CONSTITUTIVE EQUATIONS AND THE SOLUTION OF SOME PROBLEMS OF INTERACTING CONTINUOUS MEDIA

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ABSTRACT

CONSTITUTIVE EQUATIONS AND THE SOLUTION OF SOME PROBLEMS OF INTERACTING CONTINUOUS MEDIA

By Farhad Tabaddor

A mathematical statement, describing the incompressibility condition for a mixture of n incompressible Newtonian fluids and a linear elastic solid was obtained.

Using a thermodynamic theory of interacting media proposed by Green and Naghdi, the constitutive equations were derived for a binary mixture of an incompressible Newtonian fluid and a linear elastic solid. The similarities between these governing equations and those for a binary mixture of a compressible fluid and a linear elastic solid were discussed.

A system of field equations are formulated and the general solution for the displacements are presented for the steady-state case. A stress function solution for partial stresses of the solid was developed for steady-state plainstrain problems. These methods were applied to solve a twodimensional problem. The reduction of the present theory to Biot's consolidation theory and Darcy's law of fluid flow through porous media is discussed.

A diffusion law is given for a mixture of two ideal fluids flowing through a rigid porous material.

Finally, following the main theory and employing the notion of hidden coordinates of irreversible thermodynamics, the constitutive equation for a binary mixture of a Newtonian fluid and a viscoelastic solid is derived.

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CONTINUOUS MEDIA

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LIST OF SYMBOLS

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(r) _X i	=	Component of the reference position vector $(r) \stackrel{\downarrow}{X}$ of a particle of the rth substance of the mixture.
(r) _x i	=	Component of the current position vector $(r) \stackrel{?}{x}$ of a particle of the rth substance of the mixture.
(r) _v i	=	Component of the velocity vector $(r) \stackrel{?}{V} of$ a particle of the rth substance of the mixture.
(r) _m i	=	Component of the diffusive force vector of the rth substance of the mixture.
¯ Γ	=	Density of the isolated rth substance.
ρ _r	=	Initial density of the rth substance in the mixture.
ρr	=	Current density of the rth substance in the mixture.
ρ	=	Current density of the mixture.
(r) _F i	=	Component of the body force per unit mass of the rth component of the mixture.
(r) _a i	=	Component of the acceleration vector of the rth component of the mixture.
(r) _σ ik	=	Component of the partial stress tensor of the rth component of the mixture.
^u i	=	Component of the velocity vector \vec{U} of the solid in a fluid-solid mixture.
^a i	=	Component of the acceleration vector of the solid in a fluid-solid mixture.
^d ij	=	Component of the rate of deformation tensor of the solid in a fluid-solid mixture.
Г _{іј}	=	Component of the vorticity tensor of the solid in a fluid-solid mixture.
ω _i	=	Component of the displacement vector of the solid in a fluid-solid mixture.

- F_i = Component of the body force per unit mass of solid in a fluid-solid mixture.
- v_i = Component of the velocity vector \vec{v} of the fluid in a fluid-solid mixture.
- g_i = Component of the acceleration vector of the fluid in a fluid-solid mixture.
- f = Component of the rate of deformation tensor of the
 fluid in a fluid-solid mixture.
- Λ_{ij} = Component of the vorticity tensor of the fluid in a fluid-solid mixture.
 - G₁ = Component of the body force per unit mass of the fluid in a fluid solid mixture.
 - n_{ν} = Component of outward unit normal.
 - U = Internal energy per unit mass of the mixture.
 - T = Temperature of the mixture.
 - S = Entropy per unit mass of the mixture.
 - r = Heat supply function per unit mass of the mixture.
 - h = Flux of heat across area, per unit area and unit time.
 - m_{i} = Mass supply function of the ith substance.
 - q_{k} = Heat flux across x_{k} coordinates.
 - A = Helmholtz free energy per unit mass of the mixture.
- e = Component of strain tensor of the solid in the mixture.
- $P_0 = Porosity factor.$
 - R = Ratio of pores compressibility to total compressibility.
- η_{α} = Difference between current and initial density of the α th substance in the mixture.
- δ_{ik} = Kronecker delta.
 - p = Hydrostatic pressure.

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k = Permeability coefficient of the solid.

$$(\alpha)$$
 f_{ij} = Component of the rate of deformation of the α th component of the mixture.

- $(\alpha)_{\Gamma_{ij}} = Component of the vorticity tensor of the <math>\alpha$ th component of the mixture.
 - ξ_i = Generalized coordinates.
 - Q_i = Generalized forces.

σ_{ik} = Component of the stress tensor of the solid in a fluid-solid mixture.

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CHAPTER I

INTRODUCTION

The theory of mixtures, or heterogeneous media, has received a great deal of interest and attention in recent years.

This branch of continuum physics is concerned with that kind of medium which consist of two or more constituents, where each constituent is a simple medium with its own physical and chemical properties in the absence of the others. When the components mix together, some changes will appear in both the physical and chemical properties of each constituent. The field of interacting media deals with the necessary modifications to these properties, and formulates the proper field and constitutive equations for the media under consideration.

The problem is of great importance both from a theoretical and practical viewpoint. The wide application of this branch is rather evident from the nature of the problem. As examples, consider the diffusion problem, seepage of water and other fluids through deformable or undeformable porous media, absorption of oils by plastics, water by fibers, and many others. In what follows the basic works in this area are briefly reviewed.

1.1 Fick's Law of Diffusion

In analogy to heat conduction, and on no other grounds, Fick assumed that in a binary mixture, the rate of transfer of the diffusing substance through a unit area of the section is proportional to the concentration gradient measured normal to the section

$$F = -D\frac{\partial C}{\partial x}$$
(1.1-1)

where

F = rate of transfer per unit area,

C = concentration of diffusing substance,

D = diffusion coefficient.

The fundamental diffusion equation is derived by substituting the above constitutive law into the continuity equation of diffusing substance.

$$\frac{\partial C}{\partial t} = D\nabla^2 C \qquad (1.1-2)$$

Although the above relation was proposed for a binary mixture, Onsager (Ann. N.Y. Acd. Sci. 46,251, 1946) suggested the direct generalization of Fick's law. The above concept is not supported by principles of continuum mechanics in general, and the many limitations and conceptual difficulties arising from Fick's hypothesis suggests an entirely different approach to the problem.

A full account of linear theories of diffusion has been given by Truesdell [42,43] and the various theories of diffusion are analyzed. The assumption under which Fick's law is usually applied is that there is no mass exchange, that the pressure and total density are constant, and that there is no mean motion.

The derivation of Fick's law of diffusion has also been considered by Adkins [1], Green and Adkins [24], and Mills [30] from different approaches, and the basic assumptions of the classical theory are justified under special conditions.

1.2 Truesdell's Hydrodynamics Approach

A general theory of mixtures, or alternatively heterogeneous media was constructed by Truesdell [42]. The basic assumptions of the theory are as follows:

A. Each space point x may be occupied simultaneously by several different particles. This assumption was first suggested by Fick and Stefan. The validity of this assumption is as good as the assumption of continuity of the matter.

B. We can assign distinct kinematic quantities such as velocity and acceleration and mechanical quantities such as forces and stresses to each substance at a space point x.

C. Since the diffusion involves relative motion of different particles, a transfer of momentum between the components is involved, while the total momentum of media is conserved as a whole.

D. To account for diffusion phenomena, the body force acting on each constituent can be subdivided into an

extraneous body force, which is the same as for a single component, and a diffusive force. The theory calls for a constitutive equation for the diffusive force which satisfies all the necessary invariance requirements.

E. "Each component is considered as being subject to a partial stress whose action upon any closed diaphragm is equipollent to the action of all constituents exterior to the diaphragm upon the material of the constituent under consideration within the diaphragm" [42]. Consequently the total stress is the sum of the partial stresses.

Truesdell developed a general framework for heterogeneous media, and, based on his general theory, he gave a comprehensive analysis of four different approaches to the diffusion problem. Some aspects of the linear theory were examined. Using Truesdell's approach, Adkins [1,2,3,4] has developed a non linear theory.

In the following, the general framework of the problem is outlined. We refer the motion to a fixed system of Cartesian coordinates X_i , and denote the initial position of each substance as ${}^{(r)}X_i$, where r denotes the rth substance.

The position of a particle ${}^{(r)}x_i$ at time t is denoted by ${}^{(r)}x_i$ where

$$(r)_{x_{i}} = (r)_{x_{i}}((r)_{x_{j}}, t) \qquad r = 1, 2...n \qquad (1.2-1)$$

We can express (1.2-1) in alternative form

$$(r)_{i} x_{i} = (r)_{i} x_{i} ((r)_{j} x_{j}, t) \qquad r = 1, 2...n \qquad (1.2-2)$$

For the above deformations to be possible in real materials we must have

$$\frac{\partial (\mathbf{r}) \mathbf{x}_{i}}{\partial (\mathbf{r}) \mathbf{x}_{j}} > 0 \qquad \left| \frac{\partial (\mathbf{r}) \mathbf{x}_{j}}{\partial (\mathbf{r}) \mathbf{x}_{i}} \right| > 0 \qquad (1.2-2a)$$

The summation convention does not apply to r.

The substance $(r)_{i} X_{i}$ has a velocity $(r)_{i} v_{i}$ at time t

where

$$(r)_{i} = \frac{D^{(r)} x_{i}}{Dt}$$
 (1.2-3)

where $\frac{D^{(r)}}{Dt}$ denotes differentiation with respect to time holding ${}^{(r)}x_i$ fixed.

If the density of substance S_r is ρ_r at $(r) x_i$ then the density of the mixture is ρ where

$$\rho = \sum_{r=1}^{n} \rho_r \qquad (1.2-4)$$

and the mean velocity \overline{v}_{m} of the mixture is defined to be

$$\rho \overline{\mathbf{v}}_{m} = \sum_{r=1}^{n} \rho_{r} (r) \overline{\mathbf{v}}_{m}$$
(1.2-5)

If $\psi = \psi({(r)}x_i,t)$ is any scalar or tensor function we observe that

where $\partial/\partial t$ denotes partial differentiation with respect to time holding the spatial coordinates constant. If we define the operator D/Dt by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{v}_{m} \frac{\partial}{\partial x_{m}}$$
(1.2-6)

then

The diffusive velocity $\int_{m}^{r} u_{m}$ of the substance S_{r} is

defined to be:

$${}^{(r)}\overline{u}_{m} = {}^{(r)}v_{m} - \overline{v}_{m}$$
 (1.2-8)

thus

$$\rho^{(\mathbf{r})}\overline{u}_{m} = \rho^{(\mathbf{r})}v_{m} - \rho\overline{v}_{m} = \sum_{s=1}^{n}\rho_{s}^{(\mathbf{r})}v_{m} - \sum_{s=1}^{n}\rho_{s}^{(s)}v_{m}$$
$$= \sum_{s=1}^{n}\rho_{s}^{(\mathbf{r})}v_{m}^{-(s)}v_{m}^{(s)} \qquad (1.2-9)$$

so

$$\sum_{r=1}^{n} \rho_r^{(r)} \overline{u}_m = \sum_{r=1}^{n} (\rho_r^{(r)} v_m - \rho_r \overline{v}_m) = \sum_{r=1}^{n} \rho_r^{(r)} v_m - \sum_{r=1}^{n} \rho_r \overline{v}_m$$
$$= \rho \overline{v}_m - \rho \overline{v}_m = 0 \qquad (1.2-10)$$

If we exclude the possibility of mass generation or mass dissipation by chemical reactions, adsorption or similar processes, the continuity equations can be written for each component of the mixture in the form,

$$\frac{\partial \rho_{\mathbf{r}}}{\partial t} + \frac{\partial}{\partial \mathbf{x}_{\mathbf{i}}} (\rho_{\mathbf{r}}^{(\mathbf{r})} \mathbf{v}_{\mathbf{i}}) \equiv \frac{D \rho_{\mathbf{r}}}{D t} + \rho_{\mathbf{r}} \frac{\partial^{(\mathbf{r})} \mathbf{v}_{\mathbf{i}}}{\partial \mathbf{x}_{\mathbf{i}}} = 0 \qquad (1.2-11)$$

By adding the n equations we obtain

$$\frac{D\rho}{Dt} + \rho \frac{\partial \overline{v}_i}{\partial x_i} = 0 \qquad (1.2-12)$$

(If there exists a mass supply, equations (1.2-11) should be modified; see [42].)

We define $(r)_{\pi_i}$, the supply of momentum of rth substance, to be

$$\rho^{(\mathbf{r})}\pi_{\mathbf{i}} = \rho_{\mathbf{r}}^{(\mathbf{r})}F_{\mathbf{i}} - {}^{(\mathbf{r})}a_{\mathbf{i}}^{\mathbf{i}} + \frac{\partial^{(\mathbf{r})}\sigma_{\mathbf{i}\mathbf{k}}}{\partial\mathbf{x}_{\mathbf{k}}} \qquad (1.2-13)$$

where ${(r)}_{F_i}$ is the body force per unit density of rth component and

$${}^{(r)}a_{i} = \frac{{}^{(r)}D^{(r)}v_{i}}{Dt}$$
 (1.2-14)

If the momentum was conserved for each component, we would have had

$$(r) \pi_{i} = 0$$
 for $r = 1, 2...n$

Since we assume that total momentum of the mixture is conserved, it follows that

$$\sum_{r=1}^{n} {(r) \atop r}_{i} = 0 \qquad (1.2-15)$$

From the sum of n equations (1.2-13), and considering (1.2-15) we obtain

$$\frac{\partial \sigma_{ik}}{\partial x_{k}} + \rho F_{i} = \rho a_{i} \qquad (1.2-16)$$

where

$$F_{i} = \sum_{r=1}^{n} \rho_{r}^{(r)} F_{i} \text{ and } \rho_{a_{i}} = \sum_{r=1}^{n} \rho_{r}^{(r)} a_{i} \qquad \sigma_{ik} = \sum_{r=1}^{n} (r) \sigma_{ik}$$

To complete the theory, the constitutive equations for stresses and the supply of momentum should be postulated. The equations (1.2-11-13-15) plus the constitutive equations are 13n partial differential equations for 13ⁿ unknowns, namely the densities, stresses, displacement, and diffusive forces (together with appropriate boundary and initial conditions).

Based on the above development Adkins [1-4] discussed the invariance requirements and the restrictions imposed upon the constitutive equations. In the first paper he assumed the stresses to be a function only of quantities specifying the motion of the substance under consideration and its concentration, and that there is no contribution from the presence of the other components. He has also applied the theory to a number of steady-state problems of non-Newtonian fluids and flow of fluids through rigid plates and some wave propagations. Later he discussed the case where the stresses might depend upon the velocity gradients of other components. He also has applied the theory to the problems of fluid flow through elastic solids, where attention is confined to steady-state problems. Adkins has also [4] discussed the diffusion through Aeolotropic highly elastic solids. Essentially the same approach was employed by Kelly [45] for chemically reacting media. Hayday [28] came up with essentially the same results with a somewhat different degree of generality. His development can be considered as an alternative approach to Truesdell's formalism. The main axioms were presented in integral form.

1.3 Thermodynamic Theories

In the works cited so far, equations of mass, momentum and energy balance are postulated for each component of the mixture. No account was taken of possible thermodynamic restrictions which might be imposed upon these equations.

In order to improve the theory and to remove some of conceptual difficulties, Green and Naghdi [25] developed a rather new approach to the problem. Instead of postulating equations of mass, momentum, and energy balance for each constituent, they proposed a single energy equation and an entropy production inequality for the whole continuum, allowing for chemical and thermal reactions. By systematic application of invariance requirements, they derived the equations of mass and momentum, which are basically the same as those proposed by Truesdell. However differences occur in the other parts of the theory.

As will be seen, the diffusive force comes into the picture in a natural manner. The partial stresses, although defined essentially the same as those of Truesdell, are not necessarily symmetric, in spite of the absence of couple or multipolar stresses. This immediately suggests that the interaction between components of the mixture occurs not only through the body forces but also through the body couples. Truesdell did not consider the case of non symmetric partial stresses, which in effect implies that the moment of momentum is not a conserved quantity for each constituent. The full

use of energy and entropy production inequality equation imposes restrictions on constitutive equations. The theory was derived for the case of a binary mixture and was first applied to the mixture of two Newtonian fluids. The theory postulates a temperature T and an internal energy U for the mixture, but not for the components. For many cases, this does not introduce a serious restriction.

Green and Steel [26] applied this theory to derive the constitutive equations for a mixture of a Newtonian fluid and an elastic solid, and also the mixture of two elastic solids. Crochet and Naghdi [22] discussed the problem of a mixture of an elastic solid and a fluid. In a paper by Mills [30] the problem of the mixture of two incompressible Newtonian fluids was discussed and the incompressibility condition and the resulting modifications presented. A law similar to Fick's law of diffusion was derived for a binary mixture of incompressible fluids.

The problems of wave propagation in a mixture of an elastic solid and a Newtonian fluid and also in two elastic solids have been considered by Steel [41]. In another paper [40] Steel considered the plain strain problems of two elastic solids and presented the solutions in the complex plane. As an example, an infinite body initially containing a circular hole with a special loading condition, is solved by use of complex potentials. Some uniqueness theorems were presented by Atkin, <u>et al</u>. [5] for the linear case, and a set of sufficient subsidiary conditions are stated.

