c) \to \pi_1(M_1)$ $i = 1,2$ is not a monomorphism. Now since $T|_{M_1} : M_1 \to M_2$ there can be no other fixed points and one has that T is a reflection across F_c. However, using Stallings loop theorem [4] and Dehn's lemma there exists a disc D in M_1 and TD in M_2 such that $[\partial D] \neq e \in \pi_1(F_c)$. Adjoining M_1 to M_2 along F_c gives $D \cup TD$ an invariant 2-sphere S which is non-bounding and thus since there is no $S^1 \times S^2$ summand S separates M into two non-trivial manifolds $\bar{M}_1$ and $\bar{M}_2$. "Gluing" a 3-cell to $\bar{M}_1$ along S and extending the involution by extending the reflection on S and doing the same to $\bar{M}_2$ yields two non-trivial manifolds which induce by connected summing the manifold M and involution T.

Case 2: F_c is two sided but does not separate.

By Stallings [8] there is a loop α in F_c which bounds a disc D such that $F_c \cap D = \alpha$. By general position arguments assume $\dot{D} \cap T\dot{D} = \dot{\cup} S_i$, S_i an embedded 1-sphere termed an intersection circle. It will now be shown how D can be modified to give a disc D' such that $D' \cap F_c = \alpha$ and $D' \cap TD' = \alpha$. Let B be an intermost circle in D of $\dot{D} \cap T\dot{D}$.

Case A: $TB \cap B = \emptyset$.

Near TB choose a "small" regular neighborhood N of TB in D. The outer rim gives a curve B'. There is a disc $\bar{D}$ in M "close" to that bounded by B in D considered embedded in M such that $T\partial\bar{D} = B'$. Form the disc D' by removing the interior of the disc in D which is bounded by B' and attaching $T\bar{D}$. Then $\dot{D}' \cap T\dot{D}'$ has one less intersection circle than $\dot{D} \cap T\dot{D}$.

Case B: $TB \cap B = B$.

Consider a small annulus N of B in D considered in the zero section of a regular neighborhood of D, embedded in R^3. TN either lies on one side of N or it does not.

Subcase 1: TN lies on one side of N. Then on the other side one can put in an annulus $\tilde{N}$ sufficiently close to N such that $\tilde{N} \cap N = \partial\tilde{N}$ and B is in the annulus $\bar{N}$ in N bounded by $\partial\tilde{N}$. Removing the interior of $\bar{N}$ and "gluing" in $\tilde{N}$ gives an annulus $N' \ni TN' \cap N' = \emptyset$. Thus B is removed as an intersection circle.

Subcase 2: TN lies on both sides of N. Then change the disc by using TD_B in place of D_β when D_β is the disk in D bounded by B. Now use subcase 1.

Thus it is possible to modify the disc D to eliminate all intersection circles, so assume D has no intersection circles. Note that $D \cup TD$ is now an invariant sphere and that it does not bound a 3-cell since ∂D is not null homotopic in F_c. Proceed as in case 1 to show the involution is induced.

Case B: F_c is one sided.

Subcase 1: F_c is the only 2-dimensional fixed point set component of $F(T)$. Opening M along F_c gives a manifold $\tilde{M}$ with boundary $\tilde{N}$ which double covers F_c. The hypotheses that $F_c \not\# P_2$ implies that there are no elements of order 2 in $\pi_1(F_c)$, so by the Van Kampen theorem applied to an invariant regular neighborhood N and a fattened complement, one has $i_*\pi_1(\partial N) \to \pi_1(M-\dot{N})$ a non-monomorphism. Thus $i_*\pi_1(\tilde{N}) \to \pi_1(\tilde{M})$ is not a monomorphism. Removing any isolated fixed points if necessary, one has an induced involution $T : \tilde{M} \to \tilde{M}$ which is free. The orbit space is a manifold OM with boundary OB and $i_*\pi_1(OB) \xrightarrow{i_*} \pi_1(OM)$ is not a monomorphism. Thus one has a disc D' with boundary $\partial D'$ which is not null homotopic in OB. Lifting to $M-\dot{N}$ gives two discs D and TD. Using the techniques of case 2 above it can be assumed that $D \cap TD = \emptyset$. Now "sew" up M along $\tilde{N}$ to give M and a non-boundary invariant 2-sphere. Proceed again as in case 1 to show that involution is induced.

