OSCILLATION PROPERTIES OF A DELAY DIFFERENTIAL EQUATION OF ORDER 2n

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

RAYMOND D. TERRY

1972

This is to certify that the

thesis entitled

Oscillation Properties of a Delay Differ-

ential Equation of Order 2n

presented by

Raymond D. Terry

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Major professor

Date May 19, 1972

O-7639

ABSTRACT

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The main purpose of this thesis is to provide criteria for the oscillation of solutions of certain nonlinear delay differential equations of even order. In the first four sections we consider the equation

$$(1.1) \qquad D^n[r(t)D^n y](t) + f(t, y_\tau(t))y_\tau(t) = 0.$$

It is assumed that: (1) $y_\tau(t) = y(t - \tau(t))$; (2) the delay $\tau(t)$ is nonnegative and bounded; and (3) the function $f(t,u)$ is continuous, nonnegative, odd and monotone in u.

In chapter one a classification of solutions according to types $B_j (j = 0,\dots,n-1)$ is introduced. When $r(t) \equiv 1$ this classification coincides with the one introduced by Kiguradze (Dokl. Akad. Nauk SSR, 144 (1962), 33-36). It is first demonstrated that a nonoscillatory solution $y(t)$ of (1.1) is necessarily of type B_j for some $j = 0,\dots,n-1$.

Chapter two provides integral criteria for the nonexistence of solutions of type B_j; conditions for the oscillation of all solutions of (1.1) follow immediately. The major result of this section is Theorem 2.4.

In chapter three we let $r(t) \equiv 1$ in (1.1) and consider the asymptotic properties of the resulting equation (3.1). A necessary and sufficient condition is given for the existence of a solution of (3.1) which is asymptotic to t^{2n-1}.

In chapter four Lyapunov's direct method is used to obtain nonoscillation criteria when the conditions on $f(t,u)$ are weakened. The results derived here agree with those obtained in section two.

Chapter five deals with a more general nonlinear delay equation of order 2n:

$$(5.1) \qquad D^n[r(t)D^n y](t) + \sum_{i=1}^{N} f_i(t)F_i[\mu^*(t\epsilon_{2n} - \tau_i(t))] = 0$$

The results of this chapter depend on the assumption of either (i) the existence of an index j for which F_j has some degree of superhomogeneity; or (ii) the existence of two indices j,k for which $x_k^{-1}F_j$ has prescribed monotone properties.

Oscillation Properties of a Delay Differential Equation of Order 2n

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1972

675809

ACKNOWLEDGEMENTS

The author wishes to express his deep gratitude to Dr. Pui-Kei Wong for his encouragement and patience during the writing of this thesis.

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Chapter 1. Introduction

The main purpose of this paper is to discuss the oscillatory and nonoscillatory behavior of solutions of the 2n-th order delay differential equation[1]

$$(1.1) \qquad D^n[r(t)D^n y](t) + y_\tau(t) f(t, y_\tau(t)) = 0,$$

where $y_\tau(t) = y[t - \tau(t)]$, $0 \le \tau(t) \le T$, and $0 < m \le r(t) \le M$. Throughout chapters two and three $f(t,u)$ is assumed to satisfy the following three hypotheses:

(i) $f(t,u)$ is a continuous real-valued function on $[0,\infty) \times R$;

(ii) for each fixed $t \in [0,\infty)$, $f(t,u) < f(t,v)$ for $0<u<v$; and

(iii) for each fixed $t \in [0,\infty)$, $uf(t,u) > 0$ for $u \neq 0$.

In chapter four, these restrictions on $f(t,u)$ shall be relaxed and replaced by others as indicated there.

Existence and uniqueness theorems for solutions of (1.1) are well known, cf. [3], chapter 1. The basic initial value problem is usually stated in terms of a first order system

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[1]For typographical reasons the operator notation will be used consistently with the possible exception of an occasional y' or y''. We have

$$(D^s y)(t) = D^s y(t) = y^{(s)}(t) = \frac{d^s y}{dt^s} \text{ and}$$

$$D^s y_\tau(t) = (D^s y)(t - \tau(t)) = y^{(s)}(t - \tau(t)).$$

$$y'(t) = f[t, y(t), y_T(t)], \quad t \geq t_o$$

(1.2) $$y(t) = \phi(t), \quad t_o - T \leq t \leq t_o,$$

where $y(t)$ is an m-vector, f a given continuous m-vector, and $\phi(t)$ is a given continuous m-vector function on $[t_o - T, t_o]$. By converting (1.1) into a first order system of the form (1.2), existence and uniqueness results for (1.2) may then be applied to (1.1). Briefly, a solution of (1.1) for $t \geq t_o$ is uniquely determined by $(2n-1)$ continuous initial functions $\phi_k(t)$ satisfying $D^k y(t) = \phi_k(t)$ for $t_o - T \leq t \leq t_o$, where we usually require that $\phi_k(t_o) = D^k y(t_o + 0)$, $k = 0, 1, \ldots, 2n-1$. Throughout this paper a solution of (1.1) is understood to mean a solution which can be continued indefinitely.

