STUDIES ON THE NONLINEAR VIBRATIONS OF SYSTEMS WITH ONE AND TWO DEGREE-OF-FREEDOM

Thesis for the Degree of Ph. D.

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DAVID OWEN SWINT

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ABSTRACT

STUDIES ON THE NONLINEAR VIBRATIONS OF

SYSTEMS WITH ONE AND TWO DEGREE-OF-FREEDOM

BY

David Owen Swint

This study presents an ultraspherical polynomial (U.P.) approximate method for representing natural vibrations for one and two degree-of-freedom spring-mass systems possessing nonlinear restoring forces. The nonlinear equations of motion are approximated by a set of linear equations over appropriate intervals of amplitude.

General multilinear U.P. relations are developed for approximating nonlinear restoring forces. The special cases of odd restoring forces, $f(x) = \alpha x + \beta x^3$, sin x, and sinh x, are examined in detail, and used in the one degree-of-freedom system to obtain approximate expressions for period-amplitude relations. In general, for these odd functions considered, the bilinear U.P. approximate method gave improved period-amplitude relations as compared to previously published approximate period-amplitude relations obtained from linear U.P. approximations.

In examining the two degree-of-freedom system a special case is treated which shows that where the linear U.P. method predicts a certain solution, the bilinear U.P. method does not. This special case is solved exactly by a finite difference technique and is shown to support the bilinear prediction.

Finally, the finite difference method, which was developed for solving the two degree-of-freedom system exactly, is applied to other

cases where the linear U.P. method predicts more than the usual two modes of vibration (superabundant modes). The finite difference method not only reveals the region over which these superabundant modes are present but also shows how these modes approach, for large amplitudes, limiting values predicted previously by other authors.

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By

David Owen Swint

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I. INTRODUCTION

1.1 Background Survey.

Nonlinear vibration problems occuring in engineering are usually difficult to solve exactly and only a relatively few have been so solved. Approximate methods used to obtain solutions of nonlinear systems vary widely. Broadly speaking, such nonlinear vibration problems are grouped as those which are nearly linear (slightly nonlinear) and those which are strongly nonlinear. The bulk of work to date has been carried out on the former group primarily for two reasons. First, it is reasonable to suppose that certain phenomena known to exist in the related linear system are only slightly changed in the slightly nonlinear case; and second, the majority of physical systems falls into this group.

Some approximate methods for obtaining solutions of slightly nonlinear vibration problems are the iteration method, the classical perturbation method, and the method of variation of parameters. The iteration method [2] usually consists in adopting the linear solution as a first approximation. Upon substituting this first approximation into the system of equations, a second approximation is obtained. This process can be repeated to obtain greater refinements provided that it is convergent.

The classical perturbation method is based on generating solutions to nonlinear vibration problems from known linear solutions which lie close to the nonlinear ones. This is accomplished by expanding the desired quantity in a power series with respect to some small parameter and converting the nonlinear problem into a set of linear problems.

The method of the variation of parameters [2] consists in adopting solutions which appear to be simple harmonic solutions, but the amplitude and phase of which are assumed to be slowly varying functions of time. The amplitude and phase are the solutions of a new set of nonlinear (auxiliary) equations. These auxiliary equations are integrated approximately by utilizing the property that the quantities of interest vary only slowly with time.

This research is concerned with yet another approximate technique of analysis--that of expanding the nonlinear terms of the problem in terms of ultraspherical polynomials. This expansion is made over an appropriate interval which best corresponds to the actual motion of the nonlinear system and linearization achieved by truncating the expanded series after the linear term. Denman introduced this approach first in 1959 and since then, with his co-workers, has intensively examined and extensively used this method in a number of studies [3,4,5,6,7]. The problems proposed in this dissertation are described below in Section 1.5 and are primarily motivated by the work of Denman and his co-workers. It is thus proper at this point to review in Sections 1.2 and 1.3 first the properties of the ultraspherical polynomials and the use made of these polynomials primarily by Denman and his co-workers. After this a two-line approximate method used by Ergin is presented in Section 1.4. In view of the work to date Section 1.5 gives a brief overview of the organization of this dissertation.

1.2 Properties of the Ultraspherical Polynomials.

The ultraspherical polynomials (U.P.) [20] are sets of polynomials orthogonal on the interval (-1,1) with respect to the weight function $(1-x^2)^{\lambda-1/2}$, each set corresponding to a value of $\lambda>-1/2$. They may be obtained from

$$P_n^{(\lambda)}(x) = A_n^{(\lambda)} (1-x^2)^{-\lambda+1/2} \left(\frac{d}{dx}\right)^n (1-x^2)^{\lambda+n-1/2}$$

where

$$A_n^{(\lambda)} = \frac{(-1)^n \, \Gamma(\lambda+1/2) \, \Gamma(n+2\lambda)}{2^n \, n! \, \Gamma(2\lambda) \, \Gamma(n+\lambda+1/2)} \qquad \lambda \neq 0$$

$$A_n^{(\lambda)} = \frac{(-1)^n \, 2^n \, n!}{(2^n)\,!} \qquad \lambda = 0$$

Here $A_n^{(\lambda)}$ is a normalizing factor, n is the degree of the U.P., and λ is an index, i.e., a parameter identifying a particular subset of polynomials.

A function, f(x), defined on the interval [-A,A] may be expanded in terms of these polynomials as

$$f(x) = \sum_{n=0}^{\infty} a_n^{(\lambda)} P_n^{(\lambda)}(t)$$

where $x = At$ and $-A \leq x \leq A$, $-1 \leq t \leq 1$, with coefficients $a_n^{(\lambda)}$ given as

$$a_n^{(\lambda)} = \frac{\int_{-1}^{1} f(At)\, P_n^{(\lambda)}(t)\, (1-t^2)^{\lambda-1/2}\, dt}{\int_{-1}^{1} [P_n^{(\lambda)}(t)]^2\, (1-t^2)^{\lambda-1/2}\, dt}$$

Some frequently used subsets of the ultraspherical polynomials are: $\lambda=0$, Chebyshev polynomials of the first kind; $\lambda=1/2$, Legendre polynomials; $\lambda=1$, Chebyshev polynomials of the second kind; and $\lambda\to\infty$, the powers of x. Appendix A contains additional properties of the ultraspherical polynomials.

1.3 Applications of U.P. to Nonlinear Vibration Problems.

The following is a brief summary of some of the problems in which the U.P. method has been applied.

1.3.1 Problems of Single Degree-of-Freedom.

The governing equation of motion for the free oscillation of the single degree-of-freedom system in Figure 1.1 is

$$\ddot{x} + f(x) = 0$$

where $f(x) = \omega_o^2 \sin x$ and $\omega_o = \sqrt{mg/\ell}$

Exact solution. The solution for the period can be expressed exactly in terms of elliptic integrals [4] as

$$\tau = \frac{4}{\omega_o} \int_o^{\pi/2} \frac{d\phi}{(1-k^2 \sin^2\phi)^{1/2}} = \frac{4}{\omega_o} K(k,\pi/2)$$

where $K(k,\pi/2)$ denotes the complete elliptic integral of the first kind and $k = \sin(A/2)$, with A being the amplitude of motion.

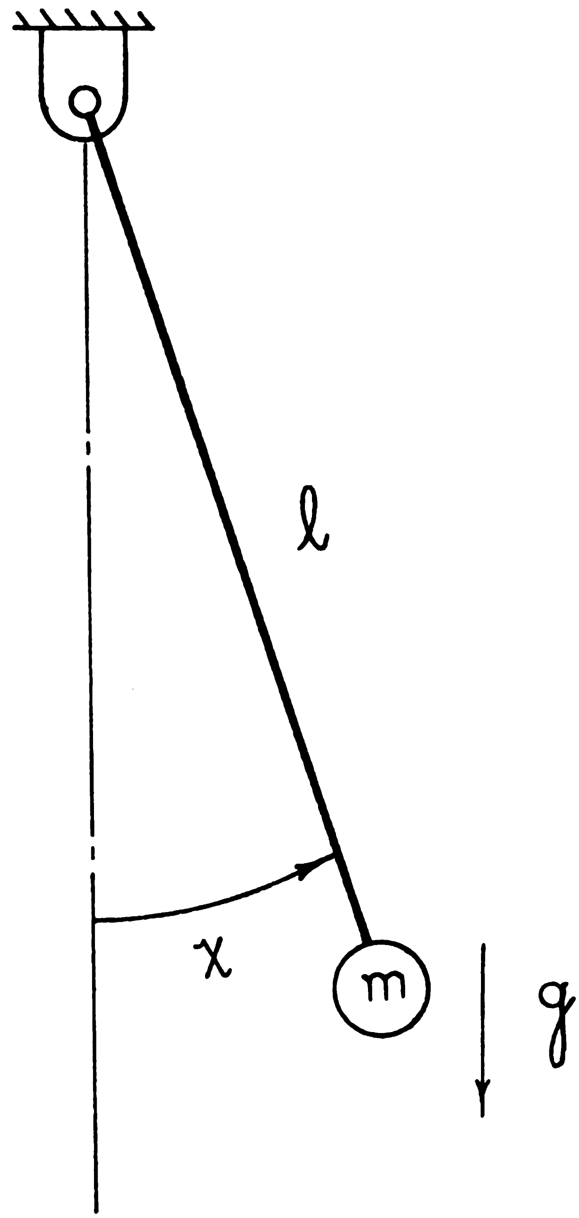


Figure 1.1 A Single Degree-of-Freedom System

The above can be rewritten nondimensionally as

$$\tau/\tau_o = \frac{2}{\pi} K(k,\pi/2)$$

where

$$\tau_o = \frac{2\pi}{\omega_o}$$

<u>One-line approximation.</u> Denman and his co-workers [3,4,5,6,7] introduced U.P. in solving single degree-of-freedom, free and forced, vibrating systems. The nonlinear function was expanded in terms of U.P. and truncated after the linear term. The vibration period was amplitude dependent, a fact not present when the same function is expanded by the Taylor series and truncated after the linear term. This is illustrated by the examples that follow.

i) <u>Taylor approximation.</u> Replace sin x by the linear term of the Taylor series expansion as $\sin x \simeq x$ and the equation of motion becomes $\ddot{x} + \omega_o^2 x = 0$. The approximate frequency relation is given by $\omega^* = \omega_o$ or $\tau^*/\tau_o = (2\pi/\omega_o)/\ (2\pi/\omega_o) = 1$, where * indicates an approximation.

ii) <u>U.P. approximation.</u> Replace sin x by the linear term of the ultraspherical polynomial expansion as

$$\sin x = a_1^{(\lambda)} P_1^{(\lambda)}\left(\frac{x}{A}\right) = \left[\left(\frac{2}{A}\right)^{(\lambda+1)} \Gamma(\lambda+2) \quad J_{\lambda+1} \quad (A)\right] x$$

where Γ is the Gamma function and J_n is an ordinary Bessel function of order n. The equation of motion becomes

$$\ddot{x} + \omega_o^2 \left[\left(\frac{2}{A}\right)^{\lambda+1} \Gamma(\lambda+2) \; J_{\lambda+1} \; (A) \right] x = 0$$

The approximate frequency-amplitude relation is given by

$$\omega^* = \omega_o \left[\left(\frac{2}{A}\right)^{\lambda+1} \Gamma(\lambda+2) \; J_{\lambda+1} \; (A) \right]^{1/2}$$

Thus

$$\frac{\tau^*(\lambda,A)}{\tau_o} = \frac{2\pi/\omega^*}{2\pi/\omega_o} = \frac{\omega_o}{\omega^*} = \left[\left(\frac{2}{A}\right)^{\lambda+1} \Gamma(\lambda+2) \; J_{\lambda+1} \; (A) \right]^{-1/2}$$

or,

for $\lambda = 0$,
$$\frac{\tau^*(0,A)}{\tau_o} = \left[\left(\frac{2}{A}\right) J_1 \; (A) \right]^{-1/2}$$

for $\lambda = 1/2$,
$$\frac{\tau^* \; (1/2,A)}{\tau_o} = \left[\frac{3\sqrt{\pi/2} \; J_{3/2} \; (A)}{(A)^{3/2}} \right]^{-1/2}$$

iii) Graphical approximation. A simplified graphical approximation as suggested by Denman and Liu [6] is now summarized here. This graphical approximation has been shown to be related to the one-line U.P. method. This procedure is illustrated below for the function $f(x) = \sin x$ as:

1. Plot the nonlinear function $f(x)$ of the system as $f(x)$ vs x.
2. Select an amplitude $x = A$ and draw a straight line from the origin to an ordinate at A, such that the maximum error due to this straight-line approximation to $f(x)$ in $(0,A)$ is minimized as shown in Figure 1.2 for $A = 0$ and $A = 3\pi/4$.

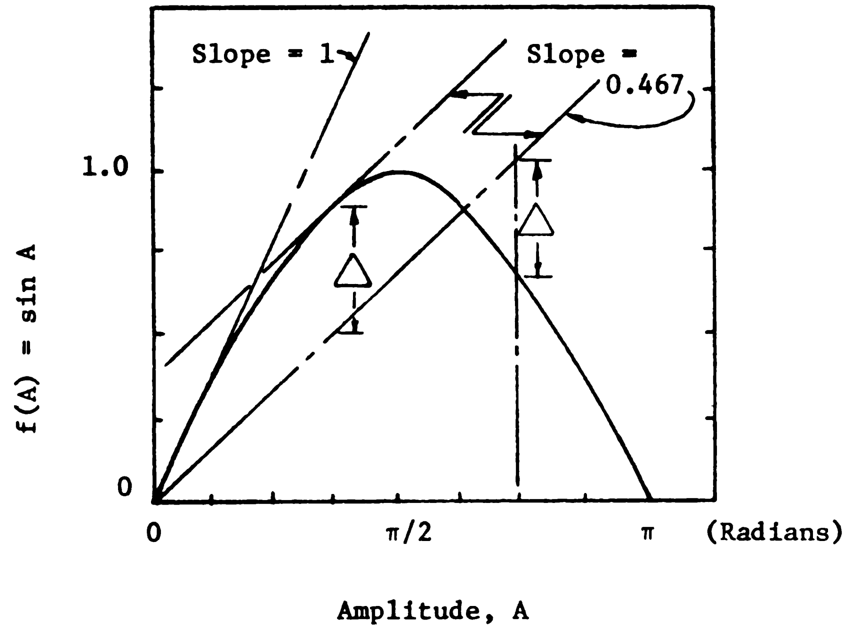


Figure 1.2 Graphical Approximation to the Non-linear Function Sin x

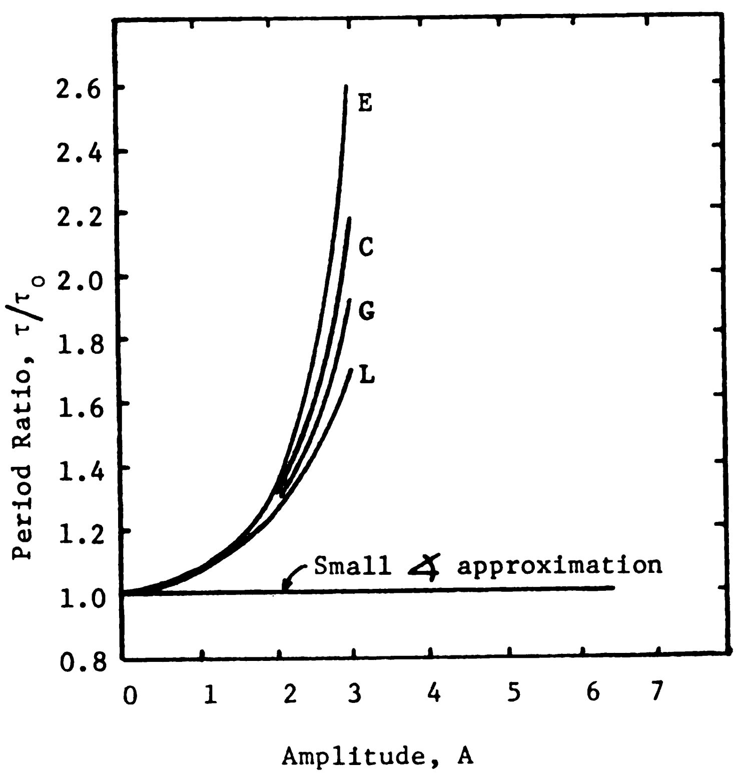


Figure 1.3 **Period-Amplitude Curves for the Free Vibration of a Single Degree-of-Freedom System With Sin x being the Nonlinear Restoring Force**

3. The slope of this straight line is an approximate equivalent "spring constant" corresponding to this value of A. From this the approximate period ratio τ^*/τ_o can be obtained as:

$$\ddot{x} + kx = 0 \text{ where } \omega = \sqrt{k} \begin{cases} \omega_o = \sqrt{k_o} = \sqrt{1} & \text{(slope @ A = 0)} \\ \omega^* = \sqrt{k} = \sqrt{0.467} & \text{(slope @ A } 3\pi/4) \end{cases}$$

Therefore,

$$\tau^*/\tau_o = \frac{2\pi/\sqrt{0.467}}{2\pi/1} = \frac{1}{0.689} = 1.46$$

This value is then plotted as a point on the τ/τ_o versus A graph in Figure 1.3.

4. The process is repeated for various values of the amplitude A, and the graph of τ^*/τ_o versus A is obtained.

The graph τ/τ_o versus A in Figure 1.3 contains, in addition to the graphical approximation C, the exact result E, the linear Chebyshev approximation C, the linear Legendre approximation L, and linear Taylor approximation T; to the function sin x representing a "soft" restoring force. The graphical method is quite simple and can be used whether f(x) is expressed in terms of simple functions or numerically as a load-displacement plot.

1.3.2 <u>Problems of Multiple Degree-of-Freedom</u>.

In addition to the use of U.P. in the study of single degree-of-freedom problems, Liu [7] examined a two degree-of-freedom, free and forced (with damping), vibrating systems represented in Figure 1.4. The coupling spring between the two masses was of the cubic type. This nonlinear spring was then linearized (one-line approximation) in

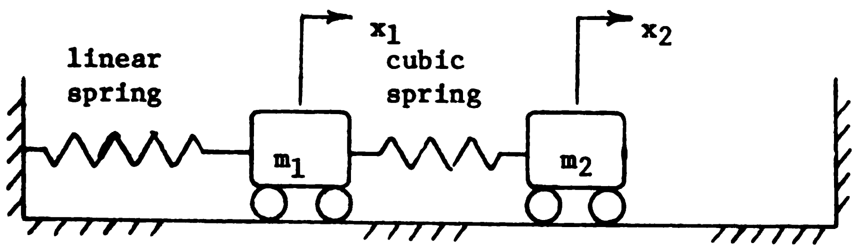


Figure 1.4 An Unsymmetric Coupled Spring-Mass System

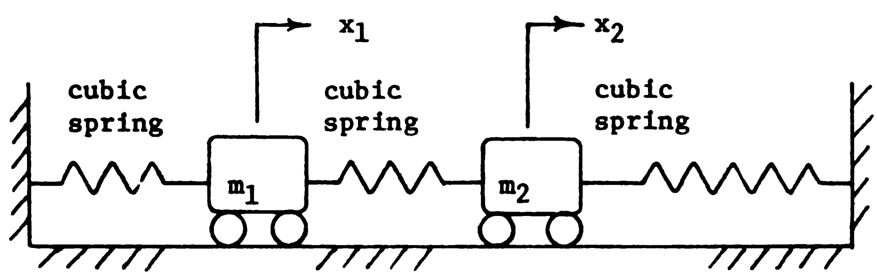


Figure 1.5 A Symmetric Coupled Spring-Mass System

terms of ultraspherical polynomials. Frequency-amplitude expressions were obtained, tabulated, and compared with numerical results. More recently, Anand [18] examined a symmetric two-mass system using a method which will be shown in this research to be essentially a one-line U.P. approximate method (Figure 1.5). Aside from the efforts of Liu and Anand, however, very little has been done in the application of the U.P. approximate method to multiple degree-of-freedom systems.

1.3.3 Problems Involving Continuous Systems.

Blotter [8] has applied the ultraspherical polynomial (one-line) method to systems governed by nonlinear partial differential equations in one space variable and one time variable. He assumed an autonomous system with the nonlinear term being the nonlinear forcing functions. A linear mode of deflection was assumed and this allowed him to linearize the nonlinear forces. This method was then applied to typical systems of strings, bars, circular membranes and plates on nonlinear foundations and with immovable end supports vibrating at large amplitudes. He found that if the Chebyshev polynomials ($\lambda=0$) are used, the frequency-amplitude relationship agrees exactly with the first order perturbation solutions.

1.4 Two-Line Approximation to Nonlinear Vibration Problems.

Ergin [9] introduced a two-line segment (bilinear) approximation for the nonlinear restoring force to obtain solutions to a number of single degree-of-freedom, transient-load problems. Ergin approximated the nonlinear function by two straight-line segments, each of which was determined by its slope and a point through which it passes. The problem reduced then to solving the same number of linear equations

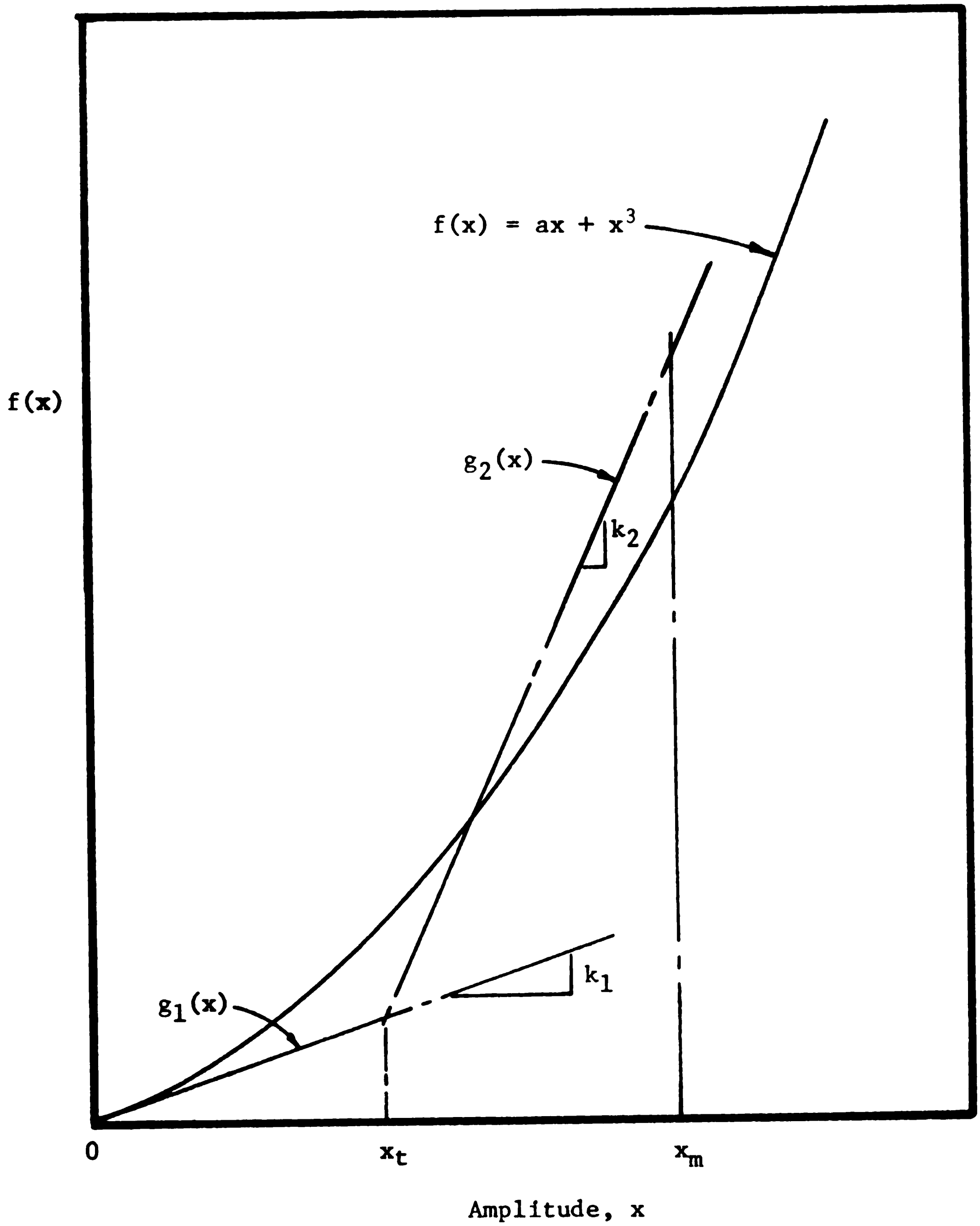


Figure 1.6 Ergin's Bilinear Approximation to the Nonlinear Function $f(x) = ax + x^3$

as there were line segments with the proper matching of displacements and velocities at the transition points. For simplicity Ergin chose the slope k_1 of the first line segment as the slope of the function at the origin, i.e., linear Taylor approximation, (Figure 1.6). The remaining task was then to establish the location of the transition point x_t between the two line segments and the slope k_2 of the second line segment by minimizing the mean square error. Ergin's two-line approach has provided the motivation in this dissertation to construct, as a natural extension, a two-line ultraspherical polynomial approximate method.

1.5 Organization of Dissertation.

In Chapter II a general development of the U.P. approximation to a nonlinear restoring force is presented, giving two bilinear approximations over some appropriate interval containing the equilibrium point. The bilinear U.P. approximation is then shown to degenerate into a one-line (linear) U.P. approximation as the transition amplitude point x_t connecting the two lines either approaches zero or the maximum amplitude x_m. Next, a mean square error minimizing method is used to generate a similar bilinear approximation. This is shown to agree with the bilinear U.P. approximation when certain conditions are satisfied. Three examples of odd, nonlinear restoring forces $f(x) = \alpha x + \beta x^3$, $\sin x$, and $\sinh x$ are then approximated by this bilinear U.P. approximation.

In Chapter III one degree-of-freedom free, undamped, nonlinear vibration problems are solved for the three nonlinear restoring forces whose bilinear approximations were derived in Chapter II. Also included in Chapter III is the one degree-of-freedom forced, undamped, nonlinear vibration problem involving a step function excitation with the restoring force being $f(x) = x + x^3$.

Chapter IV treats a two degree-of-freedom free, undamped, symmetric nonlinear vibration problem with cubic nonlinear restoring forces. An approximate technique which was developed by Anand and which yielded more than two modes of vibration (superabundant modes) is shown to be essentially the one-line U.P. approximation. A special case in which Anand's development [18] predicts these superabundant modes is then solved analytically using a bilinear U.P. analysis. The bilinear U.P. method shows no evidence that such a superabundant mode exists. An exact solution by a finite difference method of this special case is then shown to agree with the bilinear U.P. prediction. Also included in Chapter IV are additional exact solutions of the two degree-of-freedom problem considered. The results then enable us to argue that the superabundant modes for Anand's problem do approach, for large amplitudes, Rosenberg's [15] straight-line superabundant modes.

A brief summary of results as well as conclusions are contained in Chapter V.

II. BILINEAR ULTRASPHERICAL POLYNOMIAL APPROXIMATIONS

In this chapter we consider nonlinear functions of one space variable represented by two bilinear approximations over some appropriate interval which includes the equilibrium point. Next, the simplified case of the U.P. bilinear approximation of an odd function is considered and is compared against another bilinear approximation obtained by a mean square error minimizing method. Finally, three odd functions $\alpha x + \beta x^3$, sin x, and sinh x serve to illustrate the procedure leading to their U.P. bilinear approximation.

2.1 Ultraspherical Polynomial Approximation to a Nonlinear Function.

In this section we consider the nonlinear function G(x) shown in Figure 2.1. One may think of G(x) as representing some restoring force in the study of vibrations of physical systems. By equating G(x) to zero and solving for the real roots, one equilibrium point, x_o say, can be found. On either side of the equilibrium point a particle will in general experience a different spring force. For the case above, the spring force is approximated bilinearly on each side of the equilibrium point over the total interval from α to β as follows.

In the interest of simplifying the algebra, the origin is shifted horizontally to x_o. Next, the two slope parameters k_1 and k_3 are computed for the intervals $[0, |z_{m_1}|]$ and $[-|z_{m_3}|, 0]$ respectively, where $|z_{m_1}| = |x_{t_1} - x_o|$, $|z_{m_3}| = |x_o - x_{t_2}|$ and x_{t_1} and x_{t_2} are suitably chosen transition points. To obtain the line k_2,

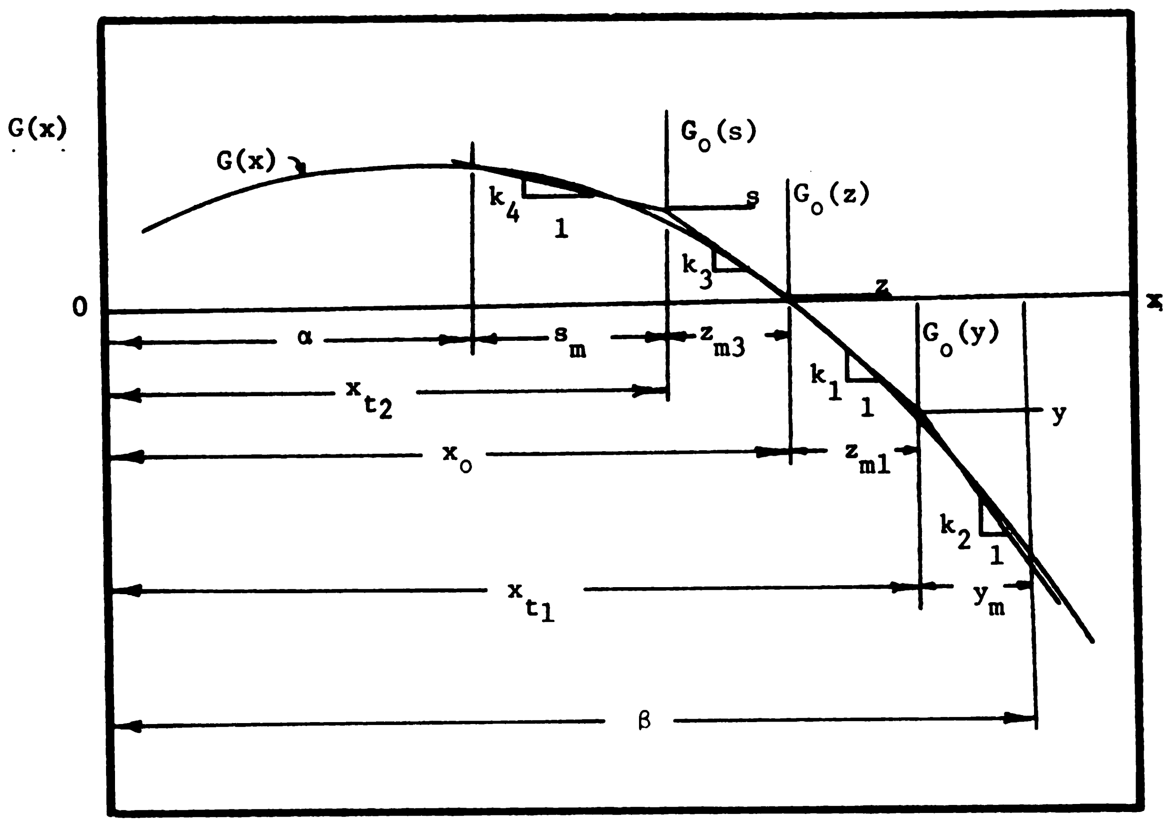


Figure 2.1 Nonlinear Function of One Space Variable Approximated by Multiple Linear Approximations

the origin is shifted horizontally and vertically to x_{t_1}. The line k_2 is then computed for the interval $[0, y_m]$ where $|y_m| = |\beta - x_{t_1}|$. This is followed by yet another shift in the origin horizontally and vertically to x_{t_2}, where the line k_4 is computed for the interval $[-|s_m|, 0]$, $|s_m| = |x_{t_2} - \alpha|$. In this way four linear functions are constructed, each computed by the one-line method. In each case, the shifted nonlinear restoring function is expanded in the ultraspherical polynomials $\left\{P_n^{(\lambda)}(t)\right\}$, where each $P_n^{(\lambda)}(t)$ is of degree n in t, and the polynomials are orthogonal on the appropriate symmetric interval. This is motivated by an approach by Howard [10] which concerns the construction of an odd function over the appropriate half interval.

As an illustration, the shifted function on $[0, |z_{m_1}|]$ is $G_o(z) = G(z + x_o)$ where $x = z + x_o$ (Figure 2.2). In order to expand $G_o(z)$ on $[0, |z_{m_1}|]$ in polynomials orthogonal on the symmetric interval $[-|z_{m_1}|, |z_{m_1}|]$, we construct an odd function which coincides with $G_o(z)$ on $[0, |z_{m_1}|]$.

A function G(x) continuous on $[-\xi, \xi]$ has the ultraspherical polynomial expansion

$$G(x) = \sum_{n=0}^{\infty} a_n^{(\lambda)} P_n^{(\lambda)}\left(\frac{x}{\xi}\right) = \sum_{n=0}^{\infty} a_n^{(\lambda)} P_n^{(\lambda)}(t) = a_o^{(\lambda)} P_o^{(\lambda)}(t)$$

$$+ a_1^{(\lambda)} P_1^{(\lambda)}(t) + a_2^{(\lambda)} P_2^{(\lambda)}(t) + a_3^{(\lambda)} P_3^{(\lambda)}(t) + \cdots .$$

where $-1 \leq t \leq 1$ and $(-\xi \leq x \leq \xi)$ with the coefficients $a_n^{(\lambda)}$ written as

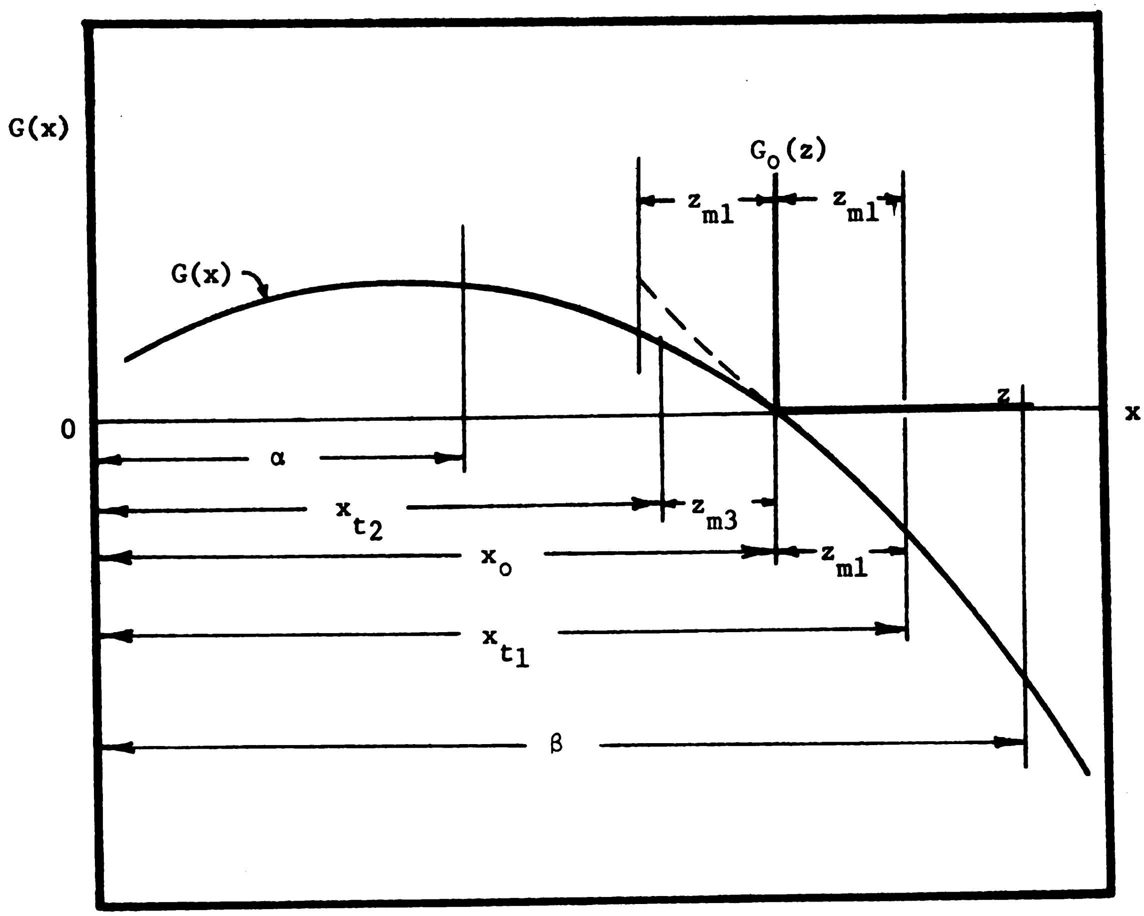


Figure 2.2 A Nonlinear Function G(x) Represented Between x_o and x_{t1} by an Odd Function About z=0 ($x=x_o$)

$$a_n^{(\lambda)} = \frac{\int_{-1}^{1} G(\xi t)\, P_n^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_{-1}^{1} [P_n^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt}$$

and where the weight function $\omega(\lambda,t)$ is defined as $\omega(\lambda,t) = (1-t^2)^{\lambda-1/2}$ and $\lambda>-1/2$.

