PERSONAL INCOME TAXES AND THE CAPITAL ASSET PRICING MODEL: SOME THEORETICAL RESULTS

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Ronald F. Singer

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ABSTRACT

PERSONAL INCOME TAXES AND THE CAPITAL ASSET PRICING MODEL: SOME THEORETICAL RESULTS

By

Ronald Frederick Singer

The incentive effects of the taxation of capital income has long been a source of controversy. However, with few exceptions, the lines of inquiry have centered around only two aspects of the problem; substitution of a riskless asset for the "risky portfolio" and the effects of differential tax rates on dividend versus capital gains income. This study is an investigation into a third avenue of inquiry; the effect of personal income taxes on the substitution of risky assets within the investor's risky portfolio and the resultant equilibrium structure of security returns.

The study broadens the assumptions underlying the Capital Asset Pricing Model to permit investors to make decisions on the basis of after tax rather than before tax parameters, when the investor's tax liability is a strick function of total income. The resultant first order conditions are derived and a representative investor's required risk premium for a specific security is considered in a general model and in some specific cases.

It is concluded that personal income taxes do not affect investor behavior as long as marginal tax rates are non-stochastic.

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That is, the investor faces a tax function which is linear in income. Taxes are also irrelevant to investor behavior if all security rates of return possess normal probability density functions, regardless of the form of the tax function. However, if the investor faces stochastic marginal tax rates and security returns are non-normal then the required risk premium will depend on the parameters of the tax function, the degree of skewness and kurtosis of the rate of return of the investor's portfolio, as well as higher joint moments of the bivariate density function of the security and portfolio returns.

Similarly, the equilibrium structure of security returns remain unaffected by the tax function if all investors face non-stochastic marginal tax rates and/or security returns are normally distributed. However, if some investors face stochastic marginal tax rates, then the equilibrium structure of security returns will depend not only on the traditional market parameters, but also on higher moments of the investors' portfolio rates of return and higher joint moments of the security and portfolio returns.

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A DISSERTATION

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This work could not have been completed without the help and encouragement of many people. Professor Robert Rasche deserves special mention for his trust and faith before this dissertation was even begun. Who, as chairman of my doctoral committee, devoted time and energy far beyond the limits normally reserved to a thesis student. Professor Mark Ladenson and Norman Obst also deserve mention for their cooperation and suggestions.

It is impossible to separately identify all of my friends and colleagues, each of whom added, in his or her unique way, to the completion of this work. Special mention is due Steve, for his support during the most difficult times and Carol who is a good listener.

My family carried the double burden of the anxiety of time in addition to the normal frustrations involved in writing a dissertation. I mention the Millers, especially for their tolerance, my parents, for their encouragement and Jeanne - to whom this work is and always has been dedicated - who made it all worthwhile.

Finally, I should mention the Department of Economics and Finance, Baruch College for the generous grant of a leave of absence and Mrs. Noralee Burkhardt for her accurate and speedy typing of a very difficult manuscript.

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CHAPTER 1

INTRODUCTION

The incentive effects of the taxation of capital income has long been a source of controversy. However, with few exceptions, the lines of inquiry have centered around only two aspects of the problem; substitution of a riskless asset for the "risky portfolio" and the effects of differential tax rates on dividend versus capital gains income.[1] This study is an investigation into a third avenue of inquiry; the effect of personal income taxes on the substitution of risky assets within the investor's risky portfolio and the resultant equilibrium structure of security returns.

It is natural that this aspect of the incentive effects of taxation has been ignored up to now. Until recently, a theoretical model of the structure of security returns had not been developed. Only in the last decade has research by Sharpe, Lintner and Mossin provided investigators with the Capital Asset Pricing Model, a model of the equilibrium structure of security returns in the context of perfectly competitive markets.[2] This model serves as a benchmark against which the implications of various modifications of its underlying assumptions may be judged. The present study considers one possible modification of the Capital Asset Pricing Model. The traditional model is modified to accommodate personal income taxes as a relevant variable. That is, the investor is assumed to consider the distribution of after tax returns as a decision stimulant rather than before tax returns.

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In this context, specification of the tax function defines the breadth of the issue under consideration. It is assumed throughout that tax liability is a continuous function of income, and its functional form is known by the investor with certainty. All income, except capital income, is non-stochastic and known at the beginning of the period. This serves to remove any uncertainty from the investor's anticipated tax liability except that resulting from the stochastically determined return on the risky securities in his portfolio. Thus, the study concentrates on the stochastic nature of security returns and its implication on the investor's tax liability, rather than on uncertainty involved in the tax function per se.[3] Furthermore, all sources of income are assumed to be realized and taxed at the end of the period. This permits a single period analysis, avoiding the difficult multi-period decision problem. Finally, it is assumed that capital gains and dividends are treated identically for tax purposes. Hence the study abstracts from the issue of the effect of differential tax rates on investor behavior.

Since the general methodology of the present investigation is deeply rooted in the traditional Capital Asset Pricing Model, chapter 2 presents a brief review of the traditional theory. This is followed by relevant empirical tests of the traditional model and some of its modifications.

Chapter 3 contains a mathematical formulation of investor behavior in the context of a general income tax structure. The formulation extends the traditional Capital Asset Pricing Model to incorporate personal income taxes as a relevant variable. The investor is assumed to act in response to after tax rather than before tax variables. F

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variables. First order conditions in terms of the joint moments of the tax function, marginal tax rate and security returns are derived. Then key parameters are identified which describe the implications of various forms of the tax function on investor behavior.

Chapter 4 considers some special cases of the more general results. Specifically, four cases are presented assuming that the tax function may be approximated by a two degree Taylor Expansion. In these cases it is further assumed that investors possess two parameter utility functions. Different results are derived under different assumptions about the nature of the probability distribution of security returns. Finally, an L degree Taylor Expansion is assumed, when the investor possesses a three parameter utility function and security returns are normally distributed.

Chapter 5 derives market equilibrium in the context of a tax function which can be approximated by a two degree Taylor Expansion, two parameter utility functions for all investors and non-normal security returns.

The last chapter summarizes the results and provides some conclusions about the effect of income taxes on security returns.

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NOTES TO CHAPTER 1

1. Some of the notable works on the effects of taxation on risk taking per se are; Evsey Domar and Richard Musgrave, "Proportional Income Taxation and Risk Taking," Quarterly Journal of Economics, 58 (May, 1944), 388-422; Jan Mossin, "Taxation and Risk-Taking," Economica, 35 (February, 1968), 74-82; Martin Feldstein, "The Effects of Taxation on Risk Taking," Journal of Political Economy, 77 (September, 1969), 755-64; Aiko Shibata, "Effects of Taxation on Risk Taking," American Economic Review, 59 (May, 1969), 553-61; Joseph Stiglitz, "Effects of Income, Wealth and Capital Gains Taxation on Risk-Taking," Quarterly Journal of Economics, 83 (May, 1969), 263-83; Syed Ahson, "Progression and Risk Taking," Oxford Economic Papers, 26 (November, 1974), 318-28: on the effects of differential tax rates are; M. K. Richter, "Cardinal Utility, Portfolio Selection and Taxation," Review of Economic Studies, 27 (June, 1960), 152-66; Rudolph Penner, "A Note on Portfolio Selection and Taxation," Review of Economic Studies, 31 (January, 1964), 83-86; Edwin Elton and Martin Gruber, "Marginal Stockholder Tax Rates and the Clientele Effect," Review of Economics and Statistics, 52 (February, 1970), 68-74; Michael Brennan, Investor Taxes, Market Equilibrium and Corporate Finance (Unpublished Ph.D. Dissertation, MIT, June, 1970); Michael J. Brennan, "Taxes, Market Valuation and Corporate Financial Policy," National Tax Journal, 23 (December, 1970), 417-27: A notable exception is Susan J. Lepper, "Effects of Alternative Tax Structure on Individual's Holdings of Financial Assets," in Donald Hester and James Tobin (eds.), Risk Aversion and Portfolio Choice, (New York: John Wiley and Sons, 1967), 51-109.

2. William F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," Journal of Finance, 19 (September, 1964), 425-42; John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolio and Capital Budgets," Review of Economics and Statistics, 47 (February, 1965), 13-37; Jan Mossin, "Equilibrium in a Capital Asset Market," Econometrica, 34 (October, 1966), 768-83.

3. See David Myers, "Nonmarketable Assets and Capital Market Equilibrium Under Uncertainty," in Michael Jensen, Studies in the Theory of Capital Markets, (New York: Praeger, 1972), 223-48, for a methodology which may be ad pted to the investigation of a stochastic tax function.

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CHAPTER 2

THE FOUNDATIONS AND EMPIRICAL FINDINGS OF THE CAPITAL ASSET PRICING MODEL

2.1. Methodology of the Capital Asset Pricing Model

The traditional Capital Asset Pricing Model (CAPM) is an attempt to explain the structure of returns on risky securities in the context of expectee utility maximization under conditions of uncertainty.[1] The model is derived from the following assumptions:

Market Assumptions

A.1. Each individual investor is free to borrow or lend an unlimited amount at an exogenously determined "riskless" rate of interest.

A.2. Each investor can invest any fraction of his capital in any or all of a given finite set of risky securities.

A.3. The market supply, in terms of number of shares, of each risky asset is exogenously determined.

A.4. All investors make all purchases and sales at discrete points in time, the time period being identical for all investors.

A.5. All assets are traded in a single competitive market. That is, each investor's demand for any asset is sufficiently small, relative to total market demand, so that his transactions have an insignificant effect on the market price of that asset. In addition, there are no t

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Assumptions Regarding Investors

B.1. Investors have already determined the fraction of their capital they intend to use for liquidity and transactions purposes.

B.2. Each investor's decisions are made on the basis of the mean and variance of his portfolio return.

B.3. Investors are able to assess the relevant expectations, variances and covariances of each risky asset's rate of return.

B.4. All investors are "risk averse" in the sense that each prefers more expected return for a given amount of "risk" and less "risk" for a given amount of return.

B.5. All investors have identical assessments of each asset's "risk" and return. This has been called the "homogeneous expectations" assumption.[2]

In general, if the investor acts in accordance with the Von-Neumann-Morgenstern[3] axioms of expected utility maximization and either all investors possess quadratic utility functions, or all risky securities possess two parameter probability density functions, then it can be shown that assumption B.2 will hold. That is, the mean and variance (or standard deviation) of portfolio returns are the relevant arguments in the investor's utility function.[4] Thus, in this context, "risk" refers to the variance (standard deviation) of portfolio return.

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$$\tilde{R}_i = \frac{\tilde{P}_i - P_i + D_i}{P_i}$$

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Under the assumptions enumerated above, conditions for an investor's expected utility maximization are derived. Then, market equilibrium conditions are imposed, and the structure of equilibrium security returns is obtained. Specifically, the methodology and conclusions of the traditional Capital Asset Pricing Model are as follows. There exists K investors-indexed by $k(1,\ldots,k,\ldots,k)$-trading, in N different risky-assets indexed by i and j $(1,\ldots,i,\ldots,j,\ldots,N)$- in accordance with the assumptions set out above. Let,

$\tilde{P}_i$	be the random end of period price per share of asset i, $i = 1,\ldots,N$
P_i	be the non-random beginning period price of asset i
D_i	be the non-random dividend payment, in dollars, of asset i over the period
$\tilde{R}_i = \dfrac{\tilde{P}_i - P_i + D_i}{P_i}$	be the single period before tax rate of return of risky asset i
R_{N+1}	be the single period before tax rate of return on the riskless asset
X_{ik}	be the proportion of total speculative capital invested in asset i by investor k
$X_{N+1,k}$	be the proportion of total speculative capital invested in the riskless asset by investor k

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$\sigma_{ij} = \sigma(\tilde{R}_i, \tilde{R}$

$\sigma(\tilde{R}_k, \tilde{R}_j) = \quad j$

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$\sigma_{ij} = \sigma(\tilde{R}_i,\tilde{R}_j) = E[(\tilde{R}_i-\bar{R}_i)(\tilde{R}_j-\bar{R}_j)]$ be the covariance of single period rate of return between risky asset i and j, and

$\sigma(\tilde{R}_k,\tilde{R}_j) = \sum_{j=1}^{N} X_{jk}\sigma_{ij}$ be the covariance of single period rate of return between investor k's entire portfolio and asset j

$\sigma_{(\tilde{R}_k,\tilde{R}_k)} = \sum_{i=1}^{N}\sum_{j=1}^{N} X_{ik}X_{jk}\sigma_{ij}$ be the variance of single period rate of return of investor k's entire portfolio

$\tilde{R}_k = \sum_{i=1}^{N} X_{ik}\tilde{R}_i + X_{N+1,k}R_{N+1}$ be the single period rate of return of investor k's entire portfolio

In accordance with assumptions B.2 and B.4 above, investor k's utility function may be written as

$$U_k = U_k(E[\tilde{R}_k],\ \sigma_{(\tilde{R}_k,\tilde{R}_k)})$$

$$\frac{\partial U}{\partial E[\tilde{R}_k]} > 0,\ \frac{\partial U}{\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}} < 0,\ \frac{\partial^2 U}{\partial E[\tilde{R}_k]\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}} < 0 \qquad 2.1$$

The investor desires to maximize 2.1 subject to the constraint

$$\sum_{j=1}^{N+1} X_{j,k} = 1 \qquad 2.2$$

That is, the sum of the proportional holdings of all assets in the investor's portfolio must be one. Forming the Lagrangean function and maximizing:

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$$\underset{X_{ik}}{\text{Max}}\ \mathcal{L} = U_k(E[\tilde{R}_k],\ \sigma_{(\tilde{R}_k,\tilde{R}_k)} - \lambda_k(\sum_{j=1}^{N+1} X_{jk} - 1))$$

$$\frac{\partial \mathcal{L}}{\partial X_{ik}} \Rightarrow \frac{\partial U_k}{\partial E[\tilde{R}_k]} \frac{\partial E[\tilde{R}_k]}{\partial X_{ik}} + \frac{\partial U_k}{\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}} \frac{\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial X_{i,k}} - \lambda_k = 0$$

$$i = 1,\ldots,N+1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_k} \Rightarrow \sum_{j=1}^{N+1} X_{j,k} - 1 = 0\ . \qquad 2.3$$

Clearly,

$$\frac{\partial E[\tilde{R}_k]}{\partial X_{ik}} = E[\tilde{R}_i],$$

$$\frac{\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial X_{ik}} = \frac{\partial [\sum_i \sum_j X_{ik}X_{jk}\sigma_{ij}]}{\partial X_{ik}} = 2 \sum_j X_{jk}\sigma_{ij}$$

$$= 2\sigma_{(\tilde{R}_k,\tilde{R}_i)}\ . \qquad 2.4$$

Specifically, for the riskless asset,

$$\frac{\partial E[\tilde{R}_k]}{\partial X_{N+1,k}} = R_{N+1}$$

$$\frac{\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial X_{N+1,k}} = 0\ . \qquad 2.5$$

Thus,

$\frac{\partial U}{\partial E[\tilde{R}_k]} E[\tilde{R}_i$

$\frac{\partial U}{\partial E[\tilde{R}_k]} R_{N+1}$

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$\sum_{j=1}^{N+1} X_j$

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Let,

$h_{ik} = \frac{X_{ik}}{1-X_{N+1,k}}$

$\sum h_{jk} \tilde{R}_j = \tilde{R}_k^r$

$$\frac{\partial U}{\partial E[\tilde{R}_k]} E[\tilde{R}_i] + 2 \frac{\partial U}{\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}} \sigma_{(\tilde{R}_k,\tilde{R}_i)} - \lambda_k = 0 \qquad i = 1,\ldots,N$$

$$\frac{\partial U}{\partial E[\tilde{R}_k]} R_{N+1} - \lambda_k = 0$$

$$\sum_{j=1}^{N+1} X_j - 1 = 0 \qquad 2.6$$

The Lagrange multiplier may be eliminated by subtracting the $N+1^{th}$ equation from the remaining N equations, yielding after some rearranging,

$$-2 \frac{\partial U/\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial U/\partial E[\tilde{R}_k]} = \frac{E[\tilde{R}_i] - R_{N+1}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}} \qquad i = 1,\ldots,N$$

$$\sum_{j=1}^{N+1} X_{jk} - 1 = 0. \qquad 2.7$$

The Capital Asset Pricing Model is concerned with the choice of the portfolio of risky securities. In order to convert equation 2.7 into parameters involving the risky security, the following new symbols are defined.

Let,

$h_{ik} = \frac{X_{ik}}{1-X_{N+1,k}}$ be the proportion of asset i in investor k's portfolio of risky securities only. $i = 1,\ldots,N.$

$\sum_j h_{jk}\tilde{R}_j = \tilde{R}_k^r$ be the stochastically determined return of investor k's risky portfolio

$$\sigma_{(\tilde{R}^r_k,\tilde{R}^r_k)} = \sum_i \sum_j \ldots$$

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$$\sigma_{(\tilde{R}^r_k,\tilde{R}_i)} = \sum_j h \ldots$$

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Then equations

$$-2 \frac{\partial U/\partial\sigma_{(\tilde{R}_k,\tilde{R} \ldots}}{\partial U/\partial E[\tilde{R}_k]} \ldots$$

$$\sum_{j=1}^{N} X_j \ldots$$

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$$-2 \frac{\partial U/\partial\sigma_{(\tilde{R}_k,\tilde{R} \ldots}}{\partial U/\partial E[\tilde{R}_k]} \ldots$$

Setting 2.8 an

$$\frac{E[\tilde{R}_i] \ldots}{\sigma_{(\tilde{R}^r_k,\tilde{R} \ldots}}$$

$\sigma_{(\tilde{R}_k^r,\tilde{R}_k^r)} = \sum_i \sum_j h_{ik} h_{jk} \sigma_{ij}$ be the variance of investor k's risky portfolio

$\qquad = (1-X_{N+1,k})^{-2} \sigma_{(\tilde{R}_k,\tilde{R}_k)}$

$\sigma_{(\tilde{R}_k^r,\tilde{R}_i)} = \sum_j h_{jk} \sigma_{ij}$ be the covariance of the return of asset i with investor k's risky portfolio.

$\qquad = (1-X_{N+1,k})^{-1} \sigma_{(\tilde{R}_k,\tilde{R}_i)}$

Then equations 2.7 may be written as,

$$- 2 \frac{\partial U/\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial U/\partial E[\tilde{R}_k]} = \frac{E[\tilde{R}_i] - R_{N+1}}{(1-X_{N+1,k})\sigma_{(\tilde{R}_k^r,\tilde{R}_i)}} \qquad i = 1,\ldots,N$$

$$\sum_{j=1}^{N} X_{jk} = (1-X_{N+1,k}) \qquad 2.8$$

Now, equation 2.8 holds for all assets in the investor's portfolio. Thus, multiplying the numerator and denominator by h_{ik} and summing the numerator and denominator separately over all i (1,...,N) yields:

$$- 2 \frac{\partial U/\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial U/\partial E[\tilde{R}_k]} = \frac{\sum_j h_{jk}(E[\tilde{R}_i] - R_{N+1})}{[\sum_j h_{jk}\sigma_{(\tilde{R}_k^r,\tilde{R}_j)}](1-X_{N+1,k})} = \frac{E[\tilde{R}_k^r] - R_{N+1}}{\sigma_{(\tilde{R}_k^r,\tilde{R}_k^r)}(1-X_{N+1,k})} \qquad 2.9$$

Setting 2.8 and 2.9 equal to each other,

$$\frac{E[\tilde{R}_i] - R_{N+1}}{\sigma_{(\tilde{R}_k^r,\tilde{R}_i)}} = \frac{E[\tilde{R}_k^r] - R_{N+1}}{\sigma_{(\tilde{R}_k^r,\tilde{R}_k^r)}} ,$$

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$$E[\tilde{R}_i] - R_{N+1} = [\frac{E[\tilde{R}^r_k] - R_{N+1}}{\sigma_{(\tilde{R}^r_k,\tilde{R}^r_k)}}]\sigma_{(\tilde{R}^r_k,\tilde{R}_i)} \quad , \quad i = 1,\ldots,N \ . \qquad 2.10$$

Equations 2.10 describe the relationship between the required risk premium, defined as the expected return of a risky asset above the return obtainable from the riskless asset, in terms of the portfolio's expected excess return, the portfolio variance and the covariance between the portfolio return and the asset in question. Note that equations 2.9 are consistent with the familiar results that the marginal rate of substitution must be equal to the marginal rate of transformation for utility maximization to obtain. Specifically, the marginal rate of substitution between risk and return, conditional on the investor's utility function may be derived as,

$$dU_k = \frac{\partial U_k}{\partial E[\tilde{R}_k]} dE[\tilde{R}_k] - \frac{\partial U}{\partial \sigma_{(\tilde{R}_k,\tilde{R}_k)}} d\sigma_{(\tilde{R}_k,\tilde{R}_k)} = 0,$$

so that the marginal rate of substitution between risk and return is simply,

$$\frac{dE[\tilde{R}_k]}{d\sigma_{(\tilde{R}_k,\tilde{R}_k)}} = - \frac{\partial U_k/\partial\sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial U_k/\partial E[\tilde{R}_k]} \ . \qquad 2.11$$

Similarly, the marginal rate of transformation between risk and return for asset i, when the proportion of asset i in the investor's portfolio is increased by reducing the proportion of funds in th

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funds in the riskless asset is,

$$\frac{dE[\tilde{R}_k]}{d\sigma_{(\tilde{R}_k,\tilde{R}_k)}} = \frac{\dfrac{\partial E[\tilde{R}_k]}{\partial (X_{ik}-X_{N+1,k})}}{\dfrac{\partial\sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial (X_{ik}-X_{N+1,k})}}$$

$$= \frac{E[\tilde{R}_i] - R_{N+1}}{2\ \mathrm{Cov}(\tilde{R}_i,\tilde{R}_k)} \quad .$$

Similarly, the marginal rate of substitution between risk and return is equal to the marginal rate of transformation between risk and return for the risky portfolio as a whole, when funds are drawn from the riskless asset, since the expected return of the investor's portfolio may be written as,

$$E[\tilde{R}_k] = (1-X_{N+1,k})E[\tilde{R}_k^r] + X_{N+1,k}R_{N+1}$$

$$\frac{\partial E[\tilde{R}_k]}{\partial (1-X_{N+1,k})} = E[\tilde{R}_k^r] - R_{N+1},$$

and its variance may be written as,

$$\sigma_{(\tilde{R}_k,\tilde{R}_k)} = (1-X_{N+1,k})^2\sigma_{(\tilde{R}_k^r,\tilde{R}_k^r)}$$

$$\frac{\partial\sigma_{(\tilde{R}_k,\tilde{R}_k)}}{\partial (1-X_{N+1,k})} = 2(1-X_{N+1,k})\sigma_{(\tilde{R}_k^r,\tilde{R}_k^r)} \quad .$$

So that,

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$$\frac{dE[\tilde{R}_k]}{d\sigma_{(R_k,R_k)}} = \frac{E[\tilde{R}^r_k] - R_{N+1}}{2(1-X_{N+1,k})\sigma_{(\tilde{R}^r_k,\tilde{R}^r_k)}} .$$

That is, equation 2.9 insures that the marginal rate of transformation between risk and return for the portfolio as a whole, be equal to the investor's marginal rate of substitution between risk and return.[4]

Equation 2.10 reveals that the relevant measure of risk for an individual security is not the own variance of the security, but the covariance of that security with the investor's portfolio even though risk for the portfolio as a whole is measured by its own variance. This is because diversification can eliminate most of the effects of the asset's own variance on the total variance of the portfolio. Only the effects of the covariance of the asset's return with the other assets in the investor's portfolio cannot be eliminated through diversification.[5]

James Tobin has shown that the existence of a riskless asset insures that the proportionate composition of the assets in the risky portfolio is invariant to the specific shape of the utility function - as long as investors' decisions are made on the basis of the mean and variance of the portfolio returns, all investors are risk averse, and investors have homogeneous expectations of security returns (Assumptions B.2, B.4 and B.5 respectively). Thus, barring perfect positive correlation between any two or more subsets of securities, all investo

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all investors will hold the identical risky portfolio, adjusting their individual degrees of risk aversion by leveraging or lending at the risk free rate of return.[6]

Since all investors hold the same risky portfolio, in equilibrium, they all hold the "market portfolio", defined as that portfolio in which every asset outstanding is held in proportion to its total value. Thus, if the market portfolio is represented by R_M, equation 3.10 may be written as,

$$E[\tilde{R}_i] - R_{N+1} = \frac{E[\tilde{R}_M] - R_{N+1}}{\sigma_{(\tilde{R}_M,\tilde{R}_M)}} \sigma_{(\tilde{R}_M,\tilde{R}_i)}, \; i = 1,\ldots,N. \qquad 2.12$$

Now, the equilibrium risk premium for any asset may be expressed in terms of market parameters rather than individual investor parameters.

2.2 Empirical Findings in the Absence of Taxes

The Capital Asset Pricing Model, defined in 2.12 appears to lend itself to empirical verification by simple linear regression methods. That is, a cross section linear regression model can be hypothesized such that

$$\underline{Y}_i = \alpha + \beta \underline{X}_i + \underline{\tilde{\varepsilon}}_i$$

where the errors satisfy classical normal linear regression assumptions and

$$Y_i = E[\tilde{R}_i] - R_{N+1}$$

$$X_i = \sum_j X_j \sigma_{ij}$$

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Then the least-squares estimates $\hat{\beta}$ and $\hat{\alpha}$ can be used to test the hypothesis,

$$\beta = \frac{E(\tilde{R}_M) - R_{N+1}}{\sigma^2(\tilde{R}_M)} , \quad \alpha = 0 .$$

However, two immediate problems arise in practical application:

(1) calculation of X_i

(2) Y_i and β consist of expectations which are not directly observable.

Only realizations are observable. Thus it is necessary to transform the expectations into realizations or posit an expectations generating scheme from realized values.

Both of these problems are greatly facilitated by assuming all risky assets are related only through a simple relationship with an underlying market factor. Specifically, Professor Sharpe's "Diagonal Model" or Professor Fama's "Market Model"[7] assumes that all risky securities are linearly related to an index representing market movements as a whole.[8] Thus, the Market Model is specified as:

$$\begin{aligned} \tilde{R}_i &= a_i + b_i\tilde{I} + \tilde{\varepsilon}_i \qquad i = 1,\ldots,N . \\ E[\tilde{\varepsilon}_i] &= 0 \\ E[\tilde{\varepsilon}_i\tilde{\varepsilon}_j] &= 0 \quad i \neq j \\ &= \sigma^2_{\tilde{\varepsilon}} \quad i = j \\ E[\tilde{\varepsilon}_i,\tilde{I}] &= 0 \end{aligned} \qquad 2.13$$

a_i, b_i are constants, $\tilde{R}_i$, $\tilde{I}$, $\tilde{\varepsilon}$ are normally distributed random variables.

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$$\tilde{R}_M = \sum_j X_j \tilde{R}_j = \sum_j [X_j a_j + X_j b_j \tilde{I} + X_j \tilde{\varepsilon}_j],$$

where X_j is the proportion of asset j in the market portfolio. It is possible to scale $\tilde{I}$ so that $\sum_j X_j b_j = 1$, $\sum_j X_j a_j = 0$, i.e. scale $\tilde{I}$ so that $E[\tilde{R}_M] = E[\tilde{I}]$. Then:

$$\sigma^2_{(\tilde{R}_M)} = E\{[\tilde{I} + \sum_j X_j \tilde{\varepsilon}_j - E[\tilde{I}]]^2\}$$

$$= \sigma^2_{\tilde{I}} + \sum_j X_j^2 \sigma^2_{\tilde{\varepsilon}_j}$$

$$\sigma_{ij} = E\{[b_i[\tilde{I} - E(\tilde{I})] + \tilde{\varepsilon}_i][b_j[\tilde{I} - E[\tilde{I}]] + \tilde{\varepsilon}_j]\}$$

$$= b_i b_j \sigma^2_{\tilde{I}} \qquad i \neq j$$

$$= b_i^2 \sigma^2_{\tilde{I}} + \sigma^2_{\tilde{\varepsilon}_i} \qquad i = j \qquad 2.14$$

$$\sum_j X_j \sigma_{ij} = b_i \sigma_I^2 + X_i \sigma^2_{\tilde{\varepsilon}_i} \qquad \text{since} \qquad \sum_j X_j b_j = 1 .$$

Substituting 2.13 into the Capital Asset Pricing Model, equation 2.12 yields:

$$E[\tilde{R}_i] - R_{N+1} = [E(\tilde{R}_M) - R_{N+1}]\left[\frac{b_i \sigma^2_{\tilde{I}} + X_i \sigma^2_{\tilde{\varepsilon}_i}}{\sigma^2_{(\tilde{R}_M)}}\right] \qquad 2.15$$

Empirical evidence suggests that $\sigma^2_{\tilde{\varepsilon}_i}$ is approximately equal to $\sigma^2_{(\tilde{R}_m)}$ and for N sufficiently large, X_i will be small (on average $\frac{1}{N}$) relative to b_i (on average 1). Similarly $\sum_j X_j^2 \sigma^2_{\tilde{\varepsilon}_j}$ will be small relative to $\sigma^2_{\tilde{I}}$ in 2.14. Thus, $\sigma^2_{\tilde{R}_M} \simeq \sigma^2_{\tilde{I}}$, and substituting these approximations into 2.15 yields:

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$$E[\tilde{R}_i] - R_{N+1} = E[\tilde{R}_M - R_{N+1}]b_i \ . \tag{2.16}$$

Professor Jensen[10] has succeeded in expressing 2.15 in <u>ex-post</u> realizations rather than <u>ex-ante</u> expectations. The following derivation differs slightly from Jensen's for the sake of continuity of the presentation. Taking the expectation of 2.13: $E[\tilde{R}_i] = a_i + b_i E[\tilde{I}]$, solve for a_i and substitute back into 2.13. This yields:

$$\tilde{R}_i = E[\tilde{R}_i] + b_i[\tilde{I} - E(\tilde{I})] + \tilde{\varepsilon}_i, \tag{2.17}$$

and

$$\sum_j X_j\tilde{R}_j = E[\sum_j X_j\tilde{R}_j] + \sum_j X_j b_j[\tilde{I} - E[\tilde{I}]] + \sum_j X_j\tilde{\varepsilon}_j \ ,$$

or

$$\tilde{R}_M = E[\tilde{R}_M] + [\tilde{I} - E(\tilde{I})] + \sum_j X_j\tilde{\varepsilon}_j, \tag{2.18}$$

since $\sum_j X_j b_j = 1$. Substituting 2.17 and 2.18 into 2.16:

$$\tilde{R}_i - b_i[\tilde{I} - E(\tilde{I})] - \tilde{\varepsilon}_i - R_{N+1} = [\tilde{R}_M - (\tilde{I} - E[\tilde{I}])$$

$$- \sum_j X_j\tilde{\varepsilon}_j - R_{N+1}]b_i \ .$$

Thus,

$$\tilde{R}_i - R_{N+1} = (\tilde{R}_M - R_{N+1})b_i - b_i \sum_j X_j\tilde{\varepsilon}_j + \tilde{\varepsilon}_i \ . \tag{2.19}$$

Now, $\sum_j X_j\tilde{\varepsilon}_j \simeq 0$ since $E(\tilde{\varepsilon}_j) = 0$ and independent of each other and $\sigma^2_{(\Sigma_j X_j \tilde{\beta}}$

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Thus, $\tilde{R}_i - R_{N+1} \simeq (\tilde{R}_M - R_{N+1})b_i + \tilde{\varepsilon}_i$ or, taking conditional expectations of $\tilde{R}_i$:

$$E[(\tilde{R}_i - R_{N+1}) | \tilde{R}_M = R_M, b_i] = (R_M - R_{N+1})b_i \ .$$

Thus, the model contains only ex-post realizations. All variables are observable and the hypothesis is at least potentially testable.

Note that least squares estimates from time series regression of the "Market Model", 2.13, yields

$$\hat{\beta}_i = \hat{b}_i = \frac{\sum_t (R_{it} - \bar{R}_i)(I_t - \bar{I})}{\sum_t (I_t - \bar{I})^2} = \frac{\text{Cov}(\tilde{R}_i, \tilde{I})}{\sigma^2_{(\tilde{I})}}$$

and if the index is taken as a proxy for the movement of the market in general,

$$\hat{\beta}_i = \hat{b}_i = \frac{\text{Cov}(\tilde{R}_i, \tilde{R}_M)}{\sigma^2_{(\tilde{R}_M)}} \ .$$

Substituting $\hat{b}_i$ for b_i into 2.16, the equation becomes,

$$E[\tilde{R}_i] - R_{N+1} = E[\tilde{R}_M - R_{N+1}] \frac{\text{Cov}(\tilde{R}_i, \tilde{R}_M)}{\sigma^2_{(\tilde{R}_M)}}$$

identical to the CAPM for sufficiently large number of securities.

Douglas[12] published the earliest direct test of the CAPM. He regresses the mean quarterly rates of return of a cross section sample of 6

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sample of 616 common stocks on both their variance of return and on heir covariance of return with the average return on all stocks in the sample for the given quarter (repeated for seven non-overlapping five year periods from 1926 to 1960). The CAPM predicts that Douglas' regression equation:

$$R_i = \hat{\gamma}_0 + \hat{\gamma}_1 \sigma(\tilde{R}_i, \tilde{I}) + \hat{\gamma}_2 \sigma^2_{\tilde{R}_i} + \tilde{\varepsilon}_i$$

should have coefficients consistent with:

$$\hat{\gamma}_0 = R_{N+1}$$

$$\hat{\gamma}_1 + \frac{(R_I - R_{N+1})}{\sigma^2_{\tilde{R}_M}}$$

$$\hat{\gamma}_2 = X_i \frac{R_I - R_{N+1}}{\sigma^2_{R_M}} \simeq 0$$

for a well diversified portfolio and sufficiently large N. However, Douglas finds that $\hat{\gamma}_2$ is positive and significant and $\hat{\gamma}_1$ is not significantly different from zero.

He also cites an unpublished study by John Lintner confirming the Douglas results. Lintner utilizes a two pass regression. The first, a time series regression of the annual rates of return of 301 common stocks over the ten-year period 1954-63, on the yearly average rate of return for all the stocks in the sample. The slope coefficient $\hat{\beta}_i$ in the regression equation:

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$$R_{it} = \hat{\alpha}_i + \hat{\beta}_i R_{Mt} + \tilde{\varepsilon}_{it}$$

is the estimate of b_i in the market model. He then uses these estimates and the estimate of the residual variance, $\bar{S}^2_{\tilde{\varepsilon}_{it}}$ of the time series regression as independent variables in a second pass, cross-sectional regression with the mean annual return of each security as the dependent variable.

$$\bar{R}_i = \hat{\gamma}_0 + \hat{\gamma}_1\hat{\beta}_i + \hat{\gamma}_2 \hat{\bar{S}}^2_{\tilde{\varepsilon}_i} + \tilde{\varepsilon}_i \ .$$

The results again seem to contradict the CAPM. Although $\hat{\gamma}_1$ is significnat its value is much less than $R_M - R_{N+1}$, its predicted value. Also, $\hat{\gamma}_2$ is positive and significant and $\hat{\gamma}_0$ is much greater than its predicted value, R_{N+1}.

Miller and Scholes[13] have examined possible bias in the model as presented by Lintner. Replicating Lintner's study on a different set of data, but over the identical time periods, they test for the following source of bias:

(1) Misspecification due to failure to include the risk free rate of interest in the basic estimating equation.

(2) Nonlinearity in the risk-return relationship.

(3) Possible heteroscedasticity.

(4) Errors in measurement of risk and return.

The study concludes that (1), (2) and (3) could not have caused the Lintner results. Although heteroscedasticity _per se_ does not imply bias in the coefficients, they investigate the possibility that excessive weight given observations at the high end of the risk scale, due t

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scale, due to the greater dispersion at this end of the scale, distorts the true relationship between risk and return in the specific data utilized. They find, however, that heteroscedasticity does not seem to explain the Lintner estimates. In addition, the authors indicate that the regression is nonlinear. Although a quadratic seems to fit the data well, its curvature was "in the wrong direction".[14]

However (4) seems to be a possible explanation of the Lintner-Douglas results. Bias due to the inclusion of a random variable, $\hat{\beta}$ as an explanatory variable may have reduced its coefficient, $\hat{\gamma}_1$ by as much as 64 per cent of its true value. The estimate of residual risk $\hat{S}_{e_i}$ is found to be correlated with the estimates of $\hat{\beta}_i$. Thus, the estimate may have been acting as a proxy for the non-diversifiable risk coefficient, b_i.

However, these two sources of potential bias are not found to be of sufficient magnitude to have caused the Lintner-Douglas results alone. Further tests on the index as a proxy for the "true index" and skewness in the errors of the first-pass regression are performed. The index measure is found to be satisfactory. Although skewness is not directly observable according to the investigators, a simulation indicates that this could have caused the Lintner results. It should be observed that no attempt is made to "correct for" these biases and verify the CAPM. They are merely suggested as potential explanations for the disturbing Lintner-Douglas results.

Jacob[15] replicates Lintner's two-step procedure, but explicitly includes the risk free rate (yield on 90-day Treasury bills). Us

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bills). Using a sample of 593 NYSE stocks, she regresses annual and monthly return data on an unweighted index of the price relatives of her sample, to find estimates of b_i. The cross section regression on $\hat{\beta}_i$ found the regression constant to be greater than predicted and the slope to be less than predicted. The average R^2 was only .089, leading her to conclude that the CAPM and/or the market model are not satisfactory predictors of security behavior.

Black, Jensen and Scholes[16] (B-J-S), provide tests of the CAPM designed to eliminate the bias associated with the Lintner-Douglas results. Using a large sample of NYSE securities, they divide them into ten portfolios grouped to maximize the dispersion of individual security $\hat{\beta}$'s. Using the average $\hat{\beta}$ in each group (as opposed to the individual security $\hat{\beta}$) they are able to significantly reduce bias resulting from including $\hat{\beta}$, a random variable, as an explanatory variables. By estimating a series regression on the average $\hat{\beta}$'s using data from periods <u>subsequent</u> to the period in which the $\hat{\beta}$'s were estimated, they are able to eliminate the correlation between $\hat{\beta}$ and S_e^2. These two problems associated with the Lintner result corrected, the CAPM is tested using both time series and cross-section regression techniques. Results of the time series regression over the entire period

$$R_{it} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_i + \tilde{\varepsilon}_{it} \qquad i = 1,\ldots,10,$$

suggests that the regression constant is inversely related to the β coefficient, with high β securities having negative constants and low β securities having positive constants. Similarly, the slope coeff

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slope coefficient is related to the β coefficient. Dividing the period 1931-1965 into 17 non-overlapping two year periods and estimating cross-sectional regressions on the same data they find the coefficients to be unstable over-time, the intercept increasing over time and the slope decreasing. However, each period's regression results seemed to be significant. This led them to consider a "two factor" model of form:

$$\tilde{R}_{it} = \tilde{\gamma}_{0t}(1 - \beta_j) + \tilde{\gamma}_{1t}\beta_j + \tilde{e}_{ji}$$

where $\tilde{\gamma}_{0t}$ and $\tilde{\gamma}_{1t}$ are random variables fluctuating over time, $\tilde{\gamma}_{0t}$ is called the "β factor", since its coefficient is a function of β, while the predicted value of $\tilde{\gamma}_{1t}$ is the market rate of return. Although no direct tests of the "Beta factor" are performed, they conclude that its existence has been established by their explicit estimation procedure. However, their estimate of the Beta factor required assuming both independence of the residuals and homoscedasticity. The latter has been shown to be violated at least in the Miller-Scholes data, (and there is no justification for assuming the former), making the B-J-S results suspect. In addition, although some theoretical explanations for the existence of the "β factor" have been offered, none of them explains its secular trend.[17, 18]

Fama and Macbeth[19] also attempt to test the CAPM. Using all common stocks listed on the NYSE (and not delisted over the test period), they group all securities into 20 portfolios by their individual

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individual $\hat{\beta}_i$ using monthly data over an initial 7 year period. They then recompute the portfolio average $\hat{\beta}$'s $(\hat{\beta}_p)$ using the subsequent five year data and test using subsequent four year data. The full test equation is:

$$R_{pt} = \tilde{\gamma}_{0t} + \tilde{\gamma}_{1t}\hat{\beta}_{p,t-1} + \tilde{\gamma}_{2t}\hat{\beta}^2_{p,t-1} + \tilde{\gamma}_{3t}\bar{S}_{p,t-1}(\tilde{e}_i) + \tilde{\varepsilon}_{pt},$$

$$p = 1,\ldots,20$$

$\hat{\beta}^2_{p_{t-1}}$ is the average of the squared $\hat{\beta}$'s in each portfolio, $\bar{S}_{p_{t=1}}(\tilde{e}_i)$ is the standard deviation of the least square errors of the market model used to compute $\hat{\beta}_i$. Using a sequence of cross-sectional regressions and analysing the sequence of coefficients they were unable to reject the hypothesis that:

$$E[\tilde{\gamma}_{2t}] = E[\tilde{\gamma}_{3t}] = 0$$

$$E[\tilde{\gamma}_{1t}] = E[\tilde{R}_{Mt}] - E[\tilde{R}_{zt}]$$

$$E[\tilde{\gamma}_{0t}] = R_{N+1,t},$$

where $\tilde{R}_{zt}$ is the stochastic return of a portfolio which has a zero covariance with the market portfolio. The Fama and Macbeth results seem to verify the CAPM. However, distinguishing between a random variable $\tilde{\gamma}_{jt}$, with expected value, $E[\tilde{\gamma}_{jt}] = k$ and a parameter, $\gamma_{jt} = k$ for all t, they conclude that,

> ... there are variables in addition to $\hat{\beta}_p$ that systematically affect period-by-period returns. Some of these omitted variables are apparently related to $\hat{\beta}^2_p$ and $[\bar{\sigma}_p(\tilde{e}_i)]$. But the latter

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> are almost surely proxies, since there is no economic rationale for their presence in our stochastic risk-return model.[20]

Blume and Friend,[21] also present tests of the CAPM in the light of the B-J-S results. They point out that the most generally accepted theoretical explanation of the "Beta factor" has been the absence of a true riskless asset.[22] However, replicating the B-J-S grouping and time series testing procedure, they show that the return on the Beta factor is inconsistent with its theoretical value (which would have to be close to the rate on 90-day treasury bills). They offer an alternative theoretical explanation for the existence of the Beta factor, that is, segmentation of markets between stocks and bonds. Nevertheless, no explicit test of their hypothesis was performed.