Subcase 2: F_c is not the only two dimensional fixed point set. Let F_K $K = 1,\ldots,n$ be the 2-sided 2-dimensional fixed point sets. By case 1 F_i does not separate. Open M along F_1 to obtain ${}_1M_0$ $\partial_1 M_0 = {}_0F_1 \dot{\cup} {}_0F_2$ where ${}_0F_1 \simeq F_1 \simeq {}_0F_2$. Form ${}_1M_0 \times Z$ and let ${}_1\tilde{M} = {}_1M_0 \times Z/(x,i) \not\sim_{x \in {}_0F_1} (\tilde{T}x, i+1)$ where $\tilde{T}$ is the induced involution on ${}_1M_0$. $\tilde{M}$ covers M and has an involution ${}_1\tilde{T}$ extending $\tilde{T}$ and covering the involution T on M. ${}_1\tilde{T}$ has one less 2-dimensional fixed point set component and a neighborhood of F_c can be lifted homeomorphically to a neighborhood of the fixed point set covering F_c also denoted by F_c. Note $i_*\pi_1(F_c) \to \pi_1\tilde{M}$ is not a monomorphism. One could continue this construction removing each F_c as a 2-sided fixed point set component of an involution on an open manifold $\tilde{M}$ with an involution $\tilde{T}$ covering M and T such that a neighborhood of the one sided fixed point set components lifts homeomorphically the neighborhood of the one sided fixed point component sets of $\tilde{M}$. Now open $\tilde{M}$ along the one sided components F_d, $d = 1,\ldots,\ell$ $F_d \neq F_c$. This gives a manifold $\tilde{M}_d$, $\partial\tilde{M}_d \approx \dot{\cup}\, \partial NF_d$ where NF_d is a regular neighborhood of F_d. Consider $\bar{M} = \tilde{M}_d \times Z_2/(x \times 0) \nsim (\tilde{T}(x) \times 1)$ where $x \in \dot{\cup}\, \partial NF_d$, $\bar{M}$ covers $\tilde{M}_d$ and has an involution $\bar{T}$ covering $\tilde{T}$ induced by $\tilde{T} \times$ identity on $\tilde{M}_d \times Z_2$. $\bar{T}$ has two 2-dimensional fixed point sets ${}_1F_c$, ${}_2F_c$ each p.l. homeomorphic to F_c and under the covering projection a neighborhood of ${}_iF_c$ $i = 1,2$ is homeomorphic to a neighborhood of F_c. Now $i_*\pi\,\partial N_1F_c \to \pi_1(\bar{M} - \bigcup_{i=1}^{2} (N_iF_c))$ is not a monomorphism. Hence repeating the argument in subcase 1 to get two loops $\tilde{\alpha}$ and $\bar{T}\tilde{\alpha}$ which project to 2-loops in $M - \dot{N}F_c$ α and $T\alpha$ and two discs D TD bounding α and $T\alpha$. Modify if necessary to get $D \cap TD = \emptyset$ and a non-boundary invariant separating 2-sphere.

<u>Corollary 1</u>. The inclusion in homotopy of any 2-dimensional component $\neq P_2$ of the fixed point set of an involution T on a irreducible orientable 3-manifold is a monomorphism.

Use of this corollary is made in Chapter II to consider involutions of $S^1 \times S^1 \times S^1$.

Corollary 2. If F_c is a 2-dimensional fixed point set of an involution of $M_1 \# M_2$, M_1 and M_2 are orientable 3-manifolds with the conditions of theorem 1 and if $\pi_1(F_c)$ cannot be a subgroup of $\pi_1(M_1) * \pi_1(M_2)$ then the involution is induced and F_c is induced as a connected sum between two manifolds not necessarily M_1 and M_2.

As an example of the last corollary, consider $P_3 \# P_3$ which can have the "Klein bottle" K has a fixed point set [4]. Since $\pi_1(K)$ is not a free group and has no elements of order 2, the Kurosch subgroup theorem says $\pi_1(K)$ cannot be a subgroup of $Z_2 * Z_2$, thus $i_* : \pi_1(K) \to \pi_1(P_3 \# P_3) = Z_2 * Z_2$ is not a monomorphism and by Theorem 1 the involution is induced by connected sum from the unique involution on P_3 with 2-dimensional fixed point set component. This answers the uniqueness question of K as a fixed point set component in $P_3 \# P_3$ [4].