<u>Parameter $k_1(\lambda,z_{m_1})$:</u> On the interval $[-|z_{m_1}|, |z_{m_1}|]$

$$G_o(z) = \begin{cases} -[G(-z + x_o)], & -|z_{m_1}| \le z<0 \\ G(z + x_o), & 0 < z \le |z_{m_1}| \end{cases}$$

Expanding $G_o(z)$ over this symmetric interval in terms of these polynomials and truncating after the linear term, we obtain

$$G_o(z) = a_1^{(\lambda)} \frac{z}{|z_{m_1}|} = [k_1(\lambda,z_{m_1})]\, z$$

where,

$$k_1(\lambda,z_{m_1}) = \frac{a_1^{(\lambda)}}{|z_{m_1}|} = \frac{1}{|z_{m_1}|}\left[\frac{\int_{-1}^{1} G_o(|z_{m_1}|t)\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_{-1}^{1} [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt}\right]$$

$$= \frac{1}{|z_{m_1}|}\left[\frac{\int_{-1}^{0} -[G(-|z_{m_1}|t + x_o)]\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{2\int_{0}^{1} [P_1^{(\lambda)}(t)]^2 \omega(\lambda,t)\, dt} + \frac{\int_{0}^{1} G(|z_{m_1}|t + x_o)\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{}\right] \tag{2.1}$$

By recognizing that $P_1^{(\lambda)}(-t) = -P_1^{(\lambda)}(t)$ and $\omega(\lambda,t)$ is the positive definite weight function, equation (2.1) is rewritten as,

$$k_1(\lambda, z_{m_1}) = \frac{1}{|z_{m_1}|}\left[\frac{\int_o^1 G(|z_{m_1}|t + x_o)\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_o^1 [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt}\right] \tag{2.2}$$

where $|z_{m_1}| = |x_{t_1} - x_o|$

In a similar manner the parameters k_3, k_2 and k_4 become

$$k_3(\lambda, z_{m_3}) = \frac{1}{|z_{m_3}|}\left[\frac{\int_o^1 -[G(-|z_{m_3}|t + x_o)]\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_o^1 [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt}\right] \tag{2.3}$$

where

$|z_{m_3}| = |x_o - x_{t_2}|$,

$k_2(\lambda, k_1, x_{t_1}, y_m)$

$$= \frac{1}{|y_m|}\left[\frac{\int_o^1 [G(|y_m|t + x_{t1}) - k_1(x_{t1} - x_o)]\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_o^1 [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt}\right] \tag{2.4}$$

where $|y_m| = |\beta - x_{t1}|$, and

$k_4(\lambda, k_3, x_{t_2}, s_m)$

$$= \frac{1}{|s_m|}\left[\frac{\int_o^1 -[G(-|s_m|t + x_{t_2}) - k_3(x_{t_2}-x_o)]\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_o^1 [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt}\right] \tag{2.5}$$

where $|s_m| = |x_{t_2}-\alpha|$

2.1.1 Limiting Cases.

If we allow the transition points, x_{t_1} and x_{t_2}, to simultaneously approach the extremes in amplitude, β and α , the four slope parameters k_1, k_2, k_3 and k_4 will degenerate into two slope parameters identical (for $\lambda=0$) to that obtained by Howard [10]. Applying these limiting processes to each of the four slope parameters (2.2), (2.3), (2.4) and (2.5) respectively we obtain

$$\lim_{\substack{x_{t_1}\to\beta \\ \lambda=0}} [k_1(\lambda,z_{m_1})] = \lim_{\substack{x_{t_1}\to\beta \\ \lambda=0}} \left[\frac{1}{|z_{m1}|}\;\frac{\int_o^1 G(|z_{m1}|t + x_o)\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_o^1 [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt}\right]$$

$$= \frac{1}{|\beta-x_o|}\left[\frac{\int_o^1 G(|\beta-x_o|t+x_o)\, T_1(t)\, \omega(0,t)\, dt}{\int_o^1 [T_1(t)]^2\, \omega(0,t)\, dt}\right] \tag{2.6}$$

and

$$\lim_{\substack{x_{t_2}\to\alpha \\ \lambda=0}} [k_3(\lambda, z_{m_3})] = \lim_{\substack{x_{t_2}\to\alpha \\ \lambda=0}} \left[\frac{1}{|z_{m_3}|} \cdot \frac{\int_0^1 -[G(-|z_{m_3}|\, t + x_o)]\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_0^1 [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt} \right]$$

$$= \frac{1}{|x_o-\alpha|} \left[\frac{\int_0^1 -[G(-|x_o-\alpha|\, t + x_o)]\, T_1(t)\, \omega(0,t)\, dt}{\int_0^1 [T_1(t)]^2\, \omega(0,t)\, dt} \right] \tag{2.7}$$

and

$$\lim_{\substack{x_{t_1}\to\beta \\ \lambda=0}} [k_2(\lambda, k_1, x_{t_1}, y_m)] = \lim_{\substack{x_{t_1}\to\beta \\ \lambda=0}} \left[\frac{1}{|y_m|} \frac{\int_0^1 [G(|y_m|\, t + x_{t_1}) - k_1(x_{t_1} - x_o)]\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_0^1 [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt} \right] \to \infty \tag{2.8}$$

and

$$\lim_{\substack{x_{t_2}\to\alpha \\ \lambda=0}} [k_4(\lambda,k_3,x_{t_2},s_m)] = \lim_{\substack{x_{t_2}\to\alpha \\ \lambda=0}} \left[\frac{1}{|s_m|} \cdot \right. \tag{2.9}$$

$$\left. \frac{\int_o^1 -[G(-|s_m|\ t + x_{t_2})-k_3(x_{t_2}-x_o)]\ P_1^{(\lambda)}(t)\ \omega(\lambda,t)\ dt}{\int_o^1 [P_1^{(\lambda)}(t)]^2\ \omega(\lambda,t)\ dt} \right] \to \infty$$

Upon examining equation (2.6), we find

$$\lim_{\substack{x_{t_1}\to\beta \\ \lambda=0}} [k_1(\lambda,z_{m_1})] = \frac{1}{|B|} \left[\frac{\int_o^1 G_o(|B|\ t)\ t\ (1-t^2)^{-1/2}\ dt}{\int_o^1 t^2\ (1-t^2)^{-1/2}\ dt} \right]$$

$$= \frac{1}{|B|} \left[\frac{4}{\pi} \int_o^1 G_o(|B|\ t)\ t\ (1-t^2)^{-1/2}\ dt \right] \tag{2.10}$$

where

$$G_o(|B|t) = G(|B|t + x_o), \qquad T_1(t) = t,$$

$$|B| = |\beta-x_o|, \qquad \int_{-1}^1 [T_1(t)]^2\ (1-t^2)^{-1/2}\ dt = \pi/2$$

Similarly, equation (2.7) yields,

$$\lim_{\substack{x_{t_2}\to\alpha \\ \lambda=0}} [k_3(\lambda,z_{m_3})] = \frac{1}{|A|}\left[-\frac{4}{\pi} \int_o^1 -G_o(-|A|t)\ t\ (1-t^2)^{-1/2}\ dt\right] \tag{2.11}$$

where

$$G_o(-|A|t) = G(-|A|t + x_o) \text{ and } |A| = |x_o-\alpha|$$

The limiting cases of k_1 (2.10) and of k_3 (2.11) represent the two slope parameters (Chebyshev polynomials, λ=0) that approximate G(x) over the interval $[\alpha,\beta]$. These agree exactly with Howard's result. The limiting cases of k_2 (2.8) and of k_4 (2.9) are meaningless since the respective amplitudes y_m and s_m approach zero.

2.1.2 Slope Parameters for the Case of a Nonlinear Odd Function Over a Symmetric Interval.

Restricting our consideration to nonlinear odd functions over a symmetric interval $[-x_m, x_m]$, we take the origin as the equilibrium point and equate the absolute values of the transition points about the equilibrium point. In Figure 2.1 we take G(x) as an odd function with x_o=0 to establish the equilibrium point as the origin and then observe that $|z_{m_1}| = x_t$ and $|z_{m_3}| = -x_t$ specify equal transition amplitude points. Substituting these relations into (2.2) and (2.3) we find that k_1 and k_3 yield the equivalent slopes

$$k_1 = k_3 = \frac{1}{x_t} \frac{\int_o^1 \omega\bar{g}\, t\, dt}{\int_o^1 \omega\, t^2\, dt} \tag{2.12}$$

where

$$x_t = x_{t_1} = x_{t_2}, \qquad \omega = \omega(\lambda,t) = (1-t^2)^{\lambda-1/2}$$

$$\bar{g} = G(x_t t), \qquad x = x_t t$$

Similarly, from (2.4) and (2.5) we find that k_2 and k_4 yield the equivalent slopes

$$k_2 = k_4 = \frac{1}{y_m} \quad \frac{\int_o^1 \omega \bar{\bar{g}}\, t\, dt - k_1 x_t \int_o^1 \omega\, t\, dt}{\int_o^1 \omega\, t^2\, dt} \tag{2.13}$$

where

$$y_m = x_m - x_t$$

$$\bar{\bar{g}} = G(y_m t + x_t)$$

$$x = y_m t + x_t$$

2.1.3 Slope Parameters Derived by Minimizing Methods.

In this section we shall first discuss a number of techniques for minimizing the error between the approximating function and the nonlinear function. We then apply a technique modeled after an approach by Ergin [9] to derive slope parameters for the linear and bilinear approximations which correspond to equations (2.12) and (2.13).

A variety of methods has been proposed to minimize the error between an approximating function and the actual nonlinear function [21]. Ergin investigated three methods: 1) That the work done per cycle by the bilinear and the nonlinear spring forces is the same; 2) that the mean square error between the bilinear and the nonlinear spring forces is minimized; and 3) that the spring forces are equal at the maximum displacement point. Ergin's development centers around the second approach. In this research we will investigate a fourth method which can be regarded as a generalization of Ergin's approach.

<u>Linear Approximation.</u> The approximation of a nonlinear function $f(x)$ over a symmetric interval by a polynomial $g(x)$ of any degree can be achieved by minimizing the integrated square of the error, subject to some appropriate weight function. We impose at this point one condition on these polynomials:

$p_n(t)$ is a polynomial of n^{th} degree in t. (2.14)

Thus, we express the square of the error as,

$$E^2 = \left\| f-g \right\| = \int_{-x_m}^{x_m} \omega(x)\ [f(x) - g(x)]^2\ dx \quad *$$

or,

$$E^2 = \int_{-x_m}^{x_m} \omega(x)\ \ [f(x) - a_o p_o(x) - a_1 p_1(x) - \cdots - a_k p_k(x)]^2\ dx$$

By a change of variable, $x = x_m t$, $\omega(x)$ and $p_n(x)$ are represented as $\omega(t)$ and $p(t)$ respectively since they are arbitrary at this point. Now, E^2 becomes

$$E^2 = \int_{-1}^{1} x_m\ \omega(t)\ [f(x_m t) - \sum_{n=0}^{k} a_n p_n(t)]^2\ dt$$

and by squaring the integrand, we have

$$E^2 = \int_{-1}^{1} x_m\ \omega(t)\ [f(x_m t)]^2\ dt + \sum_{n=0}^{k} a_n^2 \int_{-1}^{1} x_m\ \omega(t)\ [p_n(t)]^2\ dt$$

* Snyder [11] defines this relation as the least square norm of the difference of two functions over an interval.

$$-2 \sum_{n=0}^{k} a_n \int_{-1}^{1} x_m \omega(t)\ f\ (x_m t)\ p_n(t)\ dt$$

$$+2 \sum_{\substack{n=0 \\ n \neq \ell}}^{k} \sum_{\ell=0}^{k} a_n a_\ell \int_{-1}^{1} x_m \omega(t)\ p_n(t)\ p_\ell(t)\ dt \tag{2.15}$$

At this point we impose the orthonormal relation

$$\int_{-1}^{1} \omega(t)\ p_n(t)\ p_\ell(t) = \delta_{n\ell} \tag{2.16}$$

where $\delta_{n\ell}$ is the Kronecker delta. Now equation (2.15) can be simplified by using (2.16),

$$E^2 = \int_{-1}^{1} x_m\ \omega(t)\ [f(x_m t)]^2\ dt + \sum_{n=0}^{k} a_n{}^2\ x_m \int_{-1}^{1} \omega(t)\ [p_n(t)]^2\ dt$$

$$-2 \sum_{n=0}^{k} a_n x_m \int_{-1}^{1} \omega(t)\ f(x_m t)\ p_n(t)\ dt$$

The partial derivative of E^2 with respect to each a_n must vanish in order to make E^2 minimum. Thus

$$\frac{\partial E^2}{\partial a_n} = 0 = 2a_n x_m \int_{-1}^{1} \omega(t)\ [p_n(t)]^2\ dt - 2x_m \int_{-1}^{1} \omega(t)\ f(x_m t)\ p_n(t)\ dt$$

Hence,

$$a_n = \frac{\int_{-1}^{1} \omega(t)\ f(x_m t)\ p_n(t)\ dt}{\int_{-1}^{1} \omega(t)\ [p_n(t)]^2\ dt} \tag{2.17}$$

The set of polynomials corresponding to a particular weight function that satisfy the conditions (2.14) and (2.16) may be determined step by step.* Since $p_o(t)$ is of zero order, we write $p_o(t) = a$ and determine this constant from the normalizing condition

$$\int_{-1}^{1} \omega(t)\, a^2 dt = 1 \tag{2.18}$$

Since Schelkunoff [12] has shown that a weight function of one, $\omega(t) = 1$, yields the Legendre polynomials, a natural extension would then be to use the weight function, $\omega(\lambda,t) = (1-t^2)^{\lambda-1/2}$, associated with the ultraspherical polynomials. The Legendre polynomial is a special case of the ultraspherical polynomial when $\lambda = 1/2$. Thus, with $\omega(\lambda,t) = (1-t^2)^{\lambda-1/2}$ equation (2.17) reduces to

$$a = \left[\int_{-1}^{1} \omega(\lambda,t)\, dt\right]^{-1/2} = \Gamma(\lambda+1)/[\sqrt{\pi}\, \Gamma(\lambda+1/2)]^{1/2} = p_o(t)$$

Next, we determine $p_1(t) = b + ct$ from the orthonormal conditions,

$$\int_{-1}^{1} \omega(\lambda,t)\, p_o(t)\, p_1(t)\, dt = \int_{-1}^{1} \omega(\lambda,t)\, [a]\, [b + ct]\, dt = 0$$

and,

$$\int_{-1}^{1} \omega(\lambda,t)\, [p_1(t)]^2\, dt = 1$$

* The process is known as the Gram-Schmidt process [22].

Solving for b and c, we find

$$p_1(t) = [2\Gamma(\lambda+2)/[\sqrt{\pi}\ \Gamma(\lambda+1/2)]]^{1/2}\ t$$

Proceeding to the next polynomial and next, and next, we find a definite pattern which reduces to the form,

$$p_n(t) = \left[\frac{2\lambda^2\ \sqrt{\pi}\ \Gamma(\lambda+1/2)\ \Gamma(2\lambda+n)}{(n+\lambda)\ \Gamma(n+1)\ \Gamma(\lambda+1)\ \Gamma(2\lambda+1)}\right]^{-1/2} P_n^{(\lambda)}(t) = C_n P_n^{(\lambda)}(t) \tag{2.19}$$

After some algebraic manipulations C_n can be reduced to the form (see Appendix B for details),

$$p_n(t) = \left[\frac{2^{1-2\lambda}\ \pi\ \Gamma(2\lambda+n)}{(n+\lambda)\ (\Gamma(\lambda))^2\ \Gamma(n+1)}\right]^{-1/2} P_n^{(\lambda)}(t)$$

where the $P_n^{(\lambda)}(t)$ polynomials are called the ultraspherical polynomials. These ultraspherical polynomials are orthogonal but not normalized. The constant multiplier, C_n, is the normalizing factor.

Upon substituting equation (2.19) into equation (2.16) we obtain,

$$\int_{-1}^{1} \omega(\lambda,t)\ C_n P_n^{(\lambda)}(t)\ C_\ell\ P_\ell^{(\lambda)}(t)\ dt = \delta_{n\ell}$$

or,

$$\int_{-1}^{1} \omega(\lambda,t)\ P_n^{(\lambda)}(t)\ P_\ell^{(\lambda)}(t)\ dt = \delta_{n\ell}\ [C_n C_\ell]^{-1}$$

$$= \begin{cases} 0, \ n \neq \ell \\ \\ (C_n)^{-2} = \dfrac{2^{1-2\lambda} \pi \, \Gamma(2\lambda+n)}{(n+\lambda) \ [\Gamma(\lambda)]^2 \ \Gamma(n+1)} & \begin{matrix} n=\ell \\ \lambda \neq 0 \end{matrix} \end{cases}$$

Having shown that the ultraspherical polynomials can be generated from minimizing the integral of the squared error relationship, we now show the connection between the a_1 coefficient of the approximating polynomial and the slope parameter k.

Now, consider approximating a nonlinear odd function $f(x)$ over a symmetric interval by a linear approximation--that is, we truncate the approximating polynomial, $g(x)$, after the linear term.

$$f(x) \simeq g(x) = \sum_{n=0}^{k} a_n \, p_n \, (x)$$

letting $k = 1$,

$$g(x) = a_o p_o(x) + a_1 p_1(x)$$

or, by change of variable, $x = x_m t$

$$g(x_m t) = a_o p_o(t) + a_1 p_1(t) = a_o C_o P_o^{(\lambda)}(t) + a_1 C_1 P_1^{(\lambda)}(t)$$

Since f(x) is an odd function, only the linear term survives, therefore

$$g(x_m t) = a_1 C_1[2\lambda t] = \eta a_1 t$$

or

$$g(x) = [\eta a_1/x_m]\, x = kx \tag{2.20}$$

where $\eta = 2\lambda C_1$ and $k = \eta a_1/x_m$

From equation (2.20)

$$a_1 = \frac{\int_{-1}^{1} \omega(\lambda,t)\, f(x_m t)\, C_1 P_1^{(\lambda)}(t)\, dt}{\int_{-1}^{1} \omega(\lambda,t)\, [C_1 P_1^{(\lambda)}(t)]^2\, dt} = \frac{1}{\eta}\, \frac{\int_{o}^{1} \omega(\lambda,t)\, f(x_m t)\, t\, dt}{\int_{o}^{1} \omega(\lambda,t)\, t^2\, dt}$$

Thus,

$$g(x) = \frac{\eta}{x_m} \left[\frac{1}{\eta}\, \frac{\int_{o}^{1} \omega(\lambda,t)\, f(x_m t)\, t\, dt}{\int_{o}^{1} \omega(\lambda,t)\, t^2\, dt} \right] x$$

$$g(x) = \left[\frac{1}{x_m}\, \frac{\int_{o}^{1} \omega(\lambda,t)\, f(x_m t)\, t\, dt}{\int_{o}^{1} \omega(\lambda,t)\, t^2\, dt} \right] x = [k]\, x \tag{2.21}$$

Denman [5] obtained this general relationship for k by another method, namely, by expanding the nonlinear odd function in terms of ultraspherical polynomials and truncating after the linear term.

Based on these observations, the mean square error method was applied in generating polynomial approximations to smaller segments of a larger interval. Qualitatively, this provides a means of obtaining a multilinear polynomial approximation over the total interval and in particular, a bilinear ultraspherical polynomial approximation.

Bilinear Approximation. If we consider a general nonlinear function $f(x)$ derivable from the potential function $V(x) = \int f(x)\, dx$ and assume that the origin has been shifted to the local minimum potential point $(dV(x)/dx = 0)$, we can create two odd functions about this minimum potential point (one function that coincides with the nonlinear function for negative arguments and one that coincides for positive arguments). Having done this, we can compute the slope parameters k_1 and k_3. Similarly, by two more shifts of the origin the slope parameters, k_2 and k_4 can be found.

Now, we proceed to show under what conditions the mean square error method generates the ultraspherical polynomial approximating function. With no loss of generality we can simplify the algebra by choosing a nonlinear odd function with local minimum point at the origin.

The integral expression to be minimized is

$$E^2 = \left\| f-g \right\| = \int_{-x_m}^{x_m} \omega(x) \; [f(x)-g(x)]^2 \, dx = 2\int_{0}^{x_m} \omega(x) \; [f(x)-g(x)]^2 \, dx$$

$$= 2\left[\int_0^{x_t} \omega(x)\ [f(x)-g(x)]^2\ dx + \int_{x_t}^{x_m} \omega(x)\ [f(x)-g(x)]^2\ dx\right]$$

The first integral can be simplified by a change of variable, $x = x_m t$. The function in the second interval is shifted so that its origin is the transition point x_t. As shown in the Figure 2.3 below an odd function is created, followed by a change of variable to represent the interval as [-1,1].

Thus

$$E^2 = x_t \int_{-1}^{1} \omega(t)\ [f(x_t t) - \sum_{n=0}^{k} a_n p_n(t)]^2\ dt$$

$$+ y_m \int_{-1}^{1} \omega(t)\ [f_o(t) - \sum_{n=0}^{k} b_n p_n(t)]^2\ dt$$

where $f_o(t)$ is the odd function defined as

$$f_o(t) = \left\{\begin{array}{ll} -[f(-y_m t + x_t) - a_1\eta], & -1 \leq t < 0 \\ \\ [f(y_m t + x_t) - a_1\eta], & 0 < t \leq 1 \end{array}\right\} \tag{2.22}$$

and

C_1 = normalizing factor

$\eta = 2\lambda C_1$

$y_m = x_m - x_t$

Squaring the integrands, we have

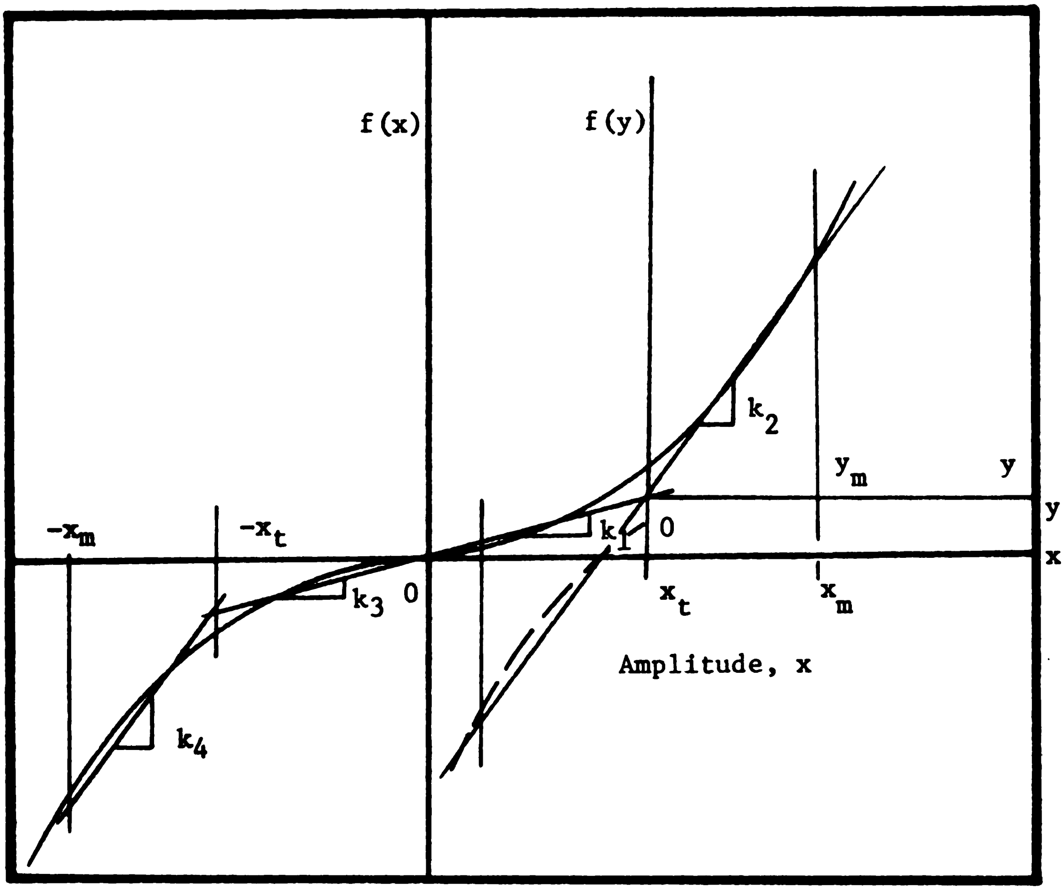


Figure 2.3 Nonlinear Function of One Space Variable Approximated by Multiple Linear Approximations

$$E^2 = \int_{-1}^{1} x_t\omega(t)[f(x_t t)]^2\,dt + \sum_{n=0}^{k} a_n^2\,x_t \int_{-1}^{1}\omega(t)[p_n(t)]^2\,dt$$

$$-2\sum_{n=0}^{k} a_n x_t \int_{-1}^{1} (t)\; f(x_t t)\; p_n(t)\,dt$$

$$+2\sum_{n=0}^{k}\sum_{\substack{\ell=0\\ n\neq\ell}}^{k} a_n a_\ell x_t \int_{-1}^{1} \omega(t)\; p_n(t)\; p_\ell(t)dt + y_m \int_{-1}^{1} \omega(t)[f_o(t)]^2\,dt$$

$$+\sum_{n=0}^{k} b_n{}^2\; y_m \int_{-1}^{1} \omega(t)[p_n(t)]^2\,dt -2\sum_{n=0}^{k} b_n y_m \int_{-1}^{1} \omega(t)f_o(t)p_n(t)\,dt$$

$$+\;2\sum_{n=0}^{k}\sum_{\substack{\ell=0\\ n\neq\ell}}^{k} b_n b_\ell y_m \int_{-1}^{1} \omega(t)p_n(t)\; p_\ell(t)\,dt \tag{2.23}$$

At this point we impose the orthonormal relation

$$\int_{-1}^{1} \omega(t)\; p_n(t)\; p_\ell(t)\,dt = \delta_{n\ell} \tag{2.24}$$

The error relation can now be simplified by substituting equations (2.22) and (2.24) into (2.23) to obtain

$$E^2 = x_t \int_{-1}^{1}\omega(t)\;\; [f(x_t t)]^2\,dt + \sum_{n=0}^{k} a_n{}^2 x_t \int_{-1}^{1} \omega(t)\;\; [p_n(t)]^2\,dt$$

$$-2\sum_{n=0}^{k} a_n x_t \int_{-1}^{1} \omega(t)\;\; f(x_t t)\;\; p_n(t)\,dt + y_m \int_{-1}^{1} \omega(t)[f_o(t)]^2\,dt$$

$$+\sum_{n=0}^{k} b_n^2\; y_m \int_{-1}^{1} \omega(t)[p_n(t)]^2\,dt -2\sum_{n=0}^{k} b_n y_m \int_{-1}^{1}\omega(t)[f_o(t)]p_n(t)\,dt \tag{2.25}$$

The partial derivatives with respect to a_n, b_n and x_t must vanish in order to minimize E^2. Taking first $\partial(E^2)/\partial a_n = 0$ for $n \neq 1$ and recognizing that $\partial f_o(t)/\partial a_n = 0$ for $n \neq 1$ we obtain,

$$\frac{\partial E^2}{\partial a_n} = 2\, a_n x_t \int_{-1}^{1} \omega(t)[p_n(t)]^2\, dt - 2x_t \int_{-1}^{1} \omega(t)\; f(x_t t)\; p_n(t)\; dt = 0$$

Thus,

$$a_n = \frac{\int_{-1}^{1} \omega(t)\; f(x_t t)\; p_n(t)\; dt}{\int_{-1}^{1} \omega(t)\; [p_n(t)]^2\; dt} = \int_{-1}^{1} \omega(t)\; f(x_t t)\; p_n(t)\; dt$$

By specifying the weight function as $\omega(t) = \omega(\lambda,t) = (1-t^2)^{\lambda-1/2}$ we have $p_n(t) = C_n P_n^{(\lambda)}(t)$ and

$$a_n = C_n^{-1}\left[\frac{\int_{-1}^{1} \omega(\lambda,t)\; f(x_t t)\; P_n^{(\lambda)}(t)\; dt}{\int_{-1}^{1} \omega(\lambda,t)\; [P_n^{(\lambda)}(t)]^2\; dt}\right] = C_n^{-1}\left[\frac{\int_{-1}^{1} \omega(\lambda,t)\; f(x_t t)\; P_n^{(\lambda)}(t)\; dt}{C_n^{-2}}\right]$$

or,

$$a_n = C_n \int_{-1}^{1} \omega(\lambda,t)\; f(x_t t)\; P_n^{(\lambda)}(t)\; dt \qquad \text{where } n \neq 1 \qquad (2.26)$$

Next, taking $\partial(E^2)/\partial a_1 = 0$ and recognizing that $\partial f_o(t)/\partial a_1 \neq 0$ we obtain

$$a_1 x_t \int_o^1 \omega(t)[p_1(t)]^2\,dt - b_1 y_m \int_o^1 \omega(t)\,\frac{\partial f_o(t)}{\partial a_1}\,p_1(t)\,dt$$

$$= x_t \int_o^1 \omega(t)\,f(x_t t)\,p_1(t)\,dt - y_m \int_o^1 \omega(t)\,f_o(t)\,\frac{\partial f_o(t)}{\partial a_1}\,dt \qquad (2.27)$$

Substituting $\partial f_o(t)/\partial a_1 = -\eta$ from (2.22) into (2.27) we find

$$a_1 x_t \int_o^1 \omega(t)\,[p_1(t)]^2\,dt + b_1 y_m \eta \int_o^1 \omega(t)\,p_1(t)\,dt$$

$$= x_t \int_o^1 \omega(t)\,f(x_t t)\,p_1(t)\,dt + y_m \eta \int_o^1 \omega(t)\,[f(y_m t + x_t) - a_1 \eta]\,dt \qquad (2.28)$$

Rewriting and using $p_1(t) = C_1 P_1^{(\lambda)}(t) = 2\lambda C_1 t = \eta t$ we obtain

$$a_1[x_t\,\eta \int_o^1 \omega(t)\,t^2\,dt + y_m \eta \int_o^1 \omega(t)\,dt] + b_1[y_m \eta \int_o^1 \omega(t)\,t\,dt]$$

$$= x_t \int_o^1 \omega(t)\,f(x_t t)\,t\,dt + y_m \int_o^1 \omega(t)\,f(y_m t + x_t)\,dt \qquad (2.29)$$

Taking $\partial(E^2)/\partial b_n = 0$ and recognizing that $\partial f_o(t)/\partial b_n = 0$ we obtain

$$b_n = \frac{\int_{-1}^1 \omega(t)\,f_o(t)\,p_n(t)\,dt}{\int_{-1}^1 \omega(t)\,[p_n(t)]^2\,dt} \qquad (2.30)$$

or, for $\omega(t) = \omega(\lambda,t) = (1-t^2)^{\lambda-1/2}$, $p_n(t) = C_n P_n^{(\lambda)}(t)$ and

$$b_n = \frac{C_n^{-1} \int_{-1}^{1} \omega(\lambda,t)\; f_o(t)\; P_n^{(\lambda)}(t)\; dt}{\int_{-1}^{1} \omega(\lambda,t)\; [P_n^{(\lambda)}(t)]^2\; dt} \tag{2.31}$$

$$= C_n^{-1} \int_{-1}^{1} \omega(\lambda,t)\; f_o(t)\; P_n^{(\lambda)}(t)\; dt$$

where $f_o(t)$ is given by (2.22).

Finally taking $\partial(E^2)/\partial x_t = 0$ we find an expression which is not linear in x_t. For this reason we allow it to assume the role of a variable parameter. It is worth noting at this point that all the b_n's are dependent upon the coefficient a_1, as is expected since a_1 and x_t established the origin about which the b_n coefficients are determined.

The a_1 and b_1 coefficients are uniquely determined by simultaneously solving equations (2.28) and (2.30). To accomplish this we rewrite (2.28) as

$$a_1[x_t\, \eta \int_o^1 \omega(t)\; t^2\; dt + y_m\, \eta \int_o^1 \omega(t)\; dt] + b_1[y_m \eta \int_o^1 \omega(t)\; t\; dt]$$

$$= x_t \int_o^1 \omega(t)\; f(x_t t)\; t\; dt + y_m \int_o^1 \omega(t)\; f(y_m t + x_t)\; dt \tag{2.32}$$

and, substituting $p_1(t) = \eta t$ and $f_o(t)$ from (2.22) into (2.30) for n=1 we obtain

$$a_1[\eta \int_o^1 \omega(t)\; t\; dt] + b_1[\eta \int_o^1 \omega(t)\; t^2\; dt] = \int_o^1 \omega(t)\; f(y_m t+x_t)\; t\; dt \tag{2.33}$$

From these equations we solve for a_1 and b_1. We are now in a position to find expressions for the two slope parameters k_1 and k_2. Recall that $k=\eta a_1/x_m$ by (2.20) for the linear approximation. In a similar manner for the bilinear approximation that $k_1=\eta a_1/x_t$ and $k_2=\eta b_1/y_m$. Thus we find for k_1

$$k_1 = \frac{x_t \int_o^1 \omega\, \bar{g} t\, dt + y_m \int_o^1 \omega \bar{\bar{g}}\, dt - y_m \int_o^1 \omega \bar{\bar{g}} t\, dt\, [\int_o^1 \omega t\, dt / \int_o^1 \omega t^2\, dt]}{x_t^2 \int_o^1 \omega t^2\, dt + y_m x_t \int_o^1 \omega dt - y_m x_t\, [[\int_o^1 \omega t\, dt]^2 / \int_o^1 \omega t^2\, dt]} \qquad (2.34)$$

and from (2.33) b_1 is expressed in terms of a_1 and k_2 becomes

$$k_2 = \frac{\int_o^1 \omega \bar{\bar{g}} t\, dt - k_1 x_t \int_o^1 \omega t\, dt}{y_m \int_o^1 \omega t^2\, dt} \qquad (2.35)$$

where

$$\bar{g} = f(x_t t)$$
$$\bar{\bar{g}} = f(y_m t + x_t)$$
$$\omega = \omega(t) = (1-t^2)^{\lambda-1/2}$$

We have shown in equation (2.21) that the slope parameter k_1 for a nonlinear function can be obtained either by an orthogonal polynomial expansion or by the mean square error minimization method, both yielding equivalent relations. In like manner, we observe that the slope parameters k_1 and k_2 for the bilinear approximation can be obtained either by an orthogonal polynomial expansion (equations (2.12) and (2.13), respectively) or by a mean square error minimization method (equations (2.34) and (2.35), respectively). Equations (2.13) and (2.35) for k_2 agree exactly and equations (2.12) and (2.34) for k_1 also agree provided two conditions are satisfied. These conditions, obtained by comparing terms in equations (2.12) and (2.34), are given as

$$\int_0^1 \omega\bar{\bar{g}}\, dt - \int_0^1 \omega\bar{\bar{g}}\, t\, dt\, [\int_0^1 \omega t\, dt / \int_0^1 \omega t^2\, dt] = 0 \tag{2.36}$$

and

$$\int_0^1 \omega dt - [[\int_0^1 \omega t\, dt]^2 / \int_0^1 \omega t^2\, dt] = 0 \tag{2.37}$$

Condition (2.37) is satisfied only when $\lambda=-.5$ as can be readily verified using Appendix D. Again from Appendix D equation (2.36) simplifies to

$$\int_0^1 \omega\bar{\bar{g}}\, dt = \int_0^1 \omega\bar{\bar{g}}\, t\, dt\, [[2\Gamma(\lambda+2)]/[\sqrt{\pi}\, \Gamma(\lambda+3/2)]] \tag{2.38}$$

and substituting $\lambda = -.5$ into (2.38) we obtain

$$\int_0^1 \omega\bar{\bar{g}}\, dt = \int_0^1 \omega\bar{\bar{g}}t\, dt \tag{2.39}$$

Therefore, we see that the conditions (2.36) and (2.37) can be replaced by the two simplified conditions--equation (2.39) and $\lambda=-.5$, the lower limit for the λ parameter. To be presented later in this research, equations (2.44) and (2.45), the limiting cases

$$\lim_{\lambda \to -.5} k_1 \qquad \text{and} \qquad \lim_{\lambda \to -.5} k_2$$

reveal that the $\lambda = -.5$ case is equivalent to a linear interpolation of the nonlinear function. Hence, when these simplified conditions are satisfied the orthogonal polynomial expansion technique presented in this research agrees exactly with the mean square error method; otherwise, it only approximates the mean square error method.

To illustrate when both conditions (2.39) and $\lambda = -.5$ are satisfied, the case of the nonlinear function $f(x) = x + x^3$ with $x_m = 2$ and $x_t = 1$ is presented in Appendix C.

The flexibility of using ultraspherical polynomials in the bilinear approximation development is removed by the condition $\lambda = -.5$. We can arrive at the k_1 (2.12) and k_2 (2.13) relations by expressing the error relationship separately for each subinterval rather than collectively. For example, setting

$$E^2 = E_1^2 + E_2^2$$

where

$$E_1^2 = \int_o^{x_t} \omega(x)\ [f(x) - g_1(x)]^2\ dx$$

$$E_2^2 = \int_{x_t}^{x_m} \omega(x)\ [f(x) - g_2(x)]^2\ dx$$

and

$$g_1(x) = \sum_{n=0}^{k} a_n p_n(x)$$

$$g_2(x) = \sum_{n=0}^{k} b_n p_n(x)$$

We can obtain the slope parameter k_1 from

$$\partial(E^2)/\partial a_1 = \partial(E_1^2)/\partial a_1 \quad \text{as } \partial(E_2^2)/\partial a_1 = 0$$

and obtain the slope parameter k_2 from

$$\partial(E^2)/\partial b_1 = \partial(E_2{}^2)/\partial b_1 \quad \text{as } \partial(E_1{}^2)/\partial b_1 = 0$$

2.2 Slope Parameters for Some Special Nonlinear Odd Function.

We now apply equations (2.12) and (2.13), which contain expressions for the two ultraspherical polynomial slope parameters, k_1 and k_2, to three specific odd functions namely, $\alpha x + \beta x^3$, sin x and sinh x. Some limiting cases will also be examined.

2.2.1 $f(x) = \alpha x + \beta x^3$.

Using the k_1 relations (2.12) we substitute into these equations

$$f(x) = \alpha x + \beta x^3$$

$$\bar{g} = f(x_t t) = \alpha x_t t + \beta(x_t t)^3$$

$$\omega = (1-t^2)^{\lambda-1/2}$$

$$x = x_t t$$

to obtain

$$k_1(\lambda,x_t) = \frac{\int_0^1 (1-t^2)^{\lambda-1/2} [\alpha x_t t + \beta(x_t t)^3]\ t\ dt}{x_t \int_0^1 (1-t^2)^{\lambda-1/2}\ t^2\ dt}$$

The numerator and the denominator are evaluated in Appendix D, with this final result

$$k_1(\lambda,x_t) = \alpha + 3\beta x_t^2/[2(\lambda+2)] \qquad (2.40)$$

Similarly, using the k_2 relation (2.13) we substitute

$$f(x) = \alpha x + \beta x^3$$

$$\bar{\bar{g}} = f((x_m-x_t)t + x_t) = \alpha[(x_m-x_t)t + x_t] + \beta[(x_m-x_t)t + x_t]^3$$

$$\omega = (1-t^2)^{\lambda-1/2}$$

$$y_m = x_m-x_t$$

to obtain

$$k_2(\lambda,y_m) = \frac{2\Gamma(\lambda + 2)\ x_t\ (\alpha-k_1 + \beta x_t^2)}{\sqrt{\pi}\ (x_m-x_t)\ \Gamma(\lambda + 3/2)} + \alpha + 3\beta x_t^2$$

$$+ \frac{6\beta\ (x_m-x_t)\ x_t\ \Gamma(\lambda +2)}{\sqrt{\pi}\ \Gamma(\lambda + 5/2)} + \frac{3\beta\ (x_m-x_t)^2}{2(\lambda + 2)} \qquad (2.41)$$

Limiting Cases. In the limit as $x_t \to x_m$ the bilinear ultraspherical polynomial approximation degenerates into the linear ultraspherical polynomial approximation. To see this, we take the limit of (2.40) as $x_t \to x_m$ and obtain,

$$\lim_{x_t \to x_m} [k_1(\lambda,x_t)] = \lim_{x_t \to x_m} [\alpha + 3\beta x_t^{\,2}/[2(\lambda+2)] = \alpha + 3\beta x_m^{\,2}/[2(\lambda+2)] \tag{2.42}$$

Hence,

$$\lim_{x_t \to x_m} [k_1(\lambda,x_t)] = k$$

So in the limit k_1 is equivalent to the one-line ultraspherical polynomial approximation.* Similarly, taking the limit of (2.41) we obtain

$$\lim_{x_t \to x_m} [k_2(\lambda,y_m) = \lim_{x_t \to x_m} \frac{2\Gamma(\lambda+2)\ x_t\ (\alpha-k_1+\beta x_t^{\,2})}{\sqrt{\pi}\ (x_m-x_t)\ \Gamma(\lambda+3/2)}$$

$$+ \alpha + 3\beta x_t^{\,2} + \frac{6\beta\ (x_m-x_t)\ x_t\ \Gamma(\lambda+2)}{\sqrt{\pi}\ \Gamma(\lambda+5/2)}$$

$$+ \frac{3\beta\ (x_m-x_t)^2}{2(\lambda+2)} \quad \to \infty$$

This limiting case for k_2 is meaningless because the interval is so small. These limiting cases for k_1 and k_2 are shown in Figure 2.4.