Recently [31] the above theory has been generalized for a multicomponent mixture by Mills. The case of n fluids and an elastic solid has been worked out in detail. As an illustration, the steady-state problem of a binary mixture of ideal gases in an isotropic rigid solid is discussed.

The formalism of Green and Naghdi [25] will be briefly reviewed for future use and the expressions and definitions identical to section 2 will be omitted. If the motion of a mixture of two components, S_1 and S_2 , is referred to fixed Cartesian coordinates and the material coordinate of S_1 is denoted by X and S_2 by Y, the position of a typical particle at time t is

$$x_{i} = x_{i}(x_{j}, t)$$
 $y_{i} = y_{i}(y_{j}, t)$ (1.3-1)

where the condition (1.2-2a) is satisfied.

We consider those particles which occupy the same position at time t so that

$$\mathbf{y}_{\mathbf{i}} = \mathbf{x}_{\mathbf{i}} \tag{1.3-2}$$

The velocity vectors at point $x_i = y_i$ at time t are

$$u_{i} = \frac{(1)_{Dx_{i}}}{Dt}$$
 $v_{i} = \frac{(2)_{Dy_{i}}}{Dt}$ (1.3-3)

where D/Dt has the same meaning as before. The acceleration vectors are denoted by a_i and g_i where

$$a_{i} = \frac{(1)_{Du_{i}}}{Dt}$$
 $v_{i} = \frac{(2)_{Dy_{i}}}{Dt}$ (1.3-4)

The densities at time t are ρ_1 and ρ_2 and the rate of deformation tensors are defined to be

$$d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$
 $f_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ (1.3-5)

The vorticity tensors are defined to be

$$\Gamma_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i}) \qquad \Lambda_{ij} = \frac{1}{2} (v_{i,j} - v_{j,i}) \qquad (1.3-6)$$

$$\rho = \rho_1 + \rho_2 \qquad \rho \overline{v}_i = \rho_1 u_i + \rho_2 v_i \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{v}_m \frac{\partial}{\partial x_m}$$

$$\rho \frac{D}{Dt} = \rho_1 \frac{(1)_D}{Dt} + \rho_2 \frac{(2)_D}{Dt} \text{ as before.}$$

If A is an arbitrary fixed closed surface enclosing a volume V and n_k is the outward unit normal, the following energy equation is postulated.

$$\frac{\partial}{\partial t} \int_{V} [(\rho_{1} + \rho_{2})U + \frac{1}{2}\rho_{1}u_{i}u_{i} + \frac{1}{2}\rho_{2}v_{i}v_{i}]dV + \int_{A} [n_{k}(\rho_{1}u_{k} + \rho_{2}v_{k})U + \frac{1}{2}\rho_{1}n_{k}u_{k}u_{i}u_{i} + \frac{1}{2}\rho_{2}n_{k}v_{k}v_{i}v_{i}]dA = \int_{V} (\rho r + \rho_{1}F_{i}u_{i} + \rho_{2}G_{i}v_{i})dV + \int_{A} (t_{i}u_{i} + p_{i}v_{i})dA - \int_{A} h dA$$
(1.3-7)

where U is the internal energy of the mixture per unit mass. F_i,G_i are externally applied body forces per unit masses of S_1 and S_2 .

 t_i, p_i are surface force vectors per unit area of A, such that $t_i u_i$ and $p_i v_i$ are the rates of work per unit area of A. r is the heat supply function per unit mass of the mixture due to external sources.

h is the flux of heat across A per unit area and unit time. If we define m_1 and m_2 to be

 $m_{1} = \frac{(1)_{D\rho_{1}}}{Dt} + \rho_{1}u_{k,k} \qquad m_{2} = \frac{(2)_{D\rho_{2}}}{Dt} + \rho_{2}v_{k,k} \quad (1.3-8)$ and apply a uniform rigid body translation to (1.3-7), assuming that ρ_{1} , ρ_{2} , U, (F_i - a_i), (G_i - g_i), t_i, p_i, h, r, not being altered by this, we obtain:

$$\int_{V} [\rho_{1}(F_{i} - a_{i}) + \rho_{2}(G_{i} - g_{i}) - m_{1}u_{i} - m_{2}v_{i}]dv$$

+
$$\int_{A} (t_{i} + p_{i})dA = 0$$
 (1.3-9)

$$m_1 + m_2 = 0$$
 (1.3-10)

If σ_{ki} and τ_{ki} are defined to be the values of t_i and p_i when the surface at a point x_i is perpendicular to x_k axis, then the application of (1.3-9) to a tetrahedron yields,

$$r_{i} + p_{i} = n_{k}(\sigma_{ki} + \tau_{ki})$$
 (1.3-11)

We refer to σ_{ki} and τ_{ki} as partial stresses. If we use (1.3-11) in (1.3-9) we obtain

$$(\sigma_{ki} + \tau_{ki}), k + \rho_1 F_i + \rho_2 G_i = \rho_1 a_i + \rho_2 g_i + m_1 u_i + m_2 v_i$$

(1.3-12)

Equation (1.3-12) is equivalent to the sum of the equations of linear momentum supply for a binary mixture as was proposed by Truesdell. With the help of equations (1.3-8, 10, 11, 12) equation (1.3-1) can be reduced to $\int_{V} [\rho r - \rho \frac{DU}{Dt} + \frac{1}{2}(\rho_{1}F_{i} - \rho_{2}G_{i})(u_{i} - v_{i}) - \frac{1}{2}(\rho_{1}a_{i} - \rho_{2}g_{i})(u_{i} - v_{i}) + \frac{1}{2}(\sigma_{ki} + \tau_{ki})(u_{i} + v_{i}), k]dV + \frac{1}{2}\int_{A} (t_{i} - p_{i})(u_{i} - v_{i})dA$ $- \int_{A} h \ dA = 0 \qquad (1.3-13)$

If q_k is the flux of heat across the x_k planes, then the application of (1.3-13) to a tetrahedron bounded by coordinate planes, and a plane with outward unit normal n_k yields

$$\overline{\pi}_{i}(u_{i} - v_{i}) - (h - n_{k}q_{k}) = 0$$
 (1.3-14)

where

$$\overline{\pi}_{i} = \frac{1}{2} [(t_{i} - p_{i}) - n_{k}(\sigma_{ki} - \tau_{ki})]$$

Using (1.3-14) and applying (1.3-13) to an arbitrary volume yields

$$\rho \mathbf{r} - \mathbf{q}_{k,k} - \rho \frac{DU}{Dt} + \pi_{i} (\mathbf{u}_{i} - \mathbf{v}_{i}) + \sigma_{ki} \mathbf{u}_{i,k} + \tau_{ki} \mathbf{v}_{i,k} = 0$$
(1.3-16)

where

$$\pi_{i} = \frac{1}{2}(\sigma_{ki} - \tau_{ki}), + \frac{1}{2}\rho_{1}(F_{i} - a_{i}) - \frac{1}{2}\rho_{2}(G_{i} - g_{i})$$
(1.3-17)

We refer to $\boldsymbol{\pi}_i$ as a diffusive force. It is further deduced that

$$\sigma_{ki} + \pi_{ki} = \sigma_{ik} + \tau_{ik}$$
 (1.3-18)

and

$$h - n_k q_k = 0$$
 (1.3-19)

Equation (1.3-16) can be written as $\rho \mathbf{r} - \mathbf{q}_{\mathbf{k},\mathbf{k}} - \rho \frac{\mathrm{DU}}{\mathrm{Dt}} + \pi_{\mathbf{i}} (\mathbf{u}_{\mathbf{i}} - \mathbf{v}_{\mathbf{i}}) + \frac{1}{2} (\sigma_{\mathbf{k}\mathbf{i}} + \sigma_{\mathbf{i}\mathbf{k}}) \mathbf{d}_{\mathbf{i}\mathbf{k}}$ $+ \frac{1}{2} (\tau_{\mathbf{k}\mathbf{i}} + \tau_{\mathbf{i}\mathbf{k}}) \mathbf{f}_{\mathbf{i}\mathbf{k}} + \frac{1}{2} (\sigma_{\mathbf{k}\mathbf{i}} - \sigma_{\mathbf{i}\mathbf{k}}) (\Gamma_{\mathbf{i}\mathbf{k}} - \Lambda_{\mathbf{i}\mathbf{k}}) = 0 \qquad (1.3-20)$

Also postulated was an entropy production inequality in the form

$$\frac{\partial}{\partial t} \int_{V} (\rho_{1} + \rho_{2}) S \, dV + \int_{A} n_{k} (\rho_{1}u_{k} + \rho_{2}v_{k}) S \, dA - \int_{V} \frac{r}{T} \, dV$$

$$+ \int_{A} \frac{h}{T} dA \ge 0 \text{ or}$$

$$\int_{V} \rho \frac{DS}{Dt} dV - \int_{V} \rho \frac{r}{T} dV + \int_{A} \frac{h}{T} dA \ge 0 \qquad (1.3-21)$$

where S is the entropy per unit mas of the mixture and T is the temperature. (S,T are not decomposed into components and in effect they are considered to be the average for the mixture.) The theory would be completed if we supplement the constitutive equations for π_i , $\overline{\pi}_i$, $(\sigma_{ki} + \sigma_{ik})$, $(\tau_{ki} + \tau_{ik})$, $(\sigma_{ki} - \sigma_{ik})$ and m_1 and m_2 with thermodynamical restrictions furnished by (1.3-21).

1.4 Flow of Fluids Through Porous Media

The special case of heterogeneous media which has received the most attention and work in the past, is the flow of fluids through porous media.

The basic hypothesis and references to classical work in this area can be found in the books written by Muskat [32] and Scheidgger [37]. A review article [38] also covers a good number of references. The basic equation in this area of work is Darcy's law. On experimental grounds only, Darcy postulated a linear relationship between velocity and pressure gradient of the fluid, while the porous medium is assumed to be rigid.

Later (1941) Biot and co-workers, adopting the Darcy law, discussed the flow of fluids through deformable elastic solids in an extensive series of papers [6-17]. A review article by Paria [36] covers the essential part of the literature. This will be discussed in detail in Chapter IV.

1.5 Further Theories

the In order to remove some of the restrictions of/former theory resulting from the assumption of a single temperature and internal energy for the mixture, Green and Naghdi [27] proposed a theory of mixtures in which all dependent variables were admitted for each constituent. Thus an energy equation and an entropy production inequality were postulated for each component of the mixture. In formulating these equations care was taken so that the suitable sums of these equations were compatible to that of the mixture. Again by full use of the invariance requirements they deduced similar equations to those shown before.

The results of the theory are in complete agreement with the former case of a binary mixture, but, in addition, the thermodynamic variables could be obtained for each component. They pointed out that in general the thermodynamic properties of the mixture are not merely related to those of its components by algebraic sums. However, if the temperatures of the components are the same as that of the mixture, the entropy and internal energy of the mixture are related to those of its components by some suitable algebraic sums.

1.6 Scope and Objectives

The main objective of this work is to establish a theoretical foundation for a binary mixture of an incompressible Newtonian fluid and a linear elastic solid based on the theory of Green and Naghdi [25]. The scope of the work is outlined below.

a) Mathematical statement of the incompressibility condition.

b) Derivation of constitutive equations, using incompressibility condition and formulating the complete field equations.

c) General displacement solution of the problem in terms of potential functions for steady-state case.

d) General solution of the problem in terms of stress function for two-dimensional steady-state case.

e) Reduction to Darcy's law for flow of fluids through a rigid porous media and Biot's equation of fluid flow through a linear elastic solid.

f) Application of the general solution to one and two dimensional problems.

g) Derivation of constitutive equation for a mixture of a Newtonian fluid and a viscoelastic solid.

h) Derivation of constitutive equations for a mixture of two ideal incompressible fluids and a linear elastic solid; derivation of a diffusion law and comparison with the existing diffusion expressions.

CHAPTER II

BINARY MIXTURE OF A NEWTONIAN FLUID AND

AN ELASTIC SOLID

2.1 General Remarks

In this chapter the mathematical statement of incompressibility for a mixture of n incompressible fluids and a linear elastic solid will be derived. The special case of this condition for a binary mixture of an incompressible fluid and a linear elastic solid will be obtained from the general expression.

The constitutive equations for such a binary mixture the will be obtained by use of/entropy production inequality and the incompressibility condition and compared with those of the compressible case. The complete field equations will also be stated.

Our interest in incompressibility is twofold; firstly it is of theoretical interest and secondly the fact that such a condition offers a great deal of simplifications in different theories of single media such as classical hydrodynamic and finite elasticity.

2.2 Basic Assumptions

The assumptions used in this chapter are listed below:

a) Both elastic solid and fluid are initially at rest under zero initial stresses.

b) The continuum is initially homogeneous and isotropic.

c) The displacement of the solid, as well as its space and time derivatives, remain small during the motion, such that we can neglect any term higher than the first in all of our fundamental equations.

d) The change in density and the velocity, as well as their space and time derivatives, of each component during the motion are small of the same order, in the sense that we can neglect any term higher than the first in all fundamental equations. The temperature remains constant throughout the deformation.

It is perhaps worth noting that one cannot speak of small quantities which are not dimensionless, since the numerical values of such quantities depend on the scale used in the problem. In such cases, the equations should be made dimensionless before any smallness argument can be applied.

2.3 Incompressibility Condition

We consider the mixture of n incompressible fluids and a linear elastic solid under the above assumptions. We suppose that the initial porosity of solid is P_0 and the initial volume concentrations of fluids within the pores of

solid is $(\alpha)_{C}$ $\alpha = 1, 2, 3...n$ it is readily seen that

$$\sum_{\alpha=1}^{n} {}^{(\alpha)}C = 1$$
 (2.3-1)

Let \overline{V}_1 be the initial volume of an element of solid and V_1 the volume of the same element at t = t, we then have,

$$v_1 = \overline{v}_1 (1 + e_{mm})$$
 (2.3-2)

where
$$e_{ij} = \frac{1}{2} \left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i} \right)$$
 (2.3-3)

and
$$\omega_i = x_i - X_i$$
 (2.3-4)

where x_i is the position of a particle of solid, whose initial position is X_i , both referred to the same fixed Cartesian coordinates. From the basic assumptions, it follows that there is a linear relation between the compressibility and volume change of the pores. With that in mind, it is easily seen that:

$$v = R(V_1 - \overline{V}_1) + P_0 \overline{V}_1 \qquad (2.3-5)$$
$$v = \overline{V}_1 (Re_{mm} + P_0)$$

or

where v is the actual volume of the pores in an element \overline{v}_1 at time t (we only consider the interconnected pores, and treat the closed pores as a part of solid). R is a constant expressing the ratio of pore compressibility to the total compressibility.

If $(\alpha)_{\overline{\rho}}$, $(\alpha)_{\overline{\rho}}$, $(\alpha)_{\rho}$, $(\alpha)_{\nu}$ are the initial density of the fluid α , the initial density of fluid α in the mixture, the density of fluid α in the mixture at t = t and the actual volume of fluid within the volume v_1 respectively, we would have

$$(\alpha) = (\alpha) = (\alpha) \rho V_1$$
 for $\alpha = 1, 2...n$ (2.3-6)

and

$$\sum_{\alpha=1}^{n} {}^{(\alpha)} v = v$$
 (2.3-7)

also

$$(\alpha)_{\overline{\rho}} = (\alpha)_{CP_{0}} (\alpha)_{\overline{\rho}}^{=} \text{ for } \alpha = 1, 2...n$$
 (2.3-8)

By use of (2.3-2,5,6,8) in (2.3-7) we obtain

$$P_{o} \sum_{\alpha=1}^{n} \frac{(\alpha)_{c}(\alpha)}{(\alpha)_{\overline{\rho}}}^{\rho} = (R - P_{o})e_{mm} + P_{o} \qquad (2.3-9)$$

The continuity equations for fluids can be written as:

$$\frac{\partial (\alpha)}{\partial t} \eta + (\alpha) \overline{\rho}(\alpha) f_{kk} = 0 \text{ for } = 1, 2...n \qquad (2.3-10)$$

where ${(\alpha)}_{ik}$ is the rate of deformation tensor for α th fluid and

$$(\alpha)$$
 $\eta = (\alpha) \rho - (\alpha) \overline{\rho}$ for $\alpha = 1, 2...n$ (2.3-11)

so

$$\frac{\partial^{(\alpha)}}{\partial t} = \frac{\partial^{(\alpha)}}{\partial t} for \ \alpha = 1, 2...n \qquad (2.3-12)$$

If we substitute (2.3-12) and (2.3-10) into the partial time derivative of (2.3-9) we obtain

$$P_{O} \sum_{\alpha=1}^{n} {}^{(\alpha)}C^{(\alpha)}f_{kk} + (R - P_{O})\frac{\partial e_{mm}}{\partial t} = 0 \qquad (2.3-13)$$

In the case of a binary mixture of a fluid and an elastic solid $\alpha = 1$ and ${}^{(1)}C = 1$, therefore the equation (2.3-13) reduces to

$$f_{kk} = \frac{P_o - R}{P_o} \frac{\partial e_{mm}}{\partial t}$$
(2.3-14)

$$0 < R \leq 1$$
 $0 < P_0 \leq 1$

where f_{ij} is the rate of deformation tensor of fluid. Biot [9] stated the incompressibility condition for a mixture of a solid modeled by rigid spheres connected by helical springs, and an incompressible fluid. Such a relation can be obtained from relation (2.3-14) by setting R = 1 and integrating the equation.

