ON A GENERALIZATION OF THE LOTOTSKY SUMMABILITY METHOD

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ABSTRACT

ON A GENERALIZATION OF THE LOTOTSKY SUMMABILITY METHOD

by Herbert B. Skerry

A. Jakimovski defined his (F,d_n) summability method as a generalization of the Lototsky method. Subsequently, G. Smith generalized the (F,d_n) method to the (f,d_n) method, and later generalized this to the (f,d_n,z_1) method. Let f be analytic at the origin and suppose z_1 is a point in its disc of convergence. Let $c_{00} = 1$, $c_{0k} = 0$ for $k > 0$, and

$$\prod_{1}^{n} \frac{f(zz_1) + d_\nu}{f(z_1) + d_\nu} = \sum_{k=0}^{\infty} c_{nk} z^k \quad \text{for} \quad n \geq 1.$$

Then the (f,d_n,z_1) method is defined by the matrix (c_{nk}). The (f,d_n), (F,d_n), and Lototsky methods are, respectively, the methods $(f,d_n,1)$, $(z,d_n,1)$, and $(z,n-1,1)$.

Chapter 1 is concerned with the generalization $(f,d_n,z_1)^*$ of (f,d_n); it is defined by the matrix (c^*_{nk}), where $c^*_{00} = 1$, $c^*_{0k} = 0$ for $k > 0$, and

$$\prod_{1}^{n} \frac{f(z) + d_\nu}{f(z_1) + d_\nu} = \sum_{k=0}^{\infty} c^*_{nk} z^k \quad \text{for} \quad n \geq 1.$$

Various properties of this method, which behaves quite differently from (f,d_n,z_1), are extracted.

Sufficient conditions for the regularity of (f,d_n) have been given by Smith, but all require that f have real, non-negative Taylor coefficients. Chapter 2 presents some sufficiency conditions for the regularity of (f,d_n,z_1)

under different restrictions on f.

Chapter 3 concerns itself with the coincidence of the (f, d_n, z_1) method with various other methods, including the Sonnenschein, Hausdorff, quasi-Hausdorff, Norlund, and Riesz methods, and the modified quasi-Hausdorff method of M. S. Ramanujan.

The last chapter deals with questions of inclusion between two (f, d_n, z_1) methods and between (f, d_n, z_1) and several other methods, e.g., the Riesz, Abel, Y, and (E, p) methods.

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INTRODUCTION

A sequence summability method S is a way of associating a unique number with each of a class of sequences. The largest class of sequences for which S performs this function is called the summability field of S. If S associates L with $x = \{x_n\}$ we say S sums x to L. If S sums every convergent sequence, then S is conservative, and if, moreover, S sums each convergent sequence to its limit, then S is regular. We will be concerned only with complex sequences.

Every complex matrix having infinitely many rows and columns defines a sequence summability method. If A is such a matrix, then for suitable sequences x the A-transform, $t = Ax$, determined by matrix multiplication is a sequence; if it converges we say A sums x to $\lim t_n$. It is clear that the summability field of A is the class of sequences whose A-transform exists and converges.

Necessary and sufficient conditions for such a matrix $A = (a_{nk})$ to be conservative are well-known ([8], p. 43). They are

(0.1) $$\sup_n \sum_k |a_{nk}| < \infty ,$$

(0.2) $$\lim_n \sum_k a_{nk} = \zeta \text{ exists,}$$

(0.3) $$\lim_n a_{nk} = a_k \text{ exists for each } k.$$

A is regular if and only if it is conservative with $\zeta = 1$ and $a_k = 0$ for each k ([8], p. 43).

Given the methods S and S', we will say S is <u>at least as strong</u> as S' if the summability field of S' is a subset of that of S. Under these conditions the two methods are <u>consistent</u> if each element of the smaller summability field is summed to the same number by both methods.

A number of the classical sequence summability methods are matrix methods. The best-known among them are probably the Cesaro methods (see [8]). In [10] Lototsky defined a method which Agnew considers to rival the Cesaro methods in importance (see [4]). This method was subsequently generalized by Jakimovski [9] to his (F,d_n) method, and this in turn was generalized by Smith [15] to the (f,d_n) method. Finally, in a paper to appear [16], Smith generalized (f,d_n) to (f,d_n,z_1). We will concern ourselves with the last three methods.

The following definitions and conventions will be used.

<u>Definition 0.4</u>: Let f be holomorphic at the origin and let $\{d_n\}_1^{\infty}$ be a sequence of complex numbers with $d_n \neq -f(z_1)$, where z_1 is in the disc of convergence of f. Let

$$(0.5) \qquad \prod_{1}^{n} (f(z) + d_k) = \sum_{k=0}^{\infty} p_{nk} z^k , \quad n \geq 1.$$

Then the method (f,d_n,z_1) is defined by the matrix $C = (c_{nk})$, where

$$(0.6) \qquad c_{0k} = \begin{cases} 1, & k = 0 \\ 0, & k > 0 \end{cases} \quad \text{and} \quad c_{nk} = \frac{p_{nk} z_1^k}{\prod_1^n (f(z_1)+d_k)}, \quad n \geq 1.$$

In terms of the above definition, the Lototsky method is the method $(z, n-1, 1)$ and (F, d_n) is the method $(z, d_n, 1)$. The (f, d_n) method is defined to be $(f, d_n, 1)$.

We will assume throughout that $z_1 \neq 0$. For convenience we will use Jakimovski's notation:

$$\prod_1^n (f(z) + d_k) = (f(z) + d_n)! \ .$$

The definition of (f, d_n, z_1) insures that the condition (0.2) holds with $\zeta = 1$, so we need only consider the remaining two conditions in any questions of regularity or conservativity that arise in connection with (f, d_n, z_1) and its various special cases.

CHAPTER I

THE METHOD $(f, d_n, z_1)^*$

We have remarked that Smith first defined and investigated the method (f, d_n) and only later generalized it to (f, d_n, z_1). The question arises as to whether some other generalization is not equally as natural and as useful. In this light, consider the

Definition 1.1: Let $f, \{d_n\}_1^{\infty}$, and z_1 be as in definition 0.4. Then the method $(f, d_n, z_1)^*$ is given by the matrix $C^* = (c^*_{nk})$, where

$$(1.2) \qquad c^*_{0k} = \begin{cases} 1, k = 0 \\ 0, k > 0 \end{cases} \quad \text{and} \quad c^*_{nk} = \frac{p_{nk}}{(f(z_1)+d_n)!}, \quad n \geq 1.$$

It seems clear that Smith inserted the factor z_1^k in the expression for c_{nk} to preserve the property $\sum_k c_{nk} = 1$ which obtains for (f, d_n). We have dropped this factor in the above definition and have accordingly lost this convenient property, but it is not obvious that anything more than convenience has been lost.

The following necessity conditions for regularity are restatements of theorems in [15] with $f(z_1)$ substituted for $f(1)$. For the sake of completeness we include the proofs, which are essentially Smith's. We will use the notation

$$(1.3) \quad \text{Re } f(z) = u, \ \text{Im } f(z) = v, \ d_n = x_n + iy_n = \rho_n e^{i\theta_n},$$

$$f(z_1) = a + ib.$$

<u>Theorem 1.4</u>: Let $d_n \neq -f(0)$ Then a necessary condition that $(f,d_n,z_1)^*$ be regular is that there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$\sum_{k=1}^{\infty}\left(1 - \left|\frac{f(0) + d_{n_k}}{f(Z_1) + d_{n_k}}\right|^2\right) = \infty .$$

<u>Proof</u>: By setting $z = 0$ in (0.5) and using (1.2), we see that

$$c^*_{no} = \prod_1^n \frac{f(0) + d_k}{f(z_1) + d_k} .$$

Since the method is regular, $\lim_n c^*_{no} = 0$. It follows immediately that

$$\lim_n |c^*_{no}|^2 = \prod_1^{\infty}\left|\frac{f(0)+d_k}{f(z_1)+d_k}\right|^2 = \prod_1^{\infty}\{1-[1-\left|\frac{f(0)+d_k}{f(z_1)+d_k}\right|^2]\}$$

$$= \prod_1^{\infty} (1 - a_k) = 0, \quad \text{where} \quad a_k < 1.$$

Clearly, an infinite number of a_k's must be positive in order that $\prod_1^{\infty}(1 - a_k) = 0$; let $\{a_{n_k}\}$ be the sequence of positive a_k's. Then $\prod_{k=1}^{\infty} (1 - a_{n_k}) = 0$, whence $\sum_{k=1}^{\infty} a_{n_k} = \infty$ as claimed.

<u>Corollary 1.5</u>: Let $d_n \neq -f(0)$ and let $f(0)$ and $f(z_1)$ be real. Then a necessary condition for the regularity of $(f,d_n,z_1)^*$ is the existence of a strictly increasing sequence $\{n_k\}$ of natural numbers satisfying

$$\sum_{k=1}^{\infty} \frac{f(0) + f(z_1) + 2x_{n_k}}{|f(z_1) + d_{n_k}|^2} = \pm \infty .$$

Proof: Since $f(0)$ and $f(z_1)$ are real we have

$$(1.6)\quad 1-\left|\frac{f(0)+d_{n_k}}{f(z_1)+d_{n_k}}\right|^2 = [f(z_1)-f(0)]\frac{f(z_1)+f(0)+2x_{n_k}}{|f(z_1)+d_{n_k}|^2} .$$

The above theorem then says that

$$[f(z_1)-f(0)]\sum_{k=1}^{\infty}\frac{f(z_1)+f(0)+2x_{n_k}}{|f(z_1)+d_{n_k}|^2} = \infty ,$$

so the result follows.

Corollary 1.7: Let $d_n \neq -f(0)$. Then a necessary condition for the regularity of $(f,d_n,z_1)^*$ is that $|f(z_1)+d_n| > |f(0)+d_n|$ for infinitely many n. Furthermore, if $f(0)$ and $f(z_1)$ are real, a necessary condition is that

$$x_n > -\tfrac{1}{2}[f(0)+f(z_1)] > -f(z_1) \quad \text{when } f(z_1) > f(0)$$

and

$$x_n < -\tfrac{1}{2}[f(0)+f(z_1)] < -f(z_1) \quad \text{when } f(z_1) < f(0)$$

for infinitely many n.

Proof: In the proof of theorem 1.4 it was observed that $0 < 1 - \left|\frac{f(0)+d_n}{f(z_1)+d_n}\right|^2 < 1$ for infinitely many n, i.e., $0 < \left|\frac{f(0)+d_n}{f(z_1)+d_n}\right|^2 < 1$ for infinitely many n. The first statement of the corollary follows. This last inequality together with (1.6) says that

$$f(z_1)+f(0)+2x_n > 0 \quad \text{when} \quad f(z_1) > f(0), \text{ i.e.,}$$

$$x_n > -\tfrac{1}{2}[f(0) + f(z_1)] > -f(z_1).$$

The remaining statement of the corollary is proved in a similar manner.

Corollary 1.8: If $d_n \neq -f(0)$, a necessary condition for the regularity of $(f,d_n,z_1)^*$ is that $f(z_1) \neq f(0)$.

Proof: This is an immediate consequence of Theorem 1.4.

Theorem 1.9: If $(f,d_n,z_1)^*$ is regular, then

$$\sum_1^\infty \frac{1}{|f(0) + d_n|} = \infty .$$

Proof: If $f(0) + d_n = 0$ for some n, the theorem follows trivially. Hence, suppose $f(0) + d_n \neq 0$. Let $\{n_k\}$ be the sequence of positive integers whose existence is assured by Corollary 1.7; then $|f(z_1)+d_{n_k}| > |f(0)+d_{n_k}|$. It follows that

$$1 - \left|\frac{f(0) + d_{n_k}}{f(z_1) + d_{n_k}}\right|^2 = \frac{|f(z_1) + d_{n_k}|^2 - |f(0) + d_{n_k}|^2}{|f(z_1) + d_{n_k}|^2}$$

$$= \frac{(|f(z_1) + d_{n_k}|-|f(0) + d_{n_k}|)(|f(z_1) + d_{n_k}|+|f(0) + d_{n_k}|)}{|f(z_1) + d_{n_k}|^2}$$

$$\leq \frac{|(f(z_1) + d_{n_k})-(f(0) + d_{n_k})|(|f(z_1) + d_{n_k}|+|f(0) + d_{n_k}|)}{|f(z_1) + d_{n_k}|^2}$$

$$= |f(z_1) - f(0)|\frac{|f(z_1) + d_{n_k}|+|f(0) + d_{n_k}|}{|f(z_1) + d_{n_k}|^2} < \frac{2|f(z_1)-f(0)|}{|f(z_1)+d_{n_k}|}$$

$$< \frac{2|f(z_1) - f(0)|}{|f(0) + d_{n_k}|} .$$ Thus if $\sum_{k=1}^{\infty} \frac{1}{|f(0) + d_{n_k}|}$ converges,

so also does $\sum_{k=1}^{\infty} \left(1 - \left|\frac{f(0) + d_{n_k}}{f(z_1) + d_{n_k}}\right|^2\right)$ in violation of Theorem 1.4.

<u>Corollary 1.10</u>: If $d_n \neq -f(0)$, then $(f, d_n, z_1)^*$ is regular only if $\sum_N^{\infty} \frac{1}{\rho_n} = \infty$ for arbitrary N.

Corollary 1.11: If $d_n \neq -f(0)$, then $(f, d_n, z_1)^*$ is regular only if $\sum_N^{\infty} \frac{1}{|f(z_1) + d_n|} = \infty$, N arbitrary.

For reference purposes we include the following three lemmas, which occur in essence in [15] either as results or as portions of proofs.

<u>Lemma 1.12</u>: Let $d_n \geq 0$ and let the power series expansion of f about the origin have real, non-negative coefficients. Then (f, d_n) is regular if and only if

$$\sum_1^{\infty} \frac{1}{f(1) + d_n} = \infty .$$

<u>Proof:</u> This is Lemma 2.2 in [15].

<u>Lemma 1.13</u>: In the notation of (1.3),

$$\left|\frac{f(z) + d_n}{f(z_1) + d_n}\right| \leq \exp\left\{\frac{\rho_n[(u-a)\cos\theta_n + (v-b)\sin\theta_n]}{|f(z_1) + d_n|^2} + \frac{u^2 + v^2 - a^2 - b^2}{2|f(z_1) + d_n|^2}\right\}$$

$$= \exp\left\{\frac{\mathrm{Re}([\overline{f(z)}-\overline{f(z_1)}]d_n)}{|f(z_1)+d_n|^2} + \frac{u^2+v^2-a^2-b^2}{2|f(z_1)+d_n|^2}\right\}.$$

Proof: Let $x = \left|\frac{f(z)+d_n}{f(z_1)+d_n}\right|^2 - 1$ and use the fact that $1 + x \le e^x$ to get the result by a short calculation.

Lemma 1.14: Let z be fixed. In the notation of (1.3), let

$$\sum_{\rho_n \neq 0} \frac{1}{\rho_n} = \infty,\ \theta_n \longrightarrow \theta,\ \rho_n \longrightarrow \infty.$$

Then

$$\prod_1^\infty \frac{f(z)+d_n}{f(z_1)+d_n} = 0 \quad \text{if } (u-a)\cos\theta+(v-b)\sin\theta = \mathrm{Re}([\overline{f(z)}-\overline{f(z_1)}]e^{i\theta}) < 0$$

and

$$\prod_1^\infty \left|\frac{f(z)+d_n}{f(z_1)+d_n}\right| = \infty \quad \text{if } (u-a)\cos\theta+(v-b)\sin\theta = \mathrm{Re}([\overline{f(z)}-\overline{f(z_1)}]e^{i\theta}) > 0.$$

Proof: Let

$$Q_n = \frac{\rho_n[(u-a)\cos\theta_n+(v-b)\sin\theta_n]}{|f(z_1)+d_n|^2} + \frac{u^2+v^2-a^2-b^2}{2|f(z_1)+d_n|^2}$$

and $Q = (u-a)\cos\theta + (v-b)\sin\theta$. It is clear that $\rho_n Q_n \longrightarrow Q$. If $Q < 0$, then there is a $K > 0$ such that $-K > \rho_n Q_n$ for all large n, so $-\frac{K}{\rho_n} > Q_n$ for large n.

By Lemma 1.13, $\left|\frac{f(z)+d_n}{f(z_1)+d_n}\right| \le \exp\{Q_n\} < \exp\{-\frac{K}{\rho_n}\}$ for large n. Then

$$\prod_N^{\infty}\left|\frac{f(z) + d_n}{f(z_1) + d_n}\right| \leq \exp\left\{-K \sum_N^{\infty} \frac{1}{\rho_n}\right\} = 0.$$

If $Q > 0$ we may use essentially the same argument, interchanging the roles of z and z_1 in Lemma 1.13, to show that $\prod_N^{\infty}\left|\frac{f(z_1) + d_n}{f(z) + d_n}\right| = 0$, whence $\prod_1^{\infty}\left|\frac{f(z) + d_n}{f(z_1) + d_n}\right| = \infty$.

<u>Theorem 1.15</u>: Let $T_n = \operatorname{Im}[(\overline{f(z_1)} - \overline{f(1)})e^{i\theta_n}]$ and let $d_n = \rho_n e^{i\theta_n} \neq -f(0)$. Suppose

(i) $a + ib = f(z_1) \neq f(1) = \alpha + i\beta$,

(ii) θ_n is bounded away from $\theta^* + 2\ell\pi$ for large n,

and either

(iii) $\liminf T_n > 0$

or

(iv) $\limsup T_n < 0$.

Then $(f, d_n z_1)^*$ is not regular.

<u>Proof</u>: Suppose the contrary. Then, in the notation of (1.2),

$$\sum_k c^*_{nk} = \prod_1^n \frac{f(1) + d_m}{f(z_1) + d_m} \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty.$$

As a consequence,

$$\left|\frac{f(1) + d_m}{f(z_1) + d_m} - 1\right| = \left|\frac{f(1) - f(z_1)}{f(z_1) + d_m}\right| \longrightarrow 0 \text{ as } m \longrightarrow \infty,$$

so $\rho_m \longrightarrow \infty$. From (ii) we may suppose that $\theta^* + \delta \leq \theta_n \leq \theta^* + 2\pi - \delta$ for large n. Since $\rho_m \longrightarrow \infty$, we may use this same branch of the argument for $\Psi_m = \arg[f(1) + d_m]$ and

$\Psi'_m = \arg[f(z_1) + d_m]$ for large m. Let $\phi_m = \arg \dfrac{f(1) + d_m}{f(z_1) + d_m}$.

The factors $\dfrac{f(1) + d_m}{f(z_1) + d_m} \longrightarrow 1$, so we may use the principal branch of the argument for $\phi_m = \Psi_m - \Psi'_m$ for large m.

The convergence of $\prod_1^\infty \dfrac{f(1) + d_m}{f(z_1) + d_m}$ implies the convergence of $\sum_1^\infty \phi_m$. For a given large m, if neither $\alpha + x_m$ nor $a + x_m$ is zero, we have, by choosing the appropriate branch of the inverse tangent function for each of Ψ_m and Ψ'_m, that $\Psi_m = \tan^{-1} \dfrac{\beta + y_m}{\alpha + x_m}$ and $\Psi'_m = \tan^{-1} \dfrac{\beta + y_m}{a + x_m}$. Then

$$\tan \phi_m = \tan (\Psi_m - \Psi'_m) = \frac{\dfrac{\beta + y_m}{\alpha + x_m} - \dfrac{b + y_m}{a + x_m}}{1 + \dfrac{\beta + y_m}{\alpha + x_m} \cdot \dfrac{b + y_m}{a + x_m}} =$$

$$\frac{(a + x_m)(\beta + y_m) - (\alpha + x_m)(b + y_m)}{(\alpha + x_m)(a + x_m) + (\beta + y_m)(b + y_m)} = Q_m \quad , \text{ so}$$

$$(1.16) \quad \phi_m = \mathrm{Tan}^{-1} Q_m = \mathrm{Tan}^{-1} \frac{(a+x_m)(\beta+y_m) - (\alpha+x_m)(b+y_m)}{(\alpha+x_m)(a+x_m) + (\beta+y_m)(b+y_m)} ,$$

where Tan^{-1} denotes the principal branch.

It can be easily shown by routine calculation that (1.16) is still valid if $\alpha + x_m = 0$ or $a + x_m = 0$. Now write

$$\rho_m Q_m = \rho_m \frac{(a+\rho_m\cos\theta_m)(\beta+\rho_m\sin\theta_m)-(\alpha+\rho_m\cos\theta_m)(b+\rho_m\sin\theta_m)}{(\alpha+\rho_m\cos\theta_m)(a+\rho_m\cos\theta_m)+(\beta+\rho_m\sin\theta_m)(b+\rho_m\sin\theta_m)}$$

$$= \rho_m \frac{(a\beta-\alpha b)+\rho_m(\beta-b)\cos\theta_m+\rho_m(a-\alpha)\sin\theta_m}{(a\alpha+b\beta)+\rho_m(a+\alpha)\cos\theta_m+\rho_m(b+\beta)\sin\theta_m+\rho_m^2}$$

$$= \frac{o(1)+(\beta-b)\cos\theta_m + (a-\alpha)\sin\theta_m}{o(1)+1} = \frac{o(1)+T_m}{o(1)+1}.$$

If (iii) holds, then for all large m we have $\rho_m Q_m \geq \delta > 0$, so, in particular, $Q_m > 0$. Then (1.16) implies $\Phi_m > 0$ and thus $\Phi_m \longrightarrow 0^+$. But now, since $Q_m \longrightarrow 0^+$ from (1.16),

$$\frac{\Phi_m}{\rho_m^{-1}} = \rho_m \mathrm{Tan}^{-1} Q_m = \frac{\mathrm{Tan}^{-1} Q_m}{Q_m} \cdot \rho_m Q_m \geq \varepsilon > 0$$

for large m, so $\rho_m^{-1} \leq \varepsilon^{-1}\Phi_m$. Then the convergence of $\Sigma\, \Phi_m$ implies that of $\Sigma\, \rho_m^{-1}$ in violation of Corollary 1.10. If, on the other hand, (iv) holds, the argument proceeds in a similar manner.

We remark here that the above proof depends only on the conservativity condition (0.2) with $\zeta \neq 0$ and on Corollary 1.10. Inspection of the proofs of this corollary and its antecedents shows that the only regularity condition used is $\lim\limits_n c^*_{no} = 0$. We may thus state

<u>Corollary 1.17</u>: Under the hypotheses of Theorem 1.15, the $(f,d_n,z_1)^*$-method cannot satisfy both $\lim\limits_n c^*_{no} = 0$ and $\lim\limits_n \sum\limits_k c^*_{nk} = \zeta \neq 0$.

The following corollary appeared in the proof of Theorem 1.15.

<u>Corollary 1.18</u>: Let $\lim\limits_n \Sigma\, c^*_{nk} = \zeta \neq 0$. Then $\rho_n \longrightarrow \infty$.

We will now prove a result which is not a direct corollary of Theorem 1.15, but which is closely allied to it.

<u>Theorem 1.19</u>: Let $d_n = \rho_n e^{i\theta_n} \neq -f(0)$, $a + ib = f(z_1) \neq f(1) = \alpha + i\beta$, and $\theta_n \longrightarrow \theta$. Then $(f,d_n,z_1)^*$ is not regular.