We consider the second limiting case as $x_t \to 0$. The bilinear ultraspherical polynomial again degenerates into the linear ultraspherical

* For the one-line ultraspherical polynomial solution one has $k(\lambda,x_m) = \alpha+3\beta x_m^{\,2}/[2(\lambda+2)]$

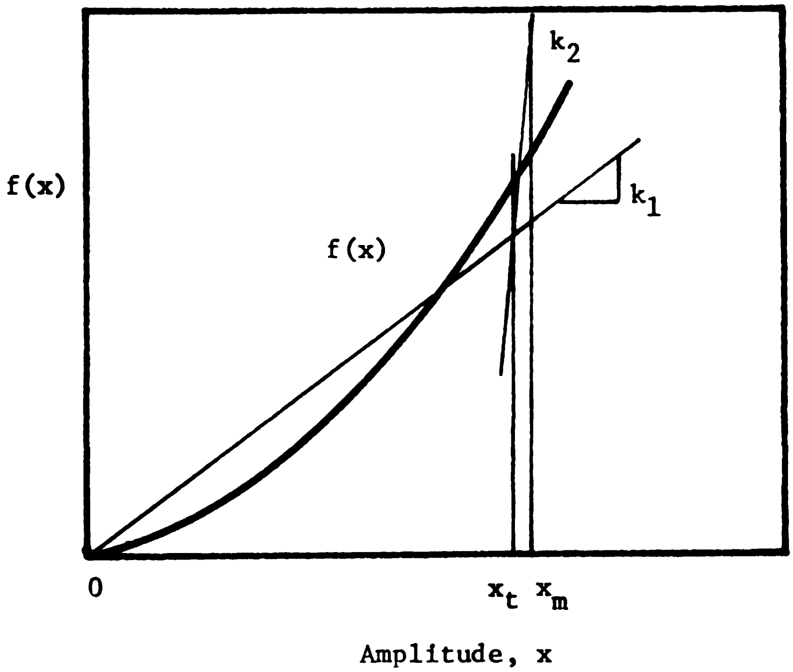


Figure 2.4 Nonlinear Function Approximated by a Bilinear Approximation as $x_t \to x_m$, $k_1 \to k$, $k_2 \to \infty$

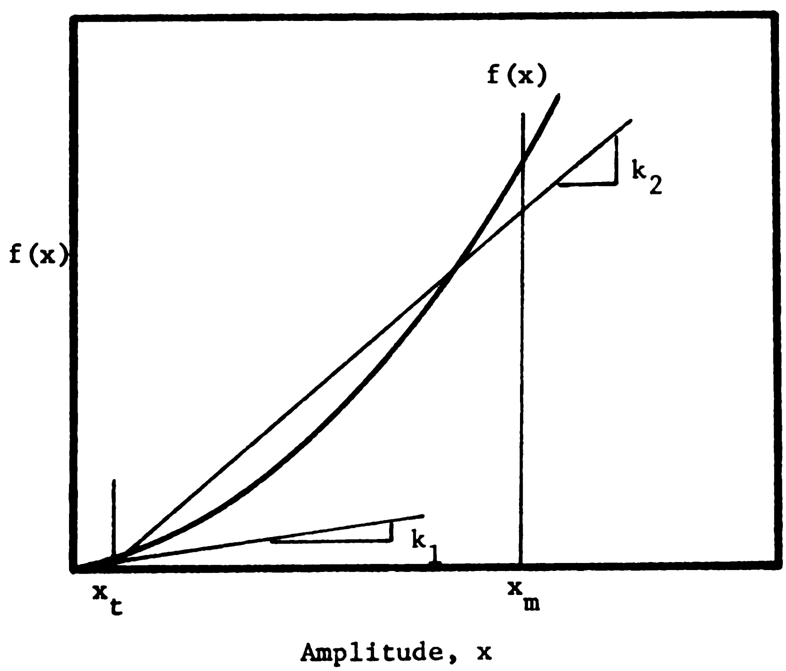


Figure 2.5 Nonlinear Function Approximated by a Bilinear Approximation as $x_t \to 0$, $k_1 \to \partial f/\partial x|_{x=0}$, $k_2 \to k$

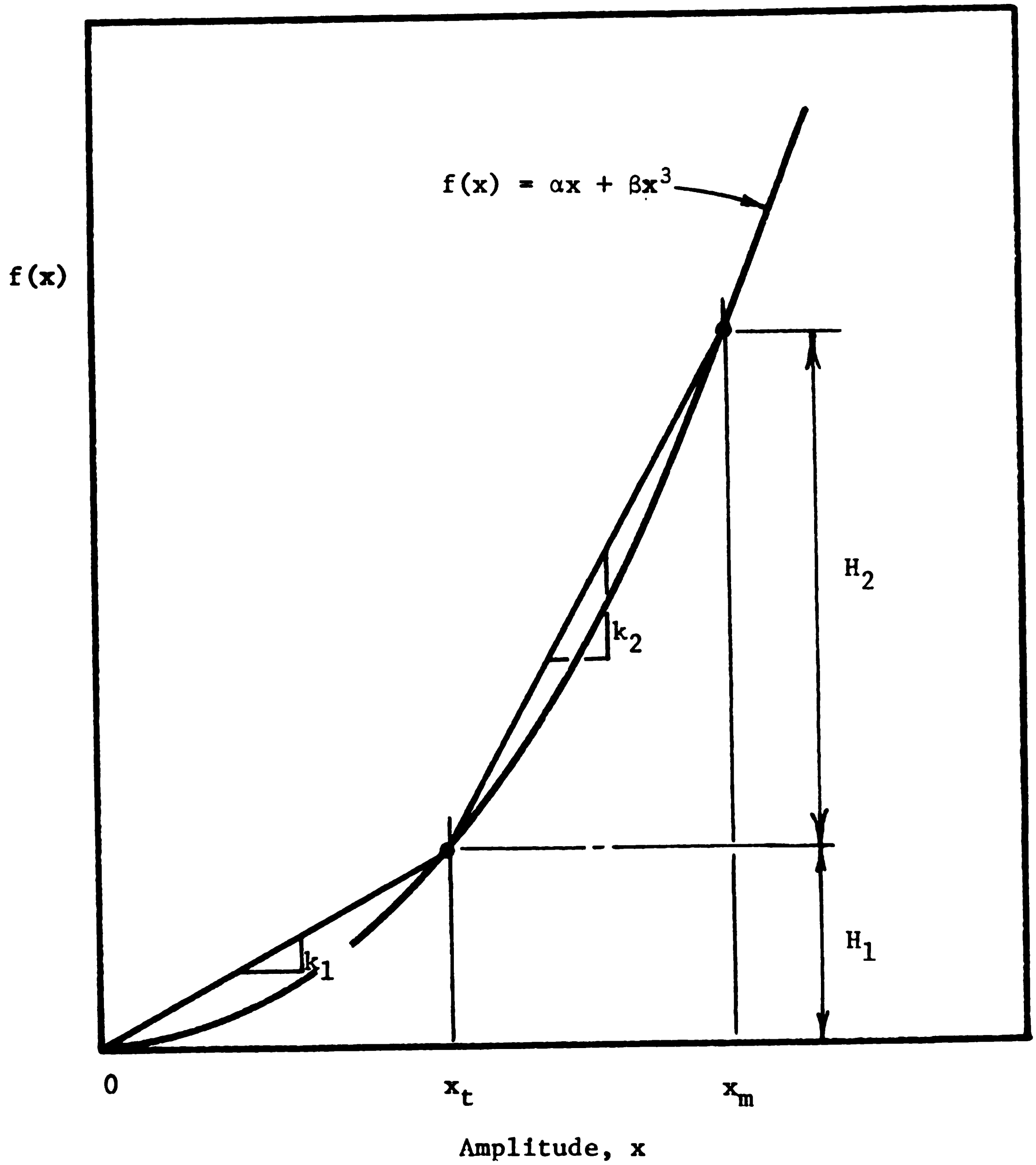


Figure 2.6 Nonlinear Function Approximated by Line Segments Joining Points Along the Curve $f(x) = \alpha x + \beta x^3$

polynomial approximation. To see this we take the limit of (2.40) and (2.41) as $x_t \to 0$ and obtain

$$\lim_{x_t \to 0} [k_1(\lambda, x_t)] = \lim_{x_t \to 0} [\alpha + 3\beta x_t^2/[2(\lambda+2)]] \;,$$

the Taylor series linear approximation (i.e., slope of the function at the origin). This limiting case for k_1 is meaningless because the interval is so small.

And,

$$\lim_{x_t \to 0} [k_2(\lambda, y_m)] = \alpha + 3\beta x_m^2/[2(\lambda+2)] \qquad (2.43)$$

These limiting cases for k_1 and k_2 are shown in Figure 2.5.

In the limiting case of $\lambda = -0.5$, k_1 and k_2 degenerate into the slopes of straight-line segments joining points along the plot of the nonlinear function. Recall that the ultraspherical polynomials are orthogonal with respect to the weight function, $\omega = (1-t^2)^{\lambda-1/2}$ for $\lambda > -.5$. So technically, the $\lambda = -.5$ case cannot be considered in an ultraspherical polynomial expansion. Taking the limit of (2.40) and (2.41) as $\lambda \to -.5$ however, we obtain

$$\lim_{\lambda \to -.5} [k_1(\lambda, x_t)] = \lim_{\lambda \to -.5} [\alpha + 3\beta x_t^2/[2(\lambda+2)]] = \alpha + \beta x_t^2 \qquad (2.44)$$

and

$$\lim_{\lambda \to -.5} [k_2(\lambda, y_m)] = \alpha + \beta(x_m^2 + x_m x_t + x_t^2) \qquad (2.45)$$

Taking values from Figure 2.6 we have

$$\frac{H_1}{x_t} = \text{slope of first line segment} = f(x)/x_t \Big|_{x=x_t}$$

$$= (1/x_t)\ [\alpha x_t + \beta x_t^3) \qquad (2.46)$$

$$= \alpha + \beta x_t^2$$

$$\frac{H_2}{x_m - x_t} = \text{slope of the second line segment}$$

$$= [1/(x_m - x_t)] \left\{ f(x) \Big|_{x=x_m} - f(x) \Big|_{x=x_t} \right\}$$

$$= [1/(x_m - x_t)]\ \ [\alpha x_m + \beta x_m{}^3 - \alpha x_t - \beta x_t{}^3]$$

$$= \alpha + \beta(x_m{}^2 + x_m x_t + x_t{}^2) \qquad (2.47)$$

Comparing equation (2.44) with (2.46) and (2.45) with (2.47) we see that the $\lambda = -.5$ bilinear limiting case is equivalent to a linear interpolation of the nonlinear function.

Dependence of k_1 and k_2 on Amplitude. For the nonlinear function $f(x) = \alpha x + \beta x^3$ with $\alpha = 1$ and $\beta = 1$, the dependence of k_1 and k_2 upon the transition amplitude x_t and the maximum amplitude x_m is shown in Figure 2.7, 2.8 and 2.9. By comparing these figures we observe that the k_2 curves approach k, the one-line U.P. approximation, for small x_t (Figure 2.7) as previously predicted by (2.43). Similarly, we observe that the k_1 curves approach k for large x_t (Figure 2.9), as previously predicted by (2.42).

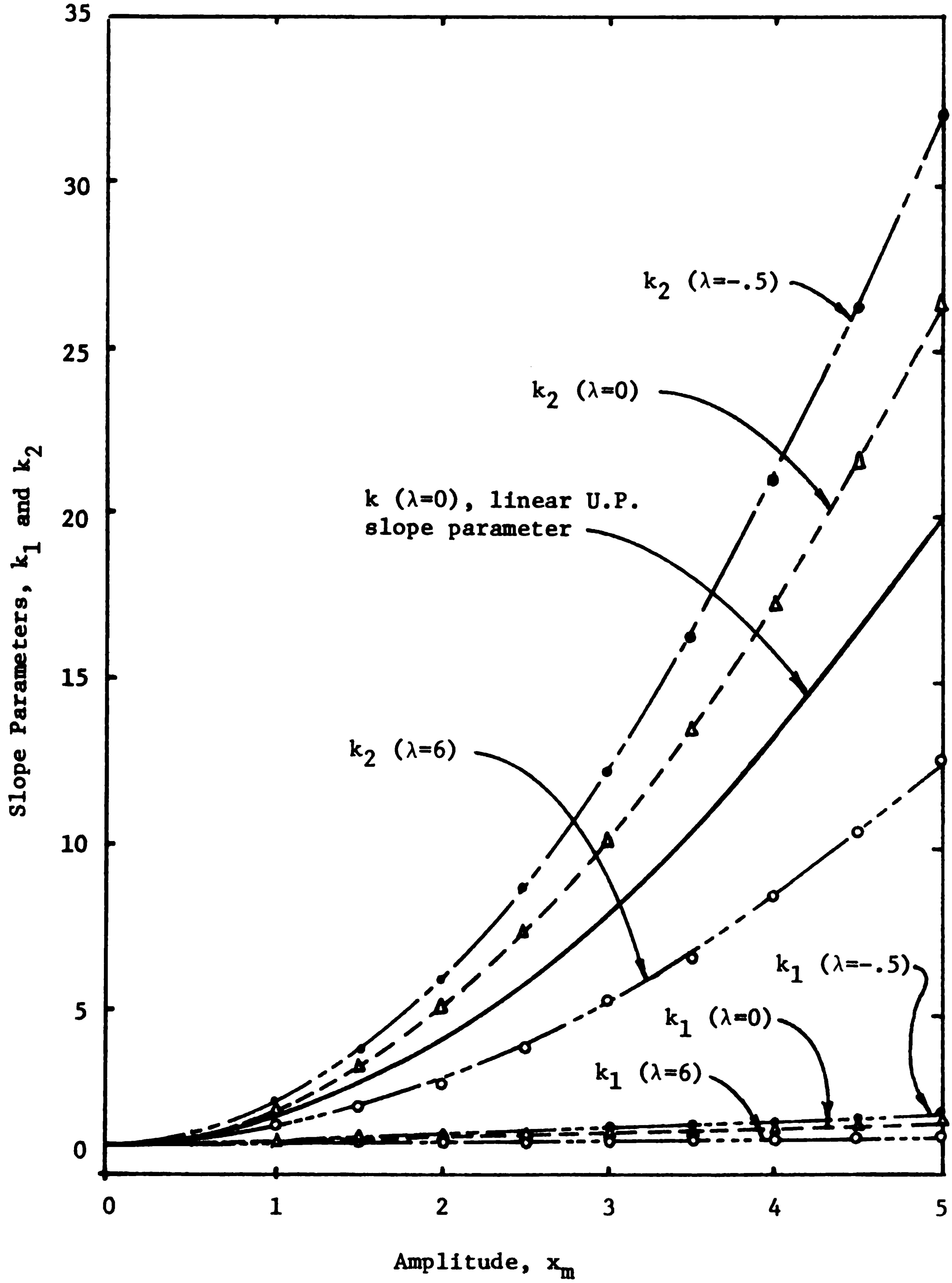


Figure 2.7 Bilinear U.P. Slope Parameters, k_1 and k_2, versus Amplitude, x_m, for the Nonlinear Function $f(x) = x + x^3$, $(x_t/x_m = 0.2)$

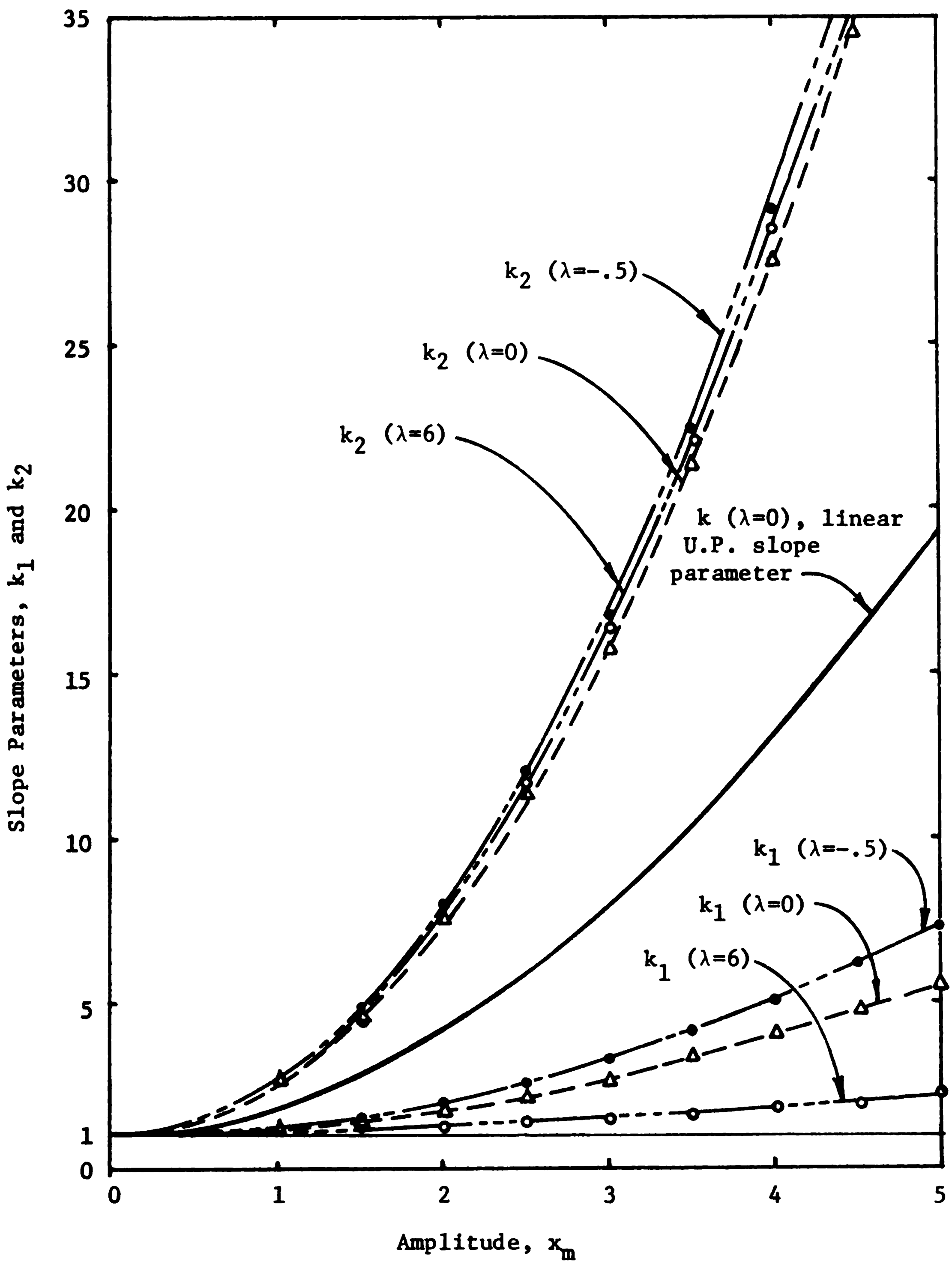


Figure 2.8 Bilinear U.P. Slope Parameters, k_1 and k_2 versus Amplitude, x_m, for the Nonlinear Function $f(x) = x + x^3$, $(x_t/x_m = 0.5)$

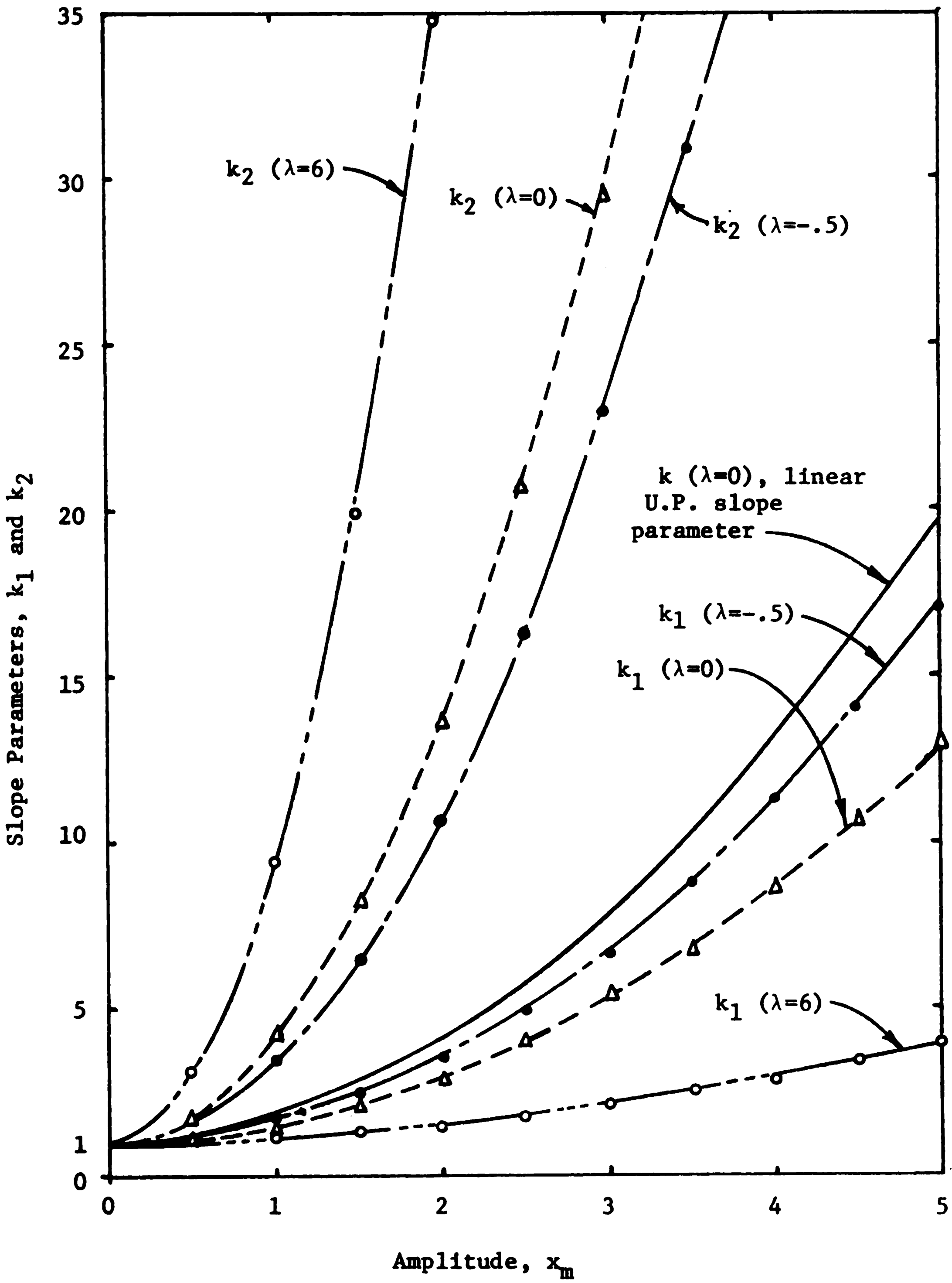


Figure 2.9 Bilinear U.P. Slope Parameters, k_1 and k_2, versus Amplitude, x_m, for the Nonlinear Function $f(x) = x + x^3$, $(x_t/x_m = 0.8)$

2.2.2 $f(x) = \sin x$.

We again substitute the nonlinear function $f(x) = \sin x$ into equation (2.12) and (2.13) to obtain

$$k_1(\lambda,x_t) = \frac{1}{x_t}\left[\frac{\int_0^1 (1-t^2)^{\lambda-1/2} [\sin(x_t t)]\, t\, dt}{\int_0^1 (1-t^2)^{\lambda-1/2}\, t^2\, dt}\right] \tag{2.48}$$

The integrals in (2.48) are evaluated in Appendix E and are substituted back in (2.48) to yield

$$k_1(\lambda,x_t) = [\Gamma(\lambda+2)/(x_t/2)^{\lambda+1}]J_{\lambda+1}(x_t) \tag{2.49}$$

Also

$$k_2(\lambda,x_m-x_t) = \frac{1}{x_m-x_t}\left[\frac{\int_0^1 (1-t^2)^{\lambda-1/2} [\sin(x_t t)]\, t\, dt - k_1 x_t \int_0^1 (1-t^2)^{\lambda-1/2}\, t\, dt}{\int_0^1 (1-t^2)^{\lambda-1/2}\, t^2\, dt}\right] \tag{2.50}$$

The integrals in (2.50) are evaluated easily following similar steps to those in Appendix E so that

$$k_2(\lambda,x_m-x_t) = \frac{-2k_1\, x_t\, \Gamma(\lambda+2)}{\sqrt{\pi}\,(x_m-x_t)\,\Gamma(\lambda+3/2)} +$$

$$+ \frac{\Gamma(\lambda+2)}{\left(\frac{x_m-x_t}{2}\right)^{\lambda+1}} \cos x_t \, J_{\lambda+1}(x_m-x_t)$$

$$+ \frac{2\Gamma(\lambda+2)}{x_m-x_t} \sin x_t \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x_m-x_t}{2}\right)^{2n}}{\Gamma(n+1/2)\,\Gamma(n+\lambda+3/2)} \tag{2.51}$$

For comparison we note that the one-line ultraspherical polynomial approximation is $[\Gamma(\lambda+2)/(x_m/2)^{\lambda+1}]\ J_{\lambda+1}(x_m)$ (2.52)

Limiting Cases: In the limit as $x_t \to x_m$ we take the limit of (2.49) and obtain

$$\lim_{x_t \to x_m} [k_1(\lambda, x_t)] = \lim_{x_t \to x_m} [\Gamma(\lambda+2)/(x_t/2)^{\lambda+1}]$$

$$= \Gamma(\lambda+2)/[x_m/2]^{\lambda+1} \quad J_{\lambda+1}(x_m) = k \tag{2.53}$$

Similarly, taking the limit of (2.51) we obtain

$$\lim_{x_t \to x_m} [k_2(\lambda, x_m - x_t)] = \lim_{x_t \to x_m} \Bigg\{ \frac{-2k_1\, x_t\, \Gamma(\lambda+2)}{\sqrt{\pi}(x_m-x_t)\ \Gamma(\lambda+3/2)}$$

$$+ \Gamma(\lambda+2)/[(x_m-x_t)/2]^{\lambda+1} \left[\cos x_t \, J_{\lambda+1}(x_m-x_t) \right]$$

$$+ \frac{2\Gamma(\lambda+2)}{x_m-x_t} \left[\sin x_t \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x_m-x_t}{2}\right)^{2n}}{\Gamma(n+1/2)\,\Gamma(n+\lambda+3/2)} \right] \Bigg\} \to \infty \tag{2.54}$$

We consider the second limiting case $x_t \to 0$. The bilinear U.P. approximation again degenerates into the linear U.P. approximation. To see this we take the limit of (2.49) and (2.51) as $x_t \to 0$ and obtain

$$\lim_{x_t \to 0} [k_1(\lambda, x_t)] = \lim_{x_t \to 0} [\Gamma(\lambda+2)/(x_t/2)^{\lambda+1}] [J_{\lambda+1}(x_t)] = 0 \qquad (2.55)$$

and

$$\lim_{x_t \to 0} [k_2(\lambda, x_m - x_t)] = (\Gamma(\lambda+2)/(x_m/2)^{\lambda+1}] \; J_{\lambda+1}(x_m) = k \qquad (2.56)$$

As shown previously for the cubic nonlinear case, the bilinear ultraspherical polynomial approximations degenerate into the linear ultraspherical polynomial approximation as seen upon comparing (2.53) and (2.56) with (2.52).

Dependence of k_1 and k_2 on Amplitude. For the nonlinear function $f(x) = \sin x$ the dependence of k_1 and k_2 on both the transition amplitude x_t and maximum amplitude x_m is shown in Figures 2.10, 2.11, and 2.12. Again, comparing these figures we observe that the k_2 curves approach k, the one-line U.P. approximation, for small x_t (Figure 2.10) as predicted by (2.56). Similarly, the k_1 curves approach k for large x_t (Figure 2.12) as predicted by (2.53).

2.2.3 $f(x) = \sinh x$.

Into the equations (2.12) and (2.13) we substitute the nonlinear function $f(x) = \sinh x$ to obtain

$$k_1(\lambda, x_t) = \frac{1}{x_t} \left[\frac{\int_0^1 (1-t^2)^{\lambda-1/2} [\sinh(x_t t)]\, t\, dt}{\int_0^1 (1-t^2)^{\lambda-1/2}\, t^2\, dt} \right] \qquad (2.57)$$

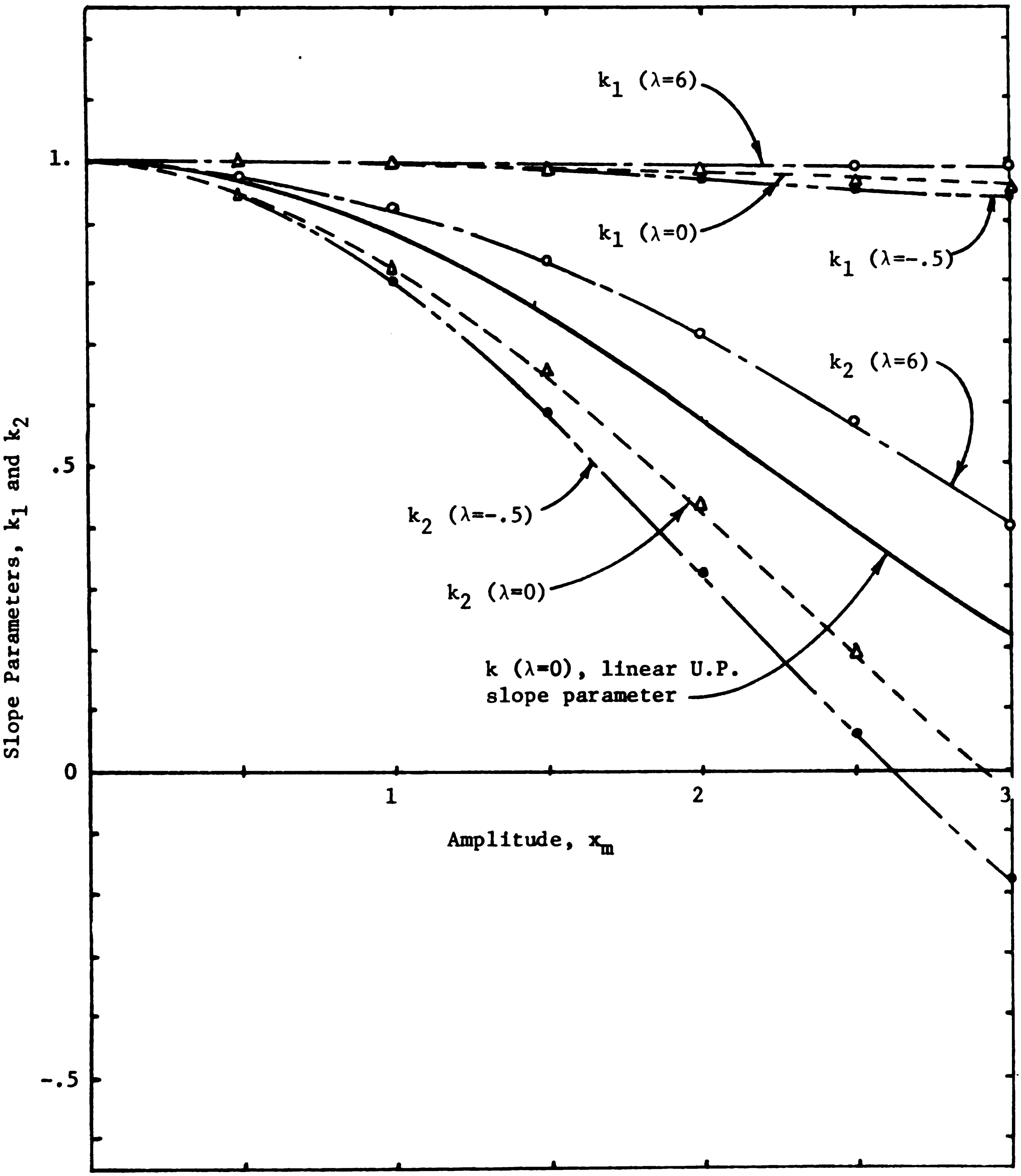


Figure 2.10 Bilinear U.P. Slope Parameters, k_1 and k_2, versus Amplitude, x_m, for the Nonlinear Function $f(x) = \sin x$, $(x_t/x_m = 0.2)$

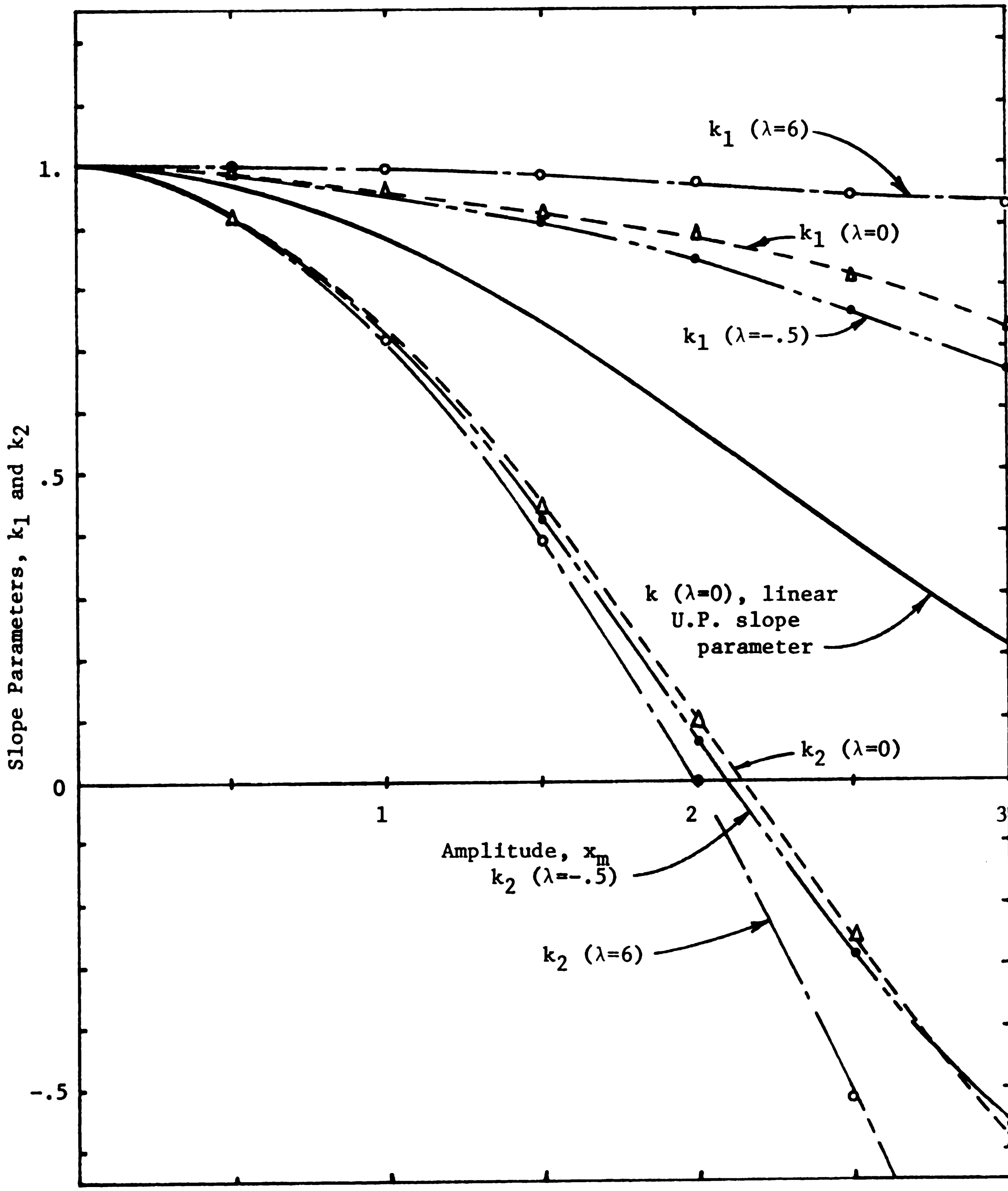


Figure 2.11 Bilinear U.P. Slope Parameters, k_1 and k_2, versus Amplitude, x_m, for the Nonlinear Function $f(x) = \sin x$, $(x_t/x_m = 0.5)$

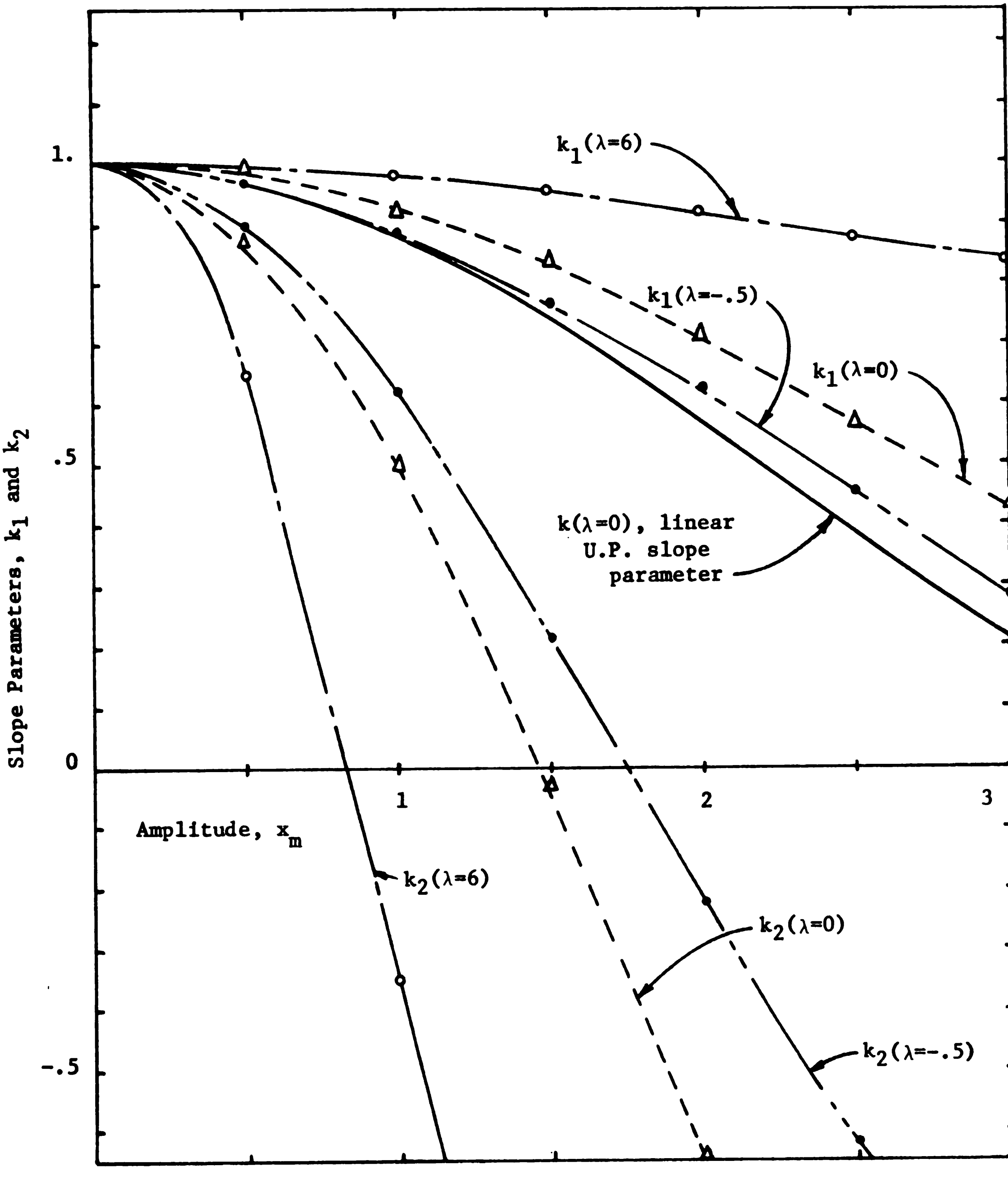


Figure 2.12 Bilinear U.P. Slope Parameters, k_1 and k_2, versus Amplitude x_m, for the Nonlinear Function $f(x) = \sin x$ ($x_t / x_m = 0.8$)

The integrals in (2.57) can be evaluated easily following similar steps to those in Appendix E, and it follows that

$$k_1(\lambda,x_t) = \frac{(\lambda+2)}{(x_t/2)^{\lambda+1}} \quad [I_{\lambda+1}(x_t)] \tag{2.58}$$

where $I_n(x) = (-i)^n J_n(ix)$ is a modified Bessel function.