2.3. Empirical Findings of the Effects of Taxes on the Structure of Security Returns and the Capital Asset Pricing Model

Brennan[23] has made the only explicit attempt to assess the impact of investor taxes on the CAPM. Deriving market equilibrium from individual utility maximization, Brennan finds that the theoretical relationship of risk and return, given a specific tax structure is:

$$E[\tilde{R}_i] = T_2 R_{N+1} + T_1 \delta_i + E[\tilde{R}_M - T_1 \delta_M - T_2 R_{N+1}]\beta_i$$

where

δ_i, δ_M are dividend yields on asset i and the market portfolio respectively (assumed to be known *ex-ante* with certainty).

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$$T_1 = \frac{T_d - T_g}{1 - T_g}\ ,\ T_2 = 1 - T_1$$

where T_d and T_g are weighted average marginal tax rates on dividend and capital gains income respectively. It is noted that if dividend and capital gains income are taxed at the same rate, then $T_1 = 0$, and $T_2 = 1$ and the traditional CAPM applies. However, where $T_d > T_g$ the slope is less than predicted by the CAPM and the intercept is larger. In addition, the investor is not indifferent between capital gains versus dividend income, preferring a security with high capital gains to a security with high dividend yield, even though the total, before tax return is the same for the two securities.

Empirical tests of the model on data from all New York Stock Exchange securities over the period 1946 to 1965 were performed. The tests consisted of constructing a set of 99 portfolios, according to dividend yields of each security. Using the data from these portfolios Brennan attempts to fit the data to the regression:

$$R_i = \hat{\gamma}_0 + \hat{\gamma}_1\beta_i + \hat{\gamma}_2\delta_i + \tilde{e}_i\ .$$

He concludes that the results obtained are not enconsistent with the hypothesis, that tax effects on security returns are important and that his model fits the data better than the traditional CAPM.

However, Brennan's main concern is with the effects of <u>differential tax rates</u> on the trade-off between dividend and capital gains income. He assumes a specific tax structure such that the marginal tax rate on capital gains income is exogenous, and independent of the level of realized capital gains, asserting this is necessary in

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necessary in order to derive "manageable expressions... for the variance of after-tax terminal wealth."[24] This assumption is equivalent to removing all uncertainty from the tax function since it is further assumed that the marginal tax rate on ordinary income depends upon labor and dividend income, know for certain at the beginning of the period.

Furthermore, Black and Scholes[25] argue that if differential tax rates do significantly affect the equilibrium value of securities, then corporations will tend to adjust their dividend policy to eliminate any differential effect. Noting that Brennan's cross-section regression techniques do not identify a causal relationship or the direction of causality, they point out that low risk securities are likely to have high dividend yields and vice versa. Thus, Brennan's regression results could have resulted from the phenomenon identified by B-J-S. That is, that low risk securities tend to have higher expected returns than the CAPM predicts because of the "Beta Factor." If low risk tends to be associated with high dividend yields, then a regression containing dividend yield as an explanatory variable would result in a spurious significant coefficient on dividend yields, whereas actually the "Beta Factot" is the significant explanatory variable.

In order to allow for an independent test of the dividend policy effect, they derive the equation,

$$E[\tilde{R}_i] = \gamma_0 + E[(\tilde{R}_M) - \gamma_0]\beta_i + \gamma_1(\frac{\delta_i - \delta_M}{\delta_M}) ,$$

by combining the B-J-S hypothesis with Brennan's hypothesis. δ_i is the dividen

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the dividend yield of asset (portfolio) i, δ_m is the dividend yield of the market portfolio. It is now possible to test the hypothses,

$\gamma_0 = R_{N+1}$ Implies no Beta Factor

$\gamma_1 = 0$ Implies no dividend effect,

Using data from 1936 to 1966, they find $\hat{\gamma}_1$ not significantly different from zero for the entire time period and for five separate subperiods. $\hat{\gamma}_0$ was significantly different from R_{N+1} for the entire period and for all subperiods except 1936-1946. Thus, they conclude that the results reported by Brennan seems to have been caused by the proxying effect of the dividend yield variable for the Beta Factor variable confirming the B-J-S results.

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NOTES TO CHAPTER 2

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CHAPTER 3

THE EFFECTS OF PERSONAL INCOME TAXES ON INVESTOR BEHAVIOR

This chapter contains a mathematical formulation of the effects of personal income taxes on investor behavior in the context of a general personal income tax structure. The formulation extends the traditional "Capital Asset Pricing Model" to permit the investor to maximize expected utility with respect to after tax rather than before tax variables.[1] First order conditions are derived assuming investor income is subject to a relatively general tax structure. Key parameters are identified which describe the functional form of the tax structure and the joint probability density function of the relevant securities. Finally, a linear and quadratic tax function is assumed and specific results are derived.

3.1. The Assumptions

Market Assumptions[2]

A.1. Each individual investor is free to borrow or lend an unlimited amount at an exogenously determined, before tax "riskless" rate of interest.

A.2. Each investor can invest any fraction of his capital in any or all of a given finite set of risky securities.

A.3. The market supply, in terms of number of shares, of each risky asset is exogenously determined.

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A.4. All investors make all purchases and sales at discrete points in time, the time period being identical for all investors.

A.5. All assets are traded in a single competitive market. That is, each investor's demand for any asset is sufficiently small relative to total market demand, so that his transactions have an insignificant effect on the market price of that asset. In addition, there is no transaction or information costs and all assets are perfectly divisible..

Assumptions Regarding Investors

B.1. Investors have already determined the fraction of their capital they intend to use for liquidity and transactions purposes. This fraction, as well as the total amount of capital held for speculative purposes is independent of the tax structure.

B.2. Each investor acts in accordance with the Von-Neumann-Morgenstern axioms of expected utility maximization and possesses a quadratic utility function with respect to terminal after tax wealth. This implies, that investor's decisions are made on the basis of the mean and variance of after tax portfolio returns.[3]

B.3. Investors are able to assess the relevant expectations, variances and covariances of each risky asset's before tax rate of return.

B.4. All investors are "risk averse" in the sense that each prefers more after tax expected return for a given amount of after tax risk and less after tax risk for a given amount of after tax return.[4]

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B.5. All investors have identical assessments of each asset's before tax risk and return.

B.6. Each investor knows the functional form and relevant parameters of the tax function facing him.

Assumptions Regarding the Tax Function

C.1. The total tax liability for any investor is a monotonically increasing, differentiable, function of the investor's income over the relevant time period.

C.2. Income is defined as the sum of income from labor and capital sources. Labor income is known with certainty over the relevant period and is exogenously determined. Capital income is the sum of capital gains, dividends and interest, in dollars, generated over the period from capital investment. At least a part of capital income is stochastically determined for a significantly large number of investors. Capital gains and dividend income is assumed, for income tax purposes, to be realized at the end of the period.

The assumptions enumerated above are designed to retain the essential features of the Capital Asset Pricing Model, modified only to permit investors to take cognizance of the tax function and behave in relation to after tax rather than before tax variables. All departures from the traditional assumptions are made for obvious reasons with the possible exception of assumption B.2. Recall that the traditional assumptions underlying the no-tax model requires only that investors' decisions be made on the basis of the mean and variance of portfolio returns.[5] It is well known that this assumption is consistent with Von-Neumann-Morgenstern axioms of expected utility maximizat

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maximization when either (i) the individual has a quadratic utility function with respect to terminal wealth, or (ii) the joint distribution of security returns belongs to a two parameter family of distributions.[6] Since the solution to the traditional model is identical regardless of the choice of the underlying justification for the mean-variance approach, further specification is not necessary.

However, if investors behave in relation to after tax portfolio distributions, it would be difficult to justify the mean-variance approach on the basis of a two parameter probability distribution of after tax security returns. Clearly, if before tax returns are normally distributed, then a severely limited family of tax functions is necessary if after tax returns are to be a member of a two parameter family of distributions. Similarly, if justification for the mean-variance approach is based on the proposition that after tax returns are normally distributed, severe restrictions on the before tax asset distributions are necessary, and it is likely that little intuitive justification could be found for adopting such an assumption. Thus, unless otherwise specified, justification for the mean-variance approach will be based on the assumption of quadratic utility functions from here on. The reader is advised that this specific specification of each investor's utility function is not without its drawbacks. Caution must be utilized in applying the specific results derived here to more general models.[7]

By retaining the essential features of the traditional model it is possible to undertake a direct comparison of results derived from the after tax model with the traditional no tax model. This facilitate

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facilitates comparison of the behavior of investors in the absence of taxes with investors subject to a personal income tax. In this way, behavior resulting from the imposition of the tax can be isolated from investor behavior in the absence of the tax. In addition, consideration of market equilibrium permits assessment of the effects of investor taxes on the equilibrium structure of security returns, compared to the structure that would exist in the absence of taxes.

3.2. Model of Investor Behavior

This section presents a model describing the behavior of K investors, indexed by $k(1,\ldots,k,\ldots,K)$ trading in N different risk assets indexed by i and j $(1,\ldots,i,\ldots,j,\ldots,N)$ in accordance with the assumptions set out in Section 3.1.[8] The following variables and parameters are defined:

$\tilde{P}_i$ is the random end of period price per share of asset i.

P_i is the non-random beginning period price of asset i.

D_i is the non-random dividend payment, in dollars of asset i over the period.

$\tilde{R}_i = \dfrac{\tilde{P}_i - P_i + D_i}{P_i}$ is the single period before tax rate of return of risky asset i.

R_{N+1} is the single period before tax rate of return on the riskless asset.

V_k^r is the total dollars investor k places in speculative capital.

X_{ik} is the proportion of total speculative capital invested in asset i by investor k.

$X_{N+1,k}$ is the proportion of total speculative capital invested in the riskless asset by investor k.

Y_k^h is the certain dollar amount investor k receives over the period.

$\tilde{Y}_k^c = V_k(\sum_{i=1}^{N} X_{ik}\tilde{R}_i + X_{N+1}R_{N+1})$ is the capital income received by individual k over the period.

$\tilde{Y}_k = \tilde{Y}_k^c + Y_k^h$ is total income received by investor k over the period.

E[ ] is the expectations operator.

$\sigma_{(\tilde{R}_i,\tilde{R}_j)} = E[(\tilde{R}_i - \bar{R}_i)(\tilde{R}_j - \bar{R}_j)]$ is the covariance of single period rate of return between risky asset i and j.

$\sigma_{(\tilde{R}_k,\tilde{R}_j)}$ is the covariance of single period rate of return between investor k's entire portfolio and asset j.

$\sigma_{(\tilde{R}_i,\tilde{R}_k)}$ is the variance of single period rate of return of investor k's entire portfolio.

and in general,

$\sigma_{(\tilde{L},\tilde{M})}$ is the covariance between random variables $\tilde{L}$ and $\tilde{M}$ for $\tilde{L} \neq \tilde{M}$ or the variance of random variable $\tilde{L}$ for $\tilde{L} = \tilde{M}$.

$\tilde{R}_k = \sum_{i=1}^{N} X_{ik}\tilde{R}_i + X_{N+1,k}R_{N+1}$ is the single period rate of return of investor k's entire portfolio.

$\tilde{T}_k = g(\tilde{Y}_k)$ is investor k's total tax liability at the end of the period, $g(\tilde{Y}_k)$ is a monotonically increasing, differentiable function of $\tilde{Y}_k$.

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$1 > g'_k > 0$ is investor k's marginal tax rate at the end of the period. Note that

$$\tilde{g}_k^{-1} = \frac{dg}{d\tilde{Y}_k} = \frac{\partial g}{\partial \tilde{Y}_k^c} = \frac{\partial g}{\partial Y_k^h}$$

$$\tilde{g}''_k = \frac{d^2 g}{d\tilde{Y}_k^2}$$

$\tilde{R}_k^t = \tilde{R}_k - \frac{1}{V_k} \tilde{T}_k$ is the after tax single period rate of return of investor k's portfolio.

Throughout, ˜ represents random variables. Where convenient superbars over a variable replaces the expectations operator and ˜ is suppressed when the meaning is clear. The term "return" replaces the phrase "single period rate of return." In Addition, since this section deals with the individual investor, the subscript k, identifying investors will be omitted when convenient. It is understood that the relevant variables should be identified for each investor.

In accordance with assumptions B.2 and B.4 above, investor k's utility frunction may be written as:

$$U_k = U_k(E[\tilde{R}_k^t],\ \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}) \qquad 3.1$$

$$\frac{\partial U}{\sigma_{E[\tilde{R}_k^t]}} > 0,\ \frac{\partial U}{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}} < 0,\ \frac{\partial^2 U}{\partial_{E[\tilde{R}_k^t]} \partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}} < 0$$

The investor desires to maximize 3.1 subject to the constraint

$$\sum_{j=1}^{N} X_{j,k} + X_{N+1,k} = 1 \qquad 3.2$$

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$$\frac{\partial L}{\partial X_i} = \frac{\partial U}{\partial E[\tilde{R}_r^t}$$

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That is the sum of the proportional holdings of all assets in the portfolio must be one. Forming the Lagrangian function and maximizing:

$$\underset{X_i}{\text{Max}}\ L = U(E[\tilde{R}_k^t],\ \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}) - \lambda_k(\sum_{j=1}^{N+1} X_j - 1),$$

$$\frac{\partial L}{\partial X_i} \Rightarrow \frac{\partial U}{\partial E[\tilde{R}_k^t]}\ \frac{\partial E[\tilde{R}_k^t]}{\partial X_i} + \frac{\partial U}{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}\ \frac{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial X_i} - \lambda = 0,\ i = 1,\ldots,N+1 \qquad 3.3$$

$$\frac{\partial L}{\partial \lambda} \Rightarrow \sum_{i=1}^{N+1} X_i - 1 = 0,$$

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_i} = \frac{\partial}{\partial X_i}\{\sum_{j=1}^{N+1} X_j\bar{R}_j - E[\frac{1}{V_k} + g(V_k \sum_{j=1}^{N+1} X_j R_j)]\} .$$

$$= \bar{R}_i - \frac{1}{V_k} E[\frac{\partial g}{\partial X_i}],$$

$$= \bar{R}_i - \frac{1}{V_k} E[g_k' \cdot \frac{\partial \tilde{Y}_k}{\partial X_i}]$$

$$= \bar{R}_i - \frac{1}{V_k} E[\tilde{g}_k' \cdot V_k\tilde{R}_i].$$

Add and subtract $E[\tilde{g}_k']E[\tilde{R}_i]$,

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_i} = \bar{R}_i - \frac{1}{V_k} \cdot V_k E[\tilde{g}_k'\tilde{R}_i] - E[\tilde{g}_k']E[\tilde{R}_i] + E[\tilde{g}_k']E[\tilde{R}_i],$$

$$= \bar{R}_i(1 - E[\tilde{g}_k']) - \partial_{(\tilde{g}_k',\tilde{R}_i)}. \qquad 3.4$$

In addition,

$$\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)} = E[(\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k)^2] - \{E[\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k]\}^2 .$$

Thus,

$$\frac{\partial\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial X_i} = E[\frac{\partial}{\partial X_i}(\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k)^2] - 2E[(\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k]E[\frac{\partial}{\partial X_i}(\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k)]$$

$$= E[2(\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k)(\tilde{R}_i - \frac{1}{V_k}V_k\tilde{g}_k'\cdot\tilde{R}_i)]$$

$$- 2E[\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k]E[\tilde{R}_i - \frac{1}{V_k}V_k\tilde{g}_k'\tilde{R}_i)]$$

$$= 2\{E[\tilde{R}_k\cdot\tilde{R}_i - \frac{1}{V_k}\tilde{T}_k\tilde{R}_i - \tilde{g}_k'\tilde{R}_k\tilde{R}_i + \frac{1}{V_k}\tilde{T}_k\tilde{g}_k'\tilde{R}_i] - E[\tilde{R}_k]E[\tilde{R}_i]$$

$$+ \frac{1}{V_k}E[\tilde{T}_k]E[\tilde{R}_i] + E[\tilde{R}_k]E[\tilde{g}_k'\tilde{R}_i] - \frac{1}{V_k}E[\tilde{T}_k]E[\tilde{g}_k'\tilde{R}_i]\} .$$

$$= 2\{E[\tilde{R}_k\tilde{R}_i] - E[\tilde{R}_k]E[\tilde{R}_i] - \frac{1}{V_k}(E[\tilde{T}_k\tilde{R}_i] - E[\tilde{T}_k]E[\tilde{R}_i])$$

$$- (E[\tilde{R}_k\tilde{g}_k'\tilde{R}_i] - E[\tilde{R}_k]E[\tilde{g}_k'\tilde{R}_i]) + \frac{1}{V_k}(E[\tilde{T}_k\tilde{R}_i\tilde{g}_k']$$

$$- E[\tilde{T}_k]E[\tilde{R}_i\tilde{g}_k'])\}$$

$$= 2\{E[(\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k)\tilde{R}_i] - E[\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k]E[\tilde{R}_i]$$

$$- E[(\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k)(\tilde{g}_k'\tilde{R}_i)]$$

$$+ E[\tilde{R}_k - \frac{1}{V_k}\tilde{T}_k]E[\tilde{g}_k'\tilde{R}_i]\}$$

$$= 2\{\sigma_{(\tilde{R}_k^t,\tilde{R}_i)} - \sigma_{(\tilde{R}_k^t,(\tilde{g}_k'\cdot\tilde{R}_i))}\} = 2\sigma_{(\tilde{R}_k^t,\tilde{R}_i(1-\tilde{g}_k'))} \qquad 3.5$$

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Equation 3.5 represents the marginal contribution of asset i to the total after tax portfolio risk and is propottion to the covariance of after tax portfolio return with the after tax marginal contribution of asset i to the total portfolio return. (That is, $\frac{\partial \tilde{R}_k^t}{\partial X_{ik}} = \tilde{R}_i(1 - \tilde{g}_k').$)

For the $N + 1^{th}$ riskless asset, 3.4 and 3.5 reduce to:

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_{N+1}} = R_{N+1}(1 - E[\tilde{g}_k']) \qquad 3.4i$$

$$\frac{\partial \sigma_{(\tilde{R}_k^t, \tilde{R}_k^t)}}{\partial X_{N+1}} = -2R_{N+1}\sigma_{(\tilde{R}_k^t, \tilde{g}_k')} \qquad 3.5i$$

Clearly, the riskless asset is no longer "riskless" in the sense that its marginal contribution to the portfolio return is no longer non-stochastic. The marginal after tax return depends upon the variability of the marginal tax rate. That is $\frac{\partial \tilde{R}_k^t}{\partial X_{N+1}} = R_{N+1}(1 - \tilde{g}_k')$, a stochastic variable. In addition, if the investor increases the proportion of the riskless asset in his total portfolio he reduces variability of that portfolio in two ways.

(i) He reduces before tax portfolio variance by reducing leverage. That is the investor increases the non-stochastic element in his portfolio.

(ii) The investor reduces the variabilityof his tax base. This in turn affects the variability of his tax bill, thereby affecting the volatility of after tax portfolio returns.

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Substituting (3.4), (3.5), (3.4i), (3.5i) into (3.3) yields:

$$\frac{\partial U}{\partial E[\tilde{R}_k^t]} \{\bar{R}_i(1 - E[\tilde{g}_k']) - \sigma_{(\tilde{g}_k',\tilde{R}_i)}\}$$

$$+ \frac{\partial U}{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}} \{2[\sigma_{(\tilde{R}_k^t,\tilde{R}_i(1 - g_k'))}] - \lambda_k = 0 \qquad i = 1,\ldots,N \qquad 3.6$$

$$\frac{\partial U}{\partial E[\tilde{R}_k^t]} \{R_{N+1}(1 - E[\tilde{g}_k']) + \frac{\partial U}{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}} \{-2R_{N+1}\sigma_{(\tilde{R}_k^t,\tilde{g}_k')}\} - \lambda_k = 0$$

The Lagrange multiplier (λ_k) may be eliminated by subtracting equation N + 1 from the remaining N equations, so that,

$$\frac{\partial U}{\partial E[\tilde{R}_k^t]} \{(\bar{R}_i - R_{N+1})(1 - E[\tilde{g}_k']) - \sigma_{(\tilde{g}_k',\tilde{R}_i)}\}$$

$$+ 2\frac{\partial U}{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}} \{\sigma_{(\tilde{R}_k^t,\tilde{R}_i(1 - \tilde{g}_k'))} + R_{N+1}\sigma_{(\tilde{R}_k^t,\tilde{g}_k')}\} = 0.$$

$$i = 1,\ldots,N \qquad 3.7$$

Solving for the marginal rate of substitution between after tax risk and return yields,

$$-\frac{\partial U_k/\partial \sigma_{\tilde{R}_k^t,\tilde{R}_k^t}}{\partial U_k/\partial E[\tilde{R}_k^t]} = \frac{1}{2}\,\frac{(\bar{R}_i - R_{N+1})(1 - E[\tilde{g}_k']) - \sigma_{(\tilde{g}_k',\tilde{R}_i)}}{\sigma_{(\tilde{R}_k^t,\tilde{R}_i(1-\tilde{g}'_k))} + R_{N+1}\sigma_{(\tilde{R}_k^t,\tilde{g}_k')}} \, . \qquad 3.8$$

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Since, on the margin, an increase in the proportion of funds held in one asset requires a decrease in the proportion of funds held in at least one other asset, assumed here to be the $(N + 1)^{th}$ security, the marginal rate of transformation between risk and return for risky security i is:

$$- \frac{d\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)} / d(X_i - X_{N+1})}{dE[\tilde{R}_k^t] / d(X_i - X_{N+1})} = \frac{\dfrac{\partial E[\tilde{R}_k^t]}{\partial (X_i - X_{N+1})}}{\dfrac{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial (X_i - X_{N+1})}} , \qquad 3.9$$

$$dx_j = 0, \; j \neq i$$

which is the right hand side of 3.8. Thus the familiar results hold. Utility maximization requires the marginal rate of substitution (between after tax risk and return) be equal to the marginal rates of transformation (between after tax risk and return) for each security in the investor's portfolio.

In addition, since 3.8 holds for all securities held by investor k,

$$\frac{(\bar{R}_i - R_{N+1})(1-E[\tilde{g}_k']) - \sigma_{(\tilde{g}_k',\tilde{R}_i)}}{\sigma_{\tilde{R}_k^t,\tilde{R}_i(1-\tilde{g}_k')} + R_{N+1}\sigma_{\tilde{R}_k^t,\tilde{g}_k'}} = \frac{(\bar{R}_j - R_{N+1})(1-E[\tilde{g}_k']) - \sigma_{(\tilde{g}_k',\tilde{R}_j)}}{\sigma_{\tilde{R}_k^t,\tilde{R}_j(1-\tilde{g}_k')} + R_{N+1}\sigma_{\tilde{R}_k^t,\tilde{g}_k'}} \qquad 3.10$$

for all $i,j = 1,\ldots,N$

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$$\frac{\tilde{R}_i - R_{N+1}}{\sigma(\tilde{R}_k, \tilde{R}_i)} =$$

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That is, the marginal rates of transformation between after tax risk and return will be equal for each risky asset in the investor's portfolio. However, solving for the marginal rate of transformation of before tax risk and return[10] of asset i yields:[11]

$$\frac{\bar{R}_i - R_{N+1}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}} = \left[\frac{R_j - R_{N+1}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}} - \frac{\sigma_{(\tilde{g}_k',\tilde{R}_j)}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}(1-E[\tilde{g}_k'])}\right] \left[\frac{1-(1/V_k)\frac{\sigma_{(\tilde{R}_k,\tilde{R}_i)}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}} - \frac{\sigma_{(\tilde{R}_k^t,\tilde{g}_k'\tilde{R}_i)}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}} + R_{N+1}\frac{\sigma_{(\tilde{R}_k^t,\tilde{g}_k')}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}}}{1-(1/V_k)\frac{\sigma_{(\tilde{T}_k,\tilde{R}_j)}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}} - \frac{\sigma_{(\tilde{R}_k^t,\tilde{g}_k'\tilde{R}_j)}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}} + R_{N+1}\frac{\sigma_{(\tilde{R}_k^t,\tilde{g}_k')}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}}}\right]$$

$$+ \frac{\sigma_{(\tilde{g}_k',\tilde{R}_i)}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}(1-E[\tilde{g}_k'])}$$

That is,

$$MRT^i_{(\sigma,\ E[\tilde{R}])} = MRT^j_{(\sigma,\ E[\tilde{R}])}[A] + C,$$

where,

$$A = \frac{1 - \frac{1}{V_k}\frac{\sigma_{(\tilde{T}_k,\tilde{R}_i)}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}} - \frac{\sigma_{(\tilde{R}_k^t,\tilde{g}_k'\tilde{R}_i)}}{\sigma_{(\tilde{R}_i,\tilde{R}_i)}} + R_{N+1}\frac{\sigma_{(\tilde{R}_k^t,\tilde{g}_k')}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}}}{1 - \frac{1}{V_k}\frac{\sigma_{(\tilde{T}_k,\tilde{R}_j)}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}} - \frac{\sigma_{(\tilde{R}_k^t,\tilde{g}_k')}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}} + R_{N+1}\frac{\sigma_{(\tilde{R}_k^t,\tilde{g}_k')}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}}}$$

$$C = -\frac{\sigma_{(\tilde{g}_k',\tilde{R}_j)}}{\sigma_{(\tilde{R}_k,\tilde{R}_j)}(1 - E[\tilde{g}_k'])}[A] + \frac{\sigma_{(\tilde{g}_k',\tilde{R}_i)}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}(1 - E[\tilde{g}_k'])}$$

Thus, equality of after tax marginal rates of transformation between risk and return, requires that before tax marginal rates of transformation be unequal and not proportional to one another.

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yields:

$$\frac{(\bar{R}_i - R_{\dots}}{(\bar{R}^t_k, \tilde{R}_i \dots} = \frac{\sum_{j=1}^{N} [h \dots}{\sum_{j=1}^{N} [h \dots}$$

If $h_{jk \dots}$

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Since expression 3.10 holds for all assets in the investor's portfolio, the numerator and denominator of the right hand side can be summed separately without destroying the equality. Multiplying the numerator and denominator by h_{jk} and sum over all j $(j = 1,\dots,N)$ yields:

$$\frac{(\bar{R}_i - R_{N+1})(1 - E[\tilde{g}'_k]) - \sigma_{(\tilde{g}'_k,\tilde{R}_i)}}{\sigma_{((\tilde{R}^t_k,\tilde{R}_i(1-\tilde{g}'_k))} + R_{N+1}\sigma_{(\tilde{R}^t_k,\tilde{g}'_k)}} = \frac{\sum_{j=1}^{N}[h_{jk}(\bar{R}_j - R_{N+1})(1 - E[\tilde{g}'_k]) - h_{jk}\sigma_{(\tilde{g}'_k,\tilde{R}_j)}]}{\sum_{j=1}^{N}[h_{jk}\sigma_{(\tilde{R}^t_k,\tilde{R}_j(1 = \tilde{g}'_k))} + h_{jk}R_{N+1}\sigma_{\tilde{R}^t_k,\tilde{g}'_k}} \qquad 3.12$$

If h_{jk} is defined as the proportion of asset j in investor k's <u>risky</u> portfolio, that is, $h_{jk} = \frac{X_{jk}}{1 - X_{N+1,k}}$, then clearly,

$$\sum_{j=1}^{N} h_{jk} = 1$$

$$\sum_{j=1}^{N} h_{jk}(\bar{R}_j - R_{N+1}) = \bar{R}^r_k - R_{N+1} \qquad 3.13$$

$$\sum_{j=1}^{N} h_{jk}\sigma_{\tilde{g}'_k,\tilde{R}_j} = \sigma_{\tilde{g}'_k,\tilde{R}^r_k}$$

$$\sum_{j=1}^{N} h_{jk}\sigma_{(\tilde{R}^t_k,\tilde{R}_j(1-\tilde{g}'_k))} = \sigma_{(\tilde{R}^t_k,\tilde{R}^r_k(1-\tilde{g}'_k))},$$

where

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$$\sum_{j=1}^{N} h_{jk} R_j = R_k^r, \text{ the return on investor } k\text{'s risky portfolio.}$$

Carrying out the summation in 3.12 yields:

$$\frac{(\bar{R}_i - R_{N+1})(1 - E[\tilde{g}_k']) - \sigma_{(\tilde{g}_k', \tilde{R}_i)}}{\sigma_{(\tilde{R}_k^t, \tilde{R}_i(1-\tilde{g}_k'))} + R_{N+1}\sigma_{(\tilde{R}_k^t, \tilde{g}_k')}}$$

3.14

$$= \frac{(\bar{R}_k^r - R_{N+1})(1 - E[\tilde{g}_k']) - \sigma_{(\tilde{g}_k', \tilde{R}_k^r)}}{\sigma_{(\tilde{R}_k^t, \tilde{R}_k^r(1-\tilde{g}_k'))} + R_{N+1}\sigma_{(\tilde{R}_k^t, \tilde{g}_k')}} \quad i = 1,\ldots,N \ .$$

The N equations in 3.14 together with the last equation in 3.3, gives $N + 1$ equations in $N + 1$ unknowns $(X_{1,k},\ldots,X_{N+1,k})$. Thus, it is possible, at least in principle, to solve for the proportion of each asset in the investor's optimal portfolio.

In order to consider the effects of different tax functions on the investor's behavior, it is convenient to modify the notation. Since the random variables $\tilde{R}_k^r$, $\frac{\tilde{T}_k}{V_k}$, $\tilde{R}_k$ and $\tilde{g}_k'$ can be considered as four jointly distributed random variables, having a joint probability density function, it is possible to define the notation in terms of the joint moments,

$$u_{pqmn} = E[(\tilde{R}_k^r - \bar{R}_k^r)^p (\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})^q (\tilde{R}_i - \bar{R}_i)^m (\tilde{g}_k' - \bar{g}_k)^n]$$

$$p,q,m,n = 0,1,\ldots \qquad 3.15$$

Then 3.14 can be written as:[12]

$$\frac{1 - \frac{1}{(1-X_{N-}}}{1 - \frac{1}{(1-X_{N+}}}$$

$$B^* = \frac{C}{B} =$$

$$\frac{u_{1001}(u_{01}}{(1-g_K^{-1})(u_{1}}$$

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$$\frac{1-\frac{1}{(1-X_{N+1,k})}[\frac{u_{0110}}{u_{1010}}-\frac{1}{(1-g_k^{-1})}(\frac{u_{1011}(1-X_{N+1,k})-u_{0111}}{u_{1010}})-\frac{1}{(1-g_k^{-1})^2}(u_{1001}u_{0011}(1-X_{N+1,k})-u_{0101}u_{0011})]}{1-\frac{1}{(1-X_{N+1,k}}\frac{u_{1100}}{u_{2000}}-\frac{1}{(1-g_k^{-1})}(\frac{u_{20001}(1-X_{N+1,k})-u_{1101}}{u_{2000}})-\frac{1}{(1-g_k^{-1})^2}(\frac{u_{1001}^2(1-X_{N+1,k})-u_{0101}u_{1001}}{u_{2000}})]}$$

$$B^* = \frac{C}{B} =$$

$$\frac{u_{1001}(u_{0111}-(1-X_{N+1,k})u_{1011})-u_{0011}(u_{1101}-(1-X_{N+1,k})u_{2001})}{(1-g_k^{-1})[u_{1101}-(1-X_{N+1,k})u_{2001}+(1-g_k^{-1})(1-X_{N+1,k})u_{2000}-u_{1100})]-[(1-X_{N+1,k})u_{1001}^2-u_{0101}u_{1001}]}+$$

$$\frac{(1-g_k^{-1})[u_{1001}((1-X_{N+1,k})u_{1010}-u_{0110})-u_{0011}((1-X_{N+1,k})u_{2000}-u_{1100})]}{(1-g_k^{-1})[u_{1101}-(1-X_{N+1,k})u_{2001}+(1-g_k^{-1})((1-X_{N+1,k})u_{2000}-u_{1100})]-[(1-X_{N+1,k})u_{1001}^2-u_{0101}u_{1001}]}$$

Solving for the required risk premium of asset i, $(\bar{R}_i - R_{N+1})$ yields:[13]

$$(\bar{R}_i - R_{N+1}) = [\bar{R}_k^r - R_{N+1}]\,\frac{A}{B} - \frac{C}{B} \qquad 3.17$$

$$A = (1-g_k')[u_{0111}-(1-X_{N+1,k})u_{1011}+(1-g_k')((1-X_{N+1,k})u_{1010}-u_{0110})]$$

$$- [(1-X_{N+1,k})u_{1001}u_{0011} - u_{0101}u_{0011}] .$$

$$B = (1-g_k')[u_{1101}-(1-X_{N+1,k})u_{2001} + (1-g_k')((1-X_{N+1,k})u_{2000}-u_{1100})]$$

$$- [(1-X_{N+1,k})u_{1001}^2 - u_{0101}u_{1001}].$$

$$C = u_{1001}(u_{0111}-(1-X_{N+1,k})u_{1011})-u_{0011}(u_{1101}-(1-X_{N+1,k})u_{2001})$$

$$+ (1-E[\tilde{g}_k'])[u_{1001}((1-X_{N+1,k})u_{1010}-u_{0110})-u_{0011}((1-X_{N+1,k})u_{2000}-u_{1100})].$$

Equation 3.17 can be put into a form analogous to the traditional Capital Asset Pricing Model, by dividing the numerator and denominat

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denominator of $\frac{A}{B}$ by $(1-g_k')^2(1-X_{N+1,k})u_{2000}$ and rearranging slightly.[14] This yields:

$$\bar{R}_i - R_{N+1} = [(\bar{R}_{k_r} - R_{N+1})\frac{u_{1010}}{u_{2000}}][A^*] - B^*, \qquad 3.18$$

where,

$$A^* = \frac{A\ u_{2000}(1 - \bar{g}_k')^2(1 - X_{N+1,k})}{B\ u_{1010}(1 - \bar{g}_k')^2(1 - X_{N+1,k})} =$$

$$\frac{(\bar{R}_i - R_{N+1})(1-E[\tilde{g}_k'] - u_{0011}}{(u_{0111}-(1-X_{N+1,k})u_{1011}) + (1-\bar{g}_k')[(1-X_{N+1,k})u_{1010}-u_{0110}] - (\bar{R}_i - R_{N+1})((1-X_{N+1,k})u_{1001} - u_{0101})} \cdot$$

$$\frac{(\bar{R}_k^r - R_{N+1})(1-E[\tilde{g}_k']) - u_{1001}}{(u_{1101}-(1-X_{N+1,k})u_{2001}+(1-\bar{g}_k')[(1-X_{N+1,k})u_{2000}-u_{1100}] - (\bar{R}_k^r - R_{N+1})((1-X_{N+1,k})u_{1001} - u_{0101})} .$$

(each firm is assumed to have a body of stockholders who find that security more desirable than other securities.)[16] However, unlike the clientele effect assumed by Miller and Modigliani, the clientele effect identified here does not merely depend on the firm's dividend policy, but is determined by all the risk characteristics of the firm including, but not limited to the firm's dividend policy.[17]

In addition, Miller and Modigliani hypothesize that if investors are rational, there will be no clientele effect unless there are differential tax rates on dividende versus capital gains income (or some other imperfection such as transaction costs). The model presented here treats dividend and capital gains identically for tax purposes.

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One additional major concludion of the traditional Capital Asset Pricing Model does not hold under the more general model. By virtue of Tobin's separation theorem, under the no-tax model, the choice of an investor's risky portfolio is independent of the specific parameters of his utility function.[18] This conclusion results from the fact that, the first order conditions for utility maximization are proportional to the investor's risk aversion parameter.[19] Thus, the investor's proportional holdings of any risky asset is independent of his utility function. As a result, all investors hold the identical risky portfolio, adjusting the degree of risk each takes on by holding more or less of the riskless asset, or borrowing at the risk free rate.

Put another way, an investor's required risk premium for any risky asset, can be written as:[20]

$$\bar{R}_i - R_{N+1} = (E[\tilde{R}_M] - R_{N+1}) \frac{\sigma(\tilde{R}_M, \tilde{R}_i)}{\sigma(\tilde{R}_M, \tilde{R}_M)} ;$$

where, R_M is the return on the market portfolio. Clearly this is independent of the individual's utility function, and specifically is independent of the proportion of the investor's total capital devoted to risky assets, $(1 - X_{N+1,k})$. On the other hand, equation 3.18 is not necessarily independent of the proportion of investor k's total capital placed at risk. The right hand side of 3.18 contains $(1 - X_{N+1,k})$ explicitly. In addition, for a general tax function, the investor's total tax liability and marginal tax rates will, in

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will, in part, depend on the proportionate holdings of the riskless asset. Therefore, the moments on the right hand side of 3.18 will not be independent of the investor's proportional holdings of the riskless asset. That is, all investors will not necessarily hold the same risky portfolio, even though all investors have identical assessments of the joint probability distribution of all assets and identical tax parameters. This should agree with ad hoc intuitive reasoning since Tobin's separation theorem requires the existence of a riskless asset.[21] It has been shown above that an asset which has a before tax riskless rate of return, will not be riskless after taxes and thus, a necessary condition for the separation theorem does not exist in the after tax model.[22]

3.3. The Analytical Theory of Regression and Correlation and its Implications on the Present Model

Equation 3.18 describes the investor's equilibrium equation. It involves parameters contained in the conventional, before-tax model, but also includes higher and mixed moments of the joint density function of $\tilde{R}_k^r$ and $\tilde{R}_i$ as well as moments involving the investor's tax liability $(\frac{T_k}{V_k})$ and marginal tax rate (g_k'). Clearly, these expressions will depend not only on the joint distribution of the securities themselves, but also on the specific form of the tax function. In order to consider the implication of various forms of the tax function it is convenient to review the analytical theory of regression and correlation. It will be shown below that regression and correlation theory provide a means by which the salient features of the tax function may be described. Use of this methodology along with the investor's maximizing

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equation permits consideration of alternative tax functions and their effects on the required risk premium of a risky asset.