Oscillation theory of ordinary differential equations originated with the fundamental investigations of Sturm in the nineteenth century. In recent decades, the subject has been broadened in many directions, and the study of oscillations of functional differential equations is one of them. In the paragraphs below we shall state some basic definitions and notions needed in the sequel.

A solution $y(t)$ of (1.1) is said to be oscillatory on $[0, \infty)$ if for each $t_o > 0$, there exists a $T_o > t_o$ such that $y(T_o) = 0$; it is called nonoscillatory otherwise. Following Kiguradze [5] we say that a solution $y(t)$ is of type A_j if

$$D^k y(t) \ge 0, \quad k = 0,1,\ldots,2j+1 \quad \text{and}$$
$$(-)^{k+1} D^k y(t) \ge 0, \quad k = 2j+2,\ldots,2n$$

for all t sufficiently large. In an analogous manner we shall say that $y(t)$ is of type B_j if the derivatives of y and y_1 have certain sign properties, where $y_1(t) = r(t)D^n y(t)$. Specifically, if n is even and $j \le (n-2)/2$ or n is odd and $j \le (n-3)/2$, we require that

$$D^k y(t) \ge 0, \quad k = 0,\ldots, 2j+1;$$
$$(-)^{k+1} D^k y(t) \ge 0, \quad k = 2j+2,\ldots,n; \quad \text{and}$$
$$(-)^{n+k+1} D^k y_1(t) \ge 0, \quad k = 0,\ldots,n$$

If n is even and $j \ge n/2$ or n is odd and $j \ge (n+1)/2$, we require that

$$D^k y(t) \ge 0, \quad k = 0,\ldots,n;$$
$$D^k y_1(t) \ge 0, \quad k = 0,\ldots,2j-n+1; \quad \text{and}$$
$$(-)^{n+k-1} D^k y_1(t) \ge 0, \quad k = 2j-n+2,\ldots,n$$

Finally, if n is odd and $j = (n-1)/2$, $y(t)$ will be of type B_j if

$$D^k y(t) \ge 0, \quad k = 0,\ldots,n \quad \text{and}$$
$$(-)^k D^k y_1(t) \ge 0, \quad k = 0,\ldots,n.$$

When $r(t) \equiv 1$, these definitions reduce to the definition of an A_j-solution. In [5] Kiguradze proved a fundamental lemma which we state as follows:

Lemma 1.1. Let $u(t)$ be a continuous nonnegative function on $(0,\infty)$ with continuous derivatives up to order $2n$ inclusive which do not change sign on this interval. If $D^{2n}u(t) \le 0$, then there exists a number $0 \le p \le 1$ such that

$$D^k u(t) \ge 0, \quad k = 0,\ldots,\ell$$

$$(-)^{k+1} D^k u(t) \ge 0, \quad k = \ell+1,\ldots,2n$$

where $\ell = 2p+1$. Furthermore, $0 \le D^{\ell} n(t) \le \frac{\ell!}{t^{\ell}} u(t)$.

In view of this result, all nonoscillatory solutions of (1.1) with $r(t) \equiv 1$ are of type A_j, $j = 0,\ldots,n-1$. In the general case, we argue as follows: Suppose $y(t)$ is a nonoscillatory solution of (1.1), which we may assume to be nonnegative because of (iii). First of all, no two successive derivatives of y or y_1 can be negative. To see this we suppose $D^k y$ and $D^{k+1} y$ are negative for large t, then there are constants $C_o > 0$ and $t_o > 0$ for which $D^k y$ is a negative decreasing function on $[t_o,\infty)$ and $D^k y(t) < -C_o$ for $t \ge t_o$. Hence,[2]

$$D^{k-1}y(t) - D^{k-1}y(t_o) = \int_{t_o}^{t} D^k y\,ds < -C_o \int_{t_o}^{t} ds = -C_o(t-t_o),$$

which implies that $\lim_{t\to\infty} D^{k-1}y(t) = -\infty$. Proceeding inductively and using the fact that $D^k y$ and $D^{k-1} y$ are eventually negative, we conclude that $y(t) < 0$ for large t, which is a contradiction. A similar argument establishes the claim for

[2] During an integration the variable in a differential expression will be suppressed if there is no resulting ambiguity.