<u>Proof</u>: By Corollaries 1.10 and 1.18 the result follows immediately unless $\sum_N^\infty \frac{1}{\rho_n} = \infty$ for arbitrary N and $\rho_n \longrightarrow \infty$. Thus suppose these conditions met. Then Lemma 1.14 gives

(1.20) $\quad (\alpha - a) \cos \theta + (\beta - b) \sin \theta = 0$,

assuming $(f,d_n,z_1)^*$ is regular. If also

(1.21) $\quad (\beta - b) \cos \theta + (a - \alpha) \sin \theta = 0$

and $\theta \neq \pm \frac{\pi}{2} + 2\ell\pi$, then $\tan \theta = -\frac{\beta - b}{a - \alpha}$. But from (1.20), $\tan \theta = -\frac{\alpha - a}{\beta - b}$, so $\frac{\beta - b}{a - \alpha} = \frac{\alpha - a}{\beta - b}$ and $(\beta - b)^2 = -(\alpha - a)^2$. It follows that $\alpha = a$ and $\beta = b$, violating the hypothesis. On the other hand, if $\theta = \pm \frac{\pi}{2} + 2\ell\pi$ and (1.21) holds, then it follows from (1.20) and 1.21) together that $\alpha = a$ and $\beta = b$, again violating the hypotheses. Hence, the assumption of regularity implies that (1.21) cannot be true, so $\lim_n T_n = \lim_n [(\beta - b)\cos \theta_n + (a - \alpha) \sin \theta_n] =$

$$= (\beta - b) \cos \theta + (a - \alpha) \sin \theta,$$ is either positive or negative. Theorem 1.15 now gives a contradiction.

This last theorem is in marked contrast to Lemma 1.12, so $(f,d_n,z_1)^*$ behaves quite differently from (f,d_n,z_1) if $f(z_1) \neq f(1)$.

We have seen that (1.20) is a consequence of regularity if $d_n \neq -f(0)$, and, in fact, it is a consequence of the conservativity conditions $\lim_n c^*_{n0} = 0$ and $\lim_n \sum_k c^*_{nk} = \zeta \neq 0$. If (1.20) is interpreted as a dot product of vectors, it says

Corollary 1.22: Let $d_n \neq -f(0)$ and suppose $\lim_n c^*_{n0} = 0$ and $\lim_n \sum_k c^*_{nk} = \zeta \neq 0$. Then $f(z_1)$ and $f(1)$ lie on a normal to the ray $\arg z = \theta$.

In order to consider the relationship between $(f,d_n,z_1)^*$ and $(f,d_n,z_2)^*$, let $\gamma_n = (f(z_1) + d_n)!$ and $\delta_n = (f(z_2) + d_n)!$, and suppose the elements of the matrices corresponding to the two methods are, respectively, c^*_{nk} and b^*_{nk}. From (1.2),

$$(1.21) \quad b^*_{nk} = \frac{\gamma_n}{\delta_n} c^*_{nk}.$$

From this it is clear that if a sequence is $(f,d_n,z_1)^*$-summable to s, it is $(f,d_n,z_2)^*$-summable if and only if $L = \lim_n \frac{\gamma_n}{\delta_n}$ exists; in that event, it is summable to Ls. If $L \neq 0$, summability $(f,d_n,z_2)^*$ of a sequence to s implies summability $(f,d_n,z_1)^*$ to $L^{-1}s$, so the summability fields of the two methods are the same.

The following definition is Agnew's [3].

Definition 1.22: Given the sequence-to-sequence transforms S and T and the sequence x, let $Sx = \{S_n\}$ and $Tx = \{T_n\}$. Then the transforms S and T are equiconvergent for the class C of sequences if

$\lim(S_n - T_n) = 0$ for every $x \in C$.

Let x be a sequence. Then $\sum_k (c^*_{nk} - b^*_{nk})x_k =$

$(1 - \frac{\gamma_n}{\delta_n}) \sum_k c^*_{nk} s_k$, so if $L = 1$ the two methods are equiconvergent on the class of sequences for which the $(f,d_n,z_1)^*$ transform is bounded whether or not these sequences are summable by either of the methods. In particular, if $(f,d_n,z_1)^*$ satisfies (0.1), then the methods are equiconvergent on at least the space S_B of bounded sequences.

Theorem 1.23: Let $\prod_1^n \frac{f(z_1) + d_k}{f(z_2) + d_k} \longrightarrow L$. Then a sequence which is $(f,d_n,z_1)^*$-summable to s is $(f,d_n,z_2)^*$-summable to Ls. If $L \neq 0$, the summability fields are the same. If $L = 1$, the methods are consistent on their common summability field and are equiconvergent for all sequences for which either transform is bounded. In particular, if either transform satisfies the conservativity condition (0.1), then the methods are equiconvergent at least on S_B.

Corollary 1.24: A necessary condition that $(f,d_n,z_1)^*$ be conservative with $\zeta \neq 0$ (in the notation of (0.2)) is that (f,d_n) be conservative and have the same summability field.

Proof: Let the $(f,d_n,z_1)^*$ matrix be (c^*_{nk}). Then

$$\lim_n \sum_k c^*_{nk} = \lim_n \prod_1^n \frac{f(1) + d_m}{f(z_1) + d_m} = \zeta \neq 0, \text{ so } \lim_n \prod_1^n \frac{f(z_1) + d_m}{f(1) + d_m} = \zeta^{-1}.$$

Then Theorem 1.23 says that the summability fields are the same.

Agnew [1] formulated the following

Definition 1.25: A sequence summation method is multiplicative with multiplier L if every sequence convergent to s is summed to Ls by the method.

It is known [8] that necessary and sufficient conditions for a matrix $A = (a_{nk})$ to be multiplicative are the conservativity conditions (0.1), (0.2), and (0.3) with $a_k = 0$. The multiplier is then ζ.

Corollary 1.26: Necessary conditions that $(f, d_n, z_1)^*$ be multiplicative with non-zero multiplier ζ are that (f, d_n) be regular and have the same summability field.

Proof: In the notation of the above corollary and its proof, the relation $\lim \prod_1^n \frac{f(z_1) + d_m}{f(1) + d_m} = \zeta^{-1} \neq 0$ implies, by Theorem 1.23, that (f, d_n) is multiplicative with multiplier 1, i.e., it is regular, and it has the same field of summation.

Corollaries 1.24 and 1.26 show, in effect, that for a given f and a given sequence $\{d_n\}$, the entire class of conservative $(f, d_n, z_1)^*$ methods for which $\zeta \neq 0$ is no stronger than the single method (f, d_n).

With the machinery now at hand we can deal with the question of regularity for a large class of $(f, d_n, z_1)^*$ methods. We need the following lemma (Theorem 4.6 in [15]).

Lemma 1.27: Let $f(z) = az^m$, where $a > 0$ and m is a positive integer. Let α be given with $0 < \alpha < \frac{\pi}{2}$, and suppose there exist $\varepsilon > 0$ and $N > 0$ such that if $d_n = \rho_n e^{i\theta_n}$, then $\pi \geq \theta_n > \alpha$ and $\rho_n > \varepsilon$ for all $n > N$. Then the method (f, d_n) is not regular.

Theorem 1.28: Let $f(z) = az^m$, where $a > 0$ and m is a positive integer. Let α be given with $0 < \alpha < \frac{\pi}{2}$, and suppose there exist $\varepsilon > 0$ and $N > 0$ such that if $d_n = \rho_n e^{i\theta_n}$, then $\pi \geq \theta_n > \alpha$ and $\rho_n > \varepsilon$ for $n > N$. It follows that $(f, d_n, z_1)^*$ is not regular or even multiplicative with non-zero multiplier.

Proof: Were the contrary true, Corollary 1.26 would imply the regularity of (az^m, d_n), thereby violating Lemma 1.27.

We will now pass to considerations of a different nature.

We need the following lemma, proved by Agnew in [2] (Lemma 3.1).

Lemma 1.29: For every $n \geq 0$ let $\sum_{\nu=0}^{\infty} |a_{n\nu}| < \infty$, and let $\overline{\lim_n} \sum_{\nu=0}^{\infty} |a_{n\nu}| = A < \infty$. Then if $\lim_n a_{n\nu} = 0$ for each ν it follows that $\overline{\lim_n} |\sum_{\nu=0}^{\infty} a_{n\nu} s_\nu| \leq A \overline{\lim_\nu} |s_\nu|$.

Moreover, A is the smallest such constant in the sense that there is a bounded sequence not converging to 0 for which the equality holds.

<u>Corollary 1.30</u>: Let

$$Q_n = \left| \prod_1^n \frac{f(z_1) + d_\nu}{f(z_2) + d_\nu} - 1 \right| \sum_{\nu=0}^{\infty} |c^*_{n\nu}| = O(1),$$

let $\lim_n (\prod_1^n \frac{f(z_1) + d_\nu}{f(z_2) + d_\nu} - 1)c^*_{n\nu} = 0$ for each ν, and let $\{s_\nu\}$ be bounded. Let $\{t_n^{(1)}\}$ and $\{t_n^{(2)}\}$ be the $(f,d_n z_1)^*$ and $(f,d_n,z_2)^*$ transforms, respectively, of $\{s_\nu\}$. Then if $Q = \overline{\lim_n}\, Q_n$ it follows that $\overline{\lim_n} |t_n^{(2)} - t_n^{(1)}| \le Q\, \overline{\lim_\nu} |s_\nu|$. Moreover, Q is the smallest such constant in the sense that there is a bounded sequence not converging to 0 for which the equality holds.

<u>Proof</u>:
$$t_n^{(2)} - t_n^{(1)} = \sum_{\nu=0}^{\infty} b^*_{n\nu} s_\nu - \sum_{\nu=0}^{\infty} c^*_{n\nu} s_\nu$$
$$= \sum_{\nu=0}^{\infty} (b^*_{n\nu} - c^*_{n\nu}) s_\nu .$$

By (1.21), $t_n^{(2)} - t_n^{(1)} = (\frac{\gamma_n}{\delta_n} - 1)\sum_\nu c^*_{n\nu} s_\nu$. An application of Lemma 1.29 now gives the result.

In view of Theorem 1.23, we give a set of conditions under which the $(f,d_n,z_1)^*$-transform is bounded on S_B; in fact, the conditions are sufficient for (0.1) to hold for the $(f,d_n,z_1)^*$ matrix.

<u>Lemma 1.31</u>: In the notation (1.3), let $\rho_n \longrightarrow \infty$ and $\sum_{\rho_n \neq 0} \frac{1}{\rho_n} < \infty$. Then if f is holomorphic on the closed unit disc, the $(f,d_n,z_1)^*$ transform is bounded on S_B.

<u>Proof</u>: Let the $(f,d_n,z_1)^*$ matrix have entries c^*_{nk}, and let $\{s_k\} \in S_B$. Then, in view of (1.2) and (0.5),

$$c^*_{nk} = \frac{1}{(f(z_1) + d_n)!} \cdot \frac{1}{2\pi i} \int_C \frac{(f(t) + d_n)!}{t^{k+1}}\, dt ,$$

where C is the curve $|t| = r > 1$. If we let

$M_n(r) = \sup_{t\in C} \left| \prod_1^n \frac{f(t) + d_k}{f(z_1) + d_k} \right|$, then Cauchy's estimate gives

$|c^*_{nk}| \leq \frac{M_n(r)}{r^k}$, so

$$(1.32) \quad |\sum_k c^*_{nk} s_k| \leq O(1) \sum_k |c^*_{nk}| \leq O(1)\cdot M_n(r) \sum_0^\infty \frac{1}{r^k} =$$

$$\frac{rO(1)\cdot M_n(r)}{r - 1} = O(1)\cdot M_n(r).$$

But $M_n(r) = \sup_{t\in C} \left| \prod_1^n \frac{f(t) + d_k}{f(z_1) + d_k} \right| = \sup_{t\in C} \prod_1^n \left| 1 + \frac{f(t) - f(z_1)}{f(z_1) + d_k} \right|$

$$\leq \prod_1^n \left(1 + \frac{O(1)}{|f(z_1) + d_k|}\right) = O(1),$$ since the convergence of $\sum_{\rho_n \neq 0} \frac{1}{\rho_n}$ implies the convergence of the infinite product. The result follows.

One class of $(f,d_n,z_1)^*$ methods is easily discerned.

<u>Theorem 1.33</u>: In the notation of (1.3), let $\rho_n \longrightarrow \infty$, $\theta_n \longrightarrow \theta$, and $\sum\limits_{\rho_n \neq 0} \frac{1}{\rho_n} = \infty$. Let f be holomorphic on the closed unit disc and let C be the circle about the origin of radius $r > 1$. Then $(f, d_n, z_1)^*$ sums every bounded sequence to zero if

$$\sup_{t \in C} \operatorname{Re}\left([\overline{f(t)} - \overline{f(z_1)}]\, e^{i\theta} \right) < 0.$$

<u>Proof:</u> By (1.32), $|\sum\limits_k c^*_{nk}\, s_k| \leq O(1) \cdot M_n(r)$. We may show $M_n(r) \longrightarrow 0$ by proceeding in a manner analogous to that used in the proof of Lemma 1.14.

Let $f(t) = u(t) + iv(t)$. Define

$$Q_k(t) = \frac{\rho_k[(u(t)-a)\cos\theta_k + (v(t)-b)\sin\theta_k]}{|f(z_1) \quad d_k|^2} +$$

$$\frac{u^2(t) + v^2(t) - a^2 - b^2}{2|f(z_1) + d_k|^2}.$$

Now, $u(t)$ and $v(t)$ are bounded on C and $\rho_k \longrightarrow \infty$, so $\rho_k Q_k(t) \longrightarrow (u(t) - a)\cos\theta + (v(t) - b)\sin\theta$ uniformly on C. Since C is compact, there is a $t_n \in C$ such that $M_n(r) = \left| \prod\limits_1^n \frac{f(t_n) + d_k}{f(z_1) + d_k} \right|$. From the uniform convergence, $\rho_k Q_k(t_n) \longrightarrow (u(t_n) - a)\cos\theta + (v(t_n) - b)\sin\theta$ $= \operatorname{Re}([f(t_n) - f(z_1)]\, e^{-i\theta}) \leq \sup\limits_{t \in C} \operatorname{Re}([f(t) - f(z_1)]e^{-i\theta}) =$ $-2\delta < 0$, so that for $k > N$ we have $\rho_k Q_k(t_n) < -\delta$, or $Q_k(t_n) < -\delta/\rho_k$, and N is independent of n. Then, by

Lemma 1.13,

$$\left|\frac{f(t_n) + d_k}{f(z_1) + d_k}\right| \leq \exp\{Q_k(t_n)\} < \exp\{-\frac{\delta}{\rho_k}\} \quad \text{for} \quad k > N.$$

It follows that $M_n(r) = \left|\prod_1^N \frac{f(t_n) + d_k}{f(z_1) + d_k}\right| \cdot \prod_{N+1}^n \left|\frac{f(t_n) + d_k}{f(z_1) + d_k}\right| \leq$

$\left|\prod_1^N \frac{f(t_n) + d_k}{f(z_1) + d_k}\right| \cdot \exp\{-\delta \sum_{N+1}^n \frac{1}{\rho_k}\}$, whence $\lim_{n \to \infty} M_n(r) = 0$.

To show that the hypotheses of Theorem 1.33 can be satisfied, consider the example $f(z) = \exp\{ze^{-i \arg z_1}\}$. Let $\alpha = \arg z_1$ and let $t = re^{i\phi}$. Simple calculations show that $f(t) = e^{r\cos(\phi-\alpha)}[\cos(r\sin(\phi-\alpha)) + i\sin(r\sin(\phi-\alpha))]$, so $u = \operatorname{Re} f(t) = e^{r\cos(\phi-\alpha)} \cdot \cos(r\sin(\phi-\alpha)) \leq e^r$ for all $t \in C$. On the other hand, $f(z_1) = e^{|z_1|} = a$. Then, if $\theta = 0$,

$$\sup_{t\in C} \operatorname{Re}([f(t) - f(z_1)]e^{-i\theta}) = \sup_{t\in C}(u - a) \leq e^r - e^{|z_1|} < 0$$

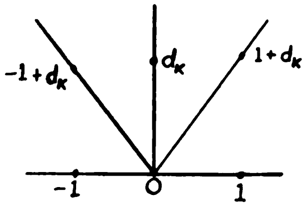
if $r < |z_1|$. Hence, we have only to choose $1 < r < |z_1|$ and $\theta = 0$ to have the above choice of f satisfy the conditions of Theorem 1.33.

We conclude this chapter with two examples. In view of Theorems 1.15, 1.19, and 1.28, the question as to whether or not there is a regular $(f,d_n,z_1)^*$ method for which $f(z_1) \neq f(1)$ arises naturally. The first example exhibits such a method. The second example shows that Theorems 1.15, 1.19, and 1.28 fail if the regularity in the conclusion is weakened to conservativity.

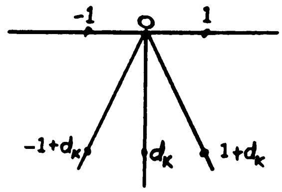
<u>Example 1.34</u>: The following result is due to Miracle ([12], Theorem 2.1):

Let $\{\lambda_n\}_1^\infty$ be a sequence of positive numbers such that $\Sigma\, \lambda_n^{-1} = \infty$. Let $d_{2n-1} = i\sqrt{\lambda_n}$ and $d_{2n} = -i\sqrt{\lambda_n}$. Then (z, d_n) is regular.

Let d_k be defined as in Miracle's theorem and suppose $\lambda_n \longrightarrow \infty$ monotonically. Let $P_k = \dfrac{1 + d_k}{-1 + d_k}$ and $\phi_k = \text{Arg } P_k$, where Arg denotes the principal branch. Let $\Psi_k = \text{Arg }(1 + d_k)$ and $\Psi_k' = \text{Arg }(-1 + d_k)$. We may suppose λ_1 is so large that $\phi_k = \Psi_k - \Psi_k'$ for every k. From the monotonicity of $\{\lambda_n\}$, it is geometrically clear that



<u>k odd</u>



<u>k even</u>

$|\phi_k| \to 0$ monotonically. Moreover, $d_{2m} = -d_{2m-1}$, so

$\Psi_{2m}' = \text{Arg}(-1 + d_{2m}) = \text{Arg}[-(1 + d_{2m-1})] = \Psi_{2m-1} - \pi$, and

$\Psi_{2m} = \text{Arg}(1 + d_{2m}) = \text{Arg}[-(-1 - d_{2m})] = \text{Arg}[-(-1 + d_{2m-1})]$

$= \Psi_{2m-1}' - \pi$. Then $\phi_{2m} = \Psi_{2m} - \Psi_{2m}' = \Psi_{2m-1}' - \Psi_{2m-1} = -\phi_{2m-1}$.

It follows that $s_{2n} = \sum_1^{2n} \phi_k = 0$, and $s_{2n-1} = s_{2n} - \phi_{2n} = -\phi_{2n} \to 0$, so $\sum_1^\infty \phi_k = 0$. As a consequence, if λ_1 is assumed to be sufficiently large, we have $\prod_1^\infty P_k = 1$. Now,

(z,d_n) is regular by Miracle's theorem, and Theorem 1.23 gives the regularity of $(z,d_n,-1)^*$.

Example 1.35: Let $f(z) = z$. Then from (0.5) follows

$$p_{nk} = \sum_{\substack{1 \le j_1 < j_2 < \\ \cdots < j_{n-k} \le n}} d_{j_1} d_{j_2} \cdots d_{j_{n-k}} \quad \text{if } n > k, \text{ and } p_{nn} = 1.$$

Similarly, if d_k is replaced by $|d_k|$, then

$$\prod_1^n (z + |d_k|) = \sum_{k=0}^n q_{nk} z^k,$$

where

$$q_{nk} = \sum_{\substack{1 \le j_1 < j_2 < \\ \cdots < j_{n-k} \le n}} |d_{j_1} d_{j_2} \cdots d_{j_{n-k}}| \quad \text{if } n > k, \text{ and } q_{nn} = 1.$$

It follows that $|p_{nk}| \le q_{nk}$, so $\sum_{k=0}^n |p_{nk}| \cdot |z|^k \le$

$\sum_{k=0}^n q_{nk} |z|^k = \prod_1^n (|z| + |d_k|)$. Then, letting $z = 1$,

$\sum_{k=0}^n |p_{nk}| \le \prod_1^n (1 + |d_k|)$. From this it is easily seen that

$\sum_{k=0}^n |c_{nk}| \le \prod_1^n \dfrac{1 + |d_k|}{|1 + d_k|}$ if c_{nk} is the typical element in the (z,d_n) matrix.

Now consider the (z,in^2) method. For this method,

$$\sum_{k=0}^n |c_{nk}| \le \prod_1^n \frac{1 + k^2}{|1+ik^2|} \le \prod_1^n \frac{1 + k^2}{k^2} = \prod_1^n \left(1+\frac{1}{k^2}\right) < \prod_1^\infty \left(1+\frac{1}{k^2}\right) < \infty,$$

so (0.1) is true, and, of course, (0.2) follows automatically for any (f,d_n) method. Now,

$$c_{nk} = \frac{1}{2\pi i}\int_C \frac{1}{t^{k+1}} \prod_1^n \frac{t+i\nu^2}{1+i\nu^2}\,dt = \frac{1}{2\pi i}\int_C \frac{1}{t^{k+1}} \prod_1^n \left(1 + \frac{t-1}{1+i\nu^2}\right)dt,$$

where C is a circle about the origin. Since the product $\prod_1^\infty \left(1 + \frac{t-1}{1+i\nu^2}\right)$ converges absolutely and uniformly on the compact set C, it is clear that $\lim_n c_{nk}$ exists, i.e., (0.3) is valid. Thus (z,in^2) is conservative, although Corollary 1.10 shows it is not regular. The product

$$\prod_1^\infty \frac{1+i\nu^2}{-1+i\nu^2} = \prod_1^\infty \left(1 + \frac{2}{-1+i\nu^2}\right)$$ is absolute convergent, hence convergent, so Theorem 1.23 says $(z,in^2,-1)^*$ is also conservative. But this method satisfies the hypotheses of Theorems 1.15, 1.19, and 1.28. In connection with Theorem 1.28 it should be observed that although the method is conservative, it is not multiplicative. By (0.5) $p_{no} = \prod_1^n d_k$, so from (1.2) we get $c^*_{no} = \prod_1^n \frac{d_k}{-1+d_k}$. Then

$$\left|\frac{1}{c^*_{no}}\right| = \left|\prod_1^n \frac{-1+d_k}{d_k}\right| = \prod_1^n \left|1 - \frac{1}{d_k}\right| \le \prod_1^n \left(1 + \frac{1}{|d_k|}\right) =$$

$$\prod_1^n \left(1 + \frac{1}{k^2}\right) < \prod_1^\infty \left(1 + \frac{1}{k^2}\right) < \infty$$

for each $n \ge 1$, so $\lim_n c^*_{no} \ne 0$.

CHAPTER II

REGULARITY CONDITIONS FOR (f, d_n, z_1)

All of the sufficiency conditions for regularity given in [15] by Smith require that the power series expansion of f about the origin have real, non-negative coefficients. We now give some sufficiency conditions which do not so require. Of course there are compensating additional hypotheses. We will need the following theorem of Bajsanski [5] and Clunie and Vermes [7]:

<u>Theorem 2.1</u>: Let f be holomorphic on the disc $|z| < R$, $R > 1$. Let $|f(z)| < 1$ for $|z| = 1$ except at a finite number of points ζ at which $|f(\zeta)| = 1$. Then, if $f^n(z) = \sum_k a_{nk} z^k$, it follows that $\sum_k |a_{nk}| = O(1)$ if and only if $\mathrm{Re} A_\zeta \neq 0$ for each such ζ, where $A_\zeta [i(z-1)]^{p(\zeta)}$ is the lead term of the Taylor expansion about 1 of $h_\zeta(z) - z^{\alpha(\zeta)}$, and where $h_\zeta(z) = \dfrac{f(\zeta z)}{f(\zeta)}$ and $\alpha(\zeta) = h'_\zeta(1)$.

