A DEVELOPMENT OF THE CHARACTER TABLES FOR CERTAIN CLASSES OF UNITARY AND LINEAR GROUPS

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ABSTRACT

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By

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The classical groups $U(n,q^2)$ and $GL(n,q)$ occur as infinite families of groups indexed by a dimension n and a prime power q. It is convenient to develop what might be called 'abstract' character tables whose entities are written as functions of n and q and which describe the characters for the entire family of such groups. It is too difficult to work with both n and q arbitrary, so n is fixed and the character table which holds for all q is found. In 1955 a method for constructing the character table of $GL(n,q)$ for a given n and arbitrary q was developed by Green [2]. Since then the only 'abstract' character tables constructed have been those for $U(2,q^2)$, $U(3,q^2)$ by Ennola [1] and $Sp(4,q)$, q odd, by Srinivasan [3].

In this paper ten abstract character tables are developed, representing the group families $SL(n,q)$, $PSL(n,q)$, $SU(n,q^2)$, $PSU(n,q^2)$ $n = 2,3$ and $PSL(4,q)$ $d = 1$. Of these, seven have never been published. The tables for these groups are of particular value because they often appear as important subgroups in other larger groups such as many of the sporadic simple groups, or

are themselves simple, or are used to construct other simple groups.

This paper has several purposes:

1. The character tables are made available in the sense that the range of all parameters are given explicitly so that the user can easily generate the desired table for any specific q without searching through the paper for the definition of the various entries in the table. This degree of explicitness is not present for the three tables now in print and for this reason they have been included in this paper.

2. A standardized notation is used for all the tables which should facilitate comparisons and other inter-connecting uses.

3. The procedure discussed in section III, together with the main theorem developed in section II should enable one to more easily work out a specific character table for any of the groups SL, PSL, SU, PSU not covered by this paper.

4. A very interesting and potentially important conjecture made by Ennola [1] relating the generalized character tables of $GL(n,q)$ and $U(n,q^2)$ is extended, in section IV, to the special and projective special groups. It is demonstrated that a change of $q \to -q$ in the character table for $SL(n,q)$ or $PSL(n,q)$ will yield the table for $SU(n,q^2)$, $PSU(n,q^2)$ respectively for the case $n = 2,3$. The converse, of course, is also valid for these cases.

The main theorems of the paper, proved in Chapter II, are the following:

<u>TH</u> If HK is a central product of groups H, K where K is abelian, then every irreducible character of HK remains irreducible when restricted to H. From this theorem, we can

obtain the following Theorem for $SL(n,q)$ which is a striking improvement on Clifford's Theorem.

Th Any irreducible character χ of $GL(n,q)$ when restricted to $SL(n,q)$ is either irreducible or splits into t conjugate, irreducible characters of $SL(n,q)$ of multiplicity 1 where $t|d$ and $d = (n,q-1)$. χ has $\frac{(q-1)}{t}$ associates in $GL(n,q)$ relative to $SL(n,q)$ if χ is reducible and $q-1$ associates if irreducible. (An identical theorem holds for $SU(n,q^2)$ with $(q-1)$ replaced by $(q+1)$.) This is the main working theorem of the paper as it tells us just how characters of GL will split when restricted to SL. Clifford's theorem only says that the number of components, t, will be a factor of $(q-1)$ which is of no help. But the above theorem says that $t = 2$ or 3 for $n = 2$ or 3 respectively.

In section III the technique used to formulate the character tables is discussed. It is shown that each conjugacy class of SL can be indexed by a Jordan canonical form.

The characters of $GL(n,q)$ are restricted down to SL and the inner product $(\chi,\chi) = \frac{1}{|SL|} \sum_{g \in SL} \chi(g)\overline{\chi(g)}$ is calculated. If $(\chi,\chi) = 1$ the restricted character is irreducible. Otherwise, it is reducible and splits up as indicated by the theorem just discussed.

Once the character table for SL is determined we can easily generate the table for PSL, since every character of SL which is constant on $Z(SL)$ is an irreducible character of $PSL = SL/Z(SL)$, and every irreducible character of PSL can be so obtained.

In the sections V through XV, the character tables are developed. Many of the routine calculations are done once in detail and merely mentioned or entirely omitted in later sections.

In section XVI the results obtained for PSL(4,q) d = 2 are given and some space is devoted to discussing the problems which prevented further progress, problems which were primarily attributable to the complexity of the table for GL(4,q).

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I. INTRODUCTION

1.1 History

Towards the end of the 19th century G. Frobenius developed the foundations for the theory of group representation in a series of papers dating from 1895 to 1911. In 1911 Burnside published his book *Theory of Groups of Finite Order* in which he independently rediscovered some results of Frobenius. More importantly he applied this embryonic theory and obtained some surprising results in group theory regarding the solvability of certain groups which gained more serious attention for this new concept. After this initial success, however, the theory of group representation lay relatively dormant for nearly 25 years. It appeared that the applications of this theory as developed up to that point had been exhausted.

This disinterest was not universal. Physicists quickly grasped the importance of representation theory and soon began using it to advantage. Since representation theory reduces the abstract properties of groups to numbers, they found that this enabled them to apply group theory to any system possessing symmetry that was too complex to handle by classical analysis. Such fields as thermodynamics, crystallography, wave equations, quantum mechanics, molecular and nuclear structure are but a few of the many areas where representation has played an important role. Thus the theory of representations developed steadily in the area of applications long after the pure mathematics of its source ceased to offer much promise. Indeed, during the period of 1905 to 1955 the growth of new developments in group theory proceeded at a very slow pace. Felix Klein felt this was due to the abstract nature of group theory which isolated it from physical phenomenon which motivated so much mathematics of that time and as a result only a certain type of mathematician was attracted to the field. He felt this homogeneity of the researchers was largely responsible for its slow rate of development, unmarked by imaginative leaps forward. If this was the reason for the decline in activity in group theory then the successes won by representation theory in the physical sciences may well have lessened the stigma of abstractness and helped in restoring vitality to the field of group theory.

1.2 Recent Interest

In the period 1925-55 a major advance was made by R. Brauer in the study of modular representations. In 1955 when activity in group theory suddenly increased, Brauer's work provided some powerful tools with which to study group structure. Thompson and Feit in their long paper on groups of odd order used modular theory with telling effect and thus rekindled interest in representation theory. M. Suzuki has used considerable character theory in his work on the classification of simple groups by the structure of involution normalizers.

At the present time significant advances in group theory are being made in the area of classifying all simple groups and here representation theory continues to play a large role since nearly all theorems on groups using characters establish the existance of normal subgroups. The first step in this classification problem was to find all simple groups of order less than some large number. Initially all the simple groups which were found could be placed in one of several classes of groups, all groups of a class sharing some common properties. However, this convenient pattern was broken when, starting in 1964, people began finding some large simple groups which didn't fit into any of the known classes. These groups became known as the 'sporadic' simple groups and they attracted considerable attention. At this date the character tables for nearly all these sporadic groups are known, together with some of their larger subgroups.

1.3 The Problem

There are several reasons for wanting to know the character table for a group. The most obvious reason is to be able to study better the structure of the group. Sometimes one has the character table of a group and desires to know some of its larger subgroups (see p. 66 [2]). Methods are available to find the character table of such subgroups (see p. 150 [16]). One may then be able to recognize this subgroup's character table as being the same as some known table. Thus a good stockpile of character tables would aid in identifying subgroups. The classical groups turn up often as subgroups and so their tables are particularly valuable.

Wales' paper [22] furnishes an excellent example of the current manner in which characters of groups, in this case PSL(2,17), can be used to obtain surprising results.

Another use for character tables lies in using them to test out various conjectures which might arise. For this purpose one would like to have tables for a wide variety of groups.

Sometimes groups are constructed by extensions of several groups. If the tables of these subgroups are known, it is often possible to obtain the character table of the constructed group. Very frequently the groups used in this construction are of the type discussed in this paper.

This paper deals with the character tables for $PSL(n,q)$, $PSU(n,q^2)$, $SL(n,q)$ and $SU(n,q^2)$ for $n = 2,3$ and some results for $n = 4$. The case of $n = 2$ was originally done by Frobenius [12], Schur [18] and later independently by Jordan [15]. This case is redone here for several reasons. The methods used by these earlier researchers are considerably different and more complex than those used in this paper. Also the resulting tables are written in such a manner that in order to use them, one would have to go back through the paper to find the definitions and range of values for the many parameters. In addition, the difference in notation from one table to another would make comparisons very difficult. The table for $PSL(3,q)$ was done by Brinkmann [3] but the results were never published. Thus it was considered advisable to work out all the tables for $n = 2,3$ whether they had been done before or not.

The final result is a set of 10 character tables, only 3 of which have been previously published, each of which describes an infinite class of finite groups and all unified by common notation with the range of parameters explicitly given to facilitate using the tables. The procedure used to obtain these tables is quite likely the most straight forward one possible and yet little can be done on the cases for $n = 4$, not because the method breaks down but because the character tables become so large and complex that the details are exhausting.

The character tables developed here are of a somewhat unusual nature and a word or two should be said about them. Sometimes it is possible to find a character table which can serve as the table for a whole class of groups by changing some parameters. Apart from the economy of such an 'abstract' character table it is valuable in that it provides a strong linkage among the groups. There have been few such tables developed and so it seems that a few more additional tables would be of value.

One of the most valuable classes of groups for which character tables are available is the class of symmetric groups S_n (also A_n). Although no abstract character table has been developed for the entire class an equivalent formulation has been obtained, namely, a comparatively simple construction method for building the character table for any of the specified groups.

In 1896 Frobenius determined the generalized character table for the groups $GL(2,q)$. In 1949 Steinberg [19] worked out the tables of $GL(n,q)$ for $n = 2,3,4$ in a particularly straight forward fashion. These results and some later work done by

Steinberg led to the development by Green [14] in 1955 of a method to calculate the characters of $GL(n,q)$ for any n. In 1963 Ennola [6] found the characters of $U(n,q^2)$ for $n = 2,3$ and recently Srinivasan [21] obtained the table for $Sp(4,q)$ q odd. This sums up all the results on 'abstract' tables as of this date.

1.4 General Theory

In the following short section an attempt is made to run quickly over the key definitions and theorems of character theory which are employed in this paper or necessary for its understanding.

DEFINITION. A representation of a group G is a homomorphism $\rho: G \to GL(n,\mathbb{C})$ where $GL(n,\mathbb{C})$ is the group of all $n \times n$, nonsingular matrices over the complex numbers; the group operation is ordinary matrix multiplication.

Every group has at least one representation, the most trivial one being $\rho_1: G \to 1$. The totality of all representations of a given group can be narrowed down by defining the following equivalence relation:

DEFINITION. If ρ and σ are n-dimensional representations of group G and there exists a nonsingular matrix A s.t. $\sigma(g) = A\,\rho(g)A^{-1}$ for all $g \in G$ then we say ρ and σ are equivalent.

The next step is to show that certain representations of G are the 'building blocks' from which all the non-equivalent representations can be constructed.

DEFINITION. A representation ρ is reducible if it is equivalent to a representation with matrices of the form $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$

$$\text{i.e.}\quad \rho: g \to \left(\begin{array}{c|c} \sigma_1(g) & F(g) \\ \hline 0 & \sigma_2(g) \end{array}\right)$$

It follows that $\sigma_1(g)$ and $\sigma_2(g)$ are also representations of G. If a representation is not reducible then it is said to be <u>irreducible</u>.

<u>THEOREM</u> 1.1 If the representation ρ (over $\mathbb{C}$) is equivalent to a reducible representation σ

$$\text{i.e.}\quad \sigma: g \to \left(\begin{array}{c|c} \sigma_1(g) & F(g) \\ \hline 0 & \sigma_2(g) \end{array}\right) \quad \text{then } \rho \text{ is equivalent to}$$

the direct sum of the representation σ_1 and σ_2.

From this theorem it follows that:

<u>THEOREM</u> 1.2 If ρ is a representation for G then ρ is equivalent to the direct sum of irreducible representations of G, for ρ over $\mathbb{C}$.

This key theorem says that once we know all the irreducible representations of a group G, we have all the possible representations since they can be constructed by adding together irreducible representations.

The next surprising theorem tells us that the number of irreducible representations of a finite group is strictly limited.

<u>THEOREM</u> 1.3 The number of non-equivalent irreducible representations of G equals the number of conjugacy classes of G.

It would be very cumbersome, and as it turns out, unnecessary to write down all the irreducible representations of a given group. To avoid doing this we look for some way to designate each of these representations. This leads to the next definition.

<u>DEFINITION</u>. If ρ is a representation of G, then the <u>character</u>

χ of ρ is the mapping from G to $\mathbb{C}$ defined by:

$\chi(g) = \text{trace } \rho(g)$.

If $g_1, g_2 \in G$ are elements of the same conjugacy class and ρ is a representation of G, then $\rho(g_1)$ is similar to $\rho(g_2)$ and since similar matrices have the same trace we can state:

<u>THEOREM</u> 1.4 Any character χ of G is constant on conjugacy classes of G.

Also:

<u>THEOREM</u> 1.5 Characters of equivalent representations are identical.

<u>THEOREM</u> 1.6 If the representation ρ of G is reducible and equal to the sum $\rho_1 + \rho_2 + \ldots + \rho_k$ of irreducible represen-ations, then $\chi_\rho = \chi_{\rho_1} + \chi_{\rho_2} + \ldots + \chi_{\rho_k}$. In particular, every character of G is the sum of irreducible characters of G.

The above theorems mean that we can identify a representation ρ by specifying its character χ_ρ which is an r-row column vector

$$\begin{matrix} \chi(g_1) \\ \chi(g_2) \\ \vdots \\ \chi(g_r) \end{matrix}$$

each entry being the trace of a representative matrix $\rho(g_i)$ from each conjugacy class of G. We call $\chi(1)$ the <u>degree</u> of the character. <u>Theorem 6</u> says we need only specify the irreducible characters of the group since all the others are simply integral linear combinations of these. Thus we can write down a square matrix for a group G called the <u>character</u> <u>table</u> for G. The columns of this table are the n irreducible characters of G.

Customarily the identity character $\begin{matrix}1\\1\\\vdots\\1\end{matrix}$ is written as the first column (or row) and the first conjugacy class, i.e. the first row (or column) is the identity of the group. The ij^{th} entry of the table is the value $\chi^j(g_i)$, g_i being an element of the i^{th} conjugacy class and χ^j being the j^{th} irreducible character of G.

It should be noted in passing that $G \cong H$ implies G and H have the same character table but the converse is not true. The quaternion group and the octic group are the smallest counter examples.

An amazing number of relationships exist between the rows and columns of a character table. In fact, these relationships are so restrictive that in some cases it is possible to construct the character table for a given simple group knowing no more than the order of the group. There is no universal procedure to develop a character table. After one has armed himself with a thorough knowledge of the various character relationships and some special number theory concepts, he can then proceed only on intuition and experience. No computer program has yet been devised to calculate character tables for all groups of order ≤ 200 and yet tables for groups of this size can be done reasonably quickly by hand. Frame's paper [9] displays in detail many of the techniques used to generate a character table of a group from that of a subgroup. Although individuals have found character tables of groups with orders as large as $|G| = 47{,}377{,}612{,}800$ the most notable success measured in terms of magnitude has been the generation of the table for Conway's group of order $2^{21}\cdot 3^9\cdot 5^4\cdot 7\cdot 11\cdot 13\cdot 23$ by computer.

The method used some of the unique properties of the group and in no way represented a general procedure which would work for other groups.

Before the more useful properties of the character table are stated, the following definition is necessary.

__DEFINITION__. Let χ_1, χ_2 be characters of group G. Let

$$(\chi_1, \chi_2) \overset{\text{def}}{=} \sum_i \frac{\chi_1(g_i)\overline{\chi_2(g_i)}}{|N(g_i)|}$$

where g_i is an element from the i^{th} conjugacy class and $|N(g_i)|$ is the order of the centralizer $N(g_i)$ of g_i.

In the following let χ_i, χ_j be irreducible characters of G and θ be a reducible character.

__THEOREM__ 1.7 (Orthogonality relations on columns)

(i). $(\chi_i, \chi_j) = \delta_{ij}$ iff χ_i, χ_j are irreducible characters.

(ii). $(\chi_i, \theta) = a_i$ where a_i is the number of times the irreducible character χ_i is contained in θ.

(iii). $(\theta, \theta) = \sum_i a_i^2$ = sum of squares of the multiplicities of all the irreducible characters appearing as components of θ.

The above relations enable one to determine if a character is irreducible or not and in the case of a reducible character, to break it up into its irreducible components. Note the following: (i) implies that if $(\Psi\ ,\Psi\ ) = 1$ then Ψ_i is irreducible. If $(\theta,\theta) = 2$ or 3 then (iii) implies that θ is composed of 2 or 3 irreducible characters respectively (this case comes up often in this paper).

THEOREM 1.8 (Row Orthogonality)

$$\sum_{i=1}^{k} \chi^i(g)\overline{\chi^i(h)} = \begin{cases} 0 & \text{if } g,h \text{ are not in the same conjugacy class} \\ |N(g)|, & \text{if } g,h \text{ are in the same conjugacy class} \end{cases}$$

Note: A special case of the above is:

$$\sum_i \chi^i(1)\chi^i(1) = \sum_i (\text{degree of } i^{th} \text{ character})^2 = |G| .$$

The next major topic is the question of how characters of the group are related to characters of its subgroups.

If χ is an irreducible character of G we can easily get a character $\chi|_H$ of $H \le G$ by restricting χ to H. $\chi|_H(g) = \chi(g)$ for g being any element in a conjugacy class of H. $\chi|_H$ is not necessarily irreducible and often it isn't. If Ψ is an irreducible character of $H \le G$ we can obtain a character Ψ^G of G by a process called inducing, which is formulated by: $\dfrac{\Psi_j^G}{|N_j^G|} = \displaystyle\sum_{C_\lambda^H \subset C_j^G} \dfrac{\chi_\lambda^H}{|N_\lambda^H|}$ where C_j^G is the j^{th} conjugacy class of G and C_λ^H are all conjugacy classes of H which are contained in the conjugacy class C_j^G. Again we note that the induced character Ψ^G need not be irreducible in G. The induced and restricted characters of H and G are bound together by the following theorem:

THEOREM 1.9 (Frobenius' Reciprocity Theorem)

$$(\chi|_H,\Psi)_H = (\chi,\Psi^G)_G .$$

This theorem says that the number of times the restricted character $\chi|_H$ contains Ψ as a component, equals the number of times that the induced character Ψ^G contains χ as a component.

Thus if $H \leq G$, and one has available the character table for $G(H)$, it may be possible to find the table for $H(G)$ by restricting (inducing) all characters of $G(H)$ to $H(G)$ and using the orthogonality properties and Frobenius' Theorem to determine how to split up all the reducible characters into their irreducible components.

The preceding has been a summary of character theory which is applicable to all groups. In the following section some theorems are discussed which apply only to special groups.

II. A MODIFICATION OF CLIFFORD'S THEOREM

In this section we introduce some results known collectively as Clifford's Theorem, concerning characters of normal subgroups. The particular structure of $SL(n,q)$ is analyzed and a theorem is obtained which enables us to extend Clifford's Theorem and achieve even more information concerning the characters of $GL(n,q)$ restricted to $SL(n,q)$.

2.1 Clifford's Theorem

Nearly every textbook which contains a section on Clifford's Theorem, including most of Clifford's paper [4], states the results in terms of irreducible G-modules and G-submodules. In this section we restate these theorems in terms of characters. Such a change in viewpoint can be found in Feit [8] and Lomont [17].

The following are various definitions, notations and concepts used in the discussion of Clifford's Theorem. Let G be a group and suppose ρ is an irreducible representation of G and χ its character.

DEFINITION. If $\sigma \in \mathrm{Aut}(G)$ then the representation ρ^σ defined by: $\rho^\sigma(g) \overset{\mathrm{def}}{=} \rho(g^\sigma)\ \forall\ g \in G$ is said to be conjugate to ρ.

Likewise the character, $\chi^\sigma(g) \overset{\mathrm{def}}{=} \chi(g^\sigma)\ \ \forall\ g \in G$ is conjugate to the character χ of G.

It can be shown that if χ is an irreducible character, then χ^{σ} is also. If $H \triangleleft G$ and χ is an irreducible character of G then $\chi|_H$, $\chi^g|_H$ are said to be <u>conjugate relative to G</u>. Conjugate characters should not be confused with characters χ, χ' which are complex conjugate characters defined by $\chi'(g) = \overline{\chi(g)}$.

<u>DEFINITION</u>. Let $H < G$. Two irreducible representations of G, ρ_1 and ρ_2 are <u>associates</u> if they have an irreducible component in common when restricted to H.

The following theorems are all modifications of, or an outgrowth of Clifford's work [4].

<u>THEOREM</u> 2.1 Clifford's Theorem

If $H \triangleleft G$ and ρ is any irreducible representation of G, then the restriction of ρ to H denoted by ρ_H is either irreducible or reducible into irreducible components of the same degree. If ρ_H' is any such irreducible component of ρ_H, then all other components are conjugates of ρ_H' relative to G and in addition, every such conjugate of ρ_H' appears as a component of ρ_H.

This theorem says that if a character χ_H splits in H then it splits into characters of equal degree and which are identical on all classes of H which are complete classes of G. The sets of values of the characters over the other classes of H are identical but the values are permuted over these classes. Now we will take up a theorem which says something about the number of components and their multiplicities.

Suppose χ_H is reducible. Then associated with this character is a subgroup G^* s.t. $H < G^* < G$. If G/H is cyclic then the following theorem provides some interesting information.

THEOREM 2.2 Clifford's Theorem

Let H be a normal subgroup of G such that G/H is cyclic of order k. Then the irreducible components of χ_H each have multiplicity 1. The number of components of χ is m, and the number of distinct associates of χ relative to H is k/m, where $m = [G:G^*]$ and $H \le G^* \le G$.

Since the scalar matrices of GL form a cyclic group, GL/SL, we know that if a character of GL splits in SL it will split into m pieces of multiplicity 1, where m divides $[GL:SL] = (q-1)$. Also, since the number of associates is $(q-1)/m$, this means that if χ_{SL} is irreducible (i.e. $m = 1$) then there are $(q-1)$ characters of GL that restrict down to the same character of SL. If χ_{SL} is reducible into m components $\Psi_1,\ldots,\Psi_m$ then there are $(q-1)/m$ characters of GL which, when restricted to SL, also split into the same components.

2.2 A Theorem Applicable to $SL(n,q)$, $SU(n,q)$

Let χ be an irreducible character of $GL(n,q)$. By Clifford's Theorem 2.2, if $\chi|_{SL}$ is reducible then it splits into m conjugate irreducible components of multiplicity 1 where m divides $[GL:SL] = (q-1)$. In this section we prove that m divides $d = (n,q-1)$, and this determines m exactly for all the groups considered in this paper.

DEFINITION. Let H,K,M be groups with $M \le Z(H)$ and suppose there exists an isomorphism θ of M into $Z(K)$. Then if we identify M with its image $\theta(M)$, there exists a group G of the form $G = HK$ with $M = H \cap K \le Z(G)$

such that H centralizes K. Such a group G is said to be a <u>central product of H and K w.r.t M</u>. In diagram form:

$$\begin{array}{ccc} H & & K \\ | & & | \\ Z(H) & & Z(K) \\ | & & | \\ M & \cong & \theta(M) \end{array}$$

<u>Lemma 2.3</u> If $H \times K$ is the direct product of a group H with a group K and Ψ,θ are irreducible characters of H,K respectively then the character χ defined by $\chi(hk) = \Psi(h)\cdot\theta(k)$ is an irreducible character of $H \times K$. Conversely, every irreducible character of $H \times K$ is equivalent to such a product of characters of H and K.

pf: see Littlewood [16]

<u>Lemma 2.4</u> If $H \times K$ is a direct product of a group H with an abelian group K and χ is an irreducible character of $H \times K$, then $\chi|_H$ is an irreducible character of H.

Pf: By <u>lemma 2.3</u> $\chi(g) = \Psi(h)\cdot\theta(k)$ where $g = hk$, $h \in H$, $k \in K$ and Ψ,θ are irreducible characters of H,K respectively. K is abelian so all its characters are of degree 1. $\therefore$ $\chi(h) = \chi(h\cdot 1) = \Psi(h)\cdot\theta(1) = \Psi(h)\cdot 1 = \Psi(h)$ for all $h \in H$. Thus $\chi|_H = \Psi$ which is an irreducible character of H.

<u>Lemma 2.5</u> If HK is a central product, then there exists a homomorphism $f:H \times K \rightarrow HK$

pf: Gorenstein [13]

<u>Lemma 2.6</u> Let f be a homomorphism $f:G \rightarrow G'$. If χ is an irreducible character of $G/\mathrm{Ker}\, f$ then χ' defined by $\chi'(g) = \chi(f(g))$ is an irreducible character of G.

Pf: Consider (χ',χ'):

$$(\chi',\chi') = \frac{1}{|G|} \sum_{g\in G} \chi'(g)\overline{\chi'(g)} = \frac{1}{|G|} \sum_{g\in G} \chi(f(g))\overline{\chi(f(g))}$$

$$= \frac{|\mathrm{Ker}\ f|}{|G|} \sum_{g\in G/\mathrm{Ker}\ f} \chi(g)\overline{\chi(g)}$$

$$= \frac{1}{|G/\mathrm{Ker}\ f|} \sum_{g\in G/\mathrm{Ker}\ f} \chi(g)\overline{\chi(g)} = (\chi,\chi) = 1$$

since χ is an irreducible character of $G/\mathrm{Ker}\ f$

$\therefore$ $(\chi'\chi') = 1$ implies χ' is an irreducible character of G.

Note: Lemma 2.4 and 2.6 seem to be familiar results but don't appear anywhere in standard references.

THEOREM 2.7 If χ is an irreducible character of a central product HK, where K is abelian, then $\chi|_H$ is an irreducible character of H.

Pf: By Lemma 2.5 there is a homomorphism $f: H \times K \to HK$ such that $HK \cong H \times K/\mathrm{Ker}\ f$.

Let χ' be defined by: $\chi'(g) = \chi(f(g))$.

By Lemma 2.6 χ' is an irreducible character of $H \times K$.

Since H is a subgroup of both $H \times K$ and HK and $\chi'(g) = \chi(f(g))$ then $\chi'(h) = \chi(h)$ for all $h \in H$.

This implies $\chi'|_H = \chi|_H$.

But $\chi'|_H$ is irreducible by Lemma 2.4 so $\chi|_H$ is an irreducible character of H.

This theorem says that every irreducible character of a central product HK with K abelian is irreducible upon restriction to H and so no splitting of characters can take place.

We now study the structure of $SL(n,q)$ and use Theorem 2.7 to obtain our major goal.

<u>DEFINITION</u>. Let $M(d) = \{A \in GL(n,q) \mid \det A = \rho^{dk} \ k = 1,\ldots,\frac{q-1}{d}\}$ where $d = (n,q-1)$ and ρ is a primitive element of $GF(q)$.

We note that every scalar matrix of $GL(n,q)$ is contained in $M(d)$ since $\det \rho^k I = \rho^{kn} = \rho^{(kn/d)d}$.

<u>Lemma 2.8</u> $M(d) \lhd GL(n,q)$.

Pf: Let $A \in GL$, $B \in M(d)$

$\det ABA^{-1} = \det A \det B \det A^{-1} = \det B = \rho^{dk}$

thus $ABA^{-1} \in M(d) \Rightarrow M(d) \lhd GL$

<u>Lemma 2.9</u> $GL/M(d) \cong \sigma(d)$, a cyclic group of order d.

Pf: Let $f: GL(n,q) \overset{\text{onto}}{\to} \sigma(d)$

where $f(A) = (\det A)^{(q-1)/d}$

Clearly f is a homomorphism and $f(GL) \cong \sigma(d)$. It is easy to show that the $\operatorname{Ker} f = M(d)$ so by the 1st Isomorphism Theorem $GL/\operatorname{Ker} f \cong f(GL)$ i.e. $GL/M(d) \cong \sigma(d)$.

<u>Lemma 2.10</u> $M(d) = SL \cdot Z(M(d))$ is a central product.

Pf: $Z(M(d)) = \{\text{scalar matrices } A \in GL\}$

$|Z(M(d))| = d(\frac{q-1}{d}) = (q-1)$

$$Z(SL) = \left\{ \begin{pmatrix} \rho^{(\frac{q-1}{d})k} & & \\ & \ddots & \\ & & \rho^{(\frac{q-1}{d})k} \end{pmatrix} k = 1,\ldots,d \right\} \qquad |Z(SL)| = d$$

Consider:

$$\begin{array}{ccc} Z(M(d)) & & SL \\ | & & | \\ M & \cong & Z(SL) \end{array}$$

(i) $M(d) = SL \cdot Z(M(d))$

Pf: Let $A \in SL$, $B \in Z(M(d))$, then $\det(AB) = \det A \cdot \det B = 1 \cdot \rho^{dk} \Rightarrow AB \in M(d) \Rightarrow SL \cdot Z(M(d)) \le M(d)$.

But $\left|SL\cdot Z(M(d))\right| = \dfrac{|SL|\cdot|Z(M(d))|}{|Z(SL)|} = \dfrac{\frac{|GL|}{(q-1)}\cdot(q-1)}{d} = |M(d)|$

$\therefore \quad SL\cdot Z(M(d)) = M(d)$.

(ii) $SL \cap Z(M(d)) = Z(SL)$

Pf: Let $A \in SL \cap Z(M(d))$

then $\det A = 1$ and A is a scalar matrix so $A \in Z(SL)$.

(iii) $Z(M(d))$ centralizes SL

Pf: Every matrix of $Z(M(d))$ is scalar and so commutes with every element of SL.

Conditions (i), (ii), (iii) $\Rightarrow$ $M(d)$ is a central product of SL and $Z(M(d))$.

We now put this all together to obtain the following results.

<u>THEOREM</u> 2.11 Every irreducible character χ of $GL(n,q)$ when restricted to $SL(n,q)$ is either irreducible or splits into t conjugate, irreducible characters of SL of multiplicity 1 where t divides d. χ has $\frac{(q-1)}{t}$ associates in GL w.r.t. SL.