The incompressibility condition (2.3-14) reduces to

$$E_{kk} = 0$$
 (2.3-15)

in the following cases:

a) steady-state case where $\frac{\partial e_{mm}}{\partial t} = 0$ b) the solid is rigid so $e_{mm} = 0$

2.4 Constitutive Equations

For a binary mixture of a linear Newtonian fluid and a non-linear elastic solid, Green and Steel [26] postulated the following constitutive equations.

$$m_1 = 0 \qquad m_2 = 0 \qquad (2.4-1)$$

$$A = A\left(\frac{\partial x_{i}}{\partial x_{j}}, \rho_{2}, T\right)$$
 (2.4-2)

$$S = S\left(\frac{\partial x_{i}}{\partial x_{j}}, \rho_{2}, T\right)$$
(2.4-3)

where

$$A = U - TS$$
 (2.4-4)

is Helmholtz free energy

$$\frac{1}{2}(\sigma_{ki} + \sigma_{ik}) = A_{ik} + A_{ikrs}f_{rs} + A_{ikj}(u_j - v_j) \quad (2.4-5)$$

$$\frac{1}{2}(\tau_{ki} + \tau_{ik}) = B_{ik} + B_{ikrs}f_{rs} + B_{ikj}(u_j - v_j) \quad (2.4-6)$$

$$\frac{1}{2}(\sigma_{ki} - \sigma_{ik}) = -\frac{1}{2}(\tau_{ki} - \tau_{ik}) = D_{ki} + D_{kirs}f_{rs} + D_{kij}(u_j - v_j)$$
(2.4-7)

$$\pi_{i} = a_{i} + a_{irs} f_{rs} + a_{ij} (u_{j} - v_{j}) \qquad (2.4-8)$$

$$q_{k} = \beta_{k} \left(T, \frac{\partial T}{\partial y_{i}}, \rho_{2}, \frac{\partial x_{r}}{\partial x_{s}} \right) + \beta_{kj} \left(T, \frac{\partial T}{\partial y_{i}}, \rho_{2} \frac{\partial x_{r}}{\partial x_{s}} \right) (u_{j} - v_{j})$$

$$(2.4-9)$$

$$\overline{\pi}_{i} = \overline{a}_{i} + \overline{a}_{irs} f_{rs} + \overline{a}_{ij} (u_{j} - v_{j})$$

$$(2.4-10)$$

where all the coefficients depend on $\frac{\partial \mathbf{x}_{r}}{\partial \mathbf{x}_{s}}$, ρ_{2} , T and $\overline{\rho}_{1}$. The dependence of the constitutive equations on the acceleration is excluded simply because we then would have to include non linear terms in f_{rs} , Λ_{rs} and $(\mathbf{u}_{i} - \mathbf{v}_{i})$ to satisfy the invariance conditions. The above coefficients should also satisfy suitable symmetry conditions such as

$$B_{ikrs} = B_{kirs} = B_{iksr}$$
(2.4-11)

By a usual invariance argument, they deduced that the dependency on $\frac{\partial \mathbf{x}_r}{\partial \mathbf{X}_q}$ should appear as a function of \mathbf{E}_{pq} where $\mathbf{E}_{pq} = \frac{\partial \mathbf{x}_i}{\partial \mathbf{X}_p} \frac{\partial \mathbf{x}_i}{\partial \mathbf{X}_q}$ (2.4-12)

If the coefficients A_{ik} are assumed to depend on displacement gradients only through ρ_1 , then they become isotropic functions of ρ_1 , ρ_2 and T thus:

$$B_{ikrs} = \lambda \delta_{ik} \delta_{rs} + \mu (\delta_{ir} \delta_{ks} + \delta_{is} \delta_{kr}) \qquad (2.4-13)$$

where δ_{ij} is the Kroneckerdelta and λ , μ are scalar functions of ρ_1 , ρ_2 and T. By use of (2.4-10 and (1.3-19) into (1.3-14) they obtained
$$\overline{a}_{i}(u_{i} - v_{i}) + \overline{a}_{irs}f_{rs}(u_{i} - v_{i}) + \overline{a}_{ij}(u_{i} - v_{i})(u_{j} - v_{j}) = 0$$
(2.4-14)

It is easily deduced that

$$\overline{a}_{i} = 0$$
 $\overline{a}_{irs} = 0$ $\overline{a}_{(ij)} = 0$ (2.4-15)
where $\overline{a}_{(ij)}$ is the symmetric part of \overline{a}_{ij} .

<u>Thermodynamic Consideration</u>.--Using (1.3-19) in (1.3-21) and then applying to an arbitrary volume it was obtained that

$$\rho T_{Dt}^{DS} - \rho r + q_{k,k} - \frac{q_k^T r_k}{T} \ge 0$$
 (2.4-16)

With the help of (1.3-8,20) and (2.4-1-9), this equation becomes

$$- \rho \left(S + \frac{\partial A}{\partial T} \right) \frac{DT}{Dt} + \left[A_{ik} - \frac{1}{2} \rho \frac{\partial x_{i}}{\partial x_{r}} \frac{\partial x_{k}}{\partial x_{s}} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right) \right] d_{ik}$$

$$+ \left(B_{ik} + \rho \rho_{2} \frac{\partial A}{\partial \rho_{2}} \delta_{ik} \right) f_{ik} + \left[a_{i} - \rho_{1} \frac{\partial A}{\partial \rho_{2}} \frac{\partial \rho_{2}}{\partial y_{i}} + \frac{1}{2} \rho_{2} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{rs}} \right) \right] d_{ik}$$

$$+ \frac{\partial A}{\partial e_{sr}} \left(\frac{\partial e_{rs}}{\partial x_{i}} \right) \left[(u_{i} - v_{i}) + (B_{rsi} + a_{irs}) f_{rs} (u_{i} - v_{i}) \right] d_{ik}$$

$$+ A_{ikrs} d_{ik} f_{rs} + A_{ikj} d_{ik} (u_{j} - v_{j}) + a_{ij} (u_{i} - v_{i}) (u_{j} - v_{j}) d_{ik}$$

$$+ B_{ikrs} f_{ik} f_{rs} + D_{ki} (\Gamma_{ik} - \Lambda_{ik}) + D_{kij} (\Gamma_{ik} - \Lambda_{ik}) (u_{j} - v_{j}) d_{ik} d_{ij} d_{ij$$

+
$$D_{kirs} f_{rs} (\Gamma_{ik} - \Lambda_{ik}) - \frac{F_{k}T'k}{T} - \frac{F_{kj}T'k(u_j - v_j)}{T} \ge 0$$
 (2.4-17)

The inequality (2.4-17) should hold for arbitrary values of d_{ik} , f_{ik} , $u_i - v_i$ and $\Gamma_{ik} - \Lambda_{ik}$, provided that the incompressibility condition (2.3-14) is satisfied. We rewrite the incompressibility condition in the following form:

$$f_{11} + f_{22} + f_{33} + a_1(d_{11} + d_{22} + d_{33}) = 0$$
 (2.4-18)

where
$$a_1 = \frac{R - P_0}{P_0}$$

and (2.4-17) as

 $N_{ik}f_{ik} + M_{ik}d_{ik} + rest of the terms \ge 0$ (2.4-17a) where

$$N_{ik} = B_{ik} + \rho \rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ik} \qquad (2.4-19)$$

$$M_{ik} = A_{ik} - \frac{1}{2} \rho \frac{\partial x_i}{\partial x_r} \frac{\partial x_k}{\partial x_s} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right)$$
(2.4-20)

Eliminating
$$f_{11}$$
 from (2.4-17) by use of (2.4-18) gives
 $f_{22}(N_{22} - N_{11}) + f_{33}(N_{33} - N_{11}) + f_{12}N_{12} + d_{11}(M_{11} - a_1N_{11})$
 $+ d_{22}(M_{22} - a_1N_{11}) + d_3(M_{33} - a_1N_{11}) + d_{12}M_{12} + \cdots > 0$
(2.4-21)

To satisfy the inequality we should have

$$S = -\frac{\partial A}{\partial T} \qquad (2.4-22)$$

$$N_{ik} = \overline{p} \delta_{ik} \qquad (2.4-23)$$

$$M_{ik} = a_1 \overline{p} \sigma_{ik} \qquad (2.4-24)$$

$$a_{i} = \rho_{1} \frac{\partial A}{\partial \rho_{2}} \frac{\partial \rho_{2}}{\partial y_{i}} - \frac{1}{2} \rho_{2} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right) \qquad (2.4-25)$$

$$D_{ki} = 0$$
 $D_{kij} = 0$ $D_{kirs} = 0$ (2.4-26)

$$A_{ikj} = 0$$
 $A_{ikrs} = 0$ (2.4-27)

where $\overline{p} = N_{11}$ is a scalar function. Now using (2.4-22-27) in (2.4-17) yields $a_{(ij)}(u_i - v_i) + (B_{rsi} + a_{irs})f_{rs}(u_i - v_i) + B_{ikrs}f_{ik}f_{rs} \ge 0$ (2.4-28)

The above equation imposes some inequality conditions on $a_{(ij)}$, $(B_{rsi} + a_{irs})$ and B_{ikrs} . Substituting the last results

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into constitutive equations and assuming that B_{ikrs} , B_{ikj} , a_{ij} are functions of $\frac{\partial x_i}{\partial X_j}$ through ρ_1 only, we obtain the following:

$$B_{ikj} = a_{irs} = 0 \qquad a_{ij} = a \delta_{ij} \qquad (2.4-29)$$

$$\sigma_{ik} = \sigma_{ki} = \frac{1}{2} \rho \frac{\partial x_i}{\partial x_r} \frac{\partial x_k}{\partial x_s} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right) + a_1 \overline{p} \delta_{ik} \quad (2.4-30)$$

$$\pi_{i} = \rho_{1} \frac{\partial A}{\partial \rho_{2}} \frac{\partial \rho_{2}}{\partial y_{i}} - \frac{1}{2}\rho_{2} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right) \frac{\partial e_{rs}}{\partial x_{i}} + a(u_{i} - v_{i})$$

$$(2.4-32)$$

where we have $\mu \ge 0$ $\lambda + \frac{2}{3}\mu \ge 0$ $a \ge 0$ (2.4-33)

Further Reduction for Linear Small Theory.--In order to obtain the main constitutive equations subject to the basic assumptions of this chapter, we adopt an expression similar to the one used by Green and Steel [26] for the Helmholtz free energy, in the form

$$\overline{\rho}A = A_0 + \alpha_1 e_{mm} + \alpha_2 \eta + \frac{1}{2} \alpha_4 e_{mm} e_{nn} + \alpha_5 e_{mn} e_{mn}$$
$$+ \frac{1}{2} \alpha_6 \eta^2 + \alpha_8 e_{mm} \eta \qquad (2.4-34)$$

where $\overline{\rho} = \overline{\rho}_1 + \overline{\rho}_2$, the constants A_0 , α_1 depend on initial densities of each substance. If we substitute (2.4-34) into (2.4-30-31) and retain the linear terms only we obtain:

$$\sigma_{ik} = \alpha_1 \delta_{ik} + \left(\alpha_4 - \frac{\alpha_1 \overline{\rho}_1}{\overline{\rho}}\right) e_{mm} \delta_{ik} + 2\left(\alpha_1 + \alpha_5\right) e_{ik}$$

$$+ \left(\alpha_8 + \frac{\alpha_1}{\overline{\rho}}\right) n \delta_{ik} + a_1 \overline{p} \delta_{ik} \qquad (2.4-35)$$
and
$$T_{in} = -\left\{\overline{\rho}, \alpha_{in} - \overline{p} + \left[\overline{\rho}, \alpha_{in} + \left(\overline{\rho}, + \overline{\rho}\right), \frac{\alpha_2}{2}\right] n + \overline{\rho}, \left(\alpha_{in} - \frac{\overline{\rho}_1 \alpha_2}{2}\right) e_{in}\right\}$$

$$\tau_{ik} = -\left\langle \overline{\rho}_{2}\alpha_{2} - \overline{p} + \left[\overline{\rho}_{2}\alpha_{6} + (\overline{\rho}_{2} + \overline{\rho}) \frac{z}{\rho} \right] \eta + \overline{\rho}_{2} \left(\alpha_{8} - \frac{1}{\overline{\rho}} \right) e_{mm} \right\rangle$$

$$\delta_{ik} + \lambda f_{rr} \delta_{ik} + 2\mu f_{ik} \qquad (2.4-36)$$

From (2.3-9) it is easily seen that:

$$\eta = \overline{\rho}_2 \frac{R - P_0}{P_0} e_{mm} \qquad (2.4-37)$$

Substituting for η in constitutive equations and applying the assumption of zero initial stresses, yields:

$$\sigma_{ik} = \left(\alpha_4 + \overline{\rho}_2 \frac{A - P_0}{P_0} \alpha_8\right) e_{mm} \delta_{ik} + 2\alpha_5 e_{ik} + a_1 \overline{p} \delta_{ik}$$
(2.4-38)

$$\tau_{ik} = -\left\{ \overline{\rho}_{2} \alpha_{8} + \overline{\rho}_{2}^{2} a_{1} \alpha_{6} \right\} e_{mm} + \overline{p} + \lambda f_{rr} \delta_{ik} + 2\mu f_{ik} \qquad (2.4-39)$$

$$\pi_{i} = a(u_{i} - v_{i})$$
 (2.4-40)

Now if we define p to be

$$- \mathbf{p} = \frac{1}{a_1} \left[a_1 \overline{\mathbf{p}} - a_1 (\overline{\rho}_2 \alpha_8 + \overline{\rho}_2^2 a_1 \alpha_6) e_{mm} \right] \quad (2.4-41)$$

the constitutive equations can be written in the form:

$$\sigma_{ik} = -a_1 p \delta_{ik} + 2a_2 e_{ik} + a_3 e_{mm} \delta_{ik} \qquad (2.4-42)$$

$$\tau_{ik} = -p \, \delta_{ik} + \lambda f_{rr} \delta_{ik} + 2\mu f_{ik} \qquad (2.4-43)$$

where

$$a_2 = \alpha_5$$

$$a_3 = \alpha_4 + \overline{\rho}_2 a_1 \alpha_8 + a_1^2 \overline{\rho}_2^2 \alpha_6 + a_1 \overline{\rho}_2 \alpha_8$$
 (2.4-44)

and where p is equivalent to thermodynamic pressure defined for simple media.

2.5 Field Equations

In this section we summarize the results obtained so far and state the field equations. Combining (1.3-12) and (1.3-17) and considering (2.4-1,40), we obtain the equations of motion:

$$\frac{\partial \sigma_{ki}}{\partial x_k} + \overline{\rho}_1 F_i - a(u_i - v_i) = \overline{\rho}_1 a_i \qquad (2.5-1)$$

$$\frac{\partial^{\tau} \mathbf{k}_{\mathbf{i}}}{\partial \mathbf{x}_{\mathbf{k}}} + \overline{\rho}_{2} \mathbf{G}_{\mathbf{i}} + \mathbf{a} (\mathbf{u}_{\mathbf{i}} - \mathbf{v}_{\mathbf{i}}) = \overline{\rho}_{2} \mathbf{g}_{\mathbf{i}}$$
(2.5-2)

where

~

$$a_{i} = \frac{\partial^{2} \omega_{i}}{\partial t^{2}}, g_{i} = \frac{\partial v_{i}}{\partial t}, u_{i} = \frac{\partial \omega_{i}}{\partial t}$$
 (2.5-3)

The constitutive equations are

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$$\sigma_{ik} = \sigma_{ki} = -a_1 p \delta_{ik} + 2a_2 e_{ik} + a_3 e_{mm} \delta_{ik} \qquad (2.5-4)$$

$$\tau_{ik} = \tau_{ki} = -p \, \delta_{ik} + \lambda f_{rr} \, \delta_{ik} + 2\mu f_{ik} \qquad (2.5-5)$$

$$\pi_{i} = a(u_{i} - v_{i}) \qquad (2.5-6)$$

and the strain and rate of deformation tensors are

$$f_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$
 (2.5-7)

$$e_{ij} = \frac{1}{2} \left(\frac{\partial \omega_{i}}{\partial x_{j}} + \frac{\partial \omega_{j}}{\partial x_{i}} \right)$$
(2.5-8)

$$f_{kk} = a_1 \frac{\partial e_{mm}}{\partial t}$$
 (2.5-9)

Equations (2.5-1-9) are 43 equations for 43 unknowns, namely σ_{ki} , τ_{ki} , π_i , a_i , g_i , p, u_i , ω_i , f_{ij} and e_{ij} . The system is mathematically complete provided that the appropriate boundary conditions are given.