Also

$$k_2(\lambda,x_m-x_t) = \frac{-2k_1x_t \quad \Gamma(\lambda+2)}{\sqrt{\pi}\ (x_m-x_t)\Gamma(\lambda+3/2)} + [\Gamma(\lambda+2)/(x_m-x_t/2)^{\lambda+1}]\cdot$$

$$\cosh x_t \ I_{\lambda+1}(x_m-x_t)$$

$$+\frac{2\Gamma(\lambda+2)}{x_m-x_t} \sinh x_t \sum_{n=0}^{\infty} \frac{[(x_m-x_t)/2]^{2n}}{\Gamma(n+1/2)\Gamma(n+\lambda+3/2)} \tag{2.59}$$

We note also that the one-line ultraspherical polynomial linear approximation is

$$k = [\Gamma(\lambda+2)/(x_m/2)^{\lambda+1}] \qquad I_{\lambda+1}(x_m) \tag{2.60}$$

<u>Limiting Cases.</u> In the limit as $x_t \to x_m$ we take the limit of (2.58) and obtain

$$\lim_{x_t\to x_m} [k_1(\lambda,x_t)] = \lim_{x_t\to x_m} [\Gamma(\lambda+2)/(x_t/2)^{\lambda+1}] \quad I_{\lambda+1}(x_t)$$

$$= [\Gamma(\lambda+2)/(x_m/2)^{\lambda+1}] \quad I_{\lambda+1}(x_m) = k \tag{2.61}$$

Similarly, taking the limit of (2.59) we obtain

$$\lim_{x_t \to x_m} [k_2(\lambda, x_m - x_t)] \to \infty$$

We consider the second limiting case $x_t \to 0$. The bilinear U.P. approximation again degenerates into the linear U.P. approximation. To see this we take the limit of (2.58) and (2.59) as $x_t \to 0$ and obtain

$$\lim_{x_t \to 0} [k_1(\lambda, x_t)] = \lim_{x_t \to 0} [\Gamma(\lambda+2)/(x_t/2)^{\lambda+1} = 0$$

and

$$\lim_{x_t \to 0} [k_2(\lambda, x_m - x_t)] = [\Gamma(\lambda+2)/(x_m/2)^{\lambda+1}]\, I_{\lambda+1}(x_m) = k \quad (2.62)$$

<u>Dependence of k_1 and k_2 on Amplitude</u>. For the nonlinear function $f(x) = \sinh x$ the dependence of k_1 and k_2 on both the transition amplitude x_t and maximum amplitude x_m is shown on Figures 2.13, 2.14, and 2.15. Again, comparing these figures we observe that the k_2 curves approach k, the one-line U.P. approximation, for small x_t (Figure 2.13) as predicted by (2.62). Similarly, the k_1 curves approach k for large x_t (Figure 2.15) as predicted by (2.61).

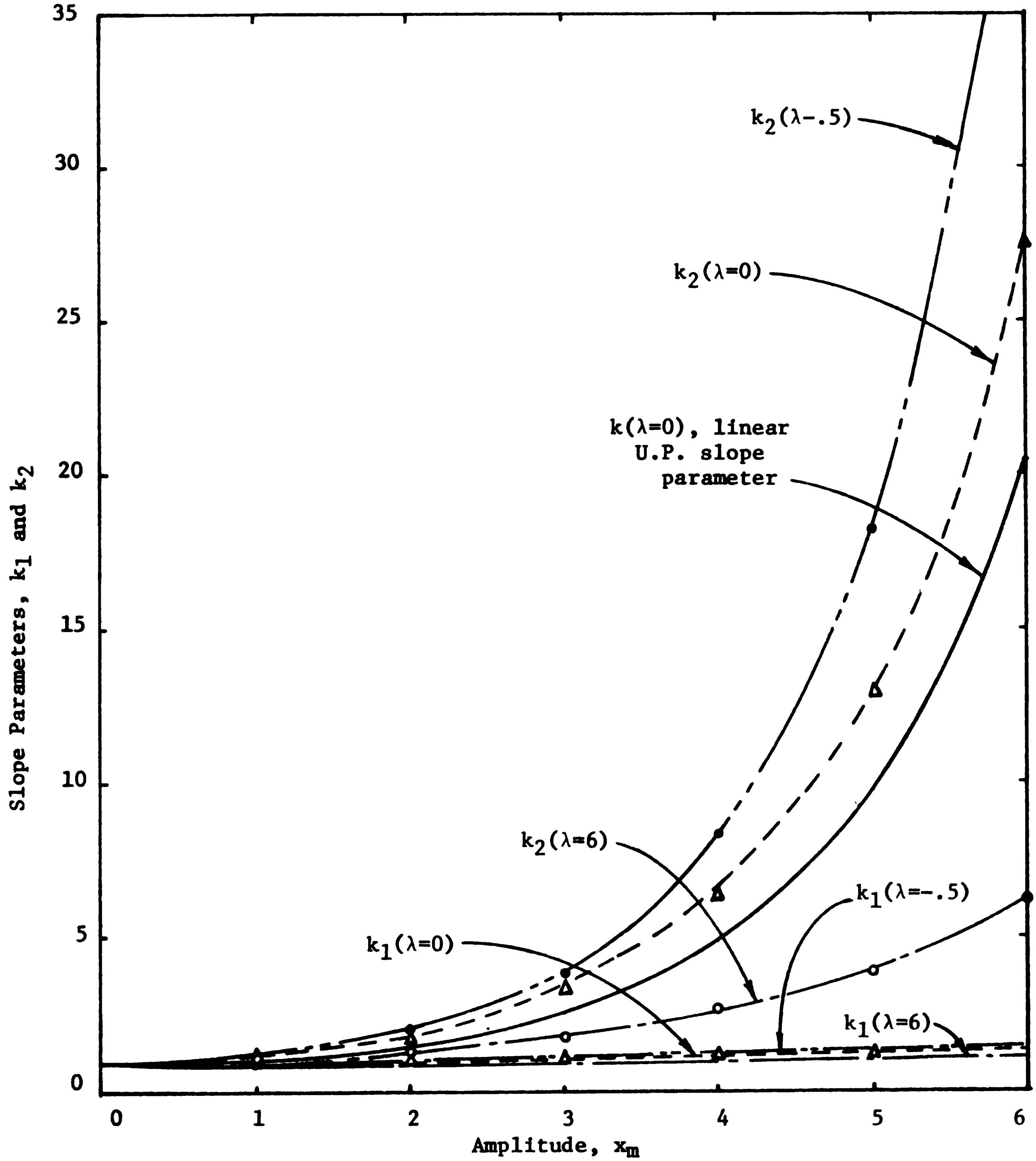


Figure 2.13 Bilinear U.P. Slope Parameters k_1 and k_2, versus Amplitude, x_m, for the Nonlinear Function $f(x) = \sinh(x)$ $(x_t/x_m = 0.2)$

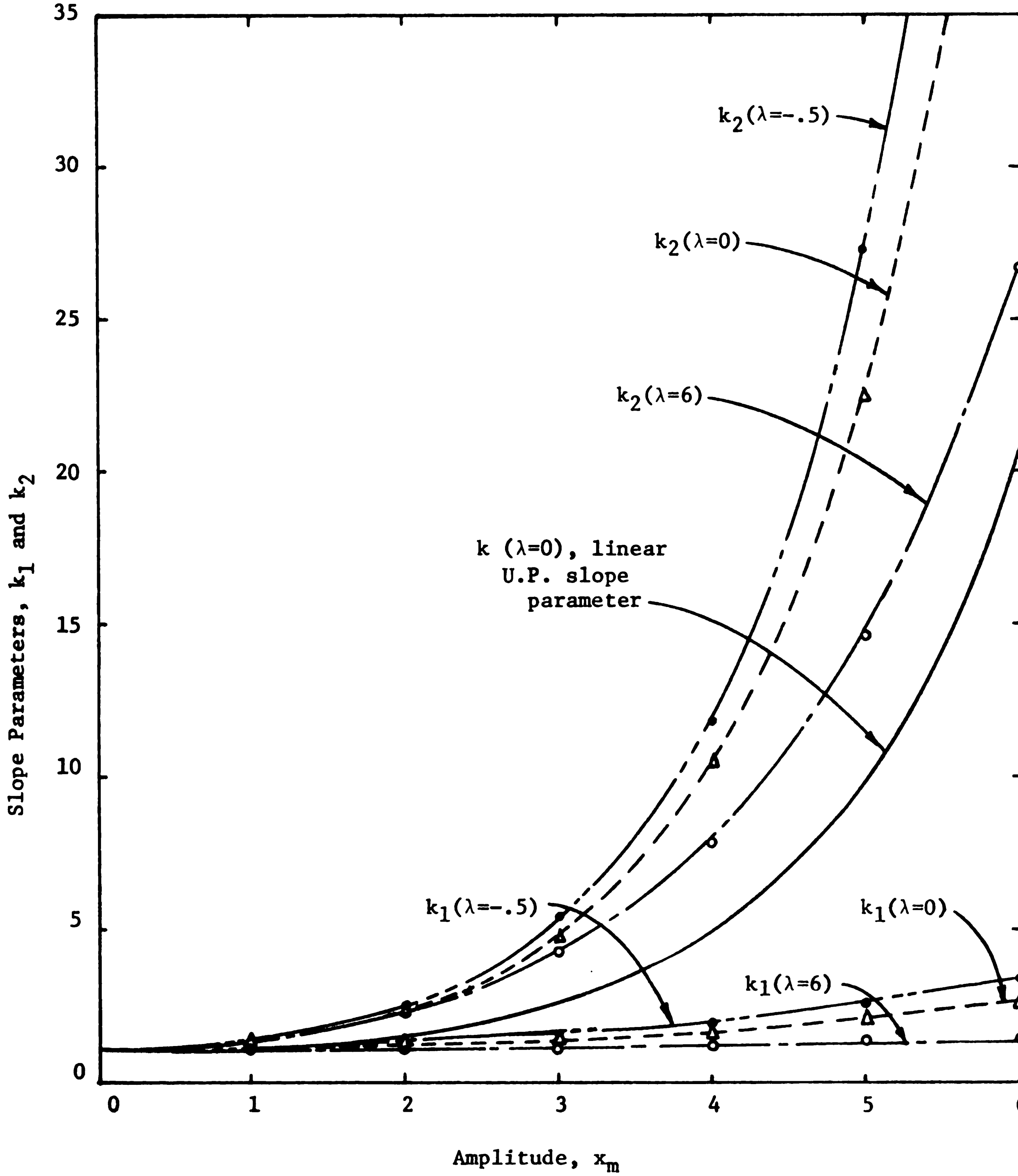


Figure 2.14 Bilinear U.P. Slope Parameters, k_1 and k_2, versus Amplitude, x_m, for the Nonlinear Function $f(x) = \sinh x$ ($x_t/x_m = 0.5$)

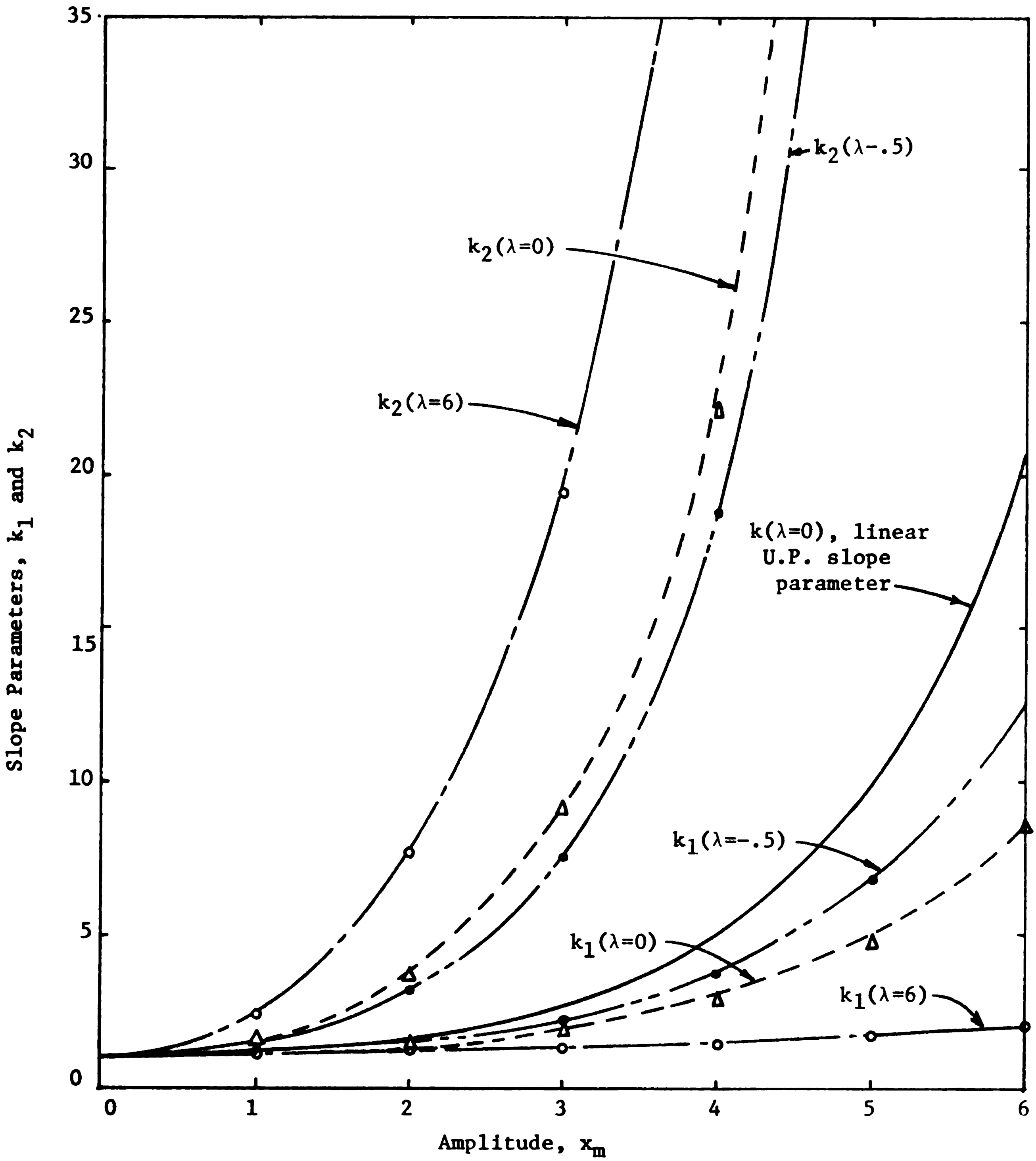


Figure 2.15 Bilinear U.P. Slope Parameters, k_1 and k_2, versus Amplitude, x_m, for the Nonlinear Function $f(x) = \sinh x$ $(x_t/x_m = 0.8)$

III. SINGLE DEGREE-OF-FREEDOM SYSTEMS

In this chapter, both free and forced vibrations of nonlinear undamped, single degree-of-freedom systems are examined. The non-linearity occurs in the restoring forces. The objective here is to show how improved period-amplitude relations are possible with bilinear U.P. approximation method.

3.1 Free Vibrations.

The governing equation is of the form

$$\ddot{x} + f(x) = 0,$$

complete with initial conditions

$$x(0) = x_m \text{ and } \dot{x}(0) = 0$$

When the nonlinear force $f(x)$ is approximated bilinearly, the equation of motion reduces to the two linear differential equations

$$\ddot{x}_1 + k_1 x_1 = 0, \qquad |x| \leq x_t \tag{3.1}$$

and

$$\ddot{x}_2 + k_2 x_2 + x_t(k_1 - k_2) = 0, \qquad x_t \leq |x| \tag{3.2}$$

where x_t is the transition point as discussed in Chapter II. The conditions, at time $t = 0$, are then

$$x_2(0) = x_m \text{ and } \dot{x}_2(0) = 0 \tag{3.3}$$

and x_1 and x_2 are matched at x_t, say at $t = t_t$.

$$x_1(t_t) = x_2(t_t) \text{ and } \dot{x}_1(t_t) = \dot{x}_2(t_t) \tag{3.4}$$

Equation (3.2) is multiplied by $\dot{x}_2$ and integrated once to yield an expression for $\dot{x}_2$

$$\dot{x}_2 = \left[C_2 - 2x_2x_t(k_1-k_2) - k_2 x_2^2 \right]^{1/2} \tag{3.5}$$

where

$$C_2 = (k_1-k_2)\ [2x_tx_m] + k_2x_m^2$$

and the initial conditions (3.3) have been used. Similarily, when equation (3.1) is multiplied by $\dot{x}_1$ and is integrated and the conditions (3.4) are used, one finds

$$\dot{x}_1 = [C_1 - k_1x_1^2]^{1/2} \tag{3.6}$$

where

$$C_1 = (k_1-k_2)\ [2x_tx_m - x_t^2] + k_2x_m^2$$

The period of motion, τ, may be computed by using equations (3.5) and (3.6) as

$$\tau = 4 \left[\int_0^{x_t} \frac{dx_1}{\dot{x}_1} + \int_{x_t}^{x_m} \frac{dx_2}{\dot{x}_2} \right]$$

$$4 \left[\int_0^{x_t} [C_1 - k_1x_1^2]^{-1/2} dx_1 + \int_{x_t}^{x_m} [C_2 - 2x_2x_t(k_1-k_2) - k_2x_2^2]^{-1/2} dx_2 \right] \tag{3.7}$$

The right hand side above can be integrated out explicitly and the results expressed in closed form. Assuming $k_1>0$, then for $k_2<0$, $k_2 = 0$ and $k_2>0$ respectively, we have, after some manipulations, For $k_2 < 0$:

$$\frac{\tau}{\tau_o} = \frac{1}{\sqrt{k_1}}\left[1 + \frac{2}{\pi}\left[-\tan^{-1}\left\{\frac{-\sqrt{-k_2}}{\sqrt{k_1}\,x_t}\left(x_m - \frac{b}{(-k_2)}\right)\left[\left(\frac{x_t - \frac{b}{(-k_2)}}{x_m - \frac{b}{(-k_2)}}\right)^2 - 1\right]^{1/2}\right\}\right.\right.$$

$$\left.\left. + \frac{\sqrt{k_1}}{\sqrt{-k_2}}\cosh^{-1}\left(\frac{x_t - \frac{b}{(-k_2)}}{x_m - \frac{b}{(-k_2)}}\right)\right]\right] \tag{3.8}$$

For $k_2 = 0$:

$$\frac{\tau}{\tau_o} = \frac{1}{\sqrt{k_1}}\left[1 + \frac{2}{\pi}\left\{\left[\frac{2(x_m - x_t)}{x_t}\right]^{1/2} - \tan^{-1}\left[\frac{2(x_m - x_t)}{x_t}\right]^{1/2}\right\}\right] \tag{3.9}$$

For $k_2 > 0$:

$$\frac{\tau}{\tau_o} = \frac{1}{\sqrt{k_1}}\left[1 + \frac{2}{\pi}\left[-\tan^{-1}\left\{\frac{\sqrt{k_2}}{\sqrt{k_1}\,x_t}\left(x_m + \frac{b}{k_2}\right)\left[1 - \left(\frac{x_t + \frac{b}{k_2}}{x_m + \frac{b}{k_2}}\right)^2\right]^{1/2}\right\}\right.\right.$$

$$\left.\left. + \frac{\sqrt{k_1}}{\sqrt{k_2}}\cos^{-1}\left(\frac{x_t + \frac{b}{k_2}}{x_m + \frac{b}{k_2}}\right)\right]\right] \tag{3.10}$$

where

$$b = x_t(k_1 - k_2), \quad \tau_o = 2\pi/\omega_o = 2\pi(k_1)^{-1/2}$$

For purposes of comparison other period-amplitude relations are for the same system also given in Table 3.1. The one-line U.P. approximate period, τ, is calculated using values for k previously given in Chapter II.

Comparison of Results. The dimensionless, one-line U.P. and exact period- amplitude relations given Table 3.1 are now compared against the bilinear U.P. approximate method in Figures 3.1, 3.2, and 3.3 for the nonlinear functions $x + x^3$, sin x, and sinh x, respectively. Relations derived previously in Chapter II for k_1 and k_2 are substituted into the appropriate τ/τ_o relation derived by the bilinear U.P. method— equation (3.8), (3.9) or (3.10)-- to yield the corresponding period-amplitude relationship. An amplitude ratio $x_t/x_m = 0.5$ was assumed. The symbols UP1 and UP2 refer to the linear and the bilinear U.P. approximate methods, respectively. The parameter is varied between $\lambda = -.5$ and $\lambda = 6$ to show qualitatively how the results are effected. These figures show how insensitive the bilinear U.P. results are to changes in λ , while for the linear U.P. results the contrary is true.

A closer look at Figures 3.1, 3.2, and 3.3 reveals further insight into the magnitude of the error by the linear and the bilinear U.P. methods. By looking at one maximum amplitude value x_m from any of these figures the error between the exact and linear U.P. is fixed. However the bilinear U.P. method is also dependent upon the amplitude ratio x_t/x_m. Quantitatively, Figures 3.4 through 3.10 give for a particular maximum amplitude x_m a measure of the error between the

Nonlinear Restoring Force $f(x)$	τ_o	Dimensionless Period-Amplitude Relations		
		Exact τ_e/τ_o	Linear τ_ℓ/τ_o	One Line U.P. τ/τ_o
$x + x^3$	2π	$\dfrac{2K(k_1)}{\pi(1+x^2_m)^{1/2}}$ where $k_1=\sin\theta=\left[\dfrac{x_m^{\ 2}}{2(1+x_m^{\ 2})}\right]^{1/2}$	1	$\left[1+\dfrac{3x_m^{\ 2}}{2(\lambda+2)}\right]^{-1/2}$
$\sin x$	2π	$\dfrac{2}{\pi}K(k)$ where $k = \sin\left(\dfrac{x_m}{2}\right)$	1	$\left[\left(\dfrac{2}{x_m}\right)^{\lambda+1}\Gamma(\lambda+2)\ J_{\lambda+1}(x_m)\right]^{-1/2}$
$\sinh x$	2π	$\dfrac{2}{\pi}\operatorname{sech}\left(\dfrac{x_m}{2}\right)K(\gamma)$ where $\gamma=\tanh\left(\dfrac{x_m}{2}\right)$	1	$\left[\left(\dfrac{2}{x_m}\right)^{\lambda+1}\Gamma(\lambda+2)I_{\lambda+1}(x_m)\right]^{-1/2}$

Table 3.1 Dimensionless Period-Amplitude Relations for Various Nonlinear Single Degree, Free Vibration Problems.

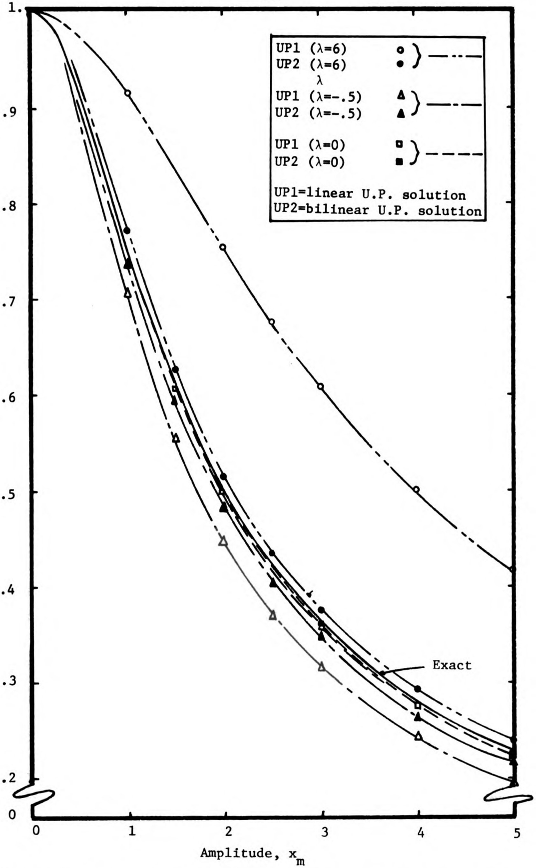


Figure 3.1 One Degree-of-Freedom, Free Vibration System Period Ratio τ/τ_0 versus Amplitude x_m for $f(x) = x + x^3$ (x_t/x_m=.5)

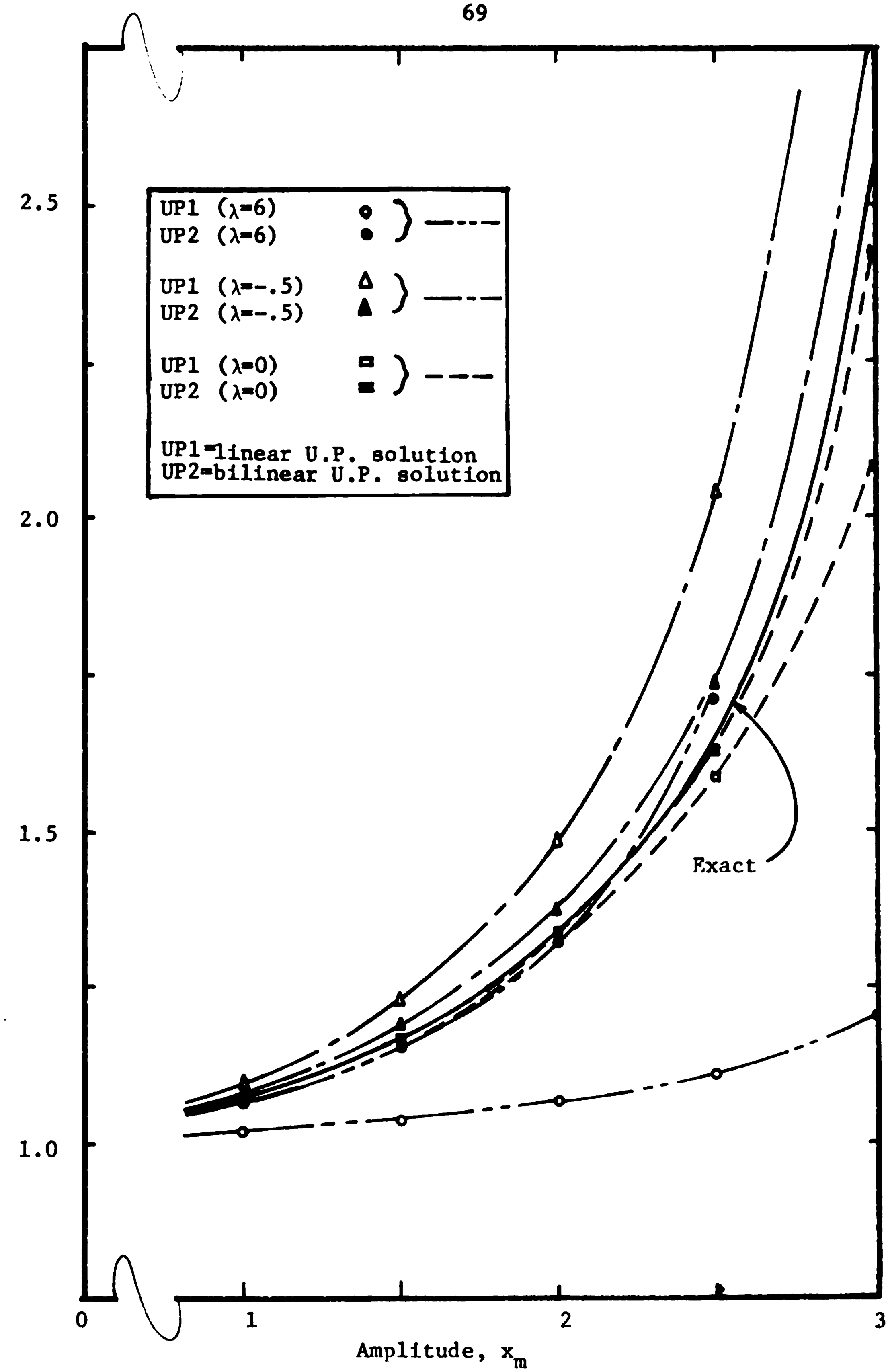


Figure 3.2 One Degree-of-Freedom, Free Vibration System Period Ratio τ/τ_o versus Amplitude x_m for $f(x) = \sin x$ (x_t/x_m=.5)

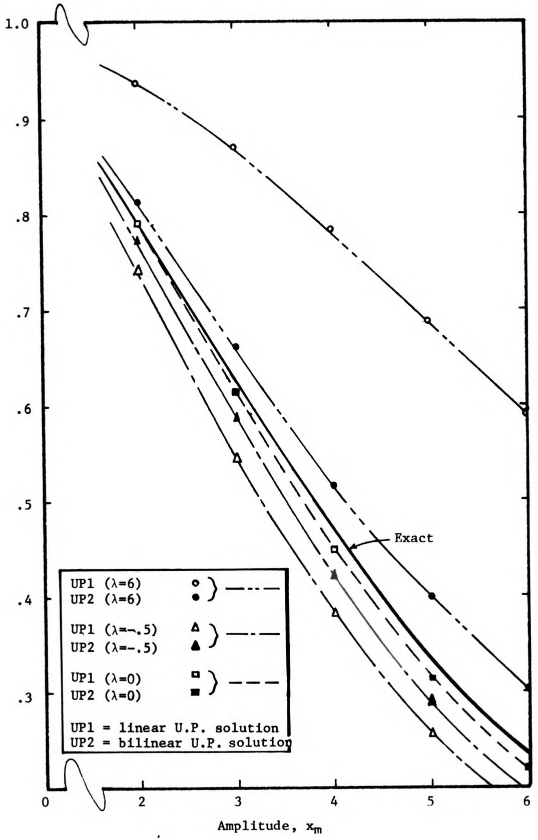


Figure 3.3 One Degree-of-Freedom, Free Vibration System Period Ratio τ/τ_o versus Amplitude x_m for $f(x) = \sinh x$ $(x_t/x_m = .5)$

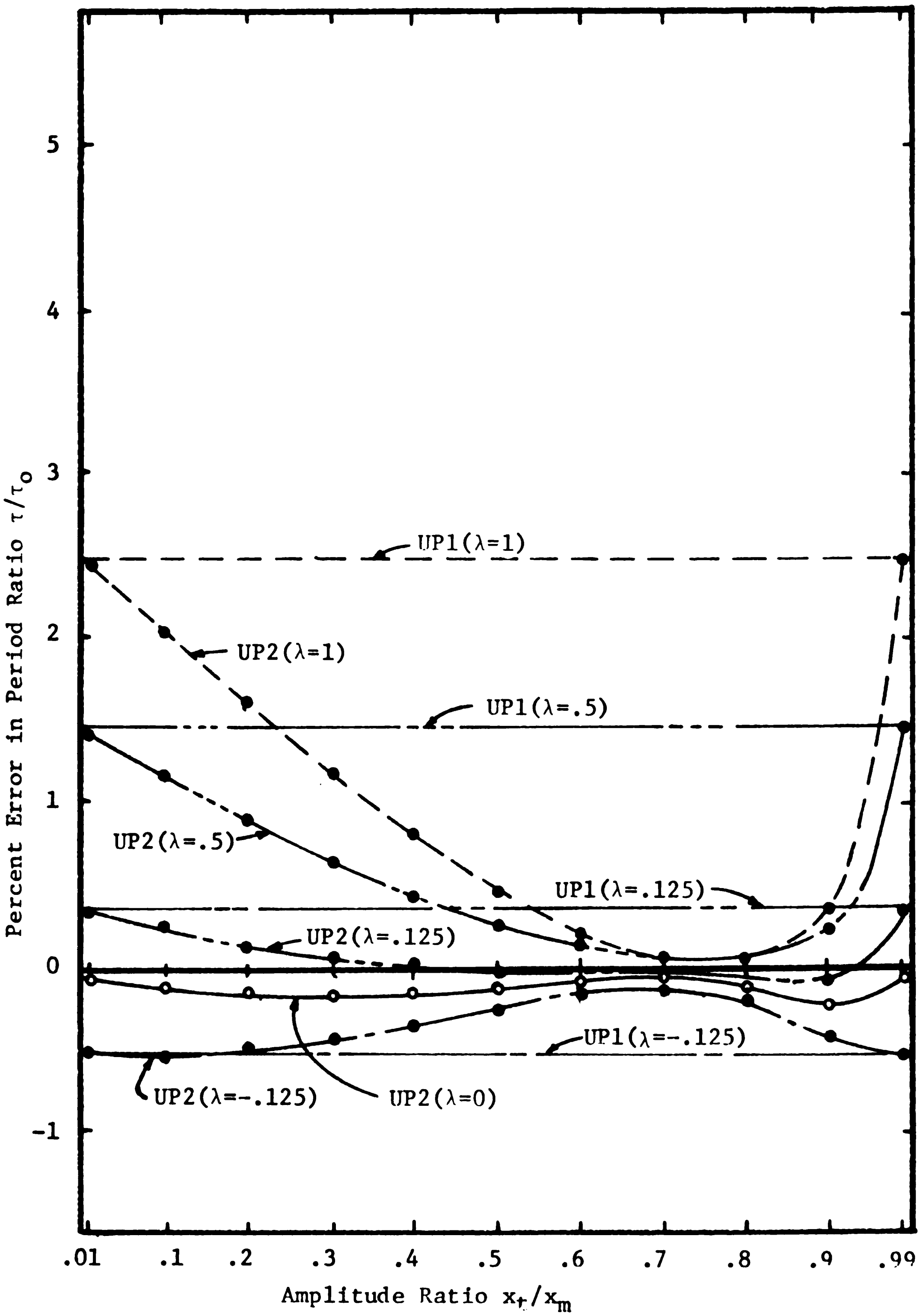


Figure 3.4 One Degree-of-Freedom, Free Vibration System Error Plot of Period Ratio τ/τ_o versus Amplitude Ratio x_t/x_m for $f(x) = x + x^3$ (x_m=0.5)

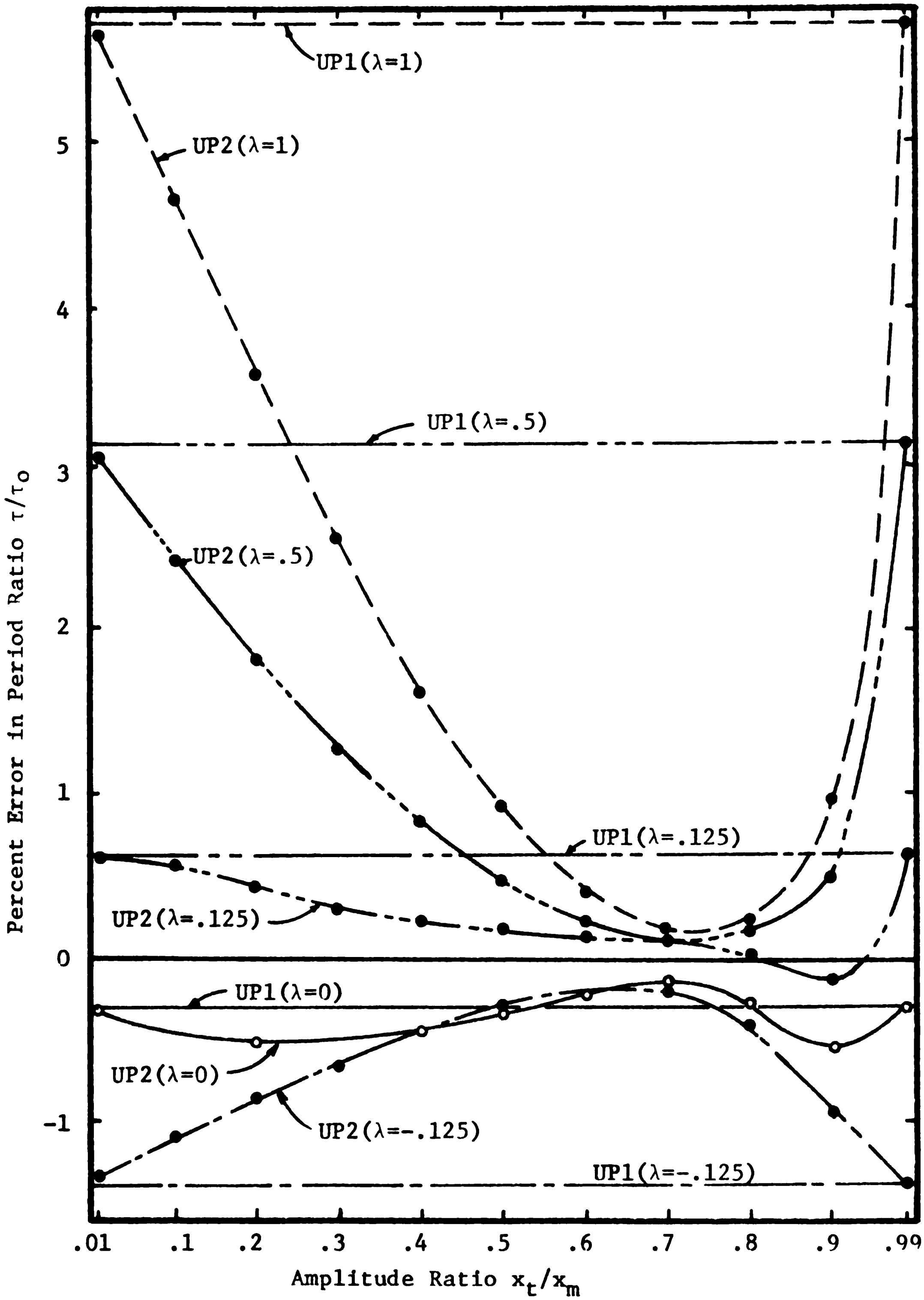


Figure 3.5 One Degree-of-Freedom, Free Vibration System Error Plot Period Ratio τ/τ_o versus Amplitude Ratio x_t/x_m for $f(x) = x + x^3$ (x_m=1.0)