Pf:

$$\begin{tikzcd}
GL \arrow[d, dash, "d"'] & & SL \times Z(M(d)) \arrow[dll, "f"'] \arrow[dd, dash] \\
M(d) = SL\cdot Z(M(d)) \arrow[drr, dash, "\frac{q-1}{d}"'] & & \\
& & SL
\end{tikzcd}$$

Suppose χ is an irreducible character of $GL(n,q)$. Now $M(d) \triangleleft GL$ by <u>Lemma 2.8</u> and $GL/M(d)$ is cyclic by <u>Lemma 2.9</u> so by <u>Clifford's Theorem 2.2</u> $\chi|_{M(d)}$ is either irreducible in $M(d)$ or it splits into t conjugate irreducible characters $\chi_1,\ldots,\chi_t$ of multiplicity 1 where t divides $[GL:M(d)] = d$. Since by <u>Lemma 2.10</u>, $M(d)$ is a central product of SL and the abelian group $Z(M(d))$, then by <u>Theorem 2.7</u> every irreducible character of $M(d)$ is also an irreducible character of SL under restriction.

We see from this theorem that any splitting undergone by the characters of GL takes place in restricting from GL to M(d). This means that the characters split into t conjugates where $t|d$. This is marked improvement over Clifford's Theorem which requires only that $t|(q-1)$.

We thus have the immediate Corollary:

Corollary 2.12 The characters of $GL(2,q)$, $GL(3,q)$, $GL(4,q)$, $d = 2$ must either be irreducible when restricted to SL or split into 2,3, or 2 irreducible conjugate characters respectively. Every character χ of $GL(2,q)$, $GL(3,q)$, $GL(4,q)$, $d = 2$ has $\frac{q-1}{t}$ associates w.r.t. SL where $t = 1$ if $\chi|_{SL}$ is irreducible or $t = 2,3$, or 2 respectively if $\chi|_{SL}$ is reducible.

Theorems 2.11 and 2.12 can be restated for the characters of $U(n,q^2)$ restricted down to $SU(n,q^2)$. We need only define M(d) by: $M(d) = \{A \in U(n,q^2) | \det A = \rho^{dk(q-1)} \quad k = 1,\ldots,\frac{q+1}{d}\}$ where $d = (n,q+1)$ and ρ is a primitive element of $GF(q^2)$.

III. TECHNIQUE

In this section we describe the procedures by which the character tables are obtained by restriction from $GL(n,q)$ and $U(n,q^2)$.

3.1 Determination of Conjugacy Classes for the Linear Groups

Dickson [5] did considerable work on determining the conjugacy classes of the general linear groups. The following is a brief description of how the class structure is found.

The elements of $GL(n,q)$ are matrices, with each of which there is associated a characteristic equation. A familiar theorem tells us that if matrix A is similar to matrix B then the characteristic equation of A equals the characteristic equation of B. Thus we can find all possible characteristic equations and corresponding to them we can associate a conjugacy class of GL having that equation. However, since the converse of the above theorem is not true, then some of the characteristic equations will correspond to several different conjugacy classes having that same equation. We can get a full separation by writing down the possible Jordan canonical forms for each type of characteristic equation in an appropriate extension field. Each canonical form corresponds to exactly one conjugacy class and every class has such a canonical form.

In [14] Green shows that if $A \in GL(n,q)$ and A has characteristic polynomial $f_1^{k_1} f_2^{k_2} \dots f_N^{k_N}$ where $f_1, \dots, f_N$ are distinct irreducible polynomials over $GF(q)$ then for every possible partition $\nu_1, \dots, \nu_N$ of the exponents $k_1, \dots, k_N$ there corresponds a distinct canonical form and thus a distinct conjugacy class.

This means that the number of conjugacy classes of $GL(n,q)$ can be determined by counting the number of possible partitions. The final result is a generating function which for a given n is a polynomial in q with constant rational coefficients expressing the number of classes for $GL(n,q)$.

For example, if we want to find the conjugacy class structure of $GL(3,q)$ we first write down the possible factorizations of the characteristic equation of degree 3 and then for each factorization we write down the possible Jordan forms.

<u>Types of characteristic polynomials</u>	<u>Jordan Canonical Form</u>
irreducible cubic	$\begin{pmatrix} \tau & & \\ & \tau^q & \\ & & \tau^{q^2} \end{pmatrix}$
quadratic factor, linear factor	$\begin{pmatrix} \rho^k & & \\ & \sigma^\ell & \\ & & \sigma^{q\ell} \end{pmatrix}$
3 equal linear factors	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^k \end{pmatrix} \begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^k \end{pmatrix} \begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & 1 & \rho^k \end{pmatrix}$
2 equal linear factors	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^\ell \end{pmatrix} \begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^\ell \end{pmatrix}$
3 different linear factors	$\begin{pmatrix} \rho^k & & \\ & \rho^\ell & \\ & & \rho^m \end{pmatrix}$

$\rho \in GF(q)$

$\sigma \in GF(q^2)$

$\tau \in GF(q^3)$

The number of conjugacy classes can now be easily counted by considering the valid range of the exponents on the primitive field elements.

Grouping the conjugacy classes in sets according to the possible Jordan Canonical forms is not only a convenient way to count the classes, but also serves several other purposes. First of all, the values which the exponents k,l,m,n assume can be used to determine the character value for each class, so that great economy of notation results; each whole set of conjugacy classes requires only one entry in each character. Thus the character table of $GL(3,q)$ is composed of only 8 rows, one for each canonical form. Also, it turns out, as first noted by Steinberg in [19], that there is a 1-1 correspondence between the sets of characters of $GL(n,q)$ having the same degree and the sets of conjugacy classes of the same canonical form. The order of a given set of conjugacy classes is equal to the order of the corresponding set of characters. Thus for the group $GL(3,q)$ we see that all the characters are of only 8 different degrees and it is particularly convenient to set up the character table so as to exhibit this 1-1 correspondence between classes and characters. We remark that this 1-1 correspondence between sets of classes and characters doesn't hold in SL and SU if $d \neq 1$.

Dickson determined the order of the centralizer of an element for each conjugacy class type of $GL(n,q)$ $n = 2,3,4$, by counting the number of matrices commuting with a given canonical form.

Finding the conjugacy class structure of SL(n,q) is accomplished by determining what canonical forms have determinant 1. Since $SL \lhd GL$, then SL is composed of complete conjugacy classes of GL. Thus every conjugacy class of SL has the same order as it did in GL with the exception of the classes of GL which split in SL. These splitting classes were determined by Dickson for $n = 2,3$.

3.2 Determination of Conjugacy Classes for the Unitary Groups

In [23] Wall obtains results on the conjugacy classes of some classical groups, the unitary group, $U(n,q^2)$, included. His main results for $U(n,q^2)$ are:

THEOREM. (i) $X \in GL(n,q^2)$ is similar to an element of $U(n,q^2)$ iff $X \sim X^{*-1}$ ('$\sim$' indicates similarity and '$*$' indicates the conjugate transpose)

(ii) Two elements of $U(n,q^2)$ are conjugate in $U(n,q^2)$ iff they are similar in $GL(n,q^2)$

(iii) The number of conjugacy classes in $U(n,q^2)$ is the coefficient of t^n in $\prod_{\lambda=1}^{\infty} \frac{1 + t^\lambda}{1 - q\, t^\lambda}$.

He also gives a formula by which the order of each conjugacy class can be calculated.

The above statements say that all the elements of $U(n,q^2)$ which are similar to some $A \in GL(n,q^2)$ such that $A \sim (A^*)^{-1}$, form exactly one conjugacy class.

In [7] Ennola refines the formula for calculating the order of the conjugacy classes. He also shows that if $A \in GL(n,q^2)$, and $A \sim (A^*)^{-1}$ and ε is an eigenvalue of A, then $\varepsilon^{q^k+1} = 1$

where k is odd. Thus we can use Jordan Canonical forms with marks of $GF(q^{2k})$, fitting the above conditions, to represent the conjugacy classes of $U(n,q^2)$. We use the same type Jordan forms employed as class representatives for $GL(n,q)$ except now we use elements such as ρ^k, σ^k, τ^k where $\rho = \sigma^{(q-1)}$, $\tau = \tau_1^{(q^3-1)}$ and σ, τ_1 are primitive elements of $GF(q^2)$, $GF(q^6)$ respectively.

As in $GL(n,q)$ we assemble the conjugacy classes of $U(n,q^2)$ into sets, each set corresponding to a Jordan Canonical form. We likewise place the irreducible characters into sets according to their degrees. As before, there is a 1-1 correspondence between the sets of classes and sets of characters in $U(n,q^2)$.

3.3 Development of the Character Tables

The character tables for $GL(n,q)$ $n = 2,3$ are given in Steinberg [19] and for $U(n,q^2)$ $n = 2,3$ in Ennola [6]. In the following we shall describe the procedure for finding the character table for $SL(n,q)$ and $PSL(n,q)$, with the understanding that the same procedure works for the unitary case; just replace GL, SL, PSL by U, SU, PSU throughout the discussion.

The conjugacy classes of $GL(n,q)$ are represented by all the possible Jordan Canonical Forms. We first find the conjugacy class representatives for $SL(n,q)$ by selecting only those representatives of GL with determinant 1. Then the irreducible characters χ_i of GL are restricted down to SL one at a time by using only the values $\chi_i(g)$ where g is from a conjugacy class of SL. The inner product $(\chi_i|_{SL},\chi_i|_{SL})$ is calculated.

If $(\chi_i|_{SL}, \chi_i|_{SL}) = 1$ then $\chi_i|_{SL}$ is irreducible and we are done with this character. Otherwise, we know it splits into d irreducible components $\Psi_1, \ldots, \Psi_d$ which are all equal on those classes of SL which are complete classes of GL. The values of the Ψ_i on the other classes can be later determined, after all the characters of GL have been restricted down to SL, by using the orthogonality relations Theorems 1.7 and 1.8 and Gaussian sums.

After the character table for SL is found, we can obtain the table of PSL = SL/Z(SL) by using the following well known theorems due to Frobenius.

<u>THEOREM</u> 3.1 If $H \triangleleft G$ and X is an irreducible character of G, then X is a character of G/H if and only if it has equal values for any two elements of G which are equivalent modulo H.

<u>THEOREM</u> 3.2 If $H \triangleleft G$ then every character of G/H is a character of G.

To determine the conjugacy classes of PSL we simply fuse into a single class of PSL all the classes of SL which are equivalent under multiplications by scalar matrices from Z(SL).

Picking out a set of classes of SL which are all equivalent to one class in PSL, we then select the characters of SL which are constant over this set of classes. These are the characters of PSL.

If one is working with a specific group character table then this procedure for finding the character table of a known subgroup works rather well. The restricted characters can be tested for reducibility and the use of the character orthogonality relations and Frobenius' reciprocity theorems would enable one to split the reducible characters into irreducible components.

It works well because the job of taking inner products (χ,χ) is not too arduous a numerical calculation. However, if one is working with a 'generalized' character table such as $GL(n,q)$ or $U(n,q^2)$ the entries are not numerical but are polynomials in q and variable roots of unity, so that the task of calculating inner products becomes extremely tedious. It would be almost hopeless if many of the restricted characters were reducible with many components. Theorems 2.11 and 2.12 are thus very important because they show that the splitting is of a very simple nature, and thereby make this process feasible. The procedure can be best understood by following through one of the table derivations in detail as done in one of the first sections of the paper.

IV. ENNOLA'S CONJECTURE

In [20] Steinberg noted that if the conjugacy classes of $GL(n,q)$ are partitioned into sets according to the type of Jordan canonical form to which they were similar, and the irreducible characters were also put into sets according to their degrees, then there was a 1-1 correspondence between the conjugacy class sets and the character sets. Also, the corresponding sets had the same order. Ennola noticed that if this same partitioning is done on $U(n,q^2)$ then the number of resulting sets is the same as for $GL(n,q)$.

In a general character table for $GL(n,q)$ and $U(n,q^2)$ we write down only one character for each such set of irreducible characters and this character has only a single entry for each set of conjugacy classes. Thus the character tables for $U(n,q^2)$ and $GL(n,q)$ contain the same number of 'generalized' entries. Ennola conjectured for all n, and proved for $n = 2,3$, that if q is everywhere replaced by $-q$ in the character table for $GL(n,q)$ and the resulting character changed in sign if the degree function is negative, then the character table $U(n,q^2)$ results. He used Brauer's characterization theorem [1] to prove it for these two cases but the complexities of $GL(n,q)$ prevent one from carrying forward his method. If this conjecture is true, then Green's method for constructing characters for $GL(n,q)$

would serve for $U(n,q^2)$ also. One would only have to use $-q$ in place of q in all the class functions constructed and change the sign if the degree becomes negative. Since $GL(n,\mathbb{C})$ is closely related to U_n (GL is a topological product of U and the space of positive definite Hermitian matrices) we might expect some connection between their character tables, but not such a simple one.

Ennola's conjecture appears so trivial that one would expect an equally simple proof. However, like many other theorems involving character relations, such a simple proof is not forthcoming. It is quite possible that the most direct proof would be through the use of Lie theory. One interesting observation which indicates some difficulty is that a permutation character of GL is <u>not</u> transformed into a permutation character of U. Thus some reducible characters split up differently than their images under the $q \to -q$ transformation.

In this paper we extend Ennola's conjecture to SL and PSL and show that it holds for all cases under consideration.

A result which is similar to the above conjecture, but which is proveable is the following:

<u>THEOREM</u> 4.1 $C_\lambda^{U_n}(q) = (-1)^n C_\lambda^{GL}(-q)$

where $C_\lambda^{U_n}(q)$ = the polynomial in q which expresses the order of the λ^{th} conjugacy class of $U(n,q^2)$

$C_\lambda^{GL}(q)$ = the polynomial in q which expresses the order of the λ^{th} conjugacy class of $GL(n,q)$

and the λ^{th} conjugacy class of GL has the same type canonical representative as the λ^{th} class of $U(n,q^2)$.

This says that $q \to -q$ will transform the polynomials expressing the conjugacy class orders of GL into those for $U(n,q^2)$. Comparing Lemma 2.4 in [14] and Definition 2 in [6] gives the above theorem. It appears that Ennola used this to obtain his tables for $U(2,q^2)$ and $U(3,q^2)$ but he never explicitly mentions it. We shall use this theorem in Chapter 15.

V. THE CHARACTER TABLE FOR $SL(2,q)$ $d = 1$

In this section we develop the character for $SL(2,q)$ $d = 1$, $q = 2^K$. This is the most trivial case because for $d = 1$ $GL(2,q) = SL(2,q) \times Z(GL)$ and every character of GL is irreducible upon restriction to SL. This one table serves for all the linear and unitary special and projective special groups since $SL(2,q) \cong PSL(2,q) \cong PSU(2,q^2) \cong SU(2,q^2)$ for $d = 1$.

5.1 Character Tables for $GL(2,q)$ and $U(2,q^2)$

All the character tables developed in the next three chapters are obtained from the table for $GL(2,q)$ which is found in Steinberg [19]. Although the table for $U(2,q)$, as found in [6], is not used, we furnish it for completeness. It is convenient to combine these two character tables as in Table 1 on the following page. This also demonstrates Ennola's conjecture regarding these two character tables.

Table 1. Character Tables for GL(2,q) and $U(2,q^2)$

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_1^{(k)}$	$\begin{pmatrix} \rho^k & \\ & \rho^k \end{pmatrix}$	$k=1,\cdots,(q-\delta)$	$q-\delta$	$q(q-\delta)^2(q+\delta)$
$C_2^{(k)}$	$\begin{pmatrix} \rho^k & \\ 1 & \rho^k \end{pmatrix}$	$k=1,\cdots,(q-\delta)$	$q-\delta$	$q(q-\delta)$
$C_3^{(k,l)}$	$\begin{pmatrix} \rho^k & \\ & \rho^l \end{pmatrix}$	$k,l=1,\cdots,(q-\delta)$ $k<l$	$\frac{1}{2}(q-\delta)(q-1-\delta)$	$(q-\delta)^2$
$C_4^{(k)}$	$\begin{pmatrix} \sigma^k & \\ & \sigma^{\delta qk} \end{pmatrix}$	$k=1,\cdots,(q^2-1)$ $k\not\equiv 0 \pmod{q+\delta}$ $C^{(k)}=C^{(q\delta k)}$	$\frac{1}{2}(q-1+\delta)(q-\delta)$	$(q+\delta)(q-\delta)$

parameters	$t=1,\cdots,(q-\delta)$		$t,u=1,\cdots,(q-\delta)$ $t<u$	$t=1,\cdots,q^2-1$ $(t)=(t\delta q)$ $t\not\equiv 0 \pmod{q+\delta}$
number of characters	$q-\delta$	$q-\delta$	$\frac{1}{2}(q-\delta)(q-1-\delta)$	$\frac{1}{2}(q-1+\delta)(q-\delta)$
class \ char.	$\chi_1^{(t)}$	$\chi_q^{(t)}$	$\chi_{q+\delta}^{(t,u)}$	$\chi_{(q-\delta)}^{(t)}$
$C_1^{(k)}$	ε^{2tk}	$q\varepsilon^{2tk}$	$(q+\delta)\varepsilon^{(t+u)k}$	$(q-\delta)\varepsilon^{tk}$
$C_2^{(k)}$	ε^{2tk}	0	$\delta\varepsilon^{(t+u)k}$	$-\delta\varepsilon^{tk}$
$C_3^{(k,l)}$	$\varepsilon^{t(k+l)}$	$\delta\varepsilon^{t(k+l)}$	$\delta(\varepsilon^{tk+ul}+\varepsilon^{uk+tl})$	0
$C_4^{(k)}$	$\varepsilon^{\delta tk}$	$-\delta\varepsilon^{\delta tk}$	0	$-\delta(\eta^{tk}+\eta^{\delta tkq})$

<u>for GL(2,q)</u>:

$\delta = +1$

ρ = primitive element of GF(q)

σ = primitive element of $GF(q^2)$

$\epsilon^{q-1} = 1,\ \eta^{q^2-1} = 1,\ \eta^{q+1} = \epsilon$

<u>for $U(2,q^2)$</u>:

$\delta = -1$

$\rho = \sigma^{q-1}$

σ = primitive element of $GF(q^2)$

$\epsilon^{q+1} = 1,\ \eta^{q^2-1} = 1,\ \eta^{q-1} = \epsilon$

5.2 Conjugacy Class Structure

For $d = 1$ $(q = 2^K)$, $SL(2,q) \cong PSL(2,q)$, so we know that no conjugacy classes of $GL(2,q)$ will split in $SL(2,q)$. The class structure is determined by observing the canonical representatives of GL in Table 1. and selecting only those whose determinant is 1. These classes are then counted. Since $SL \triangleleft GL$ every conjugacy class of SL is of the same order as it was in GL. If $|N_\lambda|$ = order of the centralizer for λ^{th} conjugacy class and $|C_\lambda|$ = order of λ^{th} conjugacy class, then $|N_\lambda^{SL}| = \frac{|SL|}{|C_\lambda|} = \frac{|GL|/(q-1)}{|C_\lambda|} = \frac{1}{q-1} \quad \frac{|GL|}{|C_\lambda|} = \frac{1}{q-1}|N_\lambda^{GL}|$. Thus we divide the order of all the centralizers in GL by $(q-1)$ to get the centralizer orders for SL.

Only the classes $C_3^{(k,\ell)}$ and $C_4^{(k)}$ require any calculation.

$$C_3^{(k,\ell)} \begin{pmatrix} \rho^k & 0 \\ 0 & \rho^\ell \end{pmatrix} \begin{cases} k,\ell = 1,\dots,q-1 & \text{is equivalent to} \\ k \neq \ell \\ (k+\ell) = (q-1) \\ C^{(k,\ell)} = C^{(\ell,k)} \end{cases}$$

$$C_3^{(k)} \begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix} \quad k = 1,\dots,\tfrac{1}{2}(q-2) .$$

Since $1 = \sigma^k \cdot \sigma^{kq} = \sigma^{k(q+1)} \Rightarrow k \equiv 0 (\text{mod } q-1)$ we get:

$$C_4^{(k)} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^{kq} \end{pmatrix} \begin{cases} k = (q-1),2(q-1),\dots,(q+1)(q-1) & \text{or} \\ k \not\equiv 0 (\text{mod } q+1) \\ C^{(k)} = C^{(kq)} \end{cases}$$

$$\begin{cases} k = (q-1),2(q-1),\dots,q(q-1) \\ C^{(k)} = C^{(kq)} \end{cases} .$$

Now let $k = k'(q-1)$ where $k' = 1,\ldots,q$

so $\sigma^k = \sigma^{k'(q-1)}$

also $\sigma^{k'(q-1)(q+1)} = 1$

$\therefore \sigma^{k'q(q-1)}\sigma^{k'(q-1)} = 1$

$\therefore \sigma^{k'q(q-1)} = \sigma^{-k'(q-1)}$

so $\quad \sigma^{kq} = \sigma^{k'q(q-1)} = \sigma^{-k'(q-1)}$

thus an equivalent form for $C_4^{(k)}$ is: $\begin{pmatrix} \sigma^{k(q-1)} & \\ & \sigma^{-k(q-1)} \end{pmatrix}$ $k = 1,\ldots,q$ $C^{(k)} = C^{(-k)}$

or $k = 1,\ldots,\frac{1}{2}q$. The resulting list of conjugacy classes appears in Table 2.

5.3 Calculation of Characters

Each of the characters of GL are now restricted down to SL.

(i) $\chi^{(t)}|_{SL} = \Psi_1 = (\varepsilon^{2(q-1)t},\varepsilon^{2(q-1)t},\varepsilon^{t(k-k)},\varepsilon^{k(q-1)t}) = (1,1,1,1)\ \forall\ t$

(ii) $\chi_q^{(t)}|_{SL} = \Psi_q = (q\varepsilon^{2(q-1)t},0,\varepsilon^{t(k-k)},-\varepsilon^{tk(q-1)}) = (q,0,1,-1)\ \forall\ t$.

There is no need to test for irreducibility since SL is a quotient group of GL for $d = 1$ and so by Frobenius' Theorem 3.1 we know that all the characters of GL restricted to SL are irreducible.

(iii) $\chi_{q+1}^{(t)}|_{SL} = \Psi_{q+1}^{(t)} = ((q+1)\varepsilon^{(t+u)(q-1)},\varepsilon^{(t+u)(q-1)},\varepsilon^{tk-uk} + \varepsilon^{-tk+uk},0)$

$= (q+1,1,\varepsilon^{k(t-u)} + \varepsilon^{-k(t-u)},0)$

let $t = (t-u)$ then $\Psi_{q+1}^{(t)} = (q+1,1,\varepsilon^{kt} + \varepsilon^{-kt},0)$

so $t = 1,\ldots,\frac{1}{2}(q-2)$

(iv) $\chi_{q-1}^{(t)}|_{SL} = \Psi_{q-1}^{(t)} = ((q-1)\eta^{t(q-1)(q+1)},-\eta^{t(q-1)(q+1)},0,-(\eta^{k(q-1)t}$

$+ \eta^{ktq(q-1)})) = ((q-1),-1,0,-(\theta^{kt} + \theta^{-kt}))$ where

$\theta^{q+1} = 1$. Thus t runs over the values $1,\ldots,\frac{1}{2}q$.

The resulting character table appears below.

Table 2. Character Table for $SL(2,q) \cong PSL(2,q) \cong PSU(2,q^2) \cong SU(2,q^2)$, $d = 1$, q even.

conjugacy class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		1	$q(q+1)(q-1)$
C_2	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$		1	q
$C_3^{(k)}$	$\begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$	$k=1,\cdots,\frac{1}{2}(q-2)$	$\frac{1}{2}(q-2)$	$q-1$
$C_4^{(k)}$	$\begin{pmatrix} \sigma^{k(q-1)} & \\ & \sigma^{-k(q-1)} \end{pmatrix}$	$k=1,\cdots,\frac{1}{2}q$	$\frac{1}{2}q$	$q+1$

parameters			$t=1,\cdots,\frac{1}{2}(q-2)$	$t=1,\cdots,\frac{1}{2}q$ $(t)=(-t)$
number of characters	1	1	$\frac{1}{2}(q-2)$	$\frac{1}{2}q$
class \ char	ψ_1	ψ_q	$\psi_{q+1}^{(t)}$	$\psi_{q-1}^{(t)}$
C_1	1	q	$q+1$	$q-1$
C_2	1	0	1	-1
$C_3^{(k)}$	1	1	$\varepsilon^{tk}+\varepsilon^{-tk}$	0
$C_4^{(k)}$	1	-1	0	$-\theta^{tk}-\theta^{-tk}$

ρ = primitive element of $GF(q)$ $\quad \varepsilon^{q-1} = 1$

σ = primitive element of $GF(q^2)$ $\quad \theta^{q+1} = 1$

VI. CHARACTER TABLE FOR $SL(2,q)$ $d = 2$

In this section the character table for $SL(2,q) \simeq SU(2,q^2)$ $d = 2$ is developed. In some respects this is a difficult table to handle since there is a sign fluctuation throughout the table which depends on whether $q = \pm 1 \pmod 4$.

6.1 Conjugacy Class Structure

(i) Consider classes of the form $\begin{pmatrix} \rho^k & \\ & \rho^k \end{pmatrix}$. Since $2 \mid (q-1)$ then $k = (q-1)$ and $k = \frac{1}{2}(q-1)$ will be the only forms with determinant 1. In SL the 1-1 correspondence between sets of classes of the same canonical form and sets of characters with the same degree does not exist and so we will separate $C_1^{(q-1)}$ and $C_1^{(\frac{1}{2}(q-1))}$ into classes labeled simply C_1', C_1''.

(ii) Consider the classes of the form $\begin{pmatrix} \rho^k & \\ 1 & \rho^k \end{pmatrix}$. Again the only ones in SL are $k = q-1$, $\frac{1}{2}(q-1)$. We know from Schur [18] that this type of class will split into $\begin{pmatrix} \rho^k & \\ 1 & \rho^k \end{pmatrix}$, $\begin{pmatrix} \rho^k & \\ \rho & \rho^k \end{pmatrix}$. We shall label the form $\begin{pmatrix} \rho^k & \\ 1 & \rho^k \end{pmatrix}$ $k = \frac{1}{2}(q-1), (q-1)$ as $C_2'^{(k)}$ and the form $\begin{pmatrix} \rho^k & \\ \rho & \rho^k \end{pmatrix}$ $k = \frac{1}{2}(q-1), (q-1)$ as $C_2''^{(k)}$.

(iii) In order for $\begin{pmatrix} \rho^k & \\ & \rho^\ell \end{pmatrix}$ to have determinant 1 the following conditions must hold on the indices:

$\ell,k = 1,\dots,q-1$

$k+\ell = q-1$

$k,\ell \neq q-1,\ \tfrac{1}{2}(q-1)$

$C^{(k)} = C^{(\ell)}$

which is equivalent to $C_3^{(k)}$ $\begin{pmatrix} \rho^k & \\ & \rho^{-k} \end{pmatrix}$ $k = 1,\dots,\tfrac{1}{2}(q-3)$.

(iv) The indices on $\begin{pmatrix} \sigma^k & \\ & \sigma^{kq} \end{pmatrix}$ are $k = (q-1), 2(q-1), \dots, q(q-1)$

$k \neq \tfrac{1}{2}(q+1)(q-1)$

$C^{(k)} = C^{(kq)}$.

This is equivalent to $\begin{pmatrix} \sigma^{k(q-1)} & \\ & \sigma^{-k(q-1)} \end{pmatrix}$ $k = 1,\dots,\tfrac{1}{2}(q-1)$.

The above calculations give us the classes as listed in Table 3.

Table 3. Conjugacy Class Structure for SL(2,q) d = 2

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_1'^{(k)}$	$\begin{pmatrix} \rho^k & \\ & \rho^k \end{pmatrix}$	$k=q-1$	1	$q(q^2-1)$
$C_1''^{(k)}$	$\begin{pmatrix} \rho^k & \\ & \rho^k \end{pmatrix}$	$k=\frac{1}{2}(q-1)$	1	$q(q^2-1)$
$C_2'^{(k)}$	$\begin{pmatrix} \rho^k & 0 \\ 1 & \rho^k \end{pmatrix}$	$k=\frac{1}{2}(q-1),(q-1)$	2	2q
$C_2''^{(k)}$	$\begin{pmatrix} \rho^k & 0 \\ \rho & \rho^k \end{pmatrix}$	$k=\frac{1}{2}(q-1),(q-1)$	2	2q
$C_3^{(k,\ell)}$	$\begin{pmatrix} \rho^k & \\ & \rho^\ell \end{pmatrix}$	$k,\ell=1,\cdots,q-1$ $k+\ell=q-1$ $k,\ell\neq q-1,\frac{1}{2}(q-1)$ $C^{(k)}=C^{(\ell)}$	$\frac{1}{2}(q-3)$	q-1
$C_4^{(k)}$	$\begin{pmatrix} \sigma^k & \\ & \sigma^{kq} \end{pmatrix}$	$k=(q-1),2(q-1),\cdots$ $\cdots,q(q-1)$ $k\neq \frac{1}{2}(q+1)(q-1)$ $C^{(k)}=C^{(qk)}$	$\frac{1}{2}(q-1)$	q+1

ρ,σ = primitive elements of $GF(q)$, $GF(q^2)$ respectively

6.2 Character Table

The characters of GL are again restricted down to SL as in Section 5.3, except that since some of the characters may now be reducible, the inner product (Ψ,Ψ) must be calculated on all the characters.

(i) $\quad \chi_1^{(t)}|_{SL} = (\varepsilon^{2t(q-1)}, \varepsilon^{2t(\frac{q-1}{2})}, \varepsilon^{2kt\ \substack{k=q-1 \\ k=\frac{1}{2}(q-1)}}, \varepsilon^{2kt\ \substack{k=q-1 \\ k=\frac{1}{2}(q-1)}},$

$\varepsilon^{t(k-k)}, \varepsilon^{tk(q-1)})$

$= (1,1,1,1,1,1,1,1) = \Psi_1$

(ii) $\quad \chi_q^{(t)} = (q,q,0,0,0,0,1,-1) \times \chi_1^{(t)}$ in GL.