2.6 The Case of Compressible Fluid

The constitutive equations for a binary mixture of a compressible Newtonian fluid and a linear elastic solid were obtained by Green and Steel [26]. Under the assumptions of this chapter those constitutive equations become:

$$\sigma_{ik} = \alpha_4 e_{mm} \delta_{ik} + 2\alpha_5 e_{ik} + \alpha_8 \eta \delta_{ik} \qquad (2.6-1)$$

$$\tau_{ik} = \{-\overline{\rho}_2 \alpha_6 \eta - \overline{\rho}_2 \alpha_8 e_{mm} + \lambda f_{rr} \} \delta_{ik} + 2\mu f_{ik} \quad (2.6-2)$$

If we define p to be

$$p = \overline{\rho}_2 \alpha_6 \eta + \overline{\rho}_2 \alpha_8 e_{mm} \qquad (2.6-3)$$

then the constitutive equations can be written as

$$\tau_{ik} = -p\delta_{ik} + \lambda f_{rr} \delta_{ik} + 2\mu f_{ik} \qquad (2.6-4)$$

$$\sigma_{ik} = \overline{a}_{1} p \delta_{ik} + 2\overline{a}_{2} e_{ik} + \overline{a}_{3} e_{mm} \delta_{ik} \qquad (2.6-5)$$

where

$$\overline{a}_1 = \frac{\alpha_8}{\overline{\rho}_2 \alpha_6}$$
 $\overline{a}_2 = \alpha_5$ $\overline{a}_3 = \alpha_4 - \frac{\alpha_8}{\alpha_6}$ (2.6-6)

Here again p is equivalent to a thermodynamic pressure. In analogy to the theory of simple Newtonian fluids, it is seen that in the case of a compressible fluid the hydrostatic pressure p has an equation of state of the form (2.6-3), while in the case of incompressibility, p introduces a new unknown to the system of equations together with an additional equation namely the incompressibility condition.

It is seen that the constitutive equation of the solid is coupled with that of the fluid through the thermodynamic pressure, while the partial stresses of the fluid are coupled with those of the solid by the solid dilatation through equation of state or through the incompressibility condition. We keep in mind that the coefficients in these equations are not numerically the same as those of the corresponding single media, however we notice that all the equations must reduce to those for a single elastic solid or fluid when ρ_2 or ρ_1 vanishes respectively.

It is perhaps worth noting that the velocity gradients of the solid were not included in the constitutive equations; however, if the partial stresses are assumed to depend on those variables as in [5] and [22] then the suitable modifications should be made.

2.7 <u>The Physical Interpretation of the Coefficients</u> of Constitutive Equations for Partial Stresses

The equations (2.4-38,39) show that the seven coefficients α_4 , α_5 , α_6 , α_8 , a_1 , λ and μ are to be determined. However in the incompressible case, the knowledge of five coefficients a_1 , a_2 , a_3 , λ and μ is sufficient. This reduction of number of independent constants is due to the fact that η is related to e_{mm} through the equation (2.4-37) and hence the constants α_6 and α_8 in the equation (2.4-34) can be absorbed by α_4 and α_5 .

In order to relate these coefficients to those of the solid and fluid components, let us first consider a medium under some external forces exerted by a highly permeable agent such that the fluid pressure remains zero on the boundary at all times. After the transient part of the motion, the system arrives at an equilibrium state where the fluid pressure as well as the fluid velocity are zero. The constitutive equations (2.5-4) or (2.6-5) hold during the motion as well as in equilibrium state. We conclude that

$$\overline{a}_2 = a_2 = G_e$$

 $a_3 = \overline{a}_3 = \lambda_e$

where G_e and λ_e are the shear modulus and Lame's constant of the saturated porous elastic material.

As it is pointed out in [13] a dry porous medium might not exhibit the same elastic properties as that of the saturated one. As an example they cited the case where elastic properties result from surface forces of a capillary nature at the interfaces of the fluid and solid. Whenever such differences are negligible the above coefficients are the ordinary properties of porous elastic solid and independent of the fluid components. The constant a_1 is also a property of solid where a knowledge of initial porosity P_0 and compressibility coefficients R of non-porous part is required. Again it is conceivable to assume that these two constants may also depend on the fluid properties and the porosity may depend on the penetrability of the fluid.

The physical interpretation of λ and μ is the same as of the single fluid; however, the numerical values of λ and

 $\boldsymbol{\mu}$ might differ because of the presence of the solid.

Finally, in the compressible case the constant \overline{a}_1 remains to be determined which in turn requires the knowledge of α_6 and α_8 . In order to determine these two constants use should be made of the equation of state (2.6-3). We may apply a constant fluid pressure p to a medium confined in a rigid boundary such that $e_{mm} = 0$ and α_6 would be found directly. Then a pressure p may be applied to the medium keeping the solid boundary tractions zero. The above two tests will supply the values of α_6 and α_8 .

CHAPTER III

METHODS OF SOLUTION

The purpose of this chapter is to furnish the general solution of the system of equations for the steady-state case by means of displacement and stress functions. First the equations of Chapter II will be reduced to a system of differential equations in terms of displacements of the solid and the velocity vector of the fluid and the hydrostatic pressure only.

A general method of solution is presented by means of scalar and vector functions satisfying harmonic, biharmonic and Helmholtz's equations. A stress function is presented to determine the partial stresses of the solid for steady-state plain-strain problems. The choice between the two different the methods of finding the partial stresses of/solid depends very much on problem at hand and especially on the prescribed boundary conditions.

For the sake of illustration, the theory is applied to one and two dimensional steady-state problems. Atkin <u>etial</u>. [5] has presented a uniqueness theorem under a set of boundary conditions. Although the present theory together with the sets of boundary conditions used in the problems of this chapter is different from that used by Atkin, a similar

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uniqueness theorem may be constructed; however we do not attempt to do so.

3.1 Displacement Equations

If we substitute the rate of deformation and strain tensors in terms of displacements and velocities into the constitutive equations for partial stresses and then write the results in tensorial form, we obtain

$$\tilde{\sigma} = -a_1 p \tilde{\delta} + a_2 (\tilde{\nabla} \tilde{\omega} + \tilde{\omega} \tilde{\nabla}) + a_3 (\tilde{\nabla} \cdot \tilde{\omega}) \tilde{\delta}$$
(3.1-1)

$$\tilde{\tau} = -\mathbf{p} \ \tilde{\delta} + \mu (\bar{\nabla} \vec{\nabla} + \bar{\nabla} \bar{\nabla}) + \lambda (\bar{\nabla} \cdot \vec{\nabla}) \tilde{\delta} \qquad (3.1-2)$$

where symbols with \approx overhead denote second order tensors and those with arrows on the top denote vectors. Substitution of the equations (3.1-1,2) into the equations of motions (2.5-1,2) and using (2.5-6) for diffusive force yields:

$$-a_{1}\vec{\nabla}p + a_{2}\vec{\nabla}\cdot(\vec{\nabla}\vec{\omega} + \vec{\omega}\vec{\nabla}) + a_{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{\omega}) - a(\vec{U} - \vec{\nabla}) = \overline{\rho}_{1}\frac{\partial^{2}\vec{\omega}}{\partial t^{2}}$$

and (3.1-3a)

 $-\vec{\nabla}p + \lambda \vec{\nabla} (\vec{\nabla} \cdot \vec{\nabla}) + \mu \vec{\nabla} \cdot (\vec{\nabla} \vec{\nabla} + \vec{\nabla} \vec{\nabla}) + a(\vec{U} - \vec{\nabla}) = \overline{\rho}_2 \frac{\partial \vec{V}}{\partial t} \quad (3.1-3b)$ where the body forces are neglected. Observing the fact that:

and

$$\vec{\nabla} \cdot \vec{\nabla} (\vec{}) = \nabla^2 (\vec{})$$

 $\vec{\nabla} \cdot () \vec{\nabla} = \vec{\nabla} (\vec{\nabla} \cdot \vec{})$

The equations (3.1-3,4) together with the incompressibility condition can be written as the following:

$$-a_{1}\vec{\nabla}p + (a_{2} + a_{3})\vec{\nabla}(\vec{\nabla}\cdot\vec{\omega}) + a_{2}\nabla^{2}\vec{\omega} - a(\vec{U} - \vec{V}) = \overline{\rho}_{1}\frac{\partial^{2}\vec{\omega}}{\partial t}$$
(3.1-4)

$$-\vec{\nabla}p + (\lambda + \mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{\nabla}) + \mu\nabla^{2}\vec{\nabla} + a(\vec{U} - \vec{\nabla}) = \overline{P}_{2}\frac{\partial V}{\partial t} \qquad (3.1-5)$$

$$\vec{\nabla} \cdot \vec{\nabla} = a_1 \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{\omega} \qquad (3.1-6)$$

In steady-state case the time derivatives of dependent variables as well as the velocity of the solid vanish, hence we have:

$$-a_{1}\vec{\nabla}p + (a_{2} + a_{3})\vec{\nabla}(\vec{\nabla}\cdot\vec{\omega}) + a_{2}\nabla^{2}\vec{\omega} + a\vec{\nabla} = 0 \qquad (3.1-7)$$

$$-\vec{\nabla}p + \mu \nabla^2 \vec{\nabla} - a\vec{\nabla} = 0 \qquad (3.1-8)$$

$$\vec{\nabla} \cdot \vec{\nabla} = 0 \tag{3.1-9}$$

where use has been made of the equation (3.1-9) in (3.1-8). The equations (3.1-7-9) constitute seven differential equations for seven unknowns, $\vec{\nabla}$, $\vec{\omega}$ and p. By application of $\vec{\nabla}$ operator to the equation (3.1-8) and making use of (3.1-9)we obtain

$$\nabla^2 \mathbf{p} = 0$$
 (3.1-10)

It is seen that the hydrostatic pressure satisfies the Laplace equation, therefore any harmonic scalar function which satisfies the required boundary conditions would be a proper expression for p.

The general solution of \vec{v} consists of the general solution of the reduced equation

$$\mu \nabla^2 \vec{v} - a \vec{v} = 0$$
 (3.1-11)

plus any particular integral of the equation (3.1-8). If we denote the general solution of (3.1-11) by $\stackrel{\rightarrow}{V}^{(r)}$ and notice that a particular solution for the equation (3.1-8) is:

$$\vec{v}$$
 particular = $-\frac{1}{a}\vec{v}p$ (3.1-12)

then the complete solution for (3.1-8) is obtained to be:

$$\vec{v} = \vec{v}^{(r)} - \frac{1}{a} \vec{v} p$$
 (3.1-13)

We observe that the general solution $\vec{V}^{(r)}$ of the reduced equation must satisfy the condition (3.1-9); this imposes some restrictions on the solution. In two dimensional case equation (3.1-9) reduces to

$$\frac{\partial \mathbf{v}_{\mathbf{x}}^{(\mathbf{r})}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}_{\mathbf{y}}^{(\mathbf{r})}}{\partial \mathbf{y}} = 0 \qquad (3.1-14)$$

It then follows that the velocity vector $\vec{v}^{(r)}$ can be derived from a scalar function ψ such that:

$$v_x^{(r)} = \frac{\partial \psi}{\partial y}$$
 $v_y^{(r)} = -\frac{\partial \psi}{\partial x}$ (3.1-15)

Substituting for $v_x^{(r)}$ and $v_y^{(r)}$ into the equation (3.1-11) yields:

$$\frac{\partial}{\partial \mathbf{y}} [\mathbf{\mu} \nabla^2 \psi - \mathbf{a} \psi] = 0 \qquad (3.1-16)$$

$$\frac{\partial}{\partial \mathbf{x}} [\mu \nabla^2 \psi - \mathbf{a} \psi] = 0 \qquad (3.1-17)$$

The above pair of equations simply imply that the expression inside the bracket is a constant. This constant can be assumed to be zero without any loss of generality in the velocity solution. Therefore the problem reduces to finding a function ψ satisfying the Helmholtz equation.

$$\mu \nabla^2 \psi - a \psi = 0$$
 (3.1-18)

It is seen that the general solution of the velocity field consists of the linear combination of two scalar functions which satisfy the Laplace and Helmholtz equations. In order to obtain a general solution for the displacements of solid, we add equation (3.1-7) and (3.1-8)

$$-(a_{1} + 1)\vec{\nabla}p + (a_{2} + a_{3})\vec{\nabla}(\vec{\nabla}\cdot\omega) + a_{2}\nabla^{2}\vec{\omega} + \mu\nabla^{2}\vec{\nabla} = 0$$
(3.1-19)

Equation (3.1-19) shows that the vector function $a_2 \nabla^2 \overset{2}{\omega} + \mu \nabla^2 \overset{2}{\nabla}$ is irrotational (referring to Helmholtz's representation) and hence can be expressed as the divergence of a scalar function $\overline{\phi}$,

$$a_2 \nabla^2 \overset{*}{\omega} + \mu \nabla^2 \overset{*}{\nabla} = \overset{*}{\nabla} \overline{\phi} \qquad (3.1-20)$$

where without loss of generality and under sufficient smoothness and integrability conditions we can find another scalar function ϕ such that

$$\overline{\phi} = \nabla^2 \phi \qquad (3.1-21)$$

Observing that

$$\vec{\nabla} \nabla^2 () = \nabla^2 \vec{\nabla} () \qquad (3.1-22)$$

we obtain,

$$\nabla^{2} [a_{2}\vec{\omega} + \mu \vec{\nabla} - \vec{\nabla}\phi] = 0 \qquad (3.1-23)$$

Let

$$a_2\vec{\omega} + \mu\vec{\nabla} - \vec{\nabla}\phi = \vec{\Psi} \qquad (3.1-24)$$

where $ec{\psi}$ is a vector function which satisfies

$$\nabla^2 \overline{\psi} = 0 \tag{3.1-25}$$

The general solution of $\vec{\omega}$ is

$$\vec{\omega} = \frac{1}{a_2} \left(\vec{\psi} + \vec{\nabla} \phi \right) - \frac{\mu}{a_2} \vec{\nabla}$$
(3.1-26)

Now if we apply the operator $\vec{\forall}$ to the equation (3.1-7) and make use of (3.1-9) and (3.1-10) we obtain

$$\nabla^2 \left(\vec{\nabla} \cdot \vec{\omega} \right) = 0 \qquad (3.1-27)$$

Combining (3.1-19), (3.1-20) and (3.1-21) yields:

$$-(a_{1} + 1)\vec{\nabla}p + (a_{2} + a_{3})\vec{\nabla}(\vec{\nabla}\cdot\vec{\omega}) + \vec{\nabla}\nabla^{2}\phi = 0 \qquad (3.1-28)$$

By applying $\vec{\nabla}$ operator to the equation (3.1-28) and using the relations (3.1-10) and (3.1-27) we obtain $\nabla^4 \phi = 0$ (3.1-29)

Hence ϕ is a biharmonic scalar function, and we again observe that the general solution of the system of equations (3.1-7-9) reduces to a linear combination: of the general solutions of some classical equations whose properties are rather well established. The solution to a particular problem would be obtained by choice of these functions such that they satisfy the prescribed boundary conditions. Note that ϕ and $\overline{\psi}$ are not independent and they must satisfy

$$\nabla^2 \phi = \frac{(a_1 + 1) a_2}{2a_2 + a_3} p - \frac{a_2}{2a_2 + a_3} \vec{\nabla} \cdot \vec{\psi} \qquad (3.1-30)$$

3.2 Solution of the Steady-State Plain Strain Problems by Means of Stress Function

In the steady-state plain strain case the equations of motion (2.5-1), in the absence of external body force, reduce to the following:

$$\begin{cases} \frac{\partial \sigma_{\mathbf{x}\mathbf{x}}}{\partial \mathbf{x}} + \frac{\partial \sigma_{\mathbf{x}\mathbf{y}}}{\partial \mathbf{y}} + av_{\mathbf{x}} = 0 \qquad (3.2-1)\\ \frac{\partial \sigma_{\mathbf{y}\mathbf{y}}}{\partial \mathbf{y}} + \frac{\partial \sigma_{\mathbf{x}\mathbf{y}}}{\partial \mathbf{x}} + av_{\mathbf{y}} = 0 \qquad (3.2-2) \end{cases}$$

Substituting for v_x and v_y from (3.1-15) and rearranging the terms, we obtain

$$\frac{\partial}{\partial \mathbf{x}} (\sigma_{\mathbf{x}\mathbf{x}} - \mathbf{p}) + \frac{\partial}{\partial \mathbf{y}} (\sigma_{\mathbf{x}\mathbf{y}} + \mathbf{a}\psi) = 0 \qquad (3.2-3)$$

$$\frac{\partial}{\partial y} (\sigma_{yy} - p) + \frac{\partial}{\partial x} (\sigma_{xy} - a\psi) = 0 \qquad (3.2-4)$$

In analogy to the theory of linear elasticity [23] it is readily seen that there exist two different scalar functions Φ and χ such that

$$\sigma_{xx} - p = \frac{\partial \Phi}{\partial y} \qquad \sigma_{xy} + a\psi = -\frac{\partial \Phi}{\partial x} \qquad (3.2-5)$$

$$\sigma_{yy} - p = \frac{\partial \chi}{\partial x}$$
 $\sigma_{xy} - a\psi = -\frac{\partial \chi}{\partial y}$ (3.2-6)

where Φ and χ are functions of x and y only. By adding the equations (3.2-5)₂ and (3.2-6)₂ and taking into account that $\sigma_{\chi y} = \sigma_{\chi \chi}$ we obtain

$$\frac{\partial \chi}{\partial y} = \frac{\partial \Phi}{\partial x} + 2a\psi \qquad (3.2-7)$$

The condition (3.2-7) imposes a restriction on functions Φ and χ . We observe that this condition would be fulfilled if a function θ is defined such that

$$\chi = \frac{\partial \theta}{\partial x} + a \int^{Y} \psi \, dy \qquad \Phi = \frac{\partial \theta}{\partial y} - a \int^{X} \psi \, dx \qquad (3.2-8)$$