the derivatives $D^k y_1$ and $D^{k+1} y_1$, $k \geq 1$. Of special interest is the case in which y_1 and Dy_1 are negative. Then there exist constants $C_o > 0$ and $t_o > 0$ for which y_1 is a negative decreasing function on $[t_o, \infty)$ and $y_1(t) < -C_o$ for $t \geq t_o$. Hence,

$$D^{n-1}y(t) - D^{n-1}y(t_o) = \int_{t_o}^{t} D^n y \, ds = \int_{t_o}^{t} \frac{y_1(s)}{r(s)} ds < -C_o M^{-1}(t-t_o),$$

which implies that $\lim_{t\to\infty} D^{n-1}y(t) = -\infty$. Since $r(t) > 0$, $D^n y(t) < 0$ for large t. Using this and the fact that $D^{n-1}y(t) < 0$ for large t, we proceed as in the first part of the argument to conclude that $y(t)$ is eventually negative, which is again a contradiction.

Secondly, if two successive derivatives of y or y_1 are positive, than all preceding derivatives of y or y_1 are positive. If $D^k y$ and $D^{k+1} y$ are positive for large t, then there are constants $C_1 > 0$ and $t_1 > 0$ for which $D^k y$ is a positive increasing function on $[t_1, \infty)$ and $D^k y(t) > C_1$ for $t \geq t_1$. Hence,

$$D^{k-1}y(t) - D^{k-1}\, y(t_1) = \int_{t_1}^{t} D^k y \, ds > C_1(t - t_1),$$

which implies that $\lim_{t\to\infty} D^{k-1}y(t) > +\infty$. A similar argument establishes the claim for the derivatives $D^k y_1$ and $D^{k+1} y_1$, $k \geq 1$. Now consider the case that y_1 and $D^1 y_1$ are eventually positive. Then there exist constants $C_1 > 0$ and $t_1 > 0$ for which y_1 is a positive increasing function on $[t_1, \infty)$ and $y_1(t) > C_1$ for $t \geq t_1$. Hence,

$$D^{n-1}y(t) - D^{n-1}y(t_1) = \int_{t_1}^{t} D^n y\,ds = \int_{t_1}^{t} \frac{y_1(s)}{r(s)}\,ds > C_1M^{-1}(t-t_1),$$

which implies that $\lim_{t\to\infty} D^{n-1}y(t) = +\infty$. Since $r(t) > 0$, $D^n y(t) > 0$ for large t. Using this and the fact that $D^{n-1}y(t) > 0$ for large t, we proceed as in the first part of the argument to conclude that $D^k y(t) > 0$, $k = 0,\ldots,n$.

It follows from these two observations that if y is a positive nonoscillatory solution of (1.1); then it is of type B_j for some $j = 0,\ldots,n-1$.

In chapter two integral criteria are given for the nonexistence of solutions of type B_j as well as for the oscillation of all solutions of (1.1). Criteria for the nonexistence of solutions of type A_j then follows as corollaries. Papers presenting integral criteria for the oscillation of second order delay equations are extensive. The equation

$$(1.3) \qquad y''(t) + p(t)y_\tau^\gamma(t) = 0$$

has been the subject of numerous studies. Gollwitzer [4] separated his study of (1.3) into two cases: $\gamma > 1$ or $\gamma < 1$, see also Wong [10]. Bradley [1] has also recently considered the case $\gamma = 1$. It is of interest to study (1.1) as one generalization of (1.3).

Chapter three provides a necessary and sufficient condition for the existence of a nonoscillatory solution of (1.1) with $r(t) \equiv 1$ having prescribed asymptotic behavior. Parallel results for the case of fourth order linear equations may be found in Leighton and Nehari [7] while that of a class of nonlinear fourth order equations is in Wong [11].

In chapter four Lyapunov's direct method is used to obtain nonoscillation criteria when conditions (ii) and (iii) are replaced by weaker assumptions. This method was employed recently by Yoshizawa [12] to study the oscillatory behavior of a nonlinear second order differential equation. In this paper we show that his method is applicable to equations of order $2n$ with retarded arguments.

Chapter five deals with a more general nonlinear equation of order $2n$ in which the function $f(t,y_\tau(t))$ is replaced by a sum of products of functions of the form:

$$f_1(t)\, F_i\, (y_{\tau_{i,1}}(t), Dy_{\tau_{i,2}}(t), \ldots, D^{n-1}y_{\tau_{i,n}}(t), y_{1\tau_{i,n+1}}(t), \ldots, D^{n-1}y_{1\tau_{i,2n}}(t)).$$

Here the analysis is simplified by the separation of f into a function of t and a function of the derivatives of y and y_1, where the variables have been retarded. Ostensibly, the problem is more complicated because f has been replaced by a function of $2n+1$ variables and because there are $2n$ different delay terms. The major difference is in the assumption of (i) the existence of an index j for which F_j has some degree of super-homogeneity; or (ii) the existence of two indices j,k for which $x_k^{-1}F_j$ has prescribed monotone properties. The second order case $(n = 1)$ with $r(t) \equiv 1$ was considered by Staikos and Petsoulas [9] subject to

(i) $F_j(\lambda x, \lambda y) = \lambda^{2p+1} F_j(x,y)$ for every $(x,y) \in R^2$, $\lambda \in R$ and some integer $p > 0$; (ii) $[F_i(x,0)]/x$ is nonincreasing on $(0,\infty)$. We will assume tacitly that $n \geq 2$.