Thus $\chi_q^{(t)}|_{SL} = (q,q,0,0,0,0,1,-1) = \Psi_q$

$$(\Psi_q,\Psi_q) = 2\,\frac{q^2}{q(q^2-1)} + \frac{q-3}{2}\left(\frac{1}{q-1}\right) + \frac{q-1}{2}\,\frac{(-1)^2}{q+1} = 1$$

so Ψ_q is irreducible.

(iii) $\quad \chi_{q+1}^{(t,u)}|_{SL} = \Psi_{q+1}^{(t,u)} = ((q+1)\varepsilon^{(t+u)(q-1)}, (q+1)\varepsilon^{(t+u)(\frac{q-1}{2})},$

$\varepsilon^{(t+u)k}\ k=\frac{1}{2}(q-1),(q-1); \varepsilon^{(t+u)k}\ k=\frac{1}{2}(q-1),\ (q-1);$

$\varepsilon^{tk-uk} + \varepsilon^{uk-tk}, 0)$

$= ((q+1),(q+1)(-1)^{t+u},(-1)^{t+u},1,(-1)^{t+u},1,$

$\varepsilon^{k(t-u)} + \varepsilon^{-k(t-u)}, 0)$

replace $(t-u)$ by t; since $t-u = t+u \pmod 2$ we can also replace $(t+u)$ by t and get:

$\Psi_{q+1}^{(t)} = (q+1,(q+1)(-1)^t,(-1)^t,1,(-1)^t,1, \varepsilon^{tk} + \varepsilon^{-tk}, 0)$

$t = 1,\ldots,q-2$

$t \neq \frac{1}{2}(q-1) \qquad t = 1,\ldots,\frac{q-3}{2}$

$\Psi^{(t)} = \Psi^{(-t)}$

Now we check for reducibility:

$$(\Psi_{q+1},\Psi_{q+1}) = \frac{2(q+1)^2}{q(q^2-1)} + \frac{4}{2q} + \sum_{k=1}^{(q-3)/2} (\epsilon^{2kt} + \epsilon^{-2kt} + 2)/(q-1)$$

$$= \frac{4}{q-1} + \frac{1}{q-1} \sum_{k=1}^{(q-3)/2} (\epsilon^{2tk} + \epsilon^{-2tk}) + \frac{2(\frac{q-3}{2})}{(q-1)}$$

$$= \frac{4}{q-1} + \frac{q-3}{q-1} + \frac{1}{q-1} \sum_{k=1}^{(q-1)} (\frac{2\epsilon^{2tk}}{2}) - \frac{2}{q-1}$$

$$= 1 + \frac{1}{(q-1)} \sum_{k=1}^{(q-1)} \epsilon^{2tk}$$

$$(\Psi_{q+1},\Psi_{q+1}) = 1 + 1 = 2 \quad \text{if} \quad t = c(\tfrac{q-1}{2})$$

$$= 1 + 0 = 1 \quad \text{for} \quad t \neq c(\tfrac{q-1}{2})$$

Thus we see that $\Psi_{q+1}^{(\frac{q-1}{2})}$ is reducible and splits into two conjugate, irreducible characters for SL. $\Psi_{q+1}^{(\frac{q-1}{2})} = \Psi_{\frac{q+1}{2}} + \Psi'_{\frac{q+1}{2}}$.

Since $(t+u)$ can equal $\frac{q-1}{2}$ for $\frac{q-1}{2}$ different values of t and u we see that χ_{q+1} has $\frac{q-1}{2}$ associates in GL relative to SL which we already know by Theorem 2.12.

This means that $\Psi_{q+1}^{(t)}$ is irreducible for the remaining $\frac{1}{2}(q-1)(q-3)$ values of $t \neq \frac{1}{2}(q-1)$.

(iv) In much the same manner we get:

$$\chi_{q-1}^{(t)}|_{SL} = \Psi_{q-1}^{(t)} = (q-1,(q-1)(-1)^t,-1,-(-1)^t,-1,-(-1)^t,0,-\theta^{kt}-\theta^{-kt})$$

$$\text{where} \quad t = 1,\dots,q^2-2 \qquad \theta^{q+1} = 1$$
$$t \neq \text{mult. } (q+1)$$
$$(t) = (tq) .$$

We check for reducibility:

$$(\Psi_{q-1},\Psi_{q-1}) = \frac{2(q-1)^2}{q(q^2-1)} + \frac{4(1)}{2q} + \sum_{k=1}^{(q-1)/2} (\theta^{2kt} + \theta^{-2kt} + 2)/(q+1)$$

$$= \frac{4}{q+1} + \frac{2(\frac{q-1}{2})}{(q+1)} + \frac{1}{q+1} \sum_{k=1}^{(q+1)/2} (\theta^{2kt} + \theta^{-2kt}) - \frac{2}{q+1}$$

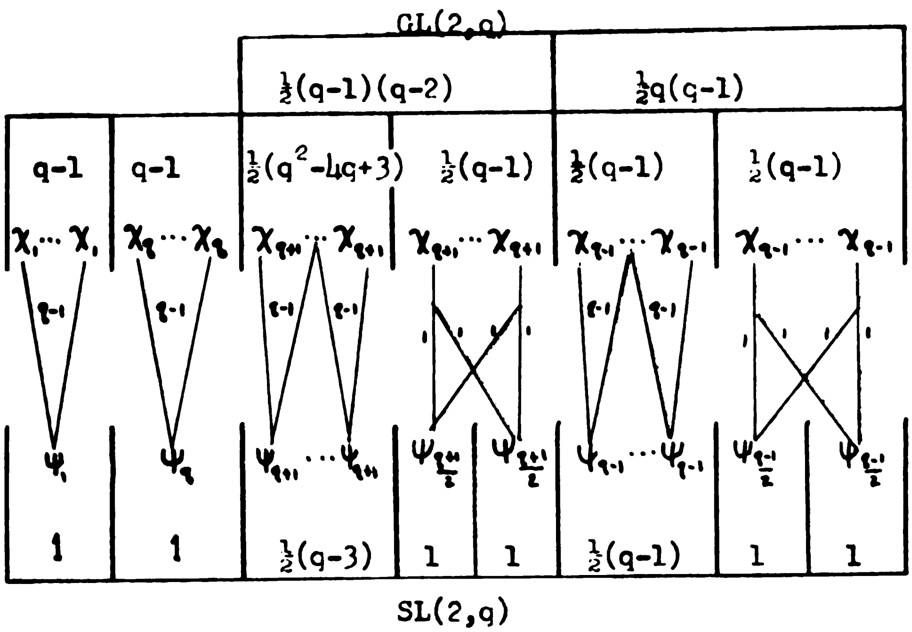
$$= 1 + \frac{1}{q+1} \sum_{1}^{q+1} \theta^{2kt}$$

$$(\Psi_{q-1},\Psi_{q-1}) = 1 + 1 = 2 \quad \text{if} \quad t = c(\tfrac{q+1}{2})$$

$$= 1 + 0 = 1 \quad \text{if} \quad t \neq (\tfrac{q+1}{2}) .$$

Thus $\Psi^{(t)}_{q-1} = \Psi_{\frac{q-1}{2}} + \Psi'_{\frac{q-1}{2}}$ for all t = multiple of $\frac{q+1}{2}$, which occurs $\frac{q-1}{2}$ times. The remaining $\frac{1}{2}(q-1)^2$ characters $\chi^{(t)}_{q-1}$ restrict down in sets of $q-1$ to the $\frac{1}{2}(q-1)$ irreducible characters, $\Psi^{(t)}_{q-1}$, of SL. The restrict table for GL to SL is given below.

Table 4. The Induce-restrict Table for GL(2,q) - SL(2,q) d = 2



At this point we know that $\chi_{q+1}^{(\frac{q-1}{2})}$ and $\chi_{q-1}^{(\frac{q+1}{2})}$ split into 2 characters of SL, $\Psi_{\frac{q+1}{2}}$, $\Psi'_{\frac{q+1}{2}}$ and $\Psi_{\frac{q-1}{2}}$, $\Psi'_{\frac{q-1}{2}}$ respectively. By Theorem 2.12 $\Psi_i(g) = \Psi'_i(g) = \frac{\chi(g)}{2}$ for $i = q+1$ and $q-1$ on all conjugacy classes of SL which are complete classes of GL. Thus $\chi_{q+1}^{(\frac{q-1}{2})}$ and $\chi_{q-1}^{(\frac{q+1}{2})}$ split in half on all classes of SL except $C_2^{\prime(k)}$, $C_2^{\prime\prime(k)}$. We now fill in the known portions of the character table and try to determine the missing values. The sign alternation on some of the characters depends on whether $\frac{q-1}{2}$ is odd or even. It is convenient to let $\delta = q(\bmod 4)$ and replace $(-1)^{\frac{q+1}{2}}$ by $\mp\delta$. Initially we will set up a separate character table for $\delta = \pm 1$ and later combine them. At this point the character table is as follows:

Table 5. Preliminary Character Table for SL(2,q) d = 2

$\delta = 1$

class \ char	Ψ_1	Ψ_q	$\Psi_{q+1}^{(t)}$	$\Psi_{q-1}^{(t)}$	$\Psi_{\frac{1}{2}(q+1)}$	$\Psi'_{\frac{1}{2}(q+1)}$	$\Psi_{\frac{1}{2}(q-1)}$	$\Psi'_{\frac{1}{2}(q-1)}$
C_1'	1	q	q+1	q-1	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
C_1''	1	q	$(q+1)(-1)^t$	$(q-1)(-1)^t$	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$	$-\frac{1}{2}(q-1)$	$-\frac{1}{2}(q-1)$
$C_2'^{(q-1)}$	1	0	1	-1	k_1+k_2	k_1-k_2	$\ell_1+\ell_2$	$\ell_1-\ell_2$
$C_2'^{(\frac{q-1}{2})}$	1	0	$(-1)^t$	$-(-1)^t$	k_3-k_4	k_3+k_4	$\ell_3-\ell_4$	$\ell_3+\ell_4$
$C_2''^{(q-1)}$	1	0	1	-1	k_1-k_2	k_1+k_2	$\ell_1-\ell_2$	$\ell_1+\ell_2$
$C_2''^{(\frac{q-1}{2})}$	1	0	$(-1)^t$	$-(-1)^t$	k_3+k_4	k_3-k_4	$\ell_3+\ell_4$	$\ell_3-\ell_4$
$C_3^{(k)}$	1	1	$\varepsilon^{tk}+\varepsilon^{-tk}$	0	$(-1)^k$	$(-1)^k$	0	0
$C_4^{(k)}$	1	-1	0	$-\theta^{kt}-\theta^{-kt}$	0	0	$-(-1)^k$	$-(-1)^k$

$\delta = -1$

class \ char	Ψ_1	Ψ_q	$\Psi_{q+1}^{(t)}$	$\Psi_{q-1}^{(t)}$	$\Psi_{\frac{1}{2}(q+1)}$	$\Psi_{\frac{1}{2}(q+1)}$	$\Psi_{\frac{1}{2}(q-1)}$	$\Psi_{\frac{1}{2}(q-1)}$
C_1'					$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
C_1''					$-\frac{1}{2}(q+1)$	$-\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
$C_2'^{(q-1)}$					k_1+k_2	k_1-k_2	$\ell_1+\ell_2$	$\ell_1-\ell_2$
$C_2'^{(\frac{q-1}{2})}$	(same as above)				k_3-k_4	k_3+k_4	$\ell_3-\ell_4$	$\ell_3+\ell_4$
$C_2''^{(q-1)}$					k_1-k_2	k_1+k_2	$\ell_1-\ell_2$	$\ell_1+\ell_2$
$C_2''^{(\frac{q-1}{2})}$					k_3+k_4	k_3-k_4	$\ell_3+\ell_4$	$\ell_3-\ell_4$
$C_3^{(k)}$					$(-1)^k$	$(-1)^k$	0	0
$C_4^{(k)}$					0	0	$-(-1)^k$	$-(-1)^k$

The missing entries in Table 5 can be filled in using the orthogonality properties of the character table; however, the details are rather laborious. A more elegant approach is the use of Gaussian sums.

The elements in classes $C_2'^{(k)}$, $C_2''^{(k)}$ are of order q. We know that the missing entries for $\Psi_{\frac{q+1}{2}}$, $\Psi'_{\frac{q+1}{2}}$ are all composed of sums of $\frac{1}{2}(q+1)$ q^{th} roots of unity and the entries for $\Psi_{\frac{q-1}{2}}$, $\Psi'_{\frac{q-1}{2}}$ are composed of sums of $\frac{1}{2}(q-1)$ q^{th} roots of unity.

We let ϵ be a q^{th} root of unity.

Also let $x = \sum_{i=k^2 (\mathrm{mod}\ q)} \epsilon^i$ (the exponents are quadratic residues of q)

and let $y = \sum_{i\neq k^2 (\mathrm{mod}\ q)} \epsilon^i$ (the exponents are non-residues of q).

Thus x and y are sums of $\frac{1}{2}(q-1)$ roots of unity. In particular they are Gaussian sums and the following theorem applies.

<u>THEOREM</u> 6.1 With x and y defined as above, then

$$x + y = -1$$

$$xy = \tfrac{1}{4}(1 - \delta q)$$

$$x = \tfrac{1}{2}(-1 + \sqrt{\delta q}) \quad y = \tfrac{1}{2}(-1 - \sqrt{\delta q}) \quad \text{where} \quad \delta = q\ (\mathrm{mod}\ 4) = \pm 1 .$$

For the entries of the characters $\Psi_{\frac{q-1}{2}}$, $\Psi'_{\frac{q-1}{2}}$ we use x,y as defined above if $\Psi_{\frac{q-1}{2}} + \Psi'_{\frac{q-1}{2}} = -1$ and $-x,-y$ if $\Psi_{\frac{q-1}{2}} + \Psi'_{\frac{q-1}{2}} = +1$.

If we let $x' = x+1$ and $y' = y+1$ then x',y' are sums of $\frac{1}{2}(q+1)$ roots of unity and we have the following corollary.

<u>Corollary 6.2</u> With x' and y' defined as above, then

$$x' + y' = 1$$

$$x'y' = \tfrac{1}{4}(1 - \delta q)$$

$$x' = \tfrac{1}{2}(1 + \sqrt{\delta q}) \qquad y' = \tfrac{1}{2}(1 - \sqrt{\delta q}) \ .$$

For the entries of the characters $\Psi_{\frac{q+1}{2}}, \Psi'_{\frac{q+1}{2}}$ we use x',y' if $\Psi_{\frac{q+1}{2}} + \Psi'_{\frac{q+1}{2}} = 1$ and $-x,-y$ if $\Psi_{\frac{q+1}{2}} + \Psi'_{\frac{q+1}{2}} = -1$.

Using this method the missing table entries can be easily filled in for the cases $\delta = 1$ and $\delta = -1$ and then these two cases can be merged to complete the block as it appears in Table 6.

Now that we have determined the table for SL, there is no need to use notation such as C_2', C_2'' etc. to link SL with GL. The resulting character table is more simply written as follows.

Table 6. The character Table for $SL(2,q) \cong SU(2,q^2)$ d = 2

conjugacy class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		1	$q(q^2-1)$
C_2	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$		1	$q(q^2-1)$
$C_3^{(k)}$	$\begin{pmatrix} (-1)^k & \\ 1 & (-1)^k \end{pmatrix}$	$k=1,2$	2	$2q$
$C_4^{(k)}$	$\begin{pmatrix} (-1)^k & \\ \rho & (-1)^k \end{pmatrix}$	$k=1,2$	2	$2q$
$C_5^{(k)}$	$\begin{pmatrix} \rho^k & \\ & \rho^{-k} \end{pmatrix}$	$k=1,\cdots,\frac{1}{2}(q-3)$	$\frac{1}{2}(q-3)$	$q-1$
$C_6^{(k)}$	$\begin{pmatrix} \sigma^{k(q-1)} & \\ & \sigma^{-k(q-1)} \end{pmatrix}$	$k=1,\cdots,\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$q+1$

parameters					$t=1,\cdots,\frac{1}{2}(q-3)$	$t=1,\cdots,\frac{1}{2}(q-1)$
number of characters	1	1	2	2	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-1)$
class \ char.	ψ_1	ψ_q	$\psi_{\frac{1}{2}(q+\delta)}$	$\psi_{\frac{1}{2}(q-\delta)}$	$\psi_{q+1}^{(t)}$	$\psi_{q-1}^{(t)}$
C_1	1	q	$\frac{1}{2}(q+\delta)$	$\frac{1}{2}(q-\delta)$	$q+1$	$q-1$
C_2	1	q	$\frac{1}{2}(q+\delta)$	$-\frac{1}{2}(q-\delta)$	$(-1)^t(q+1)$	$(-1)^t(q-1)$
$C_3^{(k)}$	1	0	$\frac{1}{2}(\delta\pm\sqrt{\delta q})$	$\frac{(-1)^k}{2}(-\delta\pm\sqrt{\delta q})$	$(-1)^{tk}$	$(-1)^{tk+1}$
$C_4^{(k)}$	1	0	$\frac{1}{2}(\delta\mp\sqrt{\delta q})$	$\frac{(-1)^k}{2}(-\delta\pm\sqrt{\delta q})$	$(-1)^{tk}$	$(-1)^{tk+1}$
$C_5^{(k)}$	1	1	$\frac{(-1)^k}{2}(1+\delta)$	$\frac{(-1)^k}{2}(1-\delta)$	$\omega_1^{tk}+\omega_1^{-tk}$	0
$C_6^{(k)}$	1	-1	$\frac{(-1)^{k+1}}{2}(1-\delta)$	$\frac{(-1)^{k+1}}{2}(1+\delta)$	0	$-\omega_2^{tk}-\omega_2^{-tk}$

$\delta = q \pmod 4$ $\omega_1^{q-1} = 1$ $\omega_2^{q+1} = 1$ ρ,σ = primitive elements of $GF(q)$, $GF(q^2)$ respectively

VII. CHARACTER TABLE FOR $PSL(2,q)$, $d = 2$

In this section the character table for $PSL(2,q) \cong PSU(2,q)$ $d = 2$ is constructed from the table of $SL(2,q)$ given in Table 6.

7.1 Conjugacy Class Structure

As in SL, the fact that $q = \pm 1 \pmod 4$ creates a problem not only in finding the characters themselves but also in determining the conjugacy classes. The scalar matrices of SL are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ so the classes of GL will combine in pairs to give the classes of PSL except for the self-equivalent classes (these are classes of SL which map into themselves under matrix multiplication by one of the above two scalar matrices). We note that the number of classes of type $C_5^{(k)}$ $(C_6^{(k)})$ in Table 6 is 'even' when $\delta = -1$ $(\delta = +1)$ and 'odd' when $\delta = +1$ $(\delta = -1)$, where $\delta = q \pmod 4$. Any time the number of classes of a certain type is 'odd' this means that one such class is self-equivalent. Thus there is a self-equivalent class of type $\begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$ when $\delta = +1$ and one of type $\begin{pmatrix} \sigma^{k(q-1)} & 0 \\ 0 & \sigma^{-k(q-1)} \end{pmatrix}$ when $\delta = -1$. We also note that the order of a self-equivalent class is a half of its order in SL since the class collapses. We now calculate the number of classes of type $C_4^{(k)}$:

$$\left.\begin{array}{ll} \text{if } \delta = -1 \text{ no. classes} = \tfrac{1}{2}(\frac{q-3}{2}) = \dfrac{(q-4)+1}{4} \\ \text{if } \delta = +1 \text{ no. classes} = \tfrac{1}{2}(\frac{q-3}{2} - 1) = \dfrac{(q-4)-1}{4} \end{array}\right\} = \frac{q-4-\delta}{4}$$

Similarly for $C_5^{(k)}$:

$$\left.\begin{array}{ll} \text{if } \delta = +1 \text{ no. classes} = \tfrac{1}{2}(\frac{q-1}{2}) = \dfrac{(q-2)+1}{4} \\ \text{if } \delta = -1 \text{ no. classes} = \tfrac{1}{2}(\frac{q-1}{2} - 1) = \dfrac{(q-2)-1}{4} \end{array}\right\} = \frac{q-2+\delta}{4}$$

The value of the index k for which the class $C_5^{(k)}$ or $C_6^{(k)}$ is self equivalent is:

$$\left.\begin{array}{ll} C_5^{(k)} \text{ (when } \delta = +1\text{)}: & k = \tfrac{1}{2}(\frac{q-3}{2} + 1) = \dfrac{q-1}{4} \\ C_6^{(k)} \text{ (when } \delta = -1\text{)}: & k = \tfrac{1}{2}(\frac{q-1}{2} + 1) = \dfrac{q+1}{4} \end{array}\right\} k = \frac{q-\delta}{4}$$

We can now determine the form of the conjugacy class of PSL which consists of the self-equivalent class of SL.

Let C_6' denote this class.

$$\text{if } \delta = +1 \text{ then } \begin{pmatrix} \rho^{\frac{q-1}{4}} & \\ & \rho^{-\frac{q-1}{4}} \end{pmatrix} \sim \begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix} \quad \phi \in GF(q) \quad \phi^4 = 1$$

is self-equivalent

$$\text{if } \delta = -1 \text{ then } \begin{pmatrix} \sigma^{\frac{(q-1)(q+1)}{4}} & \\ & \sigma^{-\frac{(q-1)(q+1)}{4}} \end{pmatrix} \sim \begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix}$$

$\phi \in GF(q^2) \quad \phi^4 = 1$.

Thus C_6' is of the form $\begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix}$ $\phi^4 = 1$ and $\phi \in \begin{cases} GF(q) & \text{if } \delta = 1 \\ GF(q^2) & \text{if } \delta = -1 \end{cases}$

The conjugacy class structure of PSL is given on the following page. The classes of SL which combine in PSL are indicated.

Table 7. Conjugacy Class Structure of PSL(2,q) d = 2

conjugacy class in SL	conjugacy class in PSL	number of classes	centralizer order
$C_1 \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $C_2 \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$C_1' \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	$\frac{1}{2}q(q^2-1)$
$C_3^{(k)} \quad \begin{pmatrix} (-1)^k & 0 \\ 1 & (-1)^k \end{pmatrix}$	$C_2' \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	1	q
$C_4^{(k)} \quad \begin{pmatrix} (-1)^k & 0 \\ \rho & (-1)^k \end{pmatrix}$	$C_3' \quad \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}$	1	q
$C_5^{(k)} \quad \begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$ $k=1,\cdots,\frac{1}{2}(q-3)$	$C_4'^{(k)} \quad \begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$ $k=1,\cdots,\frac{1}{4}(q-4-\delta)$	$\frac{1}{4}(q-4-\delta)$	$\frac{1}{2}(q-1)$
$C_6^{(k)} \quad \begin{pmatrix} \sigma^{k(q-1)} & 0 \\ 0 & \sigma^{-k(q-1)} \end{pmatrix}$ $k=1,\ \ ,\frac{1}{2}(q-1)$	$C_5'^{(k)} \quad \begin{pmatrix} \sigma^{k(q-1)} & 0 \\ 0 & \sigma^{-k(q-1)} \end{pmatrix}$ $k=1,\cdots,\frac{1}{4}(q-2+\delta)$	$\frac{1}{4}(q-2+\delta)$	$\frac{1}{2}(q+1)$
	$C_6' \quad \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}$ $\phi^4=1$ $\phi \in GF(q)$ if $\delta=+1$ $\phi \in GF(q^2)$ if $\delta=-1$	1	$(q-\delta)$

ρ = primitive element of GF(q)

σ = primitive element of $GF(q^2)$

7.2 Calculation of Characters

Every character of SL which is constant on the classes C_1 and C_2 is a character of PSL and even more, every character of PSL can be obtained in this way. Thus the characters χ_1, χ_q, $\chi_{\frac{q+\delta}{2}}$, $\chi_{q+1}^{(t)}$ for t even, $\chi_{q-1}^{(t)}$ for t even, are all characters of PSL.

The only difficulty is in finding the values for these characters on the class C_6 since the canonical form of C_6 and hence the character values, depend on δ.

We will concern ourselves with the character values $\chi_i(g)$ where $g \in C_6$ so $K = \frac{q-\delta}{4}$ as shown in 7.1.

(i) $$\left.\begin{aligned} \chi_q(g) &= +1 \quad (\text{if } \delta = +1) \\ & -1 \quad (\text{if } \delta = -1) \end{aligned}\right\} \chi_q(g) = \delta$$

(ii) $$\left.\begin{aligned} \chi_{q+1}^{(t)}(g) &= \omega_1^{kt} + \omega_1^{-kt} \quad (\delta = +1) \\ &= 0 \qquad\qquad\quad (\delta = -1) \end{aligned}\right\} \begin{aligned} \chi_{q+1}^{(t)}(g) &= \left(\frac{1+\delta}{2}\right)\left(\omega_1^{\left(\frac{q-1}{4}\right)t} + \omega_1^{-\left(\frac{q-1}{4}\right)t}\right) \\ &= (1+\delta)(-1)^t \end{aligned}$$

(iii) $$\left.\begin{aligned} \chi_{q-1}^{(t)}(g) &= 0 \quad (\delta = +1) \\ &= -(\omega_2^{kt} + \omega_2^{-kt}) \quad (\delta = -1) \end{aligned}\right\} \begin{aligned} \chi_{q-1}^{(t)}(g) &= \left(\frac{\delta-1}{2}\right)\left(\omega_2^{\left(\frac{q+1}{4}\right)t} + \omega_2^{-\left(\frac{q+1}{4}\right)t}\right) \\ &= (\delta-1)(-1)^t \end{aligned}$$

(iv) $$\left.\begin{aligned} \chi_{\frac{q+\delta}{2}}(g) &= (-1)^k\, \frac{1+\delta}{2} \quad (\delta = +1) = (-1)^{\frac{q-1}{4}}\left(\frac{1+1}{2}\right) = (-1)^{\frac{q-1}{4}} \\ &= (-1)^{k+1}\, \frac{1-\delta}{2} \quad (\delta = -1) = (-1)^{\frac{q+1}{4}-1}\left(\frac{1+1}{2}\right) = (-1)^{\frac{q-3}{4}} \end{aligned}\right\} = (-1)^{\left[\frac{q}{4}\right]}$$

where $\left[\frac{q}{4}\right]$ = greatest integer $\leq \frac{q}{4}$.

We can now write down the character table for PSL(2,q).

Table 8. Character Table for $PSL(2,q) \cong PSU(2,q^2)$, d = 2

conjugacy class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$		1	$\frac{1}{2}q(q^2-1)$
C_2	$\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$		1	q
C_3	$\begin{pmatrix} 1 & \\ \rho & 1 \end{pmatrix}$		1	q
$C_4^{(k)}$	$\begin{pmatrix} \rho^k & \\ & \rho^{-k} \end{pmatrix}$	$k=1,\cdots,\frac{1}{4}(q-4-\delta)$	$\frac{1}{4}(q-4-\delta)$	$\frac{1}{2}(q-1)$
$C_5^{(k)}$	$\begin{pmatrix} \sigma^{k(q-1)} & \\ & \sigma^{-k(q-1)} \end{pmatrix}$	$k=1,\cdots,\frac{1}{4}(q-2+\delta)$	$\frac{1}{4}(q-2+\delta)$	$\frac{1}{2}(q+1)$
C_6	$\begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix}$		1	$(q-\delta)$

parameters					$t=1,\cdots$ $\cdots,\frac{1}{4}(q-4-\delta)$	$t=1,\cdots$ $\cdots,\frac{1}{4}(q-2+\delta)$
number of characters	1	1	1	1	$\frac{1}{4}(q-4-\delta)$	$\frac{1}{4}(q-2+\delta)$
class \ char.	ψ_1	ψ_q	$\psi_{\frac{1}{2}(q+\delta)}$	$\psi'_{\frac{1}{2}(q+\delta)}$	$\psi_{q+1}^{(t)}$	$\psi_{q-1}^{(t)}$
C_1	1	q	$\frac{1}{2}(q+\delta)$	$\frac{1}{2}(q+\delta)$	q+1	q-1
C_2	1	0	$\frac{1}{2}(\delta+\sqrt{\delta q})$	$\frac{1}{2}(\delta-\sqrt{\delta q})$	1	-1
C_3	1	0	$\frac{1}{2}(\delta-\sqrt{\delta q})$	$\frac{1}{2}(\delta+\sqrt{\delta q})$	1	-1
$C_4^{(k)}$	1	1	$\frac{1}{2}(-1)^k(1+\delta)$	$\frac{1}{2}(-1)^k(1+\delta)$	$\omega_1^{kt}+\omega_1^{-kt}$	0
$C_5^{(k)}$	1	1	$\frac{1}{2}(-1)^{k+1}(1-\delta)$	$\frac{1}{2}(-1)^{k+1}(1-\delta)$	0	$-\omega_2^{kt}-\omega_2^{-kt}$
C_6	1	δ	$(-1)^{[q/4]}$	$(-1)^{[q/4]}$	$(-1)^t(1+\delta)$	$(-1)^t(\delta-1)$

ρ,σ = primitive elements of $GF(q)$, $GF(q^2)$ respectively

$\omega_1^{\frac{1}{2}(q-1)} = 1$ $\omega_2^{\frac{1}{2}(q+1)} = 1$ $\phi^4 = 1$ $\phi \in GF(q)$ if $\delta = +1$ $\delta = q \pmod 4$

$\phi \in GF(q^2)$ if $\delta = -1$

VIII. CHARACTER TABLE FOR $SU(3,q^2)$ $d = 1$

8.1 Character Table for $GL(3,q)$, $U(3,q^2)$

In the next six sections all the character tables will be derived from $GL(3,q)$ and $U(3,q^2)$ as given in Steinberg [19] and Ennola [6]. For convenience these are listed on the following pages.

8.2 Conjugacy Class Structure

The determination of the range of the class parameters becomes more complicated when we move up to 3×3 matrices so it is best to indicate how these calculations are made.

The fact that $d = 1$ implies $3 \nmid (q+1)$ and the condition that the determinant must be unity for all matrices in SU impose strict limitations on the values which the class parameters can take on.

The only classes of type $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$ in SU are those for $k = q + 1$.