We now observe that if θ is any arbitrary function of x and y, the Φ and χ functions derived from (3.2-8), would satisfy (3.2-7) and hence the equations of motion. Similar to the case of two dimensional elasticity, we call θ a stress function. Substituting for Φ and χ in relations (3.2-5) and (3.2-6) we obtain:

$$\sigma_{\mathbf{x}\mathbf{x}} = \mathbf{p} + \frac{\partial^2 \theta}{\partial \mathbf{y}^2} - \mathbf{a} \int \frac{\mathbf{x}}{\partial \mathbf{y}} \frac{\partial \psi}{\partial \mathbf{y}} d\mathbf{x} \qquad (3.2-9)$$

$$\sigma_{yy} = p + \frac{\partial^2 \theta}{\partial x^2} + a \int^y \frac{\partial \psi}{\partial x} dy \qquad (3.2-10)$$

$$\sigma_{xy} = -\frac{\partial^2 \theta}{\partial x \partial y}$$
(3.2-11)

Again, as in the theory of elasticity, the stress function θ is the only unknown function, but it is necessary to use the compatibility condition which puts a condition on the otherwise arbitrary stress function. We observe that the compatibility conditions of the present theory are identical to those of linear elasticity. For the compatibility conditions are nothing more than the mathematical conditions which insure the integrability of the strain-displacement equations. Since the strain-displacement relations for the solid are identical in form to those of classical elasticity, the compatibility conditions must be of the same mathematical form. We, therefore, have

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2 \frac{\partial^2 e_{xy}}{\partial x \partial y}$$
(3.2-12)

The non-vanishing stresses in this case are

$$\sigma_{xx} = -a_1 p + 2a_2 e + a_3 (e_{xx} + e_{yy})$$
 (3.2-13)

$$\sigma_{yy} = -a_1 p + 2a_2 e_{yy} + a_3 (e_{xx} + e_{yy})$$
 (3.2-14)

$$\sigma_{xy} = 2a_2 e_{xy}$$
 (3.2-15)

Finding strains in terms of stresses yields:

$$e_{yy} = \frac{1}{4a_2(a_2 + a_3)} \left[(2a_2 + a_3)(\sigma_{yy} + a_1p) - a_3(\sigma_{xx} + a_1p) \right]$$
(3.2-16)
$$e_{xx} = \frac{1}{4a_2(a_2 + a_3)} \left[(2a_2 + a_3)(\sigma_{xx} + a_1p) - a_3(\sigma_{yy} + a_1p) \right]$$
(3.2-17)

$$e_{xy} = \frac{1}{2a_2} \sigma_{xy}$$
 (3.2-18)

Substituting for e_{xx} , e_{yy} and e_{xy} into compatibility equation and making use of relation (3.1-10) yields:

$$(2a_{2} + a_{3})\left[\frac{\partial^{2}\sigma_{xx}}{\partial y^{2}} + \frac{\partial^{2}\sigma_{yy}}{\partial x^{2}}\right] - a_{3}\left[\frac{\partial^{2}\sigma_{xx}}{\partial x^{2}} + \frac{\partial^{2}\sigma_{yy}}{\partial y^{2}}\right] = 4(a_{2} + a_{3})\frac{\partial^{2}\sigma_{xy}}{\partial x^{\partial y}}$$

$$(3.2-19)$$

Substituting for σ_{xx} , σ_{yy} and σ_{xy} in terms of stress function in the above equation and simplifying the resulted equation, yields

$$\frac{1}{a} \nabla^4 \theta = \int^{\mathbf{x}} \frac{\partial^3 \psi}{\partial y^3} d\mathbf{x} - \int^{\mathbf{y}} \frac{\partial^3 \psi}{\partial \mathbf{x}^3} d\mathbf{y} \qquad (3.2-20)$$

The general solution for θ is obtained by addition of the particular solution of (3.2-20) to a biharmonic scalar function. Again the arbitrary constants should be chosen to satisfy the prescribed boundary conditions.

3.3 <u>One Dimensional Problem of</u> Infinite Plate

As an illustration, we will solve the following simple problem. Let us consider an infinite plate resting on a rigid highly permeable medium and bounded in Cartesian system by the faces $x_1 = 0$ and $x_1 = h$. A constant fluid pressure p_0 is applied to the face $x_1 = 0$. We assume that the lateral displacements can be neglected. Under the above conditions, the only non-vanishing components of the velocity and displacement vectors are

$$\omega_1 = \omega_1(\mathbf{x}_1) \qquad \mathbf{v}_1 = \mathbf{v}_1(\mathbf{x}_1) \qquad (3.3-1)$$

and the boundary conditions are:

at
$$x_1 = 0$$
 $\sigma_{11} + \tau_{11} = -p_0$ (3.3-2)

at
$$x_1 = h$$
 $\omega_1 = 0$ $p = 0$ (3.3-3)

The equation (3.1-10) reduces to:

$$\frac{\partial^2 \mathbf{p}}{\partial \mathbf{x}^2} = 0 \qquad (3.3-4)$$

Integrating the above equation and using the boundary condition $(3.4-3)_2$ yields to

$$p = C_0 (x - h)$$
 (3.3-5)

where C_0 is an arbitrary constant. From the relations (3.1-9) and (3.1-8) we conclude that

$$v_1 = -\frac{C_0}{a}$$
 (3.3-6)

Using the results (3.3-5) and (3.3-6) into (3.1-7) and integrating, gives

$$\omega_{1} = \frac{(a_{1} + 1)C_{0}}{2(2a_{2} + a_{3})} x_{1}^{2} + C_{1}x_{1} - \frac{(a_{1} + 1)C_{0}}{2(2a_{2} + a_{3})} h^{2} - C_{1}h \quad (3.3-7)$$

where use has been made of the boundary condition $(3.3-3)_1$ and C_1 is a new arbitrary constant.

The fluid stresses are obtained from (3.1-2) as follows:

$$r_{ij} = -C_0 (x - h) \delta_{ij}$$
 (3.3-8)

The solid stresses are

$$\sigma_{11} = (a_3 + 2a_2) \frac{\partial \omega_1}{\partial x_1} - a_1 p$$
 (3.3-9)

$$\sigma_{22} = \sigma_{33} = -a_1 p + a_3 \frac{\partial \omega_1}{\partial x_1}$$
 (3.3-10)

$$\sigma_{ij} = 0 \text{ if } i \neq j \qquad (3.3-11)$$

Substituting for ω_1 and p yields to

$$\sigma_{11} = C_0 (x + a_1h) + C_1(2a_2 + a_3)$$
(3.3-12)

$$\sigma_{22} = \sigma_{33} = -a_1 C_0 (x - h) + \frac{(a_1 + 1)a_3 C_0}{(2a_2 + a_3)} + C_1 a_3 (3.3-13)$$

The only remaining boundary condition (3.3-2) gives the constant C_1 in term of C_0

$$C_{1} = \frac{-p_{0} + C_{0}h(1 + a_{1})}{(2a_{2} + a_{3})} C_{0}$$
(3.3-14)

It may be seen that the solution is indeterminate within a constant C_0 . However, this indeterminacy may be removed by specifying the surface porosity at the face x = 0, and therefore prescribing the separate values of σ_{11} and p at that face.

4.3 Semi-Infinite Strip Problem

As another illustration, we consider an infinitely deep and long strip of an elastic solid with width π . We take the Cartesian system (x, y, z) as shown in the figure below.



A fluid pressure p is applied to the surface y = 0. Under the above conditions it is conceivable to assume all variables to be functions of x and y only.

We assume the following boundary condition.

at
$$y = 0$$
 $p = A_1 \cos x$ $v_x = 0$ $\sigma_{yy} = -A_2 \cos x$
 $\sigma_{xy} = 0$ (3.4-1)
at $x = \frac{\pm \pi}{2}$ $p = 0$ $v_y = f(y)$ $\sigma_{xx} = 0$ $\sigma_{xy} = 0$ (3.4-2)
at $y = \infty$ $p = 0$ $v_x = v_y = 0$ (3.4-3)

Let us observe that the solution for a general fluid pressure is similar to the present problem, because any applied fluid pressure f(x) for $+ \pi/2 \ge x \ge -\pi/2$ can be expanded in Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{C} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{C} \quad (3.4-4)$$

where 2C is the interval of Fourier expansion and not necessarily 2π , provided that f(x) satisfies the necessary requirements of Fourier expansion. We also notice that the problem of a strip with width ℓ can be converted to the above problem by a simple change of variable. Let us further comment that the constants A_1 and A_2 depend on fluid pressure and porosity factor P_0 . The equation (3.1-10) reduces to

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \qquad (3.4-5)$$

The general solution for p can be written as:

$$p = \overline{e}^{\alpha y} (A_{\alpha} \cos \alpha x + B_{\alpha} \sin \alpha x) + e^{\alpha y} (C_{\alpha} \cos \alpha x + D_{\alpha} \sin \alpha x) + \overline{e}^{\beta x} (E_{\beta} \cos \beta y + F_{\beta} \sin \beta y) + e^{\beta x} (G_{\beta} \cos \beta y + H_{\beta} \sin \beta y) For all values of α and β , A.....H are constants. Due
to the linearity of the equation, the above solutions can be
summed or integrated over the values of α and β .$$

Considering that p should be even in x and decaying in y, it is readily seen that the solution has the following form

$$p = Ae^{-\beta Y} \cos \beta x$$

Applying the boundary conditions $(3.4-1)_1$ and $(3.4-2)_1$ gives the particular solution for p to be:

$$p = A_1 e^{-Y} \cos x \qquad (3.4-8)$$

The equation (3.1-18) reduces to

$$\mu \left\{ \frac{\partial^2 \psi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}^2} + \frac{\partial^2 \psi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}^2} \right\} - a \psi(\mathbf{x}, \mathbf{y}) = 0 \qquad (3.4-9)$$

where the general solution of the above equation may be written as follows:

$$\begin{split} \psi(\mathbf{x},\mathbf{y}) &= e^{-(\alpha^2 + \frac{\mathbf{a}}{\mu})^{\frac{1}{2}}\mathbf{y}} (\mathbf{A}_{\alpha} \cos \alpha \mathbf{x} + \mathbf{B}_{\alpha} \sin \alpha \mathbf{x}) \\ &+ e^{(\alpha^2 + \frac{\mathbf{a}}{\mu})^{\frac{1}{2}}\mathbf{y}} (\mathbf{C}_{\alpha} \cos \alpha \mathbf{x} + \mathbf{D}_{\alpha} \sin \alpha \mathbf{x}) + e^{-(\beta^2 + \frac{\mathbf{a}}{\mu})^{\frac{1}{2}}\mathbf{y}} (\mathbf{E}_{\beta} \cos \beta \mathbf{y}) \\ &+ \mathbf{F}_{\beta} \sin \beta \mathbf{y}) + e^{(\beta^2 + \frac{\mathbf{a}}{\mu})^{\frac{1}{2}}\mathbf{y}} (\mathbf{G}_{\beta} \cos \beta \mathbf{y} + \mathbf{H}_{\beta} \sin \beta \mathbf{y}) \quad (3.4-10) \\ \end{split} \\ \end{split} \\ \end{split} \\ \end{split} \\ \end{split} \\ \end{split}$$

$$\psi = B_{\alpha} e^{-(\alpha^{2} + \frac{a}{\mu})^{\frac{1}{2}} y} \sin \alpha x + \int_{0}^{\infty} C_{\gamma} \cos \gamma y \sinh (\gamma^{2} + \frac{a}{\mu})^{\frac{1}{2}} x d\gamma$$
(3.4-11)

where the second term is just a different form obtained from the combination of some of the terms in the above general solution. This term is retained in anticipation of its necessity to satisfy the boundary condition $(3.4-2)_2$.

The velocity components of the fluid are obtained from (3.1-13), where use has been made of (3.4-11) and (3.4-9), and are as follows

$$v_{x} = \frac{1}{a} A_{1} e^{-y} \sin x - B_{\alpha} (\alpha^{2} + \frac{a}{\mu})^{\frac{1}{2}} e^{-(\alpha^{2} + \frac{a}{\mu})^{\frac{1}{2}}y} \sin \alpha x$$

$$- \int_{0}^{\infty} C_{\gamma} \gamma \sin \gamma y \sin \sqrt{\gamma^{2} + \frac{a}{\mu}} x d\gamma \qquad (3.4-12)$$

$$v_{y} = \frac{A_{1}}{a} e^{-y} \cos x - B_{\alpha} \alpha e^{-(\alpha^{2} + \frac{a}{\mu})^{\frac{1}{2}}y} \cos \alpha x$$

$$- \int_{0}^{\infty} C_{\gamma} (\gamma^{2} + \frac{a}{\mu})^{\frac{1}{2}} \cos \gamma y ch \sqrt{\gamma^{2} + \frac{a}{\mu}} x d\gamma \qquad (3.4-13)$$

The application of boundary condition $(3.4-1)_2$ implies that

$$\alpha = 1 \text{ and } B_{\alpha}(1 + \frac{a}{\mu})^{\frac{1}{2}} = \frac{A_1}{a}$$
 (3.4-14)

The only condition on the fluid velocity remaining to be satisfied is $(3.4-2)_2$. The constant C_Y should be chosen such that

$$f(y) = -\int_{0}^{\infty} C_{\gamma} (\gamma^{2} + \frac{a}{\mu})^{\frac{1}{2}} ch \sqrt{\gamma^{2} + \frac{a}{\mu}} \frac{\pi}{2} d\gamma \qquad (3.4-15)$$

In order to avoid mathematical complications, we take the case where f(y) = 0 and therefore $C_{\gamma} = 0$. The velocity components become

$$v_{x} = \frac{A_{1}}{a} \left[e^{-y} - e^{-(1 + \frac{a}{\mu})^{\frac{1}{2}} y} \right] \sin x \qquad (3.4-16)$$

$$A_{1} \left[e^{-y} - e^{-(1 + \frac{a}{\mu})^{\frac{1}{2}} y} \right] \left[\sin x + (3.4-16)$$

$$v_{y} = \frac{A_{1}}{a} \left[e^{-Y} - (1 + \frac{a}{\mu})^{-\frac{1}{2}} e^{-(1 + \frac{a}{\mu})^{2}Y} \right] \cos x \qquad (3.4-17)$$

The fluid stresses are

$$\tau_{\mathbf{x}\mathbf{x}} = -\mathbf{p} + 2\mu \frac{\partial \mathbf{v}_{\mathbf{x}}}{\partial \mathbf{x}}$$
(3.4-18)

$$\tau_{yy} = -p + 2\mu \frac{\partial v_y}{\partial y}$$
(3.4-19)

$$\tau_{\mathbf{x}\mathbf{y}} = \mu \left(\frac{\partial \mathbf{v}_{\mathbf{x}}}{\partial \mathbf{y}} + \frac{\partial \mathbf{v}_{\mathbf{y}}}{\partial \mathbf{x}} \right)$$
(3.4-20)

Substituting for v_x and v_y and p from (3.4-8), (3.4-16) and (3.4-17), yields

$$\tau_{xx} = A_1 \left[\left(\frac{2\mu}{a} - 1 \right) e^{-y} - \frac{2\mu}{a} e^{-(1 + \frac{a}{\mu})} \right] \cos x \quad (3.4-21)$$

$$\tau_{yy} = -A_1 \left[\left(\frac{2\mu}{a} + 1 \right) e^{-y} - \frac{2\mu}{a} e^{-(1 + \frac{a}{\mu})} \right] \cos x \quad (3.4-22)$$

$$\tau_{xy} = \frac{\mu A_1}{a} \left\{ -2e^{-y} + \left[(+\frac{a}{\mu}) \frac{1}{2} + (1 + \frac{a}{\mu})^{-\frac{1}{2}} \right] e^{-(1 + \frac{a}{\mu})} \right\} \sin x \quad (3.4-23)$$

Since the boundary conditions of the solid part are such that the surface tractions are all known, then the problem can be solved with the help of a stress function. The displacement solution may be ignored unless it is required. Obtaining ψ from (3.5-11)

$$\psi = \frac{A_1}{a} (1 + \frac{a}{\mu})^{-\frac{1}{2}} e^{-(1 + \frac{a}{\mu})^{\frac{1}{2}}} sin x \qquad (3.4-24)$$

the equation (3.2-20) becomes

$$\nabla^4 \theta = A_1 he^{-(1 + \frac{a}{\mu})^2 y} \cos x$$
 (3.4-25)

where

h =
$$(1 + \frac{a}{\mu}) - (1 + \frac{a}{\mu})^{-1}$$

The solution of the above equation is

$$\theta = \theta_{h} + A_{l}h \frac{\mu^{2}}{a^{2}} e^{-(1 + \frac{a}{\mu})^{\frac{1}{2}}} \cos x$$
 (3.4-26)

where θ_h is a biharmonic scalar function of x and y and the second term is the particular solution of (3.4-25). We see that the above problem is equivalent to the similar strip problem of classical elasticity under identical loading except for the extra term coming from the interaction in the form of a body force. We conclude that the whole machinery of classical elasticity is applicable to the solution of the steady-state problems of the present theory for solid part, provided that the effect of interaction is treated as a prescribed body force.

Since the elasticity solution of the semi-infinite strip, under the most general loading on the finite face and stress free elsewhere, has been solved, [29] we convert the present problem into two parts.