The "regression curve" is defined as the relationship between the conditional expectation of one jointly distributed random variable and the value of the other random variable.[23] For example, if X and Y are two jointly distributed random variables, then

$$E[Y|X] = M(x) \qquad 3.19$$

is defined as the regression curve of Y on X. Specifically, regression curves for the pairs of random variables, $\tilde{R}_k^r$, $\tilde{R}_i$, $\frac{\tilde{T}_k}{V_k}$, $\tilde{g}_k'$ can be written as,

$$E[\frac{\tilde{T}_k}{V_k}|\tilde{R}_k^r = R_k^r] = M_{21}(R_k^r) \qquad 3.20(i)$$

$$E[\frac{\tilde{T}_k}{V_k}|\tilde{R}_i = R_i] = M_{23}(R_i) \qquad 3.20(ii)$$

$$E[\tilde{g}_k'|\tilde{R}_k^r = R_k^r] = M_{41}(R_k^r) \qquad 3.20(iii)$$

$$E[\tilde{g}_k'|\tilde{R}_i = R_i] = M_{43}(R_i) \qquad 3.20(iv)$$

$$E[\frac{\tilde{T}_k}{V_k}|\tilde{g}_k' = g_k'] = M_{24}(g_k') \qquad 3.20(v)$$

Recall that the present model assumes that an investor's tax liability is a strictly monotonic, known function of income, and all the components of income except the stochastically determined return on the risky asset are known with certainty.[24] Thus, the relationship between $\frac{\tilde{T}_k}{V_k}$ and $\tilde{R}_k^r$ and the relationship between $\tilde{g}_k'$ and

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$\tilde{R}_k^r$ are deterministic (as opposed to stochastic) relationships. That is, for any value of $\tilde{R}_k^r$; $\frac{\tilde{T}_k}{V_k}$ and $\tilde{g}_k'$ respectively can assume one and only one value. Similarly, $\frac{\tilde{T}_k}{V_k}$ and $\tilde{g}_k'$ are solely determined by the same stochastic variable, $\tilde{R}_k^r$. On the other hand, the relationships between $\tilde{R}_i$; and $\frac{\tilde{T}_k}{V_k}$ and $\tilde{g}_k'$ respectively are stochastic. That is, knowledge of the specific value of $\tilde{R}_i$ does not result in certain knowledge of $\frac{\tilde{T}_k}{V_k}$ and $\tilde{g}_k'$, although the variables are not necessarily independent. Mathematically, the whole mass of the distribution of $\tilde{T}_k$ and $\tilde{g}_k'$ is situated on the regression curves $M_{21}(R_k^r)$, $M_{41}(R_k^r)$, and $M_{24}(\tilde{g}_k')$ respectively, but this is not true for $M_{23}(R_i)$ and $M_{43}(R_i)$.[25]

The difference between the former set of regression curves and the latter can be considered in the light of the source of variation of the dependent variable. That is, consider two jointly distributed random variables X, Y possessing a continuous regression curve $E[Y|X] = M(x)$. Then the variance of Y can be represented as the sum of two components, the variance of Y around the regression curve and the variance of the means of Y, conditional on the value of X. That is,

$$\begin{aligned} Var[Y] &= E[(Y - E[Y])^2] \\ &= E[(\{Y - E[Y|X]\} + \{E[Y|X] - E[Y]\})^2] \qquad 3.21 \\ &= E[\{Y - E[Y|X]\}^2] + E[(Y|X - E[Y])^2] \\ &= E[\{Y - E[Y|X]\}^2] + Var\{E[Y|X]\} \end{aligned}$$

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since the cross product, $2E[(\tilde{Y} - E[\tilde{Y}|X])]E[\{(\tilde{Y}|X) - E[\tilde{Y}]\}] = 0.$[26] Then, if Y is strictly a function of X, the variance around the regression curve $(E[\{Y - E[Y|X]\}^2])$ is zero. That is,

$$\begin{aligned} Var[Y] &= Var\{E[Y|X]\} \\ &= Var\{(Y|X)\}, \end{aligned} \qquad 3.22$$

or the variation in Y is due solely to the variation in the random variable X.

On the other hand, if the regression of Y on X is not exact, that is, a stochastic relationship exists between Y and X, then the total variance of Y will be the sum of the two terms on the right hand side of 3.21. The total variance of Y will be greater than the variance in Y due to X alone to the extent that Y varies around the regression curve $(E[\{\tilde{Y} - E[\tilde{Y}|X]\}^2])$.

The correlation ratio, developed by Karl Pearson,[27] and defined as

$$\eta^2_{Y,X} = \frac{Var\{E[Y|X]\}}{Var\ [Y]} = 1 - \frac{E[\{Y - E[Y|X]\}^2]}{Var[Y]} \qquad 3.23$$

is an indication of the extent of the dependence of Y on X. If Y is functionally dependent on X then $E[\{Y - E[Y|X]\}^2] = 0$, and $\eta^2_{Y,X} = 1$. On the other hand, if Y and X are independent, then

$$E[\{Y - E[Y|X]\}^2] = E[\{Y - E[Y]\}^2] = Var\ [Y] \qquad 3.24$$

and $\eta^2_{Y,X} = 0$. If $0 < \eta^2_{Y,X} < 1$, then Y and X are stochastically related.[28]

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A second statistic, the correlation coefficient (ρ), describes the extent to which one variable is linearly related to another. It is well known that for any jointly distributed random variables Y and X, the straight line of "closest fit" to the mass in the (X,Y) distribution is $Y = \hat{\alpha} + \hat{\beta}X$ where $\hat{\alpha}$ and $\hat{\beta}$ are chosen such that

$$E[(Y - \hat{\alpha} - \hat{\beta}X)^2] = \sigma_Y^2(1 - \rho^2_{(Y,X)}) \qquad 3.24$$

is a minimum. That is, the correlation coefficient indicates the extent to which Y varies about the "best fitting" regression line.[29] Then,

$$\sigma_y^2(1 - \rho^2_{Y,X}) = E[(Y - \hat{\alpha} - \hat{\beta}X)^2] = E(Y - E[Y|X] + E[Y|X] - \hat{\alpha} - \hat{\beta}X)^2]$$

$$= E[(Y - E[Y|X])^2 + E[(Y|X - \hat{\alpha} - \hat{\beta}X)^2],$$

since the cross product term is zero. Thus,

$$(1 - \rho^2_{(Y,X)}) = \frac{E[(Y - E[Y|X])^2]}{\sigma_Y^2} + \frac{E[(Y|X - \hat{\alpha} - \hat{\beta}X)^2]}{\sigma_Y^2} \qquad 3.25$$

Comparing 3.25 with 3.23:

$$(1 - \rho^2_{Y,X}) = 1 - \eta^2_{Y,X} + \frac{E[(Y|X - \hat{\alpha} - \hat{\beta}X)^2]}{\sigma_Y^2}$$

$$\rho^2_{(Y,X)} = \eta^2_{(Y,X)} - \frac{E[(Y|X - \hat{\alpha} - \hat{\beta}X)^2]}{\sigma_Y^2}$$

Clearly, if the regression curve of Y on X is linear, then the second term on the right hand side of 3.26 is zero and the correlation coeffic

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coefficient is equal to the correlation ratio. Specifically, if Y is a linear function of X, then $\rho^2_{(Y,X)} = \eta^2_{(Y,X)} = 1$. $\rho^2_{(Y,X)} < \eta^2_{(Y,X)} = 1$ if the variables are functionally related but the regression curve of Y on X is non-linear. $0 < \rho^2_{(Y,X)} = \eta^2_{(Y,X)} < 1$ if there is a stochastic relationship between Y and X and the regression curve is linear. $0 < \rho^2_{(Y,X)} < \eta^2_{(Y,X)} < 1$ if there is a stochastic relationship and the regression curve of Y on X is non-linear. In summary, $\rho^2_{(Y,X)}$ indicates the extent to which there is a linear relationship and $\eta^2_{(Y,X)}$ a relationship, not necessarily linear.[30]

Returning to the present model, since $\tilde{T}_k$ and $\tilde{R}^r_k$ are functionally related, then $\eta^2_{(\tilde{T}_k,\tilde{R}^r_k)} = 1$, and thus,

$$0 \leq \rho^2_{(2,1)} = \rho^2_{(1,2)} = 1 - \frac{E[\{\tilde{T}_k | R^r_k - \hat{\alpha}_{21} - \hat{\beta}_{21}(R^r_k)\}^2]}{\sigma^2_{(\tilde{T}_k)}} \leq 1.^{31} \tag{3.26}$$

Clearly, $\rho^2_{(1,2)} = 1$ if, and only if, the tax bill is a linear function of the risky portfolio's return. Since total investor income is a linear function of portfolio return,[32]

$$\tilde{Y}_k = Y_L + V_k((1 - X_{N+1,k})\tilde{R}^r_k + X_{N+1,k}R_{N+1}) , \tag{3.27}$$

then $\rho^2_{1,2} = 1$ if the tax function is linear in income. In addition, since the investor's tax bill is assumed to be a monotonically increasing function of income the covariance of taxes with portfolio return will depend upon the sign of $(1 - X_{N+1,k})$. That is, the covariance will be positive if the investor holds a predominantly long por

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long portfolio $(0 < X_{N+1,k} < 1)$, negative if the investor holds a predominantly short portfolio $(X_{N+1,k} > 1)$.[33] Therefore, from the definition of the correlation coefficient,[34]

$$|\rho_{12}| = \frac{u_{1100}}{(u_{200}u_{0200})^{1/2}} , \qquad 3.28$$

$\rho_{12} > 0$ for a long portfolio

$\rho_{12} < 0$ for a short portfolio.

That is, the sign of ρ_{12} is equal to the sign of $(1 - X_{N+1,k})$.

The relationship between the marginal tax rate and the portfolio return is only slightly more complicated. Since the total tax bill is a strict function of portfolio return, then the marginal tax rate will also be functionally related to portfolio return, except when the tax function is linear. In the latter case,the marginal tax rate will be a constant and independent of portfolio returns. Thus,

$$\eta^2_{(\tilde{g}'_k, \tilde{R}^r_k)} = \rho^2_{(41)} = \rho^2_{14} = 0 \qquad 3.29$$

Alternatively, if taxes are a strict quadratic function of income (or equivalently portfolio returns), then the marginal tax rate will be a strict linear function of portfolio returns and

$$\eta^2_{(\tilde{g}'_k, \tilde{R}^r_k)} = \rho^2_{(14)} = 1 . \qquad 3.30$$

When the marginal tax rate is increasing with income, then it will move in the same (opposite) direction as the portfolio return if the investo

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investor holds a long (short) portfolio. When the marginal tax rate is a decreasing function of income, then it will move in the opposite (same) direction as the portfolio return if the investor holds a long (short) portfolio. That is, $\rho_{(14)} = (\text{sgn } g''_k(1 - X_{N+1,k}))$.[35] If the tax function is of a more complicated form, then

$$-1 < \rho_{14} = \pm \left[1 - \frac{E[\{g'_k R^r_k - \hat{\alpha}_{41} - \hat{\beta}_{41}(R^r_k)\}^2]}{\sigma^2_{(g'_k)}}\right]^{1/2} < 1 \qquad 3.31$$

the sign again depending on the sign of $g''_k(1 - X_{N+1,k})$.

It has already been pointed out that $\tilde{T}_k$ and $\tilde{g}'_k$ are functionally related.[36] However, if the tax function is linear, the marginal tax rate is non-stochastic. Clearly, in this case $\rho_{24} = 0$. However, if the tax function is non-linear, then the correlation coefficient will be such that,

$$-1 < \rho_{14} = \pm \left[1 - \frac{\{E[T_k | g'_k - \hat{\alpha}_{24} - \hat{\beta}_{24} g'_k]\}^2}{\sigma^2_{(\tilde{T}_k)}}\right]^{1/2} < 1 \qquad 3.32$$

The sign will be positive if the marginal tax rate increases with income, negative if it decreases with income. Thus, the sign of ρ_{14} is determined by the sign of g''_k.

The relationship between the tax parameters and the security return, R_i is stochastic rather than functional. That is, $\eta^2_{(\tilde{T}_k,\tilde{R}_i)} < 1$, $\eta^2_{(\tilde{g}'_k,\tilde{R}_i)} < 1$. The correlation coefficients will be equal to

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$$-1 < \rho_{23} = \pm \left[\eta^2_{(\tilde{T}_k,\tilde{R}_i)} - E \frac{\{(T_k|R_i) - \hat{\alpha}_{23} - \hat{\beta}_{23}R_k\}^2}{\sigma^2_{(T_k)}} \right]^{1/2} < 1 \qquad 3.33$$

$$-1 < \rho_{43} = \pm \left[\eta^2_{(\tilde{g}_k',\tilde{R}_i)} - E \frac{\{g_k'|R_i - \hat{\alpha}_{43} - \hat{\beta}_{43}R_i\}^2}{\sigma^2_{(g_k')}} \right]^{1/2} < 1. \qquad 3.34$$

That is, the correlation coefficients will diverge from $|1|$ to the extent that the true regression curve departs from zero, as well as the extent to which there is variation around the true regression curve. If the tax function is linear with respect to income, then the regression curve of $\tilde{T}_k$ on $\tilde{R}_i$ will also be linear since income is a linear function of portfolio return and the portfolio return is a linear function of the security return. That is,

$$\tilde{R}_k = \sum_{j=1}^{N} X_{jk}\tilde{R}_j + X_{N+1,k}R_{N+1} .$$

Thus, $\rho^2_{23} = \eta^2_{23} < 1$. Specifically, from the definition of the correlation coefficient $\rho_{23} = \frac{u_{0110}}{(u_{0200}u_{0020})^{1/2}}$. If the tax function is written as, say,

$$\tilde{T}_k = a + b\,\tilde{Y}^r_k, \qquad b > 0,$$

$$= a + b(Y_L + V_k(1 - X_{N+1,k})\tilde{R}^r_k + X_{N+1,k}R_{N+1})$$

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$$\rho_{23} = \frac{E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}_i - \bar{R}_i)]}{\{\frac{E[(\tilde{T}_k - \bar{T}_k)^2]}{V_k^2} E[(\tilde{R}_i - \bar{R}_i)^2]^{1/2}}$$

$$= \frac{b(1-X_{N+1,k})E[(\tilde{R}_k^r - \bar{R}_k^r)(\tilde{R}_i - \bar{R}_i)]}{|b(1-X_{N+1,k})| E[(\tilde{R}_k^r - \bar{R}_k^r)^2 E[(\tilde{R}_i - \bar{R}_i)^2\}^{1/2}}$$

$$= \frac{u_{1010}}{(u_{2000}u_{0020})^{1/2}} \text{ X (the sign of } (1-X_{N+1,k})) \qquad 3.35$$

$$= \pm \rho_{13} \text{ the sign depending on the sign of } (1-X_{N+1,k}).$$

If the tax function is non-linear with respect to income, then the regression curve of taxes on the security return will also be non-linear. The correlation coefficient will be given by equation 3.33. If the investor holds a long (short) portfolio and the security is positively (negatively) correlated with the portfolio return, then the tax bill will tend to increase (decrease) as the security return increases (decreases). Therefore the correlation coefficient will be positive. If the investor holds a long (short) portfolio and the security is negatively (positively) correlated with the portfolio return, then the tax bill will tend to move in the opposite direction with the security return and the correlation coefficient will be negative. That is, the sign of ρ_{23} will be equal to the sign of $(1 - X_{N+1,k})_{13}$.

The correlation coefficient between the security and the marginal tax rate is given by equation 3.34. Clearly, if the tax functi

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function is linear, then $\rho_{34} = 0$ since the marginal tax rate will be constant. If the tax function is non-linear, then for increasing (decreasing) marginal tax rates, the marginal tax rate will move in the same (opposite) direction as the security return, if the investor holds a long (short) portfolio and the security is positively (negatively) correlated with the portfolio return. The marginal tax rate will also move in the same (opposite) direction as the security return. Extending the above analysis, it can be easily shown that the sign of the correlation coefficient is equal to the sign $g_k''(1 - X_{N+1,k})\rho_{13}$.

In addition, ρ_{14} and ρ_{12} are the simple correlation coefficients between the marginal tax rate and the total tax bill respectively. However, since the total portfolio is simply a linear combination of each security held, they can be considered the multiple correlation coefficient between all the securities held in the portfolio and the marginal and total tax bill respectively. That is,[37]

$$\rho_{14} = \rho_{4,(\underline{j})} \qquad 3.36$$

$$\rho_{12} = \rho_{2,(\underline{j})}$$

where $\underline{j}$ represents the return on each security held by investor k $(j = 1,\ldots,N)$. On the other hand, ρ_{34} and ρ_{23} are the simple correlation coefficient between security i, (a subset of $\underline{j}$) and the marginal and total tax bill respectively.

In general, it is well known that if $x_1,\ldots,x_p$ are jointly distributed random varialbes, and if $\rho_{1\,(2,\ldots,p)}$ is the multiple correlation coefficient between x_1, and x_2 through x_p (that is

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$\rho_{1\,(2,\ldots,p)}$ is the correlation between x_1 and the "best-fitting" linear combination of $x_2,\ldots,x_p$), then $|\rho_{1\,(2,\ldots,p)}| \geq |\rho_{1j}|$, where j represents x_j, an element in $x_2,\ldots,x_p$.[38] Thus,

$$|\rho_{12}| \geq |\rho_{23}| \quad , \qquad 3.37$$

$$|\rho_{14}| \geq |\rho_{34}|$$

That is, the correlation coefficient cannot be reduced by increasing the number of explanatory variables.

The following table summarizes the results of this section.

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Table 1

The Tax, Portfolio and Security Parameters' Impact on the Ranges of the Relevant Correlation Coefficients[a]

	$(1-x_{N+1,k})$	g'_k	g''_k	ρ_{13}	ρ_{12}	ρ_{14}	ρ_{23}	ρ_{34}	ρ_{24}
1	+	+	+	+	(0,1)	(0,1]	$(0,\rho_{12}]$	$(0,\rho_{14})$	(0,1)
2	-	+	+	+	(-1,0)	[-1,0)	$[\rho_{12},0)$	$(\rho_{14},0)$	(0,1)
3	+	+	-	+	(0,1)	[-1,0)	$(0,\rho_{12}]$	$(\rho_{14},0)$	(-1,0)
4	+	+	+	-	(0,1)	(0,1]	$[-\rho_{12},0)$	$(-\rho_{14},0)$	(0,1)
5	-	+	-	+	(-1,0)	(0,1]	$[\rho_{12},0)$	$(0,\rho_{14})$	(-1,0)
6	-	+	+	-	(-1,0)	[-1,0)	$(0,-\rho_{12}]$	$(0,-\rho_{14})$	(0,1)
7	+	+	-	-	(0,1)	[-1,0)	$(\rho_{12},0)$	$(0,-\rho_{14})$	(-1,0)
8	-	+	-	-	(-1,0)	(0,1]	$(0,-\rho_{12}]$	$(-\rho_{14}.0)$	(-1,0)
9	+	+	0	+	1	0	ρ_{13}	0	0
10	+	+	0	-	1	0	ρ_{13}	0	0
11	-	+	0	+	-1	0	$-\rho_{13}$	0	0
12	-	+	0	-	-1	0	$-\rho_{13}$	0	0

[a]"("or")" represents the lower or upper range respectively, as not inclusive of the extreme value. "["or"]" represents the lower or upper range respectively as inclusive of the extreme value.

3.4 Alternative Tax Functions and Their Effects on Investor Behavior

It would be highly desirable to attempt to distinguish different forms of the tax function by a single summary statistic. For example, a natural statistic might be tax elasticity with respect to income since it is the generally accepted measure of the degree of progression (regression) of a tax function.[39] However, it is concluded below, that under the given assumption, the degree or progression (regression) *per se* is not a relevant statistic in determining the required risk premium associated with

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a given security. Unfortunately, the effect of personal taxes on the investor's behavior depends, in a complicated way, on the curvature of the tax function throughout its relevant range. Thus, a number of statistics are necessary to sufficiently describe the effects personal taxes have on investor behavior.

For reasons which will soon become obvious, it is convenient to describe, to the extent possible, the nature of the tax function, and its relationship with the security and portfolio return by the following seven statistics;

(i) g': the marginal tax rate at the point where the rate of return of the investor's portfolio is the expected rate of return.

(ii) g'': the rate of change of the marginal tax rate at that point.

(iii) ρ_{12}: The correlation coefficient between the return of the investor's portfolio and total tax liability.

(iv) ρ_{14}: The correlation coefficient between the return of the investor's portfolio and the marginal tax rate.

(v) ρ_{23}: The correlation coefficient between the total tax liability and the return of the asset in question.

(vi) ρ_{34}: The correlation coefficient between the marginal tax rate and the return of the asset in question.

(vii) ρ_{24}: The correlation coefficient between the total tax liability and the marginal tax rate.

The first two describe the curvature of the tax function at a specific point. The remaining statistics describe the curvature of the tax function throughout the range of possible returns, as well as the marginal density functions of the portfolio and the asset in question.

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The major interest of this section is to describe the implications of these seven parameters on the required risk premium for a specific security. That is, the effect of these parameters on the investor's required risk premium as represented by equation 3.18. By direct substitution of the definition of the correlation coefficient:

$$\rho_{mn} = \frac{\text{Cov}(m,n)}{(\sigma^2_{(m)}\sigma^2_{(n)})^{\frac{1}{2}}}, \quad m,n = 1,2,3,4, \qquad 3.38$$

where

1 represents the random variable $\tilde{R}^r_k$

2 represents the random variable $\tilde{T}_k/V_k$

3 represents the random variable $\tilde{R}_i$

4 represents the random variable $\tilde{g}'_k$

into 3.18, and by use of the first order approximation:

$$u_{(p,q,m,n)} \approx (g'_{(\bar{R}^r_k)})^q (g''_{(\bar{R}^r_k)} V_k)^n (1 - X_{N+1,k})^{q+n} u_{(p+q+n),0,m,0)}, \qquad 3.39$$

where the subscript $(\bar{R}^r_k)$ represents the derivative evaluated at $\tilde{R}^r_k = \bar{R}^r_k$,[40] and substituting the above expressions into equation 3.18, the required rate of return of asset i for an individual investor may be expressed in terms of the seven parameters enumerated above.

Carrying out these substitutions and rearranging slightly yields the following expressions for the analogous terms in 3.18.[41]

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$$A^* = \frac{1 - [\frac{\rho_{23}}{\rho_{13}} g' - \frac{(1-g')g''V^r u_{2010}}{\rho_{13}(1-E[\tilde{g}'])(u_{2000}u_{0020})^{\frac{1}{2}}} - \frac{\rho_{34}}{\rho_{13}} \frac{(g''V^r)^2(\rho_{14}-g'\rho_{24})u_{2000}}{(1-E[\tilde{g}'])^2}]}{1 - [\rho_{12}g' - \frac{(1-g')g''V^r u_{3000}}{(1-E[\tilde{g}'])u_{2000}} - \frac{\rho_{14}(g''V^r)^2(\rho_{14}-g'\rho_{24})u_{2000}}{(1 - E[\tilde{g}'])^2}]}, \qquad 3.40$$

$$B^* =$$

$$\frac{-\{(g'')^2(1-g')(V^r)^2[\rho_{34}(u_{2000}u_{0020})^{\frac{1}{2}}u_{3000}-\rho_{14}u_{2000}u_{2010}]}{[(g'')^2(V^r)^2\rho_{14}(\rho_{14}-\rho_{24}g')u_{2000}^2+(1-E[\tilde{g}'])(g''(1-g')V^r u_{3000}-(1-E[\tilde{g}'])(1-\rho_{12}g')u_{2000}]}$$

$$+ \frac{[\rho_{14}(\rho_{13}-g'\rho_{23})-\rho_{34}(1-g'\rho_{12})]g''V^r(u_{2000})^{3/2}(u_{0020})^{\frac{1}{2}}(1-E[\tilde{g}'])}{[(g'')^2(V^r)\rho_{14}(\rho_{14}-\rho_{24}g')u_{2000}^2+(1-E[\tilde{g}'])(g''(1-g')V^r u_{3000}-(1-E[\tilde{g}'])(1-\rho_{12}g')u_{2000}]}\}$$

By considering the implications of the form of the tax function on the seven relevant parameters, and the implications of the relevant parameters on the investor's maximizing equation 3.40, conclusions may be drawn concerning the implication of the tax function on the investor's required risk premium.

First consider a tax function such that $g''_k = 0$. Clearly, this is consistent with rows 9 through 12 in Table 3.1. With regard to the form of the tax function, $g''_k = 0$ throughout, if, and only if, the tax function is a linear function of income. That is,

$$\begin{aligned} \tilde{T}_k &= a + b\,\tilde{Y}_k & b > 0 \\ &= a' + b'\,\tilde{R}^r_k(1 - X_{N+1,k}) & b' > 0 \end{aligned} \qquad 3.41$$

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Substituting the values of the correlation coefficients into equation 3.40 yields:

$$A^* = \frac{1-g'}{1-g'}\Bigg|_{(1-X_{N+1,k})>0} = \frac{1+g'}{1+g'}\Bigg|_{(1-X_{N+1,k})<0} = 1 \qquad 3.42$$

$$B^* = 0.$$

Thus, the after tax model is identical to the before tax model and the investor may ignore the tax function. Since the investor's behavior after taxes is identical to his behavior before taxes, all the conclusions of the before tax model hold.

It is important to note that a linear tax function implies nothing, in and of itself, about the degree of progression or regression of the tax function. The linear tax function presented here will be regressive, proportional or progressive depending on whether $a \gtreqless 0$.[42] Thus, the degree of progression *per se* does not affect investor behavior.[43]

However, if the tax function is non-linear with respect to income, taxes may play an important role in determining the required risk premuium for a specific security. The ranges of the relevant parameters are given by rows 1, 2, 3 and 5, in Table 1, depending on the signs of g''_k and $(1-X_{N+1,k})$. Specifically, assume a quadratic tax function, such that

$$\tilde{T}_k = a + b\,\tilde{Y}_k + c(\tilde{Y}_k)^2$$

$$\tilde{g}'_k = b + 2c\,\tilde{Y}_k,$$

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or the regression curve of $\tilde{T}_k$ and $\tilde{g}'_k$ on $\tilde{R}^r_k$, respectively, is:

$$E[\tilde{T}_k | R^r_k] = T_k = a_{21} + b_{21}R^r_k + c_{21}(R^r_k)^2, \qquad 3.43$$

$$E[\tilde{g}'_k | R^r_k] = g'_k = a_{41} + b_{41}R^r_k \ . \qquad 3.44$$

Note that the marginal tax rate is a strict linear function of the portfolio return, so that

$$\rho_{14} = \pm 1, \qquad 3.45$$

the sign equal to the sign of $g''_k(1 - X_{N+1,k})$.

The regression curve of the marginal tax rate on the security return is also linear, since the portfolio return is a linear function of each security return. Thus, the regression curve of $\tilde{g}'_k$ on $\tilde{R}_i$ can be written as $E[\tilde{g}'_k | R_i] = a_{34} + b_{34}R_i$ and the regression coefficient of $\tilde{g}'_k$ and $\tilde{R}_i$ is

$$\rho_{34} = \frac{u_{0011}}{(u_{0020}u_{0002})^{1/2}}$$

$$= \frac{E[(\tilde{g}'_k - \bar{g}'_k)(\tilde{R}_i - \bar{R}_i)]}{\{E[(\tilde{g}'_k - \bar{g}'_k)^2]E[(\tilde{R}_i - \bar{R}_i)^2]\}^{1/2}} \ . \qquad 3.46$$

Substituting 3.44 into 3.46 yields

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$$\rho_{34} = \frac{b_{41}E[(\tilde{R}_k^r - \bar{R}_k^r)(\tilde{R}_i - \bar{R}_i)]}{|b_{41}|\{E[(\tilde{R}_k^r - \bar{R}_k^r)^2]E[(\tilde{R}_i - \bar{R}_i)^2]\}^{1/2}}$$

$$= \frac{b_{41}\, u_{1010}}{|b_{41}|(u_{2000}u_{0020})^{1/2}} \qquad 3.46$$

$$= \rho_{13}(\text{sgn}\; g_k''(1 - X_{N+1,k}))\;.$$

Assuming that $g_k''(1-X_{N+1,k}) > 0$, and substituting 3.45 and 3.46 into 3.40 yields

$$A^* = \frac{1-[\frac{\rho_{23}}{\rho_{13}}g' - \frac{(1-g')g''V^r u_{2010}}{(1-E[\tilde{g}'])\rho_{13}(u_{2000}u_{0020})^{1/2}} - \frac{(g''V^r)^2(1-g'\rho_{24})u_{2000}}{(1-E[\tilde{g}'])^2}]}{1-[\rho_{12}g' - \frac{(1-g')g''V^r u_{3000}}{(1-E[g'])u_{2000}} - \frac{(g''V^r)^2(1-g'\rho_{24})u_{2000}}{(1-E[g'])^2}]}$$

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$$B^* = -\{\frac{(g''V^r)^2(1-g')[\rho_{13}(u_{2000}u_{0020})^{1/2}u_{3000} - u_{2000}u_{2010}]}{(g''V^r)^2(1-\rho_{24}g')u_{2000}^2+(1-E[g'])(g''(1-g')V^r u_{3000}-(1-E[g'])(1-\rho_{12}g')u_{2000}}$$

$$+\frac{[(\rho_{13}-g'\rho_{23})-\rho_{13}(1-g'\rho_{12})]g''V^r(u_{2000})^{3/2}(u_{0020})^{1/2}(1-E[g'])}{(g''V^r)^2(1-\rho_{24}g')u_{2000}^2+(1-E[g'])(g''(1-g')V^r u_{3000}-(1-E[g'])(1-\rho_{12}g')u_{2000}}$$

$0 < \rho_{13} < 1$, $0 < \rho_{23} < \rho_{12} < 1$, $0 < \rho_{24} < 1$, for $g_k''(1-X_{N+1,k}) > 0$.

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$$\frac{\rho_{23}}{\rho_{13}} g' - \frac{(1-g')g''V^r u_{2010}}{(1-E[\tilde{g}'])u_{1010}} \lesseqgtr \rho_{12}g' - \frac{(1-g')g''V^r u_{3000}}{(1-E[\tilde{g}']u_{2000}} \qquad 3.48$$

The direction of this inequality depends not only on the specific parameters of the tax function, but also, in a complicated way, on the higher moments of the joint probability distribution between $\tilde{R}_k^r$ and $\tilde{R}_i$. Similarly, the determination of the sign of B^* depends on the direction of the inequality,

$$(g''V^r)^2(1-g')[\rho_{13}(u_{2000}u_{0020})^{1/2}u_{3000} - u_{2000}u_{2010}] \gtreqless (g''V^r)g'(\rho_{23} - \rho_{12})(u_{2000})^{3/2}(u_{0020})^{1/2}(1-E[\tilde{g}']) \ . \qquad 3.49$$

Thus, even for this relatively simple non-linear tax function, the specific effects of personal income taxes on the security's required risk premium is quite complicated. Knowledge of the higher moments of the security and portfolio probability distribution is necessary to arrive at unambiguous results.

3.5. Conclusions

In conclusion, personal income taxes seem to be relevant in determining investor behavior. A rational investor, acting in response to after tax variables, may modify his portfolio in the light of perceived changes in his tax function. The investor, subject to a non-linear tax function, will react to various tax

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parameters. Specifically, he will react to the marginal tax rate and changes in the marginal tax rate. In addition, moments of the joint probability density function involving total and marginal tax variables as well as security and portfolio variables may be important. Subject to the specific approximation utilized, skewness (u_{3000}) of the density function of the investor's portfolio returns and mixed moments of higher order than the covariance (u_{2010}), of the security's joint density function may be relevant parameters. These parameters are ignored by the "mean-variance" utility maximizer in the absence of taxes.

Finally, it has been found that, in general, because of the complexity of the relationship between the tax function and the assets' density function, specific conclusions must incorporate both the functional form of the tax function as well as the relevant joint density functions. However, in the special case where tax liability can be represented by a strict linear function of investor's income, taxes are irrelevant. That is, the investor's behavior is identical, in terms of the required risk premium of each risky asset, to what it would be if there were no personal income taxes.

The following chapter considers the extent to which the form of the security and portfolio joint density function determines the effects of non-linear taxes on the investor's behavior.

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NOTES TO CHAPTER 3

1. See Chapter 2, pp. 5-15, for a description of the conventional Capital Asset Pricing Model.

2. Market and investor's behavioral assumptions enumerated, follow closely those specified by John Lintner, "The Valuation of Risk Assets and the Selection of risky Investments in Stock Portfolios and Capital Budgets," The Review of Economics and Statistics, 47 (February, 1965), 15-16.

3. See page 6.

4. "Risk" means variance (or its equivalent, standard deviation) of protfolio return.

5. This assumption is generally referred to as "the mean-variance approach". See e.g. William F. Sharpe, Portfolio Theory and Capital Markets (New York: McGraw Hill, 1970), pp. 196-201.

6. James Tobi, "Liquidity Preference as Behavior Towards Risk", Review of Economic Studies, 26 (February, 1958), 75-76.

7. Specifically, objections to the use of quadratic utility functions arise since they imply that marginal utility of wealth becomes negative over a relatively narrow range, and the investor is subject to increasing absolute risk aversion as initial wealth increases. That is, risky securities are inferior goods. See eg. Lintner, op. cit. n.20; Kenneth Arrow, Aspects of the Theory of Risk Bearing, (Helsinki: Yijo Jahnssonin Saatio, 1965); John W. Pratt, "Risk Aversion in the Small and in the Large," Econometrica, 32 (January-April, 1964), 132.

8. Throughout, it is assumed that all expectations exist and are finite.

9. To evaluate the derivatives, note that, subject only to existence conditions, it can be shown that if:

   $u(t) = \int_S g(\tilde{x},t)dF(\tilde{x})$, $\underline{t}$ is a parameter
   $\underline{\tilde{x}}$ is a random variable
   $\underline{F}(\tilde{x})$ is a distribution function defined over the probability space S,

   then,

$$\frac{du(t)}{dt} = \frac{\partial}{\partial t} \int_S g(\tilde{x},t)dF(\tilde{x})$$

$$= \int_S \frac{\partial g(\tilde{x},t)}{\partial t} dF(\tilde{x})$$

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That is, the derivative of an expectation is the expectation of the derivative. Cramér, Harald, Mathematical Methods of Statistics, (Princeton, Princeton University Press: 1945), 67.

10. The before tax marginal rate of transformation between risk and return of asset i is:

$$-\left.\frac{dE[\tilde{R}_k]}{d\sigma_{(\tilde{R}_k,\tilde{R}_k)}}\right|_{dx_j=0,\,i\neq j} = \frac{\partial E[\tilde{R}_k]/\partial(X_i-X_{N+1})}{\partial\sigma_{R_kR_k}/\partial(X_i-X_{N+1})} = \frac{\bar{R}_i - R_{N+1}}{2\sigma_{\tilde{R}_k,\tilde{R}_i}}$$

11. Equation 3.11 is derived by substituting

$$\sigma_{(\tilde{R}_k{}^t,\tilde{R}_i(1-\tilde{g}_k'))} = \sigma_{(\tilde{R}_k,\tilde{R}_i)} - \frac{1}{V_k}\sigma_{(\tilde{T}_k,\tilde{R}_i)} - \sigma_{\tilde{R}_k{}^t,(\tilde{g}_k'\tilde{R}_i)}$$

into 3.8 and solving for $\dfrac{\bar{R}_i - R_{N+1}}{\sigma_{(\tilde{R}_k,\tilde{R}_i)}}$.

12. See Appendix A for a derivation of the terms in 3.12 in terms of the newly introduced notation. Also note that,

$\sum_{j=1}^{N} X_{jk}\sigma_{(\tilde{R}_j\tilde{R}_k^r)} = (1-X_{N+1,k})\sigma_{(\tilde{R}_k^r,\tilde{R}_k^r)}$, or in general,

$\sigma_{(\tilde{R}_k,\tilde{z})} = \sum_{j=1}^{N}[X_{jk}(\tilde{R}_j,\tilde{z})] = (1-X_{N+1,k})\sigma_{(\tilde{R}_k^r,\tilde{z})}$, where $\tilde{z}$ is any random variable not indexed by j.

13. See Appendix B for derivation of equation 3.17.

14. Chapter 2, equation 2.10.

15. Since it has been shown that all investors hold identical risky portfolios under the assumptions of the traditional model, Tobin, op. cit., 82, this expression

$\left[\dfrac{(\bar{R}_k^r - R_{N+1})(u_{1010})}{(u_{2000})}\right]$ has the same value for all investors and represents the required risk premium of asset i on the market as a whole.

16. M. Miller and F. Modigliani, "Dividend Policy, Growth and the Valuation of Shares," Journal of Business, 34 (October, 1961), 411-33.

17. Ibid.; See also Michael J. Brennan, "Investor Taxes, Market Equilibrium and Corporate Finance" (unpublished Doctor's dissertation, Massachusetts Institute of Technology, 1970),

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18. Tobi

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20. Ibid

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p. 34 and Brennan, "Taxes, Market Valuation and Corporate Financial Policy," National Tax Journal 23 (December, 1970), 419.

18. Tobin, loc. cit.

19. Chapter 2, p. 14.

20. Ibid.

21. Tobin, op.cit., p. 67.

22. Chapter 3, equation 3.41.

23. Maurice G. Kendall and Alan Stuart, The Advanced Theory of Statistics, II (2d. ed., New York: Hafner Publishing Co., 1963), 232.

24. Page 33.

25. Harald Cramér, Mathematical Methods of Statistics, (Princeton: Princeton University Press, 1945), 281.

26. Kendall and Stuart, op.cit., II, 297.

27. Karl Pearson, "Mathematical Contributions to the Theory of Evolution. XIV. On the General Theory of Skew Correlation and Non-linear Regression", Karl Pearson's Early Statistical Papers, (Cambridge: Cambridge University Press, 1948), 484.

28. Kendall and Stuart, op.cit., II, 296-99.

29. Cramér op.cit., p. 278.

30. Kendall and Stuart, op.cit., II, 297.

31. From the definition of $\rho_{(X,Y)} = \frac{Cov(X,Y)}{(\sigma^2_{(X)}\sigma^2_{(Y)})^{1/2}}$, it is clear that $\rho^2_{(X,Y)} \equiv \rho^2_{(Y,X)}$ for all X,Y. In addition if X and Y are two jointly distributed random variables, with correlation coefficient $\rho_{(X,Y)}$, then if

$$X' = a + bX \qquad b \neq 0$$

$$Y' = c + dY \qquad d \neq 0,$$

the correlation coefficient between X' and Y' is $\rho_{(X',Y')} = \rho_{(X,Y)} X (\text{sgn } bd)$, where sgn(bd) stands for ± 1, according as (bd) is positive or negative. Cramér, op.cit., p. 279. Thus, $\rho_{(T_k/V_k,X)} = \rho_{(T_k,X)}$, since $V_k > 0$. Hereafter $\hat{\alpha}_{..}$ and $\hat{\beta}_{..}$ will be the coefficients of the relevant "best fitting" linear regression.

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32. Page 36.

33. A short sale is defined as a security which has been borrowed and sold at the market price in the beginning of the period. A predominantly short portfolio is one in which the value of short sales exceed the value of stock holdings at the beginning of the period. See John Lintner, op.cit., 19-21 for a justification for defining short sales as a negative proportion of the total portfolio.

34. Appendix C, equation C.3.

35. From here on, it shall be assumed that the sign of g_k'' remains the same throughout its entire range.

36. See page 49.

37. The notation here follows closely that adopted by Kendall and Stuart, op.cit., II, 317-42.

38. Kendall and Stuart, op.cit., II, 232.

39. Richard A. Musgrave, The Theory of Public Finance, (New York: McGraw-Hill Book Co., 1959), 100, n.2.

40. Kendall and Stuart, The Advanced Theory of Statistics, I (2d ed.; New York: Hafner Publishing Co., 1945), 232.

41. See Appendix C for the complete derivation of equation 3.40 from equation 3.18.

42. In general, if the tax function is constrained so that the investor's tax bill is zero when his net income is zero, then the tax function will be progressive, proportional or regressive depending on whether the sign of $g_k'' \gtreqless 0$.

43. It should be noted that Brennan (December, 1970), loc cit. reaches identical conclusions. However, his assumption that the marginal tax rate on capital gains income is exogenous is equivalent to assuming a linear tax function. That is, the conclusion arrived at here is consistent with Brennan's conclusion. However, this analysis makes the linearity of the tax function explicit.

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CHAPTER 4

THE EFFECTS OF PERSONAL INCOME TAXES ON INVESTOR BEHAVIOR: SOME SPECIFIC CASES

It has been shown in the previous chapter that *á priori* conclusions with respect to investor behavior can be made only when specific tax functions and probability density functions are specified. This chapter considers specific approximations to hypothetical tax functions. After direct substitution, into the investor's behavioral equations, results are obtained, conditional on the assumptions regarding the probability law obeyed by the investor's portfolio and the security in question.

4.1 The Case of a Tax Function which can be Approximated by a Second Degree Taylor Expansion and Quadratic Utility Functions

In this section it is assumed that the investor's behavioral assumptions and the market assumptions are identical to those enumerated in the beginning of chapter 3.[1] Assumptions regarding the tax function are made more specific in that it is assumed that the tax function may be approximated by a second degree Taylor Expansion around expected income. That is, the tax function can be written as,

$$\tilde{T}_k = g_{(\bar{y}_k)} + g'_k(\tilde{Y}_k - \bar{Y}_k) + (1/2)g''_k(\tilde{Y}_k - \bar{Y}_k)^2 \qquad 4.1$$

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$\tilde{Y}_k = \bar{Y}_k$.