The classes $C_4^{(k,\ell)}$, $C_5^{(k,\ell)}$ have determinant 1 only if $2k + \ell \equiv 0 (\mathrm{mod}\ q+1)$ i.e. $\ell = -2k (\mathrm{mod}\ q+1)$. Once k is selected, ℓ is uniquely determined. We can discard the ℓ parameter and the value $k = q+1$ (since $k = -2k = \ell$ for this value). Thus for $C_4^{(k)}$, $C_5^{(k)}$ in SU the valid range is $k = 1,\ldots,q$.

Table 9. Character Table for $GL(3,q)$

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_1^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^k \end{pmatrix}$	$k=1,\cdots,q-1$	$q-1$	$q^3(q^3-1)(q^2-1)(q-1)$
$C_2^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^k \end{pmatrix}$	$k=1,\cdots,q-1$	$q-1$	$q^3(q-1)^2$
$C_3^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & 1 & \rho^k \end{pmatrix}$	$k=1,\cdots,q-1$	$q-1$	$q^2(q-1)$
$C_4^{(k,\ell)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^\ell \end{pmatrix}$	$k,\ell=1,\cdots,q-1$ $k\neq\ell$	$(q-1)(q-2)$	$q(q-1)^3(q+1)$
$C_5^{(k,\ell)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^\ell \end{pmatrix}$	$k,\ell=1,\cdots,q-1$ $k\neq\ell \pmod{q-1}$	$(q-1)(q-2)$	$q(q-1)^2$
$C_6^{(k,\ell,m)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^\ell & \\ & & \rho^m \end{pmatrix}$	$k,\ell,m=1,\cdots,q-1$ $k\neq\ell\neq m \pmod{q-1}$ $k<\ell<m$	$\frac{1}{6}(q-1)\cdot(q-2)\cdot(q-3)$	$(q-1)^3$
$C_7^{(k,\ell)}$	$\begin{pmatrix} \rho^k & & \\ & \sigma^\ell & \\ & & \sigma^{q\ell} \end{pmatrix}$	$k=1,\cdots,q-1$ $\ell=1,\cdots,q^2-1$ $\ell\neq 0 \mod q+1$ $C^{(\ell)}=C^{(q\ell)} \pmod{q^2-1}$	$\frac{1}{2}q(q-1)^2$	$(q-1)^2(q+1)$
$C_8^{(k)}$	$\begin{pmatrix} \tau^k & & \\ & \tau^{kq} & \\ & & \tau^{kq^2} \end{pmatrix}$	$k=1,\cdots,q^3-1$ $k\neq 0 \mod q^2+q+1$ $C^{(k)}=C^{(kq)}=C^{(kq^2)}$ $\pmod{q^3-1}$	$\frac{1}{3}q(q-1)(q+1)$	(q^3-1)

ρ,σ,τ = primitive elements of $GF(q)$, $GF(q^2)$, $GF(q^3)$ respectively

Table 9 (cont'd.)

parameters	$t=1,\cdots,(q-1)$			$t,u=1,\cdots,(q-1)$ $t\neq u \pmod{q-1}$		$t,u,v=1,\cdots,q-1$ $t\neq u\neq v \pmod{q-1}$ $t<u<v$	$u=1,\cdots,q^2-1$ $t=1,\cdots,q-1$ $u\neq 0 \pmod{q+1}$ $(u)=(qu) \pmod{q^2-1}$	$t=1,\cdots,q^3-1$ $t\neq 0 \bmod p$ $(t)=(tq)=(tq^2) \pmod{q^3-1}$
number of characters	$q-1$	$q-1$	$q-1$	$(q-1)(q-2)$	$(q-1)(q-2)$	$\frac{1}{6}(q-1)(q-2)\cdot(q-3)$	$\frac{1}{2}q(q-1)^2$	$\frac{1}{3}q(q-1)(q+1)$
class \ char	$\chi_1^{(t)}$	$\chi_{qs}^{(t)}$	$\chi_{q^3}^{(t)}$	$\chi_p^{(t,u)}$	$\chi_{qp}^{(t,u)}$	$\chi_{sp}^{(t,u,v)}$	$\chi_{rp}^{(t,u)}$	$\chi_{r^2s}^{(t)}$
$C_1^{(k)}$	ε^{3tk}	$qs\varepsilon^{3tk}$	$q^3\varepsilon^{3tk}$	$p\varepsilon^{(2t+u)k}$	$qp\varepsilon^{(2t+u)k}$	$sp\varepsilon^{(t+u+v)k}$	$rp\varepsilon^{(t+u)k}$	$r^2s\varepsilon^{tk}$
$C_2^{(k)}$	ε^{3tk}	$q\varepsilon^{3tk}$	0	$s\varepsilon^{(2t+u)k}$	$q\varepsilon^{(2t+u)k}$	$(2q+1)\varepsilon^{(t+u+v)k}$	$-\varepsilon^{(t+u)k}$	$-r\varepsilon^{tk}$
$C_3^{(k)}$	ε^{3tk}	0	0	$\varepsilon^{(2t+u)k}$	0	$\varepsilon^{(t+u+v)k}$	$-\varepsilon^{(t+u)k}$	ε^{tk}
$C_4^{(k,\ell)}$	$\varepsilon^{t(2k+\ell)}$	$s\varepsilon^{t(2k+\ell)}$	$q\varepsilon^{t(2k+\ell)}$	$s\varepsilon^{(t+u)k+t\ell}+\varepsilon^{2kt+u\ell}$	$s\varepsilon^{(t+u)k+t\ell}+q\varepsilon^{2kt+u\ell}$	$s\sum_{(t,u,v)}\varepsilon^{(t+u)k+v\ell}$	$r\varepsilon^{(t\ell+uk)}$	0
$C_5^{(k,\ell)}$	$\varepsilon^{t(2k+\ell)}$	$\varepsilon^{t(2k+\ell)}$	0	$\varepsilon^{(t+u)k+t\ell}+\varepsilon^{2kt+u\ell}$	$\varepsilon^{(t+u)k+t\ell}$	$\sum_{(t,u,v)}\varepsilon^{(t+u)k+v\ell}$	$-\varepsilon^{(t\ell+uk)}$	0
$C_6^{(k,\ell,m)}$	$\varepsilon^{t(k+\ell+m)}$	$2\varepsilon^{t(k+\ell+m)}$	$\varepsilon^{t(k+\ell+m)}$	$\sum_{(k,\ell,m)}\varepsilon^{uk+t(\ell+m)}$	$\sum_{(k,\ell,m)}\varepsilon^{uk+t(\ell+m)}$	$\sum_{[t,u,v]}\varepsilon^{tk+u\ell+vm}$	0	0
$C_7^{(k,\ell)}$	$\varepsilon^{t(k+\ell)}$	0	$-\varepsilon^{t(k+\ell)}$	$\varepsilon^{uk+t\ell}$	$-\varepsilon^{uk+t\ell}$	0	$-\varepsilon^{tk}(\eta^{u\ell}+\eta^{u\ell q})$	0
$C_8^{(k)}$	ε^{tk}	$-\varepsilon^{tk}$	ε^{tk}	0	0	0	0	$\gamma^{tk}+\gamma^{tkq}+\gamma^{tkq^2}$

$\varepsilon^{q-1}=1 \quad \eta^{q^2-1}=1 \quad \gamma^{q^3-1}=1 \quad r=(q-1) \quad s=(q+1) \quad p=(q^2+q+1) \quad \eta^{(q+1)}=\varepsilon=\gamma^{q^2+q+1}$

$\sum_{(x,y,z)}$ means a sum over the cyclic permutations of x,y,z, $\sum_{[x,y,z]}$ means a sum over all permutations of x,y,z

Table 10. Character Table for $U(3,q^2)$

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_1^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^k \end{pmatrix}$	$k=1,\cdots,q+1$	$q+1$	$q^3(q^3+1)(q^2-1)(q+1)$
$C_2^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^k \end{pmatrix}$	$k=1,\cdots,q+1$	$q+1$	$q^3(q+1)^2$
$C_3^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & 1 & \rho^k \end{pmatrix}$	$k=1,\cdots,q+1$	$q+1$	$q^2(q+1)$
$C_4^{(k,\ell)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^\ell \end{pmatrix}$	$k,\ell=1,\cdots,q+1$ $k\neq\ell$	$q(q+1)$	$q(q^2-1)(q+1)^2$
$C_5^{(k,\ell)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^\ell \end{pmatrix}$	$k,\ell=1,\cdots,q+1$ $k\neq\ell$	$q(q+1)$	$q(q+1)^2$
$C_6^{(k,\ell,m)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^\ell & \\ & & \rho^m \end{pmatrix}$	$k,\ell,m=1\cdots,q+1$ $k\neq\ell\neq m \pmod{q+1}$ $k<\ell<m$	$\frac{1}{6}q(q^2-1)$	$(q+1)^3$
$C_7^{(k,\ell)}$	$\begin{pmatrix} \rho^k & & \\ & \sigma^\ell & \\ & & \sigma^{-q\ell} \end{pmatrix}$	$k=1,\cdots,q+1$ $\ell=1,\cdots,(q^2-1)$ $\ell\neq 0 \bmod q-1$ $C^{(\ell)}=C^{(-q\ell)} \pmod{q^2-1}$	$\frac{1}{2}(q+1)^2(q-2)$	$(q^2-1)(q+1)$
$C_8^{(k)}$	$\begin{pmatrix} \tau^k & & \\ & \tau^{kq^2} & \\ & & \tau^{kq^4} \end{pmatrix}$	$k=1,\cdots,q^3+1$ $k\neq 0 \bmod q^2-q+1$ $C^{(k)}=C^{(kq^2)}=C^{(kq^4)}$ mod q^3+1	$\frac{1}{3}q(q^2-1)$	(q^3+1)

σ,τ_1 = primitive element of $GF(q^2)$, $GF(q^6)$ respectively, $\rho=\sigma^{(q-1)}$, $\tau=\tau_1^{(q^3-1)}$

Table 10 (cont'd.)

parameters	$t=1,\cdots,(q+1)$			$t,u=1,\cdots,(q+1)$ $t\neq u \pmod{q+1}$		$t,u,v=1,\cdots,q+1$ $t\neq u\neq v \pmod{q+1}$ $t<u<v$	$u=1,\cdots,q^2-1$ $t=1,\cdots,q+1$ $u\neq 0 \bmod q-1$ $(u)=(-qu) \pmod{q^2-1}$	$t=1,\cdots,q^3+1$ $t\neq 0 \bmod p$ $(t)=(tq^2)=(tq^4) \pmod{q^3+1}$
number of characters	$q+1$	$q+1$	$q+1$	$q(q+1)$	$q(q+1)$	$\frac{1}{6}q(q^2-1)$	$\frac{1}{2}(q+1)^2(q-2)$	$\frac{1}{3}q(q^2-1)$
char / class	$\chi_1^{(t)}$	$\chi_{qr}^{(t)}$	$\chi_{q^3}^{(t)}$	$\chi_P^{(t,u)}$	$\chi_{qP}^{(t,u)}$	$\chi_{rP}^{(t,u,v)}$	$\chi_{sP}^{(t,u)}$	$\chi_{s^2r}^{(t)}$
$C_1^{(K)}$	ε^{3tK}	$qr\varepsilon^{3tK}$	$q^3\varepsilon^{3tK}$	$P\varepsilon^{(t+2u)K}$	$qP\varepsilon^{(t+2u)K}$	$rP\varepsilon^{(t+u+v)K}$	$sP\varepsilon^{(t+u)K}$	$s^2r\,\varepsilon^{tK}$
$C_2^{(K)}$	ε^{3tK}	$-q\varepsilon^{3tK}$	0	$-r\varepsilon^{(t+2u)K}$	$q\varepsilon^{(t+2u)K}$	$(2q-1)\varepsilon^{(t+u+v)K}$	$\varepsilon^{(t+u)K}$	$-s\,\varepsilon^{tK}$
$C_3^{(K)}$	ε^{3tK}	0	0	$\varepsilon^{(t+2u)K}$	0	$-\varepsilon^{(t+u+v)K}$	$\varepsilon^{(t+u)K}$	$-\varepsilon^{tK}$
$C_4^{(K,\ell)}$	$\varepsilon^{t(2K+\ell)}$	$-r\varepsilon^{t(2K+\ell)}$	$q\varepsilon^{t(2K+\ell)}$	$-r\varepsilon^{(t+u)K+u\ell}+\varepsilon^{2uK+t\ell}$	$r\varepsilon^{(t+u)K+u\ell}+q\varepsilon^{2uK+t\ell}$	$r\sum_{(t,u,v)}\varepsilon^{(t+u)K+v\ell}$	$s\,\varepsilon^{uK+t\ell}$	0
$C_5^{(K,\ell)}$	$\varepsilon^{t(2K+\ell)}$	$\varepsilon^{t(2K+\ell)}$	0	$\varepsilon^{(t+u)K+u\ell}+\varepsilon^{2uK+t\ell}$	$-\varepsilon^{(t+u)K+u\ell}$	$-\sum_{(t,u,v)}\varepsilon^{(t+u)K+v\ell}$	$\varepsilon^{uK+t\ell}$	0
$C_6^{(K,\ell,m)}$	$\varepsilon^{t(K+\ell+m)}$	$2\varepsilon^{t(K+\ell+m)}$	$-\varepsilon^{t(K+\ell+m)}$	$\sum_{(K,\ell,m)}\varepsilon^{tK+u(\ell+m)}$	$-\sum_{(K,\ell,m)}\varepsilon^{tK+u(\ell+m)}$	$-\sum_{[t,u,v]}\varepsilon^{tK+u\ell+vm}$	0	0
$C_7^{(K,\ell)}$	$\varepsilon^{t(K-\ell)}$	0	$\varepsilon^{t(K-\ell)}$	$\varepsilon^{tK-u\ell}$	$\varepsilon^{tK-u\ell}$	0	$\varepsilon^{tK}(\eta^{u\ell}+\eta^{-qu\ell})$	0
$C_8^{(K)}$	ε^{tK}	$-\varepsilon^{tK}$	$-\varepsilon^{tK}$	0	0	0	0	$-\gamma^{tK}-\gamma^{tKq^2}-\gamma^{tKq^4}$

$\epsilon^{q+1}=1 \quad \eta^{q^2-1}=1 \quad \gamma^{q^3+1}=1 \quad r=(q-1) \quad s=(q+1) \quad p=(q^2-q+1) \quad \eta^{q-1}=\epsilon=\gamma^{q^2-q+1}$

$\sum_{(x,y,z)}$ means a sum over the cyclic permutations of x,y,z, $\sum_{[x,y,z]}$ means a sum over all permutations of x,y,z

For classes $C_6^{(k,\ell,m)}$ there are $\binom{q+1}{2}$ ways to select k and ℓ. With k,ℓ selected there is a unique value of m such that $k + \ell + m \equiv 0 \pmod{q+1}$. However there are q choices of k and ℓ which will force m to equal k or ℓ so we must discard these values. Thus the number of classes equals

$$\frac{\binom{q+1}{2} - q}{3} = \frac{1}{6}\, q(q-1).$$

In order for classes of type $C_7^{(k,\ell)}$ to have determinant unity, $\rho^k \sigma^{-\ell(q-1)} = \rho^k \rho^{-\ell} = 1$ which implies $\ell = k \pmod{q+1}$. Thus ℓ is determined once k is chosen. Now $\ell \not\equiv 0 \pmod{q-1}$ or else $\sigma^\ell = \sigma^{t(q-1)} = \rho^t$ and we are back into the classes of a type already considered.

Thus $C_7^{(k)}$ in SU is indexed by
$$k = 1,\ldots,q^2-1$$
$$k \not\equiv 0 \pmod{q-1}$$
$$C^{(k)} = C^{(-qk)} .$$

The number of such classes is $\frac{1}{2}[(q^2-1) - (q+1)] = \frac{1}{2}(q-2)(q+1)$.

In order for classes of type $C_8^{(k)}$ to have determinant 1 we have the following:

$$\tau^{k(q^3-1)(q^4+q^2+1)} = 1 = \tau^{q^6-1}$$

$$\tau_1^{k(q^2+q+1)(q^2-q+1)} = \tau_1^{q^3+1} \quad \text{which implies} \quad k \equiv 0 \pmod{q+1}.$$

Thus
$$\begin{cases} k = (q+1), 2(q+1), \ldots, (q^3+1) \\ k \not\equiv 0 \pmod{q^2-q+1} \\ C^{(k)} = C^{(kq^2)} = C^{(kq^4)} \end{cases}$$
and so the number of classes equals $\frac{1}{3}[(q^2-q+1) - 1] = \frac{1}{3}\, q(q-1)$.

The complete class structure has thus been determined and appears in Table 11.

8.3 Calculation of Characters

The characters of $U(3,q^2)$ are now restricted down to $SU(3,q^2)$ by letting $k = (q+1)$ on classes $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$; $\ell = -2k$ for $C_4^{(k,\ell)}$, $C_5^{(k,\ell)}$; and $\ell = k$ for $C_7^{(k,\ell)}$. We keep in mind that $\epsilon^{q+1} = 1$. In several cases we can eliminate one of the character variables t,u,v by a suitable substitution. In $\chi_p^{(t,u)}|_{SU}$ and $\chi_{qp}^{(t,u)}|_{SU}$ we replace $(t-u)$ by t' and in $\chi_{sp}^{(t,u)}|_{SU}$ we substitute $t' - (q-1)t$ for u and obtain a simplification of the characters.

The calculation of $\chi_{sp}^{(t,u)}|_{SU}$ is done below as an example.

$$\chi_{sp}^{(t,u)}|_{SU} = (sp\ \epsilon^{(t+u)(q+1)}, \epsilon^{(t+u)(q+1)}, \epsilon^{(t+u)(q+1)}, s\epsilon^{uk-2kt},$$

$$\epsilon^{uk-2tk}, 0, \epsilon^{tk}(\eta^{uk} + \eta^{-quk}), 0)$$

$$= (sp, 1, 1, s\epsilon^{(u-2t)k}, \epsilon^{(u-2t)k}, 0, \eta^{k[(q-1)t+u]} + \eta^{k[(q-1)t-qu]}, 0)$$

(replace u by $t' - (q-1)t$)

$$= (sp, 1, 1, s\epsilon^{t'k}, \epsilon^{t'k}, 0, \eta^{kt'} + \eta^{-kqt'}, 0) = \Psi_{sp}^{(t')} .$$

None of the characters can be reducible so inner products are not calculated. The character table for $SU(3,q^2)$ $d = 1$ appears in Table 11.

Table 11. Character Table for $SU(3,q^2) \cong PSU(3,q^2)$, d = 1

conjugacy class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$		1	q^3rs^2p
C_2	$\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix}$		1	q^3s
C_3	$\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}$		1	q^2
$C_4^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,q$	q	qrs^2
$C_5^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,q$	q	$q\ s$
$C_6^{(k,\ell,m)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^\ell & \\ & & \rho^m \end{pmatrix}$	$k,\ell,m=1,\cdots,(q+1)$ $k<\ell<m$ $k+\ell+m=0 \pmod{q+1}$	$\frac{1}{6}q(q-1)$	s^2
$C_7^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \sigma^k & \\ & & \sigma^{-kq} \end{pmatrix}$	$k=1,\cdots,(q^2-1)$ $k \not\equiv 0 \bmod q+1$ $C^{(k)}=C^{(-qk)} \pmod{q^2-1}$	$\frac{1}{2}(q-2)(q+1)$	rs
$C_8^{(k)}$	$\begin{pmatrix} \tau^k & & \\ & \tau^{kq^2} & \\ & & \tau^{kq^4} \end{pmatrix}$	$k=(q+1),2(q+1),\cdots$ $\cdots,(q^3+1)$ $k \ne$ mult. q^2-q+1 $C^{(k)}=C^{(kq^2)}=C^{(kq^4)}$	$\frac{1}{3}q(q-1)$	p

σ,τ_1 = primitive elements of $GF(q^2)$, $GF(q^6)$ respectively

$\rho = \sigma^{(q-1)}$, $\tau = \tau_1^{(q^3-1)}$

Table 11 (cont'd.)

parameters				$t=1,\cdots,q$		$t,u,v=1,\cdots,q+1$ $t<u<v$ $t+u+v=0 \pmod{q+1}$	$t=1,\cdots,q^2-1$ $t\neq 0 \pmod{q+1}$ $(t)=(tq)$	$t=(q+1),2(q+1),\cdots,(q^2+q)(q+1)$ $(t)=(tq^2)=(tq^4) \pmod{q^3-1}$
number of characters	1	1	1	q	q	$\frac{1}{6}q(q-1)$	$\frac{1}{2}(q-2)(q+1)$	$\frac{1}{3}q(q-1)$
char / class	ψ_1	ψ_{qr}	ψ_{q^3}	$\psi_p^{(t)}$	$\psi_{qp}^{(t)}$	$\psi_{rp}^{(t,u,v)}$	$\psi_{sp}^{(t)}$	$\psi_{s^2r}^{(t)}$
C_1	1	qr	q^3	p	qp	rp	sp	s^2r
C_2	1	$-q$	0	$-r$	q	$2q-1$	1	$-s$
C_3	1	0	0	1	0	-1	1	-1
$C_4^{(k)}$	1	$-r$	q	$-r\varepsilon^{kt}+\varepsilon^{-2kt}$	$r\varepsilon^{kt}+q\varepsilon^{-2kt}$	$r\sum_{(t,u,v)}\varepsilon^{k(t+u-2v)}$	$s\varepsilon^{kt}$	0
$C_5^{(k)}$	1	1	0	$\varepsilon^{kt}+\varepsilon^{-2kt}$	$-\varepsilon^{kt}$	$-\sum_{(t,u,v)}\varepsilon^{k(t+u-2v)}$	ε^{kt}	0
$C_6^{(k,l,m)}$	1	2	-1	$(\varepsilon^{tk}+\varepsilon^{lt}+\varepsilon^{mt})$	$-(\varepsilon^{tk}+\varepsilon^{lt}+\varepsilon^{mt})$	$-\sum_{[t,u,v]}\varepsilon^{kt+lu+mv}$	0	0
$C_7^{(k)}$	1	0	1	ε^{kt}	ε^{tk}	0	$\eta^{+kt}+\eta^{-kqt}$	0
$C_8^{(k)}$	1	-1	-1	0	0	0	0	$-\gamma^{kt}-\gamma^{ktq^2}-\gamma^{ktq^4}$

$r = q-1$, $s = q+1$, $p = q^2 - q+1$, $\epsilon^{q+1} = 1$, $\eta^{q^2-1} = 1$, $\gamma^{q^3+1} = 1$, $\rho = \sigma^{q-1}$, $\tau = \tau_1^{q^3-1}$

$\sum_{(x,y,z)}$ means a sum over the cyclic permutations of x,y,z, $\sum_{[x,y,z]}$ means a sum over all permutations of x,y,z

IX. CHARACTER TABLE FOR $SU(3,q^2)$, $d = 3$

9.1 Conjugacy Class Structure

(i) For classes $C_1^{(k)},\ldots,C_3^{(k)}$ the only values of k for which the determinant is unity are $\frac{1}{3}(q+1)$, $\frac{2}{3}(q+1)$, and $(q+1)$. Dickson [5] shows that classes of type $\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & 1 & \rho^k \end{pmatrix}$ in $GL(3,q)$ will split into three conjugacy classes in $SL(3,q)$. Since the classes of $U(3,q^2)$ are indexed by certain classes of $GL(3,q^2)$ this means that the corresponding classes, $C_3^{(k)}$, of $U(3,q^2)$ will split in $SU(3,q^2)$ into three classes with $1,\beta,\beta^2$ as the off diagonal entries, where β is any non-cube root of unity in $G(q^2)$.

(ii) For the classes $C_4^{(k,\ell)},C_5^{(k,\ell)}$ we require that $2k + \ell \equiv 0$ (mod $q+1$), that is, $\ell = -2k(\text{mod } q+1)$. Also $k \neq$ multiple of $\frac{1}{3}(q+1)$, since this value would result in a class of type C_1,C_2 or C_3.

(iii) For the classes $C_6^{(k,\ell,m)}$ we select k,ℓ in $\binom{q+1}{2}$ ways and m is uniquely determined since $k + \ell + m \equiv 0(\text{mod } q+1)$. However, there are $(q+1) - 3$ values of k and ℓ for which the determined m value is the same as k or ℓ. We must discard these values. We thus get the class count:

$$\frac{1}{3}\left[\binom{q+1}{2} - (q-2)\right] = \frac{1}{6}(q+1)(q-2) + 1 .$$

(iv) Since $(k-\ell) \equiv 0(\text{mod } q+1)$ for classes of type $C_7^{(k,\ell)}$ we can let $\ell = k$ and let $k = 1,\ldots,(q^2-1)$. However k

cannot be equal to a multiple of $(q-1)$ or else we would have a canonical form of one of the classes already counted. Thus we discard the $(q+1)$ possible multiples of $(q-1)$ in $1,\dots,(q^2-1)$ and get the class count:

$$\frac{1}{2}[(q^2-1) - (q+1)] = \frac{1}{2}(q-2)(q+1).$$

(v) As shown in Section VIII, k, for the classes of type $C_8^{(k)}$, must be a multiple of $(q+1)$ and not a multiple of $(q^2 - q+1)$ of which there are 3 in $1,\dots,q^3+1$. The total class count is $\frac{1}{3}[(q^2 - q+1) - 3] = \frac{1}{3}(q-2)(q+1)$.

A complete listing of these results can be found in Table 14.

9.2 Calculation of Characters

The process of restricting the characters of $U(3,q^2)$ down to $SU(3,q^2)$ is little different from that of the case $d = 1$. We let $k = \frac{1}{3}(q+1), \frac{2}{3}(q+1), (q+1)$ for classes $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$; $\ell = -2k$ for $C_4^{(k,\ell)}$, $C_5^{(k,\ell)}$; $\ell = k$ for $C_7^{(k,\ell)}$ and keep in mind that $\epsilon^{q+1} = 1$. Making these substitutions we can obtain the restricted characters. A further substitution of $t' = (t-u)$ and $t' = (t+2u)$ made in both $\chi_p^{(t,u)}|_{SU}$ and $\chi_{qp}^{(t,u)}|_{SU}$ will simplify these characters. We note that $\omega^{(t,u)} = \omega^{(t+2u)}$ if $\omega^3 = 1$. A similar simplification of $\chi_{sp}^{(t,u)}|_{SU}$ is made by letting $t' = u + (q-1)t$.

All the characters thus obtained are then tested for reducibility and it is found that only the sets of characters of type χ_{rp} and χ_{s^2r} contain any reducible characters. The inner products for these characters are calculated below.

$$(\Psi_{rp},\Psi_{rp}) = \frac{3r^2p^2}{q^3rs^2p} + \frac{3(2q-1)^2}{q^3s} + \frac{9}{3q^2} + (\frac{r^2}{qrs^2} + \frac{1}{qs})\Sigma_1 + \frac{1}{s^2}\Sigma_2$$

$$= (18 + 2\,\Sigma_1 + \Sigma_2)/s^2$$

where $\Sigma_1 = \sum_{\substack{k=1 \\ k\neq 0(s/3)}}^{s} \Big| \sum_{(t,u,v)} \epsilon^{(t+u-2v)k}\Big|^2$, $\Sigma_2 = \sum_{\substack{k,\ell,m=1 \\ k<\ell<m \\ k+\ell+m=0}} \Big| \sum_{[t,u,v]} \epsilon^{tk+u\ell+vm}\Big|^2$

In this expression the numerator is an integral multiple of $s^2=(q+1)^2$. As suggested by Steinberg, this integer is the coefficient of the q^2 terms of the numerator. Only Σ_2 will have such a term.

$$\Sigma_2 = \frac{1}{6}\sum_{[t,u,v]}\Big[\sum_{\substack{k,\ell,m=1 \\ k<\ell<m \\ k+\ell+m\equiv 0(s)}} \epsilon^{tk+u\ell+vm}\Big(\sum_{[t,u,v]} \epsilon^{-tk-u\ell-vm}\Big)\Big]$$

$$= \frac{1}{6}\sum_{[t,u,v]}\Big[\sum_{k,\ell,m=1}(1+\epsilon^{(t-v)k+(u-t)\ell+(v-u)m} + \epsilon^{(t-u)k+(u-v)\ell+(v-t)m}$$

$$+ \epsilon^{(u-v)(\ell-m)}+\epsilon^{(t-v)(k-m)}+\epsilon^{(t-u)(k-\ell)}\Big] + \text{linear function of } q$$

$$= \frac{1}{6}\sum_{[t,u,v]}\Big[s^2+\sum_{k=1}^{s}\epsilon^{(t+u-2v)k}\Big(\sum_{\ell=1}^{s}\epsilon^{(2u-t-v)\ell}\Big)+\sum_{k=1}^{s}\epsilon^{(2t-u-v)k}\Big(\sum_{\ell=1}^{s}\epsilon^{(t+u-2v)\ell}\Big)$$

$$+ \frac{1}{6}\sum_{[t+u+v]}\Big[s\sum_{\ell=1}^{s}\epsilon^{(u-v)\ell}+s\sum_{k=1}^{s}\epsilon^{(t-u)k}+s\sum_{k=1}^{s}\epsilon^{(t-v)k}\Big]+ \text{linear function of } q$$

The second term vanishes since $t \neq u \neq v$, so:

$$\Sigma_2 = s^2 + \frac{2}{3}\sum_{(t,u,v)}\Big[\sum_{k=1}^{s}\epsilon^{(t+u-2v)k}\sum_{\ell=1}\epsilon^{(2u-t-v)\ell}\Big] + \text{linear function of } q$$

Thus $(\Psi_{rp},\Psi_{rp}) = \begin{cases} 3 \text{ if } \sum \epsilon^{(t+u-2v)k}\sum \epsilon^{(2u-t-v)\ell}=s^2, \text{ which} \Rightarrow v-u=u-t=s/3 \\ 1 \text{ otherwise} \end{cases}$

$\therefore$ Ψ_{rp} splits into 3 components for $u=t+s/3$, $v=t+2s/3$, $t=1,\ldots,s/3$.

The maximum value of $\chi^{(t)}_{s^2r}|_{SU} = \Psi^{(t)}_{s^2r}$ will occur when $t = \frac{kp}{3}$ $k \neq$ multiple of 3. For such a value of t we get:

$$(\Psi_{s^2r},\Psi_{s^2r}) = \frac{3s^4r^2}{q^3rs^2p} + \frac{3s^2}{q^3s} + \frac{9}{3q^3} + \frac{9(p-3)}{3p} = 3\ .$$

Thus $\Psi^{(t)}_{r^2r}$ splits into three irreducible components when $t = \frac{p}{3}$

and a different set of three components when $t = \frac{2p}{3}$.