1. The original problem except for an additional stress distribution f(x) at face y = 0, such that we have $\sigma_{yy} = -A_2 \cos x + f(x)$ at face y = 0, where f(x) is totally arbitrary at this stage.

2. A semi-infinite elastic solid with no body force and under the traction $\sigma_{yy} = -f(x)$ at face y = 0 and stress free elsewhere.

To solve the first part let us assume
$$\theta$$
 to be
 $\theta = A_1 h \frac{\mu^2}{a^2} e^{-(1 + \frac{a}{\mu})^2 y} \cos x + C_1 e^{-y} \cos x + C_2 y e^{-y} \cos x$
 $+ \int_0^\infty E_\gamma \cos \gamma y \left[x \sin \gamma x - \frac{\pi}{2} \frac{\sin \gamma \pi}{2} \cosh \gamma x \right] d\gamma$ (3.4-27)

where C_1 and C_2 are constants and E_γ is the coefficient of Fourier integral. The stresses are found to be

$$\sigma_{xx} = \left[A_1 h \frac{\mu^2}{a^2} (1 + \frac{a}{\mu}) - A_1 \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos x + (C_1 + 1)e^{-y} \cos x + C_2 y e^{-y} \cos x - 2C_2 e^{-y} \cos x - \int_0^\infty \gamma^2 E_\gamma \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[x \operatorname{sh} \gamma x - \frac{1}{2} \right] e^{-(1 + \frac{a}{\mu})^2 y} \cos \gamma y \left[$$

$$\frac{\frac{\pi}{2} \operatorname{sh} \gamma \frac{\pi}{2}}{\operatorname{ch} \gamma \frac{\pi}{2}} \operatorname{ch} \gamma \mathbf{x} \bigg] d\gamma \qquad (3.4-28)$$

$$(3.4-28)$$

$$C_{yy} = -A_{1} \left[\ln \frac{1}{a^{2}} + (1 + \frac{1}{\mu}) \right] e^{-\mu r} \cos x - (C_{1} - 1)e^{-r} \cos x - C_{1} - r e^{-r} \cos x - c^{-r} \cos x - c$$

$$\sigma_{xy} = -A_{1}h \frac{\mu^{2}}{a^{2}} (1 + \frac{a}{\mu})^{\frac{1}{2}} e^{-(1 + \frac{a}{\mu})^{\frac{1}{2}}y} \sin x - C_{1}e^{-y} \sin x$$

- $C_{2}ye^{-y} \sin x + C_{2}e^{-y} \sin x + \int_{0}^{\infty} E_{\gamma} \gamma \sin \gamma \left[sh \gamma x + \gamma x ch \gamma x - \frac{\gamma \frac{\pi}{2} sh \gamma \frac{\pi}{2}}{ch \gamma \frac{\pi}{2}} sh \gamma x \right] d\gamma$ (3.4-30)

The boundary condition $(3.5-2)_{3'}$ is identically satisfied. We now choose f(x) to be

$$f(\mathbf{x}) = \int_{0}^{\infty} E_{\gamma} \left[2\gamma \operatorname{ch} \gamma \mathbf{x} + \gamma^{2} \mathbf{x} \operatorname{sh} \gamma \mathbf{x} - \frac{\frac{\pi}{2} \operatorname{sh} \gamma \frac{\pi}{2}}{\operatorname{ch} \gamma \frac{\pi}{2}} \gamma^{2} \operatorname{ch} \gamma \mathbf{x} \right] d\gamma$$
(3.5-31)

The application of the remaining boundary conditions yields the following relations

$$- A_{1}h \frac{\mu^{2}}{a^{2}} (1 + \frac{a}{\mu})^{\frac{1}{2}} - C_{1} + C_{2} = 0$$

$$A_{2} - A_{1}\left[h \frac{\mu^{2}}{a^{2}} (1 + \frac{a}{\mu}) - 1\right] - C_{1} + 1 = 0$$

$$A_{1}h \frac{\mu^{2}}{a^{2}} (1 + \frac{a}{\mu})^{\frac{1}{2}} e^{-(1 + \frac{a}{\mu})^{\frac{1}{2}}y} + [C_{1} - C_{2} + C_{2}y]e^{-y}$$

$$= \int_{0}^{\infty} E_{\gamma} \gamma \left(\frac{sh \gamma \frac{\pi}{2} ch \gamma \frac{\pi}{2} + \gamma \frac{\pi}{2}}{ch \gamma \frac{\pi}{2}}\right) sin \gamma y d\gamma \qquad (3.4-35)$$

The values of E_{γ} , C_1 and C_2 are obtained from the above equation.

The complete solution of the problem is obtained by superposition of the two parts, while the solution to the second part, under the prescribed traction, f(x), can be obtained by method of [29]. We do not present it here. The present problem could also have been solved by use of displacement function presented in the early part of this chapter. The later method is particularly useful when either the displacements are desired or the prescribed boundary conditions are partially or totally in terms of displacement.

CHAPTER IV

DISCUSSIONS AND EVALUATION OF SOME THEORIES

In this chapter we will briefly review the equations of fluid flow through undeformable porous media based on Darcy's law. We will further review the Biot theory for flow of fluids through deformable media, a generalization of the former theory, where a modified Darcy's law is adopted. The purpose of this chapter is to examine the above theories from the standpoint of the theory of the present work. Of interest is also Brinkman's drag theory [37], which happens to be a useful modified Darcy's law.

4.1 Darcy's Law

The first assumption throughout the classical field of fluid flow through porous media is that the solid is undeformable. Hence the pores of the media are fixed and their boundary surfaces are geometrically describable. Formally speaking, the problem is but a special case of the general problem of viscous flow of fluids between impermeable boundaries. It is quite apparent that a flow problem through such a tortuous irregular channel is mathematically so complicated that the pure hydrodynamic approach is out of the question.

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Because of the above mentioned difficulties, an empirical dynamic equation, the Darcy's law, was established. This law asserts that, macroscopically the velocity is proportional to the pressure gradient acting on the fluid

$$\vec{V} = -\frac{k}{\mu} \vec{\nabla}p \qquad (4.1-1)$$

where μ is the viscosity of the fluid and k is the permeability of the solid. The permeability k in the above equation has dimension of square length and expresses the ease of the fluid flow through porous media. The monographs and the literature on the field give detailed discussions of the permeability and the various formulas expressing it in terms of porosity and other variables.

In the presence of the body force \vec{F} we have

$$\vec{\nabla} = -\frac{k}{\mu} (\vec{\nabla}p + \vec{F}) \qquad (4.1-2)$$

and in the case that the body force is derivable from a potential function G, we have

$$\mathbf{F} = - \vec{\nabla} \mathbf{G} \tag{4.1-3}$$

and hence

$$\dot{7} = - \vec{\nabla} \phi \qquad (4.1-4)$$

where

$$\phi = \frac{k}{\mu}(p - G)$$
 (4.1-5)

The expression (4.1-4) is the generalized form of Darcy's law. This law together with the equation of state and continuity equation constitute a complete system of equations. These equations supplemented by initial and boundary conditions provide all the necessary information for the solution of any particular problem. For incompressible fluids, the equations of state and continuity reduce to

$$\rho = \text{constant} \quad \nabla \cdot \nabla = 0 \quad (4.1-6)$$

and hence

$$\nabla^2 \mathbf{p} = \nabla^2 \phi = 0 \tag{4.1-7}$$

According to the above equations, we observe that there is no distinction between steady-state and nonsteadystate problems for incompressible fluids.

In order to compare the above fundamental equation to those of the present theory, we assume the solid to be undeformable and hence $\vec{\omega} = \vec{u} \equiv 0$. The equations (3.1-5) and (3.1-6) reduce to

$$-\frac{\mu}{a}\nabla^{2}\vec{v} + \vec{v} = -\frac{1}{a}\vec{\nabla}p \qquad (4.1-8)$$

$$\vec{\nabla} \cdot \vec{\nabla} = 0 \qquad \nabla^2 p = o \qquad (4.1-9)$$

In order to reduce (4.1-8) to (4.1-4), we see that the diffusive coefficient a should be assumed as

$$a = \frac{\mu}{k} \tag{4.1-10}$$

This immediately implies that the diffusive force vanishes for ideal fluids. Substituting (4.1-10) into (4.1-8) we obtain

$$- k\nabla^2 \vec{v} + \vec{v} = -\frac{k}{\mu} \vec{\nabla} p \qquad (4.1-11)$$

We see that the above equation can be reduced to (4.1-1) if the term $k\nabla^2 \vec{\nabla}$ would be negligible compared to $\vec{\nabla}$. Since we have assumed that the velocity as well as its space derivatives are small of the same order, it is concluded that k has to be small. In the following table, the values of permeability are given for some materials.

Porous Solid	Permeability (Darcy)	Porosity Fraction
Sand	2 - 180	0.31 - 50
Sandstone	$10^{-7} - 11$	0.08 - 0.40
Brick	0.0048 - 0.22	0.12 - 0.34
Soil	0.29 - 14	0.43 - 0.54

Table 4.1--Typical values of permeability and porosity for various materials. [37]

The permeability coefficient, for the above typical materials, is small enough that the first term can be neglected. However on the other extreme if k tends to infinity the equation (4.1-11) becomes

$$-\mu\nabla^2 \vec{\nabla} = \vec{\nabla} p \qquad (4.1-12)$$

which is the equation for slow viscous flow of bluids. It is seen that the reduced equation (4.1-8) with the special choice of a from (4.1-10) includes two different extremes, namely a pure slow viscous flow of fluids, and flow of fluid through highly impermeable materials. The limitations of the resulting formulas for either case can be stated rigorously from the construction of the theory. We see that, under certain limitations and conditions, the present theory gives almost identical formulas for flow of fluids through porous media as the classical theory. These limitations can be removed without any essential difficulties from continuum approach, whereas the classical field is based strictly on empirical viewpoints for a certain range and does not provide a basis for all possible generalization. Let us recall that the theory on which this work is built covers a very wide range of heterogeneous media under quite general conditions.

Atkin <u>et al</u>. [5] has given one possible set of boundary conditions for which the problem has unique solution. He remarked on the necessity of specifying at each point of the boundary two vector boundary conditions and a scalar functionfor thermal consideration. It is easily seen from equations (4.1-8) and (4.1-9) and the remarks in [5], that in steady-state case, a vector boundary condition should be prescribed at each point of the boundary for the fluid part only in order to have a complete solution. That this is true is also apparent from purely physical considerations.

Contrary to the above remarks the classical equations do not allow to us to specify a vector boundary condition but only a scalar function. From our analysis, it is seen that this occurred because of neglect of the term $k\nabla^2 \vec{V}$. We conclude that, in the case of low permeability, the error arising from the above simplification is insignificant far from the boundary while the error might be quite serious near and at the boundaries.

The two dimensional problem of the former chapter illustrates this effect, for instance, v_y vanishes at boundary

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y = 0 according to (3.4-16) however the classical approach gives $v_x = \sin x$. However for y = 0 the dominating term is $\frac{A_1}{a} e^{-y} \sin x$, the term obtained by classical theory.

4.2 Brinkman's Theory

It may be of interest that the same equation (4.1-11) has been proposed in a series of papers by Brinkman [37]. His theory is based on the assumption that the solid particles are spheres of radius R and that they are kept in position by external forces as in a bed of closely packed particles supporting each other by contact. In the absence of the particles the stresses give rise to a force \vec{F} dV which is given by Navier-Stokes equation

$$\mathbf{F}_{1} = - \vec{\nabla} \mathbf{p} + \mu \nabla^{2} \vec{\nabla} \qquad (4.2-1)$$

The presence of the solid spheres causes a damping force $F_2 dV$ on the fluid elements. It was assumed that the damping force is proportional to the mean velocity and viscosity of the fluid and to the reciprocal of permeability so

$$\vec{F}_2 = \frac{\mu}{k} \vec{V} \qquad (4.2-2)$$

Since

$$F_1 + F_2 = 0$$
 (4.2-3)

Therefore

$$-\vec{\nabla}p + \mu \nabla^2 \vec{\nabla} - \frac{\mu}{k} \vec{\nabla} = 0 \qquad (4.2-4)$$

For high particle densities the term $\mu \nabla^2 V$ is negligible compared to $\frac{\mu}{k} \vec{\nabla}$. This implies that Darcy's law is the limiting form of equation above for low permeability. The boundary conditions are that the tangential and normal velocity at the surface of the spheres to be zero. Although he obtained the same equation as (4.1-11), the derivation and assumptions are drastically restricted, and the proof is not rigorous.

4.3 Biot's Theory

The next major extension of the classical theory of flow through porous media has been done by Biot [6,17] who considered the solid to be elastically deformable. In the following we briefly review his equations. The constitutive equations for the stresses are

$$\sigma_{ij} = 2Ne_{ij} + Me_{kk}\delta_{ij} + Q\epsilon \qquad (4.3-1)$$

$$\tau_{ij} = Qe_{kk} + L\varepsilon\delta_{ij} = \sigma\delta_{ij} \qquad (4.3-2)$$

where ε is the dilatation of the fluid defined by

$$\varepsilon = \vec{\nabla} \cdot \vec{\omega}_{\mathbf{p}}$$

where $\vec{\omega}_{p}$ is the fluid displacement vector and $\sigma = -\beta p$, where β is the fraction of fluid element per unit section and p is the fluid pressure. The equation of motion in the absence of body forces for the quasi-static theory is

$$(\sigma_{ij} + \sigma \delta_{ij}), j = 0$$
 (4.3-4)

and the modified Darcy's law is

$$\nabla \sigma = \mathbf{a}(\vec{\mathbf{v}} - \vec{\mathbf{U}})$$
 (4.3-5)

While for the dynamic theory [12], the equations become:

$$\sigma_{ij'j} = \frac{\partial}{\partial t} (\rho_{11}u_i + \rho_{12}v_i) - a(v_i - u_i) \qquad (4.3-6)$$

$$\sigma_{i} = \frac{\partial}{\partial t} \left(\rho_{12} u_{i} + \rho_{22} v_{i} \right) + a \left(v_{i} - u_{i} \right) \qquad (4.3-7)$$

where $\rho_{11} + \rho_{12} = \rho_1$, $\rho_{22} + \rho_{12} = \rho_2$, ρ_{12} is a mass coupling parameter. Here again it is seen that our theory can be reduced to Biot's theory if terms with viscosity coefficients λ and μ are eliminated from the constitutive equation for fluid. This can be justified if the viscosity is low enough such that the only dominating term in the expression for fluid stress would be the hydrostatic pressure.

In the dynamic theory, equations (4.3-6,7) can be written in the following form

$$\frac{\partial \sigma_{ij}}{\partial \mathbf{x}_{j}} = \rho_{1} \frac{\partial u_{i}}{\partial t} + \rho_{12} \frac{\partial}{\partial t} (v_{i} - u_{i}) - a(v_{i} - u_{i}) \quad (4.3-8)$$

$$\frac{\partial \sigma}{\partial \mathbf{x}_{i}} = \rho_{2} \frac{\partial v_{i}}{\partial t} - \rho_{12} \frac{\partial}{\partial t} (v_{i} - u_{i}) + a(v_{i} - u_{i}) \quad (4.3-9)$$

From the standpoint of Green and Naghdi's theory, it is deduced that the diffusive force according to Biot's formula above, is

$$\pi_{i} = \rho_{12} \frac{\partial}{\partial t} (\mathbf{v}_{i} - \mathbf{u}_{i}) - \mathbf{a}(\mathbf{v}_{i} - \mathbf{u}_{i}) \qquad (4.3-10)$$

This tells us that in the dynamic theory the diffusive force is the same as of quasi-static case plus the term $\rho_{12} \frac{\partial}{\partial t} (\mathbf{v_i} - \mathbf{u_i})$. The first term satisfies the requirement of frame indifference, but the second term does not, and hence are not allowed to appear in constitutive equations. As it is remarked in [26] the effect of acceleration cannot be introduced in linear constitutive equations. The above analysis shows that Biot's dynamic theory is incorrect to this extent. Biot [9] has extended his theory for the most general
anisotropic solid and also viscoelastic case. He and coworkers have worked out some problems based on his theory; however because of mathematical complexity, very little progress has been made.

It is seen that the above theories can be explained in the light of the continuum mechanics approach while this approach enables us to rigorously analyze the existing theories and find out the pitfalls as well as their range of validity.

CHAPTER V

MIXTURE OF TWO IDEAL FLUIDS AND AN

ELASTIC SOLID

As it is pointed out in Chapter I, the theory of Green and Naghdi [25] for a binary mixture has been generalized for n components mixture by Mills [31]. The former authors have also proposed a theory for n components mixture [27], removing some of the restriction of the former one. We, however, use Mills' result in formulating the problem of the mixture of two incompressible fluids and an elastic solid.

5.1 Formulas and Notations

We first consider a mixture of n substances which are in relative motion to each other. The equations (1.2-1-10) are all valid and we will use them in this section. The rate of deformation and vorticity tensors are respectively defined to be

where a comma denotes partial differentiation with respect to space coordinates.