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$$\frac{\partial \mathcal{L}}{\partial X_{ik}} = \frac{\partial U}{\partial E}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_k} = \sum_{j=1}^{N+1}$$

where $g_{(\bar{y}_k)}$, g'_k, g''_k are the respective functions evaluated at $\tilde{Y}_k = \bar{Y}_k$. Since

$$\tilde{Y}_k = Y_k^h + \tilde{Y}_k^c$$

$$= Y_k^h + V_k(\sum_{j=1}^{N} X_{jk}\tilde{R}_j + X_{N+1,k}R_{N+1})$$

$$= Y_k^h + V_k(1-X_{N+1,k})\tilde{R}_k^r + V_kX_{N+1,k}R_{N+1}$$

$$= Y_k^h + V_kX_{N+1,k}R_{N+1} + V_k^r\tilde{R}_k^r \qquad 4.2$$

by definition, equations 4.1 can be expressed in terms of the risky security return. That is:

$$\tilde{T}_k = g_{(\bar{Y}_k)} + g'_kV_k^r(\tilde{R}_k^r - \bar{R}_k^r) + \frac{1}{2} g''_k(V_k^r)^2(\tilde{R}_k^r - \bar{R}_k^r)^2 \qquad 4.3$$

This specific form of the tax function may be directly substituted into the investor's optimizing equations 3.3 and 3.4 above,[2] yielding:

$$\underset{X_i}{\text{Max}}\ \mathcal{L} = U[E[\tilde{R}_k^t],\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}] - \lambda_k(\sum_{j=1}^{N+1} X_{jk} - 1)$$

$$\frac{\partial\mathcal{L}}{\partial X_{ik}} \Rightarrow \frac{\partial U}{\partial E[\tilde{R}_k^t]}\frac{\partial E[\tilde{R}_k^t]}{\partial X_{ik}} + \frac{\partial U}{\partial\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}\frac{\partial\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial X_{ik}} - \lambda_k = 0\ ,$$

$$i = 1,\ldots,N+1$$

$$\frac{\partial\mathcal{L}}{\partial\lambda_k} \Rightarrow \sum_{j=1}^{N+1} X_{jk} - 1 = 0, \qquad 4.4$$

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_i} = \text{i}$$

$$\frac{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial X_i}$$

$+ (1/2)(g'$

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$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_{N+1}} =$$

$(\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t}$

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$$\frac{\partial \mathcal{L}}{\partial X_i} \Rightarrow [$$

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$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_i} = \bar{R}_i(1 - g' - (1/2)(V')^2 g''' u_{2000}) - V^r g'' u_{1010}$$

$$\frac{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial X_i} = 2(1-X_{N+1})\{[(1-g')^2 u_{1010} - (3/2)g''(1-g')V^r u_{2010}$$

$$+ (1/2)(g'')^2(V^r)^2[u_{3010}-u_{2000}u_{1010}]] - \bar{R}_i[g''(1-g')V^r u_{2000}$$

$$+ (1/2)(V^r)^2[g'''(1-g')-(g'')^2]u_{3000}-(1/4)g''g'''(V^r)^3[u_{4000}-u_{2000}^2]]$$

$$i = 1,\ldots,N$$

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_{N+1}} = R_{N+1}(1 - g' - (1/2)(V^r)^2 g''' u_{2000})$$

$$(\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}/\partial X_{N+1}) = -2(1-X_{N+1})R_{N+1}[g''(1-g')V^r u_{2000}$$

$$+ (1/2)(V^r)^2][g'''(1-g')-(g'')^2]u_{3000} - (1/4)g''g'''(V^r)^3[u_{4000}-u_{2000}^2]]. \quad 4.4$$

Substituting equations 4.4 into 4.3 yields:

$$\underset{X_i}{\text{Max}}\ \mathcal{L} = U[E[\tilde{R}_k^t], \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}] - \lambda(\sum_{j=1}^{N+1} X_j - 1)$$

$$\frac{\partial \mathcal{L}}{\partial X_i} \Rightarrow [\partial U/\partial E[\tilde{R}_k^t]][\bar{R}_i(1-g'-(1/2)(V^r)^2 g''' u_{2000}) - V^r g'' u_{1010}]$$

$$+ [\partial U/\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}][2(1-X_{N+1})\{(1-g')^2 u_{1010}-(3/2)g''(1-g')V^r u_{2010}$$

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$$+ (1/2)(g'')^2(V^r)^2[u_{3010}-u_{2000}u_{1010}] - \bar{R}_i[g''(1-g')V^r u_{2000}$$

$$+ (1/2)(V^r)^2[g'''(1-g')-(g'')^2]u_{3000}-(1/4)g''g'''(V^r)^3[u_{4000}-u_{2000}^2]\}]]$$

$$- \lambda = 0, \qquad i = 1,\ldots,N$$

$$\frac{\partial \mathcal{L}}{\partial X_{N+1}} \Rightarrow [\partial U/\partial E[\tilde{R}_k^t]][R_{N+1}(1-g'-(1/2)(V^r)^2 g''' u_{2000})]$$

$$+ [\partial U/\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}][-2(1-X_{N+1})R_{N+1}\{g''(1-g')V^r u_{2000} + (1/2)(V^r)^2$$

$$[g'''(1-g')-(g'')^2]u_{3000} - (1/4)g''g'''(V^r)^3[u_{4000}-u_{2000}^2]\}] - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \Rightarrow \sum_{j=1}^{N+1} X_j - 1 = 0 \; . \qquad 4.5$$

The Lagrange Multiplier (λ) can be eliminated by subtracting the $N+1^{th}$ equation from the remaining N equations, yielding:

$$[\partial U/\partial E[\tilde{R}_k^t]][(\bar{R}_i - R_{N+1})(1-g'-(1/2)(V^r)^2 g''' u_{2000}) - V^r g'' u_{1010}]$$

$$-2(1-X_{N+1})[\partial U/\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}][(1-g')^2 u_{1010}-(3/2)g''(1-g')V^r u_{2010}$$

$$+ (1/2)(g'')^2(V^r)^2[u_{3010}-u_{2000}u_{1010}]-(\bar{R}_i-R_{N+1})[g''(1-g')V^r u_{2000}$$

$$+ (1/2)(V^r)^2[g'''(1-g')-(g'')^2]u_{3000}-(1/4)g''g'''(V^r)^3[u_{4000}-u_{2000}^2]]]= 0$$

$$\sum_{j=1}^{N+1} X_j - 1 = 0 \qquad 4.6$$

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Equations 4.6 may be solved for the investor's after tax marginal rate of substitution between risk and return. Hence,

$$2(1-X_{N+1})\left[\frac{\partial U/\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial U/\partial E[\tilde{R}_k^t]}\right] =$$

$$\{(\bar{R}_i-R_{N+1})(1-g'-(1/2)(V^r)^2 g'''u_{2000}) - V^r g'' u_{1010}\}\{[(1-g')^2 u_{1010}$$

$$-(3/2)g''(1-g')V^r u_{2010} + (1/2)(g'')^2(V^r)^2[u_{3010}-u_{2000}u_{1010}]$$

$$-\ (\bar{R}_i-R_{N+1})[g''(1-g')V^r u_{2000} + (1/2)(V^r)^2[g'''(1-g')-(g'')^2]u_{3000}$$

$$-\ (1/4)g''g'''(V^r)^3[u_{4000}-u_{2000}^2]]^{-1}\} . \quad i = 1,\ldots,N \qquad 4.7$$

Assuming the second order conditions hold, equation set 4.7 is the investor's optimizing conditions for an investor subject to a tax function consistent with equation 4.1. To be a true optimum, this condition must hold for each risky security in the investor's portfolio, as well as the entire risky portfolio. If the equation did not hold for the risky portfolio as a whole, the investor would increase or decrease his holdings of the riskless asset by selling or buying additional risky securities until a new equilibrium was reached in which 4.7 held for each security in the investor's portfolio, as well as the risky security itself. That is, the after tax marginal rate of substitution between risk and return must be equal to the marginal rate of transformation between

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$M_1 = 1 - g$

$N_1^i = V^r g'' u$

between risk and return for each security in the investor's risky portfolio, as well as the risky portfolio itself.[5]

That is, equation 4.7 must hold for $\tilde{R}_i \equiv \tilde{R}_k^r$. Then the mixed moments in 4.7 reduces to $u_{n010} = E[(\tilde{R}_k^r - \bar{R}_k^r)^n(\tilde{R}_k^r - \bar{R}_k^r)] = u_{n+1,000}$, and substituting into 4.7:

$$2(1-X_{N+1})\left[\frac{\partial U/\partial\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial U/\partial E[\tilde{R}_k^t]}\right] =$$

$$\{(\bar{R}^r - R_{N+1})(1-g'-(1/2)(V^r)^2g'''u_{2000})-V^rg''u_{2000}\}\{[(1-g')^2u_{2000}$$

$$- (3/2)g''(1-g')V^ru_{3000} + (1/2)(g'')^2(V^r)^2[u_{4000} - u_{2000}^2]$$

$$- (\bar{R}^r- R_{N+1})[g''(1-g')V^ru_{2000} + (1/2)(V^r)^2[g'''(1-g')-(g'')^2]u_{3000}$$

$$- (1/4)g''g'''(V^r)^3[u_{4000} - u_{2000}^2]]^{-1}\} \quad 4.8$$

Setting the i^{th} equation in 4.7 equal to 4.8,

$$\frac{(\bar{R}_i - R_{N+1})M_1 - N_1^i}{P_1^i - (\bar{R}_i - R_{N+1})Q_1} = \frac{(\bar{R}_k^r - R_{N+1})M_1 - N_1^k}{P_1^k - \bar{R}_k^r - R_{N+1})Q_1} \quad 4.9$$

$$i = 1,\ldots,N$$

where,

$$M_1 = 1 - g' - (1/2)(V^r)^2g'''u_{2000}$$

$$N_1^i = V^rg''u_{1010}$$

$P_1^i = (1-g$

u_2

$Q_1 = g''(1$

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$N_1^k = (V^r g$

$P_1^k = (1-g$

Hence,[7]

$(\bar{R}_i - R_{N+}$

$A_1^* = \frac{M_1 P_1^i}{M_1 P_1^k}$

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expression

$$P_1^i = (1-g')^2u_{1010} - (3/2)g''(1-g')V^r u_{2010} + (1/2)(g'')^2(V^r)^2[u_{3010} - u_{2000}u_{1010}]$$

$$Q_1 = g''(1-g')V^r u_{2000} + (1/2)(V^r)^2[g'''(1-g') - (g'')^2]u_{3000} - (1/4)g'''g'''(V^r)^3[u_{4000} - u_{2000}^2]$$

$$N_1^k = (V^r g'' u_{2000})$$

$$P_1^k = (1-g')^2u_{2000} - (3/2)g''(1-g')V^r u_{3000} + (1/2)(g'')^2(V^r)^2 + [u_{4000} - u_{2000}^2] \ .$$

Hence,[7]

$$(\bar{R}_i - R_{N+1}) = (\bar{R}_k^r - R_{N+1})\left[\frac{M_1P_1^i - Q_1N_1^i}{M_1P_1^k - Q_1N_1^k}\right] + \left[\frac{N_1^kP_1^i - N_1^iP_1^k}{M_1P_1^k - Q_1N_1^k}\right] \ . \qquad i = 1,\dots,N \qquad 4.10$$

$$= (\bar{R}_k^r - R_{N+1})A_1^* - B_1^*$$

$$A_1^* = \frac{M_1P_1^i - Q_1N_1^i}{M_1P_1^k - Q_1N_1^k}, \quad B_1^* = \frac{N_1^kP_1^i - N_1^iP_1^k}{M_1P_1^k - Q_1N_1^k} \ .$$

Clearly, the required risk premium for a risky asset will diverge from the traditional no tax model, to the extent that $A_1^* \neq \frac{u_{1010}}{u_{2000}}$, and $B_1^* \neq 0$. Substituting the expressions for M_1, P_1^i, P_1^k, Q_1 N_1^i and N_1^k directly into equation 4.10, carrying out the algebraic operations indicated, and collecting terms, yields the following expressions for A_1^* and B_1^*[8,9]:

$A_1^* =$

$\frac{u'_{11}}{u'_{20}}$

$B_1^* = \frac{u'_{11}}{1 -}$

$\alpha_1 = (3/2$

$\beta_1 = (1/2$

$\gamma_1 = (1/2$

$\delta_1 = (3/4$

$\xi_1 = (1/2$

$z_1 = (1/4$

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$$A_1^* =$$

$$\frac{u'_{11}}{u'_{20}}\left\{\frac{1-[\alpha_1-\delta_1 u'_{20}]\frac{u'_{21}}{u'_{11}} + [\beta_1 - z_1 u'_{20}]\frac{u'_{31}}{u'_{11}} - [\gamma_1+3\beta_1]u'_{20} + [\xi_1-(2/3)\delta_1]u'_{30} + z_1 u'_{40}}{1 - [\gamma_1+3\beta_1]u'_{20} - [\alpha_1-((1/\ )\delta_1 + \xi_1)u'_{20}]\frac{u'_{30}}{u'_{20}} + \beta_1 \frac{u'_{40}}{u'_{20}}}\right\}$$

$$B_1^* = \frac{u'_{11}[-3\beta_1 \frac{u'_{21}}{u'_{11}} + \xi_1 \frac{u'_{31}}{u'_{11}} + 3\beta_1 \frac{u'_{30}}{u'_{20}} - \xi_1 \frac{u'_{40}}{u'_{20}}]}{1 - [\gamma_1-3\beta_1]u'_{20} - [\alpha_1-((1/3)\delta_1 + \xi_1)u'_{20}]\frac{u'_{30}}{u'_{20}} + \beta_1 \frac{u'_{40}}{u'_{20}}}$$

$$\alpha_1 = (3/2)\ \frac{g''}{1-g'}\ V^r$$

$$\beta_1 = (1/2)(\frac{g''}{1-g'})^2 (V^r)^2$$

$$\gamma_1 = (1/2)(\frac{g'''}{1-g'})(V^r)^2 \qquad 4.11$$

$$\delta_1 = (3/4)\frac{g'''}{(1-g')}(\frac{g''}{1-g'})(V^r)^3$$

$$\xi_1 = (1/2)(V^r)^3(\frac{g''}{1-g'})^3$$

$$z_1 = (1/4)(\frac{g'''}{1-g'})(\frac{g''}{1-g'})^2(V^r)^4$$

The required risk premium is now expressed in terms of the parameters (around $\overline{Y}_k$) of the tax function and the higher moments of the securities' joint probability density function. Thus, for a given tax function, the required risk premium will depend, not only on the traditional risk measure $(\frac{u'_{11}}{u'_{20}})$, but also on higher moments of the security and portfolio joint density function. At this poin

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$(\bar{R}_i - R_N$

this point, it is convenient to transform the moments about the mean into their respective cumulants, and consider the effects of these cumulants on the required risk premium. To this end, define the cumulants,[10]

$$\kappa_{20} = u'_{20},\ \kappa_{30} = u'_{30}$$

$$\kappa_{40} = u'_{4} - 3u'^{2}_{2} \qquad 4.12$$

$$\kappa_{11} = u'_{11},\ \kappa_{21} = u'_{21}$$

$$\kappa_{31} = u'_{31} - 3u'_{20}u'_{11} \quad \text{, so that}$$

$$u'_{20} = \kappa_{20},\ u'_{30} = \kappa_{30}$$

$$u'_{40} = \kappa_{40} + 3\kappa^{2}_{20}$$

$$u'_{11} = \kappa_{11},\ u'_{21} = \kappa_{21}$$

$$u'_{31} = \kappa_{31} + 3\kappa_{20}\kappa_{11} \quad .$$

Substituting into equations 4.11 yields:

$$(\bar{R}_i - R_{N+1}) = (\bar{R}^r_k - R_{N+1})A^*_1 - B^*_1 \qquad 4.13$$

$A_1^* =$

$\frac{\kappa_{11}}{\kappa_{20}}\{1-[\alpha_1$

$+ [\xi_1-(2/$

$- [\alpha_1-((1$

$= \frac{\kappa_{11}}{\kappa_{20}} \frac{1-[}{1 \;-}$

$B_1^* = \frac{\kappa_{11}}{1 \;-}$

so that,

$(\bar{R}_i - R_N$

$[(\bar{R}_k^r -$

$+ \frac{3B}{1}$

$$A_1^* =$$

$$\frac{\kappa_{11}}{\kappa_{20}}\{1-[\alpha_1-\delta_1\kappa_{20}]\frac{\kappa_{21}}{\kappa_{11}} +[\beta_1-z_1\kappa_{20}]\frac{\kappa_{31} + 3\kappa_{20}\kappa_{11}}{\kappa_{11}} - [\gamma_1+3\beta_1]\kappa_{20}$$

$$+ [\xi_1-(2/3)\delta_1]\kappa_{30} + z_1[\kappa_{40} + 3\kappa_{20}^2]\}\{[1-[\gamma_1-3\beta_1]\kappa_{20}$$

$$- [\alpha_1-((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}} + \beta_1\frac{(\kappa_{40} + 3\kappa_{20}^2)}{\kappa_{20}}]^{-1}\}$$

$$= \frac{\kappa_{11}}{\kappa_{20}} \frac{1-[\alpha_1-\delta_1\kappa_{20}]\frac{\kappa_{21}}{\kappa_{11}} + [\beta_1-z_1\kappa_{20}]\frac{\kappa_{31}}{\kappa_{11}} - \gamma_1\kappa_{20} + [\xi_1-(2/3)\delta_1]\kappa_{30}+z_1\kappa_{40}}{1 - \gamma_1\kappa_{20}-[\alpha_1-((1/3)\delta_1 +\xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}} + \beta_1\kappa_{40}}$$

$$B_1^* = \frac{\kappa_{11}\,[-3\beta_1 \frac{\kappa_{21}}{\kappa_{11}} + \xi_1(\frac{\kappa_{31} + 3\kappa_{20}\kappa_{11}}{\kappa_{11}}) + 3\beta_1 \frac{\kappa_{30}}{\kappa_{20}} - \xi_1(\frac{\kappa_{40} + 3\kappa_{20}^2}{\kappa_{20}})]}{1 - \gamma_1\kappa_{20} -[\alpha_1-((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}} + \beta_1\kappa_{40}}$$

so that,

$$(\bar{R}_i - R_{N+1}) =$$

$$[(\bar{R}_k^r-R_{N+1})]\frac{\kappa_{11}}{\kappa_{20}}\left\{\frac{1-[\alpha_1-\delta_1\kappa_{20}]\frac{\kappa_{21}}{\kappa_{11}} +[\beta_1-z_1\kappa_{20}]\frac{\kappa_{31}}{\kappa_{11}} -\gamma_1\kappa_{20}+[\xi_1-(2/3)\delta_1]\kappa_{30}+z_1\kappa_{40}}{1-\gamma_1\kappa_{20}-[\alpha_1-((1/3)\delta_1+\xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}} + \beta_1\kappa_{40}}\right\}$$

$$+ \frac{3\beta_1\kappa_{21} - \xi_1[\kappa_{31} - \frac{\kappa_{40}\kappa_{11}}{\kappa}] - 3\beta_1 \frac{\kappa_{30}\kappa_{11}}{\kappa_{20}}}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}] \frac{\kappa_{30}}{\kappa_{20}} + \beta_1\kappa_{40}} \qquad 4.14$$

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Equation 4.14 is the risk premium investor k requires of asset i, where the investor is subject to the specified tax function. Clearly, the mixed cumulants (moments) of each security in the investor's portfolio will affect the cumulants (moments) of the investor's entire portfolio, as well as the parameters of the tax function at $\bar{Y}_k$. However, if it is assumed the value of any security is a relatively small proportion of the value of the investor's entire portfolio, then the moments will have a negligible effect on the portfolio moments and the tax parameters. Thus, in order to consider the effect of the individual security's mixed cumulants on its required risk premium, it is possible to take the total derivative of equation 4.14, permitting only the mixed cumulants to vary. That is,

$$d(\bar{R}_i - R_{N+1}) =$$

$$[\bar{R}_k^r - R_{N+1}]\frac{\partial A_1^*}{\partial \kappa_{11}} - \frac{\partial B_1^*}{\partial \kappa_{11}}\, d\kappa_{11} + [\bar{R}_k^r - R_{N+1}]\frac{\partial A_1^*}{\partial \kappa_{21}} - \frac{\partial B_1^*}{\partial \kappa_{21}}\, d\kappa_{21}$$

$$+ \ [\bar{R}_k^r - R_{N+1}]\frac{\partial A_1^*}{\partial \kappa_{31}} - \frac{\partial B_1^*}{\partial \kappa_{31}}\, d\kappa_{31}$$

$$= \left[\frac{1}{\kappa_{20}} \; \frac{-\gamma_1 + [\xi_1 - (2/3)\delta_1]\frac{\kappa_{30}}{\kappa_{20}} + z_1\frac{\kappa_{40}}{\kappa_{20}}\}(\bar{R}_k^r - R_{N+1}) + \xi_1\frac{\kappa_{40}}{\kappa_{20}} - 3\beta_1\frac{\kappa_{30}}{\kappa_{20}}}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}} + \beta_1\kappa_{40}} \right] d(\kappa_{11})$$

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$$- \left[\frac{(\bar{R}_k^r - R_{N+1})\{-[\alpha_1 - \delta_1\kappa_{20}]\frac{1}{\kappa_{20}}\} - 3\beta_1}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}} + \beta_1\kappa_{40}}\right] d(\kappa_{21}) \qquad 4.15$$

$$+ \left[\frac{(\bar{R}_k^r - R_{N+1})[\beta_1 - z_1\kappa_{20}]\frac{1}{\kappa_{20}} - \xi_1}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}} + \beta_1\kappa_{40}}\right] d(\kappa_{31}) .$$

The cumulants in equation 4.15 have significance in that they describe the shape of the joint probability density function of the portfolio and security return, as well as the probability density function of the portfolio return alone. Specifically, κ_{20} measures the variance of portfolio returns. For a given portfolio variance, κ_{30} measures the degree and direction of skewness of the portfolio's probability density function. Positive skewness implies that the distribution is skewed to the right and the portfolio mean is greater than its median. Negative skewness implies that the distribution is skewed to the left and the mean is less than the median.[11] Similarly, κ_{40} measures the kurtosis of the portfolio distribution. For κ_{40} positive, the distribution of portfolio returns are leptokurtic, for κ_{40} negative, the distribution is platykurtic, and for κ_{40} equal to zero, the distribution is mesokurtic.[12] The following sections consider various probability distributions and their implications for the effects on the required risk premium of specific tax functions.

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4.2 The Case of a Tax Function which can be Approximated by a Second Degree Taylor Expansion, Quadratic Utility Functions, and Security Returns Having a Multivariate Normal Distribution

If the portfolio and security returns have a multivariate normal probability distribution, then the portfolio probability distribution will be symmetric and mesokurtic. That is, $\kappa_{30} = \kappa_{40} = 0$. In addition, the joint cumulants κ_{21}, and κ_{31} vanish.[13] Since the two mixed cumulants are constrained to zero by the nature of the probability density function, the total derivative, given the portfolio return, the tax parameters and the cumulants of the portfolio probability distribution, reduces to:

$$d(\bar{R}_i - R_{N+1}) = \frac{(\frac{1}{\kappa_{20}} - \gamma_1)(\bar{R}_k^r - R_{N+1})}{1 - \gamma_1\kappa_{20}} d(\kappa_{11})$$

$$= \frac{1}{\kappa_{20}} (\bar{R}_k^r - R_{N+1}) d\kappa_{11} \quad 4.16$$

Thus, the increase in the required risk premium of a security per unit of increased covariance is simply,

$$\frac{d(\bar{R}_i - R_{N+1})}{d\kappa_{11}} = \frac{1}{\kappa_{20}} (\bar{R}_k^r - R_{N+1}) \quad 4.17$$

Note that this relationship is identical to the results obtained in the conventional no-tax model. For, in the conventional model, $\bar{R}_i - R_{N+1} = \frac{\kappa_{11}}{\kappa_{20}} (\bar{R}_k^r - R_{N+1})$. Thus, Alterna

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$$\frac{d(\bar{R}_i - R_{N+1})}{d\kappa_{11}} = \frac{1}{\kappa_{20}} (\bar{R}^r_k - R_{N+1}) .$$

Alternatively, the cumulants specific to a bivariate normal distribution may be substituted directly into equation 4.14. Then, equation 4.14 reduced directly to:

$$\bar{R}_i - R_{N+1} = (\bar{R}^r_k - R_{N+1})\frac{\kappa_{11}}{\kappa_{20}}\{\frac{1 - \gamma_1\kappa_{20}}{1 - \gamma_1\kappa_{20}}\} = (\bar{R}^r_k - R_{N+1})\frac{\kappa_{11}}{\kappa_{20}} , \qquad 4.18$$

identical to the traditional no-tax model.

This result may be explained through reconsideration of the original investor's maximizing equations, 4.6. It is clear that the investor alters his behavior when subject to a tax function because the expectation of the tax function and its derivative with the portfolio and security returns involve higher moments of the joint portfolio and security probability distribution. However, in the case of multivariate normal distributions, the higher mixed moments are simply functions of the lower moments u'_{20}, u'_{11} or vanish.[14] That is, if a probability distribution is normal, the variance and covariances completely describe the distribution. Thus, the covariance between the portfolio return and the security return, the variance of the portfolio return, and the expected portfolio return are the only relevant parameters to be considered by the investor. Alternatively, in terms of cumulants, in the bivariate normal case, all cumulants κ_{rs} vanish, where $r + s > 2$.[15] Thus, all cumulants other than κ_{11} and κ_{20} vanish, leaving only those relevant to the traditional no tax model.

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4.3 The Case of a Tax Function which can be Approximated by a Second Degree Taylor Expansion, Quadratic Utility Functions, and Portfolio Returns are Mesokurtic, but Skewed

In this section it is assumed that equation 4.15 holds, but that $\kappa_{40} = 0$. Thus, equation 4.15 becomes:

$$d(\bar{R}_i - R_{N+1}) =$$

$$\frac{(\bar{R}_k^r - R_{N+1})[\frac{1}{\kappa_{20}} - \gamma_1 + [\xi_1 - (2/3)\delta_1]\frac{\kappa_{30}}{\kappa_{20}} - 3\beta_1\frac{\kappa_{30}}{\kappa_{20}}}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}}} d\kappa_{11}$$

$$+ \frac{(\bar{R}_k^r - R_{N+1})\{-[\alpha_1 - \delta_1\kappa_{20}]\frac{1}{\kappa_{20}}\} + 3\beta_1}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}}} d\kappa_{21}$$

$$+ \frac{(\bar{R}_k^r - R_{N+1})[\beta_1 - z_1\kappa_{20}]\frac{1}{\kappa_{20}} - \xi_1}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}}} d\kappa_{31} \qquad 4.19$$

By constraining two of the three derivatives to zero it is possible to consider the effect of increasing each mixed cumulant on the investors required risk premium, subject to given portfolio probability density functions, and given tax parameters. That is,

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{11}} = \left.\frac{d(\bar{R}_i - R_{N+1})}{d\kappa_{11}}\right|_{\substack{d\kappa_{21}=d\kappa_{31}=0 \\ \bar{R}_k^r,\kappa_{20},\kappa_{30},\alpha_1,\beta,\delta_1,\xi_1}}$$

$$= \frac{(\bar{R}_k^r - R_{N+1})[\frac{1}{\kappa_{20}} - \gamma_1 + [\xi_1 - (2/3)\delta_1]\frac{\kappa_{30}}{\kappa_{20}}] - 3\beta_1 \frac{\kappa_{30}}{\kappa_{20}}}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}}} \qquad 4.20i$$

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{21}} = \frac{(\bar{R}_k^r - R_{N+1})\{-[\alpha_1 - \delta_1\kappa_{20}]\}\frac{1}{\kappa_{20}} + 3\beta_1}{1 - \delta_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}}} \qquad 4.20ii$$

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{31}} = \frac{(\bar{R}_k^r - R_{N+1})[\beta_1 - z_1\kappa_{20}]\frac{1}{\kappa_{20}} - \xi_1}{1 - \gamma_1\kappa_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1)\kappa_{20}]\frac{\kappa_{30}}{\kappa_{20}}} \qquad 4.20iii$$

First consider equation 4.20i. This equation has particular significance in that, conditional on the expected portfolio return and variance, $\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{11}}$ is the additional risk premium required for an additional unit of risk when the security's risk is measured by the traditional no-tax model. Recall that the no tax model requires that,[16]

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{11}} = \frac{1}{\kappa_{20}}(\bar{R}_k^r - R_{N+1}) \ . \qquad 4.21$$

Comparing 4.20i with equation 4.21, indicates the influence specific tax parameters and the moments of the portfolio probability density function have on the investor's willingness to increase his expected return by increasing risk. To facilitate this comparison, 4.20i may be rearranged so that,

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial \kappa_{11}} =$$

$$\frac{\frac{1}{\kappa_{20}}[\bar{R}_k^r - R_{N+1}][1 - \gamma_1\kappa_{20} + \xi_1\kappa_{30} - (2/3)\delta_1\kappa_{30}] - 3\beta_1\frac{\kappa_{30}}{\kappa_{20}}}{(1 - \gamma_1\kappa_{20} + \xi_1\kappa_{30} - (2/3)\delta_1\kappa_{30}) + (\delta_1\kappa_{30} - \alpha_1\frac{\kappa_{30}}{\kappa_{20}})}$$

$$= \frac{1}{\kappa_{20}}[\bar{R}_k^r - R_{N+1}]\left[\frac{1}{1 + \frac{\delta_1\kappa_{30} - \alpha_1(\kappa_{30}/\kappa_{20})}{1 - \gamma_1\kappa_{20} + \xi_1\kappa_{30} - (2/3)\delta_1\kappa_{30}}}\right]$$

$$- \frac{3\beta_1\kappa_{30}}{1 - \gamma_1\kappa_{20} + \xi_1\kappa_{30} - (2/3)\delta_1\kappa_{30} + \delta_1\kappa_{30} - \alpha_1(\kappa_{30}/\kappa_{20})} \qquad 4.22$$

Clearly, the coefficient of $\frac{1}{\kappa_{20}}(\bar{R}_k^r - R_{N+1})$ in 4.22 is greater, equal to or less than unity depending on whether

$$\frac{\delta_1\kappa_{30} - \alpha_1(\kappa_{30}/\kappa_{20})}{1 - \gamma_1\kappa_{20} + \xi_1\kappa_{30} - (2/3)\delta_1\kappa_{30}} \lesseqgtr 0 \ . \qquad 4.23$$

That is, from equation 4.11,

$$\frac{\kappa_{30}[(3/4)(\frac{g'''}{1-g'})(\frac{g''}{1-g'})(V^r)^3 - (3/2)\frac{g''}{1-g'}V^r(\kappa_{20})^{-1}]}{1 - (3/4)(\frac{g'''}{1-g'})(V^r)^2\kappa_{20} + (1/2)(V^r)^3(\frac{g''}{1-g'})^3\kappa_{30} - (1/2)(\frac{g'''}{1-g'})(\frac{g''}{1-g'})(V^r)^3\kappa_{30}} \lesseqgtr 0$$

4.24

Dividing both numerator and denominator by $\kappa_{30}(\frac{g''}{1-g'})(V^r)^3$, 4.24 reduces to:

$$\frac{[(3/4)\kappa_{30}(V^r)^3(\frac{g''}{1-g'})][\frac{g'''}{1-g'} - \frac{2}{(V^r)^2{}_{20}}]}{1 - (3/4)(\frac{g'''}{1-g'})(V^r)^2\kappa_{20} + (1/2)(V^r)^3(\frac{g''}{1-g'})^3\kappa_{30} - (1/2)\frac{g'''}{1-g'}\frac{g''}{1-g'}(V^r)^3\kappa_{30}} \lesseqgtr 0$$

4.25

The equality holds in 4.25 if $\kappa_{30} = 0$, $g'' = 0$, or $\frac{g'''}{1-g'} = 2[(V^r)^2\kappa_{20}]^{-1}$. This specific value for $\frac{g'''}{1-g'}$ does not seem to have any economic significance except that it is clearly positive, so that given the assumptions specified in this section the coefficient of $(\bar{R}_k^r - R_{N+1})\frac{1}{\kappa_{20}}$ is unity when the tax function is linear $(g_k'' = 0)$, or it may be unity if taxes are increasing (or decreasing) at an increasing rate $(\frac{g'''}{1-g'} = 2[(V')^2\kappa_{20}]^{-1} > 0)$.

Note that,

$$\begin{aligned} (V^r)^2\kappa_{20} &= (V^r)^2 u_{20} \\ &= (V^r)^2 E[\tilde{R}_k^r - \bar{R}_k^r)^2] \\ &= E[(Y_k^h + V_k(1-X_{N+1,k})\tilde{R}_k^r - Y_k^h - V_k(1-X_{N+1,k})\bar{R}_k^r)^2]. \end{aligned}$$

That is $(V^r)^2\kappa_{20}$ is the variance of investor k's income. Similarly, $(V^r)^3\kappa_{30}$ is the degree of skewness of investor k's income. It is likely, that the variance will be much greater than its skewness, and since $\frac{g'''}{1-g'}$ and $\frac{g''}{1-g'}$ are likely to be much less than one, it can safely be assumed that the denominator in 4.25 is positive. Thus, the sign of equation 4.25 will be equal to the sign of its numerator.

Now, consider the case when the investor is subject to an increasing marginal tax rate $(g'' > 0)$, a positively skewed portfolio return $(\kappa_{30} > 0)$ and a predominantly long portfolio $(V^r > 0)$. Then, the coefficient of $[\frac{g'''}{1-g'} - \frac{2}{(V^r)^2\kappa_{20}}]$ is positive. Thus, the coefficient of $(\bar{R}_k^r - R_{N+1})\frac{1}{\kappa_{20}}$ is $\lesseqgtr 1$ depending on whether $\frac{g'''}{1-g'} \gtrless \frac{2}{(V^r)^2\kappa_{20}}$. By considering all possible values of κ_{30}, $\frac{g''}{1-g'}$, V^r and $\frac{g'''}{1-g'}$, it is possible to determine the effects on both the coefficients of $(\bar{R}_k^r - R_{N+1})\frac{1}{\kappa_{20}}$ and the second terms in 4.20i-iii. That is, it is possible to determine the extent to which κ_{11}, κ_{21} and κ_{31} affect the investor's required risk premium for various values of the specified parameters. Tables 2 through 4 summarize these results

Inspection of Table 2 indicates that it is difficult to derive sweeping generalities about the effects of taxes on the required risk premium. However, careful consideration of the results leads to some intuitively appealing conclusions. The following possible explanations for some of the results of Table 2 are not meant to be definitive. It is possible that others will arrive at entirely different conclusions. However, they do provide a possible intuitive explanation for some of the specific conclusions inferred from Table 2. The additional required risk premium per unit of additional covariance is unambiguously less than the traditional model only if the portfolio return is positively skewed (lines 2, 7, 9, 14, 18 of Table 2). Since positive skewness implies that the portfolio probability density function is such

Table 2

Values of the Slope and Intercept, A and B in the Equation

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{11}} = (\bar{R}_k^r - R_{N+1})\frac{1}{\kappa_{20}}[A] + B,$$ for Selected Values of Important Parameters, when Portfolio Distributions are Mesokurtic

	(1-g')	g"/1-g'	g"'/1-g'	V^r	κ_{30}	A	B
1	+	0	N.S.[a]	≠ 0	N.S.	1	0
2	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	+	< 1	< 0
3	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	+	> 1	< 0
4	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	-	> 1	> 0
5	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	-	< 1	> 0
6	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	+	> 1	< 0
7	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	+	< 1	< 0
8	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	+	> 1	< 0
9	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	+	< 1	< 0
10	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	-	< 1	> 0
11	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	-	> 1	> 0
12	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	-	< 1	> 0
13	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	-	> 1	> 0

Table 2 (continued)

	(1-g')	g"/1-g'	g'"/1-g'	V^r	κ_{30}	A	B
14	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	+	< 1	< 0
15	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	+	> 1	< 0
16	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	-	> 1	> 0
17	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	-	< 1	> 0
18	+	≠ 0	$\frac{2}{(V^r)^2\kappa_{20}}$	≠ 0	+	1	< 0
19	+	≠ 0	$\frac{2}{(V^r)^2\kappa_{20}}$	≠ 0	-	1	> 0

[a]N.S. means the value is not specified.

Table 3

Values of the Slope and Intercept, A and B in the Equation

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{21}} = (\bar{R}_k^r - R_{N+1})\frac{1}{\kappa_{20}}[A] + B,$$ for Selected Values of Important Parameters, when Portfolio Distributions are Mesokurtic

	(1-g')	g"/1-g'	g"'/1-g'	V^r	κ_{30}	A	B
1	+	0	N.S.[a]	$\neq 0$	N.S.	0	0
2	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	+	> 0	+
3	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	+	< 0	+
4	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	-	> 0	+
5	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	-	< 0	+
6	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	+	< 0	+
7	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	+	> 0	+
8	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	+	< 0	+
9	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	+	> 0	+
10	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	-	< 0	+
11	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	-	> 0	+
12	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	-	< 0	+
13	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	-	> 0	+

Table 3 (continued)

	(1-g')	g"/1-g'	g"'/1-g'	V^r	κ_{30}	A	B
14	+	-	$> \frac{2}{(V^r)^2 \kappa_{20}}$	-	+	> 0	+
15	+	-	$< \frac{2}{(V^r)^2 \kappa_{20}}$	-	+	< 0	+
16	+	-	$> \frac{2}{(V^r)^2 \kappa_{20}}$	-	-	> 0	+
17	+	-	$< \frac{2}{(V^r)^2 \kappa_{20}}$	-	-	< 0	+
18	+	≠ 0	$\frac{2}{(V^r)^2 \kappa_{20}}$	≠ 0	+	0	+
19	+	≠ 0	$\frac{2}{(V^r)^2 \kappa_{20}}$	≠ 0	-	0	+

[a]N.S. Means the value is not specified.

Table 4

Values of the Slope and Intercept, A and B in the Equation

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{31}} = (\bar{R}_k^r - R_{N+1})\frac{1}{\kappa_{20}}[A] + B,$$ for Selected Values of

Important Parameters when Portfolio Distributions are Mesokurtic

	(1-g')	g"/1-g'	g"'/1-g'	V^r	κ_{30}	A	B
1	+	0	N.S.[a]	≠ 0	N.S.	0	0
2	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	> 0	< 0
3	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	> 0	> 0
4	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	> 0	> 0
5	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	> 0	< 0
6	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	< 0	< 0
7	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	< 0	> 0
8	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	< 0	> 0
9	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	< 0	< 0
10	+	+	$\frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	0	< 0
11	+	+	$\frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	0	> 0
12	+	-	$\frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	0	> 0
13	+	-	$\frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	0	< 0

[a]N.S. Means the value is not specified.

that there is a relatively large probability of a return below the expected return, but very small probability of a large loss. In addition, the chances of a very large gain is quite good. Thus, it might be argued that the investor would be relatively anxious to hold securities which are highly correlated with the portfolio, in the hopes that the very high gain will be realized, and with very little chance of a very large loss. Note, that the only exception to this result (barring predominantly short portfolios) is when the marginal tax rate is declining at an increasing rate (line 8, Table 2). Clearly, since the portfolio return is skewed, the possibility of the investor actually suffering a loss is quite small. Thus, the ability to receive a loss offset at the high rates specified is minimized. On the other hand, the large probability of an abnormally small return taxed at the higher marginal rate may be sufficient inducement for the investor to avoid a security which is highly correlated with the portfolio return.

The additional required risk premium per unit of risk will be unambiguously greater than predicted by the traditional model only if portfolio returns are negatively skewed (lines 4, 11, 13, 16, and 19 of Table 2). In this case, there is a high probability that the portfolio return will be slightly greater than expected with very small probability of a very large gain. In addition, the possibility of a large loss is relatively high. Therefore the investor may be expected to avoid securities with a high covariance with his portfolio, since the possibility of a large loss is so great. Again, barring short portfolios, the exceptions are if marginal tax rates are decreasing at an increasing rate, or increasing at a decreasing rate, or at least not at a very highly increasing rate (lines 12 and 5 of Table 2 respectively). These specifications of the tax function implies that returns above the expected return will be taxed at a lower marginal rate, or at least not very much higher marginal rate than that rate which would have been levied if the portfolio return had achieved its expected return. Since there is a high probability of a return slightly in excess of that expected, an investor may be induced to hold additional securities possessing a high covariance with his portfolio, even at low expected return, because of this relatively small addition to his tax bill if his portfolio returns more than is expected.