For recent related results, see the papers of Burkowski [2], Ladas [6], Wong [10], as well as the book by Norkin [8].

Chapter 2. Integral Criteria for Oscillation

In this chapter we prove some results on the nonexistence of solutions of type B_j which in turn give rise to oscillation criteria for (1.1).

The following lemmas are useful in obtaining the proof of such a nonoscillation theorem.

<u>Lemma 2.1</u>. Let $y(t)$ be a solution of (1.1) of type B_{n-1}. Then for sufficiently large t, the following estimates are valid:

(a) $t(D^{n-1}y_1)(t) \le 2(D^{n-2}y_1)(t)$;

(b) $t(D^{n-k}y_1)(t) \le 2k(D^{n-k-1}y_1)(t)$, $k = 1,\dots,n-1$;

(c) $t\, y_1(t) \le 2nM(D^{n-1}y)(t)$; and

(d) $t(D^{n-k}y)(t) \le 2(nMm^{-1} + k)(D^{n-k-1}y)$, $k = 1,\dots,n-1$,

where $y_1(t) = r(t)(D^n y)(t)$.

<u>Proof</u>: Suppose $y(t)$ is a solution of type B_{n-1}. Then there is a $T_o > 0$ such that $D^k y$, $k = 0,\dots,n-1$ and $D^k y_1$, $k = 0,\dots,n-1$, are positive for $t > T_o$. Hence $y_\tau(t) > 0$ for $t - \tau(t) > T_o$, i.e., for $t > T_o + T = T_1$. From (1.1) we have

$$D^n[r(t)D^n y](t) = - y_\tau(t) f(t, y_\tau(t))$$

so that $D^n y_1 < 0$ for $t > T_1$. Thus $D^{n-1} y_1$ is a positive decreasing function on (T_1, ∞) and

$$(2.1)\qquad (D^{n-2}y_1)(t) \geq (D^{n-2}y_1)(t) - (D^{n-2}y_1)(T_1) = \int_{T_1}^{t} D^{n-1}y_1 ds$$

$$\geq \int_{T_1}^{t} (D^{n-1}y_1)(t)ds = (t-T_1)(D^{n-1}y_1)(t).$$

Since $t - T_1 \geq \frac{1}{2}t$ for $t \geq 2T_1$, we have $(D^{n-2}y_1)(t) \geq \frac{1}{2}t(D^{n-1}y_1)(t)$ for $t \geq 2T_1$ which proves (a).

To prove (b) we proceed inductively and suppose that

$$(2.2)\qquad (t-T_1)(D^{n-k}y_1)(t) \leq k(D^{n-k-1}y_1)(t), \quad t \geq T_1$$

for some k, $1 \leq k \leq n-2$. An integration of (2.2) yields

$$[(s-T_1)D^{n-k-1}y_1]_{T_1}^{t} - \int_{T_1}^{t} D^{n-k-1}y_1\, ds = \int_{T_1}^{t} (s-T_1)D^{n-k}y_1\, ds$$

$$\leq k\int_{T_1}^{t} D^{n-k-1}y_1\, ds.$$

Hence,

$$(t-T_1)(D^{n-k-1}y_1)(t) \leq (k+1)\int_{T_1}^{t} D^{n-k-1}y_1\, ds$$

$$(2.3)\qquad = (k+1)[(D^{n-k-2}y_1)(t) - D^{n-k-2}y_1)(T_1)]$$

$$\leq (k+1)(D^{n-k-2}y_1)(t)$$

for $t > T_1$ and $\frac{1}{2}t\,(D^{n-k-1}y_1)(t) \leq (k+1)(D^{n-k-2}y_1)(t)$. Thus (2.2) is valid for all k, $1 \leq k \leq n-1$, and (b) is proved. In particular, for $k = n-1$, (2.2) becomes

$$(2.2)'\qquad (t-T_1)(Dy_1)(t) \leq (n-1)y_1(t), \quad t \geq T_1.$$

Integrating this, one gets

$$[(s-T_1)y_1(s)]_{T_1}^{t} - \int_{T_1}^{t} y_1(s)ds = \int_{T_1}^{t} (s-T_1)Dy_1\, ds \leq (n-1)\int_{T_1}^{t} y_1(s)ds.$$