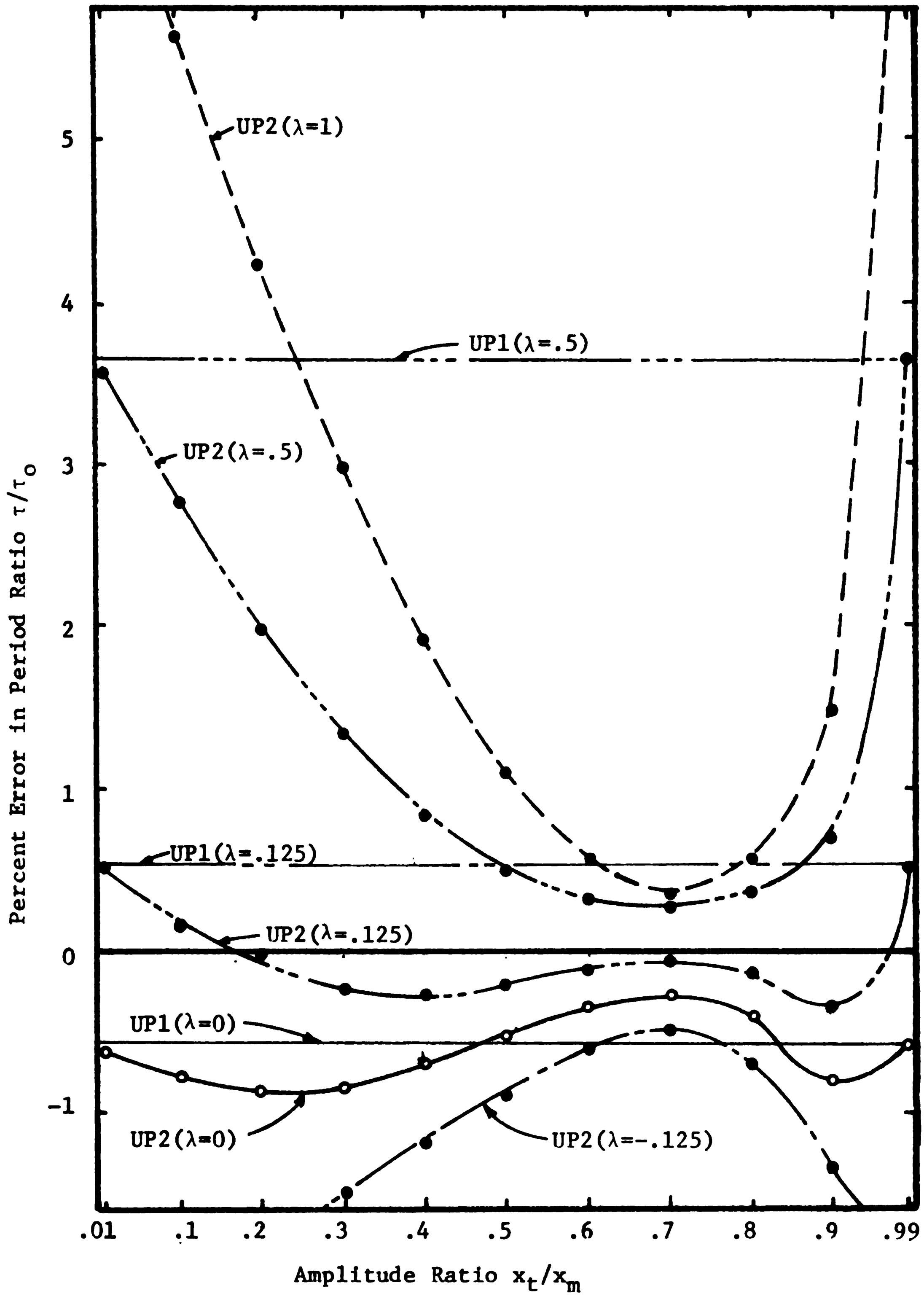


Figure 3.6 One Degree-of-Freedom, Free Vibration System Error Plot Period Ratio τ/τ_o versus Amplitude Ratio x_t/x_m for $f(x) = x + x^3$ (x_m=2.0)

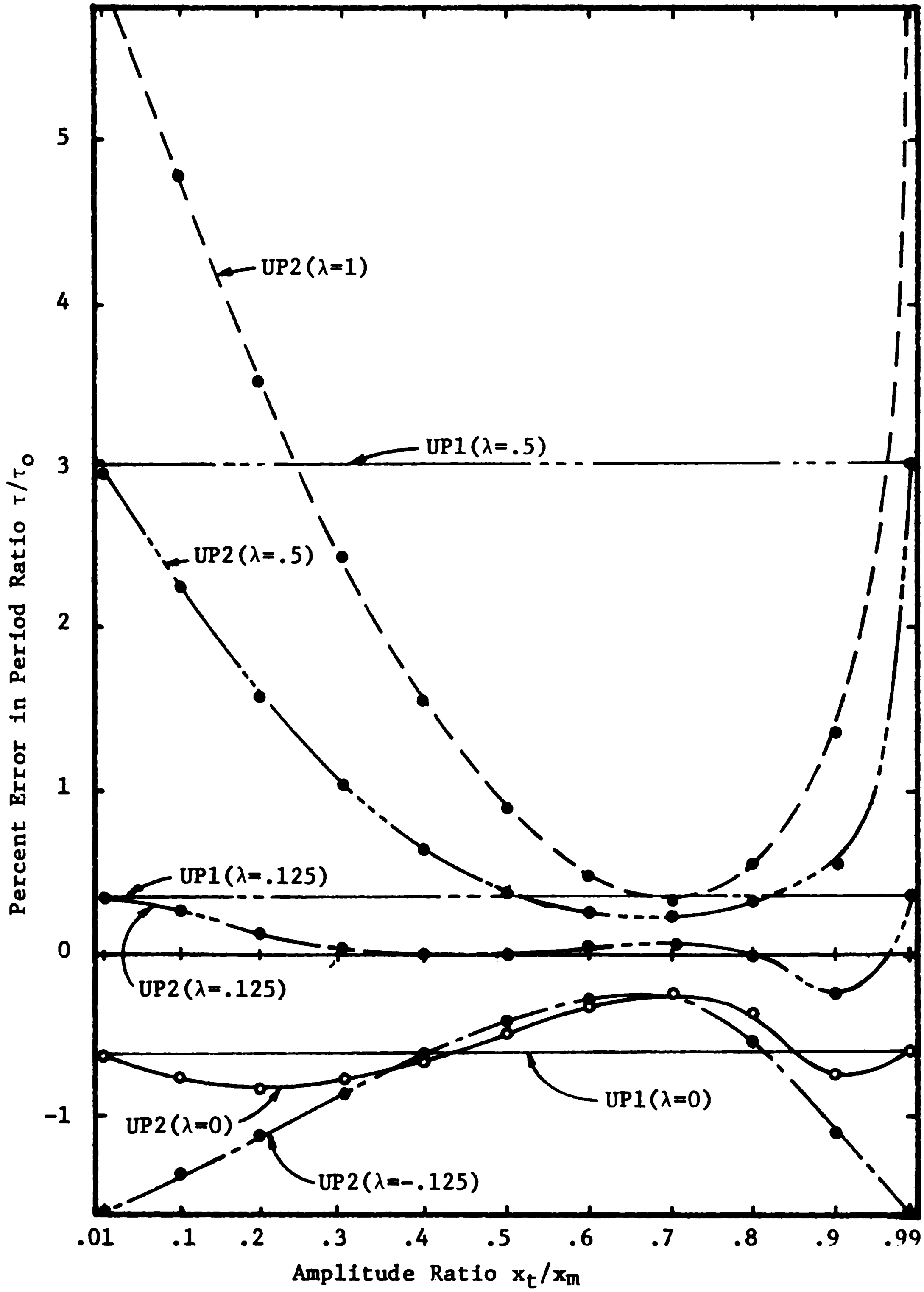


Figure 3.7 One Degree-of-Freedom, Free Vibration System Error Plot Period Ratio τ/τ_o versus Amplitude Ratio x_t/x_m for $f(x) = x + x^3$ (x_m=3.0)

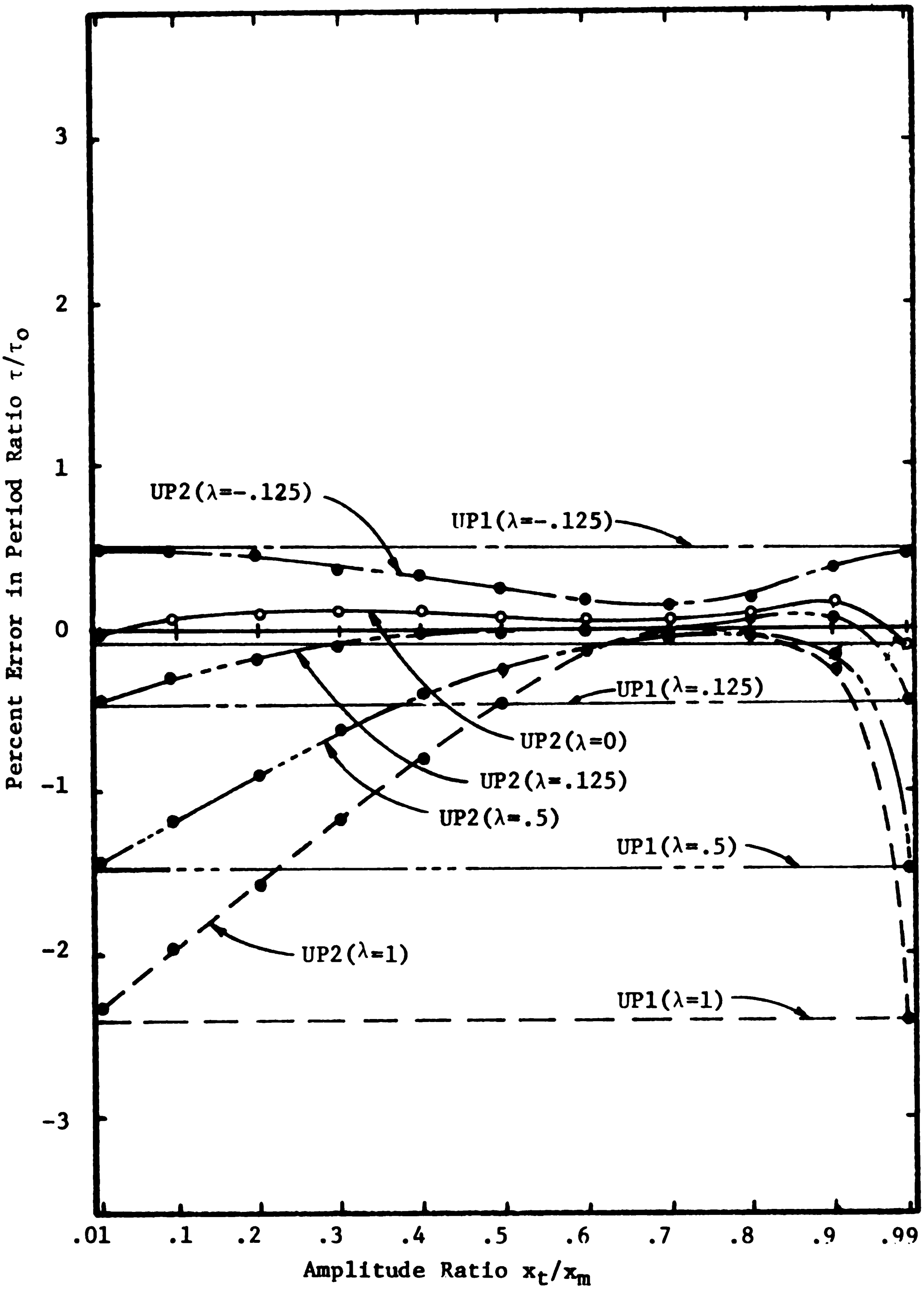


Figure 3.8 One Degree-of-Freedom, Free Vibration System Error Plot Period Ratio τ/τ_o versus Amplitude Ratio x_t/x_m for $f(x) = \sin x$ (x_m=1.0)

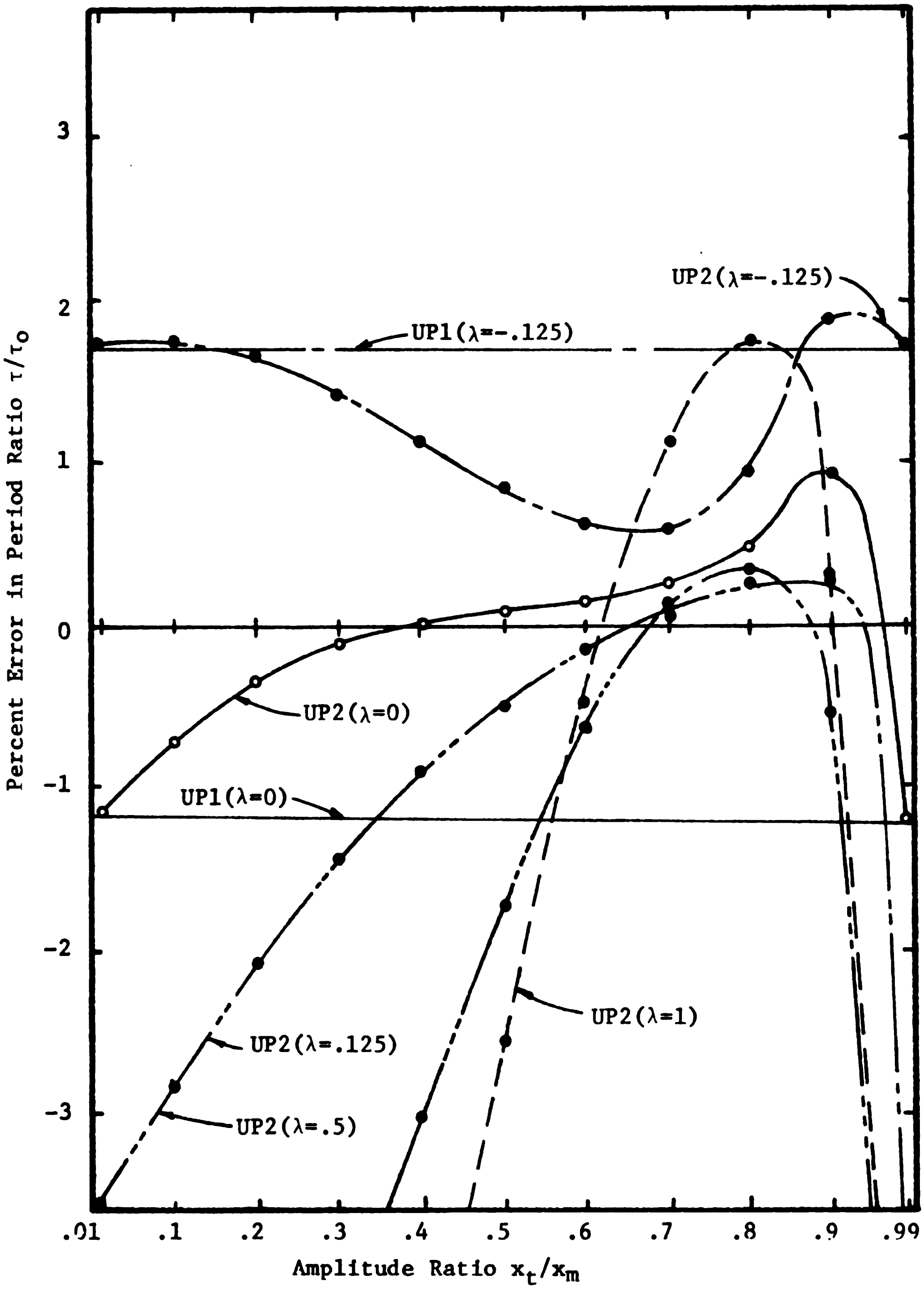


Figure 3.9 One Degree-of-Freedom, Free Vibration System Error Plot Period Ratio τ/τ_o versus Amplitude Ratio x_t/x_m for $f(x) = \sin x$ (x_m=2.0)

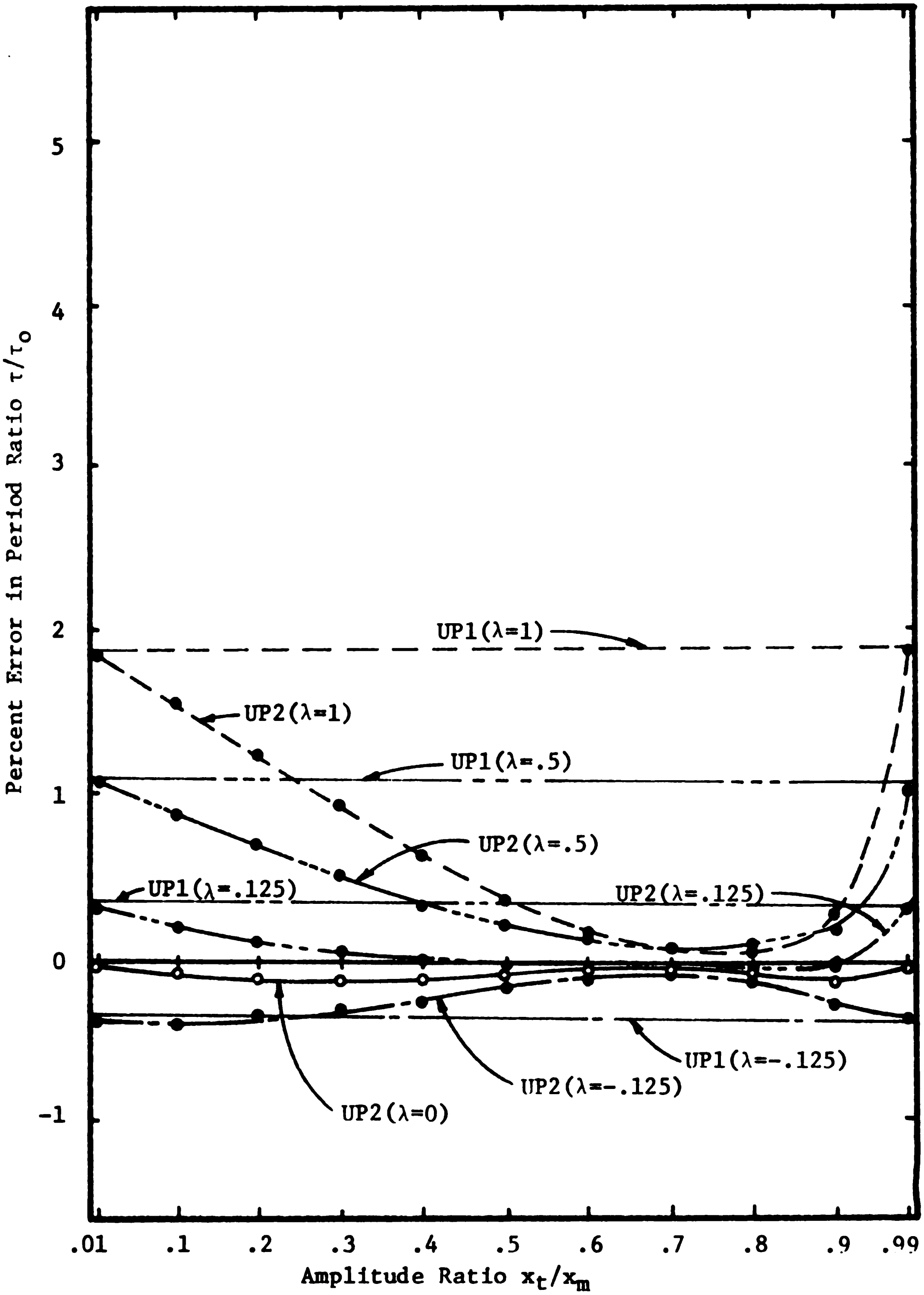


Figure 3.10 One Degree-of-Freedom, Free Vibration System Error Plot Period Ratio τ/τ_o versus Amplitude Ratio x_t/x_m for $f(x) = \sinh x$ ($x_m = 1.0$)

exact, linear U.P. and bilinear U.P. methods for a range of λ parameters and of x_t/x_m values ($0 \leq x_t/x_m \leq 1$) which includes $x_t/x_m = .5$. Figures 3.4, 3.5, 3.6 and 3.7 show this dependence for function $x + x^3$; Figures 3.8 and 3.9 for sin x; and Figure 3.10 for sinh x. The $\lambda = 0$ and $x_t/x_m = .5$ values over the maximum amplitudes considered, in general, give the better bilinear U.P. approximate results for the three nonlinear functions investigated.

3.2 Forced Vibrations.

In this section an undamped, forced vibrating system is examined. This system is solved first as a forced vibrating problem by approximating the restoring force bilinearly about its own equilibrium point (which differs from the point about which oscillation occurs). Next, this same system is solved as a free vibrating problem by approximating the equivalent nonlinear restoring force by four lines about a point called the minimum potential point where oscillation occurs. Only the case of a step function excitation and of a cubic restoring force is investigated to illustrate the procedure. One illustrative problem is solved by the former approach, yielding results comparable to the one line U.P. solution.

The governing equation is of the form

$$\ddot{x} + f(x) = F(t) = F_o u(t)$$

complete with the initial conditions

$$x(0) = 0 \text{ and } \dot{x}(0) = 0$$

where u(t) is the unit step-function ($u=0$ for $t<0$, $u = 1$ for $t>0$) and F_o is the amplitude of the applied excitation.

Solution as a Forced Vibration Problem. Like for the free vibration system the equation of motion can be approximated bilinearly by two linear differential equations

$$\ddot{x}_1 + k_1 x_1 = F(t) = F_o u(t), \qquad |x| \le x_t, \tag{3.11}$$

and

$$\ddot{x}_2 + k_2 x_2 + x_t(k_1 - k_2) = F_o u(t), \qquad |x| \ge x_t \tag{3.12}$$

where x_t is the transition point as discussed in Chapter II. The initial conditions are then

$$x_1(0) = 0 \text{ and } \dot{x}_1(0) = 0, \tag{3.13}$$

and x_1 and x_2 are matched at x_t, say at $t = t_t$

$$x_1(t_t) = x_2(t_t) \text{ and } \dot{x}_1(t_t) = \dot{x}_2(t_t) \tag{3.14}$$

Following a procedure similar to that of the free vibrating problem in Section 3.1 we obtain the period of motion, τ, as

$$\tau = 2\left[\int_o^{x_t} \frac{dx_1}{\dot{x}_1} + \int_{x_t}^{2A} \frac{dx_2}{\dot{x}_2}\right]$$

Nondimensionally, the period of motion becomes

$$\frac{\tau}{\tau_o} = \frac{1}{\pi}\left[\int_o^{x_t} \left(k_1\right)^{-1/2} [2\underset{\sim}{d}x - x^2]^{-1/2} dx + \int_{x_t}^{2A} [e + fx + gx^2]^{-1/2} dx\right] \tag{3.15}$$

where

$F_o = k_2 A - (k_1 - k_2)\ [(x_t - 4A)/4A]\ x_t$ $\qquad$ $f = 2F_o - 2x_t(k_1 - k_2)$

$\underset{\sim}{d} = F_o/k_1$ $\qquad$ $g = -k_2$

$e = (k_1 - k_2)\ x_t^2$ $\qquad$ $\tau_o = 2\pi(k_1)^{-1/2}$

The right hand side above can be integrated out explicitly and the results expressed in closed form. Assuming $k_1>0$, then for $k_2<0$, $k_2 = 0$ and $k_2>0$ respectively, we have, after some manipulations,

For $k_2<0$:

$$\frac{\tau}{\tau_o} = \frac{1}{\pi}\left[(k_1)^{-1/2}\left(\sin^{-1}\frac{x_t-\tilde{d}}{|\tilde{d}|}+\frac{\pi}{2}\right)+\frac{1}{\sqrt{g}}\log\left((e+f2A+g4A^2)^{1/2}+2A\sqrt{g}+\frac{f}{2\sqrt{g}}\right)-\frac{1}{\sqrt{g}}\log\left((e+fx_t+gx_t^{\,2})^{1/2}+x_t\sqrt{g}+\frac{f}{2\sqrt{g}}\right)\right] \tag{3.16}$$

For $k_2 = 0$:

$$\frac{\tau}{\tau_o} = \frac{1}{\pi}\left[(k_1)^{-1/2}\left(\sin^{-1}\left(\frac{x_t-\tilde{d}}{|\tilde{d}|}\right)+\frac{\pi}{2}\right)+\frac{4A}{\sqrt{k_1}\,x_t}\left(1-\frac{x_t}{2A}\right)\right] \tag{3.17}$$

For $k_2>0$:

$$\frac{\tau}{\tau_o} = \frac{1}{\pi}\left[(k_1)^{-1/2}\left(\sin^{-1}\left(\frac{x_t-\tilde{d}}{|\tilde{d}|}\right)+\frac{\pi}{2}\right)+\frac{1}{\sqrt{-g}}\sin^{-1}\left(\frac{-4Ag-f}{\sqrt{f^2-4eg}}\right)-\frac{1}{\sqrt{-g}}\sin^{-1}\left(\frac{-2x_tg-f}{\sqrt{f^2-4eg}}\right)\right] \tag{3.18}$$

The dimensionless, bilinear U.P. period-amplitude relations derived above are now used in solving the special case $f(x) = x + x^3$. The slope parameters k_1 and k_2 derived in Subsection 2.2.1 for $f(x) = x + x^3$ are now substituted into one of the above equations and the results are compared in Figure 3.11 against the linear U.P. and the exact solution, obtained by a quadrature method. As Figure 3.11 shows both the bilinear U.P. and the linear U.P. method closely approximate the exact solution. For this function $f(x) = x + x^3$, the bilinear U.P. method does show improved results over the linear U.P. method for the larger amplitudes. However, this slight improvement does not warrant the additional effort required compared to the linear U.P. method except for highly nonlinear functions.

Solution as a Free Vibration Problem. By a change of variable the governing equation is reformulated as a free vibration problem as

$$\ddot{x} + f(x) - F_o u(t) = 0$$

or

$$\ddot{x} + \overline{G}(x) = 0,$$

where

$$\overline{G}(x) = f(x) - F_o u(t)$$

Introducting the change of variable $x = z + x_o$ we obtain a formulation where $z = 0$ becomes the local minimum potential point as previously defined in Chapter II. Thus we have that

$$\ddot{z} + \overline{G}(z + x_o) = 0 \tag{3.19}$$

with the initial conditions

$$z(0) = -x_o \text{ and } \dot{z}(0) = 0$$

where x_o is the local minimum potential point obtained by setting $\overline{G}(x) = 0$ and solving for x, (see Figure 3.12).

Derivation of the Period vs Amplitude Relationship. The equation of motion (3.19) is approximated by the four linear differential equations

$$\ddot{z}_4 + k_4 z_4 + (x_{t_2} - x_o)(k_3 - k_4) = 0 \qquad -x_o \leq z \leq -|z_{m_3}| \tag{3.20}$$

and

$$\ddot{z}_3 + k_3 z_3 = 0 \qquad -|z_{m_3}| \leq z \leq 0 \tag{3.21}$$

and

$$\ddot{z}_1 + k_1 z_1 = 0 \qquad 0 \leq z \leq |z_{m_1}| \tag{3.22}$$

and

$$\ddot{z}_2 + k_2 z_2 + (x_{t_1} - x_o)(k_1 - k_2) = 0 \qquad |z_{m_1}| \leq z \leq A \tag{3.23}$$

The initial conditions are then

$$z_4(0) = -x_o \text{ and } \dot{z}_4(0) = 0 \tag{3.24}$$

and z_3 and z_4 are matched at, say $t = t_2$

$$z_3(t_2) = z_4(t_2) \text{ and } \dot{z}_3(t_2) = \dot{z}_4(t_2)$$

and z_1 and z_3 are matched at, say $t = t_o$

$$z_1(t_o) = z_3(t_o) = 0 \text{ and } \dot{z}_1(t_o) = \dot{z}_3(t_o) \tag{3.25}$$

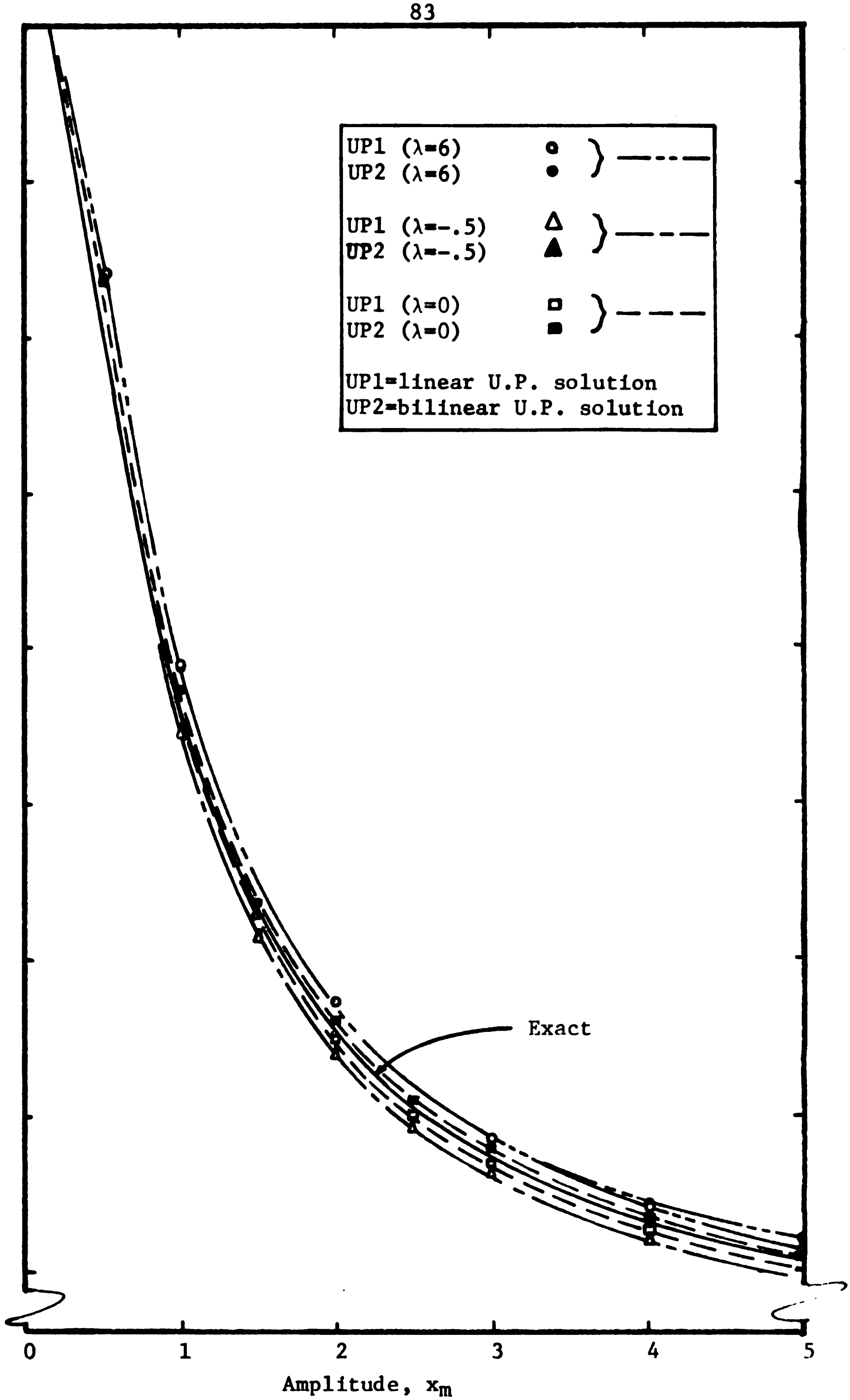


Figure 3.11 One Degree-of-Freedom, Forced Vibration System Period Ratio τ/τ_o versus Amplitude x_m for $f(x) = x + x^3$ and Unit Step Function Excitation

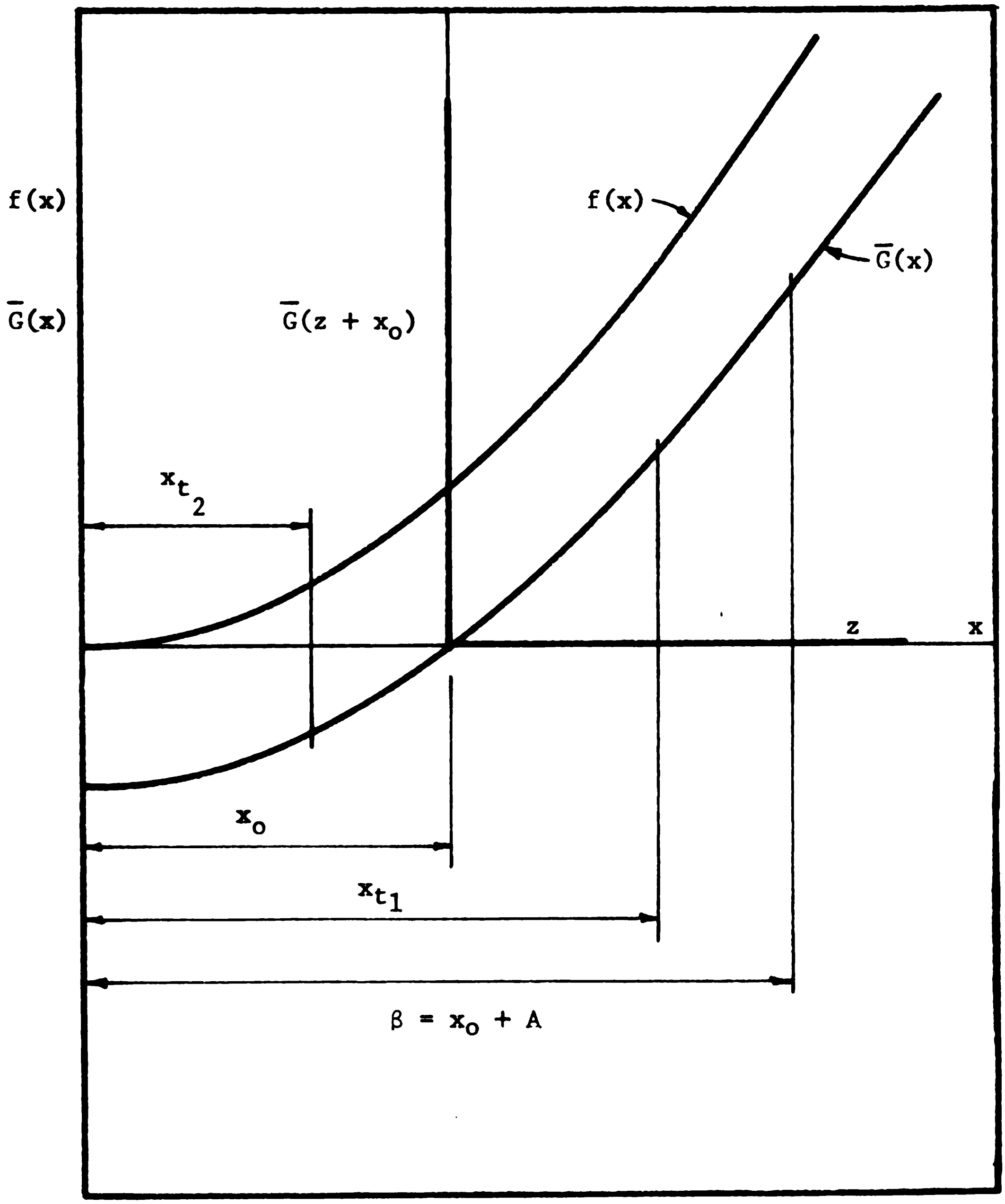


Figure 3.12 A Nonlinear Restoring Force f(x) Modified to Give a New Restoring Force $\bar{G}(x)$ With a New Equilibrium Position $x = x_o$

and z_1 and z_2 are matched at, say $t = t_1$

$$z_2(t_1) = z_1(t_1) \text{ and } \dot{z}_2(t_1) = \dot{z}_1(t_1) \tag{3.26}$$

Obtaining the first integral of each of the four linear differential equations and solving for the constant of integration, we find the period-amplitude relation as

$$\tau = 2\int_{-x_o}^{A}\frac{dz}{\dot{z}} = 2\left[\int_{-x_o}^{-|z_{m_3}|}\frac{dz_4}{\dot{z}_4} + \int_{-|z_{m_3}|}^{o}\frac{dz_3}{\dot{z}_3} + \int_{o}^{|z_{m_1}|}\frac{dz_1}{\dot{z}_1} + \int_{|z_{m_1}|}^{A}\frac{dz_2}{\dot{z}_2}\right] \tag{3.27}$$

where

$$\dot{z}_4 = [C_4 - V_4]^{1/2}$$

with

$$C_4 = k_4 x_o^2 - 2(x_{t_2} - x_o)\,(k_3 - k_4)\,x_o$$

and,

$$V_4 = k_4 z_4^2 + 2(x_{t_2} - x_o)\,(k_3 - k_4)\,z_4$$

where

$$\dot{z}_3 = [C_3 - V_3]^{1/2}$$

with

$$C_3 = C_4 - (x_{t_2} - x_o)^2 \ (k_3 - k_4)$$

and,

$$V_3 = k_3 z_3{}^2$$

where

$$\dot{z}_1 = [C_1 - V_1]^{1/2}$$

with

$$C_1 = C_3$$

and,

$$V_1 = k_1 z_1{}^2$$

where

$$\dot{z}_2 = [C_2 - V_2]^{1/2}$$

with

$$C_2 = C_1 + (x_{t_1} - x_o)^2 \ (k_1 - k_2)$$

and

$$V_2 = k_2 z_2^2 + 2(x_t - x_o) \ (k_1 - k_2) \ z_2$$

The slope parameter k_1, k_2, k_3, and k_4 are given by equations (2.2), (2.4), (2.3) and (2.5) respectively, with G(x) replaced by $\overline{G}(x)$, that is,

$$k_1(\lambda, z_{m_1}) = \frac{1}{|z_{m_1}|} \left[\frac{\int_0^1 \overline{G}\,(|z_{m_1}|\ t + x_o)\ P_1^{(\lambda)}(t)\ \ \omega(\lambda,t)\ dt}{\int_0^1 [P_1^{(\lambda)}(t)]^2\ \omega(\lambda,t)\ dt} \right] \quad (3.28)$$

where

$$|z_{m_1}| = |x_{t_1} - x_o|$$

$$k_2(\lambda, k_1, x_t, y_m) = \frac{1}{|y_m|} \left(\frac{\int_0^1 [\overline{G}(|y_m|\ t + x_{t_1}) - k_1(x_{t_1} - x_o)]\ P_1^{(\lambda)}(t)\ \omega(\lambda,t)\ dt}{\int_0^1 [P_1^{(\lambda)}(t)]^2\ \omega(\lambda,t)\ dt} \right)$$

(3.29)

where

$$|y_m| = |\beta - x_{t_1}| = |2x_o - x_{t_1}|$$

$$k_3(\lambda, z_{m_3}) = \frac{1}{|z_{m_3}|} \left[\frac{\int_0^1 -[\overline{G}(-|z_{m_3}|t + x_o)]\ P_1^{(\lambda)}(t)\ \omega(\lambda,t)\ dt}{\int_0^1 [P_1^{(\lambda)}(t)]^2\ \omega(\lambda,t\ dt} \right]$$

(3.30).

where

$$|z_{m_3}| = |x_o - x_{t_2}|$$

$$k_4(\lambda,k_3,x_{t_2},s_m) = \frac{1}{|s_m|}\left[\frac{\int_o^1 -[\bar{G}(-|s_m|t + x_{t_2})-k_3(x_{t_2}-x_o)]\, P_1^{(\lambda)}(t)\, \omega(\lambda,t)\, dt}{\int_o^1 [P_1^{(\lambda)}(t)]^2\, \omega(\lambda,t)\, dt}\right] \tag{3.31}$$

where

$$|s_m| = |x_{t_2} - \alpha| = |x_{t2}|$$

While the special case of $f(x) = x + x^3$ was not solved by the four-line U.P. method, (3.27) should give a good approximation for those cases where the restoring force is highly nonlinear.