One can continue examples with the Kurosch subgroup theorem in investigating 2-dimensional fixed point sets in connected sums of two spaces and 3-toruses.

It has been recently shown by J. Tollefson that Theorem 1 generalizes to non-orientable manifolds. Such manifolds may have a one-dimensional component in F(T) as well as a two-dimensional component. The proof of Theorem 1 above uses orientability to conclude the non existence of 1-dimensional fixed point set components.

CHAPTER II

APPLICATIONS

In this chapter the results of Theorem 1 will be used in considering involutions of $S^1 \times S^1 \times S^1$ which have a 2-dimensional component in the fixed point set. By P. Conner [2] the fixed point sets of such involutions are $S^1 \times S^1$ and $S^1 \times S^1 \dot{\cup} S^1 \times S^1$. Examples of these are $h : S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ given by $h(z_1, z_2, z_3) = (z_2, z_1, z_3)$ and $K : S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ given by $K(z_1, z_2, z_3) = (\bar{z}_1, z_2, z_3)$ respectively where of course the z_i are complex numbers such that $|z_i| = 1$. It will be shown that these are the only examples up to conjugation.

<u>Lemma</u>. If $S^1 \times S^1$ is a fixed point set component of an involution $T : S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ then $S^1 \times S^1$ is a retract of $S^1 \times S^1 \times S^1$.

<u>Proof</u>: First note that $S^1 \times S^1$ does not separate $S^1 \times S^1 \times S^1$ since from Theorem 1, $i_* \pi_1(S^1 \times S^1) \to \pi_1(S^1 \times S^1 \times S^1)$ is a monomorphism, but $\pi_1(S^1 \times S^1) = H_1(S^1 \times S^1)$, $\pi_1(S^1 \times S^1 \times S^1) = H_1(S^1 \times S^1 \times S^1)$ letting $S^1 \times S^1 = T^2$ $S^1 \times S^1 \times S^1 = T^3$ so $H_1(T^3, T^3)$ has rank 1. From the universal coefficient

theorem rank $H^1(T^3,T^2) = 1$ and since T^3-T^2 is open in T^3 it is orientable thus from Lepschetz duality the rank of $H_2(T^3-T^2)$ is 1, but it is shown in Theorem 1 that if T^2 separates T^3, it separates T^3 into two homeomorphic components thus rank $H_2(T^3-T^2)$ is even. This contradiction shows T^2 cannot separate.

Now open T^3 along T^2 by removing a small invariant regular neighborhood of T^2. This gives a manifold $\tilde{M}$ with two boundary components ∂_1, ∂_2 each homeomorphic to T^2 and $i_*\pi_1(\partial_i) \to \pi_1(\tilde{M})$ is a monomorphism for $i = 1,2$. Also $\tilde{M}$ has an involution $\tilde{T}$ such that

$$\begin{array}{ccc} M & \xrightarrow{T} & M \\ \cup & & \cup \\ \tilde{M} & \xrightarrow{\tilde{T}} & \tilde{M} \end{array}$$

notice that $\tilde{T}$ carries ∂_1 to ∂_2. Preform the following construction.

First form $\tilde{M} \times Z$, then using the relation $x \times i \sim \tilde{T}(x) \times (i+1)$ with $x \in \partial_1$ the quotient space $\frac{\tilde{M} \times Z}{\sim}$ is a connected manifold denoted by $\bar{M}$. $\bar{M}$ covers T^3 and as such $\pi_1(\bar{M})$ is an abelian group of rank 0,1 or 2. Let $Z_n = \{-n,-n+1,\ldots,-1,0,1,\ldots,n-1\}$ then froming $\tilde{M} \times Z_n$ with the relation $x \times i \sim \tilde{T}(x) \times (i+1)$