We can now construct the restrict table.

Table 12. Induce-Restrict Table for $U(3,q^2)$ - $SU(3,q^2)$ $d = 3$

$U(3,q^2)$

$\frac{1}{3}q(q+1)(q-1)$			$\frac{1}{6}q(q+1)(q-1)$	
$\frac{1}{3}(q-2)(q+1)^2$	$\frac{1}{3}(q+1)$	$\frac{1}{3}(q+1)$	$\frac{1}{6}(q-2)(q+1)^2$	$(q+1)$
$\chi_{s^2r} \cdots \chi_{s^2r}$	$\chi_{s^2r} \cdots \chi_{s^2r}$	$\chi_{s^2r} \cdots \chi_{s^2r}$	$\chi_{rp} \cdots \chi_{rp}$	$\chi_{rp} \cdots \chi_{rp}$
$q+1$ $q+1$			$q+1$ $q+1$	
$\psi_{s^2r} \cdots \psi_{s^2r}$	$\psi_{s^2r/3}$ $\psi_{s^2r/3}$ $\psi_{s^2r/3}$	$\psi_{s^2r/3}$ $\psi_{s^2r/3}$ $\psi_{s^2r/3}$	$\psi_{rp} \cdots \psi_{rp}$	$\psi_{rp/3}$ $\psi_{rp/3}$ $\psi_{rp/3}$
$\frac{1}{3}(q+1)(q-2)$	3	3	$\frac{1}{6}(q-2)(q+1)$	3

$SU(3,q^2)$

$U(3,q^2)$

$q+1$	$q+1$	$q+1$	$q(q+1)$	$q(q+1)$	$\frac{1}{2}(q-2)(q+1)^2$
$\chi_1 \cdots \chi_1$	$\chi_q \cdots \chi_q$	$\chi_{q^3} \cdots \chi_{q^3}$	$\chi_p \cdots \chi_p$	$\chi_{qp} \cdots \chi_{qp}$	$\chi_{sp} \cdots \chi_{sp}$
$q+1$	$q+1$	$q+1$	$q+1$ $q+1$	$q+1$ $q+1$	$q+1$ $q+1$
ψ_1	ψ_q	ψ_{q^3}	$\psi_p \cdots \psi_p$	$\psi_{qp} \cdots \psi_{qp}$	$\psi_{sp} \cdots \psi_{sp}$
1	1	1	q	q	$\frac{1}{2}(q+1)(q-2)$

$SU(3,q^2)$

The reducible characters Ψ_{rp} and Ψ_{s^2r} will split into thirds on all classes of SU which are complete classes of U. This leaves only the values on the classes $C_3'^{(k)}$, $C_3''^{(k)}$, $C_3'''^{(k)}$ to be determined. The missing block is:

Table 13. Partial Character Table for $SU(3,q^2)$ $d = 3$

<table>
<tr><th></th><th>$\Psi_{rp/3}$</th><th>$\Psi_{rp/3}$</th><th>$\Psi_{rp/3}$</th><th>$\Psi_{s^2r/3}$</th><th>$\Psi_{s^2r/3}$</th><th>$\Psi_{s^2r/3}$</th><th>$\Psi_{s^2r/3}$</th><th>$\Psi_{s^2r/3}$</th><th>$\Psi_{s^2r/3}$</th></tr>
<tr><td>$C_1^{(k)}$</td><td>$\frac{1}{3}rp$</td><td>$\frac{1}{3}rp$</td><td>$\frac{1}{3}rp$</td><td>$\frac{s^2}{3}r\omega^k$</td><td>$\frac{s^2}{3}r\omega^k$</td><td>$\frac{s^2}{3}r\omega^k$</td><td>$\frac{s^2}{3}r\omega^{2k}$</td><td>$\frac{s^2}{3}r\omega^{2k}$</td><td>$\frac{s^2}{3}r\omega^{2k}$</td></tr>
<tr><td>$C_2^{(k)}$</td><td>$(2q-1)$</td><td>$(2q-1)$</td><td>$(2q-1)$</td><td>$-\frac{s}{3}\omega^k$</td><td>$-\frac{s}{3}\omega^k$</td><td>$-\frac{s}{3}\omega^k$</td><td>$-\frac{s}{3}\omega^{2k}$</td><td>$-\frac{s}{3}\omega^{2k}$</td><td>$-\frac{s}{3}\omega^{2k}$</td></tr>
<tr><td>$C_3^{\prime(k)}$</td><td>a</td><td>b</td><td>c</td><td>a'</td><td>b'</td><td>c'</td><td>a''</td><td>b''</td><td>c''</td></tr>
<tr><td>$C_3^{\prime\prime(k)}$</td><td>c</td><td>a</td><td>b</td><td>c'</td><td>a'</td><td>b'</td><td>c''</td><td>a''</td><td>b''</td></tr>
<tr><td>$C_3^{\prime\prime\prime(k)}$</td><td>b</td><td>c</td><td>a</td><td>b'</td><td>c'</td><td>a'</td><td>b''</td><td>c''</td><td>a''</td></tr>
<tr><td>$C_4^{(k)}$</td><td>r</td><td>r</td><td>r</td><td colspan="6" rowspan="4">0</td></tr>
<tr><td>$C_5^{(k)}$</td><td>-1</td><td>-1</td><td>-1</td></tr>
<tr><td>$C_6^{(k,\ell)}$</td><td>1 or -2</td><td>1 or -2</td><td>1 or -2</td></tr>
<tr><td>$C_7^{(k)}$</td><td colspan="3" rowspan="2">0</td></tr>
<tr><td>$C_8^{(k)}$</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

The easiest way to fill in this block is to look at the table for $PSL(4,q)$, where only the characters $\Psi'_{rp/3}$, $\Psi''_{rp/3}$, $\Psi'''_{rp/3}$ appear. It can be shown that the values for these characters on C_3', C_3'', C_3''' must all be rational values. Thus this 3×3 block of values can be quickly found using the orthogonality relations. The values of $\Psi_{rp/3}$ on C_3', C_3'', C_3''' suggest that the reducible characters Ψ_{s^2r} split up in a similar manner. The remaining 3×6 block can then be determined using orthogonality relations.

The only remaining detail which needs to be mentioned is the values which $\Psi'_{rp/3}$, $\Psi''_{rp/3}$, $\Psi'''_{rp/3}$ have on the various classes of type $C_6^{(k,\ell)}$.

The reducible character $\Psi_{rp}^{(t,u,v)}$, $t = 1,\ldots,\frac{1}{3}(q+1)$, $u=t+\frac{1}{3}(q+1)$, $v=t+\frac{2}{3}(q+1)$, equals $-\sum_{[t,u,v]} \epsilon^{(t-v)k+(u-v)\ell}$ on $C_6^{(k,\ell)}$ $k,\ell = 1,\ldots,(q+1)$ $k \neq \ell$

Now if $k \equiv \ell \pmod 3$ then $\varepsilon^{(t-v)k+(u-v)\ell} = 1 \ \forall\, t,u,v$ permutations so $\Psi_{rp}(C_6) = -6$ otherwise $\varepsilon^{(t-v)k+(u-v)\ell} = \omega^1$ for 3 of the 6 permutations of t,u,v and the other 3 permutations give ω^2 $\therefore \Psi_{rp}(C_6) = -3(\omega' + \omega^2) = -3(-1) = +3$. From this we finally get:

$$\Psi^{(t)}_{rp/3}(C_6) = \begin{cases} -2 & \text{if } k \equiv \ell \pmod 3 \\ 1 & \text{otherwise} \end{cases}$$

The complete character table can now be written down. It appears below.

Table 14. Character Table for $SU(3,q^2)$ $d = 3$

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_1^{(k)}$	$\begin{pmatrix} \omega^k & & \\ & \omega^k & \\ & & \omega^k \end{pmatrix}$	$k=1,2,3$	3	q^3rs^2p
$C_2^{(k)}$	$\begin{pmatrix} \omega^k & & \\ 1 & \omega^k & \\ & & \omega^k \end{pmatrix}$	$k=1,2,3$	3	q^3s
$C_3'^{(k)}$	$\begin{pmatrix} \omega^k & & \\ 1 & \omega^k & \\ & 1 & \omega^k \end{pmatrix}$	$k=1,2,3$	3	$3q^2$
$C_3''^{(k)}$	$\begin{pmatrix} \omega^k & & \\ \theta & \omega^k & \\ & \theta & \omega^k \end{pmatrix}$	$k=1,2,3$	3	$3q^2$
$C_3'''^{(k)}$	$\begin{pmatrix} \omega^k & & \\ \theta^2 & \omega^k & \\ & \theta^2 & \omega^k \end{pmatrix}$	$k=1,2,3$	3	$3q^2$
$C_4^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,(q+1)$ $k \not\equiv 0 \pmod{\frac{1}{3}(q+1)}$	$q-2$	qrs^2
$C_5^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,(q+1)$ $k \not\equiv 0 \bmod \frac{1}{3}(q+1)$	$q-2$	qs
$C_6^{(k,\ell,m)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^\ell & \\ & & \rho^m \end{pmatrix}$	$k,\ell,m=1,\cdots,(q+1)$ $k<\ell<m$ $k+\ell+m \equiv 0 \pmod{q+1}$	$\frac{1}{6}(q+1)(q-2)+1$	s^2
$C_7^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \sigma^k & \\ & & \sigma^{-kq} \end{pmatrix}$	$k=1,\cdots,q^2-1$ $k \not\equiv 0 \bmod q-1$ $C^{(k)} = C^{(-kq)} \pmod{q^2-1}$	$\frac{1}{2}(q-2)(q+1)$	rs
$C_8^{(k)}$	$\begin{pmatrix} \delta^k & & \\ & \delta^{kq^2} & \\ & & \delta^{kq^4} \end{pmatrix}$	$k=1,\cdots,q^2-q$ $k \not\equiv 0 \bmod \frac{1}{3}(q^2-q+1)$ $C^{(k)} = C^{(kq^2)} = C^{(kq^4)}$	$\frac{1}{3}(q-2)(q+1)$	p

σ = primitive element of $GF(q^2)$, $\rho = \sigma^{q-1}$, $\delta^{q^2-q+1} = 1$ $\theta \in GF(q^2)$ $\theta^3 \neq 1$

Table 14 (cont'd.)

parameters				$t=1,\cdots,q$		$t=1,\cdots,q^2-1$, $t\neq 0 \pmod{q-1}$, $(t)=(-tq)$	$t,u,v=1,\cdots,q+1$, $t<u<v$, $t\neq t'+k$, $u\neq u'+k$, $v\neq v'+k$ $\pmod{q+1}$ simultaneously; $t\neq\alpha$, $u\neq t+s/3$, $v\neq t+2s/3$ simultaneously, $\alpha=1,\cdots,(q+1)/3$	$t=1,\cdots,q^2-q$, $t\neq\frac{k}{3}(q^2-q+1)$, $k\neq 0 \pmod 3$	the entries on classes C_3', C_3'', C_3''' are permuted		
number of characters	1	1	1	q	q	$\frac{1}{2}(q+1)(q-2)$	$\frac{1}{6}(q+1)(q-2)$	$\frac{1}{3}(q+1)(q-2)$	3	3	3
class \ char.	ψ_1	ψ_{qr}	ψ_q	$\psi_p^{(t)}$	$\psi_{qp}^{(t)}$	$\psi_{sp}^{(t)}$	$\psi_{rp}^{(t,u,v)}$	$\psi_{s^2r}^{(t)}$	$\psi_{rp/3}$	$\psi_{s^2r/3}$	$\psi'_{s^2r/3}$
$C_1^{(k)}$	1	qr	q^3	$p\omega^{tk}$	$qp\omega^{tk}$	$sp\omega^{tk}$	$rp\omega^{(t+u+v)k}$	$s^2r\omega^{tk}$	$\frac{1}{3}rp$	$\frac{1}{3}s^2r\omega^{k}$	$\frac{1}{3}s^2r\omega^{2k}$
$C_2^{(k)}$	1	$-q$	0	$-r\omega^{tk}$	$q\omega^{tk}$	ω^{tk}	$(2q-1)\omega^{(t+u+v)k}$	$-s\omega^{tk}$	$\frac{1}{3}(2q-1)$	$-\frac{1}{3}s\omega^{k}$	$-\frac{1}{3}s\omega^{2k}$
$C_3'^{(k)}$	1	0	0	ω^{tk}	0	ω^{tk}	$-\omega^{(t+u+v)k}$	$-\omega^{tk}$	$\frac{1}{3}(2q-1)$	$\frac{1}{3}(2q-1)\omega^{k}$	$\frac{1}{3}(2q-1)\omega^{2k}$
$C_3''^{(k)}$	1	0	0	ω^{tk}	0	ω^{tk}	$-\omega^{(t+u+v)k}$	$-\omega^{tk}$	$-\frac{1}{3}s$	$-\frac{1}{3}s\omega^{k}$	$-\frac{1}{3}s\omega^{2k}$
$C_3'''^{(k)}$	1	0	0	ω^{tk}	0	ω^{tk}	$-\omega^{(t+u+v)k}$	$-\omega^{tk}$	$-\frac{1}{3}s$	$-\frac{1}{3}s\omega^{k}$	$-\frac{1}{3}s\omega^{2k}$
$C_4^{(k)}$	1	$-r$	q	$-r\varepsilon^{tk}+\varepsilon^{-2tk}$	$r\varepsilon^{tk}+q\varepsilon^{-2tk}$	$s\varepsilon^{tk}$	$r\sum_{(t,u,v)}\varepsilon^{k(t+u-2v)}$	0	r	0	
$C_5^{(k)}$	1	1	0	$\varepsilon^{tk}+\varepsilon^{-2tk}$	$-\varepsilon^{tk}$	ε^{tk}	$-\sum_{(t,u,v)}\varepsilon^{k(t+u-2v)}$	0	-1		
$C_6^{(k,l,m)}$	1	2	-1	$\varepsilon^{tk}+\varepsilon^{tl}+\varepsilon^{tm}$	$-\varepsilon^{tk}-\varepsilon^{tl}-\varepsilon^{tm}$	0	$-\sum_{[t,u,v]}\varepsilon^{tk+ul+vm}$	0	1 or -2		
$C_7^{(k)}$	1	0	1	ε^{tk}	ε^{tk}	$\eta^{tk}+\eta^{-tkq}$	0	0	0		
$C_8^{(k)}$	1	-1	-1	0	0	0	0	$-\gamma^{tk}-\gamma^{tkq^2}-\gamma^{tkq^4}$	0	1	1

$r = q-1$, $s = q+1$, $p = q^2 - q+1$, $\varepsilon^{q+1} = 1$, $\eta^{q^2-1} = 1$, $\gamma^{p} = 1$, $\omega^3 = 1$

$\sum_{(x,y,z)}$ means a sum over the cyclic permutations of x,y,z, $\sum_{[x,y,z]}$ means a sum over all permutations of x,y,z

X. CHARACTER TABLE FOR $PSU(3,q^2)$ $d = 3$

The center of $SU(3,q^2)$ consists of the scalar matrices $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, $\begin{pmatrix} \omega & & \\ & \omega & \\ & & \omega \end{pmatrix}$, $\begin{pmatrix} \omega^2 & & \\ & \omega^2 & \\ & & \omega^2 \end{pmatrix}$ $\omega^3 = 1$. To determine the conjugacy classes of PSU we identify classes of SU which are equivalent to each other under multiplication by the above three scalar matrices since $PSU = SU/Z(SU)$. Thus, except for a single case, the classes of SU combine in sets of 3 to give the classes of PSU.

The exception lies in the set of classes of type $C_6^{(k,\ell,m)}$. The number of such classes in SL is $\frac{1}{6}(q+1)(q-2) + 1$ which is not divisible by 3 and so we know that one class is self-equivalent under scalar multiplication. This class is $C_6^{(\frac{q+1}{3}, \frac{2(q+1)}{3}, q+1)}$ which has the canonical representative $\begin{pmatrix} \omega & & \\ & \omega^2 & \\ & & 1 \end{pmatrix}$. This class is broken out and relabeled as C_9 in PSU.

To find the characters of PSL we take all characters χ of SU such that $\chi\left(\begin{pmatrix} \omega^k & & \\ & \omega^k & \\ & & \omega^k \end{pmatrix}\right)$ is constant for $k = 1,2,3$. Thus Ψ_1, Ψ_{qr}, and Ψ_{q^3} are characters of PSU. When t is a multiple of 3 then $\Psi_p^{(t)}$, $\Psi_{qp}^{(t)}$, $\Psi_{rp}^{(t)}$, $\Psi_{r^2s}^{(t)}$ are characters of PSU. All three characters $\Psi_{sp/3}$ and all $\Psi_{sp}^{(t,u,v)}$, when $(t + u + v)$ is a multiple of 3, are characters of PSL also. The table for PSL is thus very easily obtained once we have the characters of SL.

We note that the tables for $PSU(3,q^2)$ $d = 1,3$ agree with Frame's results in [10] on the number and order of the conjugacy classes and the frequency and degrees of the characters.

The number and order of the conjugacy classes for $SU(3,q^2)$ $d = 1,3$ given by Dickson [5], page 571, agree with our results for these groups.

Several character tables for specific values of q were generated and checked against existing tables for errors.

The character entry $\Psi_{rp}^{(t,u,v)}(C_9) = -\sum_{[t,u,v]} \omega^{t+2u+3v} = -\sum_{[t,u,v]} \omega^{t+2u}$ can be simplified by observing that if $t \equiv u \pmod 3$ then the condition that $t + u + v \equiv 0 \pmod{q+1}$ implies $t \equiv u \equiv v \pmod 3$. In this case $t + 2u \equiv 0 \pmod 3$ so $\Psi_{rp}^{(t,u,v)}(C_9) = -6$. Otherwise, $t + 2u \not\equiv 0 \pmod 3$ and $\Psi_{rp}^{(t,u,v)}(C_9) = -3\omega - 3\omega^2 = +3$. Thus $\Psi_{rp}^{(t,u,v)}(C_9) = -6$ or 3.

Table 15. Character Table for $PSU(3,q^2)$ $d = 3$

conj. class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$		1	$\frac{1}{3}q^3rs^2p$
C_2	$\begin{pmatrix}1&&\\1&1&\\&&1\end{pmatrix}$		1	$\frac{1}{3}q^3s$
C_3'	$\begin{pmatrix}1&&\\1&1&\\&1&1\end{pmatrix}$		1	q^2
C_3''	$\begin{pmatrix}1&&\\\theta&1&\\&\theta&1\end{pmatrix}$		1	q^2
C_3'''	$\begin{pmatrix}1&&\\\theta^2&1&\\&\theta^2&1\end{pmatrix}$		1	q^2
C_4	$\begin{pmatrix}\rho^k&&\\&\rho^k&\\&&\rho^{-2k}\end{pmatrix}$	$k=1,\cdots,\frac{1}{3}(q-2)$	$\frac{1}{3}(q-2)$	$\frac{1}{3}qrs^2$
$C_5^{(k)}$	$\begin{pmatrix}\rho^k&&\\1&\rho^k&\\&&\rho^{-2k}\end{pmatrix}$	$k=1,\cdots,\frac{1}{3}(q-2)$	$\frac{1}{3}(q-2)$	$\frac{1}{3}qs$
$C_6^{(k,\ell,m)}$	$\begin{pmatrix}\rho^k&&\\&\rho^\ell&\\&&\rho^m\end{pmatrix}$	$k,\ell,m=1,\cdots,q+1$ $k<\ell<m$ $k+\ell+m\equiv 0 \pmod{q+1}$ $k\not\equiv k', \ell\not\equiv\ell', m\not\equiv m' \pmod{\frac{1}{3}(q+1)}$ simultaneously $k\neq\frac{1}{3}(q+1)$ $\ell\neq\frac{2}{3}(q+1)$ $m\neq {}^{3}(q+1)$ simultaneously	$\frac{1}{18}(q+1)(q-2)$	$\frac{1}{3}s^2$
$C_7^{(k)}$	$\begin{pmatrix}\rho^k&&\\&\sigma^k&\\&&\sigma^{-kq}\end{pmatrix}$	$k=1,\cdots,\frac{1}{3}(q^2-1)$ $k\not\equiv 0 \bmod (q-1)$ $C^{(k)}=C^{(-kq)}$	$\frac{1}{6}(q-2)(q+1)$	$\frac{1}{3}rs$
$C_8^{(k)}$	$\begin{pmatrix}\delta^k&&\\&\delta^{kq^2}&\\&&\delta^{kq^4}\end{pmatrix}$	$k=1,\cdots,\frac{1}{3}p-1$ $C^{(k)}=C^{(kq^2)}=C^{(kq^4)}$	$\frac{1}{9}(q-2)(q+1)$	$\frac{1}{3}p$
C_9	$\begin{pmatrix}1&&\\&\omega&\\&&\omega^2\end{pmatrix}$		1	s^2

σ = primitive element of $GF(q^2)$ $\quad\rho=\sigma^{q-1}$

$\theta\in GF(q^2)$, $\theta^3\neq 1$, $\delta^{q^2-q+1}=1$, $\omega^3=1$

Table 15 (cont'd.)

parameters				t=1, ⋯ , $\frac{1}{3}(q-2)$		$t=1,\cdots,\frac{1}{3}(q^2-1)$ $t \neq 0 \bmod q-1$ $(t)=(-tq)$ $\bmod \frac{1}{3}(q^2-1)$	$t,u,v=1,\cdots,q+1$ $t<u<v$ $t\neq t',\ u\neq u',\ v\neq v',\ (\bmod \frac{q+1}{3})$ simultaneously $t\neq \alpha$, $u\neq t+s/3$ $v\neq t+\frac{2}{3}s$ simultaneously $t+u+v\equiv 0 \pmod{q+1}$ $\alpha=1,\cdots,s$	$t=1,\cdots,\frac{1}{3}p-1$ $(t)=(tq^2)=(tq^4)$ $(\bmod \frac{1}{3}p)$	permute the values on classes C_3', C_3^*, C_3^{*}
number of characters	1	1	1	$\frac{1}{3}(q-2)$	$\frac{1}{3}(q-2)$	$\frac{1}{6}(q-2)(q+1)$	$\frac{1}{18}(q-2)(q+1)$	$\frac{1}{q}(q-2)(q+1)$	3
class \ char.	ψ_1	ψ_{qr}	ψ_{q^3}	$\psi_p^{(t)}$	$\psi_{qp}^{(t)}$	$\psi_{sp}^{(t)}$	$\psi_{rp}^{(t,u,v)}$	$\psi_{s^2r}^{(t)}$	$\psi_{rp/3}$
C_1	1	qr	q^3	p	qp	sp	rp	s^2r	$\frac{1}{3}rp$
C_2	1	-q	0	-r	q	1	(2q-1)	-s	$\frac{1}{3}(2q-1)$
C_3'	1	0	0	1	0	1	-1	-1	$\frac{1}{3}(2q-1)$
C_3^*	1	0	0	1	0	1	-1	-1	$-\frac{1}{3}s$
C_3^{*}	1	0	0	1	0	1	-1	-1	$-\frac{1}{3}s$
$C_4^{(\kappa)}$	1	-r	q	$+\varepsilon^{3t\kappa}+\varepsilon^{-6t\kappa}$	$r\varepsilon^{3t\kappa}+q\varepsilon^{-6t\kappa}$	$s\varepsilon^{3t\kappa}$	$r(\varepsilon^{-3v\kappa}+\varepsilon^{-3u\kappa}+\varepsilon^{-3t\kappa})$	0	r
$C_5^{(\kappa)}$	1	1	0	$\varepsilon^{3t\kappa}+\varepsilon^{-6t\kappa}$	$-\varepsilon^{3t\kappa}$	$\varepsilon^{3t\kappa}$	$-\varepsilon^{-3v\kappa}-\varepsilon^{-3u\kappa}-\varepsilon^{-3t\kappa}$	0	-1
$C_6^{(\kappa,l,m)}$	1	2	-1	$\varepsilon^{3t\kappa}+\varepsilon^{3tl}+\varepsilon^{3tm}$	$-\varepsilon^{3t\kappa}-\varepsilon^{3tl}-\varepsilon^{3tm}$	0	$-\sum_{[t,u,v]}\varepsilon^{t\kappa+ul+vm}$	0	+1 cr-2
$C_7^{(\kappa)}$	1	0	1	$\varepsilon^{3t\kappa}$	$\varepsilon^{3t\kappa}$	$\eta^{3t\kappa}+\eta^{-3qt\kappa}$	0	0	0
$C_8^{(\kappa)}$	1	-1	-1	0	0	0	0	$-\gamma^{t\kappa}-\gamma^{t\kappa q^2}-\gamma^{t\kappa q^4}$	0
C_9	1	2	-1	3	-3	0	3 or -6	0	+1 or-2

$r = q-1,\ s = q+1,\ p = q^2 - q+1,\ \varepsilon^{q+1} = 1,\ \eta^{q^2-1} = 1,\ \gamma^{p/3} = 1\ ,\ \omega^3 = 1$

$\sum_{(x,y,z)}$ means a sum over the cyclic permutations of x,y,z, $\sum_{[x,y,z]}$ means a sum over all permutations of x,y,z

XI. CHARACTER TABLES FOR $SL(3,q)$, $PSL(3,q)$ $d = 1,3$

The structure of the character tables for the unitary and linear groups are so similar in form that the details of obtaining the characters of SL and PSL from GL differ very little from those described in Sections VIII-X for the corresponding unitary groups. Once the general method is established the differences between the unitary and linear cases can easily be handled. For this reason no calculations will be given for any of the tables in this section.

11.1 Character Table for $SL(3,q)$ $d = 1$

The calculations to determine the number of conjugacy classes of each type and the range of the class parameters are of course quite different from the calculations for $SU(3,q^2)$ but they are of the same nature. Since no restricted characters can split, the development of the character table poses no difficulty. Table 16 is the character table for $SL(3,q^2)$ $d = 1$.

Table 16. Character Table for $SL(3,q) \cong PSL(3,q)$ $d = 1$

conjugacy class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$		1	$q^3 r^2 s p$
C_2	$\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix}$		1	$q^3 r$
C_3	$\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}$		1	q^2
$C_4^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,(q-2)$	$q-2$	$qr^2 s$
$C_5^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,(q-2)$	$q-2$	qr
$C_6^{(k,\ell,m)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^\ell & \\ & & \rho^m \end{pmatrix}$	$k,\ell,m=1,\cdots,(q-1)$ $k<\ell<m$ $k+\ell+m=0 \pmod{q-1}$	$\frac{1}{6}(q-2)(q-3)$	r^2
$C_7^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \sigma^{-k} & \\ & & \sigma^{-qk} \end{pmatrix}$	$k=1,\cdots,q^2-1$ $k \neq 0 \bmod q+1$ $C^{(k)} = C^{(kq)}$	$\frac{1}{2}q(q-1)$	rs
$C_8^{(k)}$	$\begin{pmatrix} \tau^k & & \\ & \tau^{kq} & \\ & & \tau^{kq^2} \end{pmatrix}$	$k=(q-1),2(q-1),\cdots,(q^2+q)(q-1)$ $C^{(k)} = C^{(kq)} = C^{(kq^2)}$	$\frac{1}{3}q(q+1)$	p

ρ,σ,τ are primitive elements of $GF(q)$, $GF(q^2)$, $GF(q^3)$ respectively

$\rho = \sigma^{q+1} = \tau^{q^2+q+1}$ $\quad r = q - 1 \quad s = q+1 \quad p = q^2 + q+1$

Table 16 (cont'd.)

parameters				$t=1,\cdots q-2$		$t,u,v=1,\cdots,q-1$; $t<u<v$; $t+u+v=0 \pmod{q-1}$	$t=1,\cdots,q^2-1$; $t\not\equiv 0 \pmod{q+1}$; $(t)=(tq)$	$t=(q-1),2(q-1),\cdots,(q^2+q)(q-1)$; $(t)=(tq)=(tq^2)$
number of characters	1	1	1	$q-2$	$q-2$	$\frac{1}{6}(q-2)(q-3)$	$\frac{1}{2}q(q-1)$	$\frac{1}{3}q(q+1)$
class \ char	Ψ_1	Ψ_{qs}	Ψ_{q^3}	$\Psi_p^{(t)}$	$\Psi_{qp}^{(t)}$	$\Psi_{sp}^{(t,u,v)}$	$\Psi_{rp}^{(t)}$	$\Psi_{r^2s}^{(t)}$
C_1	1	qs	q^3	p	qp	sp	rp	r^2s
C_2	1	q	0	s	q	$2q+1$	-1	$-r$
C_3	1	0	0	1	0	1	-1	1
$C_4^{(K)}$	1	s	q	$s\varepsilon^{Kt}+\varepsilon^{-2Kt}$	$s\varepsilon^{Kt}+q\varepsilon^{-2Kt}$	$s\sum_{(t,u,v)}\varepsilon^{K(t+u-2v)}$	$r\varepsilon^{Kt}$	0
$C_5^{(K)}$	1	1	0	$\varepsilon^{Kt}+\varepsilon^{-2Kt}$	ε^{Kt}	$\sum_{(t,u,v)}\varepsilon^{K(t+u-2v)}$	$-\varepsilon^{Kt}$	0
$C_6^{(K,l,m)}$	1	2	1	$(\varepsilon^{Kt}+\varepsilon^{lt}+\varepsilon^{mt})$	$(\varepsilon^{Kt}+\varepsilon^{lt}+\varepsilon^{mt})$	$\sum_{[t,u,v]}\varepsilon^{tK+ul+vm}$	0	0
$C_7^{(K)}$	1	0	-1	ε^{Kt}	$-\varepsilon^{Kt}$	0	$-\eta^{-Kt}-\eta^{-Ktq}$	0
$C_8^{(K)}$	1	-1	1	0	0	0	0	$\gamma^{Kt}+\gamma^{Ktq}+\gamma^{Ktq^2}$

$\sum_{(x,y,z)}$, $\sum_{[x,y,z]}$ indicates a sum over the cyclic, over all, permutations of t,u,v respectively

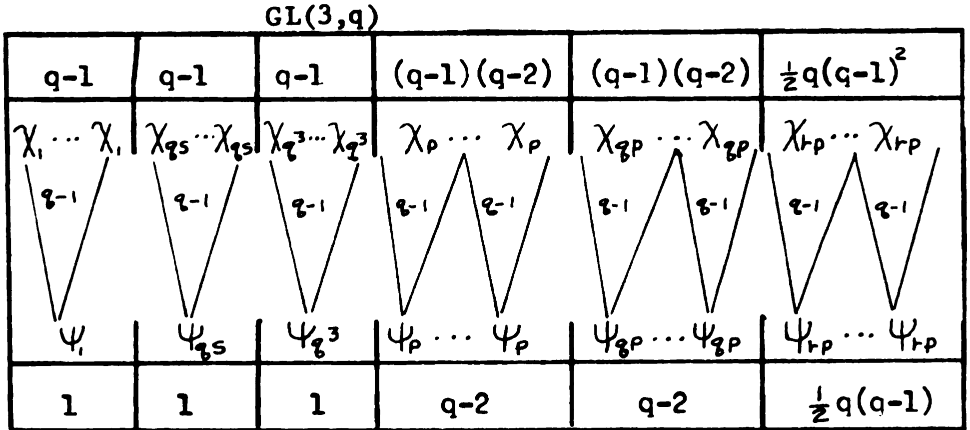
$\varepsilon^{q-1}=1$, $\eta^{q^2-1}=1$, $\gamma^{q^3-1}=1$

11.2 Character Table for SL(3,q) d = 3

The induce-restrict table for GL(3,q) to SL(3,q) given in Table 17 demonstrates the close similarity between the unitary and the linear cases.