IV. A SYMMETRIC TWO DEGREE-OF-FREEDOM NONLINEAR SYSTEM

The symmetric two degree-of-freedom problem shown in Figure 4.1 has been investigated by a number of authors [13, 14, 15, 16, 18]. Only two of these authors, however, have investigated the questions of existence of more than two "normal modes". Here the term "normal modes" in nonlinear systems is understood in the sense of Rosenberg [17] (see Appendix F). In [15] Rosenberg commented on the existence of four distinct normal modes for the two degree-of-freedom symmetric system in which the nonlinear springs are of homogeneous degree. The details are given in Appendix F. More recently, Anand [18] uncovered this multiplicity of normal modes for the symmetric two degree-of-freedom system with cubic nonlinearities.

In Section 4.1 Anand's approximation will be shown to be equivalent to a linear (one-line) ultraspherical polynomial approximation. A discussion of the six regions of the different modal patterns predicted by Anand which depend on the values of two parameters of nonlinearity is then reviewed in Section 4.2. In Section 4.3 a special case of the symmetric problem is examined which, according to Anand's approach, indicated the presence of an additional normal mode but which, according to the bilinear (two-line) ultraspherical polynomial approach as well as the exact solution, denies the existence of this additional mode. Section 4.4 contains additional calculations and discussions on this multiplicity of normal modes (superabundant modes).

4.1 Equivalence of Approximation Techniques.

Anand considered the symmetric problem shown in Figure 4.1 where

$$\left.\begin{aligned} f(x) &= k_1 x_1 + m\,\alpha_1\, x_1^3 \\ f(x_2) &= k_1 x_2 + m\,\alpha_1\, x_2^3 \\ f(x_1-x_2) &= k(x_1-x_2) + m\,\alpha\,(x_1-x_2)^3 \end{aligned}\right\} \tag{4.1}$$

The equations of motion are

$$m\ddot{x}_1 + f(x_1) + f(x_1-x_2) = 0$$

$$m\ddot{x}_2 + f(x_2) - f(x_1-x_2) = 0$$

Anand defines two new parameters, $\omega_2^2 = (k + k_1)/m$ and $\omega_1^2 = k/m$. In these parameters the equations of motion become

$$\ddot{x}_1 + \omega_2^2 x_1 + \alpha_1 x_1^3 - \omega_1^2 x_2 + \alpha(x_1-x_2)^3 = 0 \tag{4.2}$$

$$\ddot{x}_2 + \omega_2^2 x_2 + \alpha_1 x_2^3 - \omega_1^2 x_1 - \alpha(x_1-x_2)^3 = 0 \tag{4.3}$$

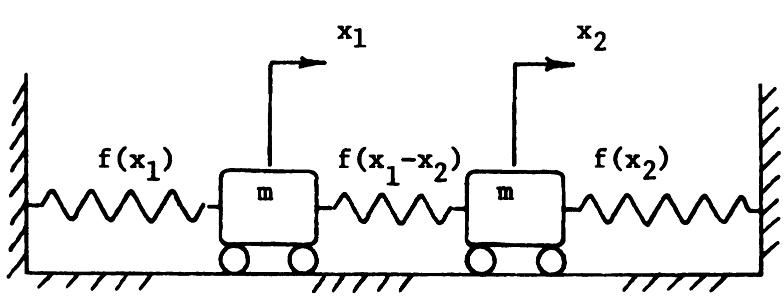


Figure 4.1 Conservative Spring-Mass Two Degree-of-Freedom System.

into which solutions of the form $x_1 = A \cos \omega t$ and $x_2 = B \cos (\omega t + \beta)$ are substituted. Since the system is conservative there is no need for assuming the phase angle β. Without loss of generality we may let $\beta = 0$ at this point. After some algebraic manipulation, terms involving $\cos \omega t$, $\sin \omega t$, $\cos 3\omega t$, and $\sin 3\omega t$ are obtained. Upon disregarding the superharmonic terms and equating the coefficients of the harmonic terms to zero one obtains

$$\frac{3}{4}(\alpha+\alpha_1)\, A^3 - \frac{9}{4}\alpha A^2 B + \frac{9}{4}\alpha A B^2 - \frac{3}{4}\alpha\, B^3 + (\omega_2{}^2-\omega^2)A - \omega_1{}^2 B = 0 \qquad (4.4)$$

$$-\frac{3}{4}\alpha A^3 + \frac{9}{4}\alpha A^2 B - \frac{9}{4}\alpha A B^2 + \frac{3}{4}(\alpha + \alpha_1)B^3 - \omega_1{}^2 A + (\omega_2{}^2-\omega^2)B = 0 \qquad (4.5)$$

Equations (4.4) and (4.5) can be obtained by yet another approximate method. If the nonlinear terms in the equations of motion, (4.2) and (4.3), are linearized in terms of ultraspherical polynomials (one-line approximations) with respect to appropriate amplitudes; and if a normal mode solution is assumed (as was assumed by Anand), then equations (4.4) and (4.5) result provided the Chebyshev polynomial ($\lambda=0$) is used. The details are presented below.

In terms of ultraspherical polynomials, the nonlinear terms of equations (4.2) and (4.3) are linearized as follows:

(i) linearize x_1, with respect to A;

(ii) linearize x_2, with respect to B; and

(iii) linearize x_1-x_2, with respect to C, where C = A−B

The resulting equations of motion are

$$\ddot{x}_1 + \omega_2{}^2 x_1 + \alpha_1 \left[\frac{3A^2}{2(\lambda+2)}\right] x_1 - \omega_1{}^2 x_2 + \alpha\left[\frac{3C^2}{2(\lambda+2)}\right](x_1-x_2) = 0$$

$$\ddot{x}_2 + \omega_2{}^2 x_2 + \alpha_1 \left[\frac{3B^2}{2(\lambda+2)}\right] x_2 - \omega_1{}^2 x_1 - \alpha\left[\frac{3C^2}{2(\lambda+2)}\right](x_1-x_2) = 0$$

Substituting a solution of the form $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A \\ B \end{Bmatrix} \cos \omega t$ into these equations yields

$$-A\omega^2 + \omega_2{}^2 A + \alpha_1 \left[\frac{3A^2}{2(\lambda+2)}\right] A - \omega_1{}^2 B + \alpha\left[\frac{3C^2}{2(\lambda+2)}\right](A-B) = 0 \qquad (4.6)$$

$$-B\omega^2 + \omega_2{}^2 A + \alpha_1 \left[\frac{3B^2}{2(\lambda+2)}\right] B - \omega_1{}^2 A - \alpha\left[\frac{3C^2}{2(\lambda+2)}\right](A-B) = 0 \qquad (4.7)$$

Substituting A−B for C in (4.6) and (4.7) yields respectively

$$\frac{3}{2(\lambda+2)} (\alpha+\alpha_1) A^3 - \frac{9}{2(\lambda+2)} \alpha A^2 B + \frac{9}{2(\lambda+2)} \alpha A B^2 - \frac{3}{2(\lambda+2)} \alpha B^3$$

$$+ (\omega_2^2-\omega^2) A - \omega_1^2 B = 0 \tag{4.8}$$

and

$$-\frac{3}{2(\lambda+2)} \alpha A^3 + \frac{9}{2(\lambda+2)} \alpha A^2 B - \frac{9}{2(\lambda+2)} \alpha A B^2 + \frac{3}{2(\lambda+2)} (\alpha+\alpha_1) B^3$$

$$-\omega_1^2 A + (\omega_2^2-\omega^2) B = 0 \tag{4.9}$$

Finally when λ is set equal to zero (the Chebyshev polynomial), equations (4.8) and (4.9) reduce to (4.4) and (4.5) as mentioned before. Thus we have shown that Anand's solution method is equivalent to the ultraspherical polynomial approximation with $\lambda = 0$.

Now, subtracting (4.9) from (4.8), or equivalently, (4.5) from (4.4) with $\lambda = 0$, we obtain

$$(3/4\ \alpha_1 + 3/2\ \alpha)\ (A^3-B^3) - (9/2\ \alpha AB + \omega^2 - \omega_2^2 - \omega_1^2)\ (A-B) = 0 \tag{4.10}$$

and, adding (4.9) and (4.8) we obtain

$$3/4\alpha_1\ (A^3 + B^3) - (\omega^2-\omega_2^2 + \omega_1^2)\ (A + B) = 0 \tag{4.11}$$

As Anand pointed out in his study, equations (4.10) and (4.11) yield three pairs of solutions for A and B which constitute the possible modes of vibration. The possible modes are summarized below.

Symmetric Case (A =B): The relation A = B represents the in-phase or symmetric mode. Dividing equation (4.11) by A + B and substituting A = B into this resulting relation yields

$$3/4\alpha_1 A^2 - (\omega^2 - \omega_2{}^2 + \omega_1{}^2) = 0 \tag{4.12}$$

This is the frequency-amplitude relation for the symmetric mode.

Antisymmetric Case (A = -B): The relation A = -B represents the out-of-phase or antisymmetric mode. Dividing equation (4.10) by A-B and substituting A = -B into this resulting relation yields

$$(3/4\alpha_1 + 6\alpha)A^2 - (\omega^2 - \omega_2{}^2 - \omega_1{}^2) = 0 \tag{4.13}$$

This is the frequency amplitude-relation for the antisymmetric mode.

Asymmetric Case (A ≠ B and A ≠ -B): The relations A ≠ B and A ≠-B represent Anand's third mode which he terms the asymmetric mode. Dividing equations (4.10) and (4.11) by A-B and A+B, respectively, we obtain

$$(3/4\alpha_1 + 3/2\alpha)\ (A^2 + AB + B^2) - (9/2\alpha\ AB + \omega^2 - \omega_2{}^2 - \omega_1{}^2) = 0 \tag{4.14}$$

$$3/4\alpha_1\ (A^2 - AB + B^2) - (\omega^2 - \omega_2{}^2 + \omega_1{}^2) = 0 \tag{4.15}$$

Solutions of equations (4.14) and (4.15) will be discussed below.

4.2 Discussion of the Regions for the Asymmetric Mode.

Anand discusses the effect the nonlinear parameters on the existence of this third mode.

Eliminating ω^2 from equations (4.14) and (4.15) we obtain

$$A^2 + B^2 - 1/\alpha\ (2\alpha - \alpha_1)AB + 4/3\alpha\ \omega_1{}^2 = 0 \tag{4.16}$$

Solving this equation for B, we get

$$B = \left(1 - \frac{\alpha_1}{2\alpha}\right)A + \left[\frac{\alpha_1}{\alpha}\left(\frac{\alpha_1}{4\alpha} - 1\right)A^2 - \frac{4}{3\alpha}\omega_1{}^2\right]^{1/2} \tag{4.17}$$

where A has been arbitrarily assumed to be larger than B.

For a real, physical solution we require that $\alpha_1 > 4\alpha$ (4.18)

which places the following restriction on A

$$A^2 > \frac{16\alpha\ \omega_1{}^2}{3\alpha_1\ (\alpha_1 - 4\alpha)} \tag{4.19}$$

By considering three cases in the $(\alpha\alpha_1)$ plane we can identify the regions where this asymmetric mode is present.

Case I. For $\alpha > 0$ and

(i) $0 < \alpha_1$ This subcase represents a system with all springs hard.* Equations (4.18) and (4.19) place a restriction on amplitude A; A cannot be too small.

* A nonlinear spring force is considered hard if its first derivative increases with increasing displacement, and soft if its first derivative decreases with increasing displacement.

(ii) $0 \leq \alpha_1 \leq 4\alpha$ Asymmetric solution does not exist.

(iii) $0 > \alpha_1$ This subcase represents a system with the coupling spring being hard and the outboard spring being soft. Equations (4.18) and (4.19) restrict the amplitude in that A cannot be too small.

Case II. For $\alpha = 0$ and

(i) $\alpha_1 > 0$ or $\alpha_1 < 0$.

Taking the limit of (4.17) as $\alpha \to 0$ we obtain

$$B \simeq \frac{-\alpha_1 A}{2\alpha} + \left[\frac{\alpha_1}{\alpha}\left(\frac{\alpha_1}{4\alpha}\right)A^2 - \frac{4}{3\alpha}\omega_1^2 \right]^{1/2}$$

Expanding the second term of (4.17) by the binomial expansion we find

$$B \simeq \frac{-\alpha_1}{2\alpha} A + \frac{\alpha_1 A}{2\alpha}\left\{ 1 - \frac{8}{3}\frac{\omega_1^2 \alpha}{A^2\alpha_1^2} + \cdots\cdots \right\}$$

As α tends to zero,

$$\lim_{\alpha \to 0} B \simeq \lim_{\alpha \to 0}\left[\frac{-\alpha_1 A}{2\alpha} + \frac{\alpha_1 A}{2\alpha} - \frac{4\omega_1^2}{3\alpha_1 A} + \cdots \right]$$

$$\simeq \frac{-4\omega_1^2}{3\alpha_1 A}$$

(ii) For $\alpha = 0$ and $\alpha_1 = 0$, Asymmetric solution does not exist.

Case III. For $\alpha < 0$ and

(i) $\alpha_1 < 0$. If $\frac{\alpha 1}{4\alpha} < 1$, $\alpha_1 > 4\alpha$ and the amplitude, A, in restricted in that A cannot be too large, i.e.

$$A^2 < \frac{16\alpha\,\omega_1^2}{3\alpha_1\,(\alpha_1 - 4\alpha)}$$

If $\frac{\alpha 1}{4\alpha} \geq 1$, $\alpha_1 \leq 4\alpha$; asymmetric solution exists regardless of the amplitude A.

(ii) $\alpha_1 = 0$ Asymmetric solution exists with no restriction on amplitude, A.

(iii) $\alpha_1 > 0$ Asymmetric solution exists with no restriction on amplitude, A.

In addition to the conditions given by equations (4.18) and (4.19), which place restrictions on the parameters α and α_1, and the amplitude A, the frequency ω must also be real in order to guarantee real asymmetric solutions. This additional requirement $\omega^2 > 0$ must therefore also be satisfied.

Cases I, II, and III are displayed below in Figure 4.2.

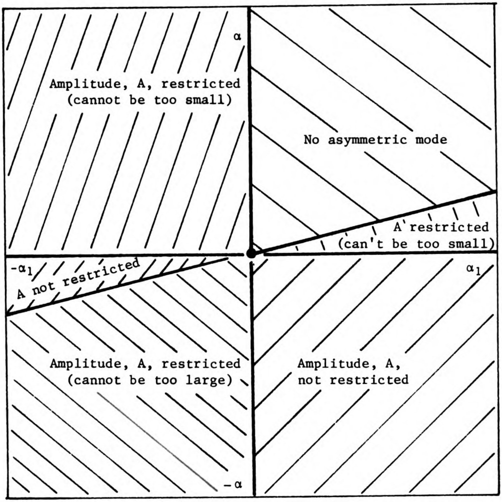


Figure 4.2 α α_1 Graph Showing the Six Regions for the Asymmetric Mode for a Two Degree, Symmetric, Free Vibration System

4.3 A Special Case of Anand's Problem.

As a test of Anand's approximate approach, a special case is examined so that the bilinear U.P. approximation method as well as the exact solution can be compared against it. To avoid undue algebraic difficulties inherent in solving the symmetric problem by the bilinear method, the coupling spring is the only nonlinear term allowed and is of the soft cubic type. In particular, we can illustrate this case in Figure 4.1 with the coefficients $m = 1$, $k_1 = 1$, $k = 1$, $\alpha_1 = 0$ and $\alpha = -1$ substituted into (4.1). The restoring forces given by (4.1) then reduce to

$$\begin{aligned} f(x_1) &= x_1 \\ f(x_2) &= x_2 \\ f(x_1-x_2) &= (x_1-x_2) - (x_1-x_2)^3 \end{aligned} \tag{4.20}$$

4.3.1 Anand's Solution.

This special case falls within Case III mentioned previously, which indicated that for the asymmetric mode no restriction is placed on the amplitude of x_1, A. However, a check still is required on the frequency, ω. Substituting the above coefficients into equation (4.15) and simplifying we find that the frequency is real,

$$\omega^2 = \omega_2{}^2 - \omega_1{}^2 = k_1/m = 1$$

In addition, substituting the parameters from this special case give for the in-phase mode the frequency relation which simplifies to the same expression as for the asymmetric mode.

For the out-of-phase mode the frequency relation simplifies to

$$\omega^2 = \omega_2{}^2 + \omega_1{}^2 + 6\alpha\ A^2 = \frac{k_1 + 2k}{m} + 6\alpha\ A^2 = 3(1-2A^2)$$

These values are plotted in (Figure 4.5) for comparison with bilinear and exact frequency relations, yet to be determined. Thus, we may conclude that for this special case, Anand's approximate method does predict an asymmetric mode. We will now apply the bilinear ultraspherical polynomial approximation to this same special case and show that the bilinear solution can be expressed in analytic form.

4.3.2 Bilinear U.P. Solution.

This special case of the symmetric two degree-of-freedom (free vibration) problem is formulated in this subsection by the two-line ultraspherical polynomial approximation. This problem is shown schematically by Figure 4.1 and the coupled equations of motion are

$$m\ddot{x}_1 = \frac{\partial U}{\partial x_1} \tag{4.21}$$

$$m\ddot{x}_2 = \frac{\partial U}{\partial x_2} \tag{4.22}$$

with $$-U = -U(x_1,x_2) = \frac{x_1{}^2}{2} + \frac{x_2{}^2}{2} + \frac{(x_1-x_2)^2}{2} - \frac{(x_1-x_2)^4}{4} \tag{4.23}$$

These equations are approximated bilinearly by equations (4.21) and (4.22) but with $U(x_1,x_2)$ approximated by

$$-U = -U(x_1,x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2} + V(x_1-x_2) \tag{4.24}$$

and

$$V(x_1-x_2) = \begin{cases} \dfrac{k_3\ (x_1-x_2)^2}{2} = \dfrac{k_3\ y^2}{2}, & y \leq y_t \qquad (4.25) \\[2ex] \dfrac{k_3(x_1-x_2)^2}{2} + 1/2\ (k_4-k_3)\ (x_1-x_2-y_t)^2 & \end{cases}$$

$$= \frac{k_3}{2}\, y^2 + 1/2(k_4-k_3)\ (y-y_t)^2, \quad y \geq y_t \tag{4.26}$$

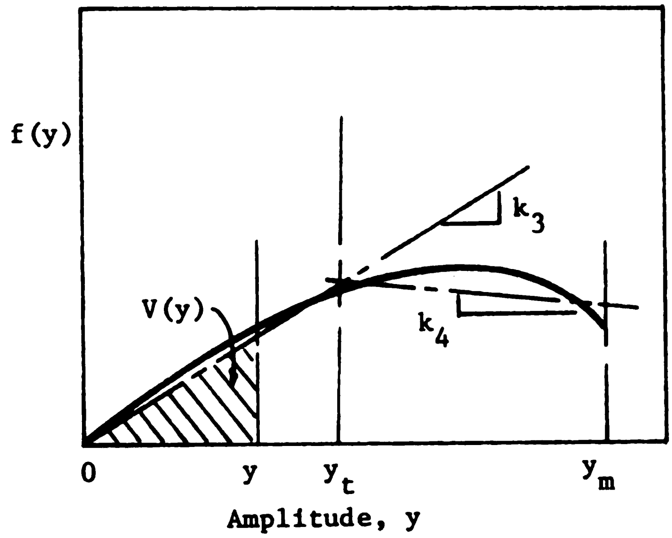
Simplified, the bilinear set of approximating equations becomes

$$\left.\begin{aligned} m\ddot{x}_1 + k_1x_1 + k_3(x_1-x_2) &= 0 \\ m\ddot{x}_2 + k_1x_2 - k_3(x_1-x_2) &= 0 \end{aligned}\right\} \quad y \leq y_t \tag{4.27}$$

$$\left.\begin{aligned} m\ddot{x}_1 + k_1x_1 + k_3(x_1-x_2) + (k_4-k_3)\ (x_1-x_2-y_t) &= 0 \\ m\ddot{x}_2 + k_1x_2 - k_3(x_1-x_2) - (k_4-k_3)\ (x_1-x_2-y_t) &= 0 \end{aligned}\right\} \quad y \geq y_t \tag{4.28}$$

The slope parameters k_3 and k_4 are the linear approximations to the nonlinear function over the appropriate intervals.

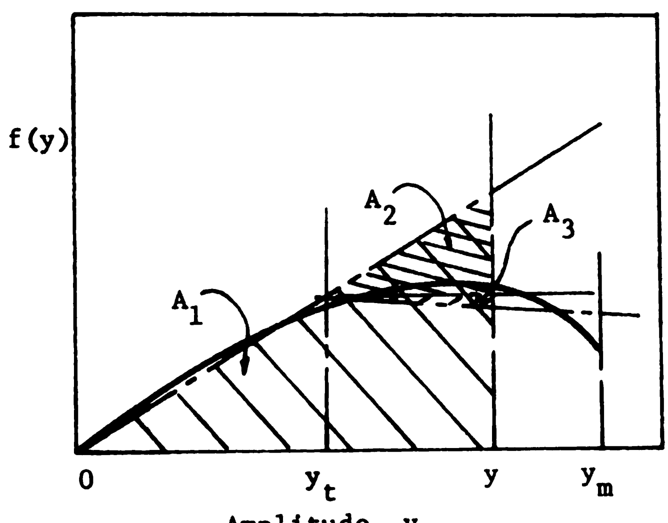
The term $V(x_1-x_2)$ is obtained by summing the areas under the force-displacement plot for the approximating slope parameters. For example, Figures 4.3 and 4.4 represent a function approximated bilinearly for the $y \leq y_t$ and $y \geq y_t$ intervals, respectively,



$$V(y) = \frac{k_3 y^2}{2}$$

$y \leq y_t$

Figure 4.3 $V = \int f(y)\, dy$ Relation for a Nonlinear Restoring Force Approximated Bilinearly ($y \leq y_t$)



$$V(y) = A_1 + A_3 - A_2$$
$$= \frac{k_3 y^2}{2} + \frac{k_4 (y-y_t)^2}{2} - \frac{k_3 (y-y_t)^2}{2}$$

(where $k_4<0$ in this case)

$y \geq y_t$

Figure 4.4 $V = \int f(y)\, dy$ Relation for a Nonlinear Restoring Force Approximated Bilinearly ($y \geq y_t$)

$$\left.\begin{aligned} m\ddot{x}_1 + k_1x_1 + k_3\,(x_1-x_2) &= 0 \qquad (4.29)\\ m\ddot{x}_2 + k_1x_2 - k_3(x_1-x_2) &= 0 \qquad (4.30)\end{aligned}\right\}\; y \le y_t$$

$$\left.\begin{aligned} m\ddot{x}_1 + k_1x_1 + k_3(x_1-x_2) + (k_4 - k_3)\,(x_1-x_2-y_t) &= 0 \qquad (4.31)\\ m\ddot{x}_2 + k_1x_2 - k_3(x_1-x_2) - (k_4-k_3)\,(x_1-x_2-y_t) &= 0 \qquad (4.32)\end{aligned}\right\}\; y \ge y_t$$

These equations can be rewritten in a simplified form by substituting $y = x_1-x_2$ and, for the $y \le y_t$ region subtracting equation (4.30) from equation (4.29) to yield the following equations

$$\left.\begin{aligned} m\ddot{y} + (k_1 + 2k_3)y &= 0 \qquad (4.33)\\ m\ddot{x}_1 + k_1x_1 + k_3y &= 0 \qquad (4.34)\end{aligned}\right\}\; y \le y_t$$

Similarly, for the $y \ge y_t$ region we can simplify these equations by subtracting equation (4.32) from equation (4.31) and by the change of variables $z = y-2F$ and $u = x_1-F$, where $F = y_t(k_4-k_3)/(k_1 + 2k_4)$

to obtain

$$\left.\begin{aligned} m\ddot{z} + (2k_4 + k_1)\,z &= 0 \qquad (4.35)\\ m\ddot{u} + k_1u + k_4z &= 0 \qquad (4.36)\end{aligned}\right\}\; y \ge y_t$$

This simplification uncouples equations (4.33) and (4.35) above.

The solutions to these four simplified equations have eight constants of integrations which are evaluated in terms of four initial conditions and four matching conditions at $y = y_t$. The initial conditions are

$$x_1(0) = A\ , \qquad \dot{x}_1(0) = 0$$

$$x_2(0) = B\ , \qquad \dot{x}_2(0) = 0$$

or, equivalently ,

$$x_1(0) = A, \qquad \dot{x}_1(0) = 0$$

$$y(0) = x_1(0)-x_2(0) = A-B = C, \text{ and } \dot{y}(0) = \dot{x}_1(0)-\dot{x}_2(0) = 0$$

The matching conditions involve equating displacements and velocities between the $y \leq y_t$ and $y \geq y_t$ regions at the transition amplitude point. Recall that y_t, the transition amplitude, is already known since the slope parameters, k_3 amd k_4, are functions of y_t. In this special case $y_t = \frac{C}{2}$ has been assumed.

$y \geq y_t$ Region:

Using these initial conditions, oscillatory motion is started in the $y \geq y_t$ region and governed by the equations of motion (4.35) and (4.36). By substituting a solution of the form

$$\begin{Bmatrix} u \\ z \end{Bmatrix} = \begin{Bmatrix} S \\ T \end{Bmatrix} \cos \omega t$$

into (4.35) and (4.36), the resulting roots of the frequency equation become

$$\omega_3 = (k_1/m)^{1/2} \text{ and } \omega_4 = [(k_1 + 2k_4)/m]^{1/2}$$

The corresponding amplitude ratios are

$$\beta_1 = \frac{T}{S} = -\left[\frac{k_1 - \omega_3^2 m}{k_4}\right] = 0$$

$$\beta_2 = \frac{T}{S} = -\left[\frac{k_1 - \omega_4^2 m}{k_4}\right] = 2$$

The above solutions are uniquely determined by normalizing the eigen-vectors. The general solution is then represented as

$$\begin{Bmatrix} u \\ z \end{Bmatrix} = \begin{Bmatrix} x_1 - F \\ y - 2F \end{Bmatrix} = a_1 \begin{Bmatrix} V_{11} \\ V_{12} \end{Bmatrix} \cos \omega_3 t + a_2 \begin{Bmatrix} V_{21} \\ V_{22} \end{Bmatrix} \cos \omega_4 t \qquad (4.37)$$

where

$$\omega_3 = (k_1/m)^{1/2} \quad \text{and} \quad \omega_4 = [(k_1 + 2k_4)/m]^{1/2}$$

and

$$V_{11} = (\beta_1^2 + 1)^{-1/2} = 1 \qquad V_{21} = (\beta_2^2 + 1)^{-1/2} = (5)^{-1/2}$$

$$V_{12} = \beta_1 (\beta_1^2 + 1)^{-1/2} = 0 \qquad V_{22} = \beta_2 (\beta_2^2 + 1)^{-1/2} = 2(5)^{-1/2}$$

Using the initial conditions

$$x_1(0) = A, \ y(0) = C$$

the constants a_1 and a_2 are obtained from (4.37) and (4.38)

$$\begin{Bmatrix} x_1(0) - F \\ y(0) - 2F \end{Bmatrix} = \begin{Bmatrix} A-F \\ C-2F \end{Bmatrix} = a_1 \begin{Bmatrix} V_{11} \\ V_{12} \end{Bmatrix} + a_2 \begin{Bmatrix} V_{21} \\ V_{22} \end{Bmatrix}$$

Since $V_{12} = 0$,

$$a_2 = (C-2F)/V_{22} = (5)^{1/2}(C-2F)/2$$

and

$$a_1 = \frac{A-F-a_2V_{21}}{V_{11}} = A - C/2$$

At $t = t_t = t_1$ and from equation (4.37)

$$y_t - 2F = a_1V_{12} \cos \omega_3 t_1 + a_2V_{22} \cos \omega_4 t_1$$

but $V_{12} = 0$, thus

$$\cos \omega_4 t_1 = \frac{y_t - 2F}{C - 2F} \qquad \text{or,} \quad t_1 = \frac{1}{\omega_4} \cos^{-1}\left[\frac{y_t - 2F}{C - 2F}\right] \tag{4.38}$$

Having completely determined the solution for the $y \geq y_t$ region, we are in a position to evaluate x_1, $\dot{x}_1$, y, and $\dot{y}$ at the transition amplitude, y_t. Thus, at $t = t_t = t_1$

$$\begin{Bmatrix} x_1(t_t) \\ y(t_t) \end{Bmatrix} = \begin{Bmatrix} x_t \\ y_t \end{Bmatrix} = a_1 \begin{Bmatrix} V_{11} \\ V_{12} \end{Bmatrix} \cos \omega_3 t_1 + a_2 \begin{Bmatrix} V_{21} \\ V_{22} \end{Bmatrix} \cos \omega_4 t_1 + \begin{Bmatrix} F \\ 2F \end{Bmatrix} \tag{4.39}$$

$$\begin{Bmatrix} \dot{x}_1(t_t) \\ \dot{y}(t_t) \end{Bmatrix} = \begin{Bmatrix} \dot{x}_t \\ \dot{y}_t \end{Bmatrix} = -\omega_3 a_1 \begin{Bmatrix} V_{11} \\ V_{12} \end{Bmatrix} \sin \omega_3 t_1 - \omega_4 a_2 \begin{Bmatrix} V_{21} \\ V_{22} \end{Bmatrix} \sin \omega_4 t_1 \tag{4.40}$$

These values for x_t, y_t, $\dot{x}_t$ and $\dot{y}_t$ will now serve as the initial conditions for the solution in the $y \leq y_t$ region.

<u>$y \leq y_t$ Region:</u> By substituting a solution of the form $\begin{Bmatrix} x_1 \\ y \end{Bmatrix} = \begin{Bmatrix} G \\ H \end{Bmatrix} e^{i\omega t}$ into equation (4.33) and (4.34), the resulting roots of the frequency equation become

$$\omega_5 = (k_1/m)^{1/2} \quad \text{and} \quad \omega_6 = [(k_1 + 2k_3)/m]^{1/2}$$

The corresponding amplitude ratios are

$$\gamma_3 = \frac{H}{G} = \frac{k_1 - m\,\omega_5^2}{k_1 + k_3 - m\omega_5^2} = 0\ , \qquad \gamma_4 = \frac{H}{G} = \frac{k_1 - m\,\omega_6^2}{k_1 + k_3 - m\omega_6^2} = 2$$

The above solutions are uniquely determined by normalizing the eigenvectors. The general solution for this region is then represented by

$$\begin{Bmatrix} x_1 \\ y \end{Bmatrix} = \begin{Bmatrix} W_{11} \\ W_{12} \end{Bmatrix} (b_1 \cos \omega_5 t + b_3 \sin \omega_5 t) + \begin{Bmatrix} W_{21} \\ W_{22} \end{Bmatrix} (b_2 \cos \omega_6 t + b_4 \sin \omega_6 t) \tag{4.41}$$

where $\omega_5 = (k_1/m)^{1/2}$ and $\omega_6 = [(k_1 + 2k_3)/m]^{1/2}$

and

$$W_{11} = (\gamma_3^2 + 1)^{-1/2} = 1 , \qquad W_{21} = (\gamma_4^2 + 1)^{-1/2} = (5)^{-1/2}$$

$$W_{12} = \gamma_3 (\gamma_3^2 + 1)^{-1/2} = 0 , \; W_{22} = \gamma_4 (\gamma_4^2 + 1)^{-1/2} = 2(5)^{-1/2}$$

Using as initial conditions the x_t, y_t, $\dot{x}_t$, and $\dot{y}_t$ previously obtained, the constants of integration b_1, b_2, and b_3 and b_4 can be evaluated.

At t = 0, we have

$$\begin{Bmatrix} x_t \\ y_t \end{Bmatrix} = \begin{Bmatrix} W_{11} \\ W_{12} \end{Bmatrix} b_1 + \begin{Bmatrix} W_{21} \\ W_{22} \end{Bmatrix} b_2$$

$$\begin{Bmatrix} \dot{x}_t \\ \dot{y}_t \end{Bmatrix} = \begin{Bmatrix} W_{11} \\ W_{12} \end{Bmatrix} \omega_5 b_3 + \begin{Bmatrix} W_{21} \\ W_{22} \end{Bmatrix} \omega_6 b_4$$

Solving for the constants we obtain

$$b_2 = (5)^{1/2} y_t/2 \tag{4.42}$$

$$b_1 = x_t - y_t/2 \tag{4.43}$$

$$b_4 = (5)^{1/2}\,\dot{y}_t/(2\omega_6) \tag{4.44}$$

$$b_3 = \frac{1}{\omega_5}\left[\dot{x}_t - \frac{\dot{y}_t}{2}\right] \tag{4.45}$$

Having completely determined the solution for the $y \leq y_t$ region, we can now establish the relationship between the initial amplitudes A and B, or equivalently, A and C. To do this we make use of the fact that for normal mode solutions the masses must simultaneously pass through the equilibrium position.

Thus, at $t = t_2$,

$$x_1(t_2) = 0 \text{ and } x_2(t_2) = 0$$

or equivalently,

$$x_1(t_2) = 0 \text{ and } y_1(t_2) = x_1(t_2) - x_2(t_2) = 0$$

Upon substituting these conditions into (4.41) we obtain

$$\begin{Bmatrix} x_1(t_2) \\ y\ (t_2) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} W_{11} \\ W_{12} \end{Bmatrix} (b_1 \cos \omega_5 t_2 + b_3 \sin \omega_5\ t_2)$$

$$+ \begin{Bmatrix} W_{21} \\ W_{22} \end{Bmatrix} (b_2 \cos \omega_6 t_2 + b_4 \sin \omega_6\ t_2) \tag{4.46}$$

Since $W_{12} = 0$, we have

$$b_2 \cos \omega_6 t_2 + b_4 \sin \omega_6\ t_2 = 0 \tag{4.47}$$

or,

$$\tan \omega_6 t_2 = -\frac{b_2}{b_4} = -\frac{\omega_6 y_t}{\dot{y}_t} \quad \text{and} \quad t_2 = \frac{1}{\omega_6} \tan^{-1}\left[\frac{-\omega_6 y_t}{\dot{y}_t}\right] \tag{4.48}$$

and,

$$0 = W_{11}\ (b_1 \cos \omega_5 t_2 + b_3 \sin \omega_5\ t_2) + W_{21}\ (b_2 \cos \omega_6 t_2 + b_4 \sin \omega_6 t_2)$$

(4.49)

However, the coefficient of W_{21} is zero by equation (4.47).

Substituting x_t, $\dot{x}_t$, and $\dot{y}_t$ from equations (4.39) and (4.40) respectively into equations (4.43) and (4.44) yields

$$b_1 = x_t - \frac{y_t}{2} = \left[A - \frac{C}{2}\right] \cos \omega_3 t_1$$

and

$$b_3 = \frac{1}{\omega_5}\left[\dot{x}_t - \frac{\dot{y}_t}{2}\right] = -\ \frac{\omega_1}{\omega_3}\left[A - \frac{C}{2}\right] \sin \omega_3 t_1 = -\left[A - \frac{C}{2}\right] \sin \omega_3 t_1$$

Thus equation (4.49) becomes

$$0 = \left[A - \frac{C}{2}\right][\cos \omega_3 t_1 \cos \omega_3 t_2 - \sin \omega_3 t_1 \sin \omega_3 t_2]$$

or,

$$0 = \left(A - \frac{C}{2}\right)[\cos \omega_3 (t_1 + t_2)] \qquad (4.50)$$

Thus, either $A - C/2 = 0$ or, $\cos \omega_3 (t_1 + t_2) = 0$

If $A - C/2 = 0$, then $A = -B$ and represents the out-of-phase mode.

The frequency-amplitude relation is then $\omega = 2\pi/\tau = 2\pi/[4(t_1 + t_2)]$

where t_1 and t_2 are given by (4.38) and (4.48), respectively. If $\cos \omega_3(t_1 + t_2) = 0$, then $\omega_3(t_1 + t_2) = \pi/2$ and the period is

$$\tau = 4(t_1 + t_2) = 4[\pi/(2\omega_3)] = 2\pi/\omega_3$$

where $\omega_3 = (k_1/m)^{1/2}$, so this represents the in-phase mode.

These frequency-amplitude relationships are plotted in Figure 4.5 along with Anand's solution obtained previously. This analytic solution shows that only the in-phase and out-of-phase modes are possible. The exact solution to this special case is next presented to resolve this dilemma between the conflicting predictions by Anand's equivalent one-line U.P. approach and the bilinear U.P. approach.

4.3.3 Exact Solution.

To set up this special case so that it can be solved exactly, the method suggested by Rosenberg is applied. The definition of normal modes as applied to nonlinear free vibration conservative systems may be found in Appendix F. Because the system of governing equations are conservative, the sum of the kinetic and potential energy is equal to the total energy for the system and is a constant, Π_0. At the equilibrium position, all of the energy is transferred into kinetic energy. At the maximum amplitudes where the velocity of each mass reverses, all of the energy is transferred into potential energy. Rosenberg has shown that if we start the motion by initially displacing the masses, the trajectory of the masses in the (x_1, x_2) plane traces out a path toward the equilibrium position. If, for a particular set of initial maximum amplitudes for the masses, the path traced out passes through the equilibrium position with all the displacements simultaneously equal to zero, then this curved path represents a modal relation and is one solution curve for the system considered. Equivalently, this same problem can be formulated in reverse by starting the motion with initial velocities specified at the equilibrium position and moving toward the maximum amplitude position. If, by assuming some initial velocity ratio between the masses, the path traced out by a finite difference method using small time steps intersects the $U_o = -U$ curve orthogonally, then the relation defining this path is called a modal relation. As already mentioned this modal relation is, in general, a curved (not straight) path. A finite difference method is formulated after this latter approach to arrive at the exact solution. This method is applied to the special case problem solved previously by the two approximating methods in Sections 4.3.1 and 4.3.2.

The finite difference method is programmed on the digital computer using small time steps, $\Delta t = .01$. Taking advantage of the symmetry of the potential function, the entire x_1x_2U- space can be represented by considering just the region between $\theta = -45^o$ and 45^o. The first step toward finding all possible normal modes for the x_1x_2U- space is to sweep through angles between -45^o and 45^o in the (x_1,x_2) plane for preset total energy valves. Specifying the total energy is equivalent to specifying the maximum amplitudes A and B for x_1 and x_2 respectively. This sweeping procedure isolates where possible modal relations exist. This is done by comparing two slope terms. One slope term is the local ratio, $\Delta x_2/\Delta x_1$, which when parallel to the gradient of the energy curve yields a modal relation. The other slope term is the slope at the corresponding point tangent to the total energy ellipse and is used as a check on the orthogonality between $\Delta x_2/\Delta x_1$ and the tangent to energy curve. When these slope parameters are orthogonal, the corresponding x_1 and x_2 values represent one point on the modal curve.