<u>Theorem 2.2</u>: Let f satisfy the conditions of Theorem 2.1, $|z_1| = 1$, $f(z_1) = 1$, and

$$\sum_1^\infty \frac{(\mathrm{Im}\sqrt{d_n})^2}{|1+d_n|^2} < \infty .$$

If $d_n = \rho_n e^{i\theta_n}$, let $\limsup_{n \to \infty, d_n \neq 0} \mathrm{Re}[(\overline{f(0)} - 1) e^{i\theta_n}] < 0$. Then (f, d_n, z_1) is regular if $\sum_1^\infty \dfrac{1}{|1 + d_n|} = \infty$.

<u>Proof</u>: $(f(z) + d_n)! = \sum_k p_{nk} z^k = \sum_{j=0}^{n} \sigma_j f^{n-j}(z)$,

where $\sigma_0 = 1$ and $\sigma_j = \sum_{\substack{1 \le \nu_1 < \\ \dots < \nu_j \le n}} d_{\nu_1} \dots d_{\nu_j}$ for $j > 0$.

Now,

$$\sum_{j=0}^{n} \sigma_j f^{n-j}(z) = \sum_{j=0}^{n} \sigma_j \sum_{k=0}^{\infty} a_{n-j,k} z^k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n} \sigma_j a_{n-j,k} \right) z^k,$$

so

(2.3) $$p_{nk} = \sum_{j=0}^{n} \sigma_j a_{n-j,k} .$$

Since $f(z_1) = 1$,

(2.4) $$(f(z_1) + d_n)! = (1 + d_n)! = \sum_0^n \sigma_j .$$

Substituting ρ_n for d_n in (2.4) gives $(1 + \rho_n)! = \sum_0^n \sigma_j'$, where $\sigma_j' = 1$ and $\sigma_j' = \sum_{\substack{1 \le \nu_1 < \\ \dots < \nu_j \le n}} \rho_{\nu_1} \dots \rho_{\nu_j}$ for $j > 0$. Then $\sum_0^n |\sigma_j| \le \sum_0^n \sigma_j' = (1 + \rho_n)!$ It follows that

$$\sum_k |c_{nk}| = \frac{1}{|1+d_n|!} \sum_k |p_{nk} z_1^k| = \frac{1}{|1+d_n|!} \sum_k |p_{nk}| =$$

$$\frac{1}{|1+d_n|!} \sum_k \left| \sum_{j=0}^{n} \sigma_j a_{n-j,k} \right| \le \frac{1}{|1+d_n|!} \sum_{j=0}^{n} |\sigma_j| \sum_k |a_{n-j,k}|$$

$$\le \frac{B}{|1+d_n|!} \sum_0^n |\sigma_j| \le B \frac{(1+\rho_n)!}{|1+d_n|!}, \text{ where } B = \sup_n \sum_k |a_{nk}|.$$

But $$\frac{1+\rho_\nu}{|1+d_\nu|} \le \frac{(1+\rho_\nu)^2}{|1+d_\nu|^2} \le \exp\left\{-1 + \frac{(1+\rho_\nu)^2}{|1+d_\nu|^2}\right\} =$$

$$\exp\left\{\frac{-|1+d_\nu|^2 + (1+\rho_\nu)^2}{|1+d_\nu|^2}\right\} = \exp\left\{\frac{4(\operatorname{Im}\sqrt{d_\nu})^2}{|1+d_\nu|^2}\right\},$$

so $\frac{(1+\rho_n)!}{|1+d_n|!} \leq \exp\{4 \sum_1^n \frac{(\mathrm{Im}\sqrt{d_\nu})^2}{|1+d_\nu|^2}\} = O(1)$. The regularity condition (0.1) is thus verified, and (0.2) is always true. We need only show that $\lim_n c_{nk} = 0$ for each k. Suppose $\limsup_{n\to\infty, d_n\neq 0} \mathrm{Re}[(\overline{f(0)} - 1)e^{i\theta_n}] = -3\delta$, $\delta > 0$. By the maximum modulus principle, $|f(0)| < 1-2\varepsilon_1$ for some $\varepsilon_1 > 0$. Let $\varepsilon_2 = \min(\varepsilon_1, \delta)$. Now let Γ be a circle about the origin so small that $|f(t) - f(0)| < \varepsilon_2$ on Γ. Then $|f(t)| \leq |f(0)| + \varepsilon_2 < 1 - \varepsilon_1$ on Γ. $|\mathrm{Re}[(\overline{f(0)} - 1)e^{i\theta_n}] - \mathrm{Re}[(\overline{f(t)} - 1)e^{i\theta_n}]| = |\mathrm{Re}[(\overline{f(0)} - \overline{f(t)})e^{i\theta_n}]| \leq |f(0)-f(t)| < \varepsilon_2$ uniformly in t and n if $d_n \neq 0$. Consequently, $\limsup_{n\to\infty, d_n\neq 0} \mathrm{Re}[(\overline{f(0)} - 1)e^{i\theta_n}] = -3\delta \Rightarrow \mathrm{Re}[(\overline{f(0)} - 1)e^{i\theta_n}] < -2\delta$ for large n, so $\mathrm{Re}[(\overline{f(t)} - 1)e^{i\theta_n}] < -2\delta + \varepsilon_2 \leq -\delta$ for large n. If we let $f(t) = u(t) + iv(t)$, then

(2.5) $\mathrm{Re}[(\overline{f(t)} - 1)e^{i\theta_n}] = (u - 1)\cos\theta_n + v\sin\theta_n < -\delta$

on Γ for $n > N$, $d_n \neq 0$.

Let $0 < \omega < 1$ and choose ε so that $0 < \varepsilon < \min\left(\frac{\varepsilon_1}{7}, \frac{\omega^2\delta}{\sqrt{2}(1+\omega)}\right)$. Let $Q_n(t) = \varepsilon^2 + u^2 + v^2 + 2\varepsilon\sqrt{(u + \rho_n\cos\theta_n)^2 + (v + \rho_n\sin\theta_n)^2} +$

$$2\rho_n[(u-1)\cos\theta_n + v\sin\theta_n].$$

Then if $t \in \Gamma$, $n > N$, and $0 < \rho_n \leq \omega$, we get

$$Q_n(t) < \varepsilon^2 + |f(t)|^2 + 6\varepsilon - 2\rho_n\delta \leq \varepsilon + |f(t)| + 6\varepsilon \leq$$

$$|f(t)| + \varepsilon_1 < 1.$$

On the other hand, if $\rho_n \geq \omega$, then

$$\frac{\varepsilon}{\delta}\sqrt{(\frac{u}{\rho_n} + \cos\theta_n)^2 + (\frac{v}{\rho_n} + \sin\theta_n)^2} < \frac{\omega^2}{\sqrt{2}(1+\omega)}\sqrt{(\frac{1}{\omega} + 1)^2(2)}$$

$$= \omega < 1,$$

so a multiplication through by $\delta\rho_n$ gives

$$\varepsilon\sqrt{(u + \rho_n\cos\theta_n)^2 + (v + \rho_n\sin\theta_n)^2} < \delta\rho_n.$$

But then,

$$Q_n(t) < \varepsilon^2 + u^2 + v^2 + 2\varepsilon\sqrt{(u + \rho_n\cos\theta_n)^2 + (v + \rho_n\sin\theta_n)^2}$$

$$- 2\rho_n\delta = \varepsilon^2 + |f(t)|^2$$

$$-2[\delta\rho_n - \varepsilon\sqrt{(u + \rho_n\cos\theta_n)^2 + (v + \rho_n\sin\theta_n)^2}]$$

$$< \varepsilon^2 + |f(t)|^2 < \varepsilon + |f(t)| < \frac{\varepsilon_1}{7} + (1 - \varepsilon_1) < 1.$$

Thus in any event, $Q_n(t) < 1$ on Γ for $n > N$, $d_n \neq 0$. But this is equivalent to $|1 + d_n|^2 > (\varepsilon + |f(t) + d_n|)^2$, or $|1 + d_n| > \varepsilon + |f(t) + d_n|$, or $-|1 + d_n| + |f(t) + d_n| < -\varepsilon$ on Γ for $n > N$, $d_n \neq 0$. This is also clearly true for all n for which $d_n = 0$. It follows immediately that

$$\left|\frac{f(t) + d_\nu}{1 + d_\nu}\right| \leq \exp\{-1 + \left|\frac{f(t) + d_\nu}{1 + d_\nu}\right|\} =$$

$$\exp\left\{\frac{-|1+d_\nu| + |f(t)+d_\nu|}{|1+d_\nu|}\right\} < \exp\left\{-\frac{\varepsilon}{|1+d_\nu|}\right\}$$

on Γ if $\nu > N$, so $\prod_1^n \left|\frac{f(t)+d_\nu}{1+d_\nu}\right| = \prod_1^N \left|\frac{f(t)+d_\nu}{1+d_\nu}\right| \cdot \prod_{N+1}^n \left|\frac{f(t)+d_\nu}{1+d_\nu}\right|$

$< O(1) \exp\left\{-\varepsilon \sum_{N+1}^n \frac{1}{|1+d_\nu|}\right\} = o(1)$ as $n \to \infty$ uniformly on Γ. Consequently, $c_{nk} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{t^{k+1}} \prod_1^n \frac{f(t)+d_\nu}{1+d_\nu}\,dt$

$$\longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Corollary 2.6: Let f satisfy the conditions of Theorem 2.1, $|z_1| = 1$, $f(z_1) = -1$, and $\sum_1^\infty \frac{(\operatorname{Im}\sqrt{-d_n})^2}{|-1+d_n|^2} < \infty$. If $d_n = \rho_n e^{i\theta_n}$, let $\limsup_{n\to\infty,\, d_n\neq 0} \operatorname{Re}\,[(\overline{f(0)}+1)e^{i\theta_n}] < 0$.

Then (f, d_n, z_1) is regular if $\sum_1^\infty \frac{1}{|-1+d_n|} = \infty$.

Proof: The above proof, with only minor changes, proves the corollary.

We remark here that inspection of the proofs above gives immediately the following two corollaries.

Corollary 2.7: Let f satisfy the conditions of Theorem 2.1, $|z_1| = 1$, $f(z_1) = 1$, and

$$\prod_1^n \frac{1+\rho_\nu}{1+d_\nu} = O(1),$$

where $d_n = \rho_n e^{i\theta_n}$. Let $\limsup_{n\to\infty,\, d_n\neq 0} \operatorname{Re}[(\overline{f(0)}-1)e^{i\theta_n}] < 0$.

Then (f,d_n,z_1) is regular if $\sum_1^{\infty} \frac{1}{|-1 + d_n|} = \infty$.

There are two cases in which Theorem 2.2 and Corollary 2.6 become particularly simple.

<u>Corollary 2.9</u>: Let f satisfy the conditions of Theorem 2.1, $|z_1| = 1$, $f(z_1) = 1$, and $d_n \geq 0$. Then (f,d_n,z_1) is regular if $\sum_1^{\infty} \frac{1}{1 + d_n} = \infty$.

<u>Corollary 2.10</u>: Let f satisfy the conditions of Theorem 2.1, $|z_1| = 1$ $f(z_1) = -1$, and $d_n \leq 0$. Then (f,d_n,z_1) is regular if $\sum_1^{\infty} \frac{1}{|-1 + d_n|} = \infty$.

It is shown in [7] that the functions

$$(2.11) \quad f(z) = \frac{1}{2 + i}(1 + iz + z^2),$$

$$(2.12) \quad f(z) = \frac{\omega z - 1}{\omega - 1} e^{\omega(z-1)}, \quad \omega = e^{i\pi/3},$$

$$(2.13) \quad f(z) = \exp\{\hat{\omega}(z - 1) - \hat{\omega}^2(z^2 - 1)\}, \ \hat{\omega} = e^{i\phi}, \ \cos\phi = \frac{1}{4},$$

all satisfy the conditions of Theorem 2.1 and have $f(1) = 1$. If $z_1 = 1$, then these functions fulfill the hypotheses of Corollary 2.9. Moreover, if f is the function (2.13), $\beta = -\sin\phi$, and $F(z) = f^{2\pi/\beta}(z)$, then an easy but tedious calculation shows that F also fulfills the conditions of Corollary 2.9 if $z_1 = e^{-2i\phi}$. If f is any function obeying the requirements of Theorem 2.1, then the maximum modulus principle implies $|f(0)| < 1$. Then $\operatorname{Re} f(0) < 1$, so if $f(0) = a + ib$, $a - 1 < 0$. Hence $\operatorname{Re}[(\overline{f(0)} - 1)e^{i\theta_n}] =$

$(a - 1)\cos \theta_n + b \sin \theta_n < 0$ if θ_n is sufficiently close to a multiple of 2π. It follows that if θ_n is so restricted and $z_1 = 1$, then the functions (2.11), (2.12), and (2.13) also meet the requirements of Theorem 2.2.

Definition 2.14: Let f be holomorphic on the disc $|z| < R$, $R > 1$, and let $f^n(z) = \sum_{k=0}^{\infty} a_{nk} z^k$ for $n \geq 0$. Then the summability method determined by the matrix (a_{nk}) is called a Sonnenschein method.

In [7] the following theorem is proved:

Theorem 2.15: The Sonnenschein method generated by f is regular if and only if either $f(z) = z^m$ for a positive integer m, or f satisfies the conditions of Theorem 2.1 and $f(1) = 1$.

The next two results are immediate consequences.

Theorem 2.16: Let f be holomorphic on the disc $|z| < R$, $R > 1$, and suppose $f(z) \neq z^m$, m a positive integer. Then if the Sonnenschein method generated by f is regular, so is the method (f, d_n), provided

$$\sum_1^{\infty} \frac{(\operatorname{Im} \sqrt{d_n})^2}{|1 + d_n|^2} < \infty, \quad \limsup_{n \to \infty,\, d_n \neq 0} \operatorname{Re}[(\overline{f(0)} - 1)e^{i\theta_n}] < 0,$$

and $\sum_1^{\infty} \frac{1}{|1 + d_n|} = \infty$.

Proof: Apply Theorem 2.2, with $z_1 = 1$, and Theorem 2.15.

Corollary 2.17: Let f be holomorphic on the disc $|z| < R$, $R > 1$. Then if the Sonnenschein method generated by f is regular, so is the method (f, d_n), provided $d_n \geq 0$ and $\sum_1^{\infty} \frac{1}{1 + d_n} = \infty$.

Proof: If $f(z) = z^m$ for some positive integer m, then Lemma 1.12 gives the conclusion. Otherwise, Corollary 2.9, with $z_1 = 1$, gives the result.

CHAPTER III

COINCIDENCE OF METHODS

__Lemma 3.1:__ The method (f,d_n,z_1) is Sonnenschein if and only if $d_n \equiv d_1$.

__Proof:__ Suppose (f,d_n,z_1) is the Sonnenschein method generated by g. Then the matrix coefficients of the two methods are the same, so

$$\prod_1^n \frac{f(zz_1) + d_k}{f(z_1) + d_k} = g^n(z)$$

for $n \geq 1$ on the intersection of their domains. In particular, when $n = 1$ we have $g(z) = \dfrac{f(zz_1) + d_1}{f(z_1) + d_1}$.

Suppose $d_1 = \ldots = d_{n-1}$ for some $n > 1$. Then

$$g^n(z) = \prod_1^{n-1} \frac{f(zz_1) + d_k}{f(z_1) + d_k} \cdot \frac{f(zz_1) + d_n}{f(z_1) + d_n} = g^{n-1}(z) \cdot \frac{f(zz_1) + d_n}{f(z_1) + d_n},$$

so $g(z) = \dfrac{f(zz_1) + d_n}{f(z_1) + d_n}$. But then $d_n = d_1$, so, by induction, $d_m \equiv d_1$. On the other hand, if $d_n \equiv d_1$, then $\prod_1^n \dfrac{f(zz_1) + d_k}{f(z_1) + d_k} = g^n(z)$, where $g(z) = \dfrac{f(zz_1) + d_1}{f(z_1) + d_1}$, so the matrix coefficients of the two methods are clearly the same. Moreover, since z_1 is in the (open) disc of convergence of f, g has radius of convergence > 1. Thus (f,d_n,z_1) is the Sonnenschein method generated by g.

Lemma 3.2: Let b and $z_1 \neq 0$ be given. Then every Sonnenschein method generated by a function g with $g(1) = 1$ can be realized as an (f,b,z_1) method for appropriate f.

Proof: Define $f(z) = g(z/z_1) - b$. Since g has radius of convergence > 1, f has radius of convergence $> |z_1|$. Then, if $d_k \equiv b$,

$$\prod_1^n \frac{f(zz_1) + d_k}{f(z_1) + d_k} = \prod_1^n \frac{g(z)}{g(1)} = g^n(z),$$

so the (f,b,z_1) coefficients are those of the Sonnenschein method.

If $g(1) \neq 1$, then the corresponding Sonnenschein method cannot be realized as an (f,d_n,z_1) method since the equality of the methods would imply that

$$g^n(z) = \prod_1^n \frac{f(zz_1) + d_k}{f(z_1) + d_k} \quad \text{for} \quad n \geq 1,$$

whence $g(1) = 1$. But any Sonnenschein method can be realized as an $(f,d_n,z_1)^*$ method if $g(z_1) = 1$ for some z_1 in the disc of convergence of g. In fact, the Sonnenschein method generated by g is easily seen to be the method $(g,o,z_1)^*$.

If the matrix of the Sonnenschein method generated by g is (a_{nk}) and $g(1) \neq 1$, then either $\lim_n \sum_k a_{nk} = \lim_n g^n(1)$ does not exist or it is zero. In the former case the method cannot be conservative, and in the latter case all constant sequences are summed to zero ([8], p. 43). Consequently, in some sense the "interesting" Sonnenschein methods are those for which $g(1) = 1$. Lemma 3.2 shows that the class of "interesting" Sonnenschein methods is a subclass of the set of (f, d_n, z_1) methods. We will now prove some results analagous to those appearing in [14] and which may be considered extensions of those results if attention is restricted to the "interesting" Sonnenschein methods.

<u>Definition 3.3</u>: If s_j is a term of a sequence, let the operator Δ be defined by $\Delta^n s_j = \sum_{k=0}^{n} (-1)^k \binom{n}{k} s_{j+k}$, $n \geq 0$.

<u>Definition 3.4</u>: The Hausdorff method (H, μ) is defined by the matrix $H = (h_{nk})$, where $h_{nk} = \binom{n}{k} \Delta^{n-k} \mu_k$ for $k \leq n$, and $h_{nk} = 0$ for $k > n$.

<u>Definition 3.5</u>: The quasi-Hausdorff method (H^*, μ) is defined by the matrix $H^* = (h^*_{nk})$, where $h^*_{nk} = \binom{k}{n} \Delta^{k-n} \mu_n$ for $k \geq n$, and $h^*_{nk} = 0$ for $k < n$.

<u>Definition 3.6</u>: The Euler method (E, p) and the circle method (T, p) are the Hausdorff and quasi-Hausdorff methods, respectively, with $\mu_n = p^n$ and $\mu_n = p^{n+1}$.

In an unpublished paper, Ramanujan has given:

<u>Definition 3.7</u>: Let the modified quasi-Hausdorff method $(\overline{H}^*, \mu)$ generated by the quasi-Hausdorff method (H^*, μ)

be given by the matrix $\overline{H}^* = (\overline{h}^*_{nk})$, where

$$\overline{h}^*_{nk} = \begin{cases} 1 & \text{for } n=k=0 \\ 0 & \text{for } n=0, k\geq 1 \\ 0 & \text{for } k=0, n\geq 1 \\ h^*_{n-1,k-1} & \text{for } n\geq 1, k\geq 1 \end{cases} \quad \text{so that } \overline{H}^* = \left[\begin{array}{c|c} 1 & 0 \quad 0 \quad 0 \ \ldots \\ \hline \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \end{matrix} & \left[\quad H^* \quad \right] \end{array}\right].$$

He has also pointed out that $\overline{H}^*(0, s_0, s_1, \ldots) = H^*(s_0, s_1, s_2, \ldots)$, so $\overline{H}^*$ is not as artificial as it may look; it arises naturally when considering the translativity problem for H^*.

In [14], the following proposition is stated:

"The circle method (T, μ_1) is the only method, regular or not, which is both quasi-Hausdorff and Sonnenschein."

Bojanic [6] subsequently pointed out that the proposition is not correct as it stands, but can be made so by replacing "circle" by "identity". In his unpublished paper Ramanujan showed that the above proposition can be made correct if it is altered so as to read:

(3.8) "The modified quasi-Hausdorff method $(\overline{H}^*, \mu)$ is Sonnenschein if and only if (H^*, μ) is the circle method."

We remark here that his proof shows that if the Sonnenschein method in question is generated by $g(z)$, then

(3.9) $g(z) = \dfrac{pz}{1 - (1 - p)z}$, where $p = \mu_0$ and $|1 - p| < 1$.

We will now prove an analog of (3.8) for (f,d_n,z_1) methods.

<u>Theorem 3.10</u>: The modified quasi-Hausdorff method $(\overline{H}^*,\mu)$, $\mu_n \neq 0$, is (f,d_n,z_1) if either

(3.11) the associated quasi-Hausdorff method (H^*,μ) is the circle method (T,p) with $|1-p| < 1$,

or

(3.12) $f(z) - f(0) = \dfrac{cpz}{z_1 - (1-p)z}$, where $\mu_n = p^{n+1}$, $|1-p| < 1$, and $d_n \equiv -f(0)$,

and only if both (3.11) and (3.12) hold.

<u>Proof</u>: Suppose $(\overline{H}^*,\mu) = (f,d_n,z_1)$, and let the (f,d_n,z_1) matrix be (c_{nk}). We first make use of the fact that $c_{nk} = \overline{h}^*_{nk}$, and then use the relation $(f(zz_1)+d_n)!$ $= (f(zz_1)+d_n)(f(zz_1)+d_{n-1})!$ to get first

$$(3.13)\qquad \frac{(f(zz_1)+d_n)!}{(f(z_1)+d_n)!} = \sum_{k=n}^{\infty} \binom{k-1}{n-1} \Delta^{k-n} \mu_{n-1} z^k, \quad n \geq 1$$

and then

$$(3.14)\qquad \frac{f(zz_1)+d_n}{f(z_1)+d_n} \sum_{k=n-1}^{\infty} \binom{k-1}{n-1} \Delta^{k-n+1} \mu_{n-2} z^k =$$

$$\sum_{k=n}^{\infty} \binom{k-1}{n-1} \Delta^{k-n} \mu_{n-1} z^k, \quad n \geq 2.$$

Setting $n = 1$ in (3.13) gives $\dfrac{f(zz_1)+d_1}{f(z_1)+d_1} = \sum_{k=1}^{\infty} \Delta^{k-1} \mu_0 z^k$,

so if $z = 0$ we have $f(0) + d_1 = 0$. Let

$f(z) = \sum_0^\infty a_k z^k$. Then from (3.14) follows, for $n \geq 2$,

$$\frac{(a_0+d_n) + \sum_1^\infty a_k z_1^k z^k}{f(z_1) + d_n} \cdot \sum_{k=n-1}^\infty \binom{k-1}{n-2} \Delta^{k-n+1} \mu_{n-2} z^k =$$

$$\sum_{k=n}^\infty \binom{k-1}{n-1} \Delta^{k-n} \mu_{n-1} z^k,$$

so equating coefficients of z^{n-1} gives $(a_0 + d_n)\mu_{n-2} = 0$, whence $d_n = -a_0 = -f(0)$. It follows immediately, by induction, that $d_n \equiv -f(0)$. But then Lemma 3.1 says that (f, d_n, z_1) is the Sonnenschein method generated by $g(z) = \dfrac{f(zz_1) - f(0)}{f(z_1) - f(0)}$. Now (3.8) implies that (H^*, μ) is the circle method, so the necessity of (3.11) has been shown along with the necessity for $d_n \equiv -f(0)$. Moreover, (3.9) gives the formula $g(z) = \dfrac{f(zz_1) - f(0)}{f(z_1) - f(0)} = \dfrac{pz}{1 - (1 - p)z}$, $|1 - p| < 1$. If $c = f(z_1) - f(0)$, then $f(zz_1) - f(0) = \dfrac{cpz}{1 - (1 - p)z}$, so $f(z) - f(0) = \dfrac{cpz}{z_1 - (1 - p)z}$. Again setting $n = 1$ in (3.13), we get $\dfrac{f(zz_1) - f(0)}{f(z_1) - f(0)} = \sum_{k=1}^\infty \Delta^{k-1} \mu_0 z^k = \dfrac{pz}{1 - (1 - p)z}$. But it is readily shown that

$$\frac{pz}{1 - (1 - p)z} = \sum_{k=1}^\infty (1 - p)^{k-1} pz^k = \sum_{k=1}^\infty \Delta^{k-1} p\, z^k, \text{ so}$$

$\Delta^{k-1}\mu_0 = \Delta^{k-1}p$ and $p = \mu_0$. Definition (3.6) now implies that $\mu_n = p^{n+1}$ for each n. Hence, (3.12) follows.

Conversely, suppose (3.11) is true. Then $\mu_n = p^{n+1}$, so $\overline{h}^*_{nk} - h^*_{n-1,k-1} = \binom{k-1}{n-1}\Delta^{k-n}\mu_{n-1} = \binom{k-1}{n-1}\Delta^{k-n}p^n$ for $n \geq 1$, $k \geq 1$. It follows that

$$\sum_{k=n}^{\infty} \overline{h}^*_{nk} z^k = \sum_{k=n}^{\infty} \binom{k-1}{n-1}\Delta^{k-n}p^n z^k = \sum_{k=n}^{\infty} \binom{k-1}{n-1}(1-p)^{k-n}p^n z^k =$$

$$\left(\frac{pz}{1-(1-p)z}\right)^n = g^n(z),$$

where $g(z) = \dfrac{pz}{1-(1-p)z}$. It is now immediate that $(\overline{H}^*,\mu)$ is the Sonnenschein method generated by g provided g has radius of convergence > 1, i.e., provided $|1-p| < 1$. (3.11) insures this condition is met. Lemma 3.2 now says that $(\overline{H}^*,\mu)$ is a method (f,b,z_1). Finally, suppose (3.12) is true. It is easily seen that $c = f(z_1) - f(0)$, so $\dfrac{f(z)-f(0)}{f(z_1)-f(0)} = \dfrac{pz}{z_1-(1-p)z}$ and thus $g(z) = \dfrac{f(zz_1)+d_m}{f(z_1)+d_m}$ $= \dfrac{f(zz_1)-f(0)}{f(z_1)-f(0)} = \dfrac{pz}{1-(1-p)z}$. Since $|1-p| < 1$, g has radius of convergence > 1, so it is clear that

$$\prod_1^n \frac{f(zz_1)+d_m}{f(z_1)+d_m} = g^n(z),$$ whence (f,d_n,z_1)

is the Sonnenschein method generated by g. We see now that

$$g^n(z) = \left(\frac{pz}{1-(1-p)z}\right)^n = \sum_{k=n}^{\infty}\binom{k-1}{n-1}(1-p)^{k-n}p^n z^k = \sum_{k=n}^{\infty}\binom{k-1}{n-1}\Delta^{k-n}p^n z^k,$$

for $n \geq 1$, $k \geq 1$, so the (f,d_n,z_1) matrix is that of the modified quasi-Hausdorff method having $\mu_n = p^{n+1}$.

Lemma 3.15: The only matrix which is both (f,d_n,z_1) and quasi-Hausdorff is I.

Proof: From definition 3.5, the quasi-Hausdorff transform of $\{s_n\}$ has the form $t_n^* = \sum_{k=n}^{\infty} \binom{k}{n} \Delta^{k-n} \mu_n s_k$. If this is also an (f,d_n,z_1) transform, then for $n = 0$ we have $\Delta^k \mu_0 = 1$ when $k = 0$ and $\Delta^k \mu_0 = 0$ when $k > 0$. It follows immediately that $\mu_0 = \mu_1 = 1$. Suppose $\mu_0 = \mu_1 = \ldots = \mu_{k-1} = 1$. Then $0 = \Delta^k \mu_0 =$

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu_j = \mu_0 \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} + (-1)^k \mu_k = \mu_0 \sum_{j=0}^{k} (-1)^j \binom{k}{j}$$

$$- \mu_0 (-1)^k + (-1)^k \mu_k = (-1)^k (\mu_k - \mu_0), \text{ so } \mu_k = \mu_0 = 1.$$

By induction, $\mu_n \equiv 1$. Then $k > n$ implies $\Delta^{k-n} \mu_n = \Delta^{k-n} \mu_0 = 0$. Clearly, $t_n^* = s_n$, so t_n^* is the identity transform.

It is remarked in [14] that the Euler method is the only one which is both Sonnenschein and Hausdorff. In the same vein is

Theorem 3.16: The Hausdorff method (H,μ) is (f,d_n,z_1) if either

(3.17) (H,μ) is the Euler method (E,p),

or

(3.18) there is a constant d_1' such that $(f,d_n,z_1) =$

$$(z,d_1', z_1) \text{ and } \mu_n = p^n, \text{ where } p = \frac{z_1}{z_1 + d_1'},$$

and only if both (3.17 and (3.18) hold.

Proof: We will first show the necessity of both (3.17) and (3.18). Since the Hausdorff matrix is triangular, it can be an (f, d_n, z_1) matrix only if f is a first degree polynomial, say $f(z) = \alpha z + \beta$. But then

$$\frac{f(zz_1) + d_k}{f(z_1) + d_k} = \frac{\alpha zz_1 + \beta + d_k}{\alpha z_1 + \beta + d_k} = \frac{zz_1 + d_k'}{z_1 + d_k'},$$

so $(f, d_n, z_1) = (z, d_n', z_1)$. Now form the analogs of (3.13) and (3.14), getting

$$(3.19) \quad \frac{(zz_1 + d_n')!}{(z_1 + d_n')!} = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \mu_k z^k, \quad n \geq 1,$$

and

$$(3.20) \quad \frac{zz_1 + d_n'}{z_1 + d_n'} \sum_{k=0}^{n-1} \binom{n-1}{k} \Delta^{n-1-k} \mu_k z^k = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \mu_k z^k, \quad n \geq 2.$$

By letting $n = k = 0$, it is seen that the equality of the matrices requires that $\mu_0 = 1$. Setting $n = 1$ in (3.19) gives $\frac{z_1}{z_1 + d_1'} = \mu_1$. For $n \geq 2$, we equate coefficients of z^n in (3.20) to get $\frac{z_1}{z_1 + d_n'} \mu_{n-1} = \mu_n$, so $\mu_n = \frac{z_1^n}{(z_1 + d_n')!}$ for each $n \geq 1$ by induction. Suppose $d_k' = d_1'$ for all $k < n$, $n \geq 2$. Equating coefficients of z^{n-1} in (3.20) gives

$$\frac{z_1}{z_1 + d_n'} (n - 1)\Delta\mu_{n-2} + \frac{d_n'}{z_1 + d_n'} \mu_{n-1} = n\Delta\mu_{n-1}, \quad n \geq 2.$$

This expression, together with the above formula for μ_n, yields $d_n' = d_{n-1}' = d_1'$ after a simple calculation. Then

$\mu_n = p^n$, where $p = \frac{z_1}{z_1 + d_1'}$, so (3.18) follows. But now definition (3.6) says that $(H,\mu) = (E,p)$, so (3.17) is true, too. Conversely, suppose (3.17) obtains, so that $\mu_n = p^n$ for every $n \geq 0$ and some p. Then

$$\sum_{k=0}^{n} \binom{n}{k}\Delta^{n-k}\mu_k z^k = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j}(-1)^j \mu_{k+j} z^k =$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k \sum_{j=0}^{n-k} \binom{n-k}{j}(-p)^j z^k = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} z^k = (pz + (1-p))^n,$$

so the method (H,μ) is the Sonnenschein method generated by $g(z) = pz + 1-p$. Lemma 3.2 now gives the desired result. If (3.18) is valid, then $\frac{(zz_1 + d_n)!}{(z_1 + d_n)!} = \left(\frac{zz_1 + d_1'}{z_1 + d_1'}\right)^n =$ $(pz + 1-p)^n = \sum_{k=0}^{n} \binom{n}{k}\Delta^{n-k}\mu_k z^k$, so the method (z,d_1',z_1) is also the method (H,μ).

<u>Definition 3.21</u>: Let $\{q_m\}_0^{\infty}$ be a complex sequence with $\lambda_n = \sum_0^n q_m \neq 0$ for $n \geq 0$. Then the Norlund method determined by the sequence $\{q_m\}$ is defined by the matrix $A = (a_{nk})$, where $a_{nk} = q_{n-k}/\lambda_n$ for $k \leq n$, and $a_{nk} = 0$ for $k > n$.

Suppose the Norlund matrix A is also an (f,d_n,z_1) matrix. Since A is triangular, we may, as in the proof of Theorem 3.16, assume $f(z) = z$. It must follow that

$$(3.22)\quad \frac{(zz_1 + d_n)!}{(z_1 + d_n)!} = \sum_{k=0}^{n} \frac{q_{n-k}}{\lambda_n} z^k \quad , \quad n \geq 1 ,$$

and from this,

$$(3.23)\quad \frac{zz_1 + d_n}{z_1 + d_n} \sum_{k=0}^{n-1} \frac{q_{n-k-1}}{\lambda_{n-1}} z^k = \sum_{k=0}^{n} \frac{q_{n-k}}{\lambda_n} z^k , \quad n \geq 2.$$

Equating constant terms in (3.22) we get, if $n = 1$,

$$(3.24)\quad q_0 d_1 = q_1 z_1 .$$

Equating coefficients of z^n in (3.23) gives

$$\frac{z_1 q_0}{\lambda_{n-1}(z_1 + d_n)} = \frac{q_0}{\lambda_n} , \text{ so } \frac{z_1 + d_n}{z_1} = \frac{\lambda_n}{\lambda_{n-1}} \quad \text{and}$$

$$1 + \frac{d_n}{z_1} = 1 + \frac{q_n}{\lambda_{n-1}} . \quad \text{Then,}$$

$$(3.25)\quad d_n \lambda_{n-1} = q_n z_1 , \quad n \geq 2.$$

Equating constant terms in (3.23), we have

$$(3.26)\quad \frac{d_n q_{n-1}}{\lambda_{n-1}(z_1 + d_n)} = \frac{q_n}{\lambda_n} , \quad n \geq 2.$$

We need two more relations:

$$(3.27)\quad \frac{q_0(d_1 + d_2)}{\lambda_1(z_1 + d_2)} = \frac{q_1}{\lambda_2} ,$$

$$(3.28)\quad \frac{z_1}{\lambda_1(z_1 + d_2)} = \frac{1}{\lambda_2} .$$

The second of these is obtained by equating the coefficients of z^2 in (3.23) with $n = 2$; the first is obtained by equating coefficients of z in (3.23) with $n = 2$ and using (3.24).

Substituting from (3.28) into (3.27) gives

$$q_0(d_1 + d_2) = q_1 z_1 . \quad \text{Then, using (3.24),}$$

we get $q_0(d_1 + d_2) = q_0 d_1$, so $d_2 = 0$. Then (3.25) implies $q_2 = 0$, and from (3.26) it is clear by recursion that $q_j = 0$ for $j \geq 2$. Then from (3.25), $d_n = 0$ for $n \geq 2$. Then the matrix in question reduces to

(3.29)
$$A = \lambda_1^{-1} \begin{pmatrix} \lambda_1 & & & & \\ q_1 & q_0 & & \bigcirc & \\ & q_1 & q_0 & & \\ & & q_1 & q_0 & \\ & \bigcirc & & \ddots & \ddots \end{pmatrix}, \text{ where } q_1 = 0 \text{ or } q_1 z_1 = q_0 d_1 \neq 0.$$

We have proved

<u>Theorem 3.30</u>: A Norlund matrix which is also an (f,d_n,z_1) matrix must have the form (3.29). If $q_1 = 0$, then the matrix is the identity, and this case arises precisely when $f(z) = z$ and $d_n \equiv 0$. The alternative case arises precisely when $f(z) = z$, $d_n = 0$ for $n \geq 2$, and $d_1 q_0 = q_1 z_1 \neq 0$.

<u>Corollary 3.31</u>: The identity matrix is the only one which is both Norlund and Sonnenschein.

<u>Proof</u>: Let $A = (a_{nk})$ be a matrix which is both Norlund and Sonnenschein, and let the generating function of A as the latter method be g. Then $g(1) = \sum_0^1 a_{1k} = \lambda_1^{-1}(q_1 + q_0) = 1$, so by Lemma 3.2, A is an (f,d_n,z_1) matrix. Then Lemma 3.1 says $d_n \equiv d_1$, whence Theorem 3.30 implies $A = I$.

Corollary 3.31 is Proposition 1 of [14].

<u>Corollary 3.32</u>: The identity is the only method which is both (f,d_n,z_1) and (C,α).

Proof: (C,α) is defined for each real α which is not a negative integer, and is the Norlund method determined by the sequence $\{q_n\}_0^\infty$, where $q_n = q_n(\alpha) = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!}$ for $n > 0$, and $q_0(\alpha) = 1$. By Theorem 3.30, $q_2(\alpha) = \frac{\alpha(\alpha+1)}{2} = 0$, so $\alpha = 0$. But $(C,0)$ is the identity.

Definition 3.33: Given a complex sequence $\{q_m\}_0^\infty$, let $\lambda_n = \sum_0^n q_m \neq 0$ for $n \geq 0$. Then the Riesz method determined by $\{q_m\}$ is defined by the matrix $A = (a_{nk})$, where $a_{nk} = q_k/\lambda_n$ for $k \leq n$, and $a_{nk} = 0$ for $k > n$.

Theorem 3.34: There is no matrix which defines both a Riesz method and an (f,d_n,z_1) method.

Proof: We proceed as before, Since the Riesz matrix is triangular, if it is also an (f,d_n,z_1) matrix, we may assume $f(z) = z$. Then it must follow that

$$\frac{(zz_1+d_n)!}{(z_1+d_n)!} = \sum_{k=0}^{n} \frac{q_k}{\lambda_n} z^k, \quad n \geq 1,$$ and from this

that

$$(3.35) \quad \frac{zz_1+d_n}{z_1+d_n} \sum_{k=0}^{n-1} \frac{q_k}{\lambda_{n-1}} z^k = \sum_{k=0}^{n} \frac{q_k}{\lambda_n} z^k, \quad n \geq 2.$$

Equating constant terms in (3.35) leads to

$$(3.36) \quad d_n q_n = \lambda_{n-1} z_1 \neq 0, \quad n \geq 2.$$

From this it is clear that $q_n \neq 0$ for $n \geq 2$, so

$d_n = \frac{\lambda_{n-1} z_1}{q_n}$. Plugging this value for d_n into (3.35) gives, after a short computation,

$$(3.37) \quad \frac{q_n q_{n-1}}{\lambda_n \lambda_{n-1}} z^n + \frac{1}{\lambda_n} \sum_{k=1}^{n-1} \left(\frac{q_n q_{k-1}}{\lambda_{n-1}} + q_k\right) z^k + \frac{q_0}{\lambda_n} = \sum_{k=0}^{n} \frac{q_k}{\lambda_n} z^k,$$

$$n \geq 2.$$

Again equating coefficients, we see that $q_n q_{k-1} = 0$ for $1 \leq k \leq n - 1$, so if $n \geq 4$ there is a $j \geq 2$ with $q_j = 0$, in violation of (3.36).

CHAPTER IV

INCLUSION RELATIONS

It is relatively easy to give an example of a conservative (f,d_n,z_1) method which is at least as strong as the $(C,1)$ method. We will give such an example below after some necessary tools have been developed. We will now concern ourselves with results in the opposite direction.

Lemma 4.1: Let f be holomorphic at the origin and let z_1 belong to its disc of convergence. Suppose $(f(z) + d_n)! = \sum_{k=0}^{\infty} p_{nk} z^k$, $n \geq 1$. Then

$$z_1 f'(z_1) \sum_1^n \frac{1}{f(z_1) + d_j} = \frac{1}{(f(z_1) + d_n)!} \sum_{k=1}^{\infty} k\, p_{nk}\, z_1^k .$$

Proof: Let $g_n(z) = (f(zz_1) + d_n)!$ Then $g_n'(z) = \sum_{k=1}^{\infty} kp_{nk}z_1^k z^{k-1}$, so $g_n'(1) = \sum_{k=1}^{\infty} kp_{nk}z_1^k$. Also, $g_n'(z) =$

$$z_1 f'(zz_1) \sum_{j=1}^{n} \prod_{\substack{1\le k\le n \\ k\neq j}} (f(zz_1) + d_k) = z_1 f'(zz_1)\cdot(f(zz_1) + d_n)!$$

$$\sum_1^n \frac{1}{f(zz_1) + d_j}, \text{ so } g_n'(1) = z_1 f'(z_1)\cdot(f(z_1)+d_n)!\sum_1^n \frac{1}{f(z_1)+d_j} .$$

The result follows.

Corollary 4.2: Let (f,d_n,z_1) be regular, $f(0)+d_n \neq 0$, and either

(4.3) $f(z_1) + d_n$ is real and has constant sign for large n,

or

(4.4) $\mathrm{Re}[f(z_1) + d_n] \equiv 0$ and $\mathrm{Im}[f(z_1) + d_n]$ has constant sign for large n.

Then if $f'(z_1) \neq 0$ it follows that

$$\frac{1}{(f(z_1) + d_n)!} \sum_{k=1}^{\infty} k\, p_{nk} z_1^k \neq o(1).$$

Proof: By Lemma 4.1 we need only show that

$\sum_1^n \frac{1}{f(z_1) + d_j} \neq o(1)$. Theorem 1 of [16] asserts that if $f(0) + d_n \neq 0$, then a necessary condition for the regularity of (f, d_n, z_1) is that $\sum_1^{\infty} \frac{1}{|f(0) + d_j|} = \infty$. But then $\sum_1^{\infty} \frac{1}{|f(z_1) + d_j|} = \infty$, too, whence so does $\left|\sum_1^{\infty} \frac{1}{f(z_1) + d_j}\right|$. The result follows.

We can now prove

Theorem 4.5: Let $f(z) = \sum_0^{\infty} a_k z^k$ have radius of convergence greater than $\rho > 0$, and let $a_k = 0$ if k is not a multiple of the integer $m > 1$. Suppose either (4.3) or (4.4), $f'(z_1) \neq 0$, $f(0) + d_n \neq 0$, and (f, d_n, z_1) to be regular. Then the summability field of (f, d_n, z_1) does not contain that of the Riesz method associated with the sequence $\{q_k\}$, with $\lambda_n = \sum_0^n q_k$, if $\liminf \frac{1}{k}\left|\frac{\lambda_k}{q_k}\right| > 0$

and $\sum\limits_{0}^{\infty} \left|\frac{\lambda_k}{q_k}\right| \cdot \left|\frac{z_1}{\rho}\right|^k$ and $\sum\limits_{0}^{\infty} \left|\frac{\lambda_k}{q_{k+1}}\right| \cdot \left|\frac{z_1}{\rho}\right|^k$ both converge.

<u>Proof</u>: Let $f^n(z) = \sum\limits_{k=0}^{\infty} a_{nk} z^k$. The only powers of z appearing in the expansion of $f^n(z)$ are multiples of m, so $a_{nk} = 0$ unless k is a multiple of m. If $(f(z) + d_n)! = \sum\limits_{k=0}^{\infty} p_{nk} z^k$, then from (2.3) it follows that $p_{nk} = 0$ if k is not a multiple of m. But then

$$\frac{p_{nk}}{q_k} - \frac{p_{n,k+1}\, z_1}{q_{k+1}} = \begin{cases} p_{nk}/q_k & \text{if } k = \ell m, \\ -p_{n,k+1} z_1/q_{k+1} & \text{if } k+1 = \ell m, \\ 0 \text{ otherwise}, \end{cases}$$

so

$$\frac{1}{|f(z_1) + d_n|!} \sum_{k=0}^{\infty} \left|\frac{p_{nk}}{q_k} - \frac{p_{n,k+1} z_1}{q_{k+1}}\right| \cdot |z_1^k \lambda_k| \geq$$

$$\frac{1}{|f(z_1) + d_n|!} \sum_{\substack{k=\ell m \\ \ell \geq 0}} \left|\frac{p_{nk}}{q_k} - \frac{p_{n,k+1} z_1}{q_{k+1}}\right| \cdot |z_1^k \lambda_k| =$$

$$\frac{1}{|f(z_1) + d_n|!} \sum_{\substack{k=\ell m \\ \ell \geq 0}} \left|\frac{p_{nk}}{q_k}\right| \cdot |z_1^k \lambda_k| =$$

$$\frac{1}{|f(z_1) + d_n|!} \sum_{k=0}^{\infty} \left|\frac{p_{nk}}{q_k}\right| \cdot |z_1^k \lambda_k|.$$ Since $\liminf \frac{1}{k}\left|\frac{\lambda_k}{q_k}\right| > 0$, there is an $\varepsilon > 0$ and $N > 0$ such that $\left|\frac{\lambda_k}{q_k}\right| \geq k\varepsilon$ for $k > N$, so $\frac{1}{|f(z_1) + d_n|!} \sum\limits_{k=N+1}^{\infty} \left|\frac{p_{nk}}{q_k}\right| \cdot |z_1^k \lambda_k| \geq$

$$\frac{\varepsilon}{|f(z_1) + d_n|!} \sum_{k=N+1}^{\infty} |kp_{nk} z_1^k| \geqslant \left| \frac{\varepsilon}{(f(z_1) + d_n)!} \sum_{k=N+1}^{\infty} kp_{nk} z_1^k \right|$$

$\neq O(1)$ by Corollary 4.2, provided we can show that

$$\left| \frac{1}{(f(z_1) + d_n)!} \sum_{k=0}^{N} kp_{nk} z_1^k \right| = O(1) \quad \text{as} \quad n \to \infty .$$

This is easily shown to be the case. For if the (f, d_n, z_1) matrix is (c_{nk}), its presumed regularity simplies that $\sum_k |c_{nk}| \leq B$ for every n. In particular,

$$|c_{nk}| = \frac{|p_{nk} z_1^k|}{|f(z_1) + d_n|!} \leq B \quad \text{for all } k \text{ and } n, \text{ so}$$

$$\left| \frac{kp_{nk} z_1^k}{(f(z_1) + d_n)!} \right| \leq kB \quad \text{for all } k \text{ and } n.$$

Then $\left| \frac{1}{(f(z_1) + d_n)!} \sum_{k=0}^{N} kp_{nk} z_1^k \right| \leq B \sum_0^N k = O(1)$. Hence,

$$\frac{1}{|f(z_1) + d_n|!} \sum_{k=0}^{\infty} \left| \frac{p_{nk}}{q_k} - \frac{p_{n,k+1} z_1}{q_{k+1}} \right| \cdot |z_1^k \lambda_k| \neq O(1) \quad \text{as}$$

$n \to \infty$, so the transformation

$$(4.6) \qquad \frac{1}{(f(z_1) + d_n)!} \sum_{k=0}^{\infty} \left(\frac{p_{nk}}{q_k} - \frac{p_{n,k+1} z_1}{q_{k+1}}\right) z_1^k \lambda_k t_k$$

is not conservative inasmuch as (0.1) fails. This fact is the crux of the argument. Suppose that $t_n = \lambda_n^{-1} \sum_0^{\infty} q_k s_k$ is the Riesz transform of $\{s_k\}$, so that $s_n = \frac{\lambda_n t_n - \lambda_{n-1} t_{n-1}}{q_n}$. Cauchy's estimate for $|p_{nk}|$ is $\frac{M_n}{\rho^k}$, and this, together with the convergence of the series $\sum \left|\frac{\lambda_k}{q_k}\right| \cdot \left|\frac{z_1}{\rho}\right|^k$ and $\sum \left|\frac{\lambda_k}{q_{k+1}}\right| \cdot \left|\frac{z_1}{\rho}\right|^k$, assures the absolute convergence of

$\sum_k \frac{p_{nk}}{q_k} \lambda_k z_1^k t_k$ and $\sum_k \frac{p_{n,k+1}}{q_{k+1}} \lambda_k z_1^k t_k$ for every bounded sequence $\{t_k\}$. In particular, if the Riesz method sums $\{s_k\}$, then $\{t_k\}$ is bounded, and we can use summation by parts to write the (f,d_n,z_1) transform of $\{s_k\}$ as

$$\frac{1}{(f(z_1) + d_n)!} \sum_{k=0}^{\infty} p_{nk} z_1^k s_k =$$

$$\frac{1}{(f(z_1) + d_n)!} \sum_{k=0}^{\infty} p_{nk} z_1^k \frac{\lambda_k t_k - \lambda_{k-1} t_{k-1}}{q_k} =$$

$$\frac{1}{(f(z_1) + d_n)!} \sum_{k=0}^{\infty} \left(\frac{p_{nk}}{q_k} - \frac{p_{n,k+1} z_1}{q_{k+1}}\right) z_1^k \lambda_k t_k .$$

But this is the transformation (4.6) and is not conservative. In view of the reversibility of the Riesz matrix, it is easily seen that the transform (4.6) will sum everything the Riesz method does if and only if the transform is conservative. The result follows.

As examples of functions which satisfy the conditions of the above theorem, we give the following. In each case $z_1 = 1$.

(4.7) $f(z) = z^m$, m an integer greater than 1,

(4.8) $f(z) = \exp\{z^m\}$, m an integer greater than 1,

(4.9) $f(z) = \cosh z$,

(4.10) $f(z) = \log \sec z$.

If f is any of these functions and $\{d_n\}$ is chosen so that $d_n \geq 0$ and $\sum_1^{\infty} \frac{1}{1+d_n} = \infty$, then Lemma 1.12 gives the regularity of (f,d_n,z_1), so all the conditions on f and $\{d_n\}$ are met.

The following is a variant of Theorem 4.5 having somewhat less elaborate hypotheses.

<u>Theorem 4.11</u>: Let f be holomorphic at the origin with radius of convergence greater than $\rho > 0$. Let

$$b_{nk} = \left| \frac{z_1^k}{(f(z_1) + d_n)!} \left[\frac{p_{nk}}{q_k} - \frac{p_{n,k+1}\, z_1}{q_{k+1}}\right] \right| \quad \text{and suppose}$$

$\lim\limits_{n} b_{nk} = 0$ for each k, $\limsup\limits_{n \to \infty} \sum\limits_{k} b_{nk} > 0$, and both

$$\Sigma \left|\frac{\lambda_k}{q_k}\right| \cdot \left|\frac{z_1}{\rho}\right|^k \quad \text{and} \quad \Sigma \left|\frac{\lambda_k}{q_{k+1}}\right| \cdot \left|\frac{z_1}{\rho}\right|^k \quad \text{converge. Then if}$$

$|\lambda_k| \to \infty$, the summability field of (f,d_n,z_1) does not contain that of the Riesz method associated with the sequence $\{q_k\}$.

<u>Proof</u>: Given $B > 0$, there is an $N > 0$ such that $k > N \Rightarrow |\lambda_k| \geq B$, so $\sum\limits_{k=0}^{\infty} b_{nk}|\lambda_k| \geq \sum\limits_{k=0}^{\infty} b_{nk}|\lambda_k| +$

$$B \sum_{k=N+1}^{\infty} b_{nk} = \sum_{k=0}^{N} (|\lambda_k|-B)b_{nk} + B\sum_{k=0}^{\infty} b_{nk} = 0(1) + B\sum_{k=0}^{\infty} b_{nk}.$$

Then, if $\limsup\limits_{n \to \infty} \sum\limits_{k} b_{nk} = \varepsilon$, $\limsup\limits_{n \to \infty} \sum\limits_{k=0}^{\infty} b_{nk}\,|\lambda_k| \geq B\varepsilon$.

Since B is arbitrary it follows that

$$\limsup_{n \to \infty} \sum_{k=0}^{\infty} b_{nk}|\lambda_k| = \infty .$$

An immediate consequence of this is that the transformation (4.6) is not conservative since (0.1) fails. The conclusion of the theorem now follows exactly as in the proof of Theorem 4.5.

Theorem 4.12: Let $f(z) = \sum_0^\infty a_k z^k$, where $a_k = 0$ if k is not a multiple of $m > 1$. Then the summability field of (f, d_n, z_1) is not contained in that of the Abel method.

Proof: The (f, d_n, z_1) matrix (c_{nk}) is defined by $c_{nk} = \frac{p_{nk}\, z_1^k}{(f(z_1) + d_n)!}$. We have already seen that for such an f, $p_{nk} = 0$ unless k is a multiple of m, so the corresponding $c_{nk} = 0$, too. Consequently, (f, d_n, z_1) will sum to zero the sequence $\{s_k\}$, where $s_k = 2^k$, $k \neq \ell m$, and $s_k = 0$, $k = \ell m$. But clearly this sequence is not Abel summable.

Corollary 4.13: Let (f, d_n, z_1) be subject to the conditions of Theorem 4.5 with $\rho > |z_1|$. Then (f, d_n, z_1) is not comparable with the Abel method or with any (C, α) method for $\alpha \geq 1$.

Proof: $(C, 1)$ is the Riesz method determined by the sequence $\{1\}$. By Theorem 4.5, the summability field of (f, d_n, z_1) does not contain that of $(C, 1)$, and hence does not contain that of (C, α) for $\alpha > 1$ or that of the Abel method. On the other hand, Theorem 4.12 says the summability field of (f, d_n, z_1) is not contained in that of the Abel method, and thus also it is not contained in that of (C, α) for any α.

The following Lemma is well-known ([17],Lemme 4).

<u>Lemma 4.14</u>: Let $\sum_{k=0}^{\infty} b_k \sum_{j=0}^{k} a_j$ converge and let $\sum_{0}^{\infty} |b_k| < \infty$. Then $\sum_{k=0}^{\infty} b_k \sum_{0}^{k} a_j = \sum_{j=0}^{\infty} a_j \sum_{k=j}^{\infty} b_k$ if and only if $\lim_{q \to \infty} \sum_{q}^{\infty} b_k \sum_{q}^{k} a_j = 0$.

<u>Definition 4.15</u>: The Y-transform of $\{s_k\}$ is the sequence $\{y_n\}$, where $y_0 = \frac{1}{2} s_0$, and $y_n = \frac{1}{2}(s_n + s_{n-1})$ for $n \geq 1$.

<u>Theorem 4.16</u>: Let f be holomorphic at the origin. Let either (4.3) or (4.4) hold, $f'(z_1) \neq 0$, and $f(0) + d_n \neq 0$. If $(f(z) + d_n)! = \sum_k p_{nk} z^k$, suppose that for each n, $(-1)^k p_{nk} z_1^k$ is real and does not change sign for $k > N$. It follows that if (f, d_n, z_1) is regular, its summability field does not contain that of the Y-method.

<u>Proof</u>: Inverting the Y-transform gives

$s_k = 2(-1)^k \sum_{0}^{k} (-1)^j y_j$. There is a $\rho > |z_1| > 0$ such that ρ is less than the radius of convergence of f, and Cauchy's estimate for p_{nk} is then M_n/ρ^k. Then for all bounded sequences $\{y_j\}$ we can apply Lemma 4.14 and write the (f, d_n, z_1) transform of $\{s_k\}$ as

$$\frac{1}{(f(z_1) + d_n)!} \sum_{k=0}^{\infty} p_{nk} z_1^k s_k =$$

$$\frac{2}{(f(z_1) + d_n)!} \sum_{k=0}^{\infty} p_{nk} z_1^k (-1)^k \sum_{j=0}^{k} (-1)^j y_j =$$

$$(4.17) \quad \frac{2}{(f(z_1) + d_n)!} \sum_{j=0}^{\infty} (-1)^j \Big(\sum_{k=j}^{\infty} (-1)^k \, z_1^k \, p_{nk}\Big) y_j .$$

To prove the theorem, it is sufficient to show that this transform is not conservative, and this will be done by violating (0.1). The regularity of (f, d_n, z_1) and (0.1) together imply that

$$\sum_k |c_{nk}| = \frac{1}{\lceil f(z_1) + d_n \rceil !} \sum_k |p_{nk} \, z_1^k| = o(1) \text{ as } n \to \infty .$$

Hence, we can invoke Lemma 4.14 again to write

$$\frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{j=0}^{\infty} \Big| \sum_{k=j}^{\infty} (-1)^k z_1^k p_{nk} \Big| =$$

$$\frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{j=0}^{N} \Big| \sum_{k=j}^{\infty} (-1)^k \, z_1^k \, p_{nk} \Big| +$$

$$\frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{j=N+1}^{\infty} \Big| \sum_{k=j}^{\infty} (-1)^k \, z_1^k \, p_{nk} \Big| =$$

$$o(1) + \frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{j=N+1}^{\infty} \sum_{k=j}^{\infty} |z_1^k \, p_{nk}| =$$

$$o(1) + \frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{k=N+1}^{\infty} (k-N) |z_1^k \, p_{nk}| =$$

$$o(1) - \frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{k=0}^{N} (k-N) |z_1^k \, p_{nk}| +$$

$$\frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{k=0}^{\infty} (k-N) |z_1^k \, p_{nk}| =$$

$$o(1) + \frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{k=0}^{\infty} (k-N) |z_1^k \, p_{nk}| =$$

$$o(1) + \frac{2}{\lceil f(z_1)+d_n \rceil !} \sum_{k=0}^{\infty} |k \, z_1^k \, p_{nk}| - \frac{2N}{\lceil f(z_1)+d_n \rceil !} \sum_{k=0}^{\infty} |z_1^k \, p_{nk}| =$$

$$o(1) + \frac{2}{|f(z_1)+d_n|} \sum_{k=0}^{\infty} |k\, z_1^k\, p_{nk}| \neq o(1) \quad \text{by Corollary 4.2,}$$

so (0.1) fails for the transformation (4.17) and the theorem is proved.

Inasmuch as the Y-method is rather weak, the possibility arises that an (f,d_n,z_1) method whose summability field does not even contain that of Y might be convergence-equivalent, i.e., that it might sum only convergent sequences. Example 4.19 below shows that this is not the case.

The following theorem is known ([15], Theorem 2.3).

<u>Theorem 4.18</u>: Suppose that $\sum_{1}^{\infty} \frac{1}{|f(1) + d_n|} = \infty$,

$\sum_{1}^{\infty} \frac{(\mathrm{Im}\sqrt{d_n})^2}{|f(1)+d_n|^2} < \infty$, and that the Taylor expansion of f about the origin has real non-negative coefficients. Then (f,d_n) is regular.

<u>Example 4.19</u>: Let $f(z) = z^3$, $z_1 = 1$, and $d_n = \frac{-1}{n^2+1}$. It is trivial to show that (4.3) holds and to see that $f'(z_1) \neq 0$ and $f(0) + d_n \neq 0$. If $f^n(z) = \sum_k a_{nk} z^k$, then

$a_{nk} = 1$ if $k = 3n$ and $a_{nk} = 0$ otherwise. Then it follows from (2.3) that $p_{nk} = 0$ if $k \neq 3\ell$ for some ℓ with $0 \leq \ell \leq n$, and $p_{nk} = \sigma_{n-k/3}$ otherwise, where $\sigma_0 = 1$ and

$\sigma_j = \sum_{\substack{1 \leq \nu_1 < \\ \cdots < \nu_j \leq n}} d_{\nu_1} \cdots d_{\nu_j}$ for $j > 0$. From the definition of

σ_j and the fact that each $d_n < 0$, it is clear that

$\operatorname{sgn} \sigma_j = (-1)^j$. Let k be any multiple of 3 for which $p_{nk} \neq 0$. Then either $p_{n,k+3} = 0$ or $p_{n,k+3} = \sigma_{n-\frac{k+3}{3}}$, so that $\operatorname{sgn} p_{n,k+3} = \operatorname{sgn} \sigma_{n-\frac{k+3}{3}} = \operatorname{sgn} \sigma_{n-\frac{k}{3}-1} = -\operatorname{sgn} \sigma_{n-\frac{k}{3}}$ $= -\operatorname{sgn} p_{nk}$. From this we have $\operatorname{sgn}[(-1)^{k+3} p_{n,k+3}] =$ $(-1)^{k+3} \operatorname{sgn} p_{n,k+3} = (-1)^{k+3}(-\operatorname{sgn} p_{nk}) = (-1)^k \operatorname{sgn} p_{nk} =$ $\operatorname{sgn}[(-1)^k p_{nk}]$, so that $(-1)^k p_{nk}$ does not change sign. Finally, Theorem 4.18 assures the regularity of (f,d_n), so the hypotheses of Theorem 4.16 are met and (f,d_n) is not at least as strong a Y. Now define the sequence $\{s_k\}$ by

$$s_k = \begin{cases} 0, & k \text{ is a multiple of } 3 \\ 1, & \text{otherwise} \end{cases}$$

This sequence is easily seen to be summable to 0 by (f,d_n), but it is not convergent, and is not even Y-summable. Thus we have

<u>Lemma 4.20</u>: There is a regular (f,d_n,z_1) method which is not comparable to the Y-method.

We will now give an example of a conservative (f,d_n,z_1) method which is at least as strong as $(C,1)$. The $(C,1)$ method is the Riesz method generated by the sequence $\{1\}$. From the proof of Theorem 4.5 we know that what we seek is a conservative (f,d_n,z_1) method for which the transformation (4.6) is conservative. Choose $f(z) = z$, $z_1 = 1$, $d_n = n^2$. Summation by parts shows that $\sum_k (k+1)(p_{nk}-p_{n,k+1}) = \sum_k p_{nk}$, so we have

$$(4.21) \quad \frac{1}{(1+d_n)!} \sum_k (k+1)(p_{nk} - p_{n,k+1}) = \frac{1}{(1+d_n)!} \sum_k p_{nk} = 1.$$

If we assume for the moment that (z,n^2) is conservative, then $\lim_n \frac{p_{nk}}{(1+d_n)!}$ must exist for each k, whence also

$$(4.22) \quad \lim_n \frac{(k+1)(p_{nk}-p_{n,k+1})}{(1+d_n)!} \quad \text{exists for each } k.$$

On any compact set, $\left| \prod_1^n \frac{t+k^2}{1+k^2} \right| = \left| \prod_1^n \left(1 + \frac{t-1}{1+k^2}\right) \right| \leq$

$\prod_1^n \left(1 + \frac{|t-1|}{1+k^2}\right) \leq \prod_1^\infty \left(1 + \frac{B}{1+k^2}\right) < \infty$, where B is a bound for $|t-1|$ on the compact set. Then, if C is a circle of radius $\rho > 1$ about the origin,

$$\frac{|p_{nk} - p_{n,k+1}|}{(1 + n^2)!} = \frac{1}{2\pi}\left| \int_C \left(\frac{1}{t^{k+1}} - \frac{1}{t^{k+2}}\right) \prod_1^n \frac{t+\nu^2}{1+\nu^2}\, dt \right| =$$

$$\frac{1}{2\pi}\left| \int_C \frac{t-1}{t^{k+2}} \prod_1^n \frac{t+\nu^2}{1+\nu^2}\, dt \right| \leq \frac{M}{2\pi} \int_C \frac{|t-1|}{\rho^{k+2}} |dt| = \frac{O(1)}{\rho^{k+1}}$$ independently

of n. It follows that

$$(4.23) \quad \frac{1}{(1+d_n)!} \sum_k (k+1)|p_{nk} - p_{n,k+1}| \leq O(1) \sum_k \frac{k+1}{\rho^{k+1}} = O(1).$$

The conditions (4.21), (4.22), and (4.23) are sufficient for the transform (4.6) to be conservative. It only remains to show that (z,n^2) is conservative. We have (0.2) with $\zeta = 1$ automatically, and because of the special choice of $f(z)$ and d_n, (0.1) is equivalent to (0.2) in this case. To show (0.3), we must show that

$$\lim_n \frac{p_{nk}}{(1 + n^2)!}$$

exists for every k. On any compact set the product

$$\prod_{1}^{\infty} \frac{t+\nu^2}{1+\nu^2} = \prod_{1}^{\infty} \left(1 + \frac{t-1}{1+\nu^2}\right)$$

converges absolutely and uniformly. Thus if C is a circle about the origin of radius >1, it is clear that

$$\lim_{n} \frac{p_{nk}}{(1+n^2)!} = \lim_{n} \frac{1}{2\pi i} \int_{C} \frac{1}{t^{k+1}} \prod_{1}^{n} \frac{t+\nu^2}{1+\nu^2}\, dt \quad \text{exists for each } k.$$

Hence, (z,n^2) is conservative.

We now turn to the question of inclusion with regard to the methods (z,d_n,z_1) and (f,d_n',z_2), that is, under what conditions the latter will contain the former.

In [11], Meir proved the following theorem:

Let $\{d_n\}_1^\infty$ and $\{d'_n\}_1^\infty$ be given with $d_n \neq -1 \neq d'_n$. Suppose $\sum_1^\infty |1 + d'_n|^{-1} = \infty$ and $0 < \frac{1 + d_k}{1 + d'_n} \leq 1$ if $n \geq n_0$ and $1 \leq k \leq n$. Then (F, d'_n) is consistent with, and at least as strong as, (F, d_n).

By generalizing his techniques we can prove a much more general theorem.

Define the linear operator E by $Es_k = s_{k+1}$, and define $E^0(s_k) = s_k$ and $E^n s_k = E(E^{n-1} s_k)$ for $n \geq 1$. Then the (z, d_n, z_1) transform may be written as

$$(4.24) \quad t_n = \prod_1^n \frac{d_k + z_1 E}{d_k + z_1} s_0 \ , \quad n \geq 0,$$

provided we define $\prod_1^0 \frac{d_k + z_1 E}{d_k + z_1} s_0 = s_0$

Since the (z, d_n, z_1) matrix is normal, it is reversible, so there are coefficients b_{nm} so that

$$(4.25) \quad s_n = \sum_{m=0}^n b_{nm} t_m \ , \quad n \geq 0.$$

Explicit formulas for these coefficients are in [9], p. 288. From (4.25) we see that $b_{nm} = 0$ if $m > n$. For convenience we also define $b_{nm} = 0$ if $n < 0$ or $m < 0$.

From (4.24) and (4.25) we get

$$(4.26) \quad E^n s_0 = s_n = \sum_{m=0}^n b_{nm} \prod_1^m \frac{d_k + z_1 E}{d_k + z_1} s_0 \ , \ n \geq 0.$$

Then

$$(4.27) \quad E^{n-1} s_0 = s_{n-1} = \sum_{m=0}^{n-1} b_{n-1,m} \prod_1^m \frac{d_k + z_1 E}{d_k + z_1} s_0 \ , \ n \geq 1.$$

Operating on (4.27) with z_1E we have

$$z_1E^n s_0 = z_1 s_n = \sum_{m=0}^{n-1} b_{n-1,m} z_1E \prod_1^m \frac{d_k + z_1E}{d_k + z_1} s_0$$

$$= \sum_{m=1}^{n} b_{n-1,m-1} z_1E \prod_1^{m-1} \frac{d_k + z_1E}{d_k + z_1} s_0$$

$$= \sum_{m=1}^{n} b_{n-1,m-1}(z_1E + d_m) \prod_1^{m-1} \frac{d_k + z_1E}{d_k + z_1} s_0$$

$$- \sum_{m=1}^{n} b_{n-1,m-1}\, d_m \prod_1^{m-1} \frac{d_k + z_1E}{d_k + z_1} s_0$$

$$= \sum_{m=1}^{n} b_{n-1,m-1}(d_m + z_1) \prod_1^{m} \frac{d_k + z_1E}{d_k + z_1} s_0$$

$$- \sum_{m=0}^{n-1} b_{n-1,m} \cdot d_{m+1} \prod_1^{m} \frac{d_k + z_1E}{d_k + z_1} s_0$$

$$= \sum_{m=0}^{n} b_{n-1,m-1}(d_m + z_1) \prod_1^{m} \frac{d_k + z_1E}{d_k + z_1} s_0$$

$$- \sum_{m=0}^{n} b_{n-1,m} \cdot d_{m+1} \prod_1^{m} \frac{d_k + z_1E}{d_k + z_1} s_0$$

$$= \sum_{m=0}^{n} \{b_{n-1,m-1}(d_m + z_1) - b_{n-1,m} \cdot d_{m+1}\} \prod_1^{m} \frac{d_k + z_1E}{d_k + z_1} s_0, \quad n \geq 1.$$

It follows from this and (4.26) that

$$(4.28) \quad b_{nm} = z_1^{-1} \{b_{n-1,m-1}(d_m + z_1) - b_{n-1,m}\, d_{m+1}\} \text{ for } m \geq 0,$$
$$n \geq 1.$$

It is clear from (4.25) that $b_{00} = 1$.

Let the (f, d_n', z_2) transform be

$$t_n' = \sum_{m=0}^{\infty} c_{nm}' s_m = \sum_{m=0}^{\infty} c_{nm}' \sum_{k=0}^{m} b_{mk} t_k, \quad n \geq 0,$$

for those sequences $\{s_m\}$ summed by (z, d_n, z_1). If the order of summation can be reversed, then

$$(4.29)\quad t'_n = \sum_{k=0}^{\infty} \left(\sum_{m=k}^{\infty} c'_{nm}\, b_{mk} \right) t_k = \sum_{k=0}^{\infty} a_{nk}\, t_k ,$$

where

$$(4.30)\quad a_{nk} = \sum_{m=k}^{\infty} c'_{nm}\, b_{mk} \quad \text{for} \quad n \geq 0,\ k \geq 0.$$

For future convenience we define $a_{nk} = 0$ if $n < 0$ or $k < 0$, and $c'_{nk} = 0$ for $k < 0$.

If (4.29) is a regular transformation, then (f, d'_n, z_2) will sum every sequence (z, d_n, z_1) does, and to the same number, that is, (f, d'_n, z_2) will be consistent with, and at least as strong as, (z, d_n, z_1). We will show that if f is a polynomial, then under suitable conditions (4.29) is regular.

Let $(z + d_n)! = \sum_{k=0}^{n} p_{nk}\, z^k$ and $(f(z) + d'_n)! =$

$$\sum_{k=0}^{\infty} p'_{nk}\, z^k.$$

Then

$$t_n = \frac{1}{(z_1 + d_n)!} \sum_{k=0}^{n} p_{nk} z_1^k s_k \text{ and } t'_n = \frac{1}{(f(z_2)+d'_n)!} \sum_{k=0}^{\infty} p'_{nk} z_2^k s_k,$$

so setting $s_k \equiv 1$ gives $t_n \equiv 1 \equiv t'_n$. Plugging these values into (4.29) gives

$$(4.31)\quad \sum_{k=0}^{\infty} a_{nk} \equiv 1, \quad n \geq 0.$$

In general, if $f(z) = \sum_{0}^{\infty} \beta_k\, z^k$, then

$$\sum_{k=0}^{\infty} p'_{n+1,k} z^k = (f(z)+d'_{n+1})! = (f(z)+d'_{n+1}) \sum_{k=0}^{\infty} p'_{nk} z^k$$

$$= \left(\sum_0^{\infty} \beta_k z^k\right)\left(\sum_{k=0}^{\infty} p'_{nk} z^k\right) + d'_{n+1} \sum_{k=0}^{\infty} p'_{nk} z^k$$

$$= \sum_{m=0}^{\infty} (q_{nm} + d'_{n+1} p'_{nm}) z^m, \text{ where } q_{nm} = \sum_{j+k=m} \beta_j p'_{nk}. \quad \text{Hence,}$$

$$(4.32) \quad p'_{n+1,m} = q_{nm} + d'_{n+1} p'_{nm}, \quad n \geq 1, \quad m \geq 0.$$

Since $c'_{nm} = \dfrac{p'_{nm} z_2^m}{(f(z_2) + d'_n)!}$ for $n \geq 1$, we may use (4.32) to derive

$$c'_{n+1,m} = \frac{p'_{n+1,m} z_2^m}{(f(z_2) + d'_{n+1})!} = \frac{q_{nm} z_2^m}{(f(z_2) + d'_{n+1})!} + \frac{d'_{n+1} p'_{nm} z_2^m}{(f(z_2) + d'_{n+1})!}$$

$$= \frac{q_{nm} z_2^m}{(f(z_2) + d'_{n+1})!} + \frac{d'_{n+1} c'_{nm}}{f(z_2) + d'_{n+1}}. \qquad \text{Then}$$

$$(4.33) \quad c'_{n+1,m} = \frac{1}{f(z_2)+d'_{n+1}} \left[z_2^m \sum_{j+\nu=m} \beta_j c'_{n\nu} z_2^{-\nu} + d'_{n+1} c'_{nm}\right]$$

for $n \geq 1$, $m \geq 0$.

From (4.30),

$$a_{n+1,k} = \sum_{m=k}^{\infty} c'_{n+1,m} b_{mk} = \sum_{m=k}^{\infty} \frac{b_{mk}}{f(z_2)+d'_{n+1}} \left[z_2^m \sum_{j+\nu=m} \beta_j c'_{n\nu} z_2^{-\nu} + d'_{n+1} c'_{nm}\right]$$

or

$$(4.34) \quad a_{n+1,k} = \frac{1}{f(z_2)+d'_{n+1}} \sum_{m=k}^{\infty} \{z_2^m b_{mk} \sum_{j+\nu=m} \beta_j c'_{n\nu} z_2^{-\nu} + d'_{n+1} a_{nk}\}$$

for $n \geq 1$, $k \geq 0$.

If the matrix $C' = (c'_{nk})$ is row finite, i.e., if f is a polynomial, then t'_n can be written in the form (4.29). It is also clear that then $\sum_k |a_{nk}| < \infty$ for each n, for suppose

$$(4.35) \qquad f(z) = \sum_0^m \beta_\nu z^\nu \ , \ \beta_m \neq 0.$$

By (4.30), $a_{nk} = \sum_{j=k}^{\infty} c'_{nj}\, b_{jk}$. But $c'_{nj} = 0$ for $j > mn$, so

$$(4.36) \quad a_{nk} = 0 \quad \text{for} \quad k > mn \quad \text{if} \quad n \geq 0 \quad \text{and} \quad k \geq 0.$$

From now on, we will suppose f is given by (4.35), and m will be reserved for the degree of f. Define $\sigma_{n+1} = d'_{n+1} + f(z_2)$. Then (4.34) gives

$$a_{n+1,k} = \frac{1}{\sigma_{n+1}}\{\sum_{\nu=k}^{\infty} z_2^\nu b_{\nu k}(\beta_0 c'_{n\nu} z_2^{-\nu} + \beta_1 c'_{n,\nu-1} z_2^{-\nu+1} + \ldots + \beta_\nu c'_{n0})$$

$$+ d'_{n+1} a_{nk}\}.$$

In view of the convention that $c'_{pq} = 0$ for $q < 0$ and the assumption that $\beta_j = 0$ for $j > m$, we can write

$$a_{n+1,k} = \frac{1}{\sigma_{n+1}}\{\sum_{\nu=k}^{\infty} z_2^\nu b_{\nu k}(\beta_0 c'_{n\nu}\, z_2^{-\nu} + \beta_1 c'_{n,\nu-1} z_2^{-\nu+1} + \ldots + \beta_m c'_{n,\nu-m} z_2^{-\nu+m})$$

$$+ d'_{n+1} a_{nk}\}$$

$$= \frac{1}{\sigma_{n+1}}\{\beta_0 \sum_{\nu=k}^{mn} c'_{n\nu} b_{\nu k} + \beta_1 z_2 \sum_{\nu=k}^{mn+1} c'_{n,\nu-1} b_{\nu k} + \ldots + \beta_m z_2^m \sum_{\nu=k}^{mn+m} c'_{n,\nu-m} b_{\nu k}$$

$$+ d'_{n+1} a_{nk}\}$$

or

$$(4.37) \quad a_{n+1,k} = \frac{1}{\sigma_{n+1}}\left(\sum_{j=0}^{m} \beta_j z_2^j \sum_{\nu=k}^{mn+j} c'_{n,\nu-j} b_{\nu k} + d'_{n+1} a_{nk}\right), \quad n \geq 1, \quad k \geq 0.$$

Here it is necessary to introduce the notation

$$(4.38) \quad \sum_{q}^{Q} d_{i_1} d_{i_2} \cdots d_{i_\ell} = \sum_{q \leq i_1 \leq i_2 \leq \cdots \leq i_\ell \leq Q} d_{i_1} d_{i_2} \cdots d_{i_\ell}, \quad \ell \geq 1,$$

and

$$(4.39) \quad \sum_{q}^{Q} d_{i_1} \cdots d_{i_0} = 1 \quad \text{for arbitrary } q \text{ and } Q.$$

We now examine $\sum_{\nu=k}^{mn+j} c'_{n,\nu-j} b_{\nu k}$. When $j = 0$ this reduces to $\sum_{\nu=k}^{mn} c'_{n\nu} b_{\nu k} = a_{nk}$. Suppose

$$(4.40) \quad \sum_{\nu=k}^{mn+j} c'_{n,\nu-j} b_{\nu k} = z_1^{-j} \sum_{r=1}^{j+1} (-1)^{r-1} \prod_{\nu=0}^{j-r} (z_1 + d_{k-\nu}) \cdot \sum_{k-j+r}^{k+1} d_{i_1} \cdots d_{i_{r-1}} \cdot a_{n,k-j+r-1} \quad \text{for } 0 \leq j \leq p,$$

where

$$(4.41) \quad \prod_{\nu=q}^{Q} (z_1 + d_{k-\nu}) = \begin{cases} 1 & \text{if } Q < q, \\ 1 & \text{if } \nu \geq k \text{ for } q \leq \nu \leq Q, \\ \prod_{\substack{q \leq \nu \leq Q \\ \nu < k}} (z_1 + d_{k-\nu}) & \text{otherwise.} \end{cases}$$

(We remark here that if $a \leq 0$, $b > 0$, then $\sum_{a}^{b} d_{i_1} \cdots d_{i_\nu} = \sum_{1}^{b} d_{i_1} \cdots d_{i_\nu}$.) Since we shall use the formula (4.28), we suppose at first that $k \geq 1$. Then

$$\sum_{\nu=k}^{mn+p+1} c'_{n,\nu-p-1} b_{\nu k} = \sum_{\nu=k}^{mn+p+1} c'_{n,\nu-p-1} \quad z_1^{-1}[b_{\nu-1,k-1}(z_1+d_k)$$

$$- b_{\nu-1,k}\, d_{k+1}] =$$

$$z_1^{-1} \ \{(z_1+d_k) \sum_{\nu=k}^{mn+p+1} c'_{n,\nu-p-1} b_{\nu-1,k-1}$$

$$- d_{k+1} \sum_{\nu=k}^{mn+p+1} c'_{n,\nu-p-1} b_{\nu-1,k}\} \ =$$

$$z_1^{-1} \ \{(z_1+d_k) \sum_{\nu=k-1}^{mn+p} c'_{n,\nu-p} b_{\nu,k-1} - d_{k+1} \sum_{\nu=k-1}^{mn+p} c'_{n,\nu-p} b_{\nu k}\} \ .$$

By the induction hypothesis (4.40), this is

$$z_1^{-1} \ \{z_1^{-p}(z_1+d_k) \sum_{r=1}^{p+1} (-1)^{r-1} \prod_{\nu=0}^{p-r} (z_1+d_{k-\nu-1}) \ \cdot$$

$$\sum_{k-1-p+r}^{k} d_{i_1} \cdots d_{i_{r-1}} \ \cdot \ a_{n,k-p+r-2} \quad -$$

$$z_1^{-p} d_{k+1} \sum_{r=1}^{p+1} (-1)^{r-1} \prod_{\nu=0}^{p-r} (z_1+d_{k-\nu}) \cdot \sum_{k-p+r}^{k+1} d_{i_1} \cdots d_{i_{r-1}} \ \cdot \ a_{n,k-p+r-1}\} =$$

$$z_1^{-p-1} \{ \sum_{r=1}^{p+1} (-1)^{r-1} \prod_{\nu=0}^{p+1-r} (z_1+d_{k-\nu}) \ \cdot \sum_{k+r-p-1}^{k} d_{i_1} \cdots d_{i_{r-1}} \cdot a_{n,k+r-p-2} \ -$$

$$\sum_{r=2}^{p+2} (-1)^{r} \prod_{\nu=0}^{p+1-r} (z_1+d_{k-\nu}) \cdot d_{k+1} \sum_{k+r-p-1}^{k+1} d_{i_1} \cdots d_{i_{r-2}} \cdot a_{n,k+r-p-2}\} =$$

$$z_1^{-p-1} \{(-1)^{p+1} d_{k+1}^{p+1} \ a_{nk} + \prod_{\nu=0}^{p} (z_1+d_{k-\nu}) \ \cdot \ a_{n,k-p-1} \quad +$$

$$\sum_{r=2}^{p+1} (-1)^{r-1} \prod_{\nu=0}^{p+1-r} (z_1+d_{k-\nu}) \cdot [\sum_{k+r-p-1}^{k} d_{i_1} \cdots d_{i_{r-1}} \quad +$$

$$d_{k+1} \sum_{k+r-p-1}^{k+1} d_{i_1} \cdots d_{i_{r-2}}] a_{n,k+r-p-2}\} =$$

$$z_1^{-p-1}\{(-1)^{p+1}d_{k+1}^{p+1}\ a_{nk} + \prod_{\nu=0}^{p}(z_1 + d_{k-\nu})\cdot\ a_{n,k-p-1} +$$

$$\sum_{r=2}^{p+1}(-1)^{r-1}\prod_{\nu=0}^{p+1-r}(z_1+d_{k-\nu})\cdot \sum_{k+r-p-1}^{k+1} d_{i_1}\cdots d_{i_{r-1}}\cdot a_{n,k+r-p-2}\} =$$

$$z_1^{-p-1}\sum_{r=1}^{p+2}(-1)^{r-1}\prod_{\nu=0}^{p+1-r}(z_1+d_{k-\nu})\cdot \sum_{k+r-p-1}^{k+1} d_{i_1}\cdots d_{i_{r-1}}\cdot a_{n,k+r-p-2},$$

so (4.40) is true for $j = p+1$ provided $k \geq 1$. By induction, (4.40) is true for $0 \leq j \leq m$ if $k \geq 1$. Now, if $k = 0$, it is very easy to prove by induction that

$$\sum_{\nu=0}^{mn+j} c'_{n,\nu-j}b_{\nu 0} = (-z_1^{-1}d_1)^j\ a_{n0} \quad \text{for} \quad 0 \leq j \leq m, \text{ so } (4.40)$$

is valid for $k \geq 0$ and $0 \leq j \leq m$.

It now follows from (4.37) that

$$(4.42)\quad a_{n+1,k} = \frac{1}{\sigma_{n+1}}\sum_{j=0}^{m}\beta_j\left(\frac{z_2}{z_1}\right)^j\sum_{r=1}^{j+1}(-1)^{r-1}\prod_{\nu=0}^{j-r}(z_1+d_{k-\nu})\cdot$$

$$\sum_{k-j+r}^{k+1} d_{i_1}\cdots d_{i_{r-1}}\cdot a_{n,k-j+r-1} + \frac{1}{\sigma_{n+1}}d'_{n+1}\ a_{nk} \quad \text{for}$$

$n \geq 1$, $k \geq 0$, under the notational conventions (4.41) and (4.38).

Now, interchanging the order of summation in (4.42), we write

$$\sum_k|a_{n+1,k}| = \frac{1}{|\sigma_{n+1}|}\sum_k\left|d'_{n+1}a_{nk}+\sum_{r=1}^{m+1}(-1)^{r-1}\sum_{j=r-1}^{m}\beta_j\left(\frac{z_2}{z_1}\right)^j\prod_{\nu=0}^{j-r}(z_1+d_{k-\nu})\cdot\right.$$

$$\left.\sum_{k-j+r}^{k+1} d_{i_1}\cdots d_{i_{r-1}}\cdot a_{n,k-j+r-1}\right|,$$

and, under the substitution $t = j+1-r$, this becomes

$$\sum_k |a_{n+1,k}| = \frac{1}{|\sigma_{n+1}|} \sum_k \Big| d'_{n+1} a_{nk} +$$

$$\sum_{r=1}^{m+1} (-1)^{r-1} \sum_{t=0}^{m+1-r} \beta_{t+r-1} \left(\frac{z_2}{z_1}\right)^{t+r-1} \prod_{\nu=0}^{t-1} (z_1 + d_{k-\nu}) \cdot$$

$$\sum_{k+1-t}^{k+1} d_{i_1} \cdots d_{i_{r-1}} \cdot a_{n,k-t} \Big| \,.$$

Again changing the order of summation inside the absolute value signs we have

$$\sum_k |a_{n+1,k}| = \frac{1}{|\sigma_{n+1}|} \sum_k \Big| d'_{n+1} a_{nk} +$$

$$\sum_{t=0}^{m} \sum_{r=1}^{m+1-t} (-1)^{r-1} \beta_{t+r-1} \left(\frac{z_2}{z_1}\right)^{t+r-1} \prod_{\nu=0}^{t-1} (z_1 + d_{k-\nu}) \cdot$$

$$\sum_{k+1-t}^{k+1} d_{i_1} \cdots d_{i_{r-1}} \cdot a_{n,k-t} \Big| =$$

$$\frac{1}{|\sigma_{n+1}|} \sum_k \Big| \Big(d'_{n+1} + \sum_{r=1}^{m+1} (-1)^{r-1} \beta_{r-1} \left(\frac{z_2}{z_1}\right)^{r-1} d_{k+1}^{r-1} \Big) a_{nk} +$$

$$\sum_{t=1}^{m} \sum_{r=1}^{m+1-t} (-1)^{r-1} \beta_{t+r-1} \left(\frac{z_2}{z_1}\right)^{t+r-1} \prod_{\nu=0}^{t-1} (z_1 + d_{k-\nu}) \cdot$$

$$\sum_{k+1-t}^{k+1} d_{i_1} \cdots d_{i_{r-1}} \cdot a_{n,k-t} \Big| \leq$$

$$\frac{1}{|\sigma_{n+1}|} \Big\{ \sum_k \Big| d'_{n+1} + \sum_{r=1}^{m+1} (-1)^{r-1} \beta_{r-1} \left(\frac{z_2}{z_1}\right)^{r-1} d_{k+1}^{r-1} \Big| \cdot |a_{nk}| +$$

$$\sum_k \Big| \sum_{r=1}^{m} (-1)^{r-1} \beta_r \left(\frac{z_2}{z_1}\right)^{r} (z_1 + d_k) \sum_{k}^{k+1} d_{i_1} \cdots d_{i_{r-1}} \Big| \cdot |a_{n,k-1}| + \cdots +$$

$$\sum_k \Big| \sum_{r=1}^{m+1-t} (-1)^{r-1} \beta_{t+r-1} \left(\frac{z_2}{z_1}\right)^{t+r-1} \prod_{\nu=0}^{t-1} (z_1 + d_{k-\nu}) \,.$$

$$\sum_{k+1-t}^{k+1} d_{i_1}\cdots d_{i_{r-1}}\Big|\cdot|a_{n,k-t}| + \ldots +$$

$$\sum_{k}\Big|\beta_m\Big(\frac{z_2}{z_1}\Big)^{m}\prod_{\nu=0}^{m-1}(z_1 + d_{k-\nu})\Big|\cdot|a_{n,k-m}|\} .$$

We recall that $a_{n\nu} = 0$ if $\nu > mn$ or if $\nu < 0$, so

$$\sum_{k}\Big|\sum_{r=1}^{m+1-t}(-1)^{r-1}\beta_{t+r-1}\Big(\frac{z_2}{z_1}\Big)^{t+r-1}\prod_{\nu=0}^{t-1}(z_1+d_{k-\nu}) \cdot$$

$$\sum_{k+1-t}^{k+1} d_{i_1}\cdots d_{i_{r-1}}\Big|\cdot|a_{n,k-t}| =$$

$$\sum_{k=t}^{mn+t}\Big|\sum_{r=1}^{m+1-t}(-1)^{r-1}\beta_{t+r-1}\Big(\frac{z_2}{z_1}\Big)^{t+r-1}\prod_{\nu=0}^{t-1}(z_1+d_{k-\nu}) \cdot$$

$$\sum_{k+1-t}^{k+1} d_{i_1}\cdots d_{i_{r-1}}\Big|\cdot|a_{n,k-t}| =$$

$$\sum_{k=0}^{mn}\Big|\sum_{r=1}^{m+1-t}(-1)^{r-1}\beta_{t+r-1}\Big(\frac{z_2}{z_1}\Big)^{t+r-1}\prod_{\nu=0}^{t-1}(z_1+d_{k+t-\nu}) \cdot$$

$$\sum_{k+1}^{k+1+t} d_{i_1}\cdots d_{i_{r-1}}\Big|\cdot|a_{nk}|.$$

Then,

$$\sum_{k}|a_{n+1,k}| \leq \frac{1}{|\sigma_{n+1}|}\sum_{k=0}^{mn}\{|d'_{n+1} + \sum_{r=1}^{m+1}(-1)^{r-1}\beta_{r-1}\Big(\frac{z_2}{z_1}\Big)^{r-1} d_{k+1}^{r-1}| +$$

$$|(z_1+d_{k+1})\sum_{r=1}^{m}(-1)^{r-1}\beta_r\Big(\frac{z_2}{z_1}\Big)^{r}\sum_{k+1}^{k+2} d_{i_1}\cdots d_{i_{r-1}}| + \ldots +$$

$$|\prod_{\nu=0}^{t-1}(z_1+d_{k+t-\nu})\sum_{r=1}^{m+1-t}(-1)^{r-1}\beta_{t+r-1}\Big(\frac{z_2}{z_1}\Big)^{t+r-1} \cdot$$

$$\sum_{k+1}^{k+1+t} d_{i_1}\ldots d_{i_{r-1}}| + \cdots +$$

$$\left|\prod_{\nu=0}^{m-1}(z_1 + d_{k+m-\nu})\cdot\beta_m\left(\frac{z_2}{z_1}\right)^m\right|\}|a_{nk}|.$$

Thus if we define

$$(4.43)\quad \phi_t^k = \prod_{\nu=0}^{t-1}(z_1+d_{k+t-\nu})\cdot\sum_{r=1}^{m+1-t}(-1)^{r-1}\beta_{t+r-1}\left(\frac{z_2}{z_1}\right)^{t+r-1}\cdot$$

$$\sum_{k+1}^{k+1+t} d_{i_1}\cdots d_{i_{r-1}},\quad 0 \leq t \leq m,$$

we have

$$(4.44)\quad \sum_k |a_{n+1,k}| \leq \frac{1}{|\sigma_{n+1}|}\sum_{k=0}^{mn}\{|d'_{n+1} + \phi_0^k| + |\phi_1^k| + \cdots +$$

$$|\phi_t^k| + \cdots + |\phi_m^k|\}|a_{nk}| \quad \text{for} \quad n \geq 1.$$

From (4.44) it is clear that if

$$(4.45)\quad \frac{1}{|\sigma_{n+1}|}\{|d'_{n+1} + \phi_0^k| + |\phi_1^k| + \cdots + |\phi_m^k|\} \leq 1 \quad \text{for each } k$$

and for all large n, then it will follow that

$$(4.46)\quad \sum_k |a_{nk}| = O(1) \quad \text{as} \quad n \to \infty.$$

In particular, if for all large n, $\dfrac{d'_{n+1} + \phi_0^k}{\sigma_{n+1}}$ and $\dfrac{\phi_t^k}{\sigma_{n+1}}$ are real and all non-negative or real and all non-positive for $1 \leq t \leq m$, then if

$$(4.47)\quad \sum_{t=0}^{m} \phi_t^k = f(z_2) \quad \text{for each } k,$$

it follows from (4.44) that $\sum_k |a_{n+1,k}| \leq \sum_k |a_{nk}|$ for all large n, so (4.46) is true. We remark here that from (4.44) it is apparent that the above conditions are needed only for $0 \leq k \leq mn$. For the moment we will assume the truth of (4.47).

Substituting $t = j-r+1$ in (4.42) gives

$$a_{n+1,k} = \frac{1}{\sigma_{n+1}} \{d'_{n+1} a_{nk} + \sum_{j=0}^{m} \beta_j \left(\frac{z_2}{z_1}\right)^j \sum_{t=0}^{j} (-1)^{j-t} \prod_{\nu=0}^{t-1} (z_1+d_{k-\nu}) \cdot$$

$$\sum_{k-t+1}^{k+1} d_{i_1} \cdots d_{i_{j-t}} \cdot a_{n,k-t}\} =$$

$$\frac{1}{\sigma_{n+1}} \{d'_{n+1} a_{nk} + \sum_{t=0}^{m} \sum_{j=t}^{m} \beta_j \left(\frac{z_2}{z_1}\right)^j (-1)^{j-t} \prod_{\nu=0}^{t-1} (z_1+d_{k-\nu}) \cdot$$

$$\sum_{k-t+1}^{k+1} d_{i_1} \cdots d_{i_{j-t}} \cdot a_{n,k-t}\}.$$

Thus if we define

$$\delta_{n0}^{k} = \frac{1}{\sigma_{n+1}} (d'_{n+1} + \sum_{j=0}^{m} (-1)^j \beta_j \left(\frac{z_2}{z_1}\right)^j d_{k+1}^{j}) \quad \text{and}$$

$$\delta_{nt}^{k} = \frac{1}{\sigma_{n+1}} (\sum_{j=t}^{m} (-1)^{j+t} \beta_j \left(\frac{z_2}{z_1}\right)^j \prod_{\nu=0}^{t-1} (z_1+d_{k-\nu}) \cdot$$

$$\sum_{k-t+1}^{k+1} d_{i_1} \cdots d_{i_{j-t}}) \quad \text{for} \quad 1 \leq t \leq m,$$

it follows that

$$(4.48) \quad a_{n+1,k} = \sum_{t=0}^{m} \delta_{nt}^{k} a_{n,k-t} \quad , \quad n \geq 1 \quad \text{and} \quad k \geq 0.$$

Letting $n = 1$ in (4.48), we get

$$a_{2k} = \delta_{10}^{k} a_{1k} + \delta_{11}^{k} a_{1,k-1} + \cdots + \delta_{1m}^{k} a_{1,k-m} .$$

Suppose

$$(4.49) \quad a_{n+1,k} = a_{1k} \prod_{\nu=1}^{n} \delta_{\nu 0}^{k} + \sum_{\nu=1}^{n} \delta_{\nu 1}^{k} a_{\nu,k-1} \cdot \prod_{j=\nu+1}^{n} \delta_{j0}^{k} + \cdots +$$

$$\sum_{\nu=1}^{n} \delta_{\nu m}^{k} a_{\nu,k-m} \cdot \prod_{j=\nu+1}^{n} \delta_{j0}^{k} ,$$

where $\prod_{j=p}^{q} \delta_{j0}^{k} = 1$ when $q < p$.

From (4.48) we get

$$a_{n+2,k} = \delta_{n+1,0}^{k} a_{n+1,k} + \sum_{t=1}^{m} \delta_{n+1,t}^{k} a_{n+1,k-t} .$$

Using (4.49), this becomes

$$a_{n+2,k} = a_{1k} \prod_{\nu=1}^{n+1} \delta_{\nu 0}^{k} + \sum_{\nu=1}^{n} \delta_{\nu 1}^{k} a_{\nu,k-1} \cdot \prod_{j=\nu+1}^{n+1} \delta_{j0}^{k} + \dots +$$

$$\sum_{\nu=1}^{n} \delta_{\nu m}^{k} a_{\nu,k-m} \cdot \prod_{j=\nu+1}^{n+1} \delta_{j0}^{k} + \sum_{t=1}^{m} \delta_{n+1,t}^{k} a_{n+1,k-t} =$$

$$a_{1k} \prod_{\nu=1}^{n+1} \delta_{\nu 0}^{k} + \sum_{\nu=1}^{n+1} \delta_{\nu 1}^{k} a_{\nu,k-1} \cdot \prod_{j=\nu+1}^{n+1} \delta_{j0}^{k} + \dots +$$

$$\sum_{\nu=1}^{n+1} \delta_{\nu m}^{k} a_{\nu,k-m} \cdot \prod_{j=\nu+1}^{n+1} \delta_{j0}^{k} .$$

By induction, (4.49) is valid for $n \geq 1$, $k \geq 0$.

Define ε_t^k by $\delta_{nt}^{k} = \dfrac{\varepsilon_t^k}{\sigma_{n+1}}$, $1 \leq t \leq m$. Also, let

$1-\delta_{n0}^{k} = \dfrac{\tau_k}{\sigma_{n+1}}$, where $\tau_k = \sigma_{n+1} - [d'_{n+1} + \sum_{j=0}^{m} (-1)^j \beta_j (\frac{z_2}{z_1})^j d_{k+1}^{j}] =$

$d'_{n+1} + f(z_2) - d'_{n+1} - f(-\frac{z_2}{z_1} d_{k+1}) = f(z_2) - f(-\frac{z_2}{z_1} d_{k+1})$ is

independent of n.

Suppose for each $k \geq 0$ there is an n_k such that for $n \geq n_k$ we have $0 \leq \delta_{n0}^{k} < 1$. Then we claim

(4.50) $\quad |a_{n+1,k}| \leq A_k (\sum_{1}^{n} |\sigma_{\nu+1}|^{-1})^{k} \cdot \exp\{-\alpha_k \sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\}$ for $n \geq 1$, $k \geq 0$, where A_k and α_k depend only on k, and $0 < \alpha_k \leq |\tau_k|$.

For every real x, $1 + x \leq e^x$, so for $k \geq 0$,

$$(4.51)\quad \left|\prod_{\nu=j}^{n} \delta^k_{\nu 0}\right| = \prod_{\nu=j}^{n} [1-(1-\delta^k_{\nu 0})] = \prod_{\nu=j}^{n} \left(1-\frac{\tau_k}{\sigma_{\nu+1}}\right) \leq$$

$$\exp\{-|\tau_k| \sum_{j}^{n} |\sigma_{\nu+1}|^{-1}\} \text{ if } j > n_k \text{ and}$$

$$\sum_{j}^{n} |\sigma_{\nu+1}|^{-1} = 0 \text{ for } j > n.$$

Suppose $n > n_k$, $1 \leq i \leq n_k$, and $q \geq 0$ is fixed. Then, by (4.51),

$$\left|\prod_{\nu=i}^{n} \delta^k_{\nu 0}\right| = \left|\prod_{\nu=i}^{n_k} \delta^k_{\nu 0}\right| \cdot \left|\prod_{\nu=n_k+1}^{n} \delta^k_{\nu 0}\right| \leq \left(\max_{1\leq j\leq n_k} \left|\prod_{\nu=j}^{n_k} \delta^k_{\nu 0}\right|\right) \cdot$$

$$\exp\{-|\tau_k| \sum_{n_k+1}^{n} |\sigma_{\nu+1}|^{-1}\} \leq Q_k \exp\{|\tau_k| \sum_{1}^{n_k} |\sigma_{\nu+1}|^{-1}\} \cdot$$

$$\exp\{-|\tau_k| \sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\} \cdot \frac{1}{|\sigma_2^{-1}|^q} \left(\sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\right)^q =$$

$$B'_k \left(\sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\right)^q \exp\{-|\tau_k| \sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\}, \text{ where } B'_k$$

depends only on k. On the other hand, if $1 \leq i \leq n \leq n_k$, then

$$\left|\prod_{\nu=i}^{n} \delta^k_{\nu 0}\right| \leq \left(\max_{1\leq \ell \leq j \leq n_k} \left|\prod_{\nu=\ell}^{j} \delta^k_{\nu 0}\right|\right) \exp\{|\tau_k| \sum_{1}^{n_k} |\sigma_{\nu+1}|^{-1}\} \cdot$$

$$\exp\{-|\tau_k| \sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\} \leq M_k \exp\{-|\tau_k| \sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\} \cdot$$

$$\frac{1}{|\sigma_2^{-1}|^q} \left(\sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\right)^q = B''_k \left(\sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\right)^q \exp\{-|\tau_k| \sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\},$$

and B_k'' depends only on k. Finally, suppose $1 \leq n < i \leq n_k$. Then

$$\left|\prod_{\nu=i}^{n} \delta_{\nu 0}^{k}\right| = 1 \leq B_k(n)\Big(\sum_1^n |\sigma_{\nu+1}|^{-1}\Big)^q \exp\{-|\tau_k|\sum_1^n |\sigma_{\nu+1}|^{-1}\}$$

if $B_k(n)$ is sufficiently large. Let $B_k''' = \max\limits_{1 \leq n \leq n_k - 1} B_k(n)$.

Now

$$\left|\prod_{\nu=i}^{n} \delta_{\nu 0}^{k}\right| \leq B_k'''\Big(\sum_1^n |\sigma_{\nu+1}|^{-1}\Big)^q \exp\{-|\tau_k|\sum_1^n |\sigma_{\nu+1}|^{-1}\} \text{ for}$$

$1 \leq n < n_k$ and $n < i \leq n_k$, and B_k''' depends only on k.

Let $B_k = \max(B_k', B_k'', B_k''')$. It follows that there is a constant B_k, independent of i, such that

$$\left|\prod_{\nu=i}^{n} \delta_{\nu 0}^{k}\right| \leq B_k\Big(\sum_1^n |\sigma_{\nu+1}|^{-1}\Big)^q \exp\{-|\tau_k|\sum_1^n |\sigma_{\nu+1}|^{-1}\} \text{ for} \tag{4.52}$$

$n \geq 1$, $k \geq 0$, if $1 \leq i \leq n_k$.

Now let $k = 0$ in (4.49) to obtain $a_{n+1,0} = a_{10} \prod\limits_{\nu=1}^{n} \delta_{\nu 0}^{0}$.

An application of (4.52) with $q = k = 0$ gives

$$|a_{n+1,0}| = |a_{10}| \cdot \left|\prod_0^n \delta_{\nu 0}^{0}\right| \leq |a_{10}| B_0 \exp\{-|\tau_0|\sum_1^n |\sigma_{\nu+1}|^{-1}\},\ n \geq 1.$$

If $A_0 = |a_{10}| B_0$ and $\alpha_0 = |\tau_0|$, we have (4.50) for $k = 0$.

Suppose (4.50) is true for columns $0, \ldots, k-1$. From (4.49) we have

$$|a_{n+1,k}| \leq \left|a_{1k} \prod_{\nu=1}^{n} \delta_{\nu 0}^{k}\right| + \sum_{\nu=1}^{n} \left|\delta_{\nu 1}^{k} a_{\nu,k-1} \prod_{j=\nu+1}^{n} \delta_{j0}^{k}\right| + \cdots +$$

$$\sum_{\nu=1}^{n} \left|\delta_{\nu m}^{k} a_{\nu,k-m} \prod_{j=\nu+1}^{n} \delta_{j0}^{k}\right| = \left|a_{1k} \prod_{\nu=1}^{n} \delta_{\nu 0}^{k}\right| + T_1 + \cdots + T_m.$$

In view of our convention regarding a_{np}, $T_r = 0$ if $r > k$. Thus we suppose $r \leq k$. Let

$$T_r = \sum_{\nu=1}^{n} |\delta_{\nu r}^{k} a_{\nu,k-r} \prod_{j=\nu+1}^{n} \delta_{j0}^{k}| \leq \sum_{\nu=1}^{n_k-1} |\delta_{\nu r}^{k} a_{\nu,k-r} \prod_{j=\nu+1}^{n} \delta_{j0}^{k}| +$$

$$\sum_{\nu=n_k}^{n} |\delta_{\nu r}^{k} a_{\nu,k-r} \prod_{j=\nu+1}^{n} \delta_{j0}^{k}| = S_1 + S_2,$$ where $S_2 = 0$ if $n_k > n$. By (4.52), $S_1 \leq n_k (\max_{\substack{1\leq\nu\leq n_k \\ 1\leq r\leq m}} |\delta_{\nu r}^{k} a_{\nu,k-r}|) B_k (\sum_{1}^{n} |\sigma_{\nu+1}|^{-1})^{k}$.

$$\exp\{-|\tau_k| \sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\} = C_k (\sum_{1}^{n} |\sigma_{\nu+1}|^{-1})^{k} \exp\{-|\tau_k| \sum_{1}^{n} |\sigma_{\nu+1}|^{-1}\}.$$

We remark here that we may assume $n_k \geq 2$, and, if it is further assumed that $\sum_{1}^{\infty} |\sigma_{\nu+1}|^{-1} = \infty$, that n_k is so large that $\sum_{1}^{n_k} |\sigma_{j+1}|^{-1} \geq 1$. Now we apply the inductive supposition (4.50) in conjunction with (4.51) to get

$$S_2 \leq \sum_{\nu=n_k}^{n} |\delta_{\nu r}^{k}| A_{k-r} (\sum_{1}^{\nu-1} |\sigma_{j+1}|^{-1})^{k-r} \exp\{-\alpha_{k-r} \sum_{1}^{\nu-1} |\sigma_{j+1}|^{-1}\} \cdot$$

$$\exp\{-|\tau_k| \sum_{\nu+1}^{n} |\sigma_{j+1}|^{-1}\} \leq D_k (\sum_{1}^{n} |\sigma_{j+1}|^{-1})^{k-r} \sum_{\nu=n_k}^{n} |\delta_{\nu r}^{k}| \cdot$$

$$\exp\{-\alpha_k (\sum_{1}^{n} |\sigma_{j+1}|^{-1} - |\sigma_{\nu+1}|^{-1})\},$$ where $D_k = \max_{1\leq r\leq m} A_{k-r}$ and $\alpha_k = \min_{1\leq r\leq m} (\alpha_{k-r}, |\tau_k|)$. But $\alpha_k |\sigma_{\nu+1}|^{-1} \leq |\tau_k| \cdot |\sigma_{\nu+1}|^{-1}$

$$= 1 - \delta_{\nu 0}^{k} \leq 1,$$ so

$$S_2 \leq eD_k\Big(\sum_1^n|\sigma_{j+1}|^{-1}\Big)^{k-r} \exp\{-\alpha_k\sum_1^n|\sigma_{j+1}|^{-1}\}\cdot\sum_{\nu=n_k}^{n}|\delta^k_{\nu r}|$$

$$= eD_k\Big(\sum_1^n|\sigma_{j+1}|^{-1}\Big)^{k-r} \exp\{-\alpha_k\sum_1^n|\sigma_{j+1}|^{-1}\}\cdot\varepsilon^k_r\sum_{\nu=n_k}^{n}|\sigma_{\nu+1}|^{-1}$$

$$\leq eD_k\varepsilon^k_r\Big(\sum_1^n|\sigma_{j+1}|^{-1}\Big)^{k-r+1} \exp\{-\alpha_k\sum_1^n|\sigma_{j+1}|^{-1}\}$$

$$\leq E_k\Big(\sum_1^n|\sigma_{j+1}|^{-1}\Big)^{k} \exp\{-\alpha_k\sum_1^n|\sigma_{j+1}|^{-1}\},$$

where $E_k = \max\limits_{1\leq r\leq m} eD_k\varepsilon^k_r$ and the inequality is independent of r. From (4.52),

$\left|a_{1k}\prod\limits_{\nu=1}^{n}\delta^k_{\nu 0}\right| \leq F_k\Big(\sum\limits_1^n|\sigma_{j+1}|^{-1}\Big)^k \exp\{-|\tau_k|\sum\limits_1^n|\sigma_{j+1}|^{-1}\}$. It now follows immediately from the estimates for S_1, S_2, and $\left|a_{1k}\prod\limits_{\nu=1}^{n}\delta^k_{\nu 0}\right|$, since they are independent of r, that (4.50) is valid for the k^{th} column and thus for every column. As a consequence we have

$$(4.53)\qquad \lim_{n\to\infty}|a_{n+1,k}| = 0 \quad \text{for} \quad k \geq 0$$

provided $\sum\limits_1^\infty|\sigma_{\gamma+1}|^{-1} = \infty$.

The formulas (4.31), (4.46), and (4.53) are the regularity conditions for the transformation (4.29).

It remains to prove (4.47). The following notation will be used throughout the proof:

$$\sum_a^b d_{i_1}\cdots d_{i_r} \equiv \sum_{\substack{a\le i_1\le i_2\le\\ \cdots\le i_r\le b}} d_{i_1}\cdots d_{i_r} \quad \text{if} \quad r \ge 1,$$

$$\sum_a^b d_{i_1}\cdots d_{i_0} = 1 = d_{i_1}\cdots d_{i_0},\ \sum_a^b d_{p_1}\cdots d_{p_0} = 1 = d_{p_1}\cdots d_{p_0},$$

$$\sum_a^b d_{p_1}\cdots d_{p_r} \equiv \sum_{\substack{a\le p_1< p_2<\\ \cdots<p_r\le b}} d_{p_1}\cdots d_{p_r} \quad \text{if} \quad r \ge 1.$$

By definition,

$$\phi_r^k = \prod_{\nu=0}^{r-1}(z_1+d_{k+r-\nu})\cdot\sum_{j=1}^{m-r+1}(-1)^{j-1}\beta_{r+j-1}\left(\frac{z_2}{z_1}\right)^{r+j-1}\cdot$$

$$\sum_{k+1}^{k+r+1} d_{i_1}\cdots d_{i_{j-1}}, \quad 0 \le r \le m. \quad \text{But}$$

$$\prod_{\nu=0}^{r-1}(z_1+d_{k+r-\nu}) = \prod_{k+1}^{k+r}(z_1+d_j) = \sum_{\nu=0}^{r} z_1^{r-\nu}\sum_{k+1}^{k+r} d_{p_1}\cdots d_{p_\nu},$$

so

$$\phi_r^k = \left(\sum_{\nu=0}^{r} z_1^{r-\nu}\sum_{k+1}^{k+r} d_{p_1}\cdots d_{p_\nu}\right)\sum_{j=1}^{m-r+1}(-1)^{j-1}\beta_{r+j-1}\left(\frac{z_2}{z_1}\right)^{r+j-1}\cdot$$

$$\sum_{k+1}^{k+r+1} d_{i_1}\cdots d_{i_{j-1}}.$$

Thus ϕ_r^k is a sum of terms of the form $(-1)^{j-1}\,T(r,\nu,j,P,I)$, where

$$(4.54)\quad T(r,\nu,j,P,I) = z_1^{r-\nu}\left(\frac{z_2}{z_1}\right)^{r+j-1}\beta_{r+j-1}(d_{p_1}\cdots d_{p_\nu})\cdot$$

$$(d_{i_1}\cdots d_{i_{j-1}}),\quad P = (p_1,\cdots,p_\nu),$$

and $I = (i_1,\ \cdots,\ i_{j-1})$, subject to

(4.55) $0 \le \nu \le r \le m$, $1 \le j \le m - r + 1$, $k + 1 \le p_1 < p_2 < \cdots < p_\nu \le k + r$, and $k + 1 \le i_1 \le i_2 \le \cdots \le i_{j-1} \le k + r + 1$.

In fact, $\phi_r^k = \Sigma(-1)^{j-1}\, T(r,\nu,j,P,I)$, where the sum is taken over all values of ν, j, P, and I subject only to the conditions (4.55). We will consider $T = T(r,\nu,j,P,I)$ and $T' = T'(r',\nu',j',P',I')$ to be distinct unless they have the same form, i.e., unless $r'-\nu' = r - \nu$, $r' + j' = r + j$, and the sequences $\{p_1', \cdots, p_{\nu'}', i_1', \cdots, i_{j'-1}'\}$ and $\{p_1, \cdots, p_\nu, i_1, \cdots, i_{j-1}\}$ are permutations of each other.

Clearly $(-1)^{j-1}\, T(r,\nu,j,P,I)$ reduces to $\beta_r z_2^r$ if $\nu = 0$ and $j = 1$, so that to prove (4.47) it is sufficient to show that

$$(4.56) \qquad \sum_{\substack{0\le r\le m \\ (\nu,j)\ne(0,1)}} (-1)^j\, T(r,\nu,j,P,I) = 0,$$

where the sum is over all r, ν, j, P, and I satisfying (4.55), except that the ordered pair $(\nu,j) \ne (0,1)$.

For convenience we shall use T for $T(r,\nu,j,P,I)$ where no confusion will result, and we will write $T \in \phi_r^k$ to indicate that T is one of the terms comprising ϕ_r^k. We will also assume $(\nu,j) \ne (0,1)$.

Lemma 4.57: Suppose $T \in \phi_r^k$, where T is given by (4.54). If $j > 1$ and there is a Ψ with $1 \le \Psi \le j - 1$ such that $i_\Psi \ne p_\mu$ for every μ in the range $1 \le \mu \le \nu$, then $T \in \phi_{r+1}^k$.

<u>Proof:</u> By (4.55), a consequence of the assumption that $j > 1$ is that $r < m$. Hence, we may set $r' = r+1$, $j' = j-1$, and $\nu' = \nu+1$. Since $k + 1 \leq i_\Psi \leq k + r + 1$, we may assume that there is a largest μ_1 such that $0 \leq \mu_1 \leq \nu$ and $p_{\mu_1} < i_\Psi$, if we define $p_0 = k$. Choose $p'_\mu = p_\mu$ if $1 \leq \mu \leq \mu_1$, $p'_{\mu_1+1} = i_\Psi$, and $p'_\mu = p_{\mu-1}$ if $\mu_1 + 2 \leq \mu \leq \nu' = \nu + 1$. Choose $i'_\mu = i_\mu$ if $1 \leq \mu \leq \Psi-1$ and $i'_\mu = i_{\mu+1}$ if $\Psi \leq \mu \leq j'-1 = j - 2$. It is easily seen that the conditions 4.55 are satisfied for ν', j', r', p'_μ, and i'_μ. Moreover, $r' - \nu' = r - \nu$, $r' + j' = r + j$, and the sets $\{p'_1, \ldots, p'_{\nu'}, i'_1, \ldots, i'_{j'-1}\}$ and $\{p_1, \ldots, p_\nu, i_1, \ldots, i_{j-1}\}$ are the same, so $T' = T$, $T' \in \Phi^k_{r'} = \Phi^k_{r+1}$. The construction of T' from T is illustrated by the diagram below:

$$\begin{array}{ll} T: & (d_{p_1} \ldots d_{p_{\mu_1}} d_{p_{\mu_1+1}} \ldots d_{p_\nu})(d_{i_1} \ldots d_{i_{\Psi-1}} d_{i_\Psi} d_{i_{\Psi+1}} \ldots d_{i_{j-1}}) \\ & \qquad \ldots \qquad\qquad \ldots \qquad\qquad\qquad \ldots \qquad\qquad \ldots \\ T': & (d_{p'_1} \ldots d_{p'_{\mu_1}} d_{p'_{\mu_1+1}} d_{p'_{\mu_1+2}} \ldots d_{p'_{\nu'}})(d_{i'_1} \ldots d_{i'_{\Psi-1}} d_{i'_\Psi} \ldots d_{i'_{j'-1}}) \end{array}$$

It should be observed that the above lemma is true even if the last hypothesis holds vacuously, i.e., if $\nu = 0$.

For convenience, define $p_0 = k$.

<u>Lemma 4.58</u>: Let T be given by (4.54) and let $T \in \Phi^k_r$. Suppose that either $\nu = 0$ or $\nu > 0$, $j > 1$, and $i_{j-1} \leq p_\nu$. Define $r'' = \max(r - \nu, p_\nu - k - 1)$. Then if

$r' \leq r$, in order that $T \in \Phi^k_{r'}$ it is necessary and sufficient that $r' \geq r''$.

<u>Proof:</u> We prove the necessity first. If $T \in \Phi^k_{r'}$, then $T = T'(r',\nu',j',P',I')$ and $r' - \nu' = r - \nu$, $r' + j' = r + j$, and $\{p'_1,\ldots,p'_{\nu'}, i'_1,\ldots,i'_{j'-1}\}$ must be a permutation of $\{p_1,\ldots,p_\nu, i_1,\ldots,i_{j-1}\}$. In particular, if $r' = r - q$, then $\nu' = \nu - q$ and $j' = j + q$. But $\nu' \geq 0$, so $q \leq \nu$, and thus $r' \geq r - \nu$. If $\nu = 0$, the proof of the necessity is complete. Thus suppose $\nu > 0$, $j > 1$, and $i_{j-1} \leq p_\nu$. Since $\nu' = \nu - q$, at least q d_{p_μ}'s must be rewritten as $d_{i'_\psi}$'s. If d_{p_ν} is so rewritten, then by (4.55), $p_\nu \leq k + r' + 1$, whence $r' \geq p_\nu - k - 1$. If d_{p_ν} is not so rewritten, then $d_{p_\nu} = d_{p'_\mu}$ for some μ, so $p_\nu \leq k + r'$, or $r' \geq p_\nu - k > p_\nu - k - 1$.

The sufficiency is trivial if $\nu = 0$, so assume $\nu > 0$. In order to prove the sufficiency of the conditions, it is enough to show that T can be re-expressed as $T'(r',\nu',j',P',I')$ with r',ν',j',p'_μ, and i'_ψ in agreement with (4.55). We will so rewrite T. Let $r' = r - q \geq r''$, and set $\nu' = \nu - q$ and $j' = j + q$. Since $r - q \geq r'' \geq r - \nu$, clearly $q \leq \nu$, so $\nu' \geq 0$. Set $p'_\mu = p_\mu$ if $1 \leq \mu \leq \nu' = \nu - q$. Let the elements of the sequence $\{p_{\nu-q+1}, p_{\nu-q+2}, \ldots, p_\nu, i_1, \ldots, i_{j-1}\}$ be partially ordered according to magnitude, and let $(i'_1, i'_2, \ldots, i'_{j'-1})$ be the resulting $(q + j - 1)$-tuple. Now, $p'_\mu \leq p'_{\nu'} = p_{\nu-q} \leq p_\nu - q \leq k + r - q = k + r'$ for each μ. By hypothesis, $r' \geq r'' \geq p_\nu - k - 1$, so $p_\nu \leq k + r' + 1$; but then $i'_\psi \leq p_\nu \leq k + r' + 1$. The other conditions of (4.55) are obviously satisfied, so $T = T' \in \Phi^k_{r'}$. The construction of T' from T is illustrated below:

$$\begin{array}{llllllllll}
T: & (d_{p_1} & \cdots & d_{p_{\nu-q}} & d_{p_{\nu-q+1}} & \cdots & d_{p_\nu}) & (d_{i_1} & \cdots & d_{i_{j-1}}) \\
 & \updownarrow & \cdots & \updownarrow & \downarrow & \cdots & \downarrow & \swarrow & \cdots & \swarrow \\
T': & (d_{p'_1} & \cdots & d_{p'_{\nu'}}) & \multicolumn{6}{l}{\text{(partially ordered by magnitude)}}
\end{array}$$

<u>Lemma 4.59</u>: Let $T = T(r^*, \nu^*, j^*, P, I)$ be given, and suppose r^* is maximal such that $T \in \Phi^k_{r^*}$. Then if $j^* > 1$, each $i_\psi = p_\mu$ for some μ.

<u>Proof</u>: It follows immediately from (4.55) that $r^* < m$. If the conclusion were false, Lemma 4.57 would say $T \in \Phi^k_{r^*+1}$, thereby violating the definition of r^*.

<u>Corollary 4.60</u>: Let $T = T(r^*, \nu^*, j^*, P, I)$ be given, and suppose r^* is maximal such that $T \in \Phi^k_{r^*}$. Then $T \in \Phi^k_r$ if and only if $r^* \geq r \geq r'' = \max(r^* - \nu^*, p_{\nu^*} - k - 1)$.

<u>Proof</u>: If $j^* > 1$, then Lemma 4.59 implies $\nu^* > 0$ and $i_{j^*-1} \leq p_{\nu^*}$. But then Lemma 4.58 says that $T \in \Phi^k_r$ if and only if $r \geq r''$. Now suppose $j^* = 1$. Since we are concerned with T's for which $(\nu, j) \neq (0,1)$, $j^* = 1$

implies $\nu^* > 0$. Then Lemma 4.58 still applies to say $T \in \Phi_r^k$ if and only if $r \geq r''$.

Now let $T = T(r,\nu,j,P,I) \in \Phi_r^k$, and let t_r be the number of distinqt subscripts among $\{p_1,\ldots,p_\nu,i_1,\ldots,i_{j-1}\}$ which are $\leq k + r$. We claim that the number of times T appears in Φ_r^k is $\binom{t_r}{\nu}$. Since r is fixed and T is fixed, so must ν and j be fixed, i.e., if $T'(r,\nu',j',P',I') \in \Phi_r^k$ and $T' = T$, then $\nu' = \nu$ and $j' = j$. Thus for any such T', $P' = (p_1', \ldots, p_\nu')$, and all the elements in this ν-tuple are distinct. The number of ways of forming such a ν-tuple from t_r distinct elements is $\binom{t_r}{\nu}$. But T' is completely determined by P' inasmuch as there is a fixed set of subscripts from which to form P' and I'. This establishes the claim.

Suppose r^* is maximal such that $T(r^*,\nu^*,j^*,P,I) \in \Phi_{r^*}^k$. If $j^* = 1$, it is obvious that $t_{r^*} = \nu^*$; if $j^* > 1$, then $r^* < m$ and Lemma 4.57 says every $i_\Psi = p_\mu$ for some μ, so again $t_{r^*} = \nu^*$. For each μ, let $p_\mu = k + \alpha_\mu$. Let $T' = T'(r',\nu',j',P',I') \in \Phi_{r'}^k$, where $r' \leq r^*$ and $T' = T$. Then $\nu' \leq \nu^*$ and $p_{\nu'}' \leq k + r'$, so we see that $t_{r'} = \nu^*$ provided $p_{\nu^*} = k + \alpha_{\nu^*} \leq k + r'$, i.e., provided $r' \geq \alpha_{\nu^*}$. Similarly, $t_{r'} = \nu^* - 1$ provided $p_{\nu^*-1} = k + \alpha_{\nu^*-1} \leq k + r'$, or $r' \geq \alpha_{\nu^*-1}$. We have shown

(4.61) $t_r = \nu^*$ for $r^* \geq r \geq \alpha_{\nu^*}$; $t_r = \nu^* - 1$ for $\alpha_{\nu^*} - 1 \geq r \geq \alpha_{\nu^*-1}$ provided $T \in \Phi_r^k$ for all such r.

Let $T(r,\nu,j,P,I) \in \Phi_r^k$. Then the sign preceding T is $(-1)^{j-1}$, so all occurrences of T in Φ_r^k have the same sign. If $T'(r-1,\nu',j',P',I') \in \Phi_{r-1}^k$ and $T' = T$, then $j' = j - 1$, so the sign preceding T' is $(-1)^{j-2}$, and the sign preceding each occurrence of T in Φ_{r-1}^k is opposite that in Φ_r^k. To prove (4.56), then, it suffices to show that for each fixed $T(r,\nu,j,P,I)$ for which $(\nu,j) \neq (0,1)$, the relation

$$\sum_{\substack{T' \in \Phi_{r'}^k \\ T'=T}} (-1)^{j_{r'}-1} \binom{t_{r'}}{\nu_{r'}} = 0 \quad \text{holds, where } T' =$$

$T'(r',\nu_{r'},j_{r'},P_{r'},I_{r'}) = T$, and the sum is over all values of r' for which $T \in \Phi_{r'}^k$. Equivalently, since $\nu_{r'}$ changes as $j_{r'}$ does, we may show

$$\sum_{\substack{T' \in \Phi_{r'}^k \\ T'=T}} (-1)^{\nu_{r'}} \binom{t_{r'}}{\nu_{r'}} = 0. \tag{4.62}$$

By Corollary 4.60, if r^* and r'' are, respectively, the largest and the smallest values of r' for which $T \in \Phi_{r'}^k$, then $r'' = r^* - \nu^*$ or $r'' = p_{\nu^*} - k - 1$. Suppose first that $r'' = p_{\nu^*} - k - 1 = \alpha_{\nu^*} - 1$. Then if $T''(r'',\nu'',j'',P'',I'') \in \Phi_{r''}^k$ and $T'' = T$, $\nu'' = \nu^* - (r^* - \alpha_{\nu^*} + 1)$, so, in view of (4.61), the relation (4.62) becomes

$$\sum_{\nu^*-(r^*-\alpha_{\nu^*})}^{\nu^*} (-1)^{\nu}\binom{\nu^*}{\nu} + (-1)^{\nu^*-(r^*-\alpha_{\nu^*}+1)} \cdot \binom{\nu^* - 1}{\nu^* - (r^*-\alpha_{\nu^*} + 1)} = 0. \tag{4.63}$$

If $\nu^* = 0$, then, since $(\nu^*, j^*) \neq (0,1)$, $j^* > 1$. But this violates Lemma 4.59 since there are i_Ψ's but no p_μ's. Thus $\nu^* > 0$, and also $\alpha_{\nu^*} \geq 1$. By (4.55), $\alpha_{\nu^*} \leq r^*$, so $\nu^* - (r^* - \alpha_{\nu^*}) \leq \nu^*$. We now need the formula

(4.64) $\binom{b}{a} = \binom{b-1}{a} \quad \binom{b-1}{a-1}$, a and b are non-negative integers, $b \geq a$.

If $r^* - \alpha_{\nu^*} = 0$, then the left side of (4.63) reduces to $(-1)^{\nu^*} + (-1)^{\nu^*-1} = 0$, and (4.63) is true. Suppose (4.63) is true for $r^* - \alpha_{\nu^*} = q$, and let $r^* - \alpha_{\nu^*} = q + 1$. Then using the supposition and (4.64), we write for the left side of (4.63),

$$\sum_{\nu^*-q-1}^{\nu^*} (-1)^\nu \binom{\nu^*}{\nu} + (-1)^{\nu^*-q-2} \begin{pmatrix} \nu^* - 1 \\ \nu^* - q - 2 \end{pmatrix} =$$

$$\left[\sum_{\nu^*-q}^{\nu^*} (-1)^\nu \binom{\nu^*}{\nu} + (-1)^{\nu^*-q-1} \begin{pmatrix} \nu^* \\ \nu^*-q-1 \end{pmatrix}\right] + (-1)^{\nu^*-q} \begin{pmatrix} \nu^*-1 \\ \nu^*-q-2 \end{pmatrix} =$$

$$(-1)^{\nu^*-q} \begin{pmatrix} \nu^*-1 \\ \nu^*-q-1 \end{pmatrix} + (-1)^{\nu^*-q-1} \begin{pmatrix} \nu^* \\ \nu^*-q-1 \end{pmatrix} + (-1)^{\nu^*-q} \begin{pmatrix} \nu^*-1 \\ \nu^*-q-2 \end{pmatrix} =$$

$$(-1)^{\nu^*-q} \begin{pmatrix} \nu^*-1 \\ \nu^*-q-1 \end{pmatrix} + (-1)^{\nu^*-q-1} \left[\begin{pmatrix} \nu^*-1 \\ \nu^*-q-1 \end{pmatrix} + \begin{pmatrix} \nu^*-1 \\ \nu^*-q-2 \end{pmatrix}\right] +$$

$$(-1)^{\nu^*-q} \begin{pmatrix} \nu^*-1 \\ \nu^*-q-2 \end{pmatrix} = 0.$$

By induction, (4.63) is true regardless of the value of $r^* - \alpha_{\nu^*}$. Now assume that $r'' = r^* - \nu^* > \alpha_{\nu^*} - 1$. Then $\nu'' = 0$, so the left side of (4.62) is

$$\sum_{0}^{\nu^*} (-1)^{\nu} \binom{\nu^*}{\nu} = (1-1)^{\nu^*} = 0.$$

Hence, (4.62) is true, and thus also (4.56).

We have proved

<u>Theorem 4.65</u>: Let $f(z) = \sum_{0}^{m} \beta_\nu z^\nu$ and let $\sigma_{n+1} = d'_{n+1} + f(z_2) \neq 0$. For $0 \leq t \leq m$, define

$$\phi_t^k = \prod_{\nu=0}^{t-1} (z_1 + d_{k+t-\nu}) \cdot \sum_{r=1}^{m+1-t} (-1)^{r-1} \beta_{t+r-1} \left(\frac{z_2}{z_1}\right)^{t+r-1} \cdot$$

$$\sum_{\substack{k+1 \leq i_1 \leq \cdots \\ \leq i_{r-1} \leq k+t+1}} d_{i_1} \cdots d_{i_{r-1}},$$

where $\sum d_{i_1} \cdots d_{i_0} = 1 = \prod_{\nu=0}^{-1} (z_1 + d_{k-\nu})$. Let $N \geq 0$ be arbitrary, and for all $n > N$, let $\frac{1}{\sigma_{n+1}} (d'_{n+1} + \phi_0^k) \geq 0$ for $0 \leq k \leq mn$ and $\frac{\phi_t^k}{\sigma_{n+1}} \geq 0$ for $1 \leq t \leq m$ and $0 \leq k \leq mn$. Furthermore, assume that for each $k \geq 0$ there exists $n_k > 0$ such that $0 \leq \frac{1}{\sigma_{n+1}} (d'_{n+1} + \phi_0^k) < 1$ for all $n \geq n_k$. Finally, let $\sum_{1}^{\infty} |d'_{n+1}|^{-1} = \infty$. Then (f, d'_n, z_2) is consistent with, and at least as strong as, (z, d_n, z_1).

We have used the fact that $\sum_{1}^{\infty} |\sigma_{n+1}|^{-1} = \infty$ if and only if $\sum_{1}^{\infty} |d'_{n+1}|^{-1} = \infty$. It should be observed that since $\phi_0^k = f(-\frac{z_2}{z_1} d_{k+1})$, the hypotheses can be cast in a slightly different form by making this substitution.

We will now show that Meir's theorem is contained in Theorem 4.65. Choose $f(z) = z$, $m = 1$, $\beta_0 = 0$, $\beta_1 = 1$, and $z_1 = z_2 = 1$. Theorem 4.65 then reduces to :

Let $1 + d'_{n+1} \neq 0$ and suppose

a) for all $n > N$, $N \geq 0$ arbitrary, $\dfrac{d'_{n+1} - d_{k+1}}{d'_{n+1} + 1} \geq 0$

for $0 \leq k \leq n$;

b) for each $k \geq 0$ there is $n_k > 0$ such that

$0 \leq \dfrac{d'_{n+1} - d_{k+1}}{d'_{n+1} + 1} < 1$ for all $n \geq n_k$;

c) for all $n > N$, $\dfrac{1 + d_{k+1}}{1 + d'_{n+1}} \geq 0$ for $0 \leq k \leq n$;

d) $\sum\limits_{1}^{\infty} |d'_{n+1}|^{-1} = \infty$.

Then (z, d'_n) is consistent with, and at least as strong as, (z, d_n).

A moment's thought confirms that the hypotheses (a), (b), and (c) are consequences of the single hypothesis

(e) for all $n > N$, $N \geq 0$ arbitrary, let $0 \leq \dfrac{d'_{n+1} - d_{k+1}}{d'_{n+1} + 1} < 1$

for $0 \leq k \leq n$.

But (e) is equivalent to

(e') for all $n > N$, $N \geq 0$ arbitrary, let $0 < \dfrac{1 + d_{k+1}}{1 + d'_{n+1}} \leq 1$

for $0 \leq k \leq n$.

Thus, substituting the (stronger) hypothesis (e') for (a), (b), and (c), Theorem 4.65 takes the form:

Let $1 + d'_{n+1} \neq 0$. For $n > N$, $N \geq 0$ arbitrary, let $0 < \frac{1 + d_{k+1}}{1 + d'_{n+1}} \leq 1$ for $0 \leq k \leq n$. Let $\sum_{1}^{\infty} |d'_{n+1}|^{-1} = \infty$. Then (z, d'_n) is consistent with, and at least as strong as, (z, d_n).

This is Meir's Theorem.

<u>Corollary 4.66</u>: Let $f(z) = \sum_{0}^{m} \beta_\nu z^\nu$, $\beta_\nu > 0$, $z_2 > 0$, $z_1 > |d_n|$, $d_n < 0$, $d'_n \geq 0$, and $\sum_{1}^{\infty} |d'_{n+1}|^{-1} = \infty$. Then (f, d'_n, z_2) is consistent with, and at least as strong as, (z, d_n, z_1).

<u>Proof</u>: $\sigma_{n+1} = d'_{n+1} + \sum_{0}^{m} \beta_\nu z_2^\nu > 0$. $\phi_0^k = f(-\frac{z_2}{z_1} d_{k+1})$ $= f(\frac{z_2}{z_1}|d_{k+1}|) > 0$. Moreover, $\prod_{\nu=0}^{t-1} (z_1 + d_{k+t-\nu}) > 0$. Also, $(-1)^{r-1} d_{i_1} \cdots d_{i_{r-1}} = |d_{i_1} \cdots d_{i_{r-1}}|$, so $(-1)^{r-1} \sum d_{i_1} \cdots d_{i_{r-1}} > 0$. It follows that $\frac{d'_{n+1} + \phi_0^k}{\sigma_{n+1}}$ and $\frac{\phi_t^k}{\sigma_{n+1}}$ are positive. Finally, $\frac{z_2}{z_1}|d_{k+1}| < z_2$, so $f(-\frac{z_2}{z_1} d_{k+1}) < f(z_2)$. Then $\frac{d'_{n+1} + \phi_0^k}{\sigma_{n+1}} = \frac{d'_{n+1} + f(-\frac{z_2}{z_1} d_{k+1})}{\sigma_{n+1}}$ $< \frac{d'_{n+1} + f(z_2)}{\sigma_{n+1}} = 1$. The hypotheses of Theorem 4.65 are satisfied, so the result follows.

<u>Corollary 4.67</u>: Let $f(z) = \sum_{0}^{m} \beta_\nu z^\nu$, $\beta_\nu > 0$, $d_n < 0$, $z_1 > |d_n| \geq \frac{z_1}{z_2} > 0$, $d'_n < 0$, $|d'_n| \leq f(1)$. Then (f, d'_n, z_2) is consistent with, and at least as strong as, (z, d_n, z_1).

<u>Proof</u>: $\sigma_{n+1} = d'_{n+1} + f(z_2) > d'_{n+1} + f(1) \geq 0$. Moreover, $\frac{z_2}{z_1}|d_{k+1}| \geq 1$, so $f(-\frac{z_2}{z_1} d_{k+1}) = \phi_0^k \geq f(1) \geq |d'_n|$, so $\dfrac{d'_{n+1} + \phi_0^k}{\sigma_{n+1}} \geq 0$. The remaining hypotheses of Theorem 4.65 follow as in the proof of Corollary 4.66, except for the condition $\sum_{1}^{\infty} \frac{1}{|d'_{n+1}|} = \infty$. But this is a trivial consequence of $|d'_n| \leq f(1)$. The result follows.

<u>Corollary 4.68</u>: Let $f(z) = \sum_{0}^{m} \beta_\nu z^\nu$, $\beta_\nu > 0$, $d_n < 0$, $f(1) \geq z_1 > |d_n| \geq 1$. Then (f, d_n, z_1) is consistent with, and at least as strong as, (z, d_n, z_1).

<u>Proof</u>: Corollary 4.67 with $z_2 = z_1$ and $d'_n = d_n$.

The above corollaries require that $d_n < 0$. Now suppose that $z_1 > 0$, $z_2 > 0$, $0 \leq d_n \leq z_1 \leq B$ with $B \geq 1$, and $d'_n \geq \max_{|z| \leq z_2} |f(z)|$. Then $\sum_{k+1}^{k+t+1} d_{i_1} \cdots d_{i_{\nu-t}} \leq B^{\nu-t}(t+1)^{\nu-t}$ $\leq B^{m-1}(m+1)^{m-1} = M$ if $1 \leq t \leq m$. For $1 \leq t \leq m$, let

$\beta_t \geq M[\max_{1 \leq j \leq m-1} (\frac{z_2}{z_1})^j] \sum_{t+1}^{m} \beta_\nu$, where $\sum_{m+1}^{m} \beta_\nu = 0$. Then

$$\phi_t^k = \prod_{\nu=0}^{t-1} (z_1 + d_{k+t-\nu}) \cdot \sum_{r=1}^{m+1-t} (-1)^{r-1} \beta_{t+r-1} \left(\frac{z_2}{z_1}\right)^{t+r-1} \cdot \sum_{k+1}^{k+t+1} d_{i_1} \cdots d_{i_{r-1}}$$

$$= \prod_{k+1}^{k+t}(z_1+d_j)\cdot\sum_{\nu=t}^{m}(-1)^{\nu-t}\beta_\nu\left(\frac{z_2}{z_1}\right)^{\nu}\cdot\sum_{k+1}^{k+t+1}d_{i_1}\cdots d_{i_{\nu-t}}$$

$$= \prod_{k+1}^{k+t}(z_1+d_j)\cdot\left[\beta_t\left(\frac{z_2}{z_1}\right)^{t}+\sum_{\nu=t+1}^{m}(-1)^{\nu-t}\beta_\nu\left(\frac{z_2}{z_1}\right)^{\nu}\cdot\sum_{k+1}^{k+t+1}d_{i_1}\cdots d_{i_{\nu-t}}\right]$$

$$\geq \left(\frac{z_2}{z_1}\right)^{t}\prod_{k+1}^{k+t}(z_1+d_j)\cdot\left[\beta_t-\sum_{\nu=t+1}^{m}\beta_\nu\left(\frac{z_2}{z_1}\right)^{\nu-t}\cdot\sum_{k+1}^{k+t+1}d_{i_1}\cdots d_{i_{\nu-t}}\right]$$

$$\geq \left(\frac{z_2}{z_1}\right)^{t}\prod_{k+1}^{k+t}(z_1+d_j)\cdot\{\beta_t - M[\max_{1\leq j\leq m-1}\left(\frac{z_2}{z_1}\right)^{j}]\sum_{t+1}^{m}\beta_\nu\}\geq 0.$$

Moreover, $\frac{z_2}{z_1}d_{k+1} = \left(\frac{d_{k+1}}{z_1}\right)z_2 \leq z_2$, so $\beta_\nu\left(\frac{z_2}{z_1}d_{k+1}\right)^{\nu} \leq \beta_\nu z_2^{\nu}$.

Then

$$\phi_0^k = f\left(-\frac{z_2}{z_1}d_{k+1}\right) = \sum_0^m \beta_\nu\left(-\frac{z_2}{z_1}d_{k+1}\right)^{\nu} < \sum_0^m \beta_\nu z_2^{\nu} = f(z_2) \text{ since } m\geq 1.$$

Thus $0 \leq \dfrac{d'_{n+1}+\phi_0^k}{\sigma_{n+1}} = \dfrac{d'_{n+1} + f\left(-\frac{z_2}{z_1}d_{k+1}\right)}{d'_{n+1} + f(z_2)} < 1$ and $\dfrac{\phi_t^k}{\sigma_{n+1}} \geq 0$.

Theorem 4.65 now yields

<u>Corollary 4.69</u>: Let $f(z) = \sum_0^m \beta_\nu z^\nu$, $z_1 > 0$, $z_2 > 0$,

$0 \leq d_n \leq z_1 \leq B$ with $B \geq 1$, $d'_n \geq \max_{|z|\leq z_2}|f(z)|$, and

$$\beta_t \geq B^{m-1}(m+1)^{m-1}[\max_{1\leq j\leq m-1}\left(\frac{z_2}{z_1}\right)^{j}]\sum_{t+1}^{m}\beta_\nu \quad \text{for } 1 \leq t \leq m,$$

where $\sum_{m+1}^{m}\beta_\nu = 0$. Then, if $\sum_1^{\infty}(d'_{n+1})^{-1} = \infty$, it follows that (f,d'_n,z_2) is consistent with, and at least as strong as, (z,d_n,z_1).

From Definition 3.6 it is easily seen that the Euler method (E,p) is defined by the transformation

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} p^k s_k = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s_k,$$ and this

in turn is seen to be the $(z, \frac{1-p}{p})$ transformation. Setting $z_1 = 1$ and $d_n \equiv \frac{1-p}{p}$ in Corollary 4.69 gives

Corollary 4.70: Let $f(z) = \sum_0^m \beta_\nu z^\nu$, $z_2 > 0$, $d_n' \geq \max_{|z| \leq z_2} |f(z)|$, and $\beta_t \geq (m+1)^{m-1} [\max_{1 \leq j \leq m-1} z_2^j] \sum_{t+1}^{m} \beta_\nu$ for $1 \leq t \leq m$, where $\sum_{m+1}^{m} \beta_\nu = 0$. Then, if $\sum_1^\infty (d_{n+1}')^{-1} = \infty$ and $\frac{1}{2} \leq p \leq 1$, it follows that (f, d_n', z_2) is consistent with, and at least as strong as, (z, d_n, z_1).

It is well-known that the (E,p) method is regular ([8], Theorem 117). Consequently, Corollary 4.70 gives

<u>Corollary 4.71</u>: Let $f(z) = \sum_0^m \beta_\nu z^\nu$, $z_1 > 0$, $d_n \geq \max_{|z| \leq z_1} |f(z)|$, and $\beta_t \geq (m+1)^{m-1} [\max_{1 \leq j \leq m-1} z_1^j] \sum_{t+1}^{m} \beta_\nu$ for $1 \leq t \leq m$, where $\sum_{m+1}^{m} \beta_\nu = 0$. Then, if $\sum_1^\infty d_{n+1}^{-1} = \infty$, it follows that (f, d_n, z_1) is regular.

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