Unfortunately, the results of Tables 3 and 4 do not seem to lend themselves to a similar intuitively appealing explanation. Although the higher moments at least theoretically, affect the investor's risk premium, there seems to be no systematic explanation for the direction of these effects. Perhaps the difficulty lies in the fact that the economic meaning of the higher mixed moments is not obvious.

4.4 The Case of a Tax Function which can be Approximated by a Second Degree Taylor Expansion, Investors Possess a Quadratic Utility Function, and Portfolio Returns Possess Non-Normal, Symmetric Probability Density Functions

This section modifies the more general case in section 4.1 by assuming that the portfolio return is distributed symmetrically about its mean, but is either leptokurtic or platykurtic. This specific case may be considered more interesting than the previous cases in the light of past empirical findings. With few exceptions[17], empirical findings indicate that security returns seem to possess symmetric but leptokurtic probability density functions.[18] That is, compared to the normal probability density, the probability of a return near its expected value is generally greater than that predicted by the normal distribution. In addition, the probability of a large gain or loss is also greater than would be expected if the probability distribution were normal.[19]

Reproducing equation 4.15 under the assumption of this section that the distribution is symmetric but non-normal ($\kappa_{30} = 0$, $\kappa_{40} \neq 0$) yields the following equations:

$$d(\bar{R}_i - R_{N+1}) =$$

$$\frac{(\bar{R}_k^r - R_{N+1})(\frac{1}{\kappa_{20}} - \gamma_1 + z_1 \frac{\kappa_{40}}{\kappa_{20}}) + \xi_1 \frac{\kappa_{40}}{\kappa_{20}}}{1 - \gamma_1\kappa_{20} + \beta_1\kappa_{40}} d\kappa_{11}$$

$$+ \frac{(\bar{R}_k^r - R_{N+1})[-(\alpha_1 - \delta_1\kappa_{20})]\frac{1}{\kappa_{20}} + 3\beta_1}{1 - \gamma_1\kappa_{20} + \beta_1\kappa_{40}} d\kappa_{21}$$

$$+ \frac{(\bar{R}_k^r - R_{N+1})[\beta_1 - z_1\kappa_{20}] \frac{1}{\kappa_{20}} - \xi_1}{1 - \gamma_1\kappa_{20} + \beta_1\kappa_{40}} d\kappa_{31} \qquad 4.26$$

Again, by constraining two of the three derivatives to zero it is possible to consider the effect of increasing each mixed cumulant on the investor's required risk premium, subject to given portfolio probability density functions and given tax parameters. Namely,

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{11}} = \frac{(\bar{R}^r_k - R_{N+1})(\frac{1}{\kappa_{20}} - \gamma_1 + z_1\frac{\kappa_{40}}{\kappa_{20}}) + \xi_1\frac{\kappa_{40}}{\kappa_{20}}}{1 - \gamma_1\kappa_{20} + \beta_1\kappa_{40}} \qquad 4.27i$$

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{21}} = \frac{(\bar{R}^r_k - R_{N+1})(-\alpha_1 + \delta_1\kappa_{20})\frac{1}{\kappa_{20}} + 3\beta_1}{1 - \gamma_1\kappa_{20} + \beta_1\kappa_{40}} \qquad 4.27ii$$

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{31}} = \frac{(\bar{R}^r_k - R_{N+1})(\beta_1 - z_1\kappa_{20})\frac{1}{\kappa_{20}} - \xi_1}{1 - \gamma_1\kappa_{20} + \beta_1\kappa_{40}} \qquad 4.27iii$$

Rearranging 4.27 slightly and using the expressions in equations 4.11 yields:

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{11}}$$

$$= \frac{(\bar{R}^r_k - R_{N+1})\frac{1}{\kappa_{20}}[1 - (1/2)\frac{g'''}{1-g'}(V^r)^2\kappa_{20} + (1/4)\frac{g'''}{1-g'}(\frac{g'''}{1-g'})^2(V^r)^4\kappa_{40}]}{1 - (1/2)\frac{g'''}{1-g'}(V^r)^2\kappa_{20} + (1/2)(\frac{g''}{1-g'})^2(V^r)^2\kappa_{40}} \qquad 4.28i$$

$$+ \frac{(1/2)(V^r)^3(\frac{g''}{1-g'})^3}{1 - (1/2)\frac{g'''}{1-g'}(V^r)^2\kappa_{20} + (1/2)(\frac{g''}{1-g'})^2(V^r)^2\kappa_{40}}$$

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{21}}$$

$$= \frac{(\bar{R}^r_k - R_{N+1})\frac{1}{\kappa_{20}}[(3/4)\frac{g'''}{1-g'}(\frac{g''}{1-g'})(V^r)^3\kappa_{20} - (3/2)\frac{g''}{1-g'}(V^r)]}{1 - (1/2)\frac{g'''}{1-g'}(V^r)^2\kappa_{20} + (1/2)(\frac{g''}{1-g'})^2(V^r)^2\kappa_{40}}$$

$$+ \frac{(3/2)(\frac{g''}{1-g'})^2(V^r)^2}{1 - (1/2)\frac{g'''}{1-g'}(V^r)^2\kappa_{20} + (1/2)(\frac{g''}{1-g'})^2(V^r)^2\kappa_{40}} \qquad 4.28ii$$

$$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial\kappa_{31}}$$

$$= \frac{\frac{1}{\kappa_{20}}(\bar{R}^r_k - R_{N+1})[(1/2)(\frac{g''}{1-g'})^2(V^r)^2 - (1/4)(\frac{g'''}{1-g'})(\frac{g''}{1-g'})^2(V^r)^4\kappa_{20}]}{1 - (1/2)\frac{g'''}{1-g'}(V^r)^2\kappa_{20} + (1/2)(\frac{g''}{1-g'})^2(V^r)^2\kappa_{40}}$$

$$- \frac{(1/2)(V^r)^3(\frac{g''}{1-g'})^3}{1 - (1/2)\frac{g'''}{1-g'}(V^r)^2\kappa_{20} + (1/2)(\frac{g''}{1-g'})^2(V^r)^2\kappa_{40}} \qquad 4.28iii$$

First, consider the case of the investor holding a portfolio with returns which possess a leptokurtic probability density function. Then the coefficient of $(\bar{R}^r_k - R_{N+1})\frac{1}{\kappa_{20}}$ will be $\lesseqgtr 1$ depending on whether $(1/4)\frac{g'''}{1-g'}(\frac{g''}{1-g'})^2(V^r)^4 \lesseqgtr \frac{1}{2}(\frac{g''}{1-g'})^2(V^r)^2$. That is, if $\frac{g'''}{1-g'} \lesseqgtr \frac{2}{(V^r)^2}$. Results for all relevant expressions with various values for g'', g''', V^r and κ_{40} are presented in the following tables.[20]

Table 5

Values of the Slope and Intercept, A and B in the Equation

$\frac{\partial(\bar{R}_i - R_{N+1})}{\partial \kappa_{11}} = (\bar{R}_k^r - R_{N+1})\frac{1}{\kappa_{20}}[A] + B$, for Selected Values of Important Parameters when Portfolio Distributions are Symmetric

	(1-g')	g''/1-g'	g'''/1-g'	V^r	κ_{40}	A	B
1	+	0	N.S.[a]	≠ 0	N.S.	1	0
2	+	+	$> \frac{2}{(V^r)^2}$	+	+	> 1	+
3	+	+	$< \frac{2}{(V^r)^2}$	+	+	< 1	+
4	+	+	$> \frac{2}{(V^r)^2}$	-	+	> 1	-
5	+	-	$> \frac{2}{(V^r)^2}$	+	+	> 1	-
6	+	+	$< \frac{2}{(V^r)^2}$	-	+	< 1	-
7	+	-	$> \frac{2}{(V^r)^2}$	-	+	> 1	+
8	+	-	$< \frac{2}{(V^r)^2}$	-	+	< 1	+
9	+	-	$< \frac{2}{(V^r)^2}$	+	+	< 1	-
10	+	+	$> \frac{2}{(V^r)^2}$	+	-	< 1	+
11	+	+	$< \frac{2}{(V^r)^2}$	+	-	> 1	+
12	+	+	$> \frac{2}{(V^r)^2}$	-	-	< 1	-
13	+	-	$> \frac{2}{(V^r)^2}$	+	-	< 1	-
14	+	+	$< \frac{2}{(V^r)^2}$	-	-	> 1	-
15	+	-	$> \frac{2}{(V^r)^2}$	-	-	< 1	+
16	+	-	$< \frac{2}{(V^r)^2}$	-	-	> 1	+

Table 5 (continued)

	(1-g')	g"/1-g'	g"'/1-g'	V^r	κ_{40}	A	B
17	+	-	$< \frac{2}{(V^r)^2}$	+	-	> 1	-
18	+	+	$\frac{2}{(V^r)^2}$	+	$\neq 0$	1	+
19	+	-	$\frac{2}{(V^r)^2}$	+	$\neq 0$	1	-
20	+	+	$\frac{2}{(V^r)^2}$	-	$\neq 0$	1	-
21	+	-	$\frac{2}{(V^r)^2}$	-	$\neq 0$	1	+

[a]N.S. means the value is not specified.

Table 5 (continued)

	(1-g')	g"/1-g'	g"'/1-g'	V^r	κ_{40}	A	B
17	+	-	$< \frac{2}{(V^r)^2}$	+	-	> 1	-
18	+	+	$\frac{2}{(V^r)^2}$	+	$\neq 0$	1	+
19	+	-	$\frac{2}{(V^r)^2}$	+	$\neq 0$	1	-
20	+	+	$\frac{2}{(V^r)^2}$	-	$\neq 0$	1	-
21	+	-	$\frac{2}{(V^r)^2}$	-	$\neq 0$	1	+

[a]N.S. means the value is not specified.

Table 6

Values of the Slope and Intercept, A and B in the Equation

$$\frac{\partial(\bar{R}_k^r - R_{N+1})}{\partial\kappa_{21}} = \frac{1}{\kappa_{20}}(\bar{R}_k^r - R_{N+1})[A] + B,$$ for Selected Values of Important Parameters when Portfolio Distributions are Symmetric

	(1-g')	g"/1-g'	g"'/1-g'	V^r	κ_{40}	A	B
1	+	0	N.S[a]	≠ 0	N.S.	0	0
2	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	+	+
3	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	-	+
4	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	-	+
5	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	-	+
6	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	+	+
7	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	+	+
8	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	+	+
9	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	-	+
10	+	+	$\frac{2}{(V^r)^2\kappa_{20}}$	+	≠ 0	0	+
11	+	+	$\frac{2}{(V^r)^2\kappa_{20}}$	-	≠ 0	0	+
12	+	-	$\frac{2}{(V^r)^2\kappa_{20}}$	+	≠ 0	0	+
13	+	-	$\frac{2}{(V^r)^2\kappa_{20}}$	-	≠ 0	0	+

[a] N.S. means that value is not specified.

Table 7

Values of the Slope and Intercept, A and B in the Equation

$\frac{\partial(\bar{R}_k^r - R_{N+1})}{\partial \kappa_{31}} = \frac{1}{\kappa_{20}} (\bar{R}_k^r - R_{N+1})[A] + B$, for Selected Values of Important Parameters when Portfolio Distributions are Symmetric

	(1-g')	g"/1-g'	g"'/1-g'	V^r	κ_{40}	A	B
1	+	0	N.S.[a]	$\neq 0$	N.S.	0	0
2	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	+	+
3	+	+	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	-	-
4	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	+	+
5	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	-	-
6	+	+	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	+	-
7	+	-	$> \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	-	+
8	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	+	N.S.	+	-
9	+	-	$< \frac{2}{(V^r)^2\kappa_{20}}$	-	N.S.	+	+
10	+	+	$\frac{2}{(V^r)^2\kappa_{20}}$	+	$\neq 0$	0	+
11	+	+	$\frac{2}{(V^r)^2\kappa_{20}}$	-	$\neq 0$	0	-
12	+	-	$\frac{2}{(V^r)^2\kappa_{20}}$	+	$\neq 0$	0	-
13	+	-	$\frac{2}{(V^r)^2\kappa_{20}}$	-	$\neq 0$	0	+

[a]N.S. means that values are not specified.

Tables 5, 6 and 7 reveal that even if security returns are symmetrically distributed, as long as the portfolio probability density function is not mesokurtic, a non-linear tax function may affect the required risk premium of the securities in the portfolio. If the possibility of predominantly short portfolios are excluded from consideration, then the required risk premium per unit of covariance is unambiguously greater than that predicted by the traditional no-tax model, only if marginal tax rates are increasing (Table 5, rows 2, 11 and 18). It could be argued that since the marginal tax rate is increasing, the investor would be hesitant to hold securities possessing a high covariance with his portfolio since the loss offset obtained for a given loss will be less than that obtained for an equal gain. This hypothesis is strengthened by the observation that the required risk premium per unit of covariance is less than predicted by the conventional model if marginal tax rates are declining (Table 5, rows 9, 13 and 19). The sign and degree of kurtosis as well as the magnitude of the third derivative of the tax function also are relevant parameters in determining the relative desirability of increased risk in the sense of increased covariance. However, these parameters do not seem to lend themselves to a reasonably simple intuitive explanation. As in the previous section, the implications of the effect of taxes on $\frac{\partial(\bar{R}_k^r - R_{N+1})}{\partial\kappa_{21}}$ and $\frac{\partial(\bar{R}_k^r - R_{N+1})}{\partial\kappa_{31}}$ do not seem to lend themselves to intuitively appealing explanations.

4.5. The Case of a Tax Function which can be Approximated by an L Degree Taylor Expansion, Investors Possess Three Parameter Utility Functions, and Security Returns Possess Multivariate Normal Probability Density Functions

In Section 4.2 it was concluded that if securities are normally distributed, then even if taxes are non-linear, the investor will behave as if capital gains and dividend income were not taxable. However, the conclusions may have depended upon the specific utility function (quadratic) or the specific approximation to the tax function (second degree Taylor Expansion) assumed in the analysis. It may be reasonable to assume that merely increasing the degree of the Taylor Expansion would not effect the general conclusions of section 4.2. It has already been pointed out that cumulants higher than the second order vanish when the distribution is joing normal.[21] However, if the tax function is non-linear the portfolio return is distributed in accordance with a normal probability density function, then the probability distribution of after tax return may be skewed. Thus, if the investor considers skewness of after tax return as a relevant parameter in his utility function, it is possible that taxes will be considered, even when security distributions are normal. To consider this possibility, the assumptions in Section 4.2 are broadened to permit skewness of after tax return to be a relevant parameter in the investor's utility function and to assume a more general L degree Taylor Expansion to approximate the tax function. Other assumptions in section 4.2 are retained. Specifically, security returns are assumed to possess a multivariate normal distribution and all expectations exist and are finite. Also, it is convenient to utilize the notation defined in section 3.2, so that the moments,

$$u_{pqmn} = E[(\tilde{R}^r_k-\bar{R}^r_k)^p(\tilde{T}_k/V_k-\bar{T}_k/V_k)^q(\tilde{R}_i - \bar{R}_i)^m(\tilde{g}'_k - \bar{g}'_k)^n] \quad p,q,m,n = 0,1,2,\ldots$$

If the degree of skewness, as well as the mean and variance of after tax return, are relevant arguments in the investor's utility function, the utility function may be symbolically represented by,

$$U = U[E[\tilde{R}^t_k],\ \sigma_{(\tilde{R}^t_k,\tilde{R}^t_k)},\ E(\tilde{R}^t_k - \bar{R}^t_k)^3] ,$$

where the arguments are defined above,[22]

$$\frac{\partial U}{\partial E[\tilde{R}^t_k]} = U_1 > 0,$$

$$\frac{\partial U}{\partial \sigma_{(\tilde{R}^t_k,\tilde{R}^t_k)}} = U_2 > 0, \qquad 4.29$$

$$\frac{\partial U}{\partial [E(\tilde{R}^t_k - \bar{R}^t_k)^3]} = U_3 \neq 0 ,$$

and the tax function may be approximated by an L degree Taylor Expansion,

$$\tilde{T}_k = g_{(\bar{Y}_k)} + \sum_{\ell=1}^{L} \frac{g^{\ell}(V^r)^{\ell}(\tilde{R}^r_k - \bar{R}^r_k)^{\ell}}{\ell!}$$

$$g'_k = g'_{(\bar{Y}_k)} + \sum_{\ell=1}^{L} \frac{g^{\ell+1}(V^r)^{\ell}(R^r_k - \bar{R}^r_k)^{\ell}}{\ell!} , \qquad 4.30$$

$g'_{(\bar{Y}_k)}$, $g_{(\bar{Y}_k)}$ are the functions evaluated at $Y_k = \bar{Y}_k$, g^{ℓ} is the ℓ^{th} derivative of the tax function with respect to income evaluated at $Y_k = \bar{Y}_k$, $\ell = 1,2,\ldots,L$.

The investor desires to maximize 4.29 subject to the constraint that $\sum_{j=1}^{N+1} X_j - 1 = 0$. Forming the Lagrangean and maximizing,

$$\underset{X_i}{\text{Max}}\ \mathcal{L} = U\{E[\tilde{R}_k^t],\ \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)},\ E(\tilde{R}_k^t - \bar{R}_k^t)^3\} - \lambda(\sum_{j=1}^{N+1} X_j - 1),$$

$$\frac{\partial\mathcal{L}}{\partial X_i} \Rightarrow U_1\frac{\partial E[\tilde{R}_k^t]}{\partial X_i} + U_2\frac{\partial\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial X_i} + U_3\frac{\partial E(\tilde{R}_k^t - \bar{R}_k^t)^3}{\partial X_i} - \lambda = 0,$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} \Rightarrow \sum_{j=1}^{N+1} X_j - 1 = 0, \quad i = 1,\ldots,N+1. \qquad 4.31$$

From equations 3.4 and 3.5 above,

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_i} = \bar{R}_i(1 - E[\tilde{g}'_k]) - \sigma_{(\tilde{g}'_k,\tilde{R}_i)} \qquad 4.32$$

$$\frac{\partial\sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial X_i} = 2\sigma_{(\tilde{R}_k^t,\tilde{R}_i(1 - \tilde{g}'_k))}, \qquad 4.33$$

and from Appendix A,

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_i} = \bar{R}_i(1 - E[\tilde{g}'_k]) = u_{0011} \qquad 4.34$$

$$\frac{\partial\sigma(\tilde{R}_k^t,\tilde{R}_k^t)}{\partial X_i} = 2\{(1 - E[\tilde{g}'])[(1 - X_{N+1})u_{1010} - u_{0110}]$$

$$+ \bar{R}_i(u_{0101} - (1-X_{N+1})u_{1001}) + u_{0111} - (1-X_{N+1})u_{1011}\} \quad . \qquad 4.35$$

In addition, from Appendix F,

$$\frac{\partial E(\tilde{R}_k^t - \bar{R}_k^t)^3}{\partial X_i} = 3\{(1-\bar{g}_k')(u_{0210}-2(1-X_{N+1})u_{1110}+(1-X_{N+1})^2u_{2010})$$

$$- \bar{R}_i(u_{0201}-2(1-X_{N+1})u_{1101}+(1-X_{N+1})^2u_{2001})$$

$$+ u_{0011}((1-X_{N+1})^2u_{2000}-2(1-X_{N+1})u_{1100} + u_{0200})$$

$$- (u_{0211} - 2(1-X_{N+1})u_{1111} + (1-X_{N+1})^2u_{2011})\} \quad .$$

$$i = 1,\ldots,N + 1 \qquad 4.36$$

Specifically for $i = N + 1$,

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_{N+1}} = R_{N+1}(1 - E[\tilde{g}']) \qquad 4.37$$

$$\frac{\partial\sigma(\tilde{R}_k^t,\tilde{R}_k^t)}{\partial X_{N+1}} = 2\{R_{N+1}(u_{0101} - (1-X_{N+1})u_{1001}) \qquad 4.38$$

$$\frac{\partial E(\tilde{R}_k^t - \tilde{R}_k^t)^3}{\partial X_{N+1}} = -3R_{N+1}(u_{0201}-2(1-X_{N+1})u_{1101} + (1-X_{N+1})^2u_{2001}). \qquad 4.39$$

Substituting equations 4.34 through 4.39 into equations 4.31 yields,

$$\frac{\partial \mathcal{L}}{\partial X_i} \Rightarrow U_1[\bar{R}_i(1-\bar{g}')-u_{0011}] + 2U_2[(1-\bar{g}')((1-X_{N+1})u_{1010}-u_{0110})$$

$$+ \bar{R}_i(u_{0101}-(1-X_{N+1})u_{1001}) + u_{0111}-(1-X_{N+1})u_{1011}]$$

$$+ 3U_3[(1-\bar{g}')(u_{0210}-2(1-X_{N+1})u_{1110}+(1-X_{N+1})^2u_{2010}$$

$$- \bar{R}_i u_{0201}-2(1-X_{N+1})u_{1101}+(1-X_{N+1})^2u_{2001})+u_{0011}((1-X_{N+1})^2u_{2000}$$

$$- 2(1-X_{N+1})u_{1100}+u_{0200})-(u_{0211}-2(1-X_{N+1})u_{1111}+(1-X_{N+1})^2u_{2011})]$$

$$- \lambda = 0, \quad i = 1,\dots,N$$

$$\frac{\partial \mathcal{L}}{\partial X_{N+1}} \Rightarrow U_1[R_{N+1}(1-g')]+2U_2[R_{N+1}(u_{0101}-(1-X_{N+1})u_{1001})]$$

$$- 3U_3[R_{N+1}(u_{0201}-2(1-X_{N+1})u_{1101}+(1-X_{N+1})^2u_{2001})] -\lambda = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \Rightarrow \sum_{j=1}^{N+1} X_j - 1 = 0 \,. \qquad 4.40$$

The Lagrange multiplier may be eliminated by subtracting the $N+1^{th}$ equation from the remaining N equations, so that:

$$U_1[(\bar{R}_i-R_{N+1})(1-\bar{g}')-u_{0011}] + 2U_2[(1-\bar{g}')((1-X_{N+1})u_{1010}-u_{0110})$$

$$+ (\bar{R}_i-R_{N+1})(u_{0101}-(1-X_{N+1})u_{1001}) + (u_{0111}-(1-X_{N+1})u_{1011})]$$

$$- 3U_3[(1-\bar{g}')(2(1-X_{N+1})u_{1110}-(1-X_{N+1})^2u_{2010}-u_{0201})+(\bar{R}_i-R_{N+1})(u_{0201}$$

$$-2(1-X_{N+1})u_{1101}+(1-X_{N+1})^2u_{2001})-u_{0011}((1-X_{N+1})^2u_{2000}-2(1-X_{N+1})u_{1100}+u_{0200})$$

$$+ (u_{0211}-2(1-X_{N+1})u_{1111} + (1-X_{N+1})^2u_{2011})] = 0. \qquad 4.41$$

Equation 4.40 may be solved for the investor's marginal rate of substitution between risk and return. That is,

$$-2\frac{U_2}{U_1} =$$

$$\frac{(\bar{R}_i - R_{N+1})(1 - \bar{g}') - u_{0011}}{(1-\bar{g}')((1-X_{N+1})u_{1010}-u_{0110})-(\bar{R}_i-R_{N+1})((1-X_{N+1})u_{1001}-u_{0101})-((1-X_{N+1})u_{1011}-u_{0111})}$$

$$+ 3\frac{U_3}{U_1}\{[(1-\bar{g}')((1-X_{N+1})^2u_{2010} + u_{0210} - 2(1-X_{N+1})u_{1110})$$

$$- (\bar{R}_i-R_{N+1})((1-X_{N+1})^2u_{2001} + u_{0201} - 2(1-X_{N+1})u_{1110})$$

$$+ u_{0011}((1-X_{N+1})^2u_{2000}-2(1-X_{N+1})u_{1100} + u_{0200})-(u_{0211}+(1-X_{N+1})^2u_{2011}$$

$$- 2(1-X_{N+1})u_{1111})][(1-\bar{g}')((1-X_{N+1})u_{1010}-u_{0110})$$

$$- (\bar{R}_i-R_{N+1})((1-X_{N+1})u_{1001}-u_{0101})-((1-X_{N+1})u_{1011}-u_{0111})]^{-1}\},$$

$$i = 1,\ldots,N \qquad 4.42$$

Equation 4.42 must hold for all securities in the investor's portfolio. Therefore, the numerator and denominator on the right hand side may be summed separately over all securities without affecting its value.

First mutliplying the numerator and denominator by h_{ik}, the value of asset i in the investor's portfolio as a proportion of the total value of the risky portfolio. Then, if the numerator and denominator are summed separately, recognizing that:

$$\sum_{j=1}^{N} h_{jk} = 1$$

$$\sum_{j=1}^{N} (\bar{R}_i - R_{N+1})h_{jk} = \bar{R}_k^r - R_{N+1}$$

$$\sum_{j=1}^{N} h_{jk}u_{n,m,1,q} = u_{n+1,m,o,q}, \text{ for all } m,q \text{ and } n, \qquad 4.43$$

equation 4.42 reduces to:

$$-2\frac{U_2}{U_1} =$$

$$\frac{(\bar{R}_k^r - R_{N+1})(1-\bar{g}') - u_{1001}}{(1-\bar{g}')((1-X_{N+1})u_{2000}-u_{1100})-(\bar{R}_k^r-R_{N+1})((1-X_{N+1})u_{1001}-u_{0101})-((1-X_{N+1})u_{2001}-u_{1101})}$$

$$+ 3\frac{U_3}{U_1}\{[(1-\bar{g}')((1-X_{N+1})^2u_{3000}+u_{1200}-2(1-X_{N+1})u_{2100})-$$

$$(\bar{R}_k^r-R_{N+1})((1-X_{N+1})^2u_{2001}+u_{0201}-2(1-X_{N+1})u_{1101})+u_{0011}((1-X_{N+1})^2u_{2000}$$

$$-2(1-X_{N+1})u_{1100}+u_{0200})-(u_{1201}+(1-X_{N+1})^2u_{3001}$$

$$-2(1-X_{N+1})u_{2101})][(1-\bar{g}')((1-X_{N+1})u_{2000}-u_{1100})$$

$$-(\bar{R}_k^r-R_{N+1})((1-X_{N+1})u_{1001}-u_{0101})-((1-X_{N+1})u_{2001}-u_{1101})]^{-1}\} . \qquad 4.44$$

Since the security and portfolio returns possess a joint normal distribution, the moments can be written as,[23]

$$u_{2r,000} = \frac{(2r)!}{2^r r!} (u_{2000})^r$$

$$u_{2r+1,000} = 0 \qquad \text{for all } r > 0$$

$$u_{2r+1,0,1,0} = \frac{(2r+1)!}{r!2^r} u_{1010}(u_{2000})^r$$

$$u_{2r,0,1,0} = 0 \qquad \text{for all } r \geq 0 . \qquad 4.45$$

By substituting equations 4.30, the specific approximations for the total and marginal tax functions, directly into the moments involving these parameters, and using expressions 4.45, all moments contained in equation 4.44 can be derived in terms of the joint moments of the portfolio and security return. Specifically,[24]

$$u_{2010} = 0 \qquad u_{0111} = (1-X_{N+1})A_5\mu_{11} \qquad u_{3001} = A_{10}\mu_2$$

$$u_{1010} = \mu_{11} \qquad u_{1101} = (1-X_{N+1})A_5\mu_2 \qquad u_{1111} = (1-X_{N+1})A_{11}\mu_{11}$$

$$u_{3000} = 0 \qquad u_{0011} = A_6\mu_{11} \qquad u_{2101} = (1-X_{N+1})A_{11}\mu_2$$

$$u_{0210} = (1-X_{N+1})^2 A_1\mu_{11} \qquad u_{1001} = A_6\mu_2 \qquad u_{0101} = (1-X_{N+1})A_{12}\mu_{20}$$

$$u_{1200} = (1-X_{N+1})^2 A_1\mu_2 \qquad u_{0200} = (1-X_{N+1})^2 A_7\mu_2$$

$$u_{1110} = (1-X_{N+1})A_2\mu_{11} \qquad u_{0110} = (1-X_{N+1})A_8\mu_{11} \qquad 4.46$$

$$u_{2100} = (1-X_{N+1})A_2\mu_2 \qquad u_{1100} = (1-X_{N+1})A_8\mu_2$$

$$u_{1011} = A_3\mu_{11} \qquad u_{0211} = (1-X_{N+1})^2 A_9\mu_{11}$$

$$u_{2001} = A_3\mu_2 \qquad u_{1201} = (1-X_{N+1})^2 A_9\mu_2$$

$$u_{0201} = (1-X_{N+1})^2 A_4\mu_2 \qquad u_{2011} = A_{10}\mu_{11}$$

$A_1,\ldots,A_{12}$ are constants, independent of the individual asset i, and for economy of notation, μ_{11} is the covariance of asset i with the risky portfolio, and μ_2 the variance of the risky portfolio. That is, all of the higher moments in equations 4.42 and 4.44 may be written as functions of the variance of the return of the risky portfolio, or the covariance between the return of the asset in question and the return of the risky portfolio. Since investor equilibrium requires that the right hand sides of 4.42 and 4.44 be equal to each other, after substitution of 4.46 into 4.42 and 4.44, investor equilibrium may be represented by:

$$\frac{(\bar{R}_i - R_{N+1})(1-\bar{g}') - A_6\mu_{11}}{(1-X_{N+1})\{(1-g')(1-A_8)\mu_{11}-(\bar{R}_i-R_{N+1})(A_6-A_{12})\mu_2-(A_3-A_5)\mu_{11}\}}$$

$$+ 3\,\frac{U_3(1-X_{N+1})}{U_1}\,\{\frac{(1-\bar{g}')(A_1-2A_2)\mu_{11}-(\bar{R}_i-R_{N+1})(A_3+A_4-2A_5)\mu_2}{(1-\bar{g}')(1-A_8)\mu_{11}-(\bar{R}_i-R_{N+1})(A_6-A_{12})\mu_2-(A_3-A_5)\mu_{11}}$$

$$+ \frac{A_6\mu_{11}\mu_2(1-2A_8+A_7)-(A_9+A_{10}-2A_{11})\mu_{11}}{(1-\bar{g}')(1-A_8)\mu_{11}-(\bar{R}_i-R_{N+1})(A_6-A_{12})\mu_2-(A_3-A_5)\mu_{11}}\}$$

$$= \frac{(\bar{R}^r-R_{N+1})(1-\bar{g}')-A_6\mu_2}{[(1-\bar{g}')(1-A_8)-\bar{R}^r-R_{N+1})(A_6-A_{12})-(A_3-A_5)]\mu_2}$$

$$+ 3\,\frac{U_3(1-X_{N+1})}{U_1}\,\{\frac{(1-\bar{g}')(A_1-2A_2)-(\bar{R}^r-R_{N+1})(A_3+A_4-2A_5)}{(1-\bar{g}')(1-A_8)-(\bar{R}^r-R_{N+1})(A_6-A_{12})-(A_3-A_5)}$$

$$+ \frac{A_6\mu_2(1-2A_8+A_7)-(A_9+A_{10}-2A_{11})}{(1-\bar{g}')(10A_8)-(\bar{R}^r-R_{N+1})(A_6-A_{12})-(A_3-A_5)}\}\ . \qquad 4.47$$

Solving for the required risk premium for security i,[25]

$$\bar{R}_i - R_{N+1} = (\bar{R}^r - R_{N+1}) \frac{M_2 P_2^i - Q_2 N_2^i}{M_2 P_2^k - Q_2 N_2^k} - \frac{N_2^k P_2^i - N_2^i P_2^k}{M_2 P_2^k - Q_2 N_2^k} ,$$

where

$$M_2 = (1-\bar{g}') + 3 \frac{U_3}{U_1} (1-X_{N+1})\{(A_3+A_4-2A_5)\mu_2$$

$$N_2^i = [A_6 + 3 \frac{U_3}{U_1} (1-X_{N+1})\{(1-\bar{g}')(A_1-2A_2)+A_6(1-2A_8+A_7)\mu_2-(A_9+A_{10}-2A_{11})]\mu_{11}$$

$$P_2^i = (1-\bar{g}')(1-A_8)-(A_3-A_5)\mu_{11}$$

$$Q_2 = (A_6 - A_{12})\mu_2$$

$$N_2^k = [A_6 + 3 \frac{U_3}{U_1} (1-X_{N+1})\{(1-\bar{g}')(A_1-2A_2)+A_6(1-2A_8+A_7)\mu_2$$

$$- (A_9+A_{10}-2A_{11})\}]\mu_2$$

$$P_2^k = (1-\bar{g}')(1-A_8)\mu_2 - (A_3-A_5)\mu_2 . \qquad 4.48$$

Clearly, $M_2P_2^i - Q_2N_2^i = \frac{\mu_2}{\mu_{11}} [M_2P_2^k - Q_2N_2^k]$, and $N_2^kP_2^i - N_2^iP_2^k = 0$, so that equation 4.52 reduces to the traditional no tax model,

$$\bar{R}_i - R_{N+1} = (\bar{R}_k^r - R_{N+1})\frac{\mu_{11}}{\mu_2} . \qquad 4.49$$

That is, it is safe to conclude that as long as securities are normally distributed, a differentiable non-linear tax function will not effect investor behavior. Section 4.2 arrives at this conclusion assuming that the investor possesses a two parameter utility function. The present section arrives at the same conclusion, even though an assumption regarding investor behavior has been broadened. That is, the investor is assumed to consider the degree of skewness of after tax return, $E(\tilde{R}_k^t - \bar{R}_k^t)^3$, as an argument in his utility function. In fact, examination of equation 4.48 reveals that this analysis may easily be extended to admit any number of moments of the probability density function of after tax returns as arguments in the investor's utility function. If the higher moments exist and are finite, then the conclusions will not differ from those derived in the present section. In addition, the section specifies a more general approximation to the tax function compared to previous sections in this chapter. Again, as long as security and portfolio returns are joint normally distributed, taxes are found to be irrelevant to the investor's decision.

This result obtains because the higher moments of a bivariate normal probability density function are simple functions of the covariances and variances of the relevant random variables, or vanish. Specifically, when the portfolio and security returns possess a bivariate normal density function, then the relevant higher mixed moments either vanish or are simple functions of the covariance. Similarly the higher moments of the marginal probability density function of the portfolio return are functions of the variance of the portfolio return or vanish. That is, knowledge of the variance and covariance of portfolio and security returns provides complete knowledge of all other joint and marginal moments.

The tax function affects investor behavior because it causes the investor to consider higher moments of the before tax distributions. However, since the variance and covariance completely determine the higher moments, the investor may ignore these higher moments and consider only the mean, variance and covariance of before tax returns. That is, the investor need only consider the portfolio variance, the covariance of security and portfolio returns, and the relevant means to choose an optimal portfolio. This is precisely the conclusion reached under the traditional no tax model. Thus, the investor's behavior is found to be consistent with the traditional Capital Asset Pricing Model, regardless of the tax function, as long as security returns possess a multivariate normal probability density function.

NOTES TO CHAPTER 4

1. Pages 32, 33.

2. Pages 37, 38.

3. See Appendix D for the derivation of the following expressions.

4. Hereafter, the subscript k, denoting individual investors, and denoting stochastic variables are deleted when convenient.

5. Alternatively, if the security probability density functions are stable under addition, since equation 4.7 must hold for each security in the investor's portfolio, the numerator and denominator of 4.7 can be multipled by the weight of each security in the investor's risky portfolio and summed over all securities giving the same results. However, the assumption of stability under summation is unnecessary to achieve the same results.

6. See Appendix E for the derivation of equations 4.9.

7. See Appendix B for an analogous derivation.

8. See Appendix E.

9. For economy of notation, u'_{np} means u_{nopo}.

10. Kendall and Stuart, op. cit., I, 70, 84.

11. Ibid., p. 85.

12. Although leptokuritc curves are generally regarded as being more peaked than the normal curve, and ptatykurtic more flat topped than the normal, this is not necessarily true. See Kendall and Stuart, op. cit., I, pp. 92, 93.

13. Kendall and Stuart, op. cit., I., p. 83

14. Ibid.

15. Ibid., p. 51.

16. See equation 2.10.

17. The notable exceptions are B. Mandelbrot, "The Variation of Some Other Speculative Prices", Journal of Business, 40 (1967), 393-413 and S.J. Press, "A Compound Event Model for Security Prices", Journal of Business, 40 (1967), 317-35. These papers indicate that security distributions are occasionally skewed.

18. M.G. Kendall, "The Analysis of Economic Time-Series - Part I: Prices", in P.H. Cootner, The Random Character of Stock Market

Prices, (Cambridge: The MIT Press, 1964), 87-99. Reprinted from Journal of the Royal Statistics Society, 96 (1953), 11-25; P.H. Cootner, "Stock Prices: Random vs. Systematic Changes", in P.H. Cootner (ed.), The Random Character of Stock Market Prices (Cambridge: The MIT Press, 1964), 231-52, reprinted from Industrial Management Review, 3 (1964), 26-45, A.B. Moore, "Some Characteristics of Changes in Common Stock Prices", in P.H. Cootner (ed.), The Random Character of Stock Market Prices (Cambridge: The MIT Press, 1964), 139-61, E.F. Fama, "The Behavior ofStock-Market Prices", Journal of Business, 38 (June, 1965), 34-105.

19. See n.12 above for a modification of this statement.

20. The denominator of equations 4.28 are again assumed always to be positive.

21. See page 85. *supra*.

22. Page 36.

23. Kendall and Stuart, *op. cit.*, I, 91.

24. See Appendix G for the derivative of these moments.

25. See Appendix B, equation B.2 for an analogous derivation.

Chapter 5

THE EFFECTS OF PERSONAL INCOME TAXES ON MARKET EQUILIBRIUM

Chapters 3 and 4 have been concerned with the affects of personal income taxes on individual investor behavior. It has been shown that under certain conditions, personal income taxes will affect the investor's demand for a specific risky asset. Presumably, this will affect the equilibrium value and therefore the expected risk premium for a specific asset on the market as a whole. This chapter considers the conditions for market equilibrium of a risky asset and determines the specific effect of personal income taxes on the risk premium for a security in terms of market equilibrium.

The following variables are defined:

n_{ik} is the number of shares of asset i willingly held by investor k.

n_{io} is the exogenously determined number of shares of asset i outstanding.

P_i is the current price of i, equivalent to the non-random beginning period price of asset i by virtue of assumption A.3 in Chapter 3.

V_i is the total current market value of asset i.

V_{ik} is the total current dollar value of investor k's holdings of asset i.

V_M is the total current market value of all assets outstanding.

$h_i \equiv \frac{V_i}{V_M}$, is the value of asset i as a proportion of the total value of all assets.

$h_{ik} \equiv \frac{V_{ik}}{V^r_k}$, is the value of asset i in investor k's portfolio as a proportion of the total value of investor k's risky portfolio.

$f_k \equiv \frac{V^r_k}{V_M}$, is the value of investor k's risky portfolio as a proportion of the total value of all risky assets outstanding.

$\sum_j h_j \tilde{R}_j \equiv \tilde{R}_M$, is the return on the "value weighted market portfolio" or the "market portfolio," defined as a portfolio with each risky asset held in proportion to its total market value.

$$\mu^k_m = E[(\tilde{R}^r_k - \bar{R}^r_k)^m],\ k = 1, \ldots, K;\ m = 1, 2, 3, 4$$

$$\mu^k_{m,1} = E[(\tilde{R}^r_k - \bar{R}^r_k)^m\ (\tilde{R}_i - \bar{R}_i)],\ k = 1, \ldots, K;\ m = 1, 2, 3, 4$$
$$i = 1, \ldots, j, \ldots, N.$$

5.1 Conditions for Market Equilibrium

In order to achieve market equilibrium, each investor must be in personal equilibrium. In addition, the total supply of each asset must equal the total demand for that asset at the current market price.[1] The former condition is satisfied, depending on the specific case and notation assumed, if equation 3.17 (or its equivalent 3.18, 3.40, 3.47) or equation 4.10 (or its equivalent 4.11, 4.14) holds for each asset i (i = 1, . . ., i, j, . . ., N) and each investor k, (k = 1, . . ., k, . . ., K). The latter condition is formally satisfied if,

$$\sum_{k=1}^{K} n_{ik} = n_{io},\ \text{for each } i = 1, \ldots, N,$$

or $P_i \sum_k n_{ik} = P_i\, n_{io}$, for each i = 1, . . ., N,

$$\sum_k P_i n_{ik} = P_i n_{io}, \text{ for each } i = 1, \ldots, N.$$

$P_i n_{io} \equiv V_i$, the total current market value of asset i.

$P_i n_{ik} \equiv V_{ik}$, the total current dollar value of asset i held by investor k.