with $x \in \partial_1$ and considering the quotient space $\frac{\tilde{M} \times Z_n}{\not\sim}$ as $\tilde{M}_n$ one has $\pi_1(\bar{M}) \simeq \varinjlim_n \pi_1(\tilde{M}_n)$. By the Van Kampen theorem, $\pi_1(\tilde{M}_n) \xrightarrow{i_*} \pi_1(\tilde{M}_{n+1})$ is a monomorphism and $\pi_1(\tilde{M}_1)$ is non-abelian unless the inclusion of the boundary ∂_1 and ∂_2 into $\tilde{M}$ is an epimorphism but $\pi_1(\tilde{M}_1)$ survives in $\pi_1(\bar{M})$ which is abelian thus the inclusion of ∂_1, ∂_2 into $\tilde{M}$ is an isomorphism in homotopy. So now by Brown [1] $\tilde{M} \underset{\text{p.l.}}{\simeq} T^2 \times I$ and one has

$$\begin{array}{ccc} T^2 \times I & \xrightarrow{\bar{T}} & T^2 \times I \\ \wr & & \wr \\ \tilde{M} & \xrightarrow{\tilde{M}} & \tilde{M} \end{array}$$

consider the relation $x \times 0 \approx \bar{T}(x) \times 1$ and form the quotient to give $K = \frac{T^2 \times I}{\approx}$.

<u>Lemma</u>. $K \simeq T^2 \times S^1$ iff $\bar{T}_{\#}: \pi_1(T^2 \times I) \to \pi_1(T^2 \times I)$ is the identity.

Proof: More generally let M be any manifold with T a p.l. homeomorphism from M to itself. Consider $\frac{M \times I}{}$ where $x \times 0 \quad T(x) \times 1$. Without loss of generality assume $x_0 = T(x_0)$ and let $\frac{x_0 \times I}{\approx} = \alpha$. One has now

$$0 \to \pi_1(M) \xrightarrow{i_\#} \pi_1\left(\frac{M \times I}{\approx}\right) \to \pi_1(S_1) \to 0$$

the claim is $i_{\#} T_{\#} B = [\alpha]^{-1} i_{\#} B[a]$ for every $B \in \pi_1(M)$.

Let D_S be the deformation retraction of $M \times I$ to $M \times \{1\}$ given by $D_S(m,t) = (m, T-S(1-t))$ and let B be a map of I into M such that $B(0) = x_0 = B(1)$.

Consider the following homotopy $h(T,S) : I \times I \to M \times I$ given by

$$h(T,S) = (x_0, 3st) \qquad \text{if } \tfrac{1}{3} S \geq t$$

$$h(t,s) = (T \circ B(\tfrac{3t-3}{3-2s}), S) \qquad \text{if } \tfrac{1}{3} S \leq t \text{ and } -\tfrac{1}{3} S+1 \geq t$$

and $$h(t,s) = (x_0, 3s(1-t)) \qquad \text{if } -\tfrac{1}{3} S+1 \leq t$$

Now $T \circ B(3t-1,1)$ is identified with $B(3t-1,\alpha)$ so $i_{\#} T_{\#} B = [\alpha]^{-1} \circ i_{\#} B \circ [\alpha]$.

In the case of the lemma if $K \simeq T^2 \times S^1$ then $\pi_1(K)$ is abelian so $i_{\#} T_{\#} B = i_{\#} B \Rightarrow T_{\#} B =$ identity and if $\bar{T}_{\#}$ is the identity then $[\alpha]$ does not act on $\pi_1(K)$ so $0 \to Z+Z \to \pi_1(K) \to Z \to 0$ splits and $\pi_1(K) = Z+Z+Z$ so K must be $T^2 \times S^1$ using Stallings fibration theorem [9].

Thus the lemma allows classification of the involution of $T^2 \times S^1$ with a 2-dimensional fixed point set component by considering actions T on $T^2 \times I$ such that $T : T^2 \times 0 \to T^2 \times 1$ and $T_{\#} : \pi_1(T^2 \times I) \to \pi_1(T^2 \times I)$ is the identity.