In view of the energy balance relation proposed by Green and Naghdi [25], the following relation is postulated by Mills as a generalization of the former one

$$\frac{\partial}{\partial t} \int_{V} \left(\rho U + \sum_{i=1}^{l} \rho_{\alpha} {}^{(\alpha)} v_{i} {}^{(\alpha)} v_{i} \right) dV + \int_{A} \left(U \sum_{\alpha=1}^{n} \rho_{\alpha} {}^{(\alpha)} v_{k} \right) v_{k} + \frac{1}{2} \sum_{\alpha=1}^{n} \rho_{\alpha} {}^{(\alpha)} v_{k} {}^{(\alpha)} v_{i} {}^{(\alpha)} v_{i} \right) n_{k} dA = \int_{V} \left(\rho r + \sum_{\alpha=1}^{n} \rho_{\alpha} {}^{(\alpha)} F_{i} {}^{(\alpha)} v_{i} \right) dV + \int_{A} \left(\sum_{\alpha=1}^{n} {}^{(\alpha)} t_{i} {}^{(\alpha)} v_{i} - h \right) dA \qquad (5.1-3)$$

where the symbols have the same meaning as those of Chapter II. By essentially the same method as that used by Green and Naghdi, namely the invariance requirement under different rigid body motions, the following relations were obtained:

$$^{(\alpha)}\sigma_{ki,k} + \rho_{\alpha} \left({}^{(\alpha)}F_{i} - {}^{(\alpha)}a_{i} \right) = {}^{(\alpha)}\pi_{i}$$
 (5.1-4)

$$\sum_{\alpha=1}^{n} {}^{(\alpha)} \pi_{i} = 0 \qquad (5.1-5)$$

If we denote the symmetric part of stress tensor by $\sigma_{(ik)}^{\sigma}$ and antisymmetric part by $\sigma_{[ik]}^{\sigma}$, we have

$$\sum_{\alpha=1}^{n} {\alpha \choose \sigma} {ki}$$
(5.1-6)

The energy equation becomes

$$\rho \frac{DU}{Dt} - \rho r + q_{k,k} - \frac{\sum_{\beta=1}^{n-1} (\beta) \pi_i \left(\gamma_i - (n) v_i \right)}{\beta = 1} - \sum_{\alpha=1}^{n-1} (\alpha) \sigma_{(ki)} (\alpha) d_{ik}$$
$$- \sum_{\sigma_{[ki]}} \left((\beta) r_{ik} - (n) r_{ik} \right) = 0 \qquad (5.1-7)$$

The entropy production inequality was postulated to

be

$$\int_{\mathbf{V}} \rho \, \frac{\mathrm{DS}}{\mathrm{Dt}} \, \mathrm{dV} - \int_{\mathbf{V}} \rho \, \frac{\mathbf{r}}{\mathrm{T}} \, \mathrm{dV} - \int \frac{\mathrm{h}}{\mathrm{T}} \, \mathrm{dA} \ge 0 \qquad (5.1-8)$$

5.2 <u>Mixture of Two Ideal Compressible</u> <u>Fluids and an Elastic Solid Under</u> <u>Isothermal Condition</u>

In view of the constitutive equations for a mixture of a Newtonian fluid and an elastic solid, the following constitutive equations were postulated as the generalization of the former ones.

$$A = A (e_{rs}, \rho_1, \rho_2)$$
 (5.2-1)

$$s = s (e_{rs}, \rho_1, \rho_2)$$
 (5.2-2)

$$^{(\alpha)}\sigma_{(ki)} = {}^{(\alpha)}A_{ik} \qquad \alpha = 1,2,3 \qquad (5.2-3)$$

$$(\alpha)_{\sigma}_{[ki]} = (\alpha)_{ik} \qquad \alpha = 1, 2, 3 \qquad (5.2-4)$$

where $\alpha = 1, 2$ is referred to fluids one and two and $\alpha = 3$ corresponds to the elastic solid.

Substitution of the above constitutive equations into the entropy production inequality and using the same argument as before yields the following

$$(\alpha)_{\sigma[ik]} = 0$$
 for $\alpha = 1, 2, 3$ (5.2-6)

$$^{(1)}\sigma_{ik} = -\rho\rho_{1} \frac{\partial A}{\partial \rho_{1}} \delta_{ik} \qquad (5.2-7)$$

$${}^{(2)}\sigma_{ik} = -\rho\rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ik}$$
 (5.2-8)

$$^{(3)}\sigma_{ik} = \rho \frac{\partial \mathbf{x}_{i}}{\partial \mathbf{X}_{r}} \frac{\partial \mathbf{x}_{k}}{\partial \mathbf{X}_{s}} \frac{\partial \mathbf{A}}{\partial \mathbf{e}_{rs}}$$
(5.2-9)

$$^{(\beta)}a_{i} = \sum_{\gamma=1}^{2} \rho_{\beta} \frac{\partial A}{\partial \rho_{\gamma}} \frac{\partial \rho_{\gamma}}{\partial x_{i}} - \rho \frac{\partial A}{\partial \rho_{\beta}} \frac{\partial \rho_{\beta}}{\partial x_{i}} + \rho_{\beta} \frac{\partial A}{\partial e_{rs}} \frac{\partial e_{rs}}{\partial x_{i}}$$
(5.2-10)

For isotropic case, the Helmholtz free energy, A, can be expanded in Taylor series to be $\overline{\rho}A = \frac{1}{2} a_4 e_{mm} e_{nn} + a_5 e_{mn} e_{mn} + a_6 e_{mm} n_1 + a_7 e_{mm} n_2 + a_8 n_1^2$ + $a_9 n_2^2 + a_{10} n_1 n_2$ (5.2-11)

where terms less than the second are not included because of the zero initial stresses.

Substituting for A in constitutive equations and retaining the linear terms only, results in the following equations:

$${}^{(1)}\sigma_{ik} = - \left\{ \overline{\rho}_{1}^{a} 6^{e}_{mm} + 2a_{8}\overline{\rho}_{1}^{\eta} + \overline{\rho}_{1}^{a} 10^{\eta} 2 \right\}^{\delta}_{ik}$$
(5.2-12)

$${}^{(2)}\sigma_{ik} = -\left\{ \overline{\rho}_{2}a_{7}e_{mm} + 2a_{9}\overline{\rho}_{2}\eta_{2} + \overline{\rho}_{2}a_{10}\eta_{1} \right\}^{\delta}ik \qquad (5.2-13)$$

$$^{(3)}\sigma_{ik} = a_4 e_{mm} \delta_{ik} + 2a_5 e_{ik} + a_6 \eta_1 \delta_{ik} + a_7 \eta_2 \delta_{ik}$$
(5.2-14)

$${}^{(1)}\pi_{i} = \kappa_{11} ({}^{(1)}v_{i} - {}^{(3)}v_{i}) + \kappa_{12} ({}^{(2)}v_{i} - {}^{(3)}v_{i})$$
(5.2-15a)

$${}^{(2)}\pi_{i} = K_{21} {}^{(1)}v_{i} - {}^{(3)}v_{i}) + K_{22} {}^{(2)}v_{i} - {}^{(3)}v_{i})$$
 (5.2-16b)

The equations of motion are

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¹⁾
$$\sigma_{ki,k} = {}^{(1)}\pi_i$$
 (5.2-16)

$$^{(2)}\sigma_{ki,k} = {}^{(2)}\pi_{i}$$
 (5.2-17)

$${}^{(3)}\sigma_{ki,k} + {}^{(1)}\pi_{i} + {}^{(2)}\pi_{i} - 0 \qquad (5.2-18)$$

In the case of rigid solid $e_{ij} = 0$, therefore

$${}^{(1)}\sigma_{ik} = \overline{\rho}_{1}(2a_{8}n_{1} + a_{10}n_{2}) \delta_{ik} \qquad (5.2-19)$$

$${}^{(2)}\sigma_{ik} = -\overline{\rho}_2 (2a_9\eta_2 + a_{10}\eta_1) \delta_{ik}$$
(5.2-20)

Substituting for partial stresses into equations of motion and making use of continuity equations, yields

$$\frac{\partial n_1}{\partial t} + \overline{\rho}_1 \nabla \cdot {}^{(1)} V = 0 \qquad (5.2-21)$$

$$\frac{\partial n_2}{\partial t} + \overline{\rho}_2 \nabla \cdot {}^{(2)} V = 0 \qquad (5.2-22)$$

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This will give us

$$2a_8 \nabla_1^2 \eta_1 + a_{10} \nabla_1^2 \eta_2 = \frac{K_{11}}{\overline{\rho_1}^2} \frac{\partial \eta_1}{\partial t} + \frac{K_{12}}{\overline{\rho_1}\overline{\rho_2}} \frac{\partial \eta_2}{\partial t}$$
(5.2-23)

$$a_{10}\nabla^{2}n_{1} + 2a_{9}\nabla^{2}n_{1} = \frac{K_{21}}{\overline{\rho_{1}\rho_{2}}} \frac{\partial n_{1}}{\partial t} + \frac{K_{22}}{\overline{\rho_{2}}^{2}} \frac{\partial n_{2}}{\partial t}$$
(5.2-24)

In steady-state case the variables are independent of time, so

$$\nabla^2 \eta_1 = 0 \qquad \nabla^2 \eta_2 = 0 \qquad (5.2-25)$$

5.3 Case of Incompressible Fluids

If the fluids are incompressible, the incompressibility condition (2.3-13) reduces to

$${}^{(1)}C{}^{(1)}d_{kk} + {}^{(2)}C{}^{(2)}d_{kk} + \frac{R - P_0}{R} \frac{\partial e_{mm}}{\partial t} = 0 \qquad (5.3-1)$$

where

$${}^{(1)}C + {}^{(2)}C = 1 \qquad \frac{{}^{(2)}C}{{}^{(1)}C} = C \qquad (5.3-2)$$

Similar to the argument made in Chapter II, the use of (5.3-1) and the entropy production inequality introduces a new unknown parameter p into constitutive equation for partial stresses. The results are

$${}^{(1)}\sigma_{ik} = - \{\overline{\rho}_{1}a_{6}e_{mm} + 2a_{8}\overline{\rho}_{1}\eta_{1} + \overline{\rho}_{1}a_{10}\eta_{2} + p\} \delta_{ik}$$
(5.3-3)
$${}^{(2)}\sigma_{ik} = - \left\langle \overline{\rho}_{2}a_{7}e_{mm} + 2a_{9}\overline{\rho}_{2}\eta_{2} + \overline{\rho}_{2}a_{10}\eta_{1} + \frac{1 - {}^{(1)}c}{{}^{(1)}c} p \right\rangle \delta_{ik}$$
(5.3-4)

$${}^{(3)}\sigma_{ik} = \left\{ a_6 \eta_1 + a_7 \eta_2 + a_4 e_{mm} \right\} \delta_{ik} + 2a_5 e_{ik}$$
(5.3-5)

In the case of undeformable solid $e_{ij} = 0$ and (5.3-3,4) becomes

$${}^{(1)}\sigma_{ik} = - \{2a_8\overline{\rho}_1\eta_1 + a_{10}\overline{\rho}_1 + p\} \delta_{ik}$$
(5.3-6)

$${}^{(2)}\sigma_{ik} = - \{2a_9\bar{\rho}_2\eta_2 + a_{10}\bar{\rho}_2\eta_1 + \frac{1 - {}^{(1)}c}{{}^{(1)}c}p\} \delta_{ik} (5.3-7)$$

Substituting equations (5.3-6,7) into equations of motion and making use of continuity equations yields

$$\nabla^{2} p + 2a_{8} \nabla^{2} \eta_{1} + a_{10} \nabla^{2} \eta_{2} = \frac{K_{11}}{\overline{\rho_{1}}^{2}} \frac{\partial \eta_{1}}{\partial t} + \frac{K_{12}}{\overline{\rho_{1}}\overline{\rho_{2}}} \frac{\partial \eta_{2}}{\partial t}$$
(5.3-8)
$$\frac{1 - \frac{(1)}{(1)} c}{(1)} \nabla^{2} p + a_{10} \nabla^{2} \eta_{1} + 2a_{9} \nabla^{2} \eta_{2} = \frac{K_{21}}{\overline{\rho_{1}}\overline{\rho_{2}}} \frac{\partial \eta_{1}}{\partial t} + \frac{K_{22}}{\overline{\rho_{2}}^{2}} \frac{\partial \eta_{2}}{\partial t}$$
(5.3-9)

The incompressibility relation (5.3-1) reduces to

$$\frac{(1)_{C}}{\overline{p_{1}}} \eta_{1} = \frac{1 - (1)_{C}}{\overline{p_{2}}} \eta_{2}$$

Eliminating p between (5.3-8) and (5.3-9) and using relation (5.3-10), we obtain:

$$B\nabla^2 \eta_{\alpha} = \frac{\partial \eta_{\alpha}}{\partial t}$$
 for $\alpha = 1,2$ (5.3-11)

where

$$B = \overline{\rho}_{1}\overline{\rho}_{2} \frac{\overline{\rho}_{1}C(2a_{8}C - a_{10}) + \overline{\rho}_{2}(a_{10}C - 2a_{9})}{C\overline{\rho}_{2}(K_{11}C + K_{12}) - \overline{\rho}_{1}(CK_{21} + K_{22})}$$

The equation (5.3-11) is the diffusion equation for a mixture of two incompressible fluids and a rigid solid. Green and Adkins [24] derived the diffusion equations for a binary mixture of compressible fluids. Later Mills [30] gave the derivation of diffusion law for a mixture of two incompressible fluids.

Equation (5.2-23) and (5.2-24), manipulated from Mills' results, are the diffusion law for a mixture of two compressible fluids flowing through a rigid body. Finally equation (5.3-11) represents the diffusion of two incompressible ideal fluids through a solid. The equations (5.2-23) and (5.2-24) or (5.3-11) may be considered as the modified Darcy's law of flow of two miscible fluids through porous rigid media.

CHAPTER VI

CONSTITUTIVE EQUATIONS FOR BINARY MIXTURE OF A NEWTONIAN FLUID AND A VISCOELASTIC

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6.1 General Remarks

In order to derive the desired constitutive equations we follow the same approach as for binary mixture of an elastic solid and a Newtonian fluid. These equations will be derived under the assumptions of Chapter II.

Before proceeding any further, it would be pertinent to consider the different approaches to the theory of viscoelasticity. The problem has been considered by many research workers where two main lines of work are of interest from thermodynamical viewpoints. The first line of activity is more or less based on Biot's linear thermodynamic theory [16-17] and the non-linear counterpart of it, where the idea of hidden coordinates has been introduced to take care of dissipation phenomena. The other line is due to Coleman [19-20], who has introduced the idea of materials with fading memory asserting that "deformation that occurred in the distant past should have less influence in determining the present stress than those occurred in the recent past." The thermodynamic aspects of the problem are discussed [19]. The

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viscoelastic materials are a special class of materials with fading memory.

In the present work the idea of hidden coordinates is employed, However the alternative of using the idea of fading memory is possible, but has not been attempted.

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6.2 Hidden Coordinates

The thermodynamic system is assumed to have n degrees of freedom defined by n state variables ξ_1, \ldots, ξ_n . These independent state variables are alternatively called generalized coordinates. These coordinates are divided into two groups of hidden and observed ones. The system is assumed to be under the action of n generalized forces Q_{i} in such a manner that $Q_i d\xi_i$ represents an incremental amount of work done on the system. The hidden variables are those whose corresponding conjugate forces are zero and are of interest only to the extent of their influence upon our observed variables. As an example, in a body under external loading, strain components are considered as observed variables and stress components as their conjugate external force, while the effect of "molecular configuration," interstitial atoms, dislocation, grain boundaries, etc., on stress-strain law can be accounted by hidden variables. The plan is to introduce the hidden coordinates in the equations of state and eliminate them from our ultimate stress-strain relationships. At this stage we are not concerned about the explicit form of ξ_i . However,

they are assumed to be functionals of observed variables history.