That is, market equilibrium requires that

$$\sum_{k=1}^{K} P_i n_{ik} \equiv \sum_{k=1}^{K} V_{ik} = V_i \equiv P_i n_{io}, \quad i = 1, \ldots, N \tag{5.1}$$

In terms of proportions, divide both sides of the equilibrium condition, 5.1 by the total market value of all shares outstanding, V_M:

$$\frac{\sum_k V_{ik}}{V_M} = \frac{V_i}{V_M} \equiv h_i \quad \text{for all } i = 1, \ldots, N. \tag{5.2}$$

5.2 Market Equilibrium in the Context of a Non-Linear Tax Function

To consider market equilibrium as a whole, equation 4.11 is used as the individual investor equilibrium. Recall that equation 4.11 represents investor equilibrium, assuming the tax function can be approximated by a second degree Taylor Expansion, and investors possess quadratic utility functions.[2] Thus, this specific analysis incorporates those same assumptions. Clearly, an analagous analysis would result from one of the other equations representing another set of underlying assumptions.

Market equilibrium requires simultaneous personal equilibrium for all investors, thus the index $k (k \equiv 1, \ldots, k, \ldots, K)$ is reinstated to identify each investor. Reproducing equation 4.11 with the newly introduced notation,

$$\bar{R}_i - R_{N+1} = (\bar{R}^r_k - R_{N+1}) [A^*_k] - B^*_k$$

$$A_k^* = \frac{\mu_{11}^k}{\mu_2^k} \left\{ \frac{1 - [\alpha_k - \delta_k \mu_2^k] \frac{\mu_{21}^k}{\mu_{11}^k} + [\beta_k - z_k \mu_2^k] \frac{\mu_{31}^k}{\mu_{11}^k}}{1 - [\gamma_k - 3\beta_k] \mu_2^k - [\alpha_k - ((1/3)\delta_k + \xi_k) \mu_2^k] \frac{\mu_3^k}{\mu_2^k} + \beta_k \frac{\mu_4^k}{\mu_2^k}} \right\}$$

$$+ \frac{[-\gamma_k - 3\beta_k] \mu_2^k + [\xi_k - (2/3)\delta_k] \mu_3^k + z_k \mu_4^k}{1 - [\gamma_k - 3\beta_k] \mu_2^k - [\alpha_k - ((1/3)\delta_k + \xi_k) \mu_2^k] \frac{\mu_3^k}{\mu_2^k} + \beta_k \frac{\mu_4^k}{\mu_2^k}},$$

$$B_k^* = \qquad 5.3$$

$$\frac{\mu_{11}^k \left[-3\beta_k \frac{\mu_{21}^k}{\mu_{11}^k} + \xi_h \frac{\mu_{31}^k}{\mu_{11}^k} + 3\beta_k \frac{\mu_3^k}{\mu_2^k} - \xi_k \frac{\mu_4^k}{\mu_2^k}\right]}{1 + [-\gamma_k - 3\beta_k] \mu_2^k - [\alpha_k - ((1/3)\delta_k + \xi_k) \mu_2^k] \frac{\mu_3^k}{\mu_2^k} + \beta_k \frac{\mu_4^k}{\mu_2^k}}$$

$$\alpha_k = (3/2)\frac{g_k''}{1-g_k'} V_k^r \qquad \delta_k = (3/4)\frac{g_k'''}{1-g_k'} \left(\frac{g_k''}{1-g_k'}\right) (V_k^r)^3$$

$$\beta_k = (1/2)\left(\frac{g_k''}{1-g_k'}\right)^2 (V_k^r)^2 \qquad \xi_k = (1/2)\ (V_k^r)^3 \left(\frac{g_k''}{1-g_k'}\right)^3$$

$$\gamma_k = (1/2)\frac{g_k'''}{1-g_k'} (V_k^r)^2 \qquad z_k = (1/4)\frac{g_k'''}{1-g_k'} \left(\frac{g_k''}{1-g_k'}\right)^2 (V_k^r)^4$$

Since 5.3 holds for all assets, it may be summed over all j . First, multiply 5.3 by h_j, the value of asset j as a proportion of the value of all risky assets and sum over j:

$$\sum_{j=1}^{N} h_j (\bar{R}_j - R_{N+1}) = (\bar{R}_k^r - R_{N+1}) \left(\sum_{j=1}^{N} h_j A_k^* \right) - \left(\sum_{j=1}^{N} h_j B_k^* \right). \tag{5.4}$$

But,

$\sum_{j=1}^{N} h_j \bar{R}_j = \bar{R}_M$, is the "market portfolio" and $\sum_{j=1}^{N} h_j = 1$, so that

5.4 reduces to

$$(\bar{R}_M - R_{N+1}) = (\bar{R}_k^r - R_{N+1}) \left(\sum_j h_j A_k^* \right) - \left(\sum_j h_j B_k^* \right),$$

$$\text{or} \quad \bar{R}_k^r - R_{N+1} = \frac{(\bar{R}_M - R_{N+1}) + (\sum_j h_j B_k^*)}{(\sum_j h_j A_k^*)} \tag{5.5}$$

Equation 5.5 may be substituted into equation 5.3, so that

$$\bar{R}_i - R_{N+1} = \left[\frac{(\bar{R}_M - R_{N+1}) + (\sum_j h_j B_k^*)}{\sum_j h_j A_k^*} \right] A_k^* - B_k^*$$

$$= \frac{(\bar{R}_M - R_{N+1}) A_k^*}{(\sum_j h_j A_k^*)} + \frac{(\sum_j h_j B_k^*) A_k^* - B_k^* (\sum_j h_j A_k^*)}{(\sum_j h_j A_k^*)} \tag{5.6}$$

$$\frac{A_k^*}{\sum_j h_j A_k^*} =$$

$$\frac{[\mu_{11}^k + [\alpha_k - \delta_k \mu_2^k]\, \mu_{21}^k + [\beta_k - z_k \mu_2^k] \mu_{31}^k + \mu_{11}^k \{ [\gamma_1 - 3\beta_1] \mu_2^k + [\xi_k - (2/3)\delta_k] \mu_3^k + z_k \mu_4^k \}}{\sum_i h_i \{ \mu_{11}^k + [\alpha_k - \delta_k \mu_2^k] \mu_{21}^k + [\beta_k - z_k \mu_2^k] \mu_{31}^k + \mu_{11}^k \{ [\gamma_1 - 3\beta] \mu_2^k + [\xi_k - (2/3)\delta_k] \mu_3^k + z_k \mu_4^k] \}}$$

5.7

$$\frac{\mu_{11}^{k} + \mu_{11}^{k}\Pi_k + \Phi_k^i}{\sum_j h_j \mu_{11}^{k} + \sum_j h_j \mu_{11}^{k}\Pi_k + \quad + \sum_j h_j \Phi_k^j}$$

Similarly,

$$\frac{A_k^* (\sum_j h_j B_k^*) - B_k^* (\sum_i h_i A_k^*)}{\sum_j h_j A_k^*} =$$

$$\frac{[\mu_{11}^{k} + \mu_{11}^{k}\Pi_k + \Phi_k^i] (\sum_j h_j B_k^*) - B_k^* [\sum_j h_j (\mu_{11}^{k} + \mu_{11}^{k}\Pi_k + \Phi_k^i]}{[\sum_j h_j (\mu_{11}^{k} + \mu_{11}^{k}\Pi_k + \Phi_k^j)]} \qquad 5.8$$

where Π_k, Φ_k^i represent their obvious equivalents in equation 5.7. Now, it is assumed that all investors have identical assessments of the expectations of all risky assets,[3] so that the left hand side of 5.6 is identical for all investors. Then, the numerator and denominator of the right hand side may be summed separately over all investors. First, multiply the numerator and denominator by f_k, the value of investor k's risky portfolio as a proportion of the total value of all risky assets outstanding. Then, carrying out the summation yields,

$$\bar{R}_i - R_{N+1} =$$

$$\frac{(\bar{R}_M - R_{N+1}) \sum_k [f_k A_k^*]}{\sum_k f_k [\sum_j h_j A_k^*]} + \frac{\sum_k f_k [(\sum_j h_j B_k^*) A_k^* - B_k^* (\sum_j h_j A_k^*)]}{\sum_k f_k [\sum_j h_j A_k^*]}. \qquad 5.9$$

From 5.9, the coefficient of $(\bar{R}_M - R_{N+1})$ is,

$$\frac{\sum_k [f_k A_k^*]}{\sum_k f_k [\sum_i h_i A_k^*]} = \frac{\sum_k [f_k \mu_{11}^k] + \sum_k [f_k \mu_{11}^k \Pi_k] + \sum_k [f_k \Phi_k^i]}{\sum_k f_k [\sum_j h_j \mu_{11}^k] + \sum_k f_k [\sum_j h_j \mu_{11}^k \Pi_k] + \sum_k f_k [\sum_j h_j \Phi_k]} \qquad 5.10$$

The first term in the numerator and denominator may easily be evaluated.

$$\sum_k [f_k \mu_{11}^k] = \sum_k \frac{V_k^r}{V_M} E[(\tilde{R}_k^r - \bar{R}_k^r)\ (\tilde{R}_i - \bar{R}_i)]$$

$$= \frac{1}{V_M} \sum_k [V_k^r\ E[(\sum_j h_{jk} (\tilde{R}_j - \bar{R}_j)\ (\tilde{R}_i - \bar{R}_i)]]$$

$$= \frac{1}{V_M} \sum_k [V_k^r \sum_j h_{jk}\ E[(\tilde{R}_j - \bar{R}_j)\ (\tilde{R}_i - \bar{R}_i)]]$$

$$= \frac{1}{V_M} \sum_j [\sum_k V_k^r h_{jk}\ E[(\tilde{R}_j - \bar{R}_j)\ (\tilde{R}_i - \bar{R}_i)]]$$

$$= \frac{1}{V_M} \sum_j [\sum_k V_{jk}\ E[(\tilde{R}_j - \bar{R}_j)\ (\tilde{R}_i - \bar{R}_i)]] \qquad 5.11$$

By virtue of the market equilibrium conditions 5.2,

$$\frac{\sum_k V_{jk}}{V_M} = \frac{V_j}{V_M} = h_j,$$

so that 5.11 can be written as,

$$\sum_k f_k \mu_{11}^k = E[(\sum_j h_j\ (\tilde{R}_j - \bar{R}_j)\ (\tilde{R}_i - \bar{R}_i)]$$

$$= E[(\tilde{R}_M - \bar{R}_M)\ (\tilde{R}_i - \bar{R}_i)]$$

$$= \sigma(\tilde{R}_M, \tilde{R}_i), \qquad 5.12$$

the covariance between asset i and the market portfolio.

Similarly,

$$\sum_k f_k[\sum_j h_j \mu_{11}^k] = \sum_k f_k \; [E[(\tilde{R}_k^r - \bar{R}_k^r) \; (\sum_j h_j \; (\tilde{R}_j - \bar{R}_j))]$$

$$= \sum_k [f_k \; E[(\tilde{R}_k^r - \bar{R}_k^r)(\tilde{R}_M - \bar{R}_M)]$$

$$= \sum_k \frac{V_k^r}{V_M} E[(\tilde{R}_k^r - \bar{R}_k^r) \; (\tilde{R}_M - \bar{R}_M)]$$

$$= \sum_k [\sum_j \frac{V_k^r}{V_M} h_{jk} \; E[(\tilde{R}_j - \bar{R}_j) \; (\tilde{R}_M - \bar{R}_M)]$$

$$= \sum_k [\sum_j \frac{V_{j,k}}{V_M} \; E[(\tilde{R}_j - \bar{R}_j) \; (\tilde{R}_M - \bar{R}_M)]$$

$$= \sum_j [\sum_k [\frac{V_{j,k}}{V_M} \; E[(\tilde{R}_j - \bar{R}_j) \; (\tilde{R}_M - \bar{R}_M)]$$

$$= \sum_j [E[(h_j \; (\tilde{R}_j - \bar{R}_j) \; (\tilde{R}_M - \bar{R}_M)]]$$

$$= E[(\tilde{R}_M - \bar{R}_M)^2] = \sigma_{(\tilde{R}_M, \tilde{R}_M)}, \qquad 5.13$$

the variance of the market portfolio.

Thus, the coefficient of $(\bar{R}_M - R_{N+1})$, given by equation 5.10 is,

$$\frac{\sigma_{(\tilde{R}_M, \tilde{R}_i)} + \sum_k [f_k \mu_{11}^k \Pi_k] + \sum_k \; [f_k \Phi_k^i]}{\sigma_{(\tilde{R}_M, \tilde{R}_M)} + \sum_k f_k [\sum_j h_j \mu_{11}^k \Pi_k] + \sum_k f_k [\sum_j h_j \Phi_k^j]} \; . \qquad 5.14$$

That is, equation 5.9, representing market equilibrium can be written

$$\bar{R}_i - R_{N+1} = (\bar{R}_M - R_{N+1}) \left[\frac{\sigma_{(\tilde{R}_M, \tilde{R}_i)} + \sum_k [f_k(\mu_{11}^k \Pi_k + \phi_k^i)]}{\sigma_{(\tilde{R}_M, \tilde{R}_M)} + \sum_k \sum_j [f_k h_j (\mu_{11}^k \Pi_k + \phi_k^j)]} \right]$$

$$+ \frac{\sum_k f_k [(\sum_j h_j B_k^*) A_k^* - B_k^* (\sum_j h_j A_k^*)]}{\sigma_{(\tilde{R}_M, \tilde{R}_M)} + \sum_k \sum_j [f_k h_j (\mu_{11}^k \Pi_k + \phi_k^j)]} \qquad \begin{array}{l} i = 1, \ldots, N \\ j = 1, \ldots, N \end{array} \qquad 5.15$$

The traditional no-tax Capital Asset Pricing Model requires that the equilibrium expected return of any risky asset i be such that,[4]

$$\bar{R}_i - R_{N+1} = \frac{(\bar{R}_M - R_{N+1}) \sigma_{(\tilde{R}_M, \tilde{R}_i)}}{\sigma_{(\tilde{R}_M, \tilde{R}_M)}} \qquad i = 1, \ldots, N. \qquad 5.16$$

Comparing equation 5.15 with the traditional results, the after tax model will diverge from the traditional model to the extent that the terms within the first bracket on the right hand side of 5.15 diverges from

and $$\frac{\sigma_{(R_M, R_i)}}{\sigma_{(R_M, R_M}}$$

$$\sum_{k=1}^{K} f_k [(\sum_i h_i B_k^*) A_k^* - B_k^* (\sum_j h_j A_k^*)] \neq 0. \qquad 5.17$$

Unfortunately, the terms in 5.17 are very complicated, involving weighted sums of the investors tax parameters, the market parameters, the investors' invested wealth relative to market wealth, and investor portfolio parameters. Comparing these terms with the results in the previous chapter, the final result bears a close resemblance to the results obtained for the individual investor, differing only by the use of market parameters derived from the market equilibrium conditions and the summations over all investors, rather than individual investor portfolio parameters.[4]

NOTES TO CHAPTER 5

1. Michael J. Brennen, "Taxes, Market Valuation and Corporate Financial Policy" National Tax Journal (December, 1970), p.

2. Page 79.

3. Page 33.

4. Page 15.

CHAPTER 6

CONCLUSION

The purpose of this study is to derive a model of investor behavior when the investor's income is subject to personal income taxation. The methodology is based on the traditional Capital Asset Pricing Model, modified to permit the investor to consider after tax, rather than before tax parameters in his decision process. It seems clear that a rational investor, subject to income tax levied on all sources of income, will consider after tax return as the relevant argument in his utility function. This study is an attempt to determine under what conditions, the inclusion of after tax rather than before tax parameters, modify the investor's desire for one risky asset relative to another.

One may argue, on _á priori_ ground that a tax function may affect investor behavior if the tax serves to distort the marginal rate of transformation between risk and return for a specific security, and/or modifies the expected wealth position of the investor. This is akin to the tax exerting a "substitution" and/or "income" effect on the investor. However it has long been recognized that, given the traditional assumptions underlying the Capital Asset Pricing Model, Tobin's separation theorem holds. That is, the composition of the investor's risky portfolio is independent of the investor's risk aversion parameter.[1] As a result, the demand for anyone risky asset relative to other assets in the investor's risky portfolio will be unaffected by a pure "income effect" unless the tax serves to break down the separation theorem. The model in Chapter 3 shows that, under certain conditions, the tax function may in fact cause the conditions for a separation theorem to be violated.

Specifically, the marginal contribution of the riskless asset to the investor's portfolio return may be represented by, $R_{N+1}(1 - \tilde{g}')$ (equation 3.5i). Thus, the riskless asset is no longer "riskless" in the sense that its marginal contribution to after tax portfolio return is no longer non-stochastic as long as the marginal tax rate is stochastically determined. Lintner has shown that the existence of a riskless asset is necessary for the separation theorem to hold.[2] Thus, the investor's choice of his risky portfolio will not necessarily be independent of the parameters of his utility function, and the tax may have an impact on the investor's relative demand for a risky asset as a result of an "income effect" exerted on the investor.

In addition, the model in chapter 3 derives the investor's conditions for utility maximization when the investor is subject to a generalized tax function and unspecified probability distribution of the security returns. In this context, it is established that the necessary conditions for investor utility maximization require that the investor equate after tax marginal rates of transformation between risk and return for each security in the investor's portfolio. This implies that before tax marginal rates of transformation, in general, will differ, at least for a generalized tax function and unspecified security probability distributions. That is, it is established that, at least in this general model, the tax function may exert a "substitution effect" on the investor's choice of the risky portfolio.

The analysis in chapter 3 then derives the required risk premium for a representative security in the investor's portfolio. It is concluded that the required risk premium is of the form,

$$\bar{R}_i - R_{N+1} = [(\bar{R}_k^r - R_{N+1})(\frac{\mu_{11}}{\mu_{20}})][A] + B, \qquad 3.1$$

where $\bar{R}_i - R_{N+1}$ is the required risk premium for risky asset i in the investor's portfolio,

μ_{11} is the covariance of asset i with the investor's risky portfolio,

μ_{20} is the variance of the investor's risky asset

$\bar{R}_k^r - R_{N+1}$ is the expected risk premium of the investor's risky portfolio

A and B are complicated functions of the moments of the joint probability density function of the investor's tax liability, marginal tax rate, and security and portfolio returns.

The first bracketed term on the right hand side of equation 3.1 is the investor's required risk premium predicted by the traditional model. Thus, the investor's required risk premium relative to the expected portfolio return and the traditional risk measure $(\frac{\mu_{11}}{\mu_{20}})$, will depend on the magnitude of A and B, functions of the relevant moments of the joint probability distribution of tax liability, the marginal tax rate, the portfolio returns and the security return.

The mixed moments in A and B may be converted into terms containing parameters of the tax function, the correlation coefficients between the tax variables and the security and portfolio returns, and the bivariate moments of the security and portfolio returns respectively. It is then concluded that as long as the tax function is linear in income, taxes have no effect on investor behavior. This conclusion results because the marginal rate of substitution between risk and return is unaffected by the imposition of a linear tax function and the conditions for the separation theorem hold since the marginal tax rate is non-stochastic. However, for a simple non-linear tax function (taxes as a quadratic function of income), the required risk premium depends, in a complicated way, on the relevant correlation coefficients, the higher moments of the bivariate probability dnesity function of portfolio and security returns, and the parameters of the tax function; as well as the parameters in the traditional, no tax model (equations 3.48 and 3.49). It is clear that in this general case, investor taxes may have an effect on the investor's required risk premium. However, the relationship depends in a complicated way, on the joint probability distribution of the security and portfolio return, as well as parameters of the tax function.

In order to derive more meaningful results, chapter 4 considers some specific cases. Sections 4.2 through 4.4 consider the case of a tax function which may be approximated by a second degree

Taylor Expansion. The additional risk premium per unit of additional covariance is then calculated. It is concluded that if security returns possess Multivariate Normal distributions, taxes still have no effect on investor behavior. However, if security returns are not Normally distributed, then investor behavior will be affected by the tax parameters.

The specific effects assuming specific tax parameters and moments of the portfolio probability distribution are summarized in tables 2 through 7. It is concluded that it is difficult to derive generalizations concerning the specific effects of the tax function on security returns. However, if the portfolio return is positively (negatively) skewed, then it is likely that the additional required risk premium per unit of additional risk will be less (greater) than predicted by the traditional no tax model. Some intuitive justification for this result is presented in section 4.3.

Similarly, if the portfolio return is symmetrically distributed, then the additional required risk premium per unit of covariance will, in general, differ from the traditional no-tax model if portfolio returns are not mesokurtic. Again, generalizations are not obvious, however, the required risk premium tend to be greater (less) than predicted by the traditional model, if marginal tax rates are increasing (decreasing). Some intuitive justification for this result is also offered in section 4.4.

Finally, a more general, L degree Taylor Expansion, with a more general three parameter utility function is assumed in section 4.5. It is concluded that as long as portfolio and security returns possess Multivariate Normal probability distributions, the tax function is irrelevant to investor behavior, even under these broader assumptions.

Thus, in general, if the tax function is linear, or if the distribution of security returns is Multivariate Normal, then taxes are irrelevant to the investor's behavior. However, if taxes are non-linear and security deistributions are not Normal, then the tax function is relevant to the investor's portfolio decision.

Chapter 5 derives the market equilibrium conditions, assuming all tax functions may be represented by a two degree Taylor Expansion, and all investors possess quadratic utility functions, with the distribution of security returns unspecified. The resultant equation,

$$\bar{R}_i - R_{N+1} = [(\bar{R}_M - R_{N+1}) \frac{\sigma_{(R_i,R_M)}}{\sigma_{(R_M,R_M)}}][A_1] + B_1 \ , \qquad 3.2$$

where

$\bar{R}_M - R_{N+1}$ is the expected risk premium of the "market protfolio"

$\sigma_{(R_i,R_M)}$ is the covariance between i the return of asset i and the market portfolio

$\sigma_{(R_M,R_M)}$ is the variance of the market portfolio.

Analogous, to equation 3.1, terms in the first bracket on the right hand side is equivalent to the risk premium predicted by the conventional model. However, A and B are complicated weighted sums of individual investor portfolio and security parameters as well as parameters of individual investor tax functions. It would seem difficult to derive further generalization from these complicated expressions.

This theoretical analysis is, of course, subject to empirical verification. Even if it is assumed that investors act in accordance with the assumptions of the model, the results do not indicate whether the tax parameters and the divergence from normality of the portfolio distributions are sufficiently important to have a significant effect on investor behavior.

Unfortunately, attempts at actual verification would likely entail insurmountable data problems. One of the convenient feature of the traditional model is that, by virtue of Tobin's separation theorem, all investors hold the market portfolio. Thus, all moments are in terms of readily observable market parameters. It has already been established, Tobin's separation theorem does not hold when the tax function is a relevant parameter. Thus, investors will choose different portfolios, and the parameters of the joint protfolio and security moments as well as the tax function will in general be specific to each investor. Thus, empirical verification requires not only the knowledge of the tax parameters for each investor, it also requires knowledge of each investor's specific portfolio.

However review of the empirical literature indicates that the traditional Capital Asset Pricing Model seems to be inconsistent with empirical findings.[3] Although some theoretical and econometric justification for these results have been offered, the consensus of most investigators is that, "the currently available empirical evidence seems to indicate that the simple version of the asset pricing model,... does not provide an adequate description of the structure of security returns."[4] This analysis offers one possible theoretical explanation for these disturbing results.

NOTES TO CHAPTER 6

1. James Tobin, "Liquidity Preference as Behavior Toward Risk", Review of Economic Studies, 25 (February, 1958), 83, 84.

2. John Lintner, "The Aggregation of Investor's Diverse Judgments and Preferences in Perfectly Competitive Security Markets", Journal of Financial and Quantitative Analysis, 4 (December, 1969), 384.

3. Section 2.2.

4. Michael Jensen, "A Review of Capital Market Theory", The Bell Journal, 3 (April, 1969), 371.

APPENDICES

APPENDIX A

$$\sigma_{(\tilde{g}'_k,\tilde{R}_i)} = E[(\tilde{R}_i - \bar{R}_i)(\tilde{g}'_k - \bar{g}'_k)] = u_{0011} \qquad \text{A.1}$$

$$\sigma_{(\tilde{R}_k{}^t,\tilde{R}_i(1-\tilde{g}'_k))} = \sigma_{(\tilde{R}_k,\tilde{R}_i)} - \sigma_{(\tilde{T}_k/V_k,\tilde{R}_i)} - \sigma_{(\tilde{R}_k,(\tilde{R}_i\tilde{g}'_k)} + \sigma_{(\tilde{T}_k/V_k,(\tilde{R}_i\tilde{g}'_k))}$$

$$= (1-X_{N+1},k)E[(\tilde{R}^r_k-\bar{R}^r_k)(\tilde{R}_i-\bar{R}_i)] + E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}_i - \bar{R}_i)]$$

$$- (1-X_{N+1},k)E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{R}_i\tilde{g}'_k - E[\tilde{R}_i\tilde{g}'_k])]$$

$$+ E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}_i\tilde{g}'_k - E[\tilde{R}_i\tilde{g}'_k])] .$$

But,

$$u_{1011} = E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{R}_i - \bar{R}_i)(\tilde{g}'_k - \bar{g}'_k)]$$

$$= E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{R}_i\tilde{g}'_k - \bar{R}_i\tilde{g}'_k - \tilde{R}_i\bar{g}'_k + \bar{R}_i\bar{g}'_k)]$$

$$= E[(\tilde{R}^r_k - \bar{R}^r_k)\tilde{R}_i\tilde{g}'_k]-\bar{R}_iE[(\tilde{R}^r_k-\bar{R}^r_k)\tilde{g}'_k]-\bar{g}'_kE[(\tilde{R}^r_k-\bar{R}^r_k)(\tilde{R}_i-\bar{R}_i)]$$

$$= E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{R}_i\tilde{g}'_k - E[\tilde{R}_i\tilde{g}'_k])] - \bar{R}_iE[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{g}'_k - \bar{g}'_k)]$$

$$- \bar{g}'_kE[(\tilde{R}^r_k - \bar{R}^r{}_k)(\tilde{R}_i - \bar{R}_i)] .$$

$$\therefore E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{R}_i\tilde{g}'_k - E[\tilde{R}_i\tilde{g}'_k])] = u_{1011} + \bar{g}'_k u_{1010} + \bar{R}_i u_{1001} .$$

Similarly,

$$u_{0111} = E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}_i - \bar{R}_i)(\tilde{g}'_k - \bar{g}'_k)] =$$

$$E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}_i\tilde{g}'_k - E[\tilde{R}_i\tilde{g}'_k])] - \bar{R}_i E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{g}'_k - \bar{g}'_k)]$$

$$- \bar{g}'_k \; E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}_i - \bar{R}_i)],$$

so that

$$E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}_i\tilde{q}'_k - E[\tilde{R}_i\tilde{g}'_k])] = u_{0111} + \bar{g}'_k u_{0110} + \bar{R}_i u_{0101} \cdot$$

Thus,

$$\sigma_{(\tilde{R}_k{}^t, \tilde{R}_i(1-\tilde{g}'_k))} = (1-X_{N+1,k})u_{1010} - u_{0110} - (1-X_{N+1,k})[u_{1011} +$$

$$\bar{g}'_k u_{1010} + \bar{R}_i u_{1001}] + u_{0111} + \bar{g}'_k u_{0110} + \bar{R}_i u_{0101}$$

$$= (1-\bar{g}'_k)[(1-X_{N+1,k})u_{1010} - u_{0110}] + \bar{R}_i(u_{0101} - (1-X_{N+1,k})U_{1001})$$

$$+ (u_{0111} - (1-X_{N+1} \quad)u_{1011}) \qquad \text{A.2}$$

$$\sigma_{(\tilde{R}_k{}^t, \tilde{g}'_k)} = E[(\tilde{R}_k - \bar{R}_k)(\tilde{g}'_k - \bar{g}'_k)] - E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{g}'_k - \bar{g}'_k)]$$

$$= (1 - X_{N+1,k})u_{1001} - u_{0101} \qquad \text{A.3}$$

$$\sigma_{(\tilde{g}'_k, \tilde{R}^r_k)} = E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{g}'_k - \bar{g}'_k)] = u_{1001}$$

$$\sigma_{(\tilde{R}^t_k,\ \tilde{R}^r_k(1-\tilde{g}'_k))} = \sigma_{(\tilde{R}_k,\tilde{R}^r_k)} - \sigma_{(\tilde{T}_k/V_k,\bar{R}^r_k)} - \sigma_{(\tilde{R}_k,(\tilde{R}^r_k\tilde{g}'_k))}$$

$$+ \sigma_{(\tilde{T}_k/V_k,(\tilde{R}^r_k\tilde{g}'_k))}$$

$$= (1-X_{N+1,k})E[(\tilde{R}^r_k - \bar{R}^r_k)^2] + E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}_i - \bar{R}_i)]$$

$$- (1-X_{N+1,k})E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{R}^r_k g'_k - E[\tilde{R}^r_k\tilde{g}'_k])]$$

$$+ E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}^r_k g'_k - E[\tilde{R}^r_k\tilde{g}'_k])] \ .$$

But,

$$u_{2001} = E[(\tilde{R}^r_k - \bar{R}^r_k)^2(\tilde{g}'_k - \bar{g}'_k)]$$

$$= E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{R}^r_k\tilde{g}'_k - \bar{R}^r_k\tilde{g}'_k - \tilde{R}^r_k\bar{g}'_k + \bar{R}^r_k\bar{g}'_k)]$$

$$= E[(\tilde{R}^r_k - \bar{R}^r_k)\tilde{R}^r_k\tilde{g}'_k] - \tilde{R}^r_k\ E[(\tilde{R}^r_k - \bar{R}^r_k)\tilde{g}'_k] - \bar{g}'_k E[(\tilde{R}^r_k - \bar{R}^r_k)^2]$$

$$\therefore\ E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{R}^r_k\tilde{g}'_k - E[\tilde{R}^r_k\tilde{g}'_k])] = u_{2001} + \bar{g}'_k u_{2000} + \bar{R}^r_k u_{1001} \ .$$

Similarly

$$u_{1101} = E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{g}'_k - \bar{g}'_k)] = E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}^r_k\tilde{g}'_k$$

$$- E[\tilde{R}^r_k\tilde{g}'_k])] - \bar{R}^r_k E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{g}'_k-\bar{g}'_k)] - \bar{g}'_k E[(\frac{\tilde{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}^r_k - \bar{R}^r_k)],$$

so that

$$E[(\frac{\ddot{T}_k}{V_k} - \frac{\bar{T}_k}{V_k})(\tilde{R}^r_k\tilde{g}'_k - E[\tilde{R}^r_k\tilde{g}'_k])] = u_{1101} + \bar{g}'_k u_{1010} + \bar{R}^r_k u_{0101} \text{ .}$$

Thus,

$$\sigma_{(\tilde{R}^t_k,\ \tilde{R}^r_k(1-\tilde{g}'_k))} =$$

$$(1-X_{N+1,k})u_{2000}-u_{1100}-(1-X_{N+1,k})[u_{2001} + \bar{g}'_k u_{2000} + \bar{R}^r_k u_{1001}]$$

$$+ u_{1101} + \bar{g}'_k u_{1100} + \bar{R}^r_k u_{0101} = (1-\bar{g}'_k)[(1-X_{N+1,k})u_{2000}-u_{1100}]$$

$$+ \bar{R}^r_k(u_{0101}-(1-X_{N+1,k})u_{1001}) + (u_{1101}-(1-X_{N+1,k})u_{2001}) \text{ .} \qquad \text{A.4}$$

$$\sigma_{(\tilde{g}'_k,\tilde{R}^r_k)} = E[(\tilde{R}^r_k - \bar{R}^r_k)(\tilde{g}'_k - \bar{g}'_k)] = u_{1001} \qquad \text{A.5}$$

APPENDIX B

Let

$M = (1 - E[\tilde{g}_k'])$

$N^i = u_{0011}$

$P^i = (u_{0111} - (1 - X_{N+1,k})u_{1011}) + (1-E[\tilde{g}_k'])((1-X_{N+1,k})u_{1010}-u_{0110})$

$Q = ((1 - X_{N+1,k})u_{1001} - u_{0101})$

$N^k = u_{1001}$

$P^k = (u_{1101} - (1-X_{N+1,k})u_{2001}) + (1-E[\tilde{g}_k'])((1-X_{N+1,k})u_{2000}-u_{1100})$

∴ 3.16 can be written as,

$$\frac{(\bar{R}_i - R_{N+1})M - N^i}{P^i - (\bar{R}_i - R_{N+1})Q} = -\frac{(\bar{R}_k^r - R_{N+1})M - N^k}{-(\bar{R}_k^r - R_{N+1})Q} \qquad \text{B.1}$$

Cross multiply:

$$[(\bar{R}_i - R_{N+1})M - N^i][P^k - (\bar{R}_k^r - R_{N+1})Q]$$

$$= [(\bar{R}_k^r - R_{N+1})M - N^k][P^i - (\bar{R}_i - R_{N+1})Q]$$

$$(\bar{R}_i - R_{N+1})MP^k - N^iP^k - (\bar{R}_i - R_{N+1})(\bar{R}_k^r - R_{N+1})MQ + (\bar{R}_k^r - R_{N+1})QN^i =$$

$$(\bar{R}_k^r - R_{N+1})MP^i - N^kP^k - (\bar{R}_i^r - R_{N+1})(\bar{R}_i - R_{N+1})MQ + (\bar{R}_i - R_{N+1})QN^k.$$

Solve for $\bar{R}_i - R_{N+1}$:

$$(\bar{R}_i - R_{N+1})[MP^k - QN^k] = (\bar{R}_k^r - R_{N+1})[MP^i - QN^i] - [N^kP_i - N^iP_k]$$

$$\therefore \bar{R}_i - R_{N+1} = (\bar{R}_k^r - R_{N+1})\frac{[MP^i - QN^i]}{[MP^k - QN^k]} - \frac{[N^kP^i - N^iP^k]}{[MP^k - QN^k]} . \qquad B.2$$

Define:

$$A = [MP^i - QN^i] = [1 - E[\tilde{g}_k']][(u_{0111} - (1 - X_{N+1,k})u_{1011})$$

$$+ (1 - E[\tilde{g}_k'])((1 - X_{N+1,k})u_{1010} - u_{0110})] - [(1 - X_{N+1,k})u_{1001} - u_{0101}]u_{0011}$$

$$= [1 - E[\tilde{g}_k']][(u_{0111} - (1 - X_{N+1,k})u_{1011}) + (1 - E[\tilde{g}_k'])((1 - X_{N+1,k})u_{1010} - u_{0110})]$$

$$- [(1 - X_{N+1,k})u_{1001}u_{0011} - u_{0110}u_{0011}].$$

$$B = [MP^k - QN^k]$$

$$= [1 - E[\tilde{g}_k']][(u_{1101} - (1 - X_{N+1,k})u_{2001}) + (1 - E[\tilde{g}_k'])((1 - X_{N+1,k})u_{2000}$$

$$- u_{1100})] - ((1 - X_{N+1,k})u_{1001} - u_{0101})u_{1001} = (1 - E[\tilde{g}_k'])[(u_{1101} - (1 - X_{N+1,k})u_{2001})$$

$$+ (1 - E[g_k'])((1 - X_{N+1,k})u_{2000} - u_{1100})] - [(1 - X_{N+1,k})u_{1001}^2 - u_{0101}u_{1001}].$$

$$C = [N^k P^i - N^i P^k]$$

$$= u_{1001}[(u_{0111}-(1-X_{N+1,k})u_{1011})+(1-E[\tilde{g}_k'])((1-X_{N+1,k})u_{1010}-u_{0110})]$$

$$- u_{0011}[(u_{1101}-(1-X_{N+1,k})u_{2001})+(1-E[\tilde{g}_k'])((1-X_{N+1,k})u_{2000}-u_{1100})]$$

$$= u_{1001}(u_{0111}-(1-X_{N+1,k})u_{1011})+(1-E[\tilde{g}_k'])u_{1001}((1-X_{N+1,k})u_{1010}-u_{0110})$$

$$- u_{0011}(u_{1101}-(1-X_{N+1,k})u_{2001})-(1-E[\tilde{g}_k'])u_{0011}((1-X_{N+1,k})u_{2000}-u_{1100})$$

$$C = u_{1001}(u_{0111}-(1-X_{N+1,k})u_{1011})-u_{0011}(u_{1101}-(1-X_{N+1,k})u_{2001})$$

$$+ (1-E[\tilde{g}_k'])[u_{1001}((1-X_{N+1,k})u_{1010}-u_{0110})-u_{0011}((1-X_{N+1,k})u_{2000}-u_{1100})].$$

APPENDIX C

$$\bar{R}_i - R_{N+1} = [(\bar{R}_k^r - R_{N+1})\frac{u_{1010}}{u_{2000}}]A^* - B^* \qquad 3.18$$

$$A^* = \frac{1 - \frac{1}{(1-X_{N+1,k})}[\gamma_i + \delta_i + z_i]}{1 - \frac{1}{(1-X_{N+1,k})}[\gamma_k + \delta_k + z_k]}, \text{ where} \qquad C.1$$

$$\gamma_i = \frac{u_{0110}}{u_{1010}}$$

$$\delta_i = \frac{-1}{(1-E[\tilde{g}_k'])}(\frac{u_{1011}(1-X_{N+1,k})-u_{0111}}{u_{1010}}) \qquad C.2$$

$$z_i = \frac{-1}{(1-E[\tilde{g}_k'])^2}(\frac{u_{1001}u_{0011}(1-X_{N+1,k})-u_{0101}u_{0011}}{u_{1010}})$$

$$\gamma_k = \frac{u_{1100}}{u_{2000}}$$

$$\delta_k = \frac{-1}{1-E[\tilde{g}'_k]}(\frac{u_{2001}(1-X_{N+1,k})-u_{1101}}{u_{2000}}),$$

$$z_k = \frac{-1}{(1-E[\tilde{g}_k'])^2}(\frac{u_{1001}^2(1-X_{N+1,k})-u_{0101}u_{1001}}{u_{2000}}) .$$

Define the correlation coefficient between any two variables as

$$\rho_{m,n} = \frac{Cov(m,n)}{(\sigma_m^2 \sigma_n^2)^{1/2}} . \qquad C.3$$

where

$m,n = 1,2,3,4$ and

1 represents $\tilde{R}_k^r$

2 represents $\tilde{T}_k/V_k$

3 represents $\tilde{R}_i$

4 represents $\tilde{g}_k'$.

In addition since the random variables $\tilde{T}_k$ and $\tilde{g}_k'$ are functionally related to $\tilde{R}_k^r$, the following moments can be approximated by:

$$u_{p,1,m,0} \simeq \frac{\partial(\tilde{T}_k/V_k)}{\partial\bar{R}_k^r} \; u_{p+1,0,m,0} ,$$

where $\frac{\partial(\tilde{T}_k/V_k)}{\partial\bar{R}_k^r}$ represents the derivative evaluated at $\tilde{R}_k^r = \bar{R}_k^r$.