Now suppose that T^2 is the only fixed point set, this implies that for classification of involutions with T^2 as fixed point one need only consider orientation reversing free involutions $\tilde{T}$ of $T^2 \times I$ with $\tilde{T}_{\#}$ the identity and considered as Z_2 action on $\pi_1(T^2)$, $\tilde{T}_*$ is the trivial action. Let Q be the quotient space of $T^1 \times I$ under $\tilde{T}$ since $\tilde{T}_*$ is trivial one has $0 \to Z \times Z \to \pi_1(Q) \to Z_2 \to 0$ with $\pi_1(Q) = Z+Z+Z_2$ or $Z+Z$. The universal covering of Q is a contractable space and there are no finite fixed point free actions on a contractable space. Thus $\pi_1(Q) = Z+Z$.

Thus Q is a 3-manifold with one boundary component ∂Q which is included monomorphically into Q. By Stallings [9] Q must be $S^1 \times M$ where M is the closed Möbius strip.

Now, if there is another free Z_2 action T_2 on $S^1 \times S^1 \times I$ which carries ∂_1 to ∂_2 with $T_{2\#}$ = identity, then letting the orbit map be P_2, one has the following diagram:

$$\begin{array}{ccccc}
S^1 \times S^1 \times I & \xrightarrow{P_1} & S \times M & \xleftarrow{P_2} & S^1 \times S^1 \times I \\
i_1 \uparrow & & i_3 \uparrow & & i_2 \uparrow \\
\partial_1 & \xrightarrow[P_1/\partial_1]{} & S^1 \times \partial M & \xleftarrow[P_2/\partial_1]{} & \partial_1
\end{array}$$

$i_{1*} = \text{identity} = i_{2*}$

p_1/∂_{1*} and p_2/∂_{1*} are isomorphisms. Thus $p_{1*}(S^1 \times S^1 \times I) = p_{2*}(S^1 \times S^1 \times I)$ and thus p_1 and p_2 are conjugate by the covering translation theorem. Thus any Z_2 action with T^2 as a fixed point set are conjugate.

Now assume that $\tilde{T}$ has T^2 as a fixed point set of $\tilde{T} : T^2 \times I \to T^2 \times I$. $i_* : \pi_1(T^2) \to \pi_1(T^2 \times I)$ is a monomorphism and hence by a similar argument to the above, T^2 separates $T^2 \times I$ into M_1, M_2 with $\tilde{T} : M_1 \to M_2$ a homeomorphism. By the Van Kampen theorem, $i_* : \pi(T^2) \to \pi(M_1)$ and $i_{1*} : \pi_1(T^2) \to \pi_1(M_2)$ is a monomorphism. But $\pi_1(T^2 \times I) = \pi_1(M_1) *_{Z+Z} \pi_1(M_2)$ and $\tilde{T}_* i_* = i_*$ so i_* must be an epimorphism. Thus $i_*\ \pi(T^2) \to \pi_1 M_1$ is an isomorphism It is possible to switch the ordering of selection of the two fixed point set components to get that the other boundary component of M_1 induces by the inclusion an isomorphism in homootpy. Thus by Brown [1], again $M_1 = T_2 \times I$ and $M_2 = T_2 \times I$ and $\tilde{T}$ is conjugate to a reflection.

It has thus been shown

<u>Theorem 2</u>. If T is an involution of $S^1 \times S^1 \times S^1$ with one 2-dimensional component it is conjugate to $h : S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ given by $h(Z_1, Z_2, Z_3) = (Z_2, Z_1, Z_3)$ and if T has two 2-dimensional components then T is conjugate to $K : S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ given by $K(Z_1, Z_2, Z_3) = (Z_1, Z_2, \bar{Z}_3)$.

The classification of the involutions of $S^1 \times S^1 \times S^1$ with 2-dimensional fixed point set components allows for an investigation of $S^1 \times K$ where K is the "Klein bottle". Suppose K is a fixed point set component of an involution $\bar{T}$ on $S^1 \times K$. Now $\pi_1(K) \xrightarrow{i_*} \pi_1(S^1 \times K)$ is a monomorphism thus lifting to the $S^1 \times S^1 \times S^1$ covering space gives a torus T^2 covering K. T^2 is 2-sided and the covering translation on T^2 reverses orientation since K is not orientable. Consider a two-sided invariant saturated regular neighborhood of T^2, say $T^2 \times [-1,1]$. With the fixed point set $T^2 \times 0$. The claim is that under the covering translation h $T^2 \times [0,1]$ goes to $T^2 \times [0,1]$. If not the regular neighborhood N of K is $S \times K$ would be orientable. However, the image of all open connected saturated sets must be all nonorientable or orientable and since $S \times K$ is nonorientable, N must be nonorientable, this contradiction yields that $h(T^2 \times [0,1]) = T^2 \times [0,1]$. Hence K is two-sided in $S \times K$. Open $S \times K$ along K to obtain a manifold $\tilde{M}$. Note