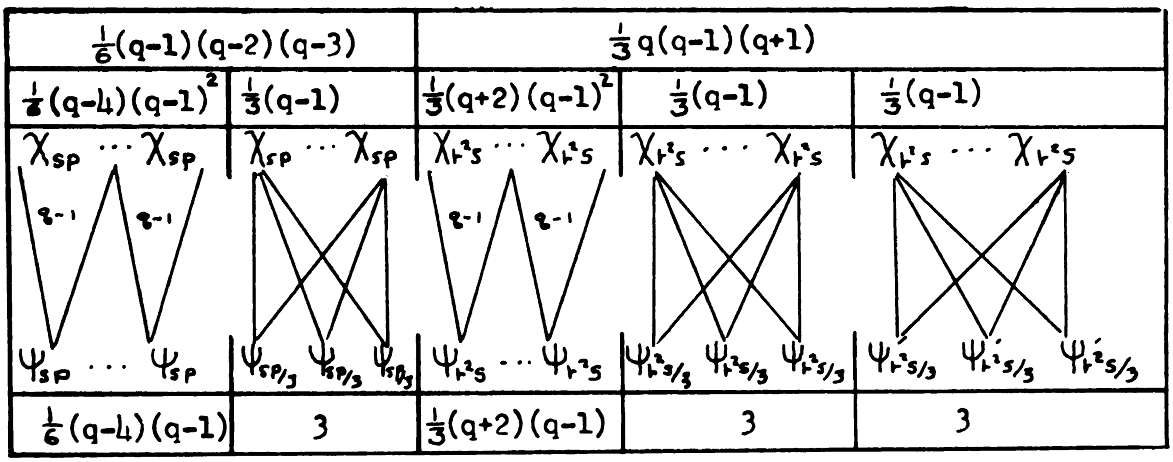
Table 17. Induce-Restrict Table for GL(3,q) - SL(3,q) d = 3

GL(3,q)



SL(3,q)

GL(3,q)



SL(3,q)

It should be noted that the 1-1 correspondence between sets of classes of similar canonical form and sets of characters of the same degree, breaks down in SL and SU. This is because the classes that split do not correspond to the characters which split. Thus, the extreme orderliness of the character tables for $GL(n,q)$ and $U(n,q^2)$ is lost in the special and projective special subgroups. The complexity becomes more pronounced with an increase in n and d since more class and character splitting occur. The character table for $SL(3,q)$ $d = 3$ appears in Table 18.

Table 18. Character Table for $SL(3,q)$ $d = 3$

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_1^{(k)}$	$\begin{pmatrix} \omega^k & & \\ & \omega^k & \\ & & \omega^k \end{pmatrix}$	$k=1,2,3$	3	q^3r^2sp
$C_2^{(k)}$	$\begin{pmatrix} \omega^k & & \\ 1 & \omega^k & \\ & & \omega^k \end{pmatrix}$	$k=1,2,3$	3	q^3r
$C_3^{\prime(k)}$	$\begin{pmatrix} \omega^k & & \\ 1 & \omega^k & \\ & 1 & \omega^k \end{pmatrix}$	$k=1,2,3$	3	$3q^2$
$C_3^{\prime\prime(k)}$	$\begin{pmatrix} \omega^k & & \\ \theta & \omega^k & \\ & \theta & \omega^k \end{pmatrix}$	$k=1,2,3$	3	$3q^2$
$C_3^{\prime\prime\prime(k)}$	$\begin{pmatrix} \omega^k & & \\ \theta^2 & \omega^k & \\ & \theta^2 & \omega^k \end{pmatrix}$	$k=1,2,3$	3	$3q^2$
$C_4^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,q-1$ $k \neq 0 \bmod \frac{1}{3}(q-1)$	$q-4$	qr^2s
$C_5^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,q-1$ $k \neq 0 \bmod \frac{1}{3}(q-1)$	$q-4$	qr
$C_6^{(k,\ell,m)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^\ell & \\ & & \rho^m \end{pmatrix}$	$k,\ell,m=1,\cdots,q-1$ $k < \ell < m$ $k+\ell+m \equiv 0 (\bmod q-1)$	$\frac{1}{6}(q-1)(q-4)+1$	r^2
$C_7^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \sigma^{-k} & \\ & & \sigma^{-qk} \end{pmatrix}$	$k=1,\cdots,q^2-1$ $k \neq 0 \bmod q+1$ $C^{(k)} = C^{(kq)} (\bmod q^2-1)$	$\frac{1}{2}q(q-1)$	rs
$C_8^{(k)}$	$\begin{pmatrix} \delta^k & & \\ & \delta^{kq} & \\ & & \delta^{kq^2} \end{pmatrix}$	$k=1,\cdots,q^2+q$ $k \neq 0 \bmod \frac{1}{3}(q^2+q+1)$ $C^{(k)} = C^{(kq)} = C^{(kq^2)} \bmod p$	$\frac{1}{3}(q+2)(q-1)$	p

ρ,σ = primitive elements of $GF(q)$, $GF(q^2)$ respectively

$\theta \in GF(q)$, $\theta^3 \neq 1$, $\delta \in GF(q^3)$, $\delta^{q^2+q+1} = 1$, $\omega^3 = 1$, $\rho = \sigma^{q+1}$

$r = q-1$, $s = q+1$, $p = q^2 + q+1$

Table 18 (cont'd.)

parameters				$t=1,\cdots,q-2$		$t=1,\cdots,q^2-1$ $t\neq 0$ (mod $q+1$) $(t)=(tq)$	$t,u,v=1,\cdots,q-1$ $t<u<v$ $t\neq t+k$ $u\neq u+k$ (mod $q-1$) $v\neq v+k$ simultaneously $t\neq\alpha$ $u\neq t+\frac{t}{3}$ $v\neq t+\frac{2t}{3}$ simultaneously $\alpha=1,\cdots(q-1)/3$	$t=1,\cdots,q^2+q$ $t\neq\frac{k}{3}(q^2+q+1)$ $k\neq 0$ (mod 3)	the entries on classes C_3',C_3'',C_3''' are permuted		
number of characters	1	1	1	$q-2$	$q-2$	$\frac{1}{2}q(q-1)$	$\frac{1}{6}(q-1)(q-4)$	$\frac{1}{3}(q-1)(q+2)$	3	3	3
class \ char.	Ψ_1	Ψ_{qs}	Ψ_{q^3}	$\Psi_p^{(t)}$	$\Psi_{qp}^{(t)}$	$\Psi_{rp}^{(t)}$	$\Psi_{sp}^{(t,u,v)}$	$\Psi_{r^2s}^{(t)}$	$\Psi_{sp/3}$	$\Psi_{r^2s/3}$	$\Psi_{r^2s/3}'$
$C_1^{(k)}$	1	qs	q^3	$p\omega^{tk}$	$qp\omega^{tk}$	$rp\omega^{tk}$	$sp\omega^{(t+u+v)k}$	$r^2s\omega^{tk}$	$\frac{1}{3}sp$	$\frac{1}{3}r^2s\omega^{k}$	$\frac{1}{3}r^2s\omega^{2k}$
$C_2^{(k)}$	1	q	0	$s\omega^{tk}$	$q\omega^{tk}$	$-\omega^{tk}$	$(2q+1)\omega^{(t+u+v)k}$	$-r\omega^{tk}$	$\frac{1}{3}(2q+1)$	$-\frac{1}{3}r\omega^{k}$	$-\frac{1}{3}r\omega^{2k}$
$C_3'^{(k)}$	1	0	0	ω^{tk}	0	$-\omega^{tk}$	$\omega^{(t+u+v)k}$	ω^{tk}	$\frac{1}{3}(2q+1)$	$\frac{1}{3}(2q+1)\omega^{k}$	$\frac{1}{3}(2q+1)\omega^{2k}$
$C_3''^{(k)}$	1	0	0	ω^{tk}	0	$-\omega^{tk}$	$\omega^{(t+u+v)k}$	ω^{tk}	$-\frac{1}{3}r$	$-\frac{1}{3}r\omega^{k}$	$-\frac{1}{3}r\omega^{2k}$
$C_3'''^{(k)}$	1	0	0	ω^{tk}	0	$-\omega^{tk}$	$\omega^{(t+u+v)k}$	ω^{tk}	$-\frac{1}{3}r$	$-\frac{1}{3}r\omega^{k}$	$-\frac{1}{3}r\omega^{2k}$
$C_4^{(k)}$	1	s	q	$s\varepsilon^{tk}+\varepsilon^{-2tk}$	$s\varepsilon^{tk}+q\varepsilon^{-2tk}$	$r\varepsilon^{tk}$	$s\sum_{(t,u,v)}\varepsilon^{k(t+u-2v)}$	0	s	0	
$C_5^{(k)}$	1	1	0	$\varepsilon^{tk}+\varepsilon^{-2tk}$	ε^{tk}	$-\varepsilon^{tk}$	$\sum_{(t,u,v)}\varepsilon^{k(t+u-2v)}$	0	1		
$C_6^{(k,l,m)}$	1	2	1	$\varepsilon^{tk}+\varepsilon^{tl}+\varepsilon^{tm}$	$\varepsilon^{tk}+\varepsilon^{tl}+\varepsilon^{tm}$	0	$\sum_{[t,u,v]}\varepsilon^{(tk+ul+vm)}$	0	-1 or 2		
$C_7^{(k)}$	1	0	-1	ε^{tk}	$-\varepsilon^{tk}$	$-\eta^{tk}-\eta^{qtk}$	0	0	0		
$C_8^{(k)}$	1	-1	1	0	0	0	0	$\gamma^{tk}+\gamma^{tkq}+\gamma^{tkq^2}$	0	1	1

$r=q-1 \quad s=q+1 \quad p=q^2+q+1 \quad \omega^3=1 \quad \varepsilon^{(q-1)}=1 \quad \eta^{q^2-1}=1 \quad \gamma^p=1$

$\sum_{(x,y,z)}$ means a sum over the cyclic permutations of x,y,z, $\sum_{[x,y,z]}$ means a sum over all permutations of x,y,z

11.3 Character Table for PSL(3,q) d = 3

The table for PSL(3,q) is quickly obtained from the table for SL(3,q) in the same manner as described previously for the unitary case. The table follows.

Table 19. Character Table for PSL(3,q) d = 3

conjugacy class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$		1	$\frac{1}{3}q^3r^2sp$
C_2	$\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix}$		1	$\frac{1}{3}q^3r$
C_3'	$\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}$		1	q^2
C_3''	$\begin{pmatrix} 1 & & \\ \theta & 1 & \\ & \theta & 1 \end{pmatrix}$		1	q^2
C_3'''	$\begin{pmatrix} 1 & & \\ \theta^2 & 1 & \\ & \theta^2 & 1 \end{pmatrix}$		1	q^2
$C_4^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,\frac{1}{3}(q-4)$	$\frac{1}{3}(q-4)$	$\frac{1}{3}qr^2s$
$C_5^{(k)}$	$\begin{pmatrix} \rho^k & & \\ 1 & \rho^k & \\ & & \rho^{-2k} \end{pmatrix}$	$k=1,\cdots,\frac{1}{3}(q-4)$	$\frac{1}{3}(q-4)$	$\frac{1}{3}qr$
$C_6^{(k,\ell,m)}$	$\begin{pmatrix} \rho^k & & \\ & \rho^\ell & \\ & & \rho^m \end{pmatrix}$	$k,\ell,m=1,\cdots,q-1$ $k<\ell<m$ $k+\ell+m \equiv 0 \pmod{q-1}$ $k\neq k'$, $\ell\neq \ell'$, $m\neq m'$ simultaneously $k\neq \frac{1}{3}(q-1)$ $\ell\neq \frac{2}{3}(q-1)$ $m=$ $(q-1)$ simultaneously	$\frac{1}{18}(q-1)(q-4)$	$\frac{1}{3}r^2$
$C_7^{(k)}$	$\begin{pmatrix} \rho^k & & \\ & \sigma^{-k} & \\ & & \sigma^{-qk} \end{pmatrix}$	$k=1,\cdots,\frac{1}{3}(q^2-1)$ $k\neq 0 \pmod{q+1}$ $C^{(k)}=C^{(kq)} \pmod{\frac{1}{3}(q^2-1)}$	$\frac{1}{6}q(q-1)$	$\frac{1}{3}rs$
$C_8^{(k)}$	$\begin{pmatrix} \delta^k & & \\ & \delta^{kq} & \\ & & \delta^{kq^2} \end{pmatrix}$	$k=1,\cdots,(\frac{p}{3}-1)$ $C^{(k)}=C^{(kq)}=C^{(kq^2)}$	$\frac{1}{9}(q+2)(q-1)$	$\frac{1}{3}p$
C_9	$\begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$		1	r^2

ρ,σ = primitive elements of $GF(q)$, $GF(q^2)$ respectively

$\theta \in GF(q)$, $\theta^3 \neq 1$, $\delta \in GF(q^3)$, $\delta^{q^2-1}=1$, $\omega^3=1$, $\rho=\sigma^{q+1}$

$r=q-1$, $s=q+1$, $p=q^2+q+1$

Table 19 (cont'd.)

char. / class	ψ_1	ψ_{qs}	ψ_{q^3}	$\psi_p^{(t)}$	$\psi_{qp}^{(t)}$	$\psi_{rp}^{(t)}$	$\psi_{sp}^{(t,u,v)}$	$\psi_{r^2s}^{(t)}$	$\psi_{sp/3}$
parameters				$t=1,\cdots,\frac{1}{3}(q-4)$		$t=1,\cdots,\frac{1}{3}(q^2-1)$ $t\not\equiv 0 \bmod q+1$ $(t)=(tq)$ $\bmod \frac{1}{3}(q^2-1)$	$t,u,v=1,\cdots,q-1$ $t<u<v$ $t+u+v\equiv 0 \pmod{q-1}$ $t\not\equiv t'$, $u\not\equiv u'$, $v\not\equiv v'$, $(\bmod \frac{q-1}{3})$ simultaneously $t\neq\alpha$, $u\neq t+{}^{h}/_{3}$, $v\neq t+{}^{2h}/_{3}$ simultaneously $\alpha=1,\cdots q-1$	$t=1,\cdots,\frac{1}{3}p-1$ $(t)=(tq)=(tq^2)$ $(\bmod \frac{1}{3}p)$	permute the values on classes $C_3', C_3'' C_3'''$
number of characters	1	1	1	$\frac{1}{3}(q-4)$	$\frac{1}{3}(q-4)$	$\frac{1}{6}q(q-1)$	$\frac{1}{18}(q-1)(q-4)$	$\frac{1}{9}(q+2)(q-1)$	3
C_1	1	qs	q^3	p	qp	rp	sp	r^2s	$\frac{1}{3}sp$
C_2	1	q	0	s	q	-1	2q+1	-r	$\frac{1}{3}(2q+1)$
C_3'	1	0	0	1	0	-1	1	1	$\frac{1}{3}(2q+1)$
C_3''	1	0	0	1	0	-1	1	1	$-\frac{1}{3}r$
C_3'''	1	0	0	1	0	-1	1	1	$-\frac{1}{3}r$
$C_4^{(k)}$	1	s	q	$s\varepsilon^{3tk}+\varepsilon^{-6tk}$	$s\varepsilon^{3tk}+q\varepsilon^{-6tk}$	$r\varepsilon^{3tk}$	$s(\varepsilon^{-3vk}+\varepsilon^{-3uk}+\varepsilon^{-3tk})$	0	s
$C_5^{(k)}$	1	1	0	$\varepsilon^{3tk}+\varepsilon^{-6tk}$	ε^{3tk}	$-\varepsilon^{3tk}$	$(\varepsilon^{-3vk}+\varepsilon^{-3uk}+\varepsilon^{-3tk})$	0	1
$C_6^{(k,l,m)}$	1	2	1	$\varepsilon^{3tk}+\varepsilon^{3tl}+\varepsilon^{3tm}$	$\varepsilon^{3tk}+\varepsilon^{3tl}+\varepsilon^{3tm}$	0	$\sum_{[t,u,v]}\varepsilon^{(tk+ul+vm)}$	0	-1 or 2
$C_7^{(k)}$	1	0	-1	ε^{3tk}	$-\varepsilon^{3tk}$	$-\eta^{3tk}-\eta^{3qtk}$	0	0	0
$C_8^{(k)}$	1	-1	1	0	0	0	0	$\gamma^{tk}+\gamma^{tkq}+\gamma^{tkq^2}$	0
C_9	1	2	1	3	3	0	6 or -3	0	-1 or 2

$r = q-1$ $s = q+1$ $p = q^2+q+1$ $\omega^3 = 1$ $\varepsilon^{(q-1)} = 1$ $\eta^{q^2-1} = 1$ $\gamma^{p/3} = 1$

$\sum_{(x,y,z)}$ means a sum over the cyclic permutations of x,y,z, $\sum_{[x,y,z]}$ means a sum over all permutations of x,y,z

XII. CHARACTER TABLE FOR PSL(4,q) $d = 1$

In his thesis, [20], Steinberg calculated the character table for $GL(4,q)$. This table has 22 types of classes and characters and many of the entries are quite complicated. Due to its size the table will not be reproduced here.

Since $SL(4,q) \cong PSL(4,q)$ for $d = 1$ there are no class or character splittings and no need to calculate inner products. In Section 12.1 the parameters on the canonical representatives are determined. In Section 12.2 a single example is given to show how these parameters are used to restrict the characters of GL down to PSL. In many cases an unnecessary parameter on the character entries can be eliminated by a suitable substitution; however, such a substitution is in some cases not readily apparent.

12.1 Conjugacy Class Structure

Considering only the diagonal elements of the canonical representatives of $GL(4,q)$, we have eleven types with which to work.

(i) $\det \begin{pmatrix} \rho^k & & & \\ & \rho^k & & \\ & & \rho^k & \\ & & & \rho^k \end{pmatrix} = 1$ implies $4k = 0 (\text{mod } q-1)$. But 2,4 do not divide $(q-1)$ so $k = (q-1)$ is the only possible value. Thus for $C_1^{(k)}, \ldots, C_5^{(k)}$ $k = (q-1)$.

(ii) $\det \begin{pmatrix} \rho^k & & & \\ & \rho^k & & \\ & & \rho^k & \\ & & & \rho^\ell \end{pmatrix} = 1$ implies $(3k + \ell) \equiv 0 (\text{mod } q-1)$ thus $\ell = -3k$. Since $k = 1, \ldots, (q-1) - 1$ there are

$(q-2)$ classes of each of the class types $C_6^{(k)}$, $C_7^{(k)}$, $C_8^{(k)}$.

(iii) $\det \begin{pmatrix} \rho^k & & & \\ & \rho^k & & \\ & & \rho^\ell & \\ & & & \rho^\ell \end{pmatrix} = 1$ implies $(2k + 2\ell) \equiv 0 (\mathrm{mod}\ q-1)$

so $\ell = -k$. Since $C^{(k)} = C^{(-k)}$ we see that $k = 1,\ldots,\frac{q-2}{2}$ and so there are $\frac{1}{2}(q-2)$ classes of each type $C_9^{(k)}$, $C_{10}^{(k)}$, $C_{11}^{(k)}$.

(iv) $\det \begin{pmatrix} \rho^k & & & \\ & \rho^k & & \\ & & \rho^\ell & \\ & & & \rho^m \end{pmatrix} = 1$ implies $(2k + \ell + m) \equiv 0 (\mathrm{mod}\ q-1)$.

We thus get $m = -(2k + \ell)$. There are $(q-1)(q-2)$ ways to select k,ℓ. But $-(2k + \ell) = k$ (which gives a class of type C_6) for $(q-2)$ choices of k,ℓ so these values must be discarded. Thus the number of classes equals

$\frac{1}{2}[(q-1)(q-2) - (q-2)] - \frac{1}{2}(q-2) = \frac{1}{2}(q-2)(q-3)$.

(v) $\det \begin{pmatrix} \rho^k & & & \\ & \rho^\ell & & \\ & & \rho^m & \\ & & & \rho^n \end{pmatrix} = 1$ implies $(k + \ell + m + n) \equiv 0 (\mathrm{mod}\ q-1)$.

Since $k < \ell < m < n$ $k,\ell,m,n = 1,\ldots,q-1$ we get $\frac{1}{24}(q-2)(q-3)(q-4)$ classes of type $C_{14}^{(k,\ell,m,n)}$.

(vi) $\det \begin{pmatrix} \rho^k & & & \\ & \rho^k & & \\ & & \sigma^\ell & \\ & & & \sigma^{q\ell} \end{pmatrix} = 1$ implies $(2k + \ell) = 0 (\mathrm{mod}\ q-1)$.

Thus $\ell = -2k$ and there are $\frac{1}{2}\, q(q-1)$ classes of type $C_{15}^{(k)}$, $C_{16}^{(k)}$.

(vii) $\det \begin{pmatrix} \rho^k & & & \\ & \rho^\ell & & \\ & & \sigma^m & \\ & & & \sigma^{mq} \end{pmatrix} = 1$ implies $(k + \ell + m) \equiv 0 (\mathrm{mod}\ q-1)$.

Thus $m = -(k + \ell)$ and $k,\ell = 1,\ldots,q^2-1$, $k \neq \ell$ and $k + \ell \neq$ multiple $(q+1)$. This gives $\frac{1}{4}\, q(q-1)(q-2)$ classes of type $C_{17}^{(k,\ell)}$.

(viii) $\det \begin{pmatrix} \sigma^k & & & \\ & \sigma^{kq} & & \\ & & \sigma^k & \\ & & & \sigma^{kq} \end{pmatrix} = 1$ implies $(2k) \equiv 0 (\mathrm{mod}\ q-1)$. But $2 \nmid (q-1)$ so $k \equiv 0\ \mathrm{mod}(q-1)$. Since $k = 1,\ldots,q^2-1$ and $k \not\equiv 0\ \mathrm{mod}(q+1)$ there are q values which are multiples of $(q-1)$. Since $C^{(k)} = C^{(kq)}$ we get $\frac{1}{2}\, q$ classes of type

$C_{18}^{(k)}, C_{19}^{(k)}$.

(ix) $\det\begin{pmatrix}\sigma^k & & & \\ & \sigma^{kq} & & \\ & & \sigma^{\ell} & \\ & & & \sigma^{\ell q}\end{pmatrix} = 1$ implies $(k+\ell) \equiv 0 \pmod{q-1}$ so $\ell = -k$. We get $\frac{1}{8}q(q-2)(q+1)$ classes of type $C_{20}^{(k)}$.

(x) $\det\begin{pmatrix}\rho^k & & & \\ & \tau^{\ell} & & \\ & & \tau^{\ell q} & \\ & & & \tau^{\ell q^2}\end{pmatrix} = 1$ implies $(k+\ell) \equiv 0 \pmod{q-1}$ so $\ell = -k$. We get $\frac{1}{3}q(q-1)(q+1)$ classes of type $C_{21}^{(k)}$.

(xi) $\det\begin{pmatrix}\omega^k & & & \\ & \omega^{kq} & & \\ & & \omega^{kq^2} & \\ & & & \omega^{kq^3}\end{pmatrix} = 1$ implies $(q^3+q^2+q+1)k \equiv 0 \pmod{q-1}$ which means $k = 0 \bmod (q-1)$. There are $\frac{1}{4}q^2(q+1)$ classes of type $C_{22}^{(k)}$.

These results are tabulated in Table 20.

12.2 Calculation of Characters

There are 22 types of Jordan canonical forms for $PSL(4,q)$ and if we look only at the diagonal of these types, as above, there are only 11 types to consider. In the same manner we can consider the characters of PSL to fall into 22 sets consisting of characters of the same degree. However some of these sets are related in the sense that the character parameters t,u,v,w run over the same range, e.g. $\chi_{sf}^{(t)}, \chi_{qs^2f}^{(t)}, \chi_{q^3sf}^{(t)}$ all contain $(q-2)$ characters. There is a rather close resemblance among these sets of characters and they behave in much the same way upon restriction to SL. Thus, in a sense, there are only eleven different type characters to handle. The details of restricting GL to SL are not of sufficient interest to warrent writing out more than one example calculation.

We consider the character $\chi_{sfp}^{(t,u,v)}$ of $GL(4,q)$. Each entry $\chi_{sfr}(g_i)$ is understood to be the value associated with the corresponding class C_i, $i = 1,\ldots,22$. As before, $s = q+1$, $r = q-1$, $f = q^2+1$, $p = q^2 + q+1$.

$$\chi_{sfp}^{(t,u,v)} = sfp\ \epsilon^{(v+u+2t)k} \qquad \text{where } t,u,v = 1,\ldots,(q-1)$$

$$(q^3 + 3q^2 + 2q+1)\epsilon^{(\ \ )k} \qquad t < u < v$$

$$s^2\epsilon^{(\ \ )k} \qquad \chi^{(t,u,v)} = \chi^{(t,v,u)}$$

$$(2q+1)\epsilon^{(\ \ )k}$$

$$\epsilon^{(\ \ )k}$$

$$sp\ \epsilon^{(t+u+v)k+t\ell} + p \sum_{(u,v)} \epsilon^{(v+2t)k+u\ell}$$

$$(2q+1)\epsilon^{(\ \ )k+t\ell} + s \sum_{(u,v)} \epsilon^{(\ \ )k+u\ell}$$

$$\epsilon^{(\ \ )k+t\ell} + \Sigma_{(u,v)}\epsilon^{(\ \ )k+u\ell}$$

$$s^2 \sum_{(u,v)} \epsilon^{(v+t)k+(u+t)\ell} + s \sum_{(k,\ell)} \epsilon^{(v+u)k+2t\ell}$$

$$s[\sum_{(u,v)} \epsilon^{(v+t)k+(u+t)\ell} + \epsilon^{2tk+(v+u)\ell}] + \epsilon^{(v+u)k+2t\ell}$$

$$\Sigma_{(u,v)}\epsilon^{(\ \ )k+(\ \ )\ell} + \Sigma_{(k,\ell)}\epsilon^{(\ \ )k+q+\ell}$$

$$s\ \Sigma_{(u,v)}(\epsilon^{[(v+t)k+u\ell+tm]} + \epsilon^{[(v+t)k+t\ell+um]})$$

$$+ s\ \epsilon^{(v+u)k+t\ell+tm} + \sum_{(u,v)} \epsilon^{2t+k+v\ell+um}$$

$$\Sigma_{(u,v)}(\qquad\qquad) + \epsilon^{[\qquad]} + \Sigma\ \epsilon^{[\qquad]}$$

$$\Sigma_{[k,\ell,m,n]}\epsilon^{vk+u\ell+tm+tn}$$

$$s\ \epsilon^{(v+u)k+t\ell}$$

$$\epsilon^{(\ \ )k+t\ell}$$

$$\epsilon^{vk+u\ell+tm} + \epsilon^{uk+v\ell+tm}$$

$\downarrow 0$

We now use the relations between the various class parameters to restrict χ_{sfp} to SL. For classes $C_1^{(k)},\ldots,C_5^{(k)}$ $k = q-1$; for $C_6^{(k,\ell)}\ldots C_8^{(k,\ell)}$ $\ell = -3k$; for $C_9^{(k,\ell)}\ldots C_{11}^{(k,\ell)}$ $\ell = -k$; for $C_{12}^{(k,\ell,m)}$, $C_{13}^{(k,\ell,m)}$ $m = -(2k+\ell)$; for $C_{14}^{(k,\ell,m,n)}$ $n = -(k+\ell+m)$; for $C_{15}^{(k,\ell)}$, $C_{16}^{(k,\ell)}$ $\ell = -2k$; and for $C_{17}^{(k,\ell,m)}$ $m = -(k+\ell)$. Since $\chi_{sfp}(g) = 0$ for $C_{18},\ldots,C_{22}$ we need not be concerned with the parameters for these classes. Using the above relations and keeping in mind that $\epsilon^{q-1} = 1$ we obtain the following restricted character.

$$\chi_{sfp}^{(t,u,v)}\big|_{SL} = sfp$$

$$(q^3 + 3q^2 + 2q + 1)$$

$$s^2$$

$$(2q+1)$$

$$\frac{1}{sp\ \epsilon^{(u+v-2t)k} + p\ \Sigma_{(u,v)}\epsilon^{(v-3u+2t)k}}$$

$$(2q+1)\epsilon^{[\quad]} + s\ \Sigma\ \epsilon^{[\quad]}$$

$$\frac{\epsilon^{[\quad]} + \Sigma\ \epsilon^{[\quad]}}{s^2\Sigma_{(u,v)}\epsilon^{(v-u)k} + s[\epsilon^{(v+u-2t)k} + \epsilon^{-(v+u-2t)k}]}$$

$$s(\Sigma\ \epsilon^{[\quad]} + \epsilon^{(v+u-2t)k}) + \epsilon^{-(\quad)}$$

$$\frac{\Sigma\ \epsilon^{[\quad]} + \epsilon^{[\quad]} + \epsilon^{-[\quad]}}{s\ \Sigma_{(u,v)}(\epsilon^{(v-t)k+(u-t)\ell} + \epsilon^{(v+t-2u)k+(t-u)\ell}) + s\ \epsilon^{(v+u-2t)k} + \Sigma_{(u,v)}\epsilon^{2(t-u)k+(v-u)\ell}}$$

$$\frac{\Sigma(\quad) + \epsilon^{[\quad]} + \Sigma\ \epsilon^{[\quad]}}{\Sigma_{(u,v)}\epsilon^{(v-t)k+(u-t)\ell} + \epsilon^{(v-t)k+(u-t)m} + \epsilon^{(v-t)\ell+(u-t)m} + \Sigma_{(k,\ell,m)}\epsilon^{(u-v)\ell-(v-t)(k+m)} + \Sigma_{(k,\ell,m)}\epsilon^{(v-u)\ell-(v-t)(k+m)}}$$

$$\frac{}{s\ \epsilon^{(v+u-2t)k}}$$

$$\begin{array}{l} \epsilon^{(\ \ )k} \\ \hline \epsilon^{(v-t)k+(u-t)\ell} + \epsilon^{(u-t)k+(v-t)\ell} \\ \hline 0 \\ \vdots \\ 0 \end{array}$$

We now observe that the substitution $t' = (u-t)$, $u' = (v-t)$ will eliminate one of the character parameters and simplify the character.

$$\chi_{sfp}^{(t',u')}\Big|_{SL} = \Psi_{sfp}^{(t,u)} = \begin{array}{l} sfp \\ (q^3 + 3q^2 + 2q+1) \\ s^2 \\ (2q+1) \\ 1 \\ \hline sp\ \epsilon^{(u+t)k} + p\ \Sigma_{(u,t)}\epsilon^{(u-3t)k} \\ (2q+1)\epsilon^{(u+t)k} + s\ \Sigma\ \epsilon^{(u-3t)k} \\ \epsilon^{(u+t)k} + s\ \Sigma\ \epsilon^{(\ \ )k} \\ \hline s^2\ \Sigma\ \epsilon^{(u-t)k} + s[\epsilon^{(u+t)k} + \epsilon^{-(u+t)k}] \\ s\ \Sigma\ \epsilon^{(u-t)k} + s\ \epsilon^{(u+t)k} + \epsilon^{-(u+t)k} \\ \Sigma_{(u,t)}\epsilon^{(u-t)k} + \epsilon^{(u+t)k} + \epsilon^{-(u+t)k} \\ \hline s\ \Sigma_{(t,u)}(\epsilon^{uk+t\ell} + \epsilon^{(u-2t)k+t\ell}) + s\ \epsilon^{(u+t)k} \\ \qquad + \Sigma_{(u,t)}\epsilon^{(u-t)\ell-2tk} \\ \Sigma_{(t,u)}(\epsilon^{uk+t\ell} + \epsilon^{(u-2t)k-t\ell}) + \epsilon^{(u+t)k} \\ \qquad + \Sigma_{(u,t)}\epsilon^{(u-t)\ell-2tk} \\ \hline \end{array} \qquad \text{where } u,t = 1,\ldots,(q-2)$$

$$\frac{\Sigma_{(u,t)}\epsilon^{uk+t\ell} + \epsilon^{uk+tm} + \epsilon^{u\ell+tm} + \Sigma_{(k,\ell,m)}\epsilon^{(t-u)\ell-u(k+m)} + \Sigma_{(k,\ell,m)}\epsilon^{(u-t)\ell-u(k+m)}}{s\ \epsilon^{(u+t)k}}$$

$$\frac{\epsilon^{(u+t)k}}{\Sigma_{(u,t)}\epsilon^{uk+t\ell}}$$

$$0$$

$$0$$

$$0$$

$$0$$

$$0$$

The remaining characters of PSL(4,q) d = 1 are developed in a similar manner. The complete character table appears in Table 20.