Thus,

$$u_{p,1,m,0} \simeq \left(\frac{\partial(\tilde{T}_k/V_k)}{\partial \kappa} \frac{\partial Y_k}{\partial\bar{R}_k^r}\right) u_{(p+1,0,m,0)}$$

$$= g'_{(\bar{R}_k^r)} (1-X_{N+1,k}) u_{(p+1,0,m,0)} .$$

Similarly,

$$u_{p,0,m,1} \simeq g''_{(\bar{R}_k^r)} V_k (1-X_{N+1,k}) u_{p+1,0,m,0},$$

and in general,

$$u_{(p,q,m,n)} \simeq (g'_{(\bar{R}_k^r)})^q (g''_{(\bar{R}_k^r)} V_k)^n (1-X_{N+1,k})^{q+n} u_{p+q+n,0,m,0}$$

where the subscript $\bar{R}_k^r$ represents the derivative evaluated at $\tilde{R}_k^r = \bar{R}_k^r$.[25]

Specifically,

$$u_{1100} = \rho_{12}(u_{2000}u_{0200})^{1/2} \simeq \rho_{12}g'_{(\bar{R}_k^r)}(1-X_{N+1,k})u_{2000}$$

$$u_{1010} = \rho_{13}(u_{2000}u_{0020})^{1/2}$$

$$u_{1001} = \rho_{14}(u_{2000}u_{0002})^{1/2} \simeq \rho_{14}g'_{(\bar{R}_k^r)}V_k(1-X_{N+1,k})u_{2000}$$

$$u_{0110} = \rho_{23}(u_{0200}u_{0020})^{1/2} \simeq \rho_{23}g'_{(\bar{R}_k^r)}(1-X_{N+1,k})(u_{2000}u_{0020})^{1/2}$$

$$u_{0101} = \rho_{24}(u_{0200}u_{0002})^{1/2} \simeq \rho_{24}g'_{(\bar{R}_k^r)}g''_{(\bar{R}_k^r)}V_k(1-X_{N+1,k})^2u_{2000}$$

$$u_{0011} = \rho_{34}(u_{0020}u_{0002})^{1/2} \simeq \rho_{34}g''_{(\bar{R}_k^r)}V_k(1-X_{N+1,k})(u_{2000}u_{0020})^{1/2}$$

$$u_{1110} \simeq g'_{(\bar{R}_k^r)}(1-X_{N+1,k})u_{2010}$$

$$u_{1101} \simeq g'_{(\bar{R}_k^r)}g''_{(\bar{R}_k^r)}V_k(1-X_{N+1,k})^2u_{3000} \qquad \text{C.4}$$

$$u_{1011} \simeq g''_{(\bar{R}_k^r)}V_k(1-X_{N+1,k})2010$$

$$u_{0111} \simeq g'_{(\bar{R}^r{}_k)}g''_{(\bar{R}_k^r)}V_k(1-X_{N+1,k})^2u_{2010}$$

$$u_{0200} \simeq (g'_{\bar{R}_k^r})^2(1-X_{N+1,k})^2u_{2000}$$

$$u_{0002} \simeq (g''_{\bar{R}_k^r})^2V_k^2(1-X_{N+1,k})^2u_{2000}$$

$$u_{2010} \simeq g'_{(\bar{R}_k^r)}(1-X_{N+1,k})u_{3000}$$

$$u_{2001} \simeq g''_{(\bar{R}_k^r)}(1-X_{N+1,k})V_ku_{3000}$$

Substituting C.3 and C.4 into C.2 yields:

$$\gamma_i = \frac{u_{0110}}{u_{1010}} \simeq \frac{\rho_{23} g'_{(\bar{R}_k^r)} (1-X_{N+1,k})(u_{2000}u_{0020})^{1/2}}{\rho_{13}(u_{2000}u_{0020})^{1/2}}$$

$$= \frac{\rho_{23}}{\rho_{13}} g'_{(\bar{R}_k^r)} (1-X_{N+1,k}) \qquad \text{C.5}$$

$$\delta_i = -\frac{1}{(1-E[\tilde{g}_k'])} \frac{u_{1011}(1-X_{N+1,k}) - u_{0111}}{u_{1010}}$$

$$\simeq \frac{-1}{(1-E[\tilde{g}_k'])}\left[\frac{g''_{(\bar{R}_k^r)} V_k (1-X_{N+1,k})^2 u_{2010} - g'_{(\bar{R}_k^r)} g''_{(\bar{R}_k^r)} V_k (1-X_{N+1,k})^2 u_{2010}}{\rho_{13}(u_{2000}u_{0020})^{1/2}}\right]$$

$$= \frac{g''_{(\bar{R}_k^r)} V_k (1-X_{N+1,k})^2 (1-g'_{(\bar{R}_k^r)}) u_{2010}}{(1-E[g_k'])\rho_{13}(u_{2000}u_{0020})^{1/2}}$$

$$z_i = \frac{-1}{(1-E[\tilde{g}_k'])^2}\left[\frac{u_{1001}u_{0011}(1-X_{N+1,k}) - u_{0101}u_{0011}}{u_{1010}}\right]$$

$$\simeq \frac{-\rho_{14}(g''_{(\bar{R}_k^r)} V_k)^2 (1-X_{N+1,k})^3 u_{2000}\rho_{34}(u_{2000}u_{0020})^{1/2}}{(1-E[\tilde{g}_k'])^2 \rho_{13}(u_{2000}u_{0020})^{1/2}}$$

$$+ \frac{\rho_{24} g'_{(\bar{R}_k^r)} (g''_{(\bar{R}_k^r)} V_k)^2 (1-X_{N+1,k})^3 u_{2000}\rho_{34}(u_{2000}u_{0020})^{1/2}}{(1-E[\tilde{g}_k'])^2 \rho_{13}(u_{2000}u_{0020})^{1/2}}$$

$$= \frac{-[\rho_{34}(g''_{(\bar{R}_k^r)} V_k)^2 (1-X_{N+1,k})^3 (\rho_{14} - \rho_{24} g'_{(\bar{R}_k^r)})] u_{2000}}{(1-E[\tilde{g}_k'])^2 \rho_{13}}$$

$$\gamma_k = \frac{u_{1100}}{u_{2000}} \simeq \frac{\rho_{12} g'_{(\bar{R}_k^r)} (1-X_{N+1,k}) u_{2000}}{u_{2000}} = \rho_{12} g'_{(\bar{R}_k^r)} (1-X_{N+1,k})$$

$$\delta_k = \frac{-1}{(1-E[\tilde{g}_k'])} [\frac{u_{2001}(1-X_{N+1,k}) - u_{1101}}{u_{2000}}]$$

$$= -[\frac{g''_{(\bar{R}_k^r)} (1-X_{N+1,k})^2 V_k u_{3000} - g'_{(\bar{R}_k^r)} g''_{(\bar{R}_k^r)} V_k (1-X_{N+1,k})^2 u_{3000}}{u_{2000}}]$$

$$= -[\frac{g''_{(\bar{R}_k^r)} (1-g'_{(\bar{R}_k^r)}) V_k (1-X_{N+1,k})^2 u_{3000}}{(1-E[\tilde{g}_k']) \quad u_{2000}}]$$

$$z_k = [\frac{-1}{(1-E[\tilde{g}_k'])^2} [\frac{u_{1001}^2 (1-X_{N+1,k}) - u_{0101} u_{1001}}{u_{2000}}]$$

$$\simeq - \frac{[(\rho_{14} g''_{(\bar{R}_k^r)} V_k)^2 (1-X_{N+1,k})^3 u_{2000}^2 - \rho_{24} g'_{(\bar{R}_k^r)} (g''_{(\bar{R}_k^r)} V_k)^2 (1-X_{N+1,k})^3 u_{2000}^2 \rho_{14})]}{(1-E[\tilde{g}_k'])^2 u_{2000}}$$

$$= -[\frac{(\rho_{14} (g''_{(\bar{R}_k^r)} V_k)^2 (1-X_{N+1,k})^3 (\rho_{14} - {}_{24} g'_{(\bar{R}_k^r)}) u_{2000}}{(1-E[\tilde{g}_k'])^2}]$$

Substituting C.5 into C.1, omitting the subscripts $\bar{R}_k^r$, k, and (N+1,k) for convenience, yields:

$$A^* = 1 - \qquad\qquad \text{C.6}$$

$$\frac{1-[\frac{\rho_{23}}{\rho_{13}} g' - \frac{(1-g')g''V^r u_{2010}}{\rho_{13}(1-E[\tilde{g}'])(u_{2000}u_{0020})^{1/2}} - \frac{\rho_{34}}{\rho_{13}} \frac{(g''V^r)^2 u_{2000}[\rho_{14}-g'\rho_{24}]}{(1-E[\tilde{g}'])^2}]}{1-[\rho_{12}g' - \frac{(1-g')g''V^r u_{3000}}{(1-E[\tilde{g}'])u_{2000}} - \frac{\rho_{14}(g''V^r)^2 u_{2000}[\rho_{14}-g'\rho_{24}]}{(1-E[\tilde{g}'])^2}]}$$

where $V^r = V_k(1-X_{N+1,k})$, the total dollar amount of capital investor k holds in risky assets.

Similarly,

$$B^* = \frac{u_{1001}(u_{0111}-(1-X_{N+1,k})u_{1011})-u_{0011}(u_{1101}-(1-X_{N+1,k})u_{2001})}{(1-E[\tilde{g}_k'])[u_{1101}-(1-X_{N+1,k})u_{20001} + (1-E[\tilde{g}_k'])((1-X_{N+1,k})u_{2000}-u_{1100})]-[(1-X_{N+1,k})u_{1001}^2-u_{0101}u_{1001}]}$$

$$+ \frac{(1-E[\tilde{g}_k'])[u_{1001}((1-X_{N+1,k})u_{1010}-u_{0110})-u_{0011}((1-X_{N+1,k})u_{2000}-u_{1100})]}{(1-E[\tilde{g}_k'])[u_{1101}-(1-X_{N+1,k})u_{2001} + (1-E[\tilde{g}_k'])((1-X_{N+1,k})u_{2000}-u_{1100})]-[(1-X_{N+1,k})u_{1001}^2-u_{0101}u_{1001}]} .$$

Again making the proper substitutions, the numerator of the first term in C.7 can be written as:

$$\rho_{14}g''V^r u_{2000}(g'g''V^r u_{2010}(1-X)-g''V^r u_{2010}(1-X))$$

$$- \rho_{34}g''V^r(u_{2000}u_{0020})^{1/2}(g'g''V^r u_{3000}(1-X)-g''V^r u_{3000}(1-X))$$

$$= \rho_{34}(g'')^2(1-g')(V^r)^2[(u_{2000}u_{0020})^{1/2}u_{3000}](1-X)$$

$$- \rho_{14}(g'')^2(1-g')(V^r)^2]U_{2000}u_{2010}](1-X) \qquad \text{C.8}$$

$$= (g'')^2(1-g')(V^r)^2]\rho_{34}(u_{2000}u_{0020})^{1/2}u_{3000} - \rho_{14}u_{2000}u_{2010}](1-X) .$$

The numerator of the second term reduces to

$$(1-E[\tilde{g}_k'])[\rho_{14}g''V^r u_{2000}((1-X)\rho_{13}(u_{2000}u_{0020})^{1/2}-\rho_{23}g'(1-X)(u_{2000}u_{0020})^{1/2})$$

$$- \rho_{34}g''V^r(u_{2000}u_{0020})^{1/2}((1-X)u_{2000}-\rho_{12}g'(1-X)u_{2000})]$$

$$= (1-E[\tilde{g}_k']) [\rho_{14} g'' V^r (u_{2000})^{3/2} (u_{0020})^{1/2} (\rho_{13} - g'\rho_{23})$$

$$- \rho_{34} g'' V^r (u_{2000})^{3/2} (u_{0020})^{1/2} (1-\rho_{12} g')] (1-X)$$

$$= [\rho_{14}(\rho_{13} - g'\rho_{23}) - \rho_{34}(1-g'\rho_{42})][g'' V^r (u_{2000})^{3/2} (u_{0020})^{1/2}](1-E[\tilde{g}'])(1-X). \quad \text{C.9}$$

The denominator becomes:

$$(1-E[\tilde{g}'])[(g'g''V^r u_{3000}(1-X) - g''V^r u_{3000}(1-X)) +$$

$$(1-E[\tilde{g}'])(u_{2000}(1-X) - \rho_{12} g' u_{2000}(1-X))] - [(1-X)(\rho_{14} g'' V^r)^2 u_{2000}^2$$

$$- \rho_{24} g' g'' V^r u_{2000} (1-X) \rho_{14} g'' V^r u_{2000}]$$

$$= -(1-E[\tilde{g}'])[g''(1-g')V^r u_{3000} - (1-E[\tilde{g}'])(1-\rho_{12} g') u_{2000}](1-X)$$

$$-[(g'')^2 (V^r)^2 \rho_{14}(\rho_{14} - \rho_{24} g') u_{2000}^2](1-X) \quad . \quad \text{C.10}$$

Thus,

$$B^* =$$

$$-\frac{(g'')^2 (1-g')(V^r)^2 [\rho_{34}(u_{2000} u_{0020})^{1/2} u_{3000} - \rho_{14} u_{2000} u_{2010}]}{[(g'')^2 (V^r)^2 \rho_{14}(\rho_{14} - \rho_{24} g') u_{2000}^2 + (1-E[\tilde{g}'])(g''(1-g')V^r u_{3000} - (1-E[\tilde{g}'])(1-\rho_{12} g'] u_{2000}]}$$

$$-\frac{[\rho_{14}(\rho_{13} - g'\rho_{23}) - \rho_{34}(1-g'\rho_{12})](g'' V^r (u_{2000})^{3/2} (u_{0020})^{1/2})(1-E[\tilde{g}'])}{(g'')^2 (V^r)^2 \rho_{14}(\rho_{14} - \rho_{24} g') u_{2000}^2 + (1-E[\tilde{g}'])(g''(1-g')V^r u_{3000} - (1-E[\tilde{g}'])(1-\rho_{12} g') u_{2000}}$$

APPENDIX D

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_{ik}} = \frac{\partial}{\partial X_{ik}}[\bar{R}_k - \frac{1}{V_k} g_{(\bar{y}_k)} -(1/2)(1-X_{N+1,k})g_k''V_k^r u_{2000}]$$

$$= \bar{R}_i - \frac{1}{V}\frac{\partial g(\bar{y})}{\partial X_i} -(1/2)(1-X_{N+1})V^r \frac{\partial g''}{\partial X_i} u_{2000} -(1/2)(1-X_{N+1})V^r g'' \frac{\partial u_{2000}}{\partial X_i}$$

$$= \bar{R}_i - \frac{1}{V}\frac{\partial g(\bar{y})}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial X_i} - (1/2)(1-X_{N+1})V^r \frac{\partial g''}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial X_i} u_{2000}$$

$$- (1/2)(1-X_{N+1})V^r g'' \frac{\partial E[\tilde{R}_k^r - \bar{R}_k^r)^2]}{\partial X_i}$$

$$= \bar{R}_i - g'\bar{R}_i - (1/2)(1-X_{N+1})V^r g''' V\bar{R}_i u_{2000}$$

$$- (1/2)(1-X_{N+1})V^r g''\cdot 2E[(\tilde{R}_k^r - \bar{R}_k^r)(\tilde{R}_i - \bar{R}_i)]$$

$$= \bar{R}_i(1-g' - (1/2)(V^r)^2 g''' u_{2000}) - V^r g'' u_{1010} \qquad \text{D.1}$$

since $\tilde{R}_k^r = \dfrac{\Sigma_j X_{jk}\tilde{R}_j}{(1-X_{N+1,k})}$

$$\frac{\partial \sigma_{(\tilde{R}_k^t,\tilde{R}_k^t)}}{\partial X_{ik}} = \frac{\partial}{\partial X_{ik}} E\{[(\tilde{R}_k-\bar{R}_k)-g_k'(1-X_{N+1,k})(\tilde{R}_k^r-\bar{R}_k^r)-(1/2)(1-X_{N+1,k})V_k^r +$$

$$g_k''[(\tilde{R}_k^r - \bar{R}_k^r)^2 - u_{2000}]]^2\}$$

$$= 2E\{[(R-\bar{R})-g'(1-X_{N+1})(R^r-\bar{R}^r)-(1/2)(1-X_{N+1})V^r g''[(R^r-\bar{R}^r)^2-u_{2000}]] +$$

$$[(R_i-\bar{R}_i)-g''V\bar{R}_i(1-X_{N+1})(R^r-\bar{R}^r)-g'(R_i-\bar{R}_i)-(1/2)(1-X_{N+1})V^r g'''V\bar{R}_i +$$

$$[(R^r-\bar{R}^r)^2-u_{2000}]-V^r g''[(R^r-\bar{R}^r)(R_i-\bar{R}_i)-E[(R^r-\bar{R}^r)(R_i-\bar{R}_i)]]]\}$$

$$= 2E\{[(1-X_{N+1})(R^r-\bar{R}^r)(1-g')-(1/2)(1-X_{N+1})V^r g''(R^r-\bar{R}^r)^2$$

$$+ (1/2)(1-X_{N+1})V^r g''u_{2000}]\times[(R_i-\bar{R}_i)(1-g')-\bar{R}_i g''V^r(R^r-\bar{R}^r)$$

$$-(1/2)\bar{R}_i(V^r)^2 g'''(R^r-\bar{R}^r)^2+(1/2)\bar{R}_i(V^r)^2 g'''u_{2000}$$

$$- V^r g''(R^r-\bar{R}^r)(R_i-\bar{R}_i) + V^r g''u_{1010}]\} \qquad \text{D.2}$$

Carrying out the multiplication in D.2 yields the sum of the following terms with their equivalents on the right hand side:

$$E\{[(1-X_{N+1})(R^r-\bar{R}^r)(1-g')][(R_i-\bar{R}_i)(1-g')]\} = (1-X_{N+1})(1-g')^2 u_{1010}$$

$$E\{[(1-X_{N+1})(R^r-\bar{R}^r)(1-g')][-\bar{R}_i g''V^r(R^r-\bar{R}^r)]\}= -\bar{R}_i g''(1-g')V^r(1-X_{N+1})u_{2000}$$

$$E\{[(1-X_{N+1})(R^r-\bar{R}^r)(1-g')][-(1/2)\bar{R}_i(V^r)^2 g'''(R^r-\bar{R}^r)^2]\}$$

$$= -(1/2)\bar{R}_i g'''(1-g')(V^r)^2(1-X_{N+1})u_{3000}$$

$$E\{[(1-X_{N+1})(R^r-\bar{R}^r)(1-g')][(1/2)\bar{R}_i(V^r)^2 g'''u_{2000}]\}$$

$$= -(1/2)\bar{R}_i g'''(1-g')(V^r)^2(1-X_{N+1})u_{1000}u_{2000} = 0$$

$$E\{[(1-X_{N+1})(R^r-\bar{R}^r)(1-g')][-V^r g''(R^r-\bar{R}^r)(R_i-\bar{R}_i)]\}$$

$$= -g''(1-g')V^r(1-X_{N+1})u_{2010}$$

$$E\{[(1-X_{N+1})(R^r-\bar{R}^r)(1-g')][V^r g'' u_{1010}]\}$$

$$= g''(1-g')V^r(1-X_{N+1})u_{1000}u_{1010} = 0$$

$$E\{[-(1/2)(1-X_{N+1})V^r g''(R^r-\bar{R}^r)^2][(R_i-\bar{R}_i)(1-g')]\}$$

$$= -(1/2)g''(1-g')V^r(1-X_{N+1})u_{2010}$$

$$E\{[-(1/2)(1-X_{N+1})V^r g''(R^r-\bar{R}^r)^2][-\bar{R}_i g'' V^r(R^r-\bar{R}^r)]\}$$

$$= (1/2)\bar{R}_i(g'')^2(V^r)^2(1-X_{N+1})u_{3000}$$

$$E\{[-(1/2)(1-X_{N+1})V^r g''(R^r-\bar{R}^r)^2][-(1/2)\bar{R}_i(V^r)^2 g'''(R^r-\bar{R}^r)^2]\}$$

$$= (1/4)\bar{R}_i g'' g'''(V^r)^3(1-X_{N+1})u_{4000}$$

$$E\{[-(1/2)(1-X_{N+1})V^r g''(R^r-\bar{R}^r)^2][(1/2)\bar{R}_i(V^r)^2 g''' u_{2000}]\}$$

$$= -(1/4)\bar{R}_i g'' g'''(1-X_{N+1})(V^r)^3 u_{2000}^2$$

$$E\{[-(1/2)(1-X_{N+1})V^r g''(R^r-\bar{R}^r)^2][-V^r g''(R^r-\bar{R}^r)(R_i-\bar{R}_i)]\}$$

$$= (1/2)(g'')^2(V^r)^2(1-X_{N+1})u_{3010}$$

$$E\{[-(1/2)(1-X_{N+1})V^r g''(R^r-\bar{R}^r)^2][V^r g'' u_{1010}]\}$$

$$= -(1/2)(g'')^2(V^r)^2(1-X_{N+1})u_{2000}u_{1010}$$

$$E\{[(1/2)(1-X_{N+1})V^r g'' u_{2000}][(R_i-\bar{R}_i)(1-g')]\}$$

$$= (1/2)(1-g')g''V^r(1-X_{N+1})u_{2000}u_{0010} = 0$$

$$E\{[(1/2)(1-X_{N+1})V^r g'' u_{2000}][-\bar{R}_i g'' V^r(R^r-\bar{R}_r)]\}$$

$$= -(1/2)\bar{R}_i(g'')^2(V^r)^2(1-X_{N+1})u_{2000}u_{1000} = 0$$

$$E\{[(1/2)(1-X_{N+1})V^r g''u_{2000}][-(1/2)\bar{R}_i(V^r)^2 g'''(R^r-\bar{R}^r)^2]\}$$

$$= -(1/4)\bar{R}_i g''g'''(V^r)^3(1-X_{N+1})u^2_{2000}$$

$$E\{[(1/2)(1-X_{N+1})V^r g''u_{2000}][(1/2)\bar{R}_i(V^r)^2 g'''u_{2000}]\}$$

$$= (1/4)\bar{R}_i g''g'''(V^r)^3(1-X_{N+1})u^2_{2000}$$

$$E\{[(1/2)(1-X_{N+1})V^r g''u_{2000}][-V^r g''(R^r-\bar{R}^r)(R_i-\bar{R}_i)]\}$$

$$= -(1/2)(g'')^2(V^r)^2(1-X_{N+1})u_{2000}u_{1010}$$

$$E\{[(1/2)(1-X_{N+1})V^r g''u_{2000}][V^r g''u_{1010}]\}$$

$$= (1/2)(g'')^2(V^r)^2(1-X_{N+1})u_{2000}u_{1010}$$

Substituting these expressions into D.2

$$\frac{\partial\sigma_{(\tilde{R}^t_k,\tilde{R}^t_k)}}{\partial X_i} = 2(1-X_{N+1})\ \{(1-g')^2u_{1010} - \bar{R}_i g''(1-g')V^r u_{2000}$$

$$-(1/2)\bar{R}_i g'''(1-g')(V^r)^2u_{3000}-g''(1-g')V^r u_{2010}-(1/2)g''(1-g')V^r u_{2010}$$

$$+(1/2)\bar{R}_i(g'')^2(V^r)^2u_{3000}+(1/4)\bar{R}_i g''g'''(V^r)^3u_{4000}-(1/4)\bar{R}_i g''g'''(V^r)^3u^2_{2000}$$

$$+(1/2)(g'')^2(V^r)^2u_{3010}-(1/2)(g'')^2(V^r)^2u_{2000}u_{1010}-(1/4)\bar{R}_i g''g'''(V^r)^3u^2_{2000}$$

$$+(1/4)\bar{R}_i g''g'''(V^r)^3u^2_{2000}$$

$$-(1/2)(g'')^2(V^r)^2u_{2000}u_{1010} + (1/2)(g'')^2(V^r)^2u_{2000}u_{1010}\}\ .$$

Collecting terms and rearranging slightly:

$$\frac{\partial\sigma_{(R_k^t,R_k^t)}}{\partial X_i} =$$

$$2(1-X_{N+1})\{(1-g')^2u_{1010}-\bar{R}_i g''(1-g')V^r u_{2000}-(3/2)g''(1-g')V^r u_{2010}$$

$$-(1/2)\bar{R}_i(V^r)^2[g'''(1-g')-(g'')^2]u_{3000}+(1/4)\bar{R}_i g''g'''(V^r)^3[u_{4000}-u^2_{2000}]$$

$$+ (1/2)(g'')^2(V^r)^2[u_{3010} - u_{2000}u_{1010}]\}$$

$$= 2(1-X_{N+1})\{(1-g')^2u_{1010}-(3/2)g''(1-g')V^r u_{2010}+(1/2)(g'')^2(V^r)^2[u_{3010}-$$

$$u_{2000}u_{1010}\} - 2(1-X_{N+1})\bar{R}_i\{g''(1-g')(V^r)u_{2000}+(1/2)(V^r)^2[g'''(1-g')$$

$$-(g'')^2]u_{3000}-(1/4)g''g'''(V^r)^3[u_{4000}-u^2_{2000}]\} \qquad \text{D.3}$$

and, specifically, for the N+1th equation,

$$\frac{\partial E[\tilde{R}_k^t]}{\partial X_{N+1}} = R_{N+1}(1-g'-(1/2)(V^r)^2g'''u_{2000}) - \qquad \text{D.4}$$

$$\frac{\partial\sigma_{(\tilde{R}_k^t,\ \tilde{R}_k^t)}}{\partial X_{N+1}} = 2(1-X_{N+1})\{-R_{N+1}g''(1-g')V^r u_{2000}-(1/2)R_{N+1}(V^r)^2[g'''(1-g')-$$

$$(g'')^2]u_{3000} + (1/4)R_{N+1}g''g'''(V^r)^3[u_{4000}-u^2_{2000}]\}$$

$$= -2(1-X_{N+1})R_{N+1}\{g''(1-g')V^r u_{2000}+(1/2)(V^r)^2[g''(1-g')-(g'')^2]u_{3000}$$

$$-(1/4)g''g'''(V^r)^3[u_{4000}-u^2_{2000}]\} \qquad \text{D.5}$$

APPENDIX E

$$A_1^* = \frac{M_1P_1^i - Q_1N_1^i}{M_1P_1^k - Q_1N_1^k} \qquad \text{E.1}$$

$$M_1 = 1 - g' - (1/2)(V^r)^2 g''' u_{2000}$$

$$N_1^i = V^r g'' u_{1010}$$

$$P_1^i = (1-g')^2 u_{1010} - (3/2)g''(1-g')V^r u_{2010} + (1/2)(g'')^2(V^r)^2[u_{3010} - u_{2000}u_{1010}]$$

$$Q_1 = g''(1-g')V^r u_{2000} + (1/2)(V^r)^2[g'''(1-g')-(g'')^2]u_{3000} - (1/4)g''g'''(V^r)^3[u_{4000} - u_{2000}^2],$$

so that

$$M_1P_1^i = [1 - g' - (1/2)(V^r)^2 g''' u_{2000}][(1-g')^2 u_{1010} - (3/2)g''(1-g') * V^r u_{2010} + (1/2)(g'')^2(V^r)^2[u_{3010} - u_{2000}u_{1010}]]$$

$$= (1-g')^3 u_{1010} - (3/2)g''(1-g')^2 V^r u_{2010} + (1/2)(g'')^2(1-g')(V^r)^2[u_{3010} - u_{2000}u_{1010}] - (1/2)g'''(1-g')^2(V^r)^2 u_{1010}u_{2000}$$
$$+ (3/4)g'''g''(1-g')(V^r)^3 u_{2000}u_{2010} - (1/4)g'''(g'')^2(V^r)^4[u_{3010}u_{2000} - u_{2000}^2 u_{1010}],$$ and

$$Q_1N_1^i = [g''(1-g')V^r u_{2000} + (1/2)(V^r)^2[g'''(1-g')-(g'')^2]u_{3000}$$

$$- (1/4)g''g'''(V^r)^3[u_{4000} - u_{2000}^2]^1[V^r g'' u_{1010}]$$

$$= (g'')^2(1-g')(V^r)^2 u_{1010}u_{2000} + (1/2)(V^r)^3[g'''g''(1-g')-(g'')^3) *$$

$$u_{3000}u_{1010} - (1/4)(g'')^2(g''')(V^r)^4[u_{4000}u_{1010} - u_{2000}^2 u_{1010}],$$

so that,

$$M_1P_1^i - Q_1N_1^i = (1-g')^3 u_{1010} - (3/2)g''(1-g')^2 V^r u_{2010}$$

$$+ (1/2)(g'')^2(1-g')(V^r)^2[u_{3010}-u_{2000}u_{1010}]-(1/2)g'''(1-g')^2(V^r)^2 *$$

$$u_{1010}u_{2000} + (3/4)g'''g''(1-g')(V^r)^3 u_{2000}u_{2010}$$

$$- (1/4)g'''(g'')^2(V^r)^4[u_{3010}u_{2000} - u_{2000}^2 u_{1010}]$$

$$- (g'')^2(1-g')(V^r)^2 u_{1010}u_{2000} - (1/2)(V^r)^3[g'''g''(1-g')-(g'')^3] *$$

$$u_{3000}u_{1010} + (1/4)(g'')^2(g''')(V^r)^4[u_{4000}u_{1010}-u_{2000}^2 u_{1010}]$$

$$= (1-g')^3 u_{1010}\{1-(3/2)\frac{g''}{(1-g')}\ V^r\ \frac{u_{2010}}{u_{1010}} + (1/2)(\frac{g''}{1-g'})^2(V^r)^2[\frac{u_{3010}}{u_{1010}} - u_{2000}]$$

$$- (1/2)(\frac{g'''}{1-g'})(V^r)^2 u_{2000} + (3/4)\frac{g'''}{(1-g')}\ (\frac{g''}{1-g'})(V^r)^3\ \frac{u_{2000}\ u_{2010}}{u_{1010}}$$

$$- (1/4)\frac{g'''}{(1-g')}(\frac{g''}{1-g'})^2(V^r)^4[\frac{u_{3010}u_{2000}}{u_{1010}} - u_{2000}^2] - (\frac{g''}{(1-g')})^2(V^r)^2 u_{2000}$$

$$- (1/2)(V^r)^3[\frac{g'''}{(1-g')}\ \frac{g''}{(1-g')} - (\frac{g''}{1-g'})^3]u_{3000} +$$

$$+ (1/4)(\frac{g''}{1-g'})^2\ \frac{g'''}{(1-g')}(V^r)^4[u_{4000} - u_{2000}^2]\} \ . \qquad E.2$$

Collecting terms,

$$= (1-g')^3 u_{1010}\{1-(3/2)(\frac{g''}{1-g'})V^r \frac{u_{2010}}{u_{1010}}$$

$$+ (1/2)(\frac{g''}{1-g'})^2 (V^r)^2 [\frac{u_{3010}}{u_{1010}} - u_{2000} - 2u_{2000}]$$

$$-(1/2)\frac{g'''}{(1-g')}(V^r)^2 u_{2000} + (3/4)\frac{g'''}{(1-g')}\frac{g''}{(1-g')}(V^r)^3 [\frac{u_{2000}u_{2010}}{u_{1010}} - (2/3)u_{3000}]$$

$$-(1/4)(\frac{g'''}{1-g'})(\frac{g''}{1-g'})^2 (V^r)^4 [\frac{u_{3010}u_{2000}}{u_{1010}} - u_{4000}] + (1/2)(V^r)^3 (\frac{g''}{1-g'})^3 u_{3000}\}$$

$$= (1-g')^3 u_{1010} \times$$

$$\{1-\alpha_1 \frac{u_{2010}}{u_{1010}} + \beta_1 [\frac{u_{3010}}{u_{1010}} - 3u_{2000}] - \gamma_1 u_{2000} + \delta_1 [\frac{u_{2000}u_{2010}}{u_{1010}} - (2/3)u_{3000}]$$

$$- z_1 [\frac{u_{3010}u_{2000}}{u_{1010}} - u_{4000}] + \xi_1 u_{3000} ,$$

where $\alpha_1 = (3/2)\frac{g''}{1-g'} V^r$

$\beta_1 = (1/2)(\frac{g''}{1-g'})^2 (V^r)^2$

$\gamma_1 = (1/2)(\frac{g'''}{1-g'})(V^r)^2$

$\delta_1 = (3/4)\frac{g'''}{(1-g')}\frac{g''}{(1-g')}(V^r)^3$

$z_1 = (1/4)(\frac{g'''}{1-g'})(\frac{g''}{1-g'})^2 (V^r)^4$

$\xi_1 = (1/2)(V^r)^3 (\frac{g''}{1-g'})^3$ E.3

Similarly, the denominator of A_1^* is,

$$M_1P_1^k - Q_1N_1^k = (1-g')^3 u_{2000} \times$$

$$\{1 - \alpha_1 \frac{u_{3000}}{u_{2000}} + \beta_1[\frac{u_{4000}}{u_{2000}} - 3u_{2000}] - \gamma_1 u_{2000} + \delta_1[(1/3)u_{3000}]$$

$$- z_1[u_{4000} - u_{4000}] + \xi_1 u_{3000}$$

$$= (1-g')^3 u_{2000}\{1-\alpha_1 \frac{u_{3000}}{u_{2000}} + \beta_1[\frac{u_{4000}}{u_{2000}} - 3u_{2000}] - \gamma_1 u_{2000} + ((1/3)\delta_1$$

$$+ \xi_1)u_{3000}\} \qquad \text{E.4}$$

(for economy of notation, u'_{np} is substituted for u_{nopo}),
Thus

$$A_1^* = \frac{M_1P_1^i - Q_1N_1^i}{M_1P_1^k - Q_1N_1^k}$$

$$= \frac{u'_{11}}{u'_{20}}\{\frac{1-[\alpha_1-\delta_1 u'_{20}]\frac{u'_{21}}{u'_{11}}+[\beta_1-z_1u'_{20}]\frac{u'_{31}}{u'_{11}}+[-\gamma_1-3\beta_1]u'_{20}+[\xi_1-(2/3)\delta_1]u'_{30}+z_1u'_{40}}{1+[-\gamma_1-3\beta_1]u'_{20}-[\alpha_1-((1/3)\delta_1+\xi_1)u'_{20}]\frac{u'_{30}}{u'_{20}}+\beta_1\frac{u'_{40}}{u'_{20}}} \qquad \text{E.4}$$

$$B_1^* = \frac{N_1^kP_1^i - N_1^iP_1^k}{M_1P_1^k - Q_1N_1^k} \qquad \text{E.5}$$

the terms N_1^k, P_1^i, N_1^i, P_1^k, M_1, Q_1 given by equation 4.9.

$$N_1^kP_1^i = [V^r g'' u'_{20}][(1-g')^2 u'_{11} - (3/2)g''(1-g')V^r u'_{21} +$$

$$(1/2)(g'')^2(V^r)^2[u'_{31} - u'_{20}u'_{11}]$$

$$= V^r(g'')(1-g')^2(u'_{20}u'_{11}) - (3/2)(g'')^2(1-g')(V^r)^2 u'_{20}u'_{21}$$

$$+ (1/2)(g'')^3(V^r)^3[u'_{20}u'_{31} - (u'_{20})^2 u'_{11}] \qquad \text{E.6}$$

$$N_1^i P_1^k = [V^r g'' u'_{11}][(1-g')^2 u'_{20} - (3/2) g''(1-g') V^r u'_{30} + (1/2)(g'')^2 (V^r)^2 +$$

$$[u'_{40} - (u'_{20})^2]$$

$$= (1-g')^2 g'' V^r u'_{11} u'_{20} - (3/2)(g'')^2 (1-g')(V^r)^2 u'_{11} u'_{30}$$

$$+ (1/2)(g'')^3 (V^r)^3 [u'_{40} - (u'_{20})^2] u'_{11} \qquad \text{E.7}$$

so that,

$$B_1^* = \frac{N_1^k P_1^i - N_1^i P_1^k}{M_1 P_1^k - Q_1 N_1^k} =$$

$$= (1-g')^3 u'_{11} u'_{20} \{V^r (\frac{g''}{1-g'}) - (3/2)(\frac{g''}{1-g'})^2 (V^r)^2 \frac{u'_{21}}{u'_{11}} + (1/2)(\frac{g''}{1-g'})^3 V^{r^3} +$$

$$[\frac{u'_{31}}{u'_{11}} - u'_{20}] - (\frac{g''}{1-g'}) V^r + (3/2)(\frac{g''}{1-g'})^2 (V^r)^2 \frac{u'_{30}}{u'_{20}}$$

$$- (1/2)(\frac{g''}{1-g'})^3 (V^r)^3 [\frac{u'_{40}}{u'_{20}} - u'_{20}]\}\{[1 + [\gamma_1 - 3\beta_1] u'_{20}$$

$$- [\alpha_1 - ((1/3)\delta_1 + \xi_1) u'_{20}] \frac{u'_{30}}{u'_{20}} + \beta_1 \frac{u'_{40}}{u'_{20}}] (1-g')^3 u'_{20}]^{-1}\}$$

$$\doteq \frac{(3\beta_1 [\frac{u'_{30}}{u'_{20}} - \frac{u'_{21}}{u'_{11}}] + \xi_1 [\frac{u'_{31}}{u'_{11}} - \frac{u'_{40}}{u'_{20}}]) u'_{11}}{[1 + [-\gamma_1 - 3\beta_1] u'_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1) u'_{20} \frac{u'_{30}}{u'_{20}} + \beta_1 \frac{u'_{40}}{u'_{20}}]}$$

$$= \frac{u'_{11} [-3\beta_1 \frac{u'_{21}}{u'_{11}} + \xi_1 \frac{u'_{31}}{u'_{11}} + 3\beta_1 \frac{u'_{30}}{u'_{20}} - \xi_1 \frac{u'_{40}}{u'_{20}}]}{1 + [-\gamma_1 - 3\beta_1] u'_{20} - [\alpha_1 - ((1/3)\delta_1 + \xi_1) u'_{20}] \frac{u'_{30}}{u'_{20}} + \beta_1 \frac{u'_{40}}{u'_{20}}}$$

APPENDIX F

$$\frac{\partial E(\tilde{R}_k^t-\bar{R}_k^t)^3}{\partial X_i} = \frac{\partial}{\partial X_i} E[(\tilde{R}_k-\bar{R}_k)^3-(\tilde{T}_k/V_k-\bar{T}_k/V_k)^3$$

$$+ 3(\tilde{T}_k/V_k-\bar{T}_k/V_k)^2(\tilde{R}_k-\bar{R}_k)-3(\tilde{R}_k-\bar{R}_k)^2(\tilde{T}_k/V_k-\bar{T}_k/V_k)]$$

$$= E[3(\tilde{R}_k-\bar{R}_k)^2(\tilde{R}_i-\bar{R}_i)-3\frac{(\tilde{T}_k-\bar{T}_k)^2}{V_k^3}\frac{\partial(\tilde{T}_k-\bar{T}_k)}{\partial X_i}$$

$$+ 3(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{R}_i-\bar{R}_i) + 6(\tilde{R}_k-\bar{R}_k)(\frac{\tilde{T}_k-\bar{T}_k}{V_k^2})\frac{\partial(\tilde{T}_k-\bar{T}_k)}{\partial X_i}$$

$$- 3\frac{(\tilde{R}_k-\bar{R}_k)^2}{V_k}\frac{\partial(\tilde{T}_k-\bar{T}_k)}{\partial X_i} - 6(\tilde{R}_k-\bar{R}_k)(\tilde{R}_i-\bar{R}_i)\frac{(\tilde{T}_k-\bar{T}_k)}{V_k}]$$

$$\frac{\partial(\tilde{T}_k-\bar{T}_k)}{\partial X_i} = \frac{\partial\tilde{T}_k}{\partial y}\frac{\partial y}{\partial X_i} - E[\frac{\partial T_k}{\partial y}\frac{\partial y}{\partial X_i}] = \tilde{g}_k' V_k \tilde{R}_i - E[\tilde{g}_k'\tilde{R}_i V_k]$$

$$= V_k[\tilde{g}_k'\tilde{R}_i - E(\tilde{g}_k'\tilde{R}_i)]$$

Therefore,

$$\frac{\partial E(\tilde{R}_k^t-\bar{R}_k^t)^3}{\partial X_i} = 3\{(1-X_{N+1})^2 u_{2010}-E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i]) + u_{0210}$$

$$+ 2E[(\tilde{R}_k-\bar{R}_k)(\frac{\tilde{T}_k-\bar{T}_k}{V_k^2})(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i])]-E[(\tilde{R}_k-\bar{R}_k)^2(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i])]$$

$$-2(1-X_{N+1})u_{1110}\}. \qquad \text{F.1}$$

But,

$$E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'\tilde{R}_i - E(\tilde{g}_k'\tilde{R}_i))] =$$

$$E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'\tilde{R}_i-E(\tilde{g}_k'\tilde{R}_i)-\bar{g}_k'\bar{R}_i + \bar{g}_k'\bar{R}_i)] =$$

$$E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)]-E[(\frac{\tilde{T}_k-\bar{\bar{T}}_k}{V_k})^2]E[\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i]$$

$$= E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)]-u_{0200}u_{0011} \quad \text{F.2}$$

Also,

$$u_{0211} = E[(\tilde{R}_i-\bar{R}_i)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'-\bar{g}_k')] = E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{R}_i\cdot\tilde{g}_k'-\bar{g}_k'\tilde{R}_i-\bar{R}_i\cdot\tilde{g}_k'+\bar{R}_i\cdot\bar{g}_k')]$$

$$= E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2((\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)+2\bar{R}_i\cdot\bar{g}_k'-\bar{g}_k'\tilde{R}_i-\bar{R}_i\cdot\tilde{g}_k')]$$

therefore

$$u_{0211} = E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)]-E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\bar{g}_k'(\tilde{R}_i-\bar{R}_i)]$$

$$- E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2\bar{R}_i(\tilde{g}_k' - \bar{g}_k') .$$

So that,

$$E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)] = u_{0211} + \bar{g}_k'u_{0210} + \bar{R}_iu_{0201} .$$

Substituting into F.2,

$$E[(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2(\tilde{g}_k'\tilde{R}_i-E(\tilde{g}_k'\tilde{R}_k)] = u_{0211}+\bar{g}_k'u_{0210}+\bar{R}_iu_{0201}-u_{0200}u_{0011} \qquad \text{F.3}$$