$$\begin{array}{ccc} T^2 & \xrightarrow{\simeq} & T^2 \times I \\ \downarrow & & \downarrow \\ \tilde{K} & \rightarrow & \tilde{M} \end{array}$$

yields immediately by Brown [1] that $\tilde{M} \underset{\text{p.l.}}{\simeq} K \times I$.

The position of the investigation is similar to Theorem 2.

<u>Theorem 3</u>. If K is a component of the fixed point set of an involution $S \times K$ then T is conjugate to either

i) a reflection $\times$ identity on $S \times K$;

ii) the induced map on $S \times K$ gotten by using the covering translation $\bar{T}$ of the double covering of K by itself. Forming $\bar{T} \times 1-t : K \times I \to K \times I$ and identifying $(X,0)$ with $(\bar{T}(X),1)$. Since $\bar{T}$ is fixed point free $\bar{T}_{\#}$ = identity, so the identification gives $S \times K$.

Proof: It has already been noted that K is 2-sided and the complement of a regular neighborhood of K is homeomorphic to $K \times I$ which is covered by $S \times S \times I$. The involution T on $K \times I$ generates an involution $\tilde{T}$ on $S \times S \times I$. The claim is that $\tilde{T}_{\#}$ = identity and thus $\tilde{T}$ is induced from an involution on $S \times S \times S$. Having

$$\begin{array}{ccc} \pi_1 S \times S \times I & \xrightarrow{\tilde{T}_{\#}} & \pi_1 S \times S \times I \\ \downarrow \rho_{\#} & & \downarrow \rho_{\#} \\ \pi_1 (K \times I) & \xrightarrow[\text{identity}]{T_{\#}} & \pi_1 K \times I \end{array}$$

substantiates the claim.

Now in Theorem 2 it is shown that $S \times S \times I$ has 2 involutions which are candidates for $\tilde{T}_{\#}$. One has T^2 as a fixed point set and $\tilde{T}$ is a reflection. T^2 covers a Klein bottle in $K \times I$ and $\bar{T}$ is a reflection giving case i. The other case is where $\tilde{T}$ and hence $\bar{T}$ is a free involution. Now as in the proof of Theorem 2, only the orbit space need by unique. In this case calling the orbit space Q. By Scott [7] Q is a line bundle over a Klein bottle. However, by Quinn [6] and the fact that Q has K as a boundary Q is the unique mapping cylinder of the double covering of K by itself.

Unfortunately there are involutions of $S^1 \times K$ with T^2 as a component of the fixed point set so a complete classification of involutions on $S^1 \times K$ cannot be claimed at this time.

BIBLIOGRAPHY

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1. E.M. Brown, "Unknotting in $M^2 \times I$", Trans. Amer. Math. Soc., 1966, p.480-505.

2. P.E. Conner, "Transformation groups on a $K(\pi,1)$ II", Mich. Math. Journal, Vol. 6, 1959, p.413-417.

3. J. Milnor, "A unique decomposition theorem for 3-manifolds", American Journal of Math., Vol. 84, 1962, p.1-7.

4. Myung M. Myung, Thesis, M.S.U., 1970.

5. C.D. Papakyriakopoulos, "On Dehn's Lemma and asphericity of knots", Annals. of Math., Vol. 66, 1957, p.1-26.

6. Joan Quinn, Thesis, M.S.U., 1970.

7. P. Scott, "On sufficiently large 3-manifolds", Quarterly Journal of Math., Vol. 23, 1972, p.159-172.

8. J. Stallings, "On the Loop Theorem", Annals. of Math., Vol. 72, 1960, p.12-19.

9. ____________, "On fibering certain 3-manifolds", Topology of 3-manifolds and related topics, Prentice-Hall 1962, p.95-100.