Table 20. Character Table for PSL(4,q) d = 1

conj. class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$		1	$q^6r^3s^2fp$
C_2	$\begin{pmatrix}1&&&\\1&1&&\\&&1&\\&&&1\end{pmatrix}$		1	q^6r^2sp
C_3	$\begin{pmatrix}1&&&\\1&1&&\\&&1&\\&&1&1\end{pmatrix}$		1	q^5rs
C_4	$\begin{pmatrix}1&&&\\1&1&&\\&1&1&\\&&&1\end{pmatrix}$		1	q^4r
C_5	$\begin{pmatrix}1&&&\\1&1&&\\&1&1&\\&&1&1\end{pmatrix}$		1	q^3
$C_6^{(k)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^k&&\\&&\rho^k&\\&&&\rho^{-3k}\end{pmatrix}$	$k=-3k$ $k=1,\cdots,q-2$	$q-2$	q^3r^3sp
$C_7^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\rho^k&\\&&&\rho^{-3k}\end{pmatrix}$			q^3r^2
$C_8^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&1&\rho^k&\\&&&\rho^{-3k}\end{pmatrix}$			q^2r
$C_9^{(k)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^k&&\\&&\rho^{-k}&\\&&&\rho^{-k}\end{pmatrix}$	$C^{(k)}=C^{(-k)}$ $k=1,\cdots\frac{1}{2}(q-2)$ $k=-k$	$\frac{1}{2}(q-2)$	$q^2r^3s^2$
$C_{10}^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\rho^{-k}&\\&&&\rho^{-k}\end{pmatrix}$	$C^{(k)}\neq C^{(-k)}$ $k=1,\cdots,q-2$ $k=-k$	$q-2$	q^2r^2s
$C_{11}^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\rho^{-k}&\\&&1&\rho^{-k}\end{pmatrix}$	$C^{(k)}=C^{(-k)}$ $k=1,\cdots,\frac{1}{2}(q-2)$ $k=-k$	$\frac{1}{2}(q-2)$	q^2r
$C_{12}^{(k,\ell)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^k&&\\&&\rho^\ell&\\&&&\rho^{-(2k+\ell)}\end{pmatrix}$	$k,\ell=1,\cdots,q-1$ $k\neq\ell \pmod{q-1}$ $C^{(\ell)}=C^{(-2k-\ell)}$	$\frac{1}{2}(q-2)(q-3)$	qr^3s
$C_{13}^{(k,\ell)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\rho^\ell&\\&&&\rho^{-(2k+\ell)}\end{pmatrix}$			qr^2
$C_{14}^{(k,\ell,m,n)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^\ell&&\\&&\rho^m&\\&&&\rho^n\end{pmatrix}$	$k,\ell,m,n=1,\cdots,q-1$ $k<\ell<m<n$ $k+\ell+m+n\equiv 0 \pmod{q-1}$	$\frac{1}{24}(q-2)(q-3)(q-4)$	r^3
$C_{15}^{(k)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^k&&\\&&\sigma^{-2k}&\\&&&\sigma^{-2kq}\end{pmatrix}$	$k=1,\cdots,q^2-1$ $k\neq 0 \pmod{q+1}$ $C^{(k)}=C^{(kq)}$	$\frac{1}{2}q(q-1)$	qr^2s^2
$C_{16}^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\sigma^{-2k}&\\&&&\sigma^{-2kq}\end{pmatrix}$			qrs
$C_{17}^{(k,\ell)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^\ell&&\\&&\sigma^{-(k+\ell)}&\\&&&\sigma^{-(k+\ell)q}\end{pmatrix}$	$k,\ell=1,\cdots,q^2-1$ $k\neq\ell$ $k+\ell\neq 0 \pmod{q+1}$	$\frac{1}{4}q(q-1)(q-2)$	r^2s

ρ,σ,τ,ω = primitive elements of $GF(q)$, $GF(q^2)$, $GF(q^3)$, $GF(q^4)$ resp.

$\rho=\sigma^{q+1}=\tau^{q^2+q+1}=\omega^{(q+1)(q^2+1)}$ $r=q-1$, $s=q+1$, $f=q^2+1$, $p=q^2+q+1$

Table 20 (cont'd.)

conj. class	canonical representative	parameters	number of classes	centralizer order
$C_{18}^{(k)}$	$\begin{pmatrix} \sigma^k & & & \\ & \sigma^{kq} & & \\ & & \sigma^k & \\ & & & \sigma^{kq} \end{pmatrix}$	$k=1,\cdots,q^2-1$ $k \neq 0 \pmod{q+1}$ $k = 0 \pmod{q-1}$	$\frac{1}{2}q$	q^2rs^2f
$C_{19}^{(k)}$	$\begin{pmatrix} \sigma^k & & & \\ 1 & \sigma^{kq} & & \\ & 1 & \sigma^k & \\ & & & \sigma^{kq} \end{pmatrix}$			q^2s
$C_{20}^{(k)}$	$\begin{pmatrix} \sigma^k & & & \\ & \sigma^{kq} & & \\ & & \sigma^{-k} & \\ & & & \sigma^{-kq} \end{pmatrix}$	$k=1,\cdots,q^2-1$ $k \neq 0 \pmod{q+1}$ $k \neq -k \pmod{q^2-1}$	$\frac{1}{8}q(q-2)(q+1)$	rs^2
$C_{21}^{(k)}$	$\begin{pmatrix} \rho^k & & & \\ & \tau^{-k} & & \\ & & \tau^{-kq} & \\ & & & \tau^{-kq^2} \end{pmatrix}$	$k=1,\cdots,q^3-1$ $k \neq 0 \pmod{q^2+q+1}$ $C^{(k)} = C^{(kq)} = C^{(kq^2)}$	$\frac{1}{3}q(q-1)(q+1)$	rp
$C_{22}^{(k)}$	$\begin{pmatrix} \omega^k & & & \\ & \omega^{kq} & & \\ & & \omega^{kq^2} & \\ & & & \omega^{kq^3} \end{pmatrix}$	$k=(q-1),2(q-1),\cdots,q^4-1$ $k \neq 0 \pmod{q^2+1}$ $C^{(k)} = C^{(kq)} = C^{(kq^2)} = C^{(kq^3)}$	$\frac{1}{4}q^2(q+1)$	sf

$$\epsilon^{q-1} = 1,\ \eta^{q^2-1} = 1,\ \theta^{q+1} = 1,\ \tau^{q^3-1} = 1,\ \phi^{(q+1)(q^2+1)} = 1$$

$$\tau^{q^2+q+1} = \eta^{q+1} = \epsilon,\ \eta^{q-1} = \theta$$

Table 20 (cont'd.)

								t=1,···,(q-2)
number of char.	1	1	1	1	1	q-2	q-2	q-2
char. / class	ψ_1	ψ_{qp}	ψ_{q^3p}	ψ_{q^2f}	ψ_{q^6}	$\psi_{sf}^{(t)}$	$\psi_{qs^2f}^{(t)}$	$\psi_{q^3sf}^{(t)}$
C_1	1	qp	q^3p	q^2f	q^6	sf	qs^2f	q^3sf
C_2	1	qs	q^3	q^2	0	p	qs^2	q^3
C_3	1	q	0	q^2	0	s	qs	0
C_4	1	q	0	0	0	s	q	0
C_5	1	0	0	0	0	1	0	0
$C_6^{(k)}$	1	p	qp	qs	q^3	$p\varepsilon^{tk}+\varepsilon^{-3tk}$	$sp\varepsilon^{tk}+qs\varepsilon^{-3tk}$	$qp\varepsilon^{tk}+q^3\varepsilon^{-3tk}$
$C_7^{(k)}$	1	s	q	q	0	$s\varepsilon^{tk}+\varepsilon^{-3tk}$	$(2q+1)\varepsilon^{tk}+q\varepsilon^{-3tk}$	$q\varepsilon^{tk}$
$C_8^{(k)}$	1	1	0	0	0	$\varepsilon^{tk}+\varepsilon^{-3tk}$	ε^{tk}	0
$C_9^{(k)}$	1	$(2q+1)$	$q(q+2)$	f	q^2	$s(\varepsilon^{tk}+\varepsilon^{-tk})$	$s^2(\varepsilon^{tk}+\varepsilon^{-tk})$	$qs(\varepsilon^{tk}+\varepsilon^{-tk})$
$C_{10}^{(k)}$	1	s	q	1	0	$\varepsilon^{tk}+s\varepsilon^{-tk}$	$s(\varepsilon^{tk}+\varepsilon^{-tk})$	$q\varepsilon^{tk}$
$C_{11}^{(k)}$	1	1	0	1	0	$\varepsilon^{tk}+\varepsilon^{-tk}$	$\varepsilon^{tk}+\varepsilon^{-tk}$	0
$C_{12}^{(k,\ell)}$	1	$(q+2)$	$(2q+1)$	6	q	$s\varepsilon^{tk}+\varepsilon^{t\ell}+\varepsilon^{-t(2k+\ell)}$	$2s\varepsilon^{tk}+s(\varepsilon^{t\ell}+\varepsilon^{-t(2k+\ell)})$	$s\varepsilon^{tk}+q(\varepsilon^{t\ell}+\varepsilon^{-t(2k+\ell)})$
$C_{13}^{(k,\ell)}$	1	2	1	1	0	$\varepsilon^{tk}+\varepsilon^{t\ell}+\varepsilon^{-t(2k+\ell)}$	$2\varepsilon^{tk}+(\varepsilon^{t\ell}+\varepsilon^{-t(2k+\ell)})$	ε^{tk}
$C_{14}^{(k,\ell,m,n)}$	1	3	3	2	1	$\varepsilon^{tk}+\varepsilon^{t\ell}+\varepsilon^{tm}+\varepsilon^{tn}$	$2(\varepsilon^{tk}+\varepsilon^{t\ell}+\varepsilon^{tm}+\varepsilon^{tn})$	$\varepsilon^{tk}+\varepsilon^{t\ell}+\varepsilon^{tm}+\varepsilon^{tn}$
$C_{15}^{(k)}$	1	q	-1	$-r$	$-q$	$s\varepsilon^{tk}$	0	$-s\varepsilon^{tk}$
$C_{16}^{(k)}$	1	0	-1	1	0	ε^{tk}	0	$-\varepsilon^{tk}$
$C_{17}^{(k,\ell)}$	1	1	-1	0	-1	$\varepsilon^{t\ell}+\varepsilon^{tk}$	0	$-(\varepsilon^{tk}+\varepsilon^{t\ell})$
$C_{18}^{(k)}$	1	-1	$-q^2$	f	q^2	0	0	0
$C_{19}^{(k)}$	1	-1	0	1	0	0	0	0
$C_{20}^{(k)}$	1	-1	-1	2	1	0	0	0
$C_{21}^{(k)}$	1	0	0	-1	1	ε^{tk}	$-\varepsilon^{tk}$	ε^{tk}
$C_{22}^{(k)}$	1	-1	1	0	-1	0	0	0

Table 20 (cont'd.)

$\psi^{(t)}=\psi^{(-t)}$	$t=1,\cdots,(q-2)$ $\psi^{(t)}\neq\psi^{(-t)}$	$\psi^{(t)}=\psi^{(-t)}$
$\frac{1}{2}(q-2)$	$q-2$	$\frac{1}{2}(q-2)$
$\psi^{(t)}_{fp}$	$\psi^{(t)}_{qfp}$	$\psi^{(t)}_{q^2fp}$
fp	qfp	q^2fp
$(2q^2+q+1)$	qp	q^2s
p	q	q^2
s	q	0
1	0	0
$p(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$	$qp\varepsilon^{2\kappa t}+p\varepsilon^{-2\kappa t}$	$qp(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$
$s(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$	$q\varepsilon^{2\kappa t}+s\varepsilon^{-2\kappa t}$	$q(\quad)$
$\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t}$	$\varepsilon^{-2\kappa t}$	0
$s^2+(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$	$s^2+q(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$	$s^2+q^2(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$
$s+(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$	$s+q\varepsilon^{-2\kappa t}$	s
$1+(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$	1	1
$s(\varepsilon^{t(\kappa+\ell)}+\varepsilon^{-t(\kappa+\ell)})+(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$	$s(\varepsilon^{t(\kappa+\ell)}+\varepsilon^{-t(\kappa+\ell)})+q\varepsilon^{-2\kappa t}+\varepsilon^{2\kappa t}$	$s(\varepsilon^{t(\kappa+\ell)}+\varepsilon^{-t(\kappa+\ell)})+q(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$
$(\quad)+(\quad)$	$(\quad)+\varepsilon^{2\kappa t}$	$\varepsilon^{t(\kappa+\ell)}+\varepsilon^{-t(\kappa+\ell)}$
$\varepsilon^{t(\kappa+\ell)}+\varepsilon^{t(\kappa+m)}+\varepsilon^{t(\kappa+n)}+\varepsilon^{-t(\kappa+\ell)}+\varepsilon^{-t(\kappa+m)}+\varepsilon^{-t(\kappa+n)}$	$\varepsilon^{t(\kappa+\ell)}+\varepsilon^{t(\kappa+m)}+\varepsilon^{t(\kappa+n)}+\varepsilon^{-t(\kappa+\ell)}+\varepsilon^{-t(\kappa+m)}+\varepsilon^{-t(\kappa+n)}$	$\varepsilon^{t(\kappa+\ell)}+\varepsilon^{t(\kappa+m)}+\varepsilon^{t(\kappa+n)}$ $\varepsilon^{-t(\kappa+\ell)}+\varepsilon^{-t(\kappa+m)}+\varepsilon^{-t(\kappa+n)}$
$\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t}$	$q\varepsilon^{2\kappa t}-\varepsilon^{-2\kappa t}$	$-q(\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t})$
$\varepsilon^{2\kappa t}+\varepsilon^{-2\kappa t}$	$-\varepsilon^{-2\kappa t}$	0
$\varepsilon^{t(\kappa+\ell)}+\varepsilon^{-t(\kappa+\ell)}$	$\varepsilon^{t(\kappa+\ell)}-\varepsilon^{-t(\kappa+\ell)}$	$-\varepsilon^{t(\kappa+\ell)}-\varepsilon^{-t(\kappa+\ell)}$
f	$-f$	f
1	1	1
$\varepsilon^{t\kappa}+\varepsilon^{-t\kappa}$	$-\varepsilon^{t\kappa}-\varepsilon^{-t\kappa}$	$\varepsilon^{t\kappa}+\varepsilon^{-t\kappa}$
0	0	0
0	0	0

Table 20 (cont'd.)

$t,u=1,\cdots,(q-2)$ $t<u$	
$\frac{1}{2}(q-2)(q-3)$	$\frac{1}{2}(q-2)(q-3)$
$\psi_{sfp}^{(t,u)}$	$\psi_{qsfp}^{(t,u)}$
sfp	$qsfp$
q^3+3q^2+2q+1	$q(2q^2+2q+1)$
s^2	qs
$2q+1$	q
1	0
$sp\varepsilon^{(t+u)\kappa}+p\sum_{(u,t)}\varepsilon^{(t-3u)\kappa}$	$sp\varepsilon^{(t+u)\kappa}+qp\sum_{(u,t)}\varepsilon^{(t-3u)\kappa}$
$(2q+1)\varepsilon^{(t+u)\kappa}+s\sum_{(u,t)}\varepsilon^{(t-3u)\kappa}$	$(2q+1)\varepsilon^{(t+u)\kappa}+q\sum_{(u,t)}\varepsilon^{(t-3u)\kappa}$
$\varepsilon^{(t+u)\kappa}+\sum_{(u,t)}\varepsilon^{(t-3u)\kappa}$	$\varepsilon^{(t+u)\kappa}$
$s^2(\varepsilon^{(u-t)\kappa}+\varepsilon^{-(u-t)\kappa})$ $+s(\varepsilon^{(u+t)\kappa}+\varepsilon^{-(u+t)\kappa})$	$s^2(\varepsilon^{(u-t)\kappa}+\varepsilon^{-(u-t)\kappa})$ $+qs(\varepsilon^{(u+t)\kappa}+\varepsilon^{-(u+t)\kappa})$
$s(\varepsilon^{(u-t)\kappa}+\varepsilon^{-(u-t)\kappa})$ $+s\varepsilon^{-(u+t)\kappa}+\varepsilon^{(u+t)\kappa}$	$s(\quad)$ $+q\varepsilon^{(u+t)\kappa}$
$(\quad)+(\quad)$	$\varepsilon^{(u-t)\kappa}+\varepsilon^{-(u-t)\kappa}$
$s\sum_{(u,t)}(\varepsilon^{t\kappa+u\ell}+\varepsilon^{(t-2u)\kappa-u\ell})$ $+s\varepsilon^{(t+u)\kappa}+\sum_{(u,t)}\varepsilon^{-2u\kappa+(t-u)\ell}$	$s\sum_{(u,t)}(\varepsilon^{t\kappa+u\ell}+\varepsilon^{(t-2u)\kappa-u\ell})$ $+s\varepsilon^{(t+u)\kappa}+q\sum_{(u,t)}\varepsilon^{-2u\kappa+(t-u)\ell}$
$\sum_{(u,t)}(\quad)$ $+\varepsilon^{(t+u)\kappa}+\sum_{(u,t)}\varepsilon^{-2u\kappa+(t-u)\ell}$	$\sum_{(u,t)}(\varepsilon^{t\kappa+u\ell}+\varepsilon^{(t-2u)\kappa-u\ell})+\varepsilon^{(t+u)\kappa}$
$\sum_{(u,t)}(\varepsilon^{u\kappa+t\ell}+\varepsilon^{u\kappa+tm}+\varepsilon^{u\ell+tm})$ $+\sum_{(\kappa,\ell,m)}\varepsilon^{(t-u)\ell-u(\kappa+m)}+\sum_{(\kappa,\ell,m)}\varepsilon^{(u-t)\ell-u(\kappa+m)}$	$\longrightarrow$
$s\varepsilon^{(u+t)\kappa}$	$-s\varepsilon^{(u+t)\kappa}$
$\varepsilon^{(u+t)\kappa}$	$-\varepsilon^{(u+t)\kappa}$
$\sum_{(\kappa,\ell)}\varepsilon^{t\kappa+u\ell}$	$-\sum_{(\kappa,\ell)}\varepsilon^{t\kappa+u\ell}$
0	0
0	0
0	0
0	0
0	0

Table 20 (cont'd.)

$t,u,v,w=1,\cdots,(q-1)$ $t<u<v<w$ $t+u+v+w \equiv 0 \pmod{q-1}$	$t=1,\cdots,q^2-1$ $t \not\equiv 0 \pmod{q+1}$ $\psi^{(t)} = \psi^{(qt)} \pmod{q^2-1}$	
$\frac{1}{24}(q-2)(q-3)(q-4)$	$\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q-1)$
$\psi_{s^2fp}^{(t,u,v,w)}$	$\psi_{rfp}^{(t)}$	$\psi_{qrfp}^{(t)}$
s^2fp	rfp	$qrfp$
$3q^3+5q^2+3q+1$	q^3-q^2-1	$-q$
$(2q+1)s$	$-f$	qr
$3q+1$	-1	$-q$
1	-1	0
$sp\sum \varepsilon_{(t,u,v,w)}^{(t+u+v+w)k}$	$rp\varepsilon^{tk}$	$rp\varepsilon^{tk}$
$(2q+1)\sum \varepsilon_{(t,u,v,w)}^{(\quad)k}$	$-\varepsilon^{tk}$	$-\varepsilon^{tk}$
$\sum \varepsilon_{(t,u,v,w)}^{(\quad)k}$	$-\varepsilon^{tk}$	$-\varepsilon^{tk}$
$s^2\sum \varepsilon_{[t,u,v,w]}^{(t+u-v-w)k}$	$r(\varepsilon^{tk}+\varepsilon^{-tk})$	$qr(\varepsilon^{tk}+\varepsilon^{-tk})$
$s\sum \varepsilon_{[t,u,v,w]}^{(\quad)k}$	$r\varepsilon^{-tk}-\varepsilon^{tk}$	$-q\varepsilon^{tk}$
$\sum \varepsilon_{[t,u,v,w]}^{(\quad)k}$	$-\varepsilon^{tk}-\varepsilon^{-tk}$	0
$s\sum \varepsilon_{[t,u,v,w]}^{(t+u-2w)k+(v-w)l}$	$r\varepsilon^{tk}$	$r\varepsilon^{tk}$
$\sum \varepsilon_{[t,u,v,w]}^{(\quad)k+(\quad)l}$	$-\varepsilon^{tk}$	$-\varepsilon^{tk}$
$\sum \varepsilon_{[k,l,m,n]}^{tk+ul+vm+wn}$	0	0
0	$r\varepsilon^{tk}-(\eta^{-2tk}+\eta^{-2tkq})$	$-r\varepsilon^{tk}-q(\eta^{-2tk}+\eta^{-2tkq})$
0	$-\varepsilon^{tk}-(\quad)$	ε^{tk}
0	$-\eta^{-(k+l)t}-\eta^{-(k+l)qt}$	$-\eta^{-(k+l)t}-\eta^{-(k+l)qt}$
0	$-f(\theta^{t}+\theta^{tq})$	$f(\theta^{t}+\theta^{tq})$
0	$-(\theta^{t}+\theta^{tq})$	$(\theta^{t}+\theta^{tq})$
0	$-\eta^{tk}-\eta^{tkq}-\eta^{-tk}-\eta^{-tkq}$	$\eta^{tk}+\eta^{tkq}+\eta^{-tk}+\eta^{-tkq}$
0	0	0
0	0	0

Table 20 (cont'd.)

$t=1,\cdots,q^2-1$ $u=1,\cdots,q-2$ $t\neq 0 \pmod{q+1}$ $\psi^{(t,u)}=\psi^{(qt,u)}$	$t=1,\cdots,q^2-1$ $t\neq 0 \pmod{q+1}$ $\psi^{(t)}=\psi^{(tq)}$	
$\frac{1}{2}q(q-1)(q-2)$	$\frac{1}{2}q$	$\frac{1}{2}q$
$\psi^{(t,u)}_{rsfp}$	$\psi^{(t)}_{r^2p}$	$\psi^{(t)}_{q^2r^2p}$
$rsfp$	r^2p	q^2r^2p
q^3-q^2-q-1	$-r$	$-q^2r$
$-s$	q^2-q+1	q^2
$-s$	$-r$	0
-1	1	0
$rp(\varepsilon^{(u+t)\kappa}+\varepsilon^{(t-3u)\kappa})$	0	0
$-(\varepsilon^{(u+t)\kappa}+\varepsilon^{(t-3u)\kappa})$	0	0
$-(\varepsilon^{(u+t)\kappa}+\varepsilon^{(t-3u)\kappa}$	0	0
$rs(\varepsilon^{(u-t)\kappa}+\varepsilon^{-(u-t)\kappa})$	r^2	r^2
$r\varepsilon^{(u-t)\kappa}-s\varepsilon^{-(u-t)\kappa}$	$-r$	$-r$
$-(\varepsilon^{(u-t)\kappa}+\varepsilon^{-(u-t)\kappa})$	1	1
$r\varepsilon^{(t\kappa+u\ell)}+r\varepsilon^{(t-2u)\kappa-u\ell}$	0	0
$-\varepsilon^{(t\kappa+u\ell)}-\varepsilon^{(t-2u)\kappa-u\ell}$	0	0
0	0	0
$-s\varepsilon^{u\kappa}(\eta^{-2t\kappa}+\eta^{-2t\kappa q})$	$-r(\theta^{t\kappa}+\theta^{t\kappa q})$	$-r(\theta^{t\kappa}+\theta^{t\kappa q})$
$-\varepsilon^{u\kappa}(\eta^{-2t\kappa}+\eta^{-2t\kappa q})$	$(\theta^{t\kappa}+\theta^{t\kappa q})$	$(\theta^{t\kappa}+\theta^{t\kappa q})$
$-(\varepsilon^{u\kappa}+\varepsilon^{u\ell})(\eta^{-(\kappa+\ell)t}+\eta^{-(\kappa+\ell)tq})$	0	0
0	$f+\frac{1}{2}[f(\theta^{2t}+\theta^{2tq})$ $-rs(\theta^{t(q^2+1)}+\theta^{tq(q^2+1)})]$	$f+\frac{1}{2}[f(\theta^{2t}+\theta^{2tq})$ $+rs(\theta^{t(q^2+1)}+\theta^{tq(q^2+1)})]$
0	$f+\frac{1}{2}[(\quad)-(\quad)]$	$f+\frac{1}{2}[(\quad)+(\quad)]$
0	$\theta^{-t\kappa}+\theta^{-t\kappa q}$	$\theta^{-t\kappa}+\theta^{-t\kappa q}$
0	0	0
0	$\frac{1}{2}(\theta^{t}+\theta^{tq}+\theta^{tq^2}+\theta^{tq^3})$	$-\frac{1}{2}(\theta^{t}+\theta^{tq}+\theta^{tq^2}+\theta^{tq^3})$

Table 20 (cont'd.)