Since $r(t) \le M$ and $D^{n-1}y(T_1) > 0$, we have

$$(t - T_1)y_1(t) \le n \int_{T_1}^{t} r(s)\ D^n\ y\ ds$$
$$\le nM[D^{n-1}y(t) - D^{n-2}y(T_1)]$$
$$\le nM(D^{n-1}y)(t)$$

for $t \ge T_1$ and $t\,y_1(t) \le 2nM(D^{n-1}y)(t)$ for $t \ge 2T_1$, which proves (c).

Furthermore, $y_1(t) = r(t)D^n y(t)$ and $r(t) \ge m$ so that for $t \ge T_1$

$$\int_{T_1}^{t} (s-T_1)y_1(s)\,ds \ge m \int_{T_1}^{t} (s-T_1)D^n y(s)\,ds$$
$$= m(t - T_1)D^{n-1}y(t) - m \int_{T_1}^{t} D^{n-1}y(s)\,ds.$$

Combining this with (2.3) one gets

$$m(t - T_1)D^{n-1}y(t) \le m \int_{T_1}^{t} D^{n-1}y(s)\,ds + nM \int_{T_1}^{t} D^{n-1}y(s)\,ds$$
$$= (m + nM) \int_{T_1}^{t} D^{n-1}y(s)\,ds$$
$$= (m + nM)\ [D^{n-2}y(t) - D^{n-1}y(T_1)]$$
$$\le (m + nM)D^{n-2}y(t).$$

It follows that for $t \ge T_1$

$$(t - T_1)D^{n-1}\,y(t) \le (nMm^{-1} + 1)D^{n-2}\,y(t)$$

and for $t \ge 2T_1$

$$t\ D^{n-1}y(t) \le 2(nMm^{-1} + 1)D^{n-2}y(t).$$

To prove the final assertion, we proceed inductively and assume that for $t \ge T_1$,

$$(t - T_1)D^{n-k}y(t) \le (nMm^{-1} + k)D^{n-k-1}y(t)$$

for some $k, 1 \le k \le n-2$. Integrating (2.5) for $t \ge T_1$ yields

$$[(s - T_1)D^{n-k-1}y]_{T_1}^{t} - \int_{T_1}^{t} D^{n-k-1}y\,ds = \int_{T_1}^{t} (s - T_1)D^{n-k}y\,ds$$

$$\le (nMm^{-1} + k)\int_{T_1}^{t} D^{n-k-1}y\,ds$$

so that

$$(t - T_1)D^{n-k-1}y(t) \le [nMm^{-1} + (k+1)]\int_{T_1}^{t} D^{n-k-1}y\,ds$$

$$\le [nMm^{-1} + (k+1)](D^{n-k-2}y(t) - D^{n-k-2}y(T_1))$$

$$\le [nMm^{-1} + (k+1)]\,D^{n-k-2}y(t)$$

for $t \ge T_1$, which implies that (d) is valid for $t \ge 2T_1$.

The investigation of similar inequalities for solutions of type B_{n-k} $(k = 2,\dots,n)$ is slightly more complicated. There are two cases.

Suppose n is an even integer and suppose $y(t)$ is a solution of (1.1) of type B_j, where $j \le \frac{n-2}{2}$. Then $2j + 2 \le n$ and the first negative derivative is $D^{2j+2}y$. By applying the same procedure as in Lemma 1.1, one obtains for $k = 1,\dots,2j+1$

(2.4a) $\quad (t - T_1)D^{2j+2-k}y(t) \le k\,D^{2j+1-k}y(t),\ t \ge T_1$ and

(2.4b) $\quad t\,D^{2j+2-k}y(t) \le 2k\,D^{2j+1-k}y(t),\ t \ge 2T_1.$

If $j > \frac{n-2}{2}$, i.e., if $j \ge n/2$, then $2j+2 \ge n+2$ and the first negative derivative is $D^{2j+2-n}y_1$. We obtain for $t \ge T_1$:

(2.5a) $(t-T_1)D^{2j+2-n-k}\, y_1(t) \le k\, D^{2j+1-n-k}\, y_1(t)$, $k=1,\ldots,2j-n+1$;

(2.5b) $(t-T_1)\, y_1(t) \le (2j-n)M\, D^{n-1}\, y(t)$;

and

(2.5c) $(t-T_1)\, D^{n-k}\, y(t) \le [(2j-n)Mm^{-1}+k]D^{n-k-1}\, y(t)$,

where $k = 1,\ldots,n-1$. Moreover, for $t \ge 2T_1$

(2.6a) $tD^{2j+2-n-k}y_1(t) \le 2kD^{2j+1-n-k}y_1(t)$, $k = 1,\ldots,2j-n-1$;

(2.6b) $ty_1(t) \le 2(2j-n)MD^{n-1}\, y(t)$;

and

(2.6c) $(t-T_1)D^{n-k}y(t) \le 2[(2j-n)Mm^{-1} + k]D^{n-k-1}\, y(t)$,

where $k = 1,\ldots,n-1$.