6.3 <u>Development of the Constitutive</u> Equations

We postulate the following constitutive equations in accordance with the equipresence principle.

$$A = A(e_{ij}, \rho_2, \xi_{\ell}, f_{ij}, d_{ij}, u_i - v_i)$$
(6.3-1)

$$S = S(e_{ij}, \rho_2, \xi_{\ell}, f_{ij}, d_{ij}, u_i - v_i)$$
(6.3-2)

$$\frac{1}{2}(\sigma_{ki} + \sigma_{ik}) = A_{ik} + A_{ikj}(u_j - v_j) + A_{ikrs}f_{rs} + \overline{A}_{ikrs}d_{rs}$$
(6.3-3)

$$\frac{1}{2}(\tau_{ki} + \tau_{ik}) = B_{ik} + B_{ikj}(u_j - v_j) + B_{ikrs}f_{rs} + \overline{B}_{ikrs}d_{rs}$$
(6.3-4)

$$\frac{1}{2}(\sigma_{ki} - \sigma_{ik}) = -\frac{1}{2}(\tau_{ki} - \tau_{ki}) = D_{ki} + D_{kij}(u_j - v_j) + D_{kirs}f_{rs}$$

$$+ \overline{D}_{ikrs}d_{rs} \qquad (6.3-5)$$

$$\pi_{i} = a_{i} + a_{ij}(u_{j} - v_{j}) + a_{irs}f_{rs} + \overline{a}_{irs}d_{rs} \qquad (6.3-6)$$

where A_{ik}depend on e_{ij} , ρ_2 , ξ_ℓ . The dependence on f_{ij} , d_{ij} , $u_i - v_i$ can be omitted from (6.3-1) and (6.3-2) by thermodynamic consideration as shown by Crochet and Naghdi [22].

$$A = A(e_{ij}, \rho_2, \xi_{\ell})$$
 (6.3-7)

$$S = S(e_{ij}, \rho_2, \xi_l)$$
 (6.3-8)

Using (1.3-20) and the entropy production inequality (2.4-16) yields

$$\frac{D\gamma}{Dt} = -\rho \frac{DA}{Dt} + \pi_{i} (u_{i} - v_{i}) + \frac{1}{2} (\sigma_{ki} + \sigma_{ik}) d_{ik} + \frac{1}{2} (\tau_{ki} + \tau_{ik}) f_{ik}$$
$$+ \frac{1}{2} (\sigma_{ki} - \sigma_{ik}) (\Gamma_{ik} - \Lambda_{ik}) \ge 0$$
(6.3-9)

Differentiating (6.2-7) gives,

$$\rho \frac{DA}{Dt} = \rho \frac{\partial A}{\partial \rho_2} \frac{D\rho_2}{Dt} + \rho \frac{\partial A}{\partial e_{rs}} \frac{De_{rs}}{Dt} + \rho \frac{\partial A}{\partial \xi_\ell} \frac{D\xi_\ell}{Dt}$$
(6.3-10)

In view of (1.3-8) and (2.4-1), equation (6.3-10) becomes

$$\rho \frac{DA}{Dt} = -\rho \rho_{2} \frac{\partial A}{\partial \rho_{2}} f_{kk} + \frac{1}{2} \frac{\partial x_{i}}{\partial x_{r}} \frac{\partial x_{j}}{\partial x_{s}} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right) d_{ij}$$

$$+ (u_{k} - v_{k}) \left[\rho_{1} \frac{\partial A}{\partial \rho_{2}} \frac{\partial \rho}{\partial y_{k}} - \frac{1}{2} \rho_{2} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right) \frac{\partial e_{rs}}{\partial x_{k}} \right]$$

$$+ \rho \frac{\partial A}{\partial \xi_{\ell}} \frac{D\xi_{\ell}}{Dt} \qquad (6.3-11)$$

With the help of (6.3-11) and (6.3-3-6), equation (6.3-9) becomes

$$\frac{D\gamma}{Dt} = \left[A_{ik} - \frac{1}{2}\rho \frac{\partial x_{i}}{\partial x_{r}} \frac{\partial x_{k}}{\partial x_{s}} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right) \right] d_{ik} + \left(B_{ik} + \rho\rho_{2} \frac{\partial A}{\partial \rho_{2}} \delta_{ik} \right) f_{ik}
+ \left[a_{i} - \rho_{1} \frac{\partial A}{\partial \rho_{2}} \frac{\partial \rho_{2}}{\partial \gamma_{i}} + \frac{1}{2} \rho_{2} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right) \frac{\partial e_{rs}}{\partial x_{i}} \right] (u_{i} - v_{i})
+ (B_{rsi} + a_{rsi}) f_{rs} (u_{i} - v_{i}) + (A_{rsi} + \overline{a}_{rsi}) drs (u_{i} - v_{i})
+ A_{ikrs} d_{ik} f_{rs} + a_{ij} (u_{i} - v_{i}) (u_{j} - v_{j}) + B_{ikrs} f_{ik} f_{rs}
+ D_{ki} (\Gamma_{ik} - \Lambda_{ik}) + D_{kij} (\Gamma_{ik} - \Lambda_{ik}) (u_{i} - v_{i}) + D_{kirs} f_{rs} (\Gamma_{ik} - \Lambda_{ik})
- \rho \frac{\partial A}{\partial \xi_{\ell}} \frac{D\xi_{\ell}}{Dt} + \overline{A}_{ikrs} d_{rs} d_{ik} + \overline{B}_{ikrs} d_{rs} f_{ik} + \overline{D}_{kirs} (\Gamma_{ik} - \Lambda_{ik})
\ge 0 \qquad (6.3-12)$$

For a given state of deformation this inequality has to be satisfied for all arbitrary values of d_{ik} , f_{ik} , $(u_i - v_i)$ $\Gamma_{ik} - \Lambda_{ik}$. Applying the above argument, we obtain:

$$A_{ik} = \frac{1}{2} \quad \frac{\partial x_i}{\partial x_r} \frac{\partial x_k}{\partial x_s} \left(\frac{\partial A}{\partial e_{rs}} + \frac{\partial A}{\partial e_{sr}} \right)$$
(6.3-13)

$$B_{ik} = -\rho \rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ik} \qquad (6.3-14)$$

$$\mathbf{a}_{\mathbf{i}} = \rho_{1} \frac{\partial \mathbf{A}}{\partial \rho_{2}} \frac{\partial \rho_{2}}{\partial \mathbf{y}_{\mathbf{i}}} + \frac{1}{2} \rho_{2} \left(\frac{\partial \mathbf{A}}{\partial \mathbf{e}_{\mathbf{rs}}} + \frac{\partial \mathbf{A}}{\partial \mathbf{e}_{\mathbf{sr}}} \right) \frac{\partial \mathbf{e}_{\mathbf{rs}}}{\partial \mathbf{x}_{\mathbf{i}}}$$
(6.3-15)

 $D_{ki} = 0$ $D_{kij} = 0$ $D_{kirs} = 0$ $\overline{D}_{kirs} = 0$ (6.3-16) In the case where the coefficients are functions of $\frac{\partial \mathbf{x}_r}{\partial \mathbf{X}_s}$ only through ρ_1 we will have

$$B_{ikj} = 0$$
 $A_{ikj} = 0$ $a_{ikj} = 0$ $\overline{a}_{ikj} = 0$ (6.3-17)

and the following relations

$$A_{ikrs} = \lambda_{1} \delta_{ik} \delta_{rs} + \mu_{1} (\delta_{ir} \delta_{ks} + \delta_{is} \delta_{kr})$$

$$\overline{A}_{ikrs} = \lambda_{2} \delta_{ik} \delta_{rs} + \mu_{2} (\delta_{ir} \delta_{ks} + \delta_{is} \delta_{kr})$$

$$B_{ikrs} = \lambda_{3} \delta_{ik} \delta_{rs} + \mu_{3} (\delta_{ir} \delta_{ks} + \delta_{is} \delta_{kr})$$

$$\overline{B}_{ikrs} = \lambda_{4} \delta_{ik} \delta_{rs} + \mu_{4} (\delta_{ir} \delta_{ks} + \delta_{is} \delta_{kr})$$
(6.3-18a)

Considering the identity $f_{rs}f_{rs} = \frac{1}{3}f_{rr}f_{ss} + f'_{rs}f'_{rs}$ where f'_{rs} is the deviatoric component of the tensor, we obtain the following inequalities

$$\mu_{3} \ge 0 \qquad \mu_{4} \ge 0 \qquad \lambda_{3} + \frac{2}{3} \mu_{3} \ge 0$$

$$\lambda_{4} + \frac{2}{3} \mu_{4} \ge 0 \qquad 4\mu_{3}\mu_{4} \ge (\mu_{1} + \mu_{2})^{2}$$

$$4(\lambda_{3} + \frac{2}{3} \mu_{3})(\lambda_{4} + \frac{2}{3} \mu_{4}) \ge [(\lambda_{1} + \lambda_{2}) + \frac{2}{3}(\mu_{1} + \mu_{2})]^{2}$$

In view of the constitutive equations proposed and worked out for the rate of entropy production by Biot [17], Schapery [39], Valanis [44] and the others, we postulate the following constitutive equations for $\frac{D\gamma}{Dt}$.

 $\frac{D}{Dt} = f(\xi_{\ell}, d_{ij}, f_{ij}, u_i - v_i, e_{ij}, \rho, \xi_{\ell}) \ge 0 \quad (6.3-19)$ where ξ_{ℓ} is the material time derivative of ℓ th hidden coordinate. If ξ_{ℓ} is expressed in terms of material coordinates, the above derivative reduces to partial derivative with respect to time.

The rate of entropy production is required to be zero in equilibrium state.

$$f(0,0,0,0,e_{ij}, \rho, \xi_{\ell}) = 0$$
 (6.3-20)

Therefore the appearance of e_{ij} , ρ , ξ_{ℓ} in the above constitutive equation is implicit and consequently (6.2-19) can be written as

$$\frac{D}{Dt} = f(\xi_{\ell}, d_{ij}, f_{ij}, u_i - v_i) \ge 0$$
 (6.3-21)

where the implicit dependence on e_{ij} , ρ , ξ_{ℓ} is understood. Assuming required smoothness of the function f, it can be expanded in Taylor series. Neglecting terms higher than the second and eliminating terms less than two on account of the assumption of zero initial stresses, yields,

$$\frac{D\gamma}{Dt} = \sum_{\alpha=1}^{\infty} (b_{\alpha} \dot{\xi}_{\ell} \dot{\xi}_{\ell} + b_{ij\alpha} d_{ij} \dot{\xi}_{\alpha} + \overline{b}_{ij\alpha} f_{ij} \dot{\xi}_{\alpha} + b_{i\alpha} (u_{i} - v_{i}) \dot{\xi}_{\alpha})$$
(6.3-22)

Comparing (6.3-22) with (6.3-12), we obtain: $b_{\alpha}\dot{\xi}_{\alpha} + b_{ij\alpha}d_{ij} + \overline{b}_{ij\alpha}f_{ij} + b_{i\alpha}(u_i - v_i) + \rho \frac{\partial A}{\partial \xi_{\ell}} = 0$ (6.3-23) Expanding A in Taylor series under the same conditions as $\mbox{D}\gamma/\mbox{D}t$

$$\overline{\rho}A = \frac{1}{2}C_{ijk\ell}e_{ij}e_{k\ell} + C_{ij\alpha}e_{ij}\xi_{\alpha} + \sum_{i}\frac{1}{2}C_{\alpha}\xi_{\alpha}\xi_{\alpha} + \frac{1}{2}b\eta^{2} + C_{ij}e_{ij}\eta$$

$$+ \overline{C}_{\alpha}\xi_{\alpha}\eta \qquad (6.3-24)$$

where $\eta = \rho_2 - \overline{\rho}_2$.

In view of (6.3-24), (6.3-23) becomes

$$\frac{\rho}{\overline{\rho}} - (C_{ij\alpha}e_{ij} + C_{\alpha}\xi_{\alpha} + \overline{C}_{\alpha}\eta) + b_{\alpha}\dot{\xi}_{\alpha} + b_{ij\alpha}d_{ij} + \overline{b}_{ij\alpha}f_{ij}$$

$$+ b_{i\alpha}(u_{i} - n_{i}) = 0 \qquad (6.3-25)$$

We write the above equation in the following form

$$\dot{\xi}_{\alpha} + \frac{C_{\alpha}}{b_{\alpha}} \xi_{\alpha} = -Q(t) \qquad (6.3-26)$$

where

$$Q(t) = \frac{1}{b_{\alpha}} \left(C_{ij\alpha} e_{ij} + \overline{C}_{\alpha} \eta + b_{ij\alpha} d_{ij} + \overline{b}_{ij\alpha} f_{ij} + b_{i\alpha} (u_i - v_i) \right)$$

$$\rho/\overline{\rho} \text{ is considered to be unity in view of the second order}$$

effect of neglected terms. The solution of equation (6.3-26)

is

$$\xi_{\alpha} = -\int_{-\infty}^{t} Q(\tau) e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)} d\tau \qquad (6.3-27)$$

or

$$\xi_{\alpha} = -\frac{b_{\alpha}}{C_{\alpha}} \left[Q(t) - \int_{-\infty}^{t} e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)} \frac{\partial Q(\tau)}{\partial t} \right] d\tau$$

Substituting (6.3-13-16) into constitutive equations (6.3-3-6), we obtain

$$\sigma_{ki} = \sigma_{ik} = \rho \frac{\partial x_i}{\partial x_r} \frac{\partial x_k}{\partial x_s} \frac{\partial A}{\partial e_{rs}} + A_{ikrs} f_{rs} + \overline{A}_{ikrs} d_{rs}$$
(6.3-28)

$$\tau_{ki} = \tau_{ik} = -\rho\rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ik} + B_{ikrs} f_{rs} + \overline{B}_{ikrs} d_{rs} \qquad (6.3-29)$$

$$\pi_{i} = \rho_{1} \frac{\partial A}{\partial \rho_{2}} \frac{\partial \rho_{2}}{\partial y_{i}} - \frac{1}{2} \rho_{2} \frac{\partial A}{\partial e_{rs}} \frac{\partial e_{rs}}{\partial x_{i}} + a(u_{i} - v_{i})$$
(6.3-30)

Substituting for A from (6.3-24) into (6.3-28-30) and retaining linear terms we obtain:

$$\sigma_{ik} = C_{ik\ell m} e_{\ell m} + \sum_{\alpha} a_{ik\alpha} \xi_{\alpha} + a_{ik} \eta + A_{ikrs} f_{rs} + \overline{A}_{kirs} d_{rs} \quad (6.3-31)$$

$$\tau_{ik} = -\overline{\rho}_{2} (b\eta + C_{\ell m} e_{\ell m} + \sum_{\alpha} \overline{C}_{\alpha} q_{\alpha}) \delta_{ik} + B_{ikrs} f_{rs} + \overline{B}_{ikrs} d_{rs} \quad (6.3-32)$$

$$\pi_{i} = a(u_{i} - v_{i})$$
 (6.3-33)

Substituting for q_{α} from (6.3-27) into constitutive equations for partial stresses yields:

$$\sigma_{ik} = \int_{-\infty}^{t} G_{ik\ell m}^{(1)}(t - \tau) \frac{\partial e_{\ell m}(\tau)}{\partial \tau} d\tau + \int_{-\infty}^{t} G_{ik\ell m}^{(2)}(t - \tau) \frac{\partial f_{\ell m}(\tau)}{\partial \tau} d\tau + \int_{-\infty}^{t} G_{ik\ell m}^{(t - \tau)} \frac{\partial d_{\ell m}(\tau)}{\partial \tau} d\tau + \int_{-\infty}^{t} G_{ik}^{(4)}(t - \tau) \frac{\partial \eta(\tau)}{\partial \tau} d\tau + \int_{-\infty}^{t} G_{ik\ell}^{(5)}(t - \tau) \frac{\partial \left(u_{\ell}(\tau) - v_{\ell}(\tau)\right)}{\partial \tau} d\tau \qquad (6.3-33)$$

$$\tau_{ik} = \left\{ \int_{-\infty}^{t} F_{\ell m}^{(1)} (t - \tau) \frac{\partial e_{\ell m}}{\partial \tau} d\tau + \int_{-\infty}^{t} F^{(4)} (t - \tau) \frac{\partial \eta(\tau)}{\partial \tau} d\tau \right. \\ \left. + \int_{-\infty}^{t} F_{\ell}^{(5)} (t - \tau) \frac{\partial (u_{\ell} - v_{\ell})}{\partial \tau} d\tau \right\} \delta_{ik} \\ \left. + \int_{-\infty}^{t} F_{ik\ell m}^{(2)} (t - \tau) \frac{\partial f_{\ell m}}{\partial \tau} d\tau + \int_{-\infty}^{t} F_{ik\ell m}^{(3)} (t - \tau) \frac{\partial d_{\ell m}}{\partial \tau} d\tau \quad (6.3-34) \right\}$$

where

$$G_{ik\ell m}^{(1)} = C_{ik\ell m} - \sum \frac{a_{ik} C_{\ell m \alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)} \right)$$

$$G_{ik\ell m}^{(2)} = A_{ik\ell m} - \sum \frac{a_{ik\alpha} \overline{b}_{\ell m \alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)} \right)$$

$$G_{ik\ell m}^{(3)} = \overline{A}_{ik\ell m} - \sum \frac{a_{ik\alpha} b_{\ell m \alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)} \right)$$

$$G_{ik}^{(4)} = a_{ik} - \sum \frac{a_{ik\alpha} \overline{C}_{\alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)} \right)$$

$$G_{ik}^{(5)} = -\sum \frac{a_{ik\alpha} b_{\ell \alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)} \right)$$

and

$$F_{\ell m}^{(1)} = -\overline{\rho}_{2}C_{\ell m} + \overline{\rho}_{2} \sum \frac{\overline{C}_{\alpha}C_{\ell m \alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)}\right)$$

$$F_{ik\ell m}^{(2)} = B_{ik\ell m} + \overline{\rho}_{2} \sum \frac{\overline{C}_{\alpha}b_{\ell m \alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)}\right) \delta_{ik}$$

$$F_{ik\ell m}^{(3)} = \overline{B}_{ik\ell m} + \overline{\rho}_{2} \sum \frac{\overline{C}_{\alpha}b_{\ell m \alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)}\right) \delta_{ik}$$

$$F^{(4)} = -\overline{\rho}b + \overline{\rho}_{2} \sum \frac{\overline{C}_{\alpha}^{2}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)}\right)$$

$$F_{\ell}^{(5)} = -\overline{\rho}_{2} \sum \frac{\overline{C}_{\alpha}b_{\ell \alpha}}{C_{\alpha}} \left(1 - e^{-\frac{C_{\alpha}}{b_{\alpha}}(t - \tau)}\right)$$

Where G⁽¹⁾_{iklm...are} relaxation functions. The above constitutive equations hold for the most general anisotropic case. The interaction was assumed to render the constitute equations for partial stresses a function of all state variables. However some of the relaxation functions might be either a constant or zero at all times. This remains to be determined through experiments and we do not discuss it any further.

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