$t,u=1,\cdots,q^2-1$ $t\neq u\neq 0 \pmod{q+1}$ $\psi^{(t,u)}=\psi^{(qt,u)}=\psi^{(t,qu)}=\psi^{(qt,qu)}$	$t=1,\cdots,q^3-1$ $t\neq 0 \bmod q^2+q+1$ $\psi^{(t)}=\psi^{(tq)}=\psi^{(tq^2)}$	$t=(q-1),2(q-1),\cdots,q^4-1$ $t\neq 0 \pmod{q^2+1}$ $\psi^{(t)}=\psi^{(tq)}=\psi^{(tq^2)}=\psi^{(tq^3)}$
$\frac{1}{8}q(q-2)(q+1)$	$\frac{1}{3}q(q-1)(q+1)$	$\frac{1}{4}q^2(q+1)$
$\psi^{(t,u)}_{\vdash^2 fp}$	$\psi^{(t)}_{\vdash^2 s^2 f}$	$\psi^{(t)}_{\vdash^3 sp}$
$\vdash^2 fp$ $-\vdash f$ $2q^2-q+1$ $-\vdash$ 1	$\vdash^2 s^2 f$ $-\vdash s$ $-\vdash s$ 1 1	$\vdash^3 sp$ $-\vdash^2 s$ $\vdash$ $\vdash$ 1
0 0 0	$\vdash^2 s\varepsilon^{t\kappa}$ $-\vdash\varepsilon^{t\kappa}$ $\varepsilon^{t\kappa}$	0 0 0
$\vdash^2(\varepsilon^{(t-u)\kappa}+\varepsilon^{-(t-u)\kappa})$ $-\vdash(\quad)$ $(\quad)$	0 0 0	0 0 0
0 0	0 0	0 0
0	0	0
$-\vdash\sum_{(u,t)}\varepsilon^{t\kappa}(\eta^{-2u\kappa}+\eta^{-2uq\kappa})$ $\sum_{(u,t)}\varepsilon^{t\kappa}(\quad)$	0 0	0 0
0	0	0
$f(\theta^u+\theta^{uq})(\theta^t+\theta^{tq})$ $(\quad)(\quad)$	0 0	$\vdash s(\theta^t+\theta^{tq})$ $-(\theta^t+\theta^{tq})$
$\sum_{(u,t)}(\eta^{t\kappa}+\eta^{t\kappa q})(\eta^{-u\kappa}+\eta^{-u\kappa q})$ 0 0	0 $\gamma^{-t\kappa}+\gamma^{-t\kappa q}+\gamma^{-t\kappa q^2}$ 0	0 0 $-\phi^t-\phi^{tq}-\phi^{tq^2}-\phi^{tq^3}$

Checking the character table for $PSL(4,q)$ posed a problem. It checked with $PSL(4,2)$, the only existing table for the case $d = 1$. However, when $q = 2$ only 11 of the 22 character types are present so this verifies only half the table. It is difficult to complete the check by generating the next larger character table and seeing if the various orthogonality properties hold, because $PSL(4,4)$ has 82 irreducible characters. The problem is surmounted in the following way. Suppose Ennola's conjecture holds. Using the conjugacy class table for $PSU(4,q^2)$ $d = 1$ given on page 98 and changing q to $-q$ in the table for $PSL(4,q)$ we should obtain the table for $PSU(4,q^2)$ $d = 1$. Having done this, we let $q = 2$ and compare the resulting table with $PSU(4,2^2)$ given in Frame [11]. The two tables checked. It very conveniently happens that $PSU(4,2^2)$ contains 8 additional character types not checked by $PSL(4,2)$. Thus we have verified 19 of the possible 22 character types. In the process we have demonstrated that Ennola's conjecture still holds. Considering the complicated nature of the characters for $PSL(4,q)$, it is nearly inconceivable that we could get the correct table for $PSU(4,2^2)$ if there was either an error in the table for $PSL(4,q)$ or if Ennola's conjecture is not valid for arbitrary q.

XIII. CONJUGACY CLASS STRUCTURE FOR $U(4,q^2)$, $PSU(4,q^2)$ $d = 1$

13.1 $U(4,q^2)$

As before, we use the results of Ennola's paper [7] to obtain a set of Jordan Canonical forms in $GL(4,q^2)$ which bear a one-to-one correspondence with the conjugacy classes of $U(4,q^2)$. We need only use simple combinatorial methods to count the number of possible canonical forms of each type. The order of the centralizers are obtained by the application of Theorem 4.1 which says that we need only change q to $-q$ in the expression for the order of the centralizer of each class in GL to obtain the centralizer order of the same type class in $U(n,q^2)$.

To ensure that the resulting conjugacy class table for $U(4,q^2)$ is correct, it was checked for $U(4,2^2)$ having order 77,760 with 60 conjugacy classes and for $U(4,3^2)$ of order 52,254,720 with 188 classes, respectively.

The details of the above work will be omitted since they are similar to ones already described.

Table 21. Conjugacy Class Structure of $U(4,q^2)$

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_1^{(k)}$	$\begin{pmatrix}\rho^k & & & \\ & \rho^k & & \\ & & \rho^k & \\ & & & \rho^k\end{pmatrix}$	$k=1,\cdots,q+1$	$q+1$	$q^6r^2s^4fp$
$C_2^{(k)}$	$\begin{pmatrix}1 & & \\ & \ddots & \\ & & \end{pmatrix}$	"	"	q^6rs^3p
$C_3^{(k)}$	$\begin{pmatrix}1 & & \\ & \ddots & \\ & 1 & \end{pmatrix}$	"	"	q^5rs^2
$C_4^{(k)}$	$\begin{pmatrix}1 & 1 & \\ & \ddots & \\ & & \end{pmatrix}$	"	"	q^4s^2
$C_5^{(k)}$	$\begin{pmatrix}1 & 1 & \\ & \ddots & \\ & 1 & \end{pmatrix}$	"	"	q^3s
$C_6^{(k,\ell)}$	$\begin{pmatrix}\rho^k & & & \\ & \rho^k & & \\ & & \rho^k & \\ & & & \rho^\ell\end{pmatrix}$	$k,\ell=1,\cdots,q+1$ $k\neq\ell$	$q(q+1)$	q^3rs^4p
$C_7^{(k,\ell)}$	$\begin{pmatrix}1 & & \\ & \ddots & \\ & & \end{pmatrix}$	"	"	q^3s^3
$C_8^{(k,\ell)}$	$\begin{pmatrix}1 & 1 & \\ & \ddots & \\ & & \end{pmatrix}$	"	"	q^2s^2
$C_9^{(k,\ell)}$	$\begin{pmatrix}\rho^k & & & \\ & \rho^k & & \\ & & \rho^\ell & \\ & & & \rho^\ell\end{pmatrix}$	$k,\ell=1,\cdots,q+1$ $k\neq\ell$ $C^{(k,\ell)}=C^{(\ell,k)}$	$\frac{1}{2}q(q+1)$	$q^2r^2s^4$
$C_{10}^{(k,\ell)}$	$\begin{pmatrix}1 & & \\ & \ddots & \\ & & \end{pmatrix}$	$C^{(k,\ell)}\neq C^{(\ell,k)}$	$q(q+1)$	q^2rs^3
$C_{11}^{(k,\ell)}$	$\begin{pmatrix}1 & & \\ & \ddots & \\ & 1 & \end{pmatrix}$	$C^{(k,\ell)}=C^{(\ell,k)}$	$\frac{1}{2}q(q+1)$	q^2s^2
$C_{12}^{(k,\ell,m)}$	$\begin{pmatrix}\rho^k & & & \\ & \rho^k & & \\ & & \rho^\ell & \\ & & & \rho^m\end{pmatrix}$	$k,\ell,m=1,\cdots,q+1$ $k<\ell<m$ $C^{(k,\ell,m)}=C^{(k,m,\ell)}$	$\frac{1}{2}q(q+1)(q-1)$	qrs^4
$C_{13}^{(k,\ell,m)}$	$\begin{pmatrix}1 & & \\ & \ddots & \\ & & \end{pmatrix}$	"	"	qs^3
$C_{14}^{(k,\ell,m,n)}$	$\begin{pmatrix}\rho^k & & & \\ & \rho^\ell & & \\ & & \rho^m & \\ & & & \rho^n\end{pmatrix}$	$k,\ell,m,n=1,\cdots,q+1$ $k<l<m<n$	$\binom{q+1}{4}$	s^4
$C_{15}^{(k,\ell)}$	$\begin{pmatrix}\rho^k & & & \\ & \rho^k & & \\ & & \sigma^\ell & \\ & & & \sigma^{-q\ell}\end{pmatrix}$	$k=1,\cdots,q+1$ $\ell=1,\cdots,q^2-1$ $\ell\not\equiv 0 \pmod{q-1}, C^{(\ell)}=C^{(-q\ell)}$	$\frac{1}{2}(q+1)^2(q-2)$	qr^2s^3
$C_{16}^{(k,\ell)}$	$\begin{pmatrix}1 & & \\ & \ddots & \\ & & \end{pmatrix}$	"	"	qrs^2
$C_{17}^{(k,\ell,m)}$	$\begin{pmatrix}\rho^k & & & \\ & \rho^\ell & & \\ & & \sigma^m & \\ & & & \sigma^{-qm}\end{pmatrix}$	$k,\ell=1,\cdots,q+1$ $k\neq\ell$ $m=1,\cdots,q^2-1$ $m\not\equiv 0 \pmod{q-1}$ $C^{(m)}=C^{(-qm)}$	$\frac{1}{4}q(q-2)(q+1)^2$	rs^3

Table 21 (cont'd.)

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_{18}^{(k)}$	$\begin{pmatrix} \sigma^k & & & \\ & \sigma^{-kq} & & \\ & & \sigma^k & \\ & & & \sigma^{-kq} \end{pmatrix}$	$k=1,\cdots,q^2-1$ $k \neq 0 \pmod{q-1}$, $C^{(k)} = C^{(-kq)}$	$\frac{1}{2}(q+1)(q-2)$	$q^2r^2s^2f$
$C_{19}^{(k)}$	$\begin{pmatrix} \sigma^k & & & \\ & \sigma^{-kq} & & \\ 1 & & \sigma^k & \\ & 1 & & \sigma^{-kq} \end{pmatrix}$	"	"	q^2rs
$C_{20}^{(k,\ell)}$	$\begin{pmatrix} \sigma^k & & & \\ & \sigma^{-kq} & & \\ & & \sigma^\ell & \\ & & & \sigma^{-\ell q} \end{pmatrix}$	$k,\ell=1,\cdots,q^2-1$ $k \neq \ell$, $k,\ell \neq 0 \pmod{q-1}$ $C^{(k,\ell)} = C^{(\ell,k)} = C^{(-kq,\ell)} = C^{(k,-q\ell)}$	$\frac{1}{8}(q+1)(q-2)(q^2-q-4)$	r^2s^2
$C_{21}^{(k,\ell)}$	$\begin{pmatrix} \rho^k & & & \\ & \tau^\ell & & \\ & & \tau^{\ell q^2} & \\ & & & \tau^{\ell q^4} \end{pmatrix}$	$k=1,\cdots,q+1$ $\ell=1,\cdots,q^3$ $\ell \neq 0 \pmod{q^2-q+1}$ $(\ell) = (\ell q^2) = (\ell q^4)$	$\frac{1}{3}q(q-1)(q+1)^2$	s^2p
$C_{22}^{(k)}$	$\begin{pmatrix} \omega^k & & & \\ & \omega^{-kq} & & \\ & & \omega^{kq^2} & \\ & & & \omega^{-kq^3} \end{pmatrix}$	$k=1,\cdots,q^4-1$ $k \neq 0 \pmod{q^2+1}$ $C^{(k)} = C^{(-kq)} = C^{(kq^2)} = C^{(-kq^3)}$	$\frac{1}{4}q^2(q^2-1)$	rsf

σ, τ_1, ω = primitive elements of $GF(q^2)$, $GF(q^6)$, $GF(q^4)$ respectively

$\rho = \sigma^{(q-1)}$, $\tau = \tau_1^{(q^3-1)}$, $r = q-1$, $s = q+1$, $p = q^2 - q+1$, $f = q^2+1$

13.2 $PSU(4,q^2)$ $d = 1$

By requiring that the determinants of the canonical representatives of $U(4,q^2)$ be unity, we can obtain the classes of $PSU(4,q^2) \cong SU(4,q^2)$ $d = 1$. The details of 'counting' the number of conjugacy classes are similar to those written out in detail for $PSL(4,q)$ and are not of sufficient interest to give here.

The resulting table was checked against $PSU(4,2^2)$ having order 25,920 with 20 classes and $PSU(4,4^2)$ of order 1,018,368,000 with 92 classes.

As mentioned in the preceding section, this table for the conjugacy classes of $PSU(4,q^2)$ together with the character table of $PSL(4,q)$ with q replaced by $-q$ convincingly appears to give the character table for $PSU(4,q^2)$ $d = 1$. At least it checks for $q = 2$.

Table 22. Conjugacy Class Structure for $PSU(4,q^2)$ d = 1

conjugacy class	canonical representative	parameters	number of classes	centralizer order
C_1	$\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$		1	$q^6r^2s^3fp$
C_2	$\begin{pmatrix}1&&&\\1&1&&\\&&1&\\&&&1\end{pmatrix}$		1	q^6rs^2
C_3	$\begin{pmatrix}1&&&\\1&1&&\\&&1&\\&&1&1\end{pmatrix}$		1	q^5rs
C_4	$\begin{pmatrix}1&&&\\1&1&&\\&1&1&\\&&&1\end{pmatrix}$		1	q^4s
C_5	$\begin{pmatrix}1&&&\\1&1&&\\&1&1&\\&&1&1\end{pmatrix}$		1	q^3
$C_6^{(k)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^k&&\\&&\rho^k&\\&&&\rho^{-3k}\end{pmatrix}$	$k=1,\cdots,q$	q	q^3rs^3p
$C_7^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\rho^k&\\&&&\rho^{-3k}\end{pmatrix}$	"	q	q^3s^2
$C_8^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&1&\rho^k&\\&&&\rho^{-3k}\end{pmatrix}$	"	q	q^2s
$C_9^{(k)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^k&&\\&&\rho^{-k}&\\&&&\rho^{-k}\end{pmatrix}$	$k=1,\cdots,q$ $k\not\equiv -k \pmod{q+1}$ $C^{(k)}=C^{(-k)}$	$\frac{1}{2}q$	$q^2r^2s^3$
$C_{10}^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\rho^{-k}&\\&&&\rho^{-k}\end{pmatrix}$	$C^{(k)}\neq C^{(-k)}$	q	q^2rs^2
$C_{11}^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\rho^{-k}&\\&&1&\rho^{-k}\end{pmatrix}$	$C^{(k)}=C^{(-k)}$	$\frac{1}{2}q$	q^2s
$C_{12}^{(k,\ell)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^k&&\\&&\rho^\ell&\\&&&\rho^{-2k-\ell}\end{pmatrix}$	$k,\ell=1,\cdots,q+1$ $k\neq\ell$	$\frac{1}{2}q(q-1)$	qrs^3
$C_{13}^{(k,\ell)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\rho^\ell&\\&&&\rho^{-2k-\ell}\end{pmatrix}$	"	$\frac{1}{2}q(q-1)$	qs^2
$C_{14}^{(k,\ell,m,n)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^\ell&&\\&&\rho^m&\\&&&\rho^n\end{pmatrix}$	$k,\ell,m,n=1,\cdots,q+1$ $k<\ell<m<n$ $k+\ell+m+n=0 \pmod{q+1}$	$\frac{1}{24}q(q-1)(q-2)$	s^3
$C_{15}^{(k)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^k&&\\&&\sigma^{2k}&\\&&&\sigma^{-2kq}\end{pmatrix}$	$k=1,\cdots,q^2-1$ $k\not\equiv 0 \pmod{q-1}$	$\frac{1}{2}(q+1)(q-2)$	qr^2s^2
$C_{16}^{(k)}$	$\begin{pmatrix}\rho^k&&&\\1&\rho^k&&\\&&\sigma^{2k}&\\&&&\sigma^{-2kq}\end{pmatrix}$	"	$\frac{1}{2}(q+1)(q-2)$	qrs
$C_{17}^{(k,\ell)}$	$\begin{pmatrix}\rho^k&&&\\&\rho^\ell&&\\&&\sigma^{(k+\ell)}&\\&&&\sigma^{-(k+\ell)q}\end{pmatrix}$	$k,\ell=1,\cdots q^2-1$ $k\neq\ell$ $k+\ell\not\equiv 0 \pmod{q-1}$ $C^{(k,\ell)}=C^{(\ell,k)}$	$\frac{1}{4}q(q+1)(q-2)$	rs^2

Table 22 (cont'd.)

conjugacy class	canonical representative	parameters	number of classes	centralizer order
$C_{18}^{(k)}$	$\begin{pmatrix} \sigma^{k} & & & \\ & \sigma^{-kq} & & \\ & & \sigma^{k} & \\ & & & \sigma^{-kq} \end{pmatrix}$	$k=1,\cdots,q^2-1$ $k\neq 0 \pmod{q-1}$ $k=0 \pmod{q+1}$	$\frac{1}{2}(q-2)$	q^2r^2sf
$C_{19}^{(k)}$	$\begin{pmatrix} \sigma^{k} & & & \\ & \sigma^{-kq} & & \\ 1 & & \sigma^{k} & \\ & 1 & & \sigma^{-kq} \end{pmatrix}$	"	$\frac{1}{2}(q-2)$	q^2r
$C_{20}^{(k)}$	$\begin{pmatrix} \sigma^{k} & & & \\ & \sigma^{-kq} & & \\ & & \sigma^{-k} & \\ & & & \sigma^{kq} \end{pmatrix}$	$k=1,\cdots,q^2-1$ $k\neq 0 \pmod{q-1}$ $C^{(k)}=C^{(-k)}=C^{(kq)}=C^{(-kq)}$	$\frac{1}{8}(q-2)(q^2-q-4)$	r^2s
$C_{21}^{(k)}$	$\begin{pmatrix} \rho^{k} & & & \\ & \tau^{-k} & & \\ & & \tau^{-kq^2} & \\ & & & \tau^{-kq^4} \end{pmatrix}$	$k=1,\cdots,q^3+1$ $k\neq 0 \pmod{q^2-q+1}$	$\frac{1}{3}q(q-1)(q+1)$	sp
$C_{22}^{(k)}$	$\begin{pmatrix} \omega^{k} & & & \\ & \omega^{-kq} & & \\ & & \omega^{kq^2} & \\ & & & \omega^{-kq^3} \end{pmatrix}$	$k=(q+1),2(q+1),\cdots,q^4-1$ $k\neq 0 \pmod{q^2+1}$	$\frac{1}{4}q^2(q-1)$	rf

σ, τ_1, ω = primitive elements of $GF(q^2)$, $GF(q^6)$, $GF(q^4)$ respectively

$\rho = \sigma^{(q-1)}$, $\tau = \tau_1^{(q^3-1)}$, $r = q-1$, $s = q+1$, $p = q^2 - q+1$, $f = q^2+1$

XIV. CHARACTER DEGREES AND FREQUENCIES FOR PSL(4,q) d = 2

The existing character tables for PSL(4,q) and PSU$(4,q^2)$ d = 2 are those having q = 2 or q = 3. These tables are of a reasonable size and most of the entries are integers. The non-integer entries, if irrational, are of the form $\frac{a \pm \sqrt{b}}{c}$ and if complex, are simple linear combinations of cube roots of unity. Thus one is easily misled into thinking that a generalized character table for these groups would involve entries on the same order of complexity as was the case for n = 2,3, that is, most of the entries would be constants or polynomials in q. These hopes are soon forgotten when we consider the table for PSL(4,q) d = 1 which we have just developed. We see for example that the entry $\Psi_{35}(C_{15}) = 2$ for PSL(4,2) corresponds to the entry $r\epsilon^{tk} - (\eta^{-2tk} + \eta^{-2tkq})$ in the generalized table and realize that the simple nature of the existing tables is due entirely to the fact that the various combinations of roots of unity, which are dependent on q, are relatively elementary when q = 2 or 3. We also note that the number of classes and characters can be expressed as polynomials in q. At the level n = 4 these polynomials are in general of a higher degree than those for n = 2 or 3. As a result, the number of classes and characters increases at a much faster rate with an increase in q. Thus we find that PSL(4,2) has 8 characters, PSL(4,3) has 29, but PSL(4,4) has 82 characters and

PSL(4,7) has 407.

The smallest projective special groups $PSL(4,q)$ for which no character tables have been developed are $PSL(4,7)$ and $PSL(4,5)$ for $d = 2,4$ respectively. These are very large groups, having over 100 conjugacy classes, and so their character tables would be rather difficult to work with even if they were available. If it were necessary to have the table or a partial table for one of these groups it would be much easier to develop it from $GL(4,q)$ with q equal to the appropriate value rather than trying to get the abstract table for arbitrary q because the calculation of inner products involving numerical values is far easier than performing the same calculation with polynomials in q and sums of $(q-1)^{st}$ roots of unity.

Aside from these practical considerations there are some theoretical difficulties which would impede progress. For the cases $d = 2,4$ both classes and characters undergo some splitting. Preliminary checks reveal this splitting is much more extensive than on the $n = 2,3$ levels. There is no correlation between the classes which split and the splitting characters. We have already seen in the case of $SL(3,q)$ $d = 3$ that classes of type C_3 split while the characters which split correspond to the classes C_6 and C_8. Thus, determining the splitting of the classes and characters are two separate problems. For the cases $n = 2,3$ we had Dickson's results to give us the class splits, but he did not work out the case for $n = 4$. In the case for $d = 2$, if a class of GL splits in SL we do not know if it splits into 2 or 4 classes.

Taking inner products to determine which characters are reducible is very difficult with the abstract characters of $GL(4,q)$ because they are so much more complicated than the previous cases considered. Also, for the case $d = 4$, characters can split into 2 or 4 components and so the inner products must be calculated with more care than previously. We finally note that after we determine which characters split, we cannot immediately fill in their values on the classes which are not complete classes of GL. Since there are many splitting classes and characters this means we are left with numerous large holes in the table to be filled in by the tedious process of using the orthogonality properties of the table.

In this section an attempt is made to at least determine the degrees and frequencies of the characters for $PSL(4,q)$ $d = 2$. We now use a different route and proceed from GL to $PGL \cong GL/Z(GL)$ to PSL.

$$\begin{tikzcd}[arrows=dash] & GL \arrow[dl, "q-1"'] \arrow[dr, "q-1"] & \\ PGL \cong GL/Z(GL) \arrow[dr, "d"'] & & SL \arrow[dl, "d"] \\ & PSL \cong SL/Z(SL) & \end{tikzcd}$$

The work is based entirely on Clifford's theorem which tells us how many associates the irreducible and reducible restricted characters have. Since $PSL \lhd PGL$ and $[PGL : PSL] = 2$ we can apply Clifford's theorem. If a character of PGL is irreducible upon restriction to PSL then it has 2 associates. If the character is reducible then it has only 1 associate and splits in half.

We now assume tentatively that if 2 divides the number of characters of PGL of the same degree, then they all restrict down in sets of 2 to irreducible characters of PSL. If 2 does not divide the number of characters of the same degree we will pull off one character and assume this character splits in PSL. We do this for $q = 3,7,11,\ldots$ and hope that a clear pattern emerges. In Steinberg's thesis [20] the frequency table for the characters of PGL(4,q) is given.

The only characters which require any special consideration are $\chi^{(t)}_{s^2fp}$ and $\chi^{(t)}_{r^2fp}$. We consider then separately.

The number of characters of degree s^2fp alternates from even to odd. This means that either the number of splitting characters alternates from 0 to 1 with every q, or the number increases by one for every q value. This situation did not occur in the tables for $n = 2,3$ so we have no precedent by which to go; however, the second alternative seems most likely.

q	number of characters of degree s^2fp in PGL $\frac{1}{24}(q-3)(q^2-6q+11)$		number of characters in PSL	number of reducible chars.	
3	0		0	0	
7	3	2	1		
		1	1	1	$n = \frac{q-3}{4}$
			1		
11	22	20	10		
		1	1	2	
			1		
		1	1		
			1		
19	172	168	84		
		1	1		
			1		
		1	1	4	
			1		
		1	1		
			1		
		1	1		
			1		

From this we conclude that of the $\frac{1}{24}(q-3)(q^2 - 6q + 11)$ characters of degree s^2fp in PGL(4,q) $d = 2$, $\frac{q-3}{4}$ of them are reducible, when restricted to PSL and the remaining $\frac{1}{24}(q^3 - 9q^2 + 23q - 15)$ will restrict down in sets of 2 to irreducible characters of PSL.

The same situation occurs for the characters of degree r^2fp except that the number of characters alternates from odd to even out of phase with the number of χ_{s^2fp}.

q	number of characters of degree r^2fp in PGL $\frac{1}{8}(q+1)(q^2-2q-1)$		number of characters in PSL	number of reducible chars.	
3		1	1	1	
			1		
7	34	32	16		
		1	1		
			1	2	$n = \frac{q+1}{4}$
		1	1		
			1		
11	147	144	72		
		1	1		
			1		
		1	1	3	
			1		
		1	1		
			1		

From this it appears that of the $\frac{1}{8}(q+1)(q^2 - 2q - 1)$ characters of degree r^2fp in PGL, $\frac{q+1}{4}$ of them are reducible when restricted to PSL.

The frequency table for $PSL(4,q^2)$ $d = 2$ is correct for $PSL(4,3)$, however we could only be sure it is correct if the table for $PSL(4,7)$ was available. The calculation $\sum_{i=1}^{k} (\Psi^i(1))^2 = |G|$ proves correct for $q = 4,7$, however this still does not indicate whether the number of splitting characters is correct because if Ψ, a character of PGL, does not split in PSL then the k associates of Ψ restrict down to $\frac{k}{2}\Psi$ characters in PSL. If $\Psi|_{PSL}$ is reducible then the k associates of Ψ restrict down to $2k$ characters of half the degree of Ψ. However

$$\underbrace{\frac{k}{2}(\Psi(1))^2}_{\text{irreducible case}} = \underbrace{2k\left(\frac{\Psi(1)}{2}\right)^2}_{\text{reducible case}}$$

so the sum of squares of character degrees is unchanged by a splitting.

The total number of characters for PSL(3,q) $d = 3$ is $\frac{1}{3}(q^2 + q + 10)$ and we get $\frac{1}{4}(2q^3 + 2q^2 + 7q + 23)$ for PSL(4,q) which appears to be of the same form.

Table 23. Character Degrees and Frequencies for PSL(4,q) $d = 2$

PGL(4,q) d=2			PSL(4,3)		PSL(4,7)		PSL(4,q) d=2	
number of characters q=3	number of characters q=7	character degree	no. char.	degree	no. of char.	degree	number of characters	character degree
2	2	1	1	1	1	1	1	1
2	2	qp	1	39	1	399	1	qp
2	2	q^2f	1	90	1	2450	1	q^2f
2	2	q^3p	1	351	1	19551	1	q^3p
2	2	q^6	1	729	1	117649	1	q^6
0	4	sf	0		2	400	$\frac{1}{2}(q-3)$	sf
0	4	qs^2f	0		2	22400	$\frac{1}{2}(q-3)$	qs^2f
0	4	q^3sf	0		2	137200	$\frac{1}{2}(q-3)$	q^3sf
0	8	sfp	0		4	22800	$\frac{1}{4}(q-3)^2$	sfp
0	8	$qsfp$	0		4	159600	$\frac{1}{4}(q-3)^2$	$qsfp$
0	$3\begin{cases}2\\1\end{cases}$	s^2fp	0 0		1 2	182400 91200	$\frac{1}{48}(q^3-9q^2+23q-15)$ $\frac{1}{2}(q-3)$	s^2fp $\frac{1}{2}s^2fp$
2	54	$rsfp$	1	1040	27	136800	$\frac{1}{8}(q-3)^3$	$rsfp$
8	112	s^2r^2f	4	640	56	115200	$\frac{1}{6}q(q^2-1)$	s^2r^2f
1	$5\begin{cases}4\\1\end{cases}$	fp	2	65	2 2	2850 1425	$\frac{1}{2}(q-3)$ 2	fp $\frac{1}{2}fp$
1	$5\begin{cases}4\\1\end{cases}$	q^2fp	2	585	2 2	139650 69825	$\frac{1}{2}(q-3)$ 2	q^2fp $\frac{1}{2}q^2fp$
2	10	qfp	1	390	5	19950	$(q-2)$	qfp
2	18	rfp	1	260	9	17100	$\frac{1}{4}(q-1)^2$	rfp
2	18	$qrfp$	1	780	9	119700	$\frac{1}{4}(q-1)^2$	$qrfp$
1	$34\begin{cases}32\\1\\1\end{cases}$	r^2fp	2	260	16 $4\begin{cases}2\\2\end{cases}$	[illegible]02600 51300	$\frac{1}{16}(q^3-q^2-5q-3)$ $\frac{1}{2}(q+1)$	r^2fp $\frac{1}{2}r^2fp$
$3\begin{cases}2\\1\end{cases}$	$7\begin{cases}6\\1\end{cases}$	r^2p	1 2	52 26	3 2	2052 1026	$\frac{1}{2}(q-1)$ 2	r^2p $\frac{1}{2}r^2p$
$3\begin{cases}2\\1\end{cases}$	$7\begin{cases}6\\1\end{cases}$	q^2r^2p	1 2	468 234	3 2	100548 50274	$\frac{1}{2}(q-1)$ 2	q^2r^2p $\frac{1}{2}q^2r^2p$
8	96	r^3sp	4	416	48	98496	$\frac{1}{8}(q-1)(q+1)^2$	r^3sp
			29		407		$\frac{1}{4}(2q^3+2q^2+7q+23)$	

Summary of Results

The primary purpose of this paper was to determine the abstract character tables for the groups $SL(n,q)$, $PSL(n,q)$, $SU(n,q^2)$, $PSU(n,q^2)$ $n = 2,3$, $PSL(4,q)$ $d = 1$ and display the tables in such a manner as to facilitate their use. The procedure developed could also be used to find additional tables for specific n and q values.

The secondary goal was to investigate the possibility of Ennola's conjecture holding for the special and projective special subgroups. The fact that the conjecture did hold for all the groups under discussion was very surprising. For example consider the groups $PSL(3,5)$ and $PSU(3,5)$. Their character tables are of completely different form because $d = 1$ for $PSL(3,5)$ whereas $d = 3$ for $PSU(3,5)$. Thus the transformation $q \to -q$ clearly does not apply to literal values since in the above case $5 \to -5$ does not change the table for $PSL(3,5)$ into the table for $PSU(3,5)$. The conjectured transformation could better be written as $q \to -q'$ where q is not necessarily q'. Even if the conjecture is true, however, the relationship between the tables of the unitary and linear groups is not completely determined because the conjugacy class structures, are not related in any known way; i.e. the polynomials giving the number of classes and irreducible characters of the linear groups cannot be transformed into the corresponding polynomials for the unitary groups. Also, the conjecture does not appear to be of any value if the tables in question are for specific values of q because the transformation only operates on the abstract symbol q and not numerical values.

The modified version of Clifford's theorem, although by no means essential to the problem of developing the character tables, was nevertheless a considerable aid. The task of determining the number of components in a reducible character becomes more laborious without the theorem because the inner products, (χ,χ), must be calculated considerably more exactly in some cases.

The theorem was also used to arrive at a conjectured table of character degress and their frequencies for PSL(4,q) d = 2 which checked for the case PSL(4,3). It would be interesting to determine if the table is also correct for PSL(4,7). This is such a large group that a check here would be very convincing.

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