We remark that if n is an odd integer and $j \le \frac{n-3}{2}$, then $2j+2 \le n-1$ so that the inequalities (2.5) are valid. If n is an odd integer and $j \ge \frac{n+1}{2}$ then $2j+2 \ge n+3$ and the inequalities (2.6) and (2.7) are valid. For $j = \frac{n-1}{2}$, $2j+2-n = 1$, so Dy_1 is the first negative derivative. We obtain for $t > T_1$

(2.7a) $(t-T_1)\, y_1(t) \le MD^{n-1}y(t)$

and

(2.7b) $(t-T_1)D^{n-k}y(t) \le (Mm^{-1}+k)D^{n-k-1}\, y(t)$, $k = 1,\ldots,n-1$.

Hence, for $t \ge 2T_1$

(2.8a) $t\, y_1(t) \le 2MD^{n-1}\, y(t)$

and

(2.8b) $tD^{n-k}\, y(t) \le 2\,(Mm^{-1}+k)D^{n-k-1}\, y(t)$, $k = 1,\ldots,n-1$.

For $T_1 = 0$ the results of Lemma 2.1 may be improved to yield

(a), (b) $t\, D^{n-k}\, y_1(t) \le k\, D^{n-k-1}\, y_1(t)$, $k = 1,\dots,n-1$;

(c) $t y_1(t) \le nMD^{n-1}\, y(t)$; and

(d) $t\, D^{n-k}\, y(t) \le (nMm^{-1} + k) D^{n-k-1} y(t)$, $k = 1,\dots,n-1$

for large t. If $r(t) \equiv 1$, $\tau(t) \equiv 0$ and $y(t)$ is a solution of type B_{n-1} on $(0,\infty)$, then $T_1 = 0$ and $m = M = 1$ so that

$$t\, D^{2n-k} y(t) \le k\, D^{2n-k-1}\, y(t), \quad k = 1,\dots,2n-1,$$

which is a special case of Kiguradze's Lemma [5]. Similarly, if $r(t) \equiv 1$, $\tau(t) \equiv 0$ and $y(t)$ is a solution of type B_j on $(0,\infty)$, then

$$t\, D^{2j+2-k}\, y(t) \le k\, D^{2j+1-k}\, y(t), \quad k = 1,\dots,2j+1.$$

<u>Lemma 2.2</u>. Let $y(t)$ be a solution of (1.1) that is ultimately positive.

(a) Suppose n is even and $j \le \frac{n-2}{2}$ or n is odd and $j \le \frac{n-1}{2}$. If $y(t)$ is a solution of type B_j, then there are constants $k > 0$ and $t_o > 0$ such that

$$\frac{D^{2j}\, y_\tau(t)}{D^{2j}\, y(t)} \ge k, \quad t \ge t_o.$$

(b) Suppose n is even and $j \ge \frac{n}{2}$ or n is odd and $j \ge \frac{n+1}{2}$. If $y(t)$ is a solution of type B_j, then there are constants $K > 0$ and $t_1 > 0$ such that

$$\frac{D^{2j-n}\ y_{1_\tau}(t)}{D^{2j-n}\ y_1(t)} \geq K, \quad t \geq t_1.$$

Remark 1. Part (a) of this lemma is analogous to one proved by Bradley [1] for the equation

$$y''(t) + p(t)y_\tau(t) = 0.$$

Since his proof depends only on the concavity of y, Lemma 2.2 follows easily for in (a), $D^{2j}\ y$ is concave and in (b), $D^{2j-n}\ y_1$ is concave.

Remark 2. The above result may, however, be obtained via Lemma 2.1 and the observations following it.