The procedure leading to a normal mode solution is to select initial velocities (V_{10} and V_{20}) at the equilibrium position (origin in Figure 4.6) in the x_1 and x_2 directions where $x_{10} = 0$ and $x_{20} = 0$. Then Δx_1 and Δx_2 are calculated using these initial velocities and the time step as

$$x_1 = V_{10}(\Delta t) \qquad \text{and } x_2 = V_{20}\ (\Delta t)$$

At point 1, a new x_1 and a new x_2 are calculated as

$$x_1 = x_{10} + \Delta x_1 \qquad \text{and } x_2 = x_{20} + \Delta x_2$$

The corresponding velocity changes are

$$\Delta V_1 = \ddot{x}_1\ (\Delta t) = \frac{1}{m}\left[\frac{\partial U(x_1,x_2)}{\partial x_1}\right](\Delta t)$$

and

$$\Delta V_2 = \ddot{x}_2(\Delta t) = \frac{1}{m}\left[\frac{\partial U(x_1,x_2)}{\partial x_2}\right](\Delta t)$$

Thus, the new velocities at point 1 become

$$V_1 = V_{10} + \Delta V_1$$

$$V_2 = V_{20} + \Delta V_2$$

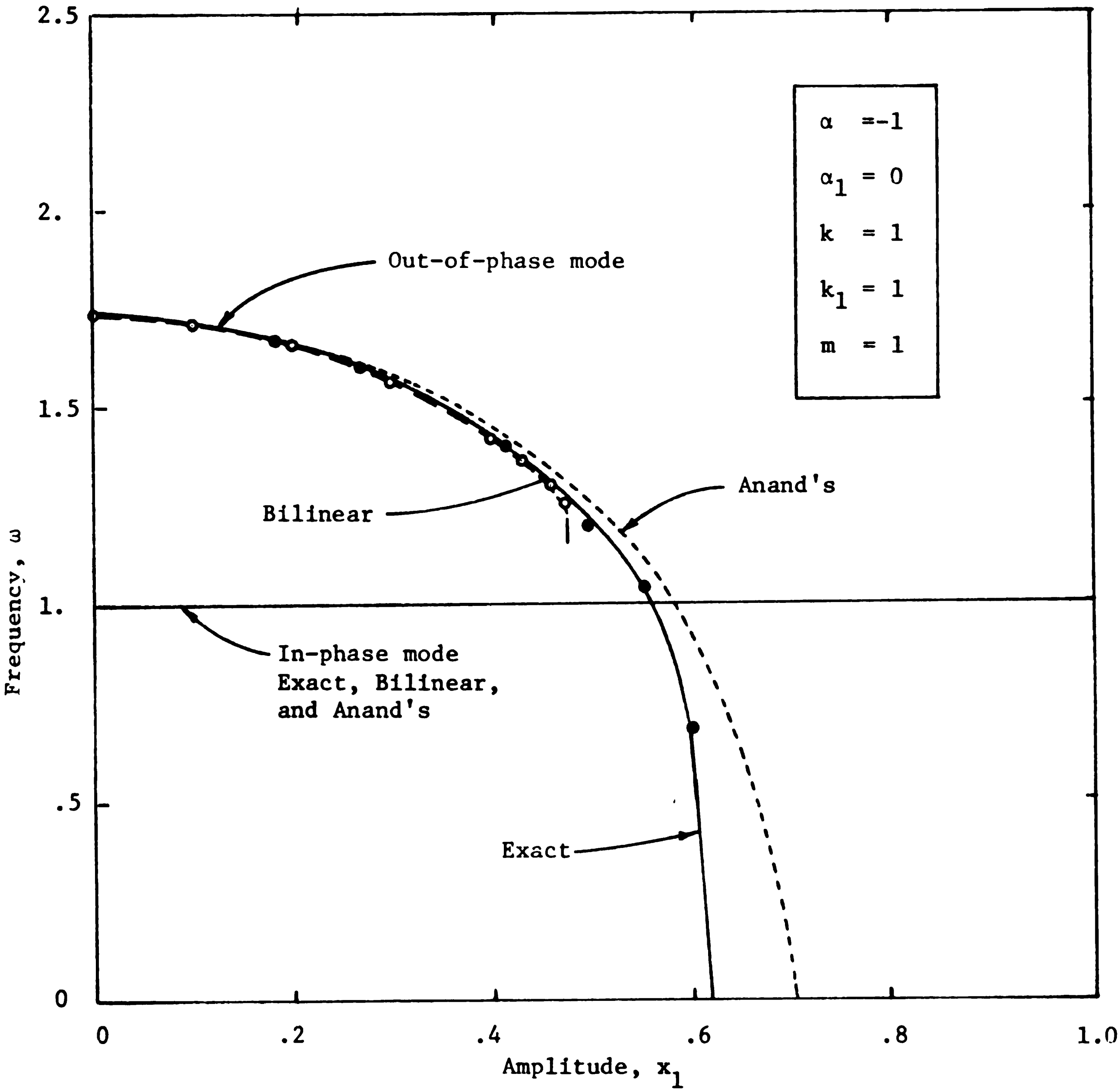


Figure 4.5 Frequency versus Amplitude Relation (α=-1, α_1=0)

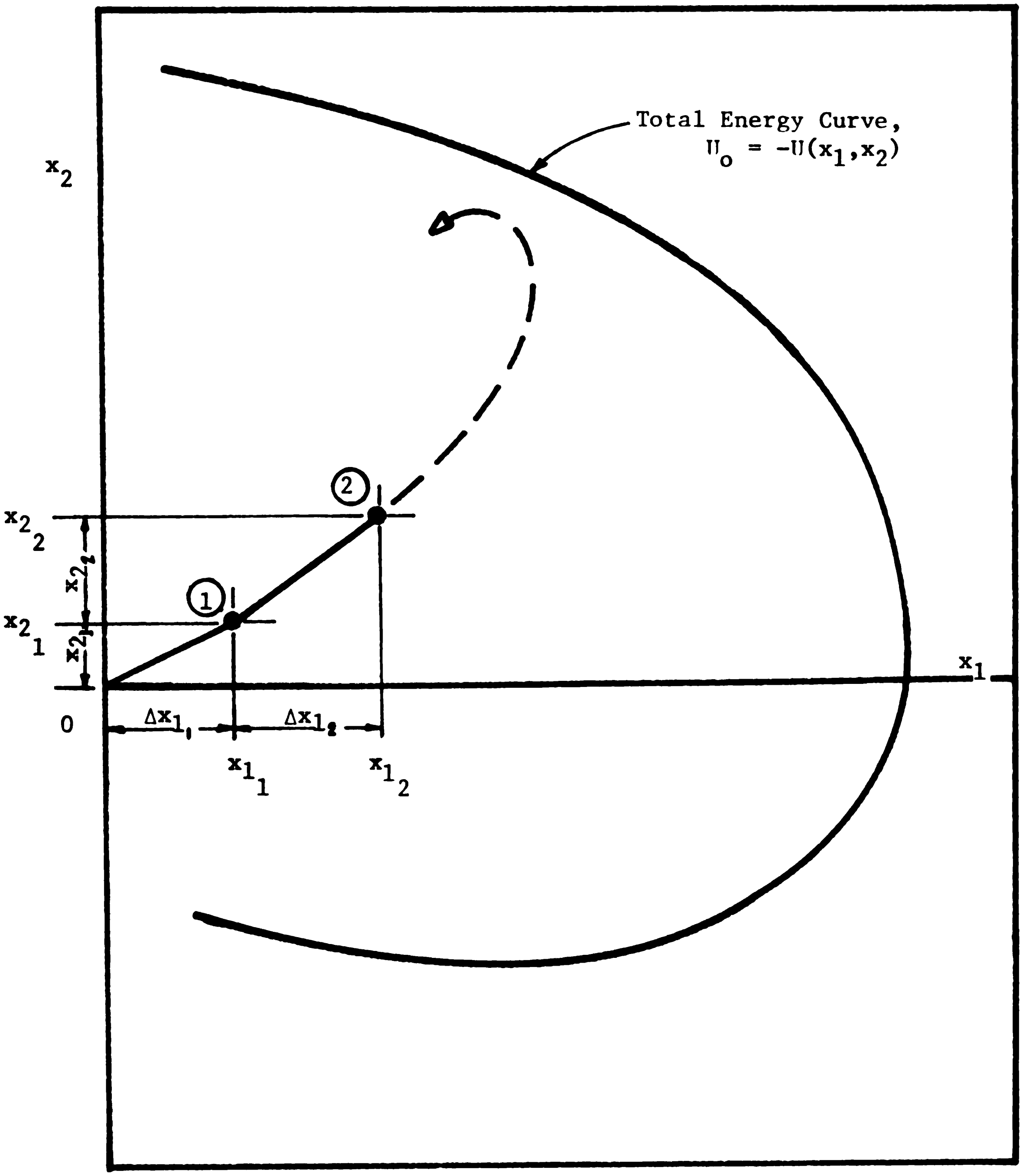


Figure 4.6 Total Energy Curve Represented in $x_1 x_2 U$-Space

At this point a check is made to determine whether V_1 has changed sign. If not, use the values at point 1 and repeat the above steps and proceed point by point until the velocities of the masses change sign. When the velocities do change sign another check is made to see whether the two slope terms are orthogonal. This procedure is repeated for the entire x_1x_2U- space, identifying all possible modes that satisfy the orthogonality requirement between the two slope terms mentioned previously. The exact frequency-amplitude results for this special case are given in Figure 4.5.

4.4 Superabundant Modes

Rosenberg shows in [15] that more than two normal modes exist for a two degree-of-freedom conservative, symmetric, homogeneous system. As explained in Appendix F a homogeneous system here is defined as one in which all of the nonlinear restoring forces have the same degree of nonlinearity. For example, such a system shown in Figure F.1 (Appendix F) with degree three, where

$$S_1 = a_3 x_1^{\,3}$$

$$S_2 = A_3(x_1 - x_2)^3$$

$$S_3 = a_3 x_2^{\,3}$$

Anand [18] has similarly shown with his approximate method the existence of the asymmetric mode. However, while Anand's problem is symmetric he allows the linear terms in the restoring forces.

Figure F.1 thus also represents Anand's system, except that the restoring forces now include linear terms in addition to cubic terms. Therefore, for Anand's system,

$$S_1 = k_1 x_1 + m\alpha_1 x_1^3$$

$$S_2 = k(x_1 - x_2) + m\alpha(x_1 - x_2)^3$$

$$S_3 = k_1 x_2 + m\alpha_1 x_2^3$$

In this section exact modal relations of the frequency-amplitude dependence are formulated for various coefficients using Rosenberg's approach as outlined in Section 4.3.3. In addition, a discussion of the limiting case of Anand's problem for large values of U_o is presented. To recreate the asymmetric modes predicted by Anand, coefficients of the restoring forces used by him are also used in the exact formulation. It will be shown that these exact modal relations will start from values predicted by Anand for small amplitudes where the linear term is predominant and approach limiting values predicted by Rosenberg as the amplitudes increase, where the cubic term is predominant.

Anand's coefficients of the restoring forces will also serve as a point of departure to obtain frequency-amplitude curves by the exact method. Figure 4.7 below represents one case where Anand discovered the asymmetric mode (Figure 4.7 appears as Figure 2 in [18]). Also, from the $\alpha\alpha_1$ graph (Figure 4.2) this case falls within the regions where the asymmetric mode is predicted. Thus, the values $\alpha = 0.32$, $\alpha_1 = 1.6$, $k_1 = .896$, $k = .1536$, and $m = 1$ are used in the exact formulation of Case I below. Besides Case I three additional cases are considered. For these cases only the α term is varied and the other terms α_1, k_1, k, and m are held fixed.

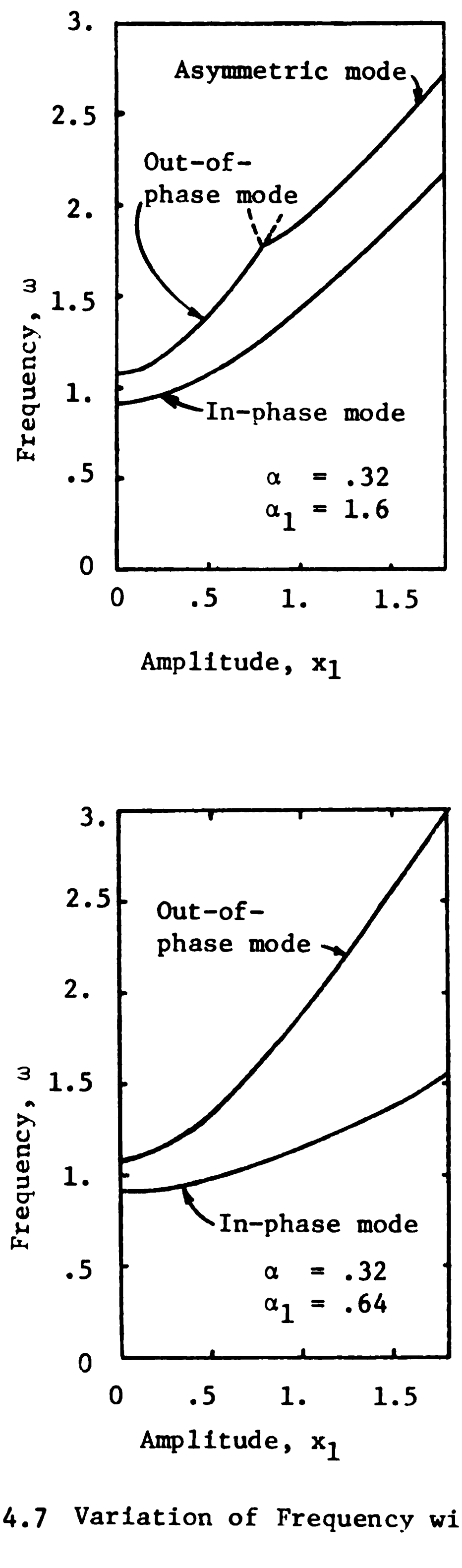


Figure 4.7 Variation of Frequency with Amplitude.

Case I (α=.32). This case represents a spring-mass system with the outboard and the inboard springs being the hard cubic type. Figure 4.8 represents the exact frequency-amplitude curves for these coefficients. In addition, the corresponding nonlinear relation between the amplitudes of vibration of the masses is given in Figure 4.9. As shown in Figure 4.9, the in-phase mode occurs along the $\theta = 45^o$ line and the out-of-phase mode occurs along the $\theta = -45^o$ line. It is of interest to note in this case that for absolute values of the amplitudes less than 0.8, only two normal modes are possible. However, for absolute values of the amplitudes greater than 0.8, the asymmetric mode branches off the out-of-phase mode. As Figure 4.2 shows, Anand does predict correctly that the amplitude could not be too small for the asymmetric mode to occur. Also, as the amplitude increases the cubic term of the restoring force is predominant and the asymmetric mode approaches a limiting value, $\theta = -20.9^o$, which is found by substituting the $m\alpha_1$ for α_3 and m for A_3 in equation (F.8).

Case II(α=0). This case represents a spring-mass system with the outboard springs being the hard cubic type and the inboard spring being linear. The exact frequency-amplitude curves are given in Figure 4.10 and the relationship between amplitudes in Figure 4.11. For absolute values of the amplitudes less than 0.36 only two normal modes exist, namely, the in-phase mode and the out-of-phase mode. However, for absolute values of the amplitudes greater than 0.36 the asymmetric mode branches off the out-of-phase mode. As Figure 4.2 shows, Anand does not correctly predict the exact result. As the absolute values of the amplitudes increase the asymmetric mode approaches the limiting value, $\theta=0^o$, again found by substituting the above coefficients into equation (F.8).

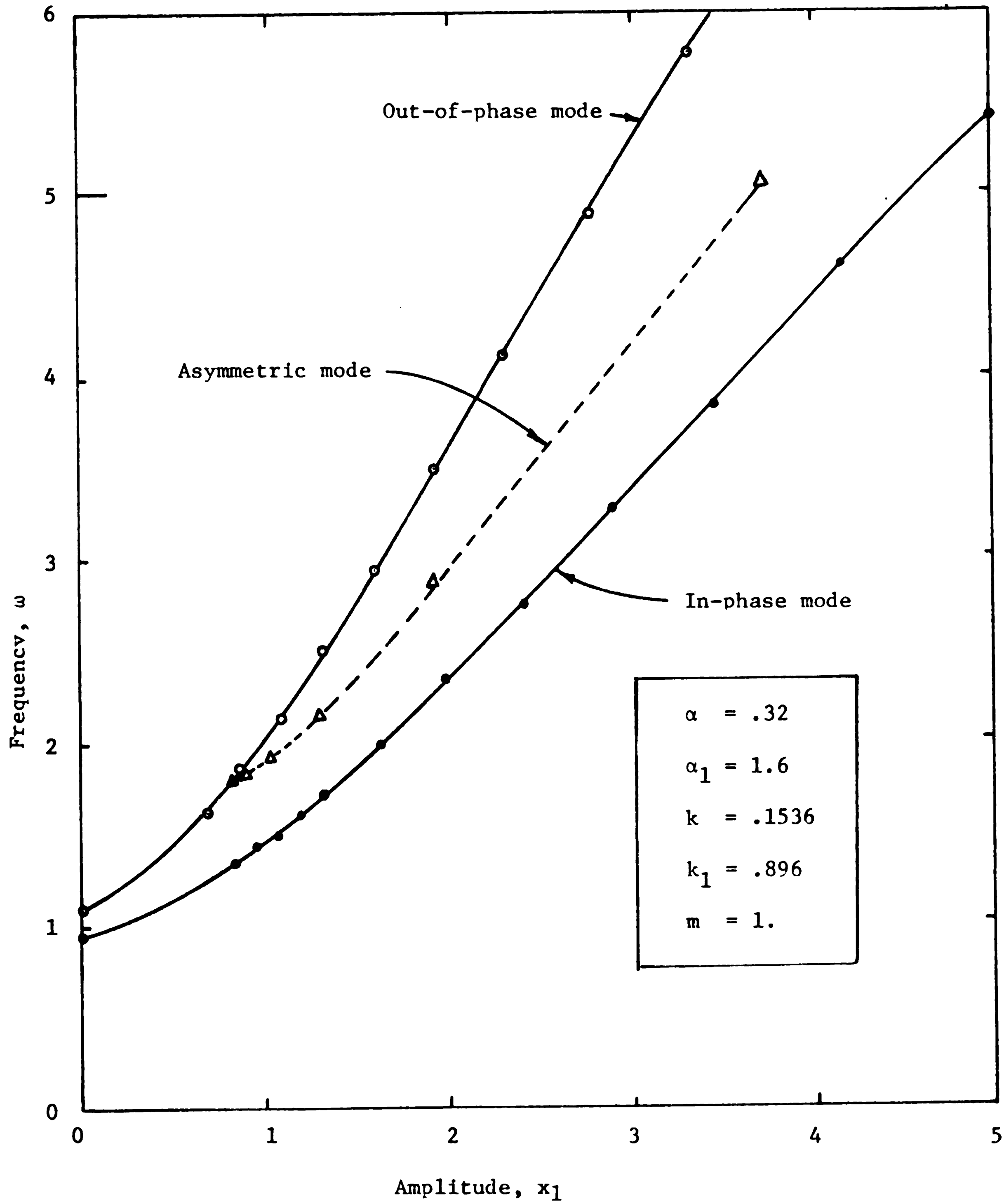


Figure 4.8 Frequency versus Amplitude Relationship (α = .32)

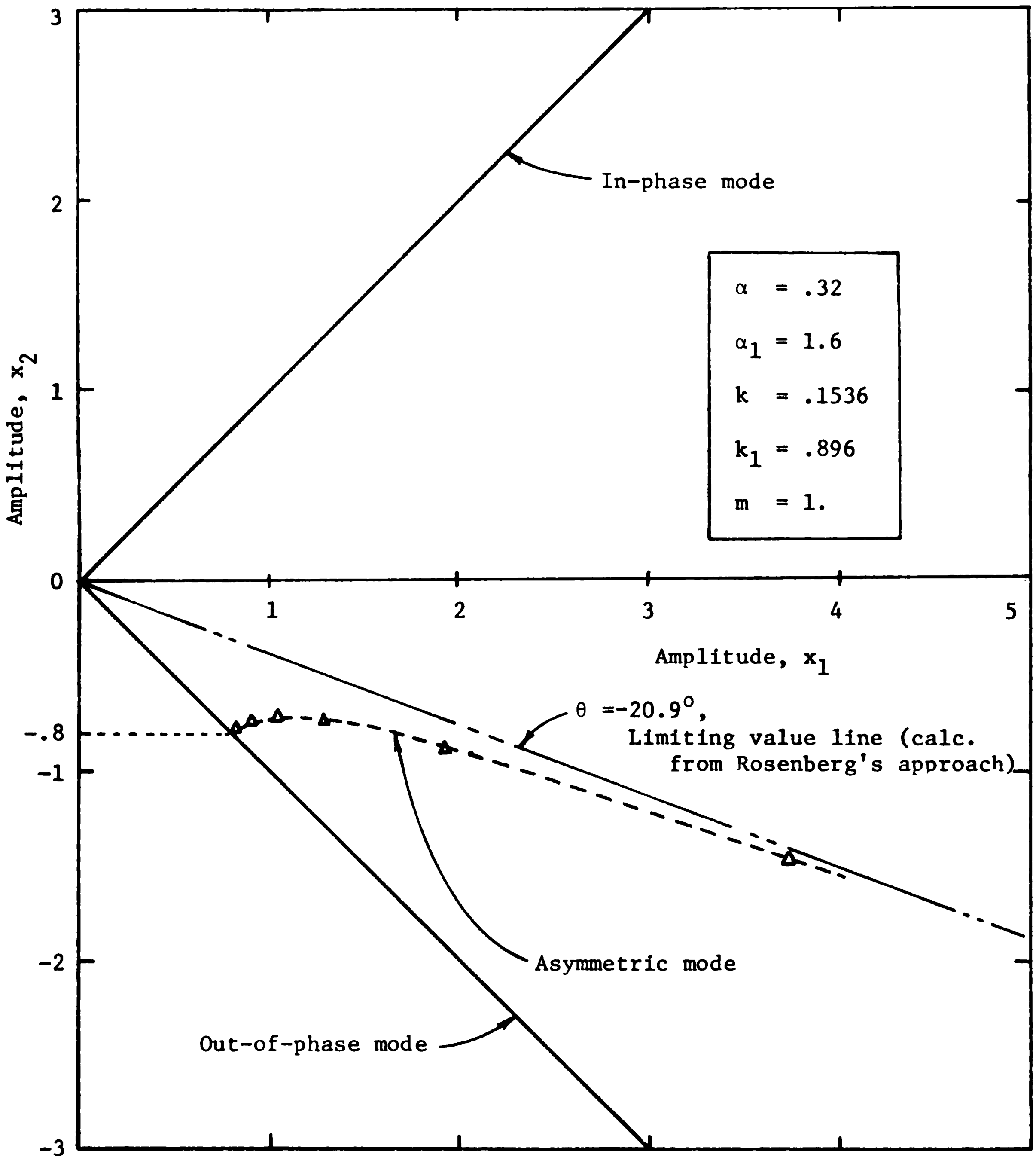


Figure 4.9 Relation Between Amplitudes of Vibration of the Masses, α= .32

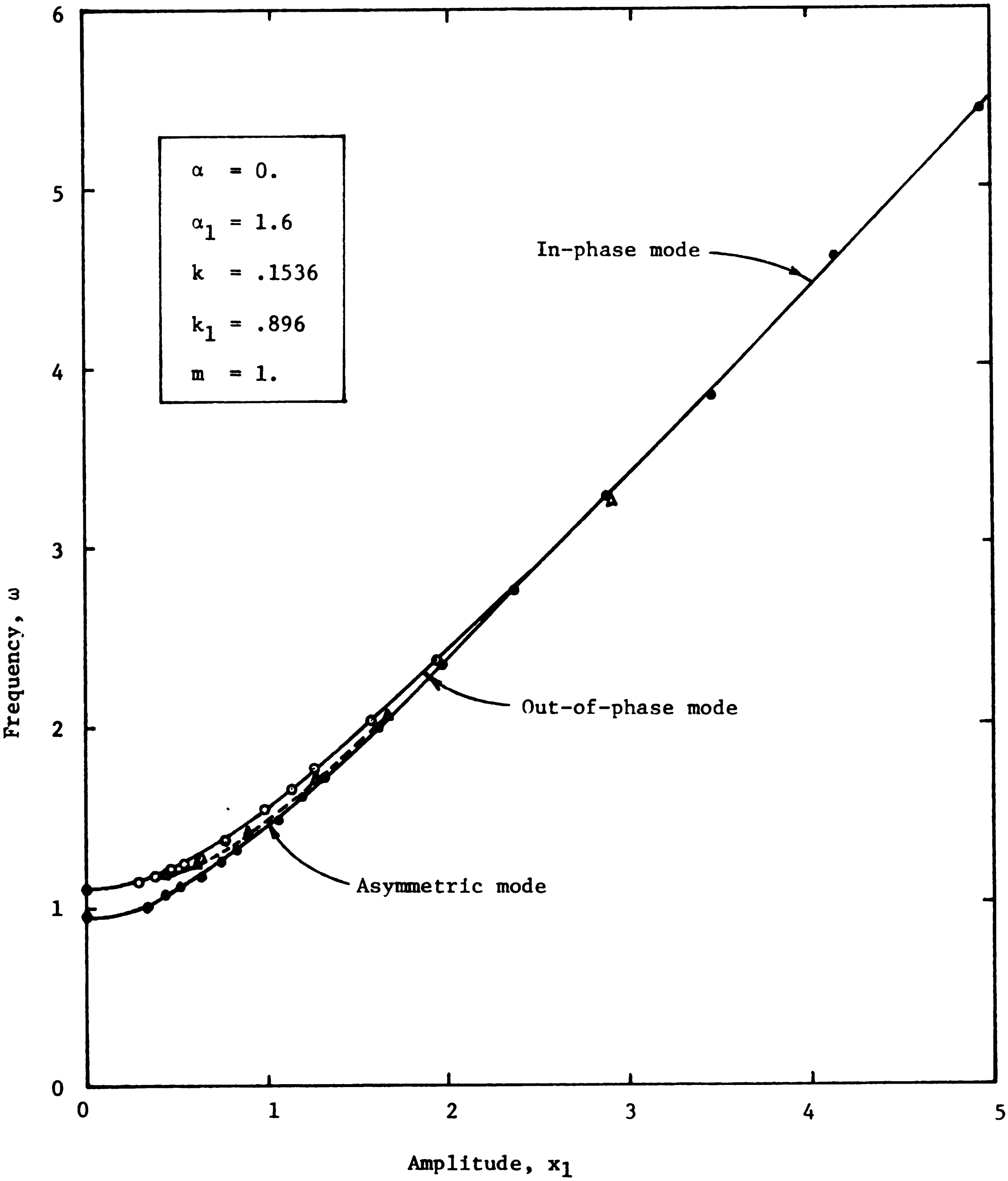


Figure 4.10 Frequency versus Amplitude Relationship (α = 0)

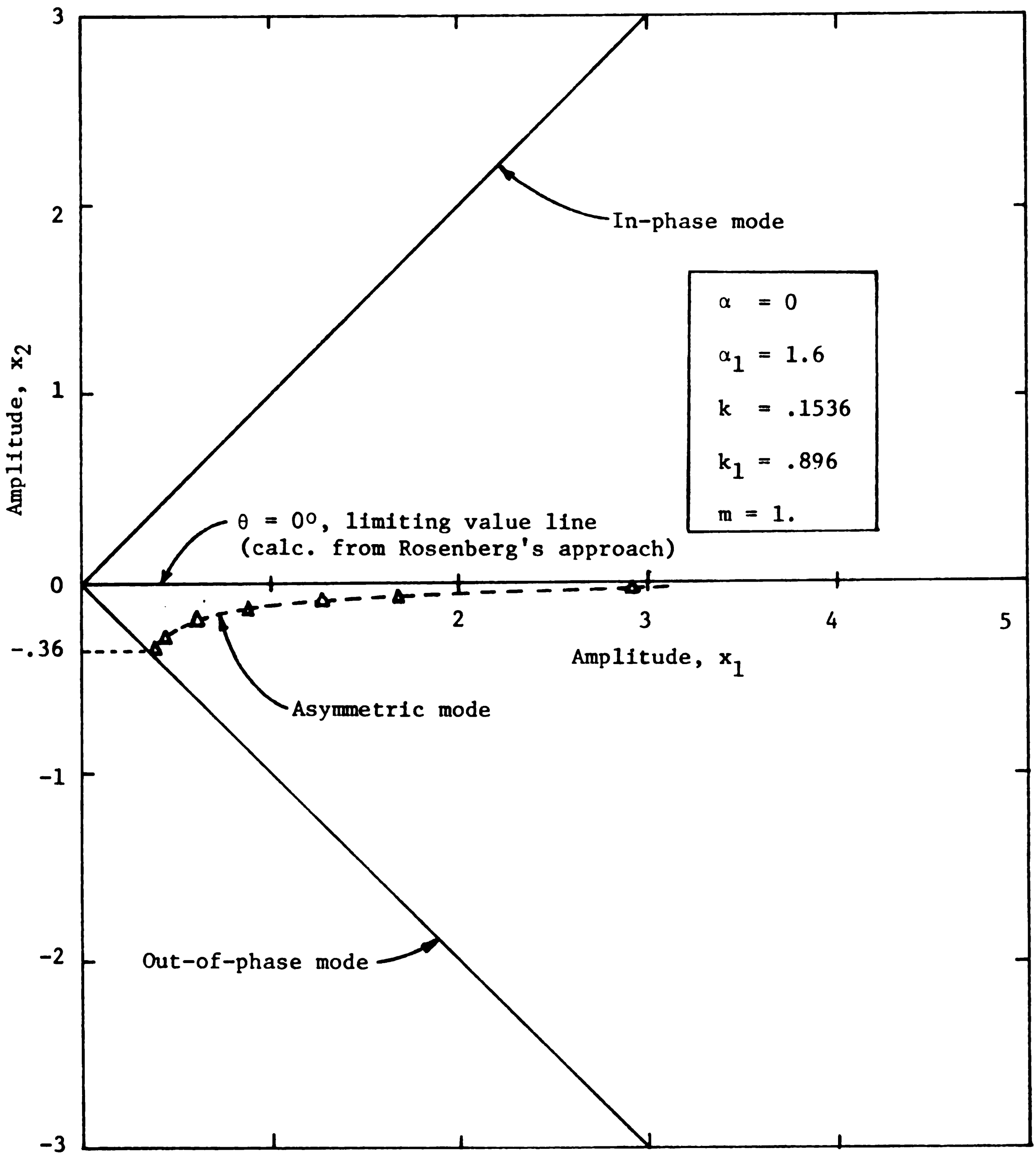


Figure 4.11 Relation Between Amplitudes of Vibration of the Masses, α= 0

Case III (α=-.2). This case represents a spring-mass system with hard, cubic outboard springs and soft, cubic inboard spring. The exact frequency amplitude curves are represented in Figure 4.12 and the relation between amplitudes in Figure 4.13. For absolute values of the amplitudes less than 0.29 only two normal modes are present—again, the in-phase and out-of-phase modes. For absolute values of the amplitudes greater than 0.29 the asymmetric mode branches again off the out-of-phase. Again, as Figure 4.2 shows, Anand predicts that the amplitude is not restricted, which, of course, does not agree with the exact result. As the absolute values of the amplitudes increase the asymmetric mode approaches limiting value, $\theta = 5.77^{\circ}$, before the amplitude becomes unbounded.

Case IV (α=-.3). This case represents a spring-mass system with hard, cubic outboard springs and soft, cubic inboard spring. The exact frequency-amplitude curves are represented in Figure (4.14) and the relationship between amplitudes in Figure (4.15). For absolute values of the amplitudes less than 0.27 only two normal modes are present—again, the in-phase and out-of-phase modes. For absolute values of the amplitudes greater than 0.27 the asymmetric mode branches off the limiting value, $\theta = 7.9^{\circ}$, before the amplitude becomes unbounded. Again, as Figure 4.2 shows, Anand's prediction does not agree with the exact result.

Limiting Cases. In this subsection we confine our discussion to Anand's system shown in Figure 4.1, with the restoring forces defined by [4.1]. We show that Rosenberg's homogeneous system with degree three does serve as a limiting case of Anand's system for large amplitudes.

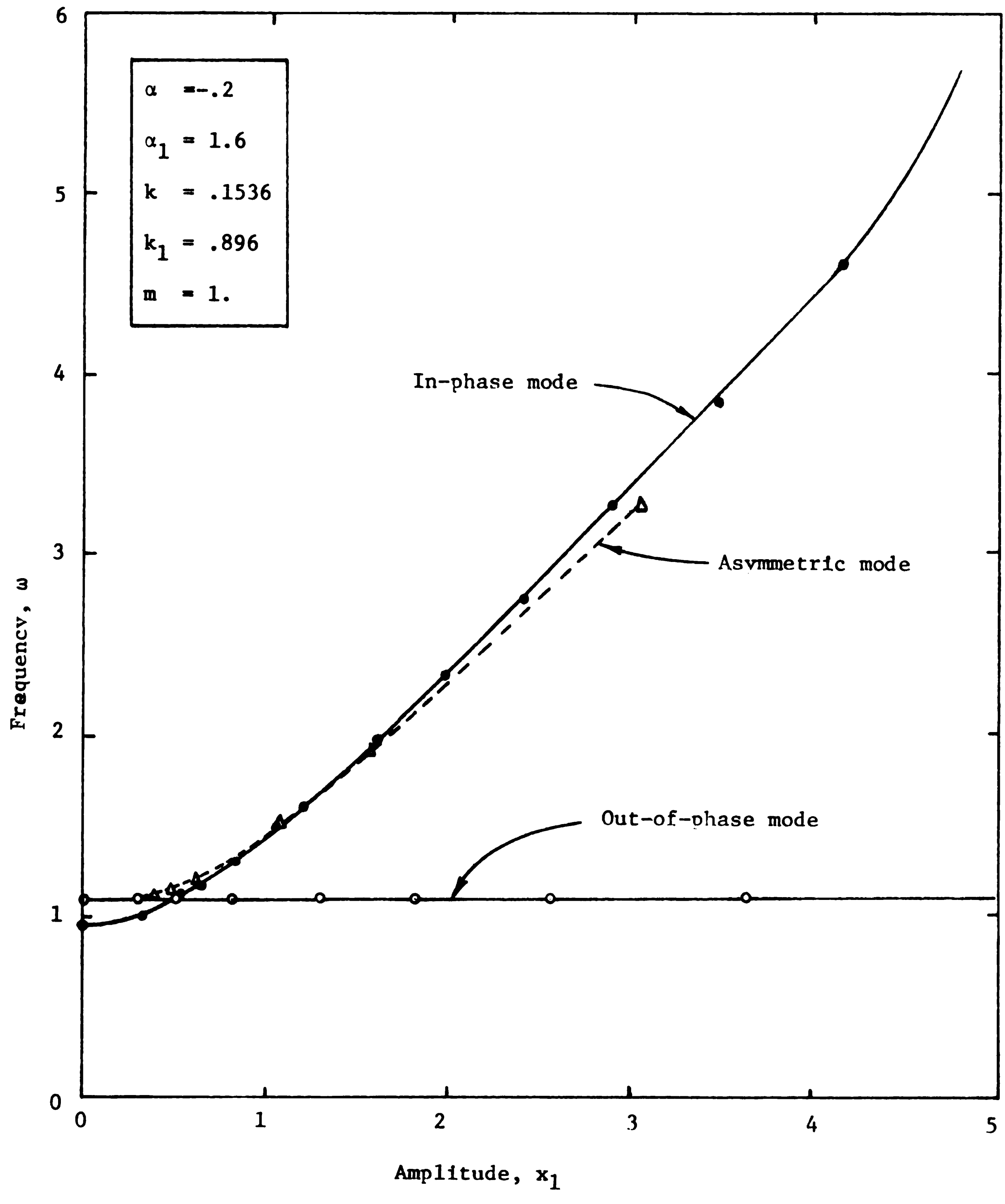


Figure 4.12 Frequency versus Amplitude Relationship (α =-.2)