Similarly,

$$E[(\tilde{R}_k-\bar{R}_k)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i])] =$$

$$(1-X_{N+1})E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{g}_k'\tilde{R}_i-E(\tilde{g}_k'\tilde{R}_i)-\bar{g}_k'\bar{R}_i+\bar{g}_k'\bar{R}_i)]$$

$$= (1-X_{N+1})\{E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)]-E[(\tilde{R}_k^r-\bar{R}_k)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})]E[\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i]\}$$

$$= (1-X_{N+1})\{E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)]-u_{1100}u_{0011}\} \qquad \text{F.4}$$

Also,

$$u_{1111} = E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{R}_i-\bar{R}_i)(\tilde{g}_k'-\bar{g}_k')]$$

$$= E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{R}_i\tilde{g}_k'-\bar{g}_k'\tilde{R}_i-\bar{R}_i\tilde{g}_k' + \bar{R}_i\bar{g}_k')]$$

$$= E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})((\tilde{R}_i\tilde{g}_k'-\bar{R}_i\bar{g}_k')+2\bar{R}_i\bar{g}_k'-\bar{g}_k'\tilde{R}_i-\bar{R}_i\tilde{g}_k')]$$

therefore

$$u_{1111} = E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{R}_i\tilde{g}_k'-\bar{R}_i\bar{g}_k')]$$

$$- \bar{g}_k' \, E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{R}_i-\bar{R}_i)]-\bar{R}_iE[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{g}_k'-\bar{g}_k')].$$

So that,

$$E[(\tilde{R}_k^r-\bar{R}_k^r)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{R}_i\tilde{g}_k'-\bar{R}_i\bar{g}_k')] = u_{1111} + \bar{R}_i u_{1101} + \bar{g}_k' u_{1110} \qquad \text{F.5}$$

Substituting F.5 into F.4,

$$E[(\tilde{R}_k-\bar{R}_k)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i])]$$

$$= (1-X_{N+1})\{u_{1111} + \bar{R}_i u_{1101} + \bar{g}_k' u_{1110} - u_{1100}u_{0011}\} \qquad \text{F.6}$$

Finally,

$$E[(\tilde{R}_k-\bar{R}_k)^2(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i])]$$

$$= E[(\tilde{R}_k-\bar{R}_k)^2(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i]-\bar{g}_k'\bar{R}_i + \bar{g}_k'\bar{R}_i)]$$

$$= (1-X_{N+1})^2 E[(\tilde{R}_k^r-\bar{R}_k^r)^2(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i]-\bar{g}_k'\bar{R}_i + \bar{g}_k'\bar{R}_i)]$$

$$= (1-X_{N+1})^2\{E(\tilde{R}_k^r-\bar{R}_k^r)^2(\tilde{g}_k'\tilde{R}_i-E[\tilde{g}_k'\tilde{R}_i])-E[(\tilde{R}_k^r-\bar{R}_k^r)^2]E[\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i]$$

$$= (1-X_{N+1})^2\{E(\tilde{R}_k^r-\bar{R}_k^r)^2(\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)] - u_{2000}u_{0011}\}. \qquad \text{F.7}$$

But,

$$u_{2011} = E[(\tilde{R}_k^r-\bar{R}_k^r)^2(\tilde{R}_i-\bar{R}_i)(\tilde{g}_k'-\bar{g}_k')]=E[(\tilde{R}_k^r-\bar{R}_k^r)^2(\tilde{R}_i\tilde{g}_k'-\bar{g}_k'\tilde{R}_i-\bar{R}_i\tilde{g}_k'+\bar{R}_i\bar{g}_k')]$$

$$= E[(\tilde{R}_k^r-\bar{R}_k^r)^2((\tilde{g}_k'\tilde{R}_i-\bar{g}_k'\bar{R}_i)+2\bar{R}_i\bar{g}_k'-\bar{g}_k'\tilde{R}_i-\bar{R}_i\tilde{g}_k')]$$

therefore

$$u_{2011} = E[(\tilde{R}_k^r - \bar{R}_k^r)^2(\tilde{g}_k'\tilde{R}_i - \bar{g}_k'\bar{R}_i)]$$

$$- \bar{g}_k'E[(\tilde{R}_k^r-\bar{R}_k^r)^2(\tilde{R}_i-\bar{R}_i)] - \bar{R}_iE[(\tilde{R}_k^r-\bar{R}_k^r)^2(\tilde{g}_k'-\bar{g}_k')] \ .$$

So that,

$$E[(\tilde{R}_k^r - \bar{R}_k^r)^2(\tilde{g}_k'\tilde{R}_i - \bar{g}_k'\bar{R}_i)] = u_{2011} + \bar{R}_iu_{2001} + \bar{g}_k'u_{2010} \qquad \text{F.8}$$

Substituting F.8 into F.7 yields;

$$E[(\tilde{R}_k-\bar{R}_k)^2(\tilde{g}_k'\tilde{R}_i - E[\tilde{g}_k'\tilde{R}_i])] =$$

$$(1-X_{N+1})^2\{u_{2011} + \bar{R}_iu_{2001} + \bar{g}_k'u_{2010} - u_{2000}u_{0011}\}. \qquad \text{F.9}$$

Substituting F.3, F.6 and F.9 into equation F.1 yields:

$$\frac{\partial E(\tilde{R}_k^t - \bar{R}_k^t)^3}{\partial X_i} =$$

$$3\{(1-X_{N+1})^2u_{2010}-(u_{0211}+\bar{g}_k'u_{0210}+\bar{R}_iu_{0201}-u_{0200}u_{0011})+u_{0210}+2(1-X_{N+1})(u_{1111}$$

$$+ \bar{R}_iu_{1101}+\bar{g}_k'u_{1110}-u_{1100}u_{0011})-(1-X_{N+1})^2(u_{2011}+\bar{R}_iu_{2001}+\bar{g}_k'u_{2010}$$

$$-u_{2000}u_{0011})-2(1-X_{N+1})u_{1110}\}. \qquad \text{F.10}$$

Equation F.10 may be rearranged slightly to

$$\frac{\partial E[(\tilde{R}_k^t - \bar{R}_k^t)^3]}{\partial X_i} =$$

$$3\{(1-\bar{g}_k')(u_{0210}-2(1-X_{N+1})u_{1110} + (1-X_{N+1})^2 u_{2010})$$

$$-\bar{R}_i(u_{0201}-2(1-X_{N+1})u_{1101} + (1-X_{N+1})^2 u_{2001})$$

$$+ u_{0011}((1-X_{N+1})^2 u_{2000}-2(1-X_{N+1})u_{1100} + u_{0200})$$

$$-(u_{0211}-2(1-X_{N+1})u_{1111}+(1-X_{N+1})^2 u_{2011})\} \qquad \text{F.11}$$

APPENDIX G

Since security and portfolio returns possess joint normal probability density functions,

$$u_{2r,000} = \frac{(2r)!}{2^r r!}(u_{2000})^r$$

$$2_{2r+1,000} = 0 \qquad \text{for all} \quad r > 0$$

G.1

$$u_{2r+1,0,1,0} = \frac{2r+1)!}{r!2^r}\, u_{1010}(u_{2000})^r$$

$$u_{2r,0,1,0} = 0 \quad \text{for all} \quad r \geq 0 .$$

In addition it is assumed that

$$\tilde{T}_k = g_{(\bar{Y}_k)} + \sum_{\ell=1}^{L} \frac{g^{\ell}(V^r)^{\ell}}{\ell!} (\tilde{R}_k^r - \bar{R}_k^r)^{\ell}$$

$$\tilde{g}_k' = g'_{(\bar{Y}_k)} + \sum_{\ell=1}^{L} \frac{g^{\ell+1}(V^r)^{\ell}}{\ell!} (\tilde{R}_k^r - \bar{R}_k^r)^{\ell} , \qquad \text{G.2}$$

$g'_{(\bar{Y}_k)}$, $g_{(\bar{Y}_k)}$ are the functions evaluated at $\tilde{Y}_k = \bar{Y}_k$, g^{ℓ} is the $\ell^{\underline{th}}$ derivative of the tax function with respect to income evaluated at $\tilde{Y}_k = \bar{Y}_k$, $\ell = 1,2,\ldots,L$. For economy of notation,

$\mu_{nm} = E[(\tilde{R}_k^r - \bar{R}_k^r)(\tilde{R}_i - \bar{R}_i)]$, so that the mixed moments can be derived by direct substitution of G.2 and the use of G.1 as follows:

$$u_{2010} = \mu_{21} = 0$$

$$u_{3000} = \mu_{30} = 0$$

$$u_{0210} = E[(\tilde{R}_i - \bar{R}_i)(\frac{\tilde{T}_k-\bar{T}_k}{V_k})^2]$$

$$= E[(\tilde{R}_i-\bar{R}_i)(g_{(\bar{Y}_k)}-g_{(\bar{Y}_k)} + \sum_{\ell=1}^{L} g^{\ell}(V^r)^{\ell-1}((\tilde{R}^r-\bar{R}^r)^{\ell}- \mu_{\ell 0}))^2]$$

$$= E[(\tilde{R}_i-\bar{R}_i) \sum_{\ell=1}^{L} \sum_{\ell'=1}^{L} \frac{g^{\ell}g^{\ell'}}{\ell!\ell'!} (V^r)^{\ell-1}(V^r)^{\ell'-1}((R^r-\bar{R}^r)^{\ell+\ell'}$$

$$+ \mu_{\ell 0}\mu_{\ell' 0}-2(\tilde{R}^r-\bar{R}^r)^{\ell}\mu_{\ell' 0})](1-X_{N+1})^2$$

$$= \sum_{\ell=1}^{L} \sum_{\ell'=1}^{L} \frac{g^{\ell}g^{\ell'}}{\ell!\ell'!}(V^r)^{\ell+\ell'-2}E[(\tilde{R}_i-\bar{R}_i)(\tilde{R}^r-\bar{R}^r)^{\ell+\ell'}$$

$$+ (\tilde{R}_i-\bar{R}_i)\mu_{\ell 0}\mu_{\ell' 0}-2(\tilde{R}^r-\bar{R}r)^{\ell}(\tilde{R}_i-\bar{R}_i)\mu_{\ell' 0}]$$

$$= \sum_{\ell=1}^{L} \sum_{\ell'=1}^{L} \frac{g^{\ell}g^{\ell'} (V^r)^{\ell+\ell'-2}(1-X_{N+1})^2}{\ell!\ell'!}[\mu_{\ell+\ell',1}-2\mu_{\ell,1}\mu_{\ell' 0}]$$

since $\mu_{01} = 0$. G.3

From G.1, for ℓ and ℓ' both even or both odd, $\mu_{\ell+\ell',1} = 0$ and $\mu_{\ell,1}\mu_{\ell' 0} = 0$. When $2p = \ell$, $2p' + 1 = \ell'$, the right hand side of G.3 may be written as

$$= \sum_{p=1}^{Q} \sum_{p'=0}^{Q'} [\frac{g^{2p}g^{(2p'+1)}(V^r)^{2(p+p')-1}(1-X_{N+1})^2}{(2p)!(2p'+1)!}(\frac{2(p+p')+1}{(p+p')!2^{p+p'}})\mu_{20}^{p+p'}]\mu_{11}$$

G.4

where Q is the largest integer such that $2Q \leq L$, and Q is the largest integer such that $2Q' + 1 \leq L$. For $2p+1 = \ell$, $2p' = \ell'$,

$$= \sum_{p=0}^{Q'} \sum_{p'=1}^{Q} [\frac{g^{2p} g^{2p'+1} (V^r)^{2(p+p')-1} (1-X_{N+1})^2}{(2p+1)!(2p')!} (\frac{2(p+p')+1)!}{(p+p')! 2^{p+p'}}$$

$$- 2\frac{(2p+1)!(2p')!}{(2p+1)!(2p')!}) \mu_{20}^{p+p'}] \mu_{11} \ . \qquad G.5$$

Thus, u_{0210} is equivalent to the sum of G.4 and G.5, or

$$u_{0210} = (1-X_{N+1})^2 A_1 \mu_{11} \qquad G.6$$

where A_1 is the sum of the coefficients of μ_{11} in G.4 and G.5.

$$u_{1200} = \sum_{j=1}^{N} h_{jk} u_{0210} = (1-X_{N+1})^2 \sum_{j=1}^{N} h_{jk} A_1 \mu_{11}$$

$$= (1-X_{N+1})^2 \sum_{j=1}^{N} h_{jk} A_1 \sigma_{(\tilde{R}_k^r, \tilde{R}_j)} \ .$$

A_1 is independent of the specific security, j, so that

$$= (1-X_{N+1})^2 A_1 \sum_{j=1}^{N} h_{jk} \sigma_{(\tilde{R}_k^r, \tilde{R}_j)}$$

$$= (1-X_{N+1})^2 A_1 \mu_{20} \qquad G.7$$

Similarly,

$$u_{1110} = E[(\tilde{R}^r - \bar{R}^r)(\tilde{R}_i - \bar{R}_i)(\frac{\tilde{T}-\bar{T}}{V})]$$

$$= (1-X_{N+1}) E[(\tilde{R}^r - \bar{R}^r)(\tilde{R}_i - \bar{R}_i) (\sum_{\ell=1}^{L} \frac{g^{\ell} (V^r)^{\ell-1}}{\ell!} ((\tilde{R}^r - \bar{R}^r)^{\ell} - \mu_{\ell 0}))]$$

$$= (1-X_{N+1}) \sum_{\ell=1}^{L} \{\frac{g^{\ell}(V^r)^{\ell-1}(1-X_{N+1})}{\ell!} E[(\tilde{R}^r-\bar{R}^r)^{\ell+1}(\tilde{R}_i-\bar{R}_i) -$$

$$(\tilde{R}^r-\bar{R}^r)(\tilde{R}_i-\bar{R}_i)\mu_{\ell 0}]\}$$

$$= (1-X_{N+1}) \sum_{\ell=1}^{L} [\frac{g^{\ell}(V^r)^{\ell-1}}{\ell!}(\mu_{\ell+1,1} - \mu_{11}\mu_{\ell 0})](1-X_{N+1}) \qquad \text{G.8}$$

From G.1, for ℓ even,

$$\mu_{\ell+1,1} = \frac{(2p+1)!}{p!2^p} \mu_{11}\mu_{20}^p \qquad \text{G.9}$$

$$\mu_{\ell 0} = \frac{(2p)!}{2^p p!} (\mu_{20})^p,$$

$$p = 2\ell .$$

For ℓ odd

$$\mu_{\ell+1,1} = \mu_{\ell 0} = 0 \qquad \text{G.10}$$

Thus,

$$u_{1110} = \sum_{p=0}^{Q'} [\frac{g^{2p}(V^r)^{2p-1}(1-X_{N+1})}{(2p)!}(\frac{(2p+1)!-(2p)!}{p!2^p})\mu_{20}^p]\mu_{11}$$

$$= (1-X_{N+1})A_2\mu_{11} . \qquad \text{G.11}$$

$$u_{2100} = (1-X_{N+1}) \sum_{j=1}^{N} h_{jk}u_{1110} = (1-X_{N+1})A_2 \sum_{j=1}^{N} h_{jk}\mu_{11}$$

$$= (1-X_{N+1})A_2\mu_{20}, \qquad \text{G.12}$$

$$u_{1011} = E[(\tilde{R}^r-\bar{R}^r)(\tilde{R}_i-\bar{R}_i)(\tilde{g}'-\bar{g}')]$$

$$= E[(\tilde{R}^r-\bar{R}^r)(\tilde{R}_i-\bar{R}_i)(\sum_{\ell=1}^{L} \frac{g^{\ell+1}(V^r)^{\ell}((R^r-\bar{R}^r)^{\ell}-\mu_{\ell 0})}{\ell!})]$$

$$= \Sigma[\frac{g^{\ell+1}(V^r)^\ell}{\ell!}E[(\tilde{R}^r-\bar{R}^r)^{\ell+1}(\tilde{R}_i-\bar{R}_i)-(\tilde{R}^r-\bar{R}^r)(\tilde{R}_i-\bar{R}_i)\mu_{\ell 0}]]$$

$$= \Sigma[\frac{g^{\ell+1}(V^r)}{\ell!}(\mu_{\ell+1,1} - \mu_{11}\mu_{20})]$$

$$= \sum_{p=1}^{Q}[\frac{g^{2p+1}(V^r)^{2p}}{(2p)!}(\frac{(2p+1)!}{2^p p!} - \frac{(2p)!}{2^p p!})\mu_{20}^p]\mu_{11} = A_3\mu_{11} \ . \qquad G.13$$

$$u_{2001} = \sum_j h_{jk}A_3\mu_{11} = A_3\mu_{20} \ . \qquad G.14$$

$$u_{0201} = E[(\frac{\tilde{T}-\bar{T}}{V})^2(\tilde{g}'-\bar{g}')]$$

$$= [(\sum_\ell \frac{g^\ell V^{\ell-1}}{\ell!}(1-X_{N+1})((\tilde{R}^r-\bar{R}^r) -\mu_{\ell 0}))^2(\sum_\ell \frac{g^{\ell+1}(V^r)}{\ell!}((\tilde{R}^r-\bar{R}^r) -\mu_{\ell 0}))]$$

$$= \sum_\ell \sum_{\ell'} \sum_{\ell''}[\frac{g^\ell(V^r)^{\ell-1}g^{\ell'}(v^r)^{\ell'-1}g^{\ell''+1}(V^r)^{\ell''}(1-X_{N+1})^2}{\ell!\ell'!\ell''!}E[((R^r-\bar{R}^r) +'$$

$$+ \mu_{\ell 0}\mu_{\ell' 0}-2\mu_{\ell 0}(\tilde{R}^r-\bar{R}^r)^{\ell'})((\tilde{R}^r-\bar{R}^r)^{\ell''} -\mu_{\ell'' 0})]]$$

$$= \sum_\ell \sum_{\ell'} \sum_{\ell''}[\frac{g^\ell g^{\ell'} g^{\ell''+1}(V^r)^{(\ell+\ell'+\ell''-2)}(1-X_{N+1})^2}{\ell'\ell'!\ell''!}(\mu_{\ell+\ell'+\ell'',0}$$

$$-2\mu_{\ell,0}\mu_{\ell'+\ell'',0}-\mu_{\ell'',0}\mu_{\ell+\ell',0} + 2\mu_{\ell,0}\mu_{\ell',0}\mu_{\ell'',0})] \qquad G.15$$

$$= (1-X_{N+1})^2\{\sum_{\substack{p\\2p=\ell}} \sum_{\substack{p'\\2p'+1=\ell'}} \sum_{\substack{p''\\2p''+1=\ell''}} [\frac{g^{2p}g^{2p'+1}g^{2p''+2}(V^r)^{2(p+p'+p'')}}{(2p)!(2p'+1)!(2p''+1)!}[\frac{2(p+p'+p''+1)!\mu_{20}^{(p+p'+p''+2)}}{2^{(p+p'+p''+2)}(p+p'+p''+2)}$$

$$-2\,\frac{(2p)!}{2^p p!}\,\mu_{20}^{p}\,\frac{2(p'+p''+1)!(\mu_{20})^{p'+p''+1}}{2^{(p'+p''+1)}(p'+p''+1)!}]]$$

$$+\sum_{\substack{p\\2p+1=\ell\\2p'=\ell'\\2p''+1=\ell''}}\sum_{p'}\sum_{p''}[\frac{g^{2p+1}g^{2p'}g^{2p''+2}(V^r)^{2(p+p'+p'')}}{(2p+1)!(2p')!(2p''+1)!}[\frac{2(p'+p+p''+1)!\,\mu_{20}^{(p+p'+p''+2)}}{2^{(p+p'+p''+2)}(p+p'+p''+2)}]$$

$$+\sum_{\substack{p\\2p+1=\ell\\2p'+1=\ell'\\2p''=\ell''}}\sum_{p'}\sum_{p''}[\frac{g^{2p+1}g^{2p'+1}g^{2p''+1}(V^r)^{2(p+p'+p'')}}{(2p+1)!(2p'+1)!(2p'')!}[\frac{(2(p'+p+p''+1))!\mu_{20}^{(p+p'+p''+2)}}{2^{(p+p'+p''+2)}(p+p'+p''+2)}$$

$$-\frac{(2p'')!}{2^{p''}(p'')!}(\mu_{20})^{p''}\,\frac{(2(p+p'+1)!}{2^{(p+p'+1)}(p+p'+1)!}\,\mu_{20}^{(p+p'+1)}]]$$

$$+\sum_{\substack{p\\\ell=2p\\\ell'=2p'\\\ell''=2p''}}\sum_{p'}\sum_{p''}[\frac{g^{2p}g^{2p'}g^{2p''+1}(V^r)^{2(p+p'+p'')}}{(2p)!(2p')!(2p'')!}[\frac{(2(p+p'+p'')!\mu_{20}^{(p+p'+p'')}}{2^{(p+p'+p'')}(p+p'+p'')!}$$

$$-\;2\,\frac{(2p)!(2p'+2p'')!}{2^{(p+p'+p'')}p!(p'+p'')!}\,\mu_{20}^{(p+p'+p'')}\;-$$

$$-\frac{(2p'')!(2(p''p'))!}{2^{(p+p'+p'')}(p'')!(p+p')!}\,\mu_{2}^{(p+p'+p'')}\;+\;2\frac{(2p)!(2p')!(2p'')!\mu_{20}^{(p+p'+p'')}}{2^{(p+p'+p'')}p!p'!p''!}]]\}$$

$$u_{0201}=(1-X_{N+1})^2A_4\mu_{20} \qquad \text{G.16}$$

$$u_{0111}=E[(\tilde{R}_i-\bar{R}_i)(\frac{\tilde{T}-\bar{T}}{V})(\tilde{g}'-\bar{g}')]$$

$$=(1-X_{N+1})E[(\tilde{R}_i-\bar{R}_i(\sum_{\ell=1}^{L}[\frac{g^{\ell}(V^r)^{\ell-1}}{\ell!}((\tilde{R}^r-\bar{R}^r)^{\ell}-\mu_{\ell 0}))(\sum_{\ell=1}^{L}[\frac{g^{\ell+1}(V^r)^{\ell}}{\ell!}((\tilde{R}^r-\bar{R}^r)^{\ell'}-\mu_{\ell' 0})])]$$

$$= (1-X_{N+1})\{\sum_{\ell}\sum_{\ell'}[\frac{g^{\ell}g^{\ell'+1}(V^r)^{\ell+\ell'-1}}{\ell!\ell'!}E[(\tilde{R}_i-\bar{R}_i)((\tilde{R}^r-\bar{R}^r)^{\ell+\ell'}$$

$$- 2\mu_{\ell 0}(\tilde{R}^r - \bar{R}^r)^{\ell'} + \mu_{\ell 0}\mu_{\ell' 0}]]\}$$

$$= (1-X_{N+1})\{\sum_{\ell}\sum_{\ell'}[\frac{g^{\ell}g^{\ell'+1}(V^r)^{\ell+\ell'-1}}{\ell!\ell'!}(\mu_{\ell+\ell',1} - 2\mu_{\ell 0}\mu_{\ell',1})]\}$$

$$= (1-X_{N+1})\{\sum_{\substack{p\ p' \\ 2p=\ell \\ 2p'+1=\ell'}}\sum[\frac{g^{2p}g^{2p'+2}V^{r2(p+p')}}{(2p)!(2p'+1)!}(\frac{(2(p+p')+1)!}{2^{(p+p')!}(p+p')!}\mu_2^{(p+p')}$$

$$- 2\frac{(2p)!(2p+1)!}{2^{p!}p!2^{p'}(p')!}\mu_2^{p+p'})\mu_{11}]\} = (1-X_{N+1})A_5\mu_{11} \qquad \text{G.17}$$

$$u_{1101} = \sum_j h_{jk}(1-X_{N+1})A_5\,\mu_{11}$$

$$= (1-X_{N+1})A_5\mu_{20} \qquad \text{G.18}$$

$$u_{0011} = E[(\tilde{R}_i-\bar{R}_i)(\sum_{\ell=1}^{L}\frac{g^{\ell+1}(V^r)^{\ell}}{\ell!}(\tilde{R}^r-\bar{R}^r)^{\ell})]$$

$$= \sum_{\ell=1}^{L}\frac{g^{\ell+1}(V^r)^{\ell}}{\ell!}\mu_{\ell 1}$$

$$= \sum_{p=0}^{Q'}\frac{g^{2(p+1)}(V^r)^{2p+1}\ (2p+1)!}{(2p+1)!(p)!2^p}\mu_{11}\mu_{20}^{p},$$

where $\ell = 2p+1$

$$u_{0011} = A_6\mu_{11} \qquad \text{G.19}$$

$$u_{1001} = \sum_j h_{jk} A_6 \mu_{11}$$

$$= A_6 \mu_{20} \qquad \text{G.20}$$

$$u_{0200} = E[(\frac{\tilde{T}-\bar{T}}{V})^2]$$

$$= (1-X_{N+1})^2 (\sum_\ell \frac{g^\ell (V^r)^{\ell-2}}{\ell!} E[(\tilde{R}^r-\bar{R}^r)^\ell - \mu_{\ell 0}])^2$$

$$= (1-X_{N+1})^2 \{\sum_\ell \sum_{\ell'} [\frac{g^\ell g^{\ell'} (V^r)^{\ell+\ell'-2}}{\ell!\ell'!} [\mu_{\ell+\ell',0} - \mu_{\ell 0}\mu_{\ell' 0}]]$$

$$= (1-X_{N+1})^2 \sum_{\substack{p\ p' \\ \ell=2p \\ \ell=2p'}} \sum [\frac{g^{2p} g^{2p'} (V^r)^{2(p+p'-1)}}{(2p)!(2p')!} [\frac{(2(p+p'))!}{2^{p+p'}(p+p')!} \mu_{20}^{(p+p')}$$

$$- \frac{(2p)!(2p')!\mu_{20}^{p+p'}}{2^{p+p'}(p)!p'!}] + \sum_{\substack{p\ p' \\ \ell=2p+1 \\ \ell'=2p'+1}} \sum [\frac{g^{2p+1} g^{2p'+1} (V^r)^{2(p+p')+1}}{(2p+1)!(2p'+1)!} [\frac{2(p+p'+1)!}{2^{p+p'+1}(p+p'+1)!} \mu_{20}^{p+p'+1}]]\}$$

$$u_{0200} = (1-X_{N+1})^2 A_7 \mu_{20} \qquad \text{G.21}$$

$$u_{2000} = \mu_{20} \qquad \text{G.22}$$

$$u_{0110} = E[(\tilde{R}_i-\bar{R}_i)(\frac{\tilde{T}-\bar{T}}{V})]$$

$$= (1-X_{N+1})\ E[(\tilde{R}_i-\bar{R}_i)(\sum_{\ell=1}^{N} \frac{g^\ell (V^r)^{\ell-1}}{\ell!} [(\tilde{R}^r-\bar{R}^r)^\ell - \mu_{\ell 0}])]$$

$$= (1-X_{N+1}) \sum_{\ell=1}^{N} \frac{g^\ell (V^r)^{\ell-1}}{\ell!} E[(\tilde{R}_i-\bar{R}_i)(\tilde{R}^r-\bar{R}^r)^\ell - \mu_{\ell 0}(\tilde{R}_i-\bar{R}_i)]$$

$$= (1-X_{N+1}) \sum_{\substack{p=0 \\ 2p+1=\ell}}^{Q'} [\frac{g^{2p+1}(V^r)^{2p}}{(2p+1)!} E[(\tilde{R}_i-\bar{R}_i)(\tilde{R}^r-\bar{R}^r)^{2p+1}]$$

$$= (1-X_{N+1})\{\sum_p[\frac{g^{2p+1}(V^r)^{2p}(2p+1)!\mu_{20}^p}{(2p+1)!2^p p!}]\mu_{11}\}$$

$$u_{0110} = (1-X_{N+1})A_8\mu_{11} \qquad G.23$$

$$u_{1100} = \sum_j[h_{jk}(1-X_{N+1})A_8\mu_{11}]$$

$$= (1-X_{N+1})A_8\mu_{20} \qquad G.24$$

$$u_{0211} = E[(\frac{\tilde{T}-T}{V})^2(\tilde{R}_i-\bar{R}_i)(\tilde{g}'-\bar{g}')]$$

$$= (1-X_{N+1})^2 E[(\tilde{R}_i-\bar{R}_i)(\sum_\ell [\frac{g^\ell(V^r)^{\ell-2}((\tilde{R}^r-\bar{R}^r)^\ell-\mu_{\ell 0})}{\ell!}])^2$$

$$(\sum_\ell [\frac{g^{\ell+1}(V^r)^{\ell-2}((\tilde{R}^r-\bar{R}^r)^\ell-\mu_{20})}{\ell!}])]$$

$$= (1-X_{N+1})^2\{\sum_\ell\sum_{\ell'}\sum_{\ell''}[\frac{g^\ell g^{\ell'} g^{\ell''+1}(V^r)^{(\ell+\ell'+\ell''-1)}}{\ell!\ell'!\ell''!}E[(\tilde{R}^r-\bar{R}^r)^{\ell+\ell'+\ell''}(\tilde{R}_i-\bar{R}_i)$$

$$+ \mu_{\ell 0}\mu_{\ell' 0}(\tilde{R}^r-\bar{R}^r)^{\ell''}(\tilde{R}_i-\bar{R}_i)-2(\tilde{R}^r-\bar{R}^r)^{\ell+\ell''}(\tilde{R}_i-\bar{R}_i)\mu_{\ell' 0}$$

$$- (\tilde{R}^r-\bar{R}^r)^{\ell+\ell'}(\tilde{R}_i-\bar{R}_i)\mu_{\ell'',0}+2(\tilde{R}^r-\bar{R}^r)^{\ell}(\tilde{R}_i-\bar{R}_i)\mu_{\ell' 0}\mu_{\ell'' 0}]]\}$$

$$= (1-X_{N+1})^2\{\sum_\ell\sum_{\ell'}\sum_{\ell''}[\frac{g^\ell g^{\ell'} g^{\ell''+1}(V^r)^{(\ell+\ell'+\ell''-4)}}{\ell!\ell'!\ell''!}(\mu_{\ell+\ell'+\ell'',1}$$

$$+ \mu_{\ell 0}\mu_{\ell' 0}\mu_{\ell'',1} - 2\mu_{\ell+\ell'',1}\mu_{\ell' 0} - \mu_{\ell+\ell',1}\mu_{\ell'' 0} + 2\mu_{\ell 1}\mu_{\ell' 0}\mu_{\ell'' 0})]\}$$

$$= (1-X_{N+1})^2 \{ \sum_{\substack{p\ p'\ p'' \\ \ell=2p \\ \ell'=2p' \\ \ell''=2p''+1}} [\frac{g^{2p} g^{2p'} g^{2(p'+1)} (V^r)^{(2(p+p'+p'')-3)}}{(2p)!(2p')!(2p''+1)!} (\frac{(2(p+p'+p'')+1)!}{(p+p'+p'')!2^{(p+p'+p'')}}$$

$$\mu_{11}\mu_{20}^{(p+p'+p'')} + \frac{(2p)!(2p')!(2p''+1)!\mu_{20}^{(p+p'+p'')}\mu_{11}}{2^{(p+p'+p'')}(p)!(p')!(p'')!}$$

$$- 2\frac{(2(p+p'')+1)!(2p')!\mu_{20}^{(p+p'+p'')}\mu_{11}}{2^{p+p'+p''}(p+p'')!p'!})]$$

$$+ \sum_{\substack{p\ p'\ p'' \\ \ell=2p \\ \ell'=2p'+1 \\ \ell''=2p''}} [\frac{g^{2p} g^{2p'+1} g^{2p''+1} (V^r)^{(2(p+p'+p''-3)}}{(2p)!(2p'+1)!(2p'')!} (\frac{(2(p+p'+p'')+1)!}{(p+p'+p'')!2^{p+p'+p''}}$$

$$\mu_{11}\mu_{20}^{(p+p'+p'')} - \frac{(2(p+p')+1)!(2p'')!\mu_{20}^{(p+p'+p'')}\mu_{11}}{2^{p+p'+p''}(p+p')!p''!})]$$

$$+ \sum_{\substack{p\ p'\ p'' \\ \ell=2p+1 \\ \ell'=2p' \\ \ell''=2p''}} [\frac{g^{2p+1} g^{2p'} g^{2p''+1} (V^r)^{(2(p+p'+p'')-3)}}{(2p'')!(2p')!(2p+1)!} (\frac{(2(p+p'+p'')+1)!}{(p+p'+p'')!2^{(p+p'+p'')}}$$

$$\mu_{11}\mu_{20}^{(p+p'+p'')} - 2\frac{(2(p+p'')+1)!(2p')!\mu_{20}^{(p+p'+p'')}\mu_{11}}{2^{p+p'+p''}(p+p'')!p'!}$$

$$- \frac{(2(p+p')+1)!(2p'')!\mu_{20}^{(p+p'+p'')}\mu_{11}}{2^{(p+p'+p'')}(p+p')!p''!}$$

$$+ 2\frac{(2p+1)!(2p')!(2p'')!\mu_{20}^{(p+p'+p'')}\mu_{11}}{2^{(p+p'+p'')}p!p'!p''!})]\}$$

$$u_{0211} = (1-X_{N+1})^2 A_9 \mu_{11} \qquad \text{G.25}$$

$$u_{1201} = \sum_j h_{jk} A_9 \mu_{11} = (1-X_{N+1})^2 A_9 \mu_{20} \qquad \text{G.26}$$

$$u_{2011} = E[(\tilde{R}^r - \bar{R}^r)^2 (\tilde{R}_i - \bar{R}_i)(\tilde{g}' - \bar{g}')]$$

$$= E[(\tilde{R}_i - \bar{R}_i)(\sum_\ell \frac{g^{\ell+1}(V^r)^\ell}{\ell!}((\tilde{R}^r - \bar{R}^r)^{\ell+2} - (\tilde{R}^r - \bar{R}^r)^2 \mu_{\ell 0}))]$$

$$= \sum_\ell [\frac{g^{\ell+1}(V^r)^\ell}{\ell!}(E[(\tilde{R}^r - \bar{R}^r)^{\ell+2}(\tilde{R}_i - \bar{R}_i) - (\tilde{R}_i - \bar{R}_i)(\tilde{R}^r - \bar{R}^r)^2 \mu_{\ell 0}])]$$

$$= \sum_\ell [\frac{g^{\ell+1}(V^r)^\ell}{\ell!}(\mu_{\ell+2,1} - \mu_{21}\mu_{\ell 0})]$$

$$= \sum_\ell [\frac{g^{\ell+1}(V^r)^\ell}{\ell!} \mu_{\ell+2,1}]$$

$$= \sum_p [\frac{g^{2p}(V^r)^{2p-1}}{(2p-1)!} \mu_{2p+1,1}], \quad \ell = 2p-1$$

$$= \sum_p [\frac{g^{2p}(V^r)^{2p-1}}{(2p-1)!} \frac{(2p+1)!}{(p)!2^p}(\mu_{20})^p]\mu_{11}$$

$$u_{2011} = A_{10}\mu_{11} \qquad \text{G.27}$$

$$u_{3001} = \sum_j h_{jk} A_{10} \mu_{11}$$

$$u_{3001} = A_{10}\mu_{20} \qquad \text{G.28}$$

$$u_{1111} = E[(\tilde{R}^r - \bar{R}^r)(\frac{\tilde{T}_k - \bar{T}_k}{V})(\tilde{R}_i - \bar{R}_i)(\tilde{g}' - \bar{g}')]$$

$$= (1-X_{N+1})\{E[(\tilde{R}^r - \bar{R}^r)(\tilde{R}_i - \bar{R}_i)(\sum_\ell \frac{g^\ell V^{\ell-1}}{\ell!}((\tilde{R}^r - \bar{R}^r)^\ell - \mu_{\ell 0}))$$

$$(\sum_{\ell} \frac{g^{\ell+1} V}{\ell!}((\tilde{R}^r-\bar{R}^r)^{\ell} - \mu_{\ell 0}))]$$

$$= (1-X_{N+1})\{\sum_{\ell}\sum_{\ell'} [\frac{g^{\ell} g^{\ell'+1}(V^r)^{\ell+\ell'-1}}{\ell!\ell'!} E((\tilde{R}^r-\bar{R}^r)^{\ell+\ell'+1}(\tilde{R}_i-\bar{R}_i)$$

$$-(\tilde{R}^r-\bar{R}^r)^{\ell+1}\mu_{\ell'0}(\tilde{R}_i-\bar{R}_i)-(\tilde{R}^r-\bar{R}^r)^{\ell'+1}(R_i-\bar{R}_i)\mu_{\ell,0}$$

$$+ (\tilde{R}^r-\bar{R}^r)(\tilde{R}_i-\bar{R}_i)\mu_{\ell 0}\mu_{\ell'0})]\}$$

$$= (1-X_{N+1})\{\sum_{\ell}\sum_{\ell'} [\frac{g^{\ell} g^{\ell'+1}(V^r)^{\ell+\ell'-1}}{\ell!\ell'!}\mu_{\ell+\ell'+1,1} - 2\mu_{\ell 0}\mu_{\ell'+1,1} + \mu_{11}\mu_{\ell 0}\mu_{\ell'0}]\}$$

$$= (1-X_{N+1})\{\sum_{\substack{p\ p'\\ \ell=2p\\ \ell'=2p'}}\sum [\frac{g^{2p} g^{2p'+1} V^{r2(p+p')-1}}{(2p)!(2p')!}(\frac{2(p+p')+1}{(p+p')!2^{p+p'}}\mu_{20}^{p+p'}\mu_{11}$$

$$- 2\frac{(2p)!(2!'+1)!}{2^p p!2^{p'}p'!}\mu_{20}^{(p+p')}\mu_{11} + \frac{(2p)!(2p')!}{2^{p+p'}(p)!p'!}\mu_{20}^{p+p'}\mu_{11})]$$

$$+ \sum_{\substack{p\ p'\\ \ell=2p+1\\ \ell=2p'+1}}\sum [\frac{g^{2p+1} g^{2p'+2}(V^r)^{2(p+p')+1}}{(2p+1)!(2p'+1)!}(\frac{(2(p+p'+1+1)!}{(p+p'+1)!2^{p+p'+1}}\mu_{20}^{p+p'+1}\mu_{11})]\}$$

$$u_{1111} = (1-X_{N+1})A_{11}\mu_{11} \qquad \text{G.29}$$

$$u_{2101} = (1-X_{N+1})\sum_j h_{jk}A_{11}\mu_{11} = (1-X_{N+1})A_{11}\mu_{20} \qquad \text{G.30}$$

$$u_{0101} = E[(\frac{\tilde{T}-\bar{T}}{V})(\tilde{g}'-\bar{g}')]$$

$$= (1-X_{N+1})\{E[(\sum_{\ell=1}^{L}\frac{g^{\ell}(V^r)^{\ell-1}}{\ell!}((\tilde{R}^r-\bar{R}^r)^{\ell}-\mu_{\ell 0}))(\sum_{\ell=1}^{L}\frac{g^{\ell+1}(V^r)^{\ell}((R^r-\bar{R}^r)^{\ell}-\mu_{20})}{\ell!})]\}$$

$$= (1-X_{N+1})\{\sum_{\ell}\sum_{\ell'}[\frac{g^{\ell}g^{\ell'+1}(V^r)^{\ell+\ell'-1}}{\ell!\ell'!}E((\tilde{R}^r-\bar{R}^r)^{\ell+\ell'}-2(\tilde{R}^r-\bar{R}^r)^{\ell'}\mu_{\ell 0}+\mu_{\ell 0}\mu_{\ell' 0})]$$

$$= (1-X_{N+1})\{\sum_{\ell}\sum_{\ell'}[\frac{g^{\ell}g^{\ell'+1}(V^r)^{\ell+\ell'-1}}{\ell!\ell'!}(\mu_{\ell+\ell',0}-\mu_{\ell' 0}\mu_{\ell 0})]\}$$

$$= (1-X_{N+1})\{\sum_{\substack{p\ p'\\ 2p=\ell\\ 2p'=\ell'}}\sum [\frac{g^{2p}g^{2p'+1}(V^r)^{2(p+p')-1}}{(2p)!(2p')!}(\frac{(2(p+p'))!}{2^{(p+p')}(p+p')!}-\frac{(2p)!(2p')!}{2^{(p+p')}p!p'!})\mu_{20}^{p+p'}]$$

$$+\sum_{\substack{p\ p'\\ \ell=2p+1\\ \ell'=2p'+1}}\sum [\frac{g^{2p+1}g^{2p'+2}(V^r)^{2(p+p')+1}}{(2p+1)!(2p'+1)!}(\frac{2(p+p'+1)!\mu_{20}^{p+p'+1}}{2^{(p+p'+1)}(p+p'+1)!})]\}$$

$$= (1-X_{N+1})A_{12}\mu_{20}\ . \qquad \text{G.31}$$

$A_1,\ldots,A_{12}$ take on the obvious values in the above equations.

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