Proof: Let $y(t)$ be a solution of type B_j where n is even and $j \leq \frac{n-2}{2}$ or n is odd and $j \leq \frac{n-3}{2}$. For $t \geq T_1$, $D^{2j+1}\ y(t) > 0$. Since $\tau(t) \geq 0$,

$$D^{2j}y_\tau(t) = D^{2j}y(t-\tau(t)) \leq D^{2j}y(t)$$

so that with the help of (2.5b), we have

$$\left|\frac{D^{2j}y_\tau(t)}{D^{2j}y(t)} - 1\right| = \frac{D^{2j}y(t) - D^{2j}y_\tau(t)}{D^{2j}y(t)} = \tau(t)\ \frac{D^{2j+1}y(s)}{D^{2j}y(t)}$$

$$\leq T\ \frac{D^{2j+1}y(s)}{D^{2j}\ y(t)} \leq \frac{2T}{s}\ \frac{D^{2j}y(s)}{D^{2j}y(t)}$$

$$\leq \frac{2T}{s},$$

where $t-\tau(t) \leq s \leq t$. Since s tends to infinity with t, we obtain

$$\lim_{t\to\infty} \frac{D^{2j}y_\tau(t)}{D^{2j}y(t)} = 1 .$$

If n is odd and $j = \frac{n-1}{2}$, we observe that for $t > T_1$, $D^{n-1}y(t)$ and $D^n y(t)$ are both positive. Since $r(t) > 0$, $y_1(t) > 0$ and $Dy_1(t) < 0$, we have by a similar argument that

$$\left|\frac{D^{2j}y_\tau(t)}{D^{2j}y(t)} - 1\right| = \frac{D^{2j}y(t) - D^{2j}y_\tau(t)}{D^{2j}y(t)} = \tau(t)\,\frac{D^n y(s)}{D^{n-1}y(t)}$$

$$\leq \frac{T\,y_1(s)}{r(s)D^{n-1}y(t)} \leq \frac{2MT}{ms}\,\frac{D^{n-1}y(s)}{D^{n-1}y(t)}$$

$$\leq \frac{2MT}{ms},$$

where $t - \tau(t) \leq s \leq t$. The conclusion then follows as in the previous case.

If n is even and $j \geq n/2$ or n is odd and $j \geq (n+1)/2$, we note that for $t > T_1$: $D^{2j-n+1}y_1(t) > 0$. Since $\tau(t) \geq 0$,

$$D^{2j-n}y_{1_\tau}(t) = D^{2j-n}y_1(t-\tau(t)) \leq D^{2j-n}y_1(t)$$

so that

$$\left|\frac{D^{2j-n}y_{1_\tau}(t)}{D^{2j-n}y_1(t)} - 1\right| = \frac{D^{2j-n}y_1(t) - D^{2j-n}y_{1_\tau}(t)}{D^{2j-n}y_1(t)} = \tau(t)\,\frac{D^{2j-n+1}y_1(s)}{D^{2j-n}y_1(t)}$$

$$\leq T\,\frac{D^{2j-n+1}y_1(s)}{D^{2j-n}y_1(t)} \leq \frac{2T}{s}\,\frac{D^{2j-n}y_1(s)}{D^{2j-n}y_1(t)}$$

$$\leq \frac{2T}{s},$$

where $t - \tau(t) \leq s \leq t$. Part (b) now follows and the lemma is proved.

Remark 3. Lemma 2.1 is clearly valid for $\tau(t)$ unbounded if, in addition, $\lim_{t\to\infty}(t-\tau(t)) = +\infty$. It is of interest to note that Lemma 2.2 is valid also even if $\tau(t)$ is unbounded provided $0 \le \tau(t) < \mu t$, where μ will be specified below. Since $t-\tau(t) \le s$, we obtain in the first case of Remark 2

$$\left|\frac{D^{2j}y_\tau(t)}{D^{2j}y(t)} - 1\right| = \tau(t)\,\frac{D^{2j+1}y(s)}{D^{2j}y(t)} \le \tau(t)\,\frac{D^{2j+1}y(s)}{D^{2j}\;y(s)}$$

$$\le \frac{\tau(t)}{s-T_1} \le \frac{\tau(t)}{t-\tau(t)-T_1} .$$

In the second case,

$$\left|\frac{D^{2j}y_\tau(t)}{D^{2j}y(t)} - 1\right| = \frac{\tau(t)y_1(s)}{r(s)D^{n-1}y(t)} \le \frac{\tau(t)}{m}\,\frac{y_1(s)}{D^{n-1}y(s)}$$

$$\le Mm^{-1}\,\frac{\tau(t)}{s-T_1} \le Mm^{-1}\,\frac{\tau(t)}{t-\tau(t)-T_1} .$$

In the third case,

$$\left|\frac{D^{2j-n}y_{1_\tau}(t)}{D^{2j-n}y_1(t)} - 1\right| = \tau(t)\,\frac{D^{2j-n-1}y_1(s)}{D^{2j-n}y_1(t)} \le \tau(t)\,\frac{D^{2j-n-1}y_1(s)}{D^{2j-n}y_1(s)}$$

$$\le \frac{\tau(t)}{s-T_1} \le \frac{\tau(t)}{t-\tau(t)-T_1} .$$

We note that $\frac{\tau(t)}{t-\tau(t)-T_1} \le 1-k$ for any $0 < k < 1$ provided $\tau(t) \le \frac{1-k}{2-k}(t-T_1)$. Moreover, for any $0 < \epsilon < \frac{1}{2}$, there is a $0 < k < 1$ satisfying

$$\frac{1-k}{2-k} = \frac{1}{2} - \epsilon .$$

Thus, in the first and third cases, if $0 \le \tau(t) \le (\frac{1}{2}-\epsilon)(t-T_1)$

for some $0 < \epsilon < \frac{1}{2}$; then there is a $0 < k < 1$ for which

$$\left|\frac{D^{2j}y_{\tau}(t)}{D^{2j}y(t)} - 1\right| \le 1 - k \quad \text{or} \quad \left|\frac{D^{2j-n}y_{1\tau}(t)}{D^{2j-n}y_1(t)} - 1\right| \le 1-k$$

respectively, which implies that $D^{2j}y_{\tau}(t) \ge kD^{2j}y(t)$ or $D^{2j-n}y_{1\tau}(t) \ge kD^{2j-n}y_1(t)$.

In a similar manner $\frac{\tau(t)}{t - \tau(t) - T_1} \le \frac{m}{M}(t - T_1)$ for any $0 < k < 1$ provided $\tau(t) \le \frac{m}{M}\frac{(1-k)}{[1+\frac{m}{M}(1-k)]}(t - T_1)$. Moreover, for any $0 < \epsilon < \frac{m}{M+m}$, there is a $0 < k < 1$ satisfying

$$\frac{m}{M}\frac{1-k}{[1+\frac{m}{M}(1-k)]} = \frac{m}{M+m} - \epsilon .$$

Thus, in the second case, if $0 \le \tau(t) \le [\frac{m}{M+m} - \epsilon](t - T_1)$ for some $0 < \epsilon < \frac{m}{M+m}$, there is a $0 < k < 1$ for which

$$\left|\frac{D^{2j}y_{\tau}(t)}{D^{2j}y(t)} - 1\right| \le 1 - k,$$

which implies that $D^{2j}y_{\tau}(t) \ge k\, D^{2j}y(t)$.

Using the two lemmas of this section, we may prove the following result.

Theorem 2.1. Suppose that for some $j = 0,1,\ldots,n-1$ and for all constants $C > 0$

$$\int^{\infty} t^{2j}f(t,C)\,dt = +\infty .$$

Then (1.1) has no solutions of type B_j.

Proof: Suppose $y(t)$ is a solution of type B_j, where n is even and $j \le (n-2)/2$ or n is odd and $j \le (n-3)/2$. Let

$$w(t) = \frac{D^{n-1}y_1(t)}{D^{2j}y(t)} \quad .$$

Then we see from (1.1) that

$$w'(t) + \frac{D^{n-1}y_1(t)\ D^{2j+1}y(t)}{[D^{2j}y(t)]^2} + \frac{y_\tau(t)}{D^{2j}y(t)}\ f(t,y_\tau(t)) = 0.$$

Since $D^{n-1}y_1(t)$ and $D^{2j+1}y(t)$ are positive for $t > T_1$

$$w'(t) + \frac{y_\tau(t)}{D^{2j}y(t)}\ f(t,y_\tau(t)) \le 0. \tag{2.9}$$

From (2.4) we obtain

$$t^{2j}\ D^{2j}\ y(t) \le 2^{2j}\ (2j+1)!\ y(t)$$

for $t > 2T_1$ so that for $t > 2T_1 + T = T_2$,

$$(t - \tau(t))^{2j}\ D^{2j}\ y_\tau(t) \le 2^{2j}\ (2j+1)!\ y_\tau(t).$$

Setting $N_1 = [2^{2j}(2j+1)!]^{-1}$ and using the fact that $0 \le \tau(t) \le T$, we can rewrite this as

$$y_\tau(t) \ge N_1(t - T)^{2j}\ D^{2j}y_\tau(t), \quad t > T_2 \ .$$

Combining this with (2.9), we have

$$w'(t) + N_1(t - T)^{2j}\ \frac{D^{2j}y_\tau(t)}{D^{2j}y(t)}\ f(t,y_\tau(t)) \le 0, \quad t > T_2 \ .$$

Since $Dy(t) > 0$ for $t > T_1$, there is a $t_0 > T_2$ and a $C > 0$ such that $y_\tau(t) \ge C$ for $t \ge t_0$. Hence

$$w'(t) + N_1(t - T)^{2j}\ \frac{D^{2j}y_\tau(t)}{D^{2j}y(t)}\ f(t,C) \le 0, \quad t \ge t_0 \ .$$

By Lemma 2.2