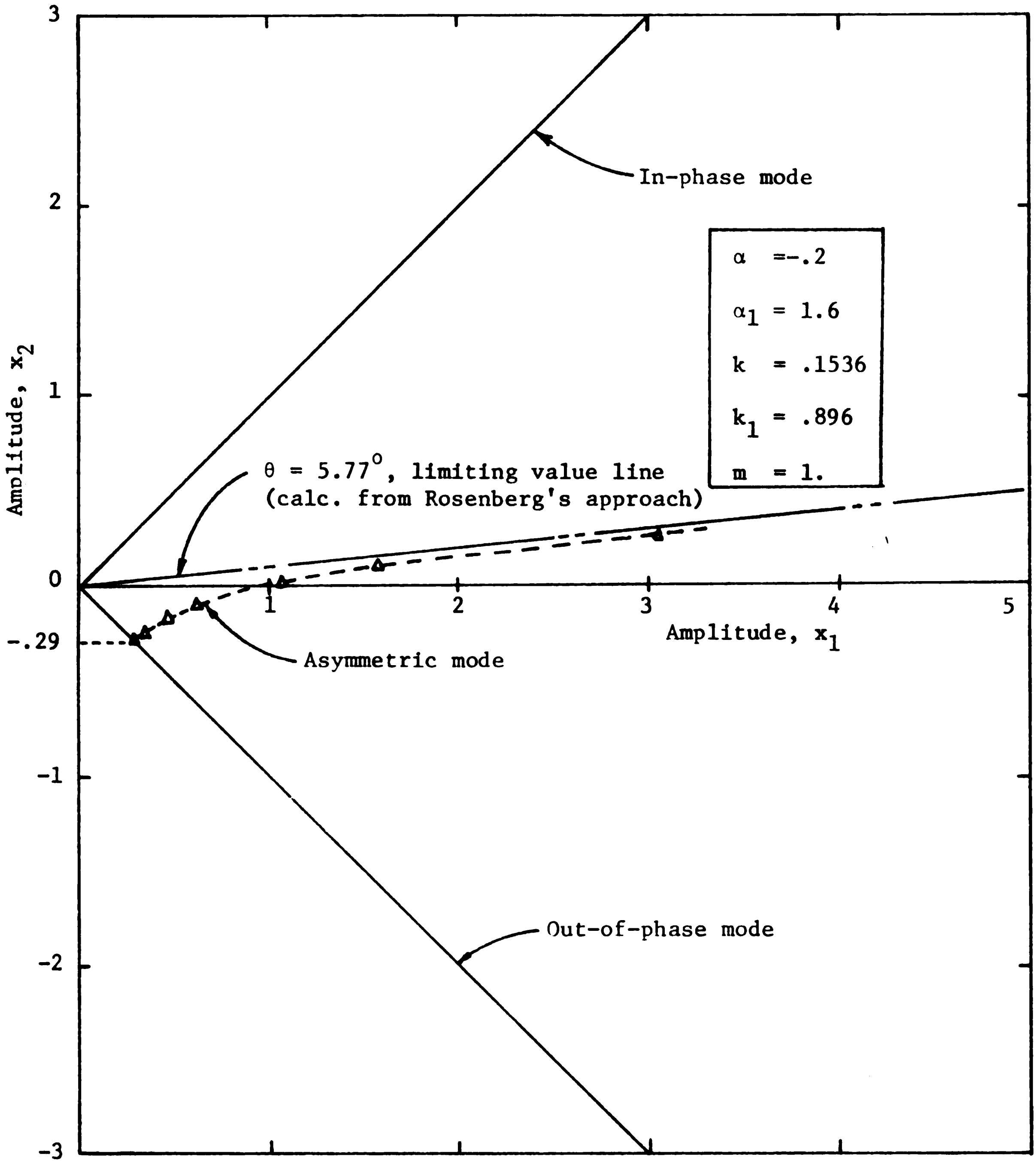


Figure 4.13 Relation Between Amplitudes of Vibration of the Masses, α=-.2

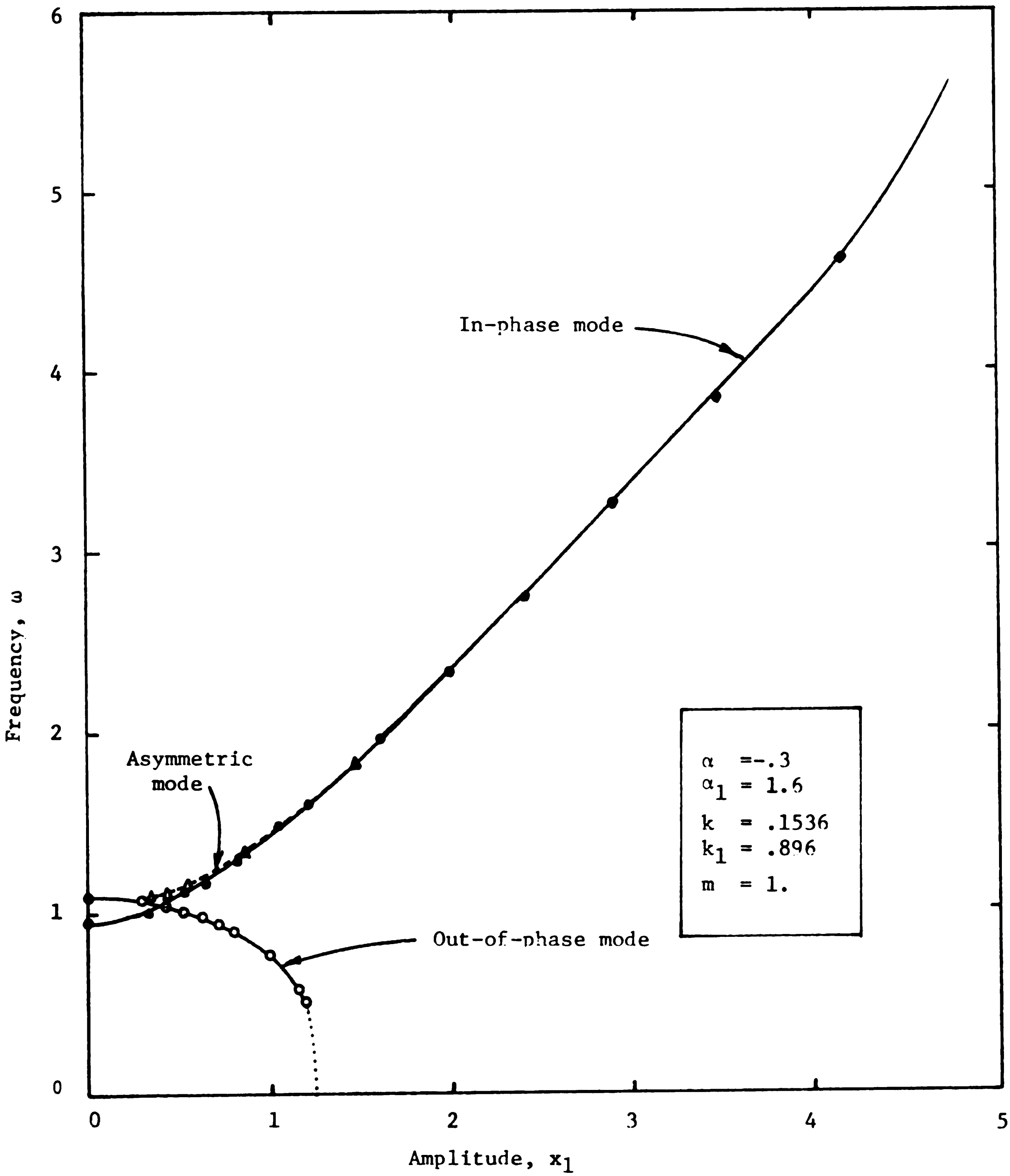


Figure 4.14 Frequency versus Amplitude Relationship (α =-.3)

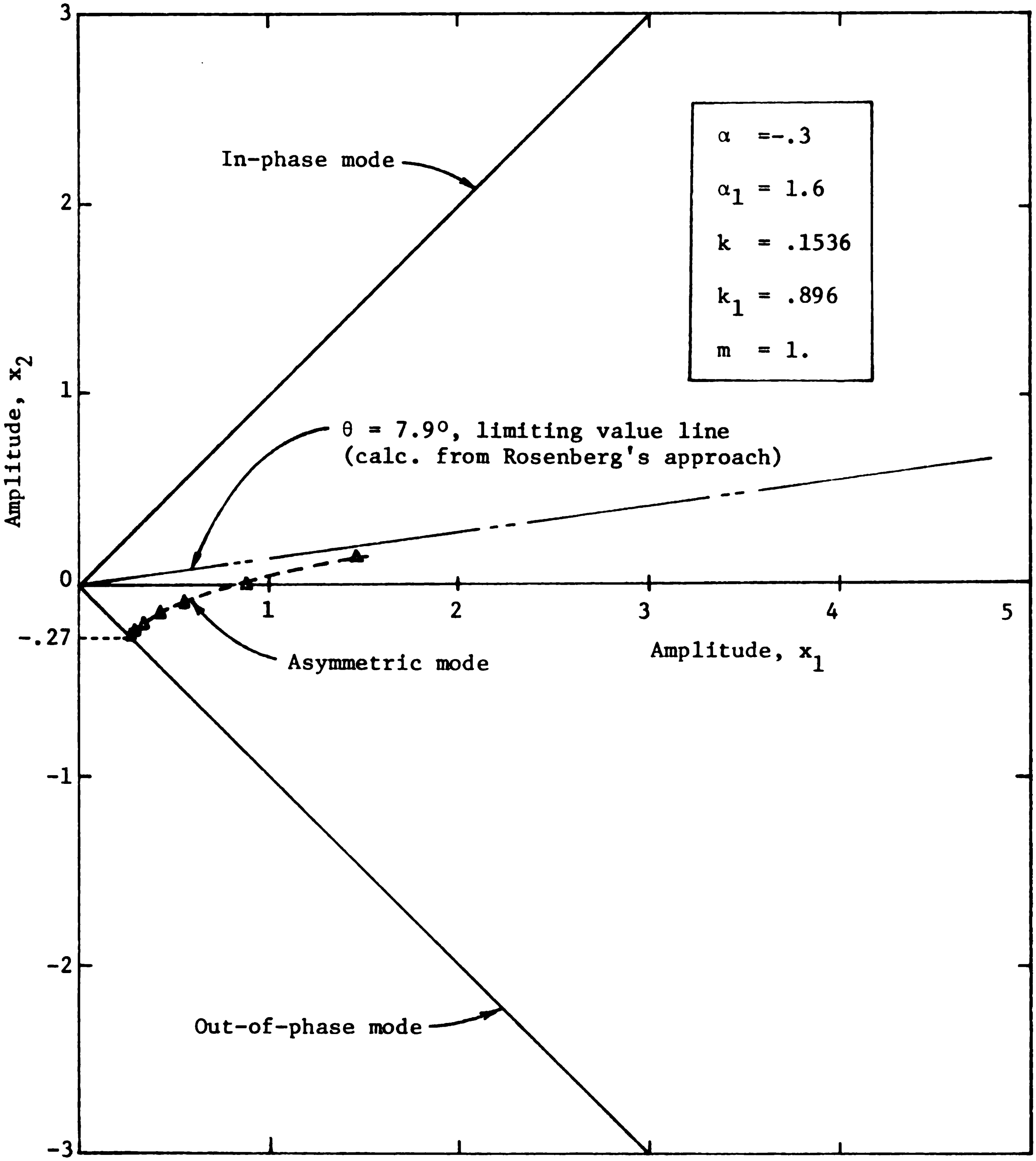


Figure 4.15 Relation Between Amplitudes of Vibration of the Masses, α=-.3

To show this Rosenberg's polar coordinate formulation of the homogeneous system is summarized in Appendix F. Likewise, Anand's system is reformulated in terms of polar coordinates, is analyzed for large amplitudes, and is compared with Rosenberg's homogeneous system.

Polar Coordinates—Anand's System. The equations of motion for this system are given by (4.2) and (4.3) or, equivalently by (F.1) and (F.2) where

$$m = m_1 = m_2$$

and

$$-U = -U(x_1,x_2) = \frac{a_1}{2}(x_1^2+x_2^2) + \frac{a_3}{4}(x_1^4+x_2^4) + \frac{A_1}{2}(x_1-x_2)^2 + \frac{A_3}{4}(x_1-x_2)^4 \tag{4.51}$$

with

$$a_1 = k_1, \quad a_3 = m\alpha_1, \quad A_1 = k, \quad \text{and} \quad A_3 = m\alpha$$

By introducing the polar coordinates

$$x_1 = r\cos\theta \quad \text{and} \quad x_2 = r\sin\theta$$

into (4.51) the potential function $-U$ becomes

$$-U = -U(r,\theta) = \frac{a_1 r^2}{2} + \frac{a_3 r^4}{4}(\cos^4\theta + \sin^4\theta) + \frac{A_1 r^2}{2}(\cos\theta - \sin\theta)^2 + \frac{A_3 r^4}{4}(\cos\theta - \sin\theta)^4 \tag{4.52}$$

Since the total energy U_o for this conservative system is constant, the potential energy $-U$ equals U_o at the maximum amplitude position; and the kinetic energy equals U_o at the equilibrium position. Thus, the derivative of (4.52) at the maximum amplitude yields

$$dU_o = \frac{\partial U}{\partial r} dr + \frac{\partial U}{\partial \theta} d\theta = 0$$

or,

$$\frac{dr}{d\theta} = -\frac{\partial U}{\partial \theta} \bigg/ \frac{\partial U}{\partial r}$$

In this case the modal relations are, in general, not straight but rather curved. However, as Rosenberg shows (Appendix F) the vanishing of the derivative of (4.52) at the maximum amplitude still defines a normal mode provided $r = r(\theta)$ intersects $-U = U_o$ orthogonally. Hence, $dr/d\theta = 0$ implies that $\partial U/\partial\theta = 0$. Thus taking the derivative (4.52) with respect to θ yields

$$\begin{aligned} \partial U/\partial\theta = \Big\{ & a_3[\cos^3\theta \sin\theta - \sin^3\theta \cos\theta] \\ & + A_3[(\sin\theta + \cos\theta)(\cos\theta - \sin\theta)^3] \\ & + A_1/r^2\,[(\cos\theta - \sin\theta)(\sin\theta + \cos\theta)] \Big\}\, r^4 \end{aligned} \tag{4.53}$$

After some algebraic and trigonometric manipulations (4.52) reduces to

$$\frac{\partial U}{\partial\theta} = 0 = A_3 r^4 \cos 2\theta \Big([a_3/(2A_3) - 1] \sin 2\theta + 1 + \frac{A_1}{A_3 r^2} \Big) \tag{4.54}$$

Equation (4.54) reduces to (F.7) when either $A_1 = 0$ (the linear coefficient of the coupling spring) or, the $A_1/(A_3r^2)$ term is small in comparison to the $[[a_3/(2A_3) -1] \sin 2\theta + 1]$ term. In this latter case, taking the limit of the bracketed term in (4.53) we obtain

$$\lim_{r\to\infty}\left[\left(\frac{\alpha_3}{2} -1\right) \sin 2\theta + 1 + \frac{A_1}{A_3r^2}\right] = \left[\left(\frac{\alpha_3}{2} -1\right) \sin 2\theta + 1\right]$$

Hence, for a positive definite potential function ($a_3/A_3 > 0$ or, $a_3/A_3 = 0$ and $A_3 > 0$) and for large amplitudes we observe that Anand's result (4.54) reduces to Rosenberg's result (F.8).

While the above discussion does not constitute a rigorous proof, we will now arrive at the same conclusion by another method that is suggested by Rosenberg.

Rosenberg shows in [14] that the solution $x_2 = x_2(x_1)$ of the system defined by (F.1), (F.2) and (4.51) is completely equivalent to finding the geodesics in the potential energy surface. The potential energy surface is a surface in the x_1x_2U-space defined by the function U. The geodesics are solutions to the differential equation

$$2(U_o + U)\eta'' + [1 +(\eta')^2][\eta' U_\xi - U_\eta] = 0 \tag{4.55}$$

where

$$\eta = (m)^{1/2} x_2 \qquad \xi = (m)^{1/2} x_1$$

$$U_\eta = \partial U/\partial\eta \qquad U_\xi = \partial U/\partial_\xi$$

and, where U_o is the total energy and U given by (4.51).

When $\eta'' = \frac{d^2\eta}{d\xi^2} = 0$, (4.55) reduces to

$\eta' = U_\eta/U_\xi$ = constant

Thus, as Rosenberg says, every straight line which intersects all lines of constant potential energy orthogonally is a modal relation since it satisfies (4.55) as well as the definition of normal mode by Rosenberg (Appendix F).

Rosenberg shows in [14] that homogeneous systems of degree one (U quadratic) and of degree three (U quartic) yield straight-line modal relations. Now, we intend to show how the potential [4.51] approaches a straight modal relation for large U_o. Thus, we split U into the quadratic and the quartic terms to obtain

$$U = U_1 + U_2 \tag{4.56}$$

where

U_1 = quadratic terms

and

U_2 = quartic terms

Dividing (4.55) by U_o and collecting terms using (4.56) we obtain

$$2\left(1 + \frac{U_1}{U_o} + \frac{U_2}{U_o}\right)\eta'' + [1 + (\eta')^2]\left[\frac{\eta' \frac{\partial U_1}{\partial \xi} - \frac{\partial U_1}{\partial \eta}}{U_o} + \frac{\eta' \frac{\partial U_2}{\partial \xi} - \frac{\partial U_2}{\partial \eta}}{U_o}\right] = 0 \tag{4.57}$$

For large U_o, $U_1 << U_2$ irregardless of the η' value. Then, (4.57) can be approximated by

$$2\left(1 + \frac{U_2}{U_o}\right)\eta'' + [1 + (\eta')^2]\left[\frac{\eta' \frac{\partial U_2}{\partial \xi} - \frac{\partial U_2}{\partial \eta}}{U_o}\right] = 0 \tag{4.58}$$

But (4.58) is just (4.55) with $U = U_2$. Thus, for large U_o we see that this case is homogeneous and of degree three. Similar arguments hold for small U_o, $U_1 >> U_2$.

V. SUMMARY AND CONCLUSIONS

Vibrations of discrete systems outside of the classical linear domain are no longer independent of amplitude. Since exact solutions to such nonlinear systems are, in general, difficult to obtain, approximate methods gain favor. An approximate method using ultraspherical polynomials (U.P.) was developed in this research and used to obtain approximate solutions to certain one and two degree-of-freedom vibration problems. The nonlinearity was assumed in the form of a restoring force. The nonlinear restoring force was approximated by expanding it in terms of ultraspherical polynomials orthogonal over the interval (-1,1) with respect to the appropriate weight function, and truncating it after the linear term. The general development of the U.P. approximation presented in this research involved calculating two bilinear approximations over some appropriate interval containing the equilibrium point.

The mean square error minimization method was also used to generate bilinear approximations. Relations were obtained which showed that $\lambda = -.5$ was one of two conditions necessary for the two methods to agree. However, this condition was shown to be merely a linear interpolation between points along the nonlinear function. In the remainder of the research the polynomial expansion method was used because of its greater flexibility in specifying the λ parameter. This method was then applied to three odd functions to illustrate the procedure leading to their U.P. bilinear approximation.

The bilinear U.P. method was used to approximate the motion of the one degree-of-freedom, undamped, free and forced, nonlinear vibrating systems. For the free vibration problem the previously obtained bilinear approximations to certain nonlinear restoring forces were used. Period-amplitude relations were graphed and revealed, in general, improved accuracy over the linear U.P. approximation. A λ parameter of $\lambda=0$ (Chebyshev polynomial) and an amplitude ratio of $x_t/x_m=.5$ were found to consistently yield the better results.

For the forced vibration problem only one case was investigated. This problem included the function $f(x) = x + x^3$ as the restoring force and the unit step-function as the exciting force. Little improvement over the linear U.P. method was obtained by using the bilinear U.P. method.

The bilinear U.P. method was then applied to the two degree-of-freedom free, undamped, symmetric nonlinear vibrating system. Anand [18] recently investigated this system using cubic restoring forces and found more than the usual two normal modes of vibration present. Normal modes for nonlinear systems were defined in the sense of Rosenberg [17]. In this research Anand's approach has been shown to be essentially the linear U.P. approximate approach. Having found this, a special case was solved by the bilinear U.P. method and its frequency-amplitude results were compared with Anand's result. Where Anand's formulation predicted the existence of this additional mode (asymmetric mode) the bilinear U.P. method denied its existence. To resolve this dilemma an exact solution was devised using a finite difference method modeled after an approach by Rosenberg [14]. The exact solution showed the bilinear U.P. prediction to be correct.

The finite difference method was applied to other cases so that the exact result could be compared to Anand's predictions. In some cases Anand's criterion did correctly predict the existence of the asymmetric mode. However, in other cases the predictions were incorrect. For the above system Rosenberg [14] has provided a more predictable method for determining these asymmetric modes.

The finite difference method developed in this research has provided a means of linking together the works of Anand and of Rosenberg, as applied to the existence of asymmetric modes.

Several avenues of research suggested by this study are as follows. The bilinear ultraspherical polynomial method could be applied to other forced vibrating systems possibly with damping present. For the symmetric two degree-of-freedom system considered here, an extension would be to generate analog computer solution solutions to compare with the digital computer results. Unsymmetric systems could also be investigated and might yield very fruitful results in understanding other unusual nonlinear phenomenon.

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TYPE POLYNOMIAL	ULTRASPHERICAL POLY.	SUBSETS OF ULTRASPHERICAL POLYNOMIALS			
		CHEBYSHEV POLY. 1st Kind $\lambda = 0$	LEGENDRE POLY. $\lambda = 1/2$	CHEBYSHEV POLY. 2nd Kind $\lambda = 1$	TAYLOR POLY. $\lambda = \infty$
Symbol	$P_n^{(\lambda)}(x)=A_n^{(\lambda)}(1-x^2)^{-\lambda+1/2}\left(\frac{d}{dx}\right)^n(1-x^2)^{n+\lambda-1/2}$	$T_n(x)=A_n^{(0)}(1-x^2)^{1/2}\left(\frac{d}{dx}\right)^n(1-x^2)^{n-1/2}$	$P_n(x)=A_n\left(\frac{d}{dx}\right)^n(1-x^2)^n=\frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n(1-x^2)^n$	$U_{n-1}(x)$ = (see [20])	
Orthogonality Relationship	Wt. fctn$=(1-x^2)^{\lambda-1/2}$ $\int_{-1}^{+1} P_n^{(\lambda)}(x)\, P_m^{(\lambda)}(x)\,(1-x^2)^{\lambda-1/2}\,dx=\begin{cases}0 & m\neq n\\ 2^{1-2\lambda}\pi\left[\Gamma(\lambda)\right]^{-2}\frac{\Gamma(n+2\lambda)}{(n+\lambda)\,\Gamma(n+1)} & m=n,\ \lambda\neq 0\end{cases}$	Wt. fctn$=(1-x^2)^{-1/2}$ $\int_{-1}^{+1}\frac{T_n(x)\,T_m(x)\,dx}{(1-x^2)^{1/2}}=\begin{cases}0 & m\neq n\\ \pi/2 & m=n\neq 0\\ \pi & m=n=0\end{cases}$	Wt. fctn=1 $\int_{-1}^{1} P_n(x)\,P_m(x)\,dx=\begin{cases}0 & m\neq n\\ \frac{2}{2n+1} & m=n\end{cases}$	Wt. fctn$=(1-x^2)^{1/2}$ $\int_{-1}^{1} U_n(x)U_m(x)(1-x^2)^{1/2}dx=\begin{cases}0 & m\neq n\\ \pi/2 & m=n\end{cases}$	
Repres. Values	$P_0^{(\lambda)}(x)=1$ $P_1^{(\lambda)}(x)=2\lambda x$ $P_2^{(\lambda)}(x)=\lambda[2(\lambda+1)x^2-1]$ $P_3^{(\lambda)}(x)=2\lambda(\lambda+1)[2/3(\lambda+2)x^3-x]$	$T_0(x)=1$ $T_1(x)=x$ $T_2(x)=2x^2-1$ $T_3(x)=4x^3-3x$	$P_0(x)=1$ $P_1(x)=x$ $P_2(x)=1/2(3x^2-1)$ $P_3(x)=1/2(5x^3-3x)$	$U_0(x)=1$ $U_4=16x^4-12x^2+1$ $U_1(x)=2x$ $U_5=32x^5-32x^3+6x$ $U_2(x)=4x^2-1$ $U_6=64x^6-80x^4+24x^2-1$ $U_3(x)=8x^3-4x$ $U_7=128x^7-192x^5+80x^3-8x$	
Recursive Relation	$nP_n^{(\lambda)}(x)=2(n+\lambda-1)x\,P_{n-1}^{(\lambda)}(x)$ $-(n+2\lambda-2)\,P_{n-2}^{(\lambda)}(x)$	$T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=0$	$(n+1)\,P_{n+1}(x)-(2n+1)x\,P_n(x)$ $+n\,P_{n-1}(x)=0$	$U_n(x)-2xU_{n-1}(x)+U_{n-2}(x)=0$	
Expansion Formula	$f(x)=\sum_{n=0}^{\infty} a_n^{(\lambda)}\,P_n^{(\lambda)}(x)$	$f(x)=\sum_{n=0}^{\infty} a_nT_n(x)$	$f(x)=\sum_{n=0}^{\infty} b_nP_n(x)$	$f(x)=\sum_{n=0}^{\infty} C_nU_n(x)$	$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)\,x^k}{k!}$
Expansion of typical Analytic Fctn.	$\sin x^{o}=a_1^{(\lambda)}P_1^{(\lambda)}\left(\frac{x}{A}\right)+a_3^{(\lambda)}P_3^{(\lambda)}\left(\frac{x}{A}\right)$ $=\left[\left(\frac{2}{A}\right)^{\lambda+1}\Gamma(\lambda+2)\,J_{\lambda+1}(A)\right]x$	$\sin x^{o}=a_1T_1\left(\frac{x}{A}\right)+a_3T_3\left(\frac{x}{A}\right)$ $=\left[\frac{2}{A}J_1(A)\right]x$	$\sin x^{o}=b_1P_1(x)+b_3P_3(x)$ $=\left[\frac{3\sqrt{\pi}/2\;J_{3/2}(A)}{(A)^{3/2}}\right]x$	$\sin x^{o}=C_1U_1(x)+C_3U_3(x)$ $=\left[\frac{8J_2(A)}{A^2}\right]x$	$\sin x^{o}=x-\frac{x^3}{3!}\approx x$

APPENDIX A

APPENDIX B

NORMALIZING COEFFICIENT, C_n

The coefficient C_n can be cast into a form generally accepted by performing the following algebraic manipulations,

$$C_n = \left[\frac{2\lambda^2 \sqrt{\pi}\ \Gamma(\lambda+1/2)\ \Gamma(2\lambda+n)}{(n+\lambda)\ \Gamma(n+1)\ \Gamma(\lambda+1)\ \Gamma(2\lambda+1)} \right]^{-1/2} = \left[D_n \right]^{-1/2} \tag{B.1}$$

$$D_n = \frac{2\lambda^2 \sqrt{\pi}\ \Gamma(\lambda+1/2)\ \Gamma(2\lambda+n)}{(n+\lambda)\ \Gamma(n+1)\ \Gamma(\lambda+1)\ \Gamma(2\lambda+1)} \tag{B.2}$$

using,

$$\Gamma(\lambda+1) = \lambda\ \Gamma(\lambda) \tag{B.3}$$

$$\Gamma(2\lambda+1) = 2\lambda\ \Gamma(2\lambda) \tag{B.4}$$

and from Gray and Mathews [19],

$$(2n)! = \frac{2^{2n}\ \Gamma(n+1/2)\ \Gamma(n+1)}{\sqrt{\pi}} \tag{B.5}$$

$$(2n)! = (2n)\ (2n-1)! = 2n\ \Gamma(2n) = \Gamma(2n+1) \tag{B.6}$$

and altering equations (B.5) and (B.6) by letting $2n = 2\lambda-1$, we obtain equation (B.5) in the form

$$(2n)! = (2\lambda-1)! = \Gamma(2\lambda) = \frac{2^{2\lambda-1}\Gamma(\lambda)\ \Gamma(\lambda+1/2)}{\sqrt{\pi}}$$

or,

$$2^{1-2\lambda}\sqrt{\pi}\ \Gamma(2\lambda) = \Gamma(\lambda)\ \Gamma(\lambda+1/2) \tag{B.7}$$

Upon substituting (B.3), (B.4), and (B.7) into (B.2),

$$D_n = \frac{2^{1-2\lambda}\ \pi\,\Gamma(2\lambda+n)}{(n+\lambda)\ [\Gamma(\lambda)]^2\Gamma(n+1)} \tag{B.8}$$

APPENDIX C

SPECIAL CASE SHOWING COMPARISON OF BILINEAR U.P. APPROXIMATION AND MEAN SQUARE ERROR METHODS

The relationships between the conditions (equations (2.36) and (2.37)) under which the bilinear ultraspherical polynomial approximation and the mean square error methods differ are compared. This is accomplished for the special case of the nonlinear function $g(x) = x + x^3$, subject to $x_m = 2$, $x_t = 1$.

The general conditions (2.36) and (2.37), are repeated here for continuity.

$$\frac{\int_o^1 \omega\bar{\bar{g}}\, dt}{\int_o^1 \omega\bar{\bar{g}}t\, dt} = \frac{\int_o^1 \omega t\, dt}{\int_o^1 \omega t^2\, dt} \tag{2.36}$$

$$\frac{\int_o^1 \omega dt}{\int_o^1 \omega t\, dt} = \frac{\int_o^1 \omega t\, dt}{\int_o^1 \omega t^2\, dt} \tag{2.37}$$

where $\omega = (1-t^2)^{\lambda-1/2}$

$\bar{g} = g(x_t t)$

$\bar{\bar{g}} = g(y_m t + x_t)$

$y_m = x_m - x_t$

Let

$$R_1 = \frac{\int_0^1 \omega t \, dt}{\int_0^1 \omega t^2 \, dt} \tag{C.1}$$

$$R_2 = \frac{\int_0^1 \omega \bar{\bar{g}} \, dt}{\int_0^1 \omega \bar{\bar{g}} \, t \, dt} \tag{C.2}$$

$$R_3 = \frac{\int_0^1 \omega \, dt}{\int_0^1 \omega \, t \, dt} \tag{C.3}$$

(C.1) and (C.3) can be readily evaluated using Appendix D. The integrals in the numerator and denominator of (C.2) can be expanded and then evaluated using Appendix D. Figure C.1 represents the values of R_1, R_2, and R_3 for various values of λ. Note that at the lower limit of the λ parameter; $\lambda = -.5$, all three R's coincide and gives the conditions when exact agreement exists between the bilinear U.P. approximation method and the mean square error method.

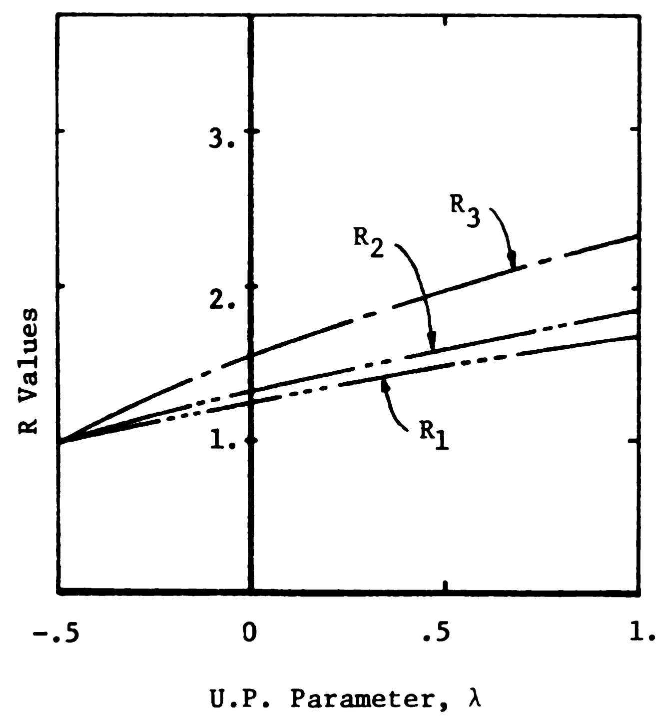

Figure C.1 Conditions comparing the bilinear U.P. method with the mean square error method plotted against the λ parameter for the function, $g(x) = x + x^3$, $x_m = 2$, $x_t = 1$.

APPENDIX D

RECURRING INTEGRALS EVALUATED

The beta function $B(m,n)$ cast into a form that recurs often in applying ultraspherical polynomial expansions.

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^1 y^{m-1}\,(1-y)^{n-1}\,dy \qquad \text{for } m,n > 0$$

let $y = x^2$,

$$B(m,n) = 2\int_0^1 x^{2m-1}\,(1-x^2)^{n-1}\,dx$$

let $s = 2m-1 \Rightarrow m = \frac{s+1}{2}$

$$\left.\begin{aligned} t &= n-1 \\ t &= \lambda-1/2 \end{aligned}\right\} \Rightarrow \quad n = t+1 = \lambda + 1/2$$

therefore,

$$\frac{\Gamma\left(\frac{s+1}{2}\right)\,\Gamma(\lambda+1/2)}{2\,\Gamma\left(\frac{s+2\lambda+2}{2}\right)} = \int_0^1 x^3(1-x^2)^{\lambda-1/2}\,dx$$

s	$\int_0^1 x^s (1-x^2)^{\lambda-1/2} dx = \dfrac{\Gamma\left(\frac{s+1}{2}\right) \Gamma(\lambda+1/2)}{2 \quad \Gamma\left(\frac{s+2\lambda+2}{2}\right)}$
0	$\dfrac{\sqrt{\pi}\ \Gamma(\lambda+1/2)}{2\ \Gamma(\lambda+1)}$
1	$\dfrac{\Gamma(\lambda+1/2)}{2\ \Gamma(\lambda+3/2)} = \dfrac{1}{2\lambda+1}$
2	$\dfrac{\sqrt{\pi}\quad \Gamma(\lambda+1/2)}{4 \quad \Gamma(\lambda+2)}$
3	$\dfrac{\Gamma(\lambda+1/2)}{2 \quad \Gamma(\lambda+5/2)} = \dfrac{2}{(2\lambda+1)\ (2\lambda+3)}$
4	$\dfrac{3\sqrt{\pi}\ \Gamma(\lambda+1/2)}{8 \quad \Gamma(\lambda+3)}$
5	$\dfrac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+7/2)} = \dfrac{8}{(2\lambda+1)\ (2\lambda+3)\ (2\lambda+5)}$
6	$\dfrac{15\sqrt{\pi}\quad \Gamma(\lambda+1/2)}{16\ \Gamma(\lambda+4)}$

APPENDIX E

EVALUATING k_1 FOR THE NONLINEAR FUNCTION, $f(x) = \sin x$

$$k_1 = \frac{1}{x_t}\left[\frac{\int_o^1 (1-t^2)^{\lambda-1/2}[\sin(x_t t)]\ t\ dt}{\int_o^1 (1-t^2)^{\lambda-1/2}\ t^2\ dt}\right] \tag{E.1}$$

By changing the variable, $t = \cos\theta$ and by using Appendix D to evaluate the denominator, k_1 becomes

$$k_1 = \frac{1}{x_t}\left[\frac{\int_o^{\pi/2} \sin^{2\lambda}(\theta)\quad [\sin(x_t \cos\theta)]\cos\theta\ d\ \theta}{\frac{\sqrt{\pi}}{4}\ \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+2)}}\right] \tag{E.2}$$

Substituting the power series expansion for $\sin(x_t \cos\theta)$ using

$\sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}\ (-1)^n}{(2n+1)\ !}$, and using the beta function relation

$$B\left(\frac{\alpha}{2},\frac{m}{2}\right) = 2\int_o^{\pi/2} \sin^{\alpha-1}(x) \quad \cos^{m-1}(x)\, dx$$

we obtain after simplifying,

$$k_1 = \frac{1}{x_t}\left[\sum_{n=0}^{\infty} \frac{(-1)^n\, x_t^{2n+1} \quad B\left[\frac{2\lambda+1}{2}, \frac{2n+3}{2}\right]}{(2n+1)! \qquad\qquad 2}\right] \frac{4}{\sqrt{\pi}} \quad \frac{\Gamma(\lambda+2)}{\Gamma(\lambda+1/2)} \tag{E.3}$$

Evaluating the beta function using Appendix D and representing the gamma functions in terms of factorials we find,

$$k_1 = \frac{1}{x_t}\left[\sum_{n=0}^{\infty} \frac{(-1)^n \quad x_t^{2n+1} \quad \Gamma(\lambda+2) \ (2n+1) \ (n-1/2)!}{(2n+1)! \quad \sqrt{\pi} \quad (n+\lambda+1)!}\right] \tag{E.4}$$

The term $(2n+1)!$ can be simplified using a relation in Gray and Mathews [19],

$$(2n+1)! = (2n+1)\ (2n)! = \frac{2^{2n}(n-1/2)! \quad n!}{\sqrt{\pi}} \tag{E.5}$$

Substituting (E.5) into (E.4) yields

$$k_1 = \frac{\Gamma(\lambda+2)}{\left(\frac{x_t}{2}\right)^{\lambda+1}} \quad \sum_{n=0}^{\infty} \frac{(-1)^n \quad \left(\frac{x_t}{2}\right)^{2n+\lambda+1}}{(n+\lambda+1)\ !\quad n!} \qquad = \frac{\Gamma(\lambda+2)}{(x_t/2)^{\lambda+1}} \ J_{\lambda+1}(x_t)$$

APPENDIX F

NORMAL MODE VIBRATIONS FOR NONLINEAR SYSTEMS

Rosenberg [17] defined normal mode vibrations for the nonlinear system in terms of a vibrations-in-unison of the physical system. A system is said to vibrate in unison if the motion satisfies all of the following conditions:

(i) all masses execute equi-periodic motion, or

$x_i(t) = x_i(t + T), \quad (i = 1, \cdots n)$

where T is a constant;

(ii) there exists a time $t = t_o$ when all masses pass through the equilibrium position, or $x_i(t_o) = 0, \quad (i = 1, \cdots\cdot n)$

(iii) there exists a time $t = t_1, \neq t_o$ when all velocities vanish, or

$\dot{x}_i(t_1) = 0, \quad (i = 1, \cdots\cdot n)$

(iv) the position of every mass at any instant of time t is uniquely determined by that of anyone of them at the same instant, or

$x_i = x_i(x_1(t)), \quad (i = 2, \cdots\cdot n)$

are all single-valued functions of x_1.

The governing equations of motion for the nonlinear system shown in figure F.1 are

$$m_1\ddot{x}_1 = \frac{\partial U}{\partial x_1} \tag{F.1}$$

$$m_2\ddot{x}_2 = \frac{\partial U}{\partial x_2} \tag{F.2}$$

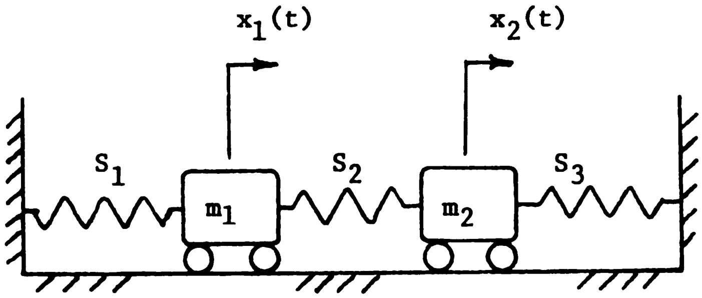

Figure F.1 Coupled Spring-Mass System

where $U = U(x_1,x_2)$ is a positive definite potential function. Consistent with above conditions a normal mode of the system in figure F.1 is defined as a function $x_2 = x_2(x_1)$ called the modal relation, which is satisfied for all time by periodic solutions $x_1 = x_1(t) = x_1(t + T)$ and $x_2 = x_2(t) = x_2(t + T)$ and where $x_2(x_1)$ is a single-valued function of x_1, in the closed domain $-U(x_1,x_2) = U_o$ of the (x_1,x_2)-plane subject to the boundary condition $x_2(0) = 0$ and which intersects the line $-U(x_1,x_2) = U_o$ orthogonally.

Rosenberg has shown that modal relations, $x_2 = x_2(x_1)$, for these nonlinear systems are, in general, not constant. However, he has shown that two classes of systems exist for which the ratios of the displacements of the two masses are identically equal to constants (i.e., $x_2 = c\ x_1$) for all time when the system vibrates in normal modes. One class is the homogenous case of degree "k" which may be entirely unsymmetric with respect to the masses as well as the anchor springs (outboard springs). The homogeneous case refers to those systems having all springs with terms of the same degree k. For example, if k=3, then all springs would have only the cubic term present. The other class is the symmetric case in which the two masses are equal and the outboard springs are equal. Rosenberg discusses one feature of the nonlinear system having two degrees-of-freedom which is not found in the linear system. This feature is that there may exist more than two normal modes. This he illustrates by choosing a symmetric homogeneous system with the potential function

$$U(x_1,x_2) = -\frac{a_3}{4}(x_1^4 + x_2^4) - \frac{A_3}{4}(x_1 - x_2)^4 \tag{F.3}$$

By introducing the polar coordinates

$$x_1 = r \cos\theta \text{ and } x_2 = r \sin\theta \tag{F.4}$$

the potential function becomes

$$U(r,\theta) = \frac{r^4}{4}[-a_3(\cos^4\theta + \sin^4\theta) \quad -A_3(\cos\theta - \sin\theta)^4] \tag{F.5}$$

The necessary and sufficient condition for the existence of straight modal relations for the positive definite potential function $U(r,\theta)$ is

$$\frac{\partial U}{\partial \theta} = \Theta_1(\theta)\Theta_2(r,\theta) \tag{F.6}$$

Equating, (F.6) to zero implies that $\Theta_1(\theta)$ is zero and its roots are the modal relations. For this illustrated case the modal relations are

$$\left([a_3/2A_3-1] \quad \sin 2\theta + 1\right) \cos 2\theta = 0 \tag{F.7}$$

Setting $\cos 2\theta = 0$, yields $\theta = \pm\pi/4$ and

setting $[a_3/2A_3 - 1] \sin 2\theta + 1 = 0$, yields the additional roots

$$\theta = -1/2 \sin^{-1} \{2/[(a_3/A_3) - 2]\} \quad \text{for } a_3/A_3 \geq 4 \tag{F.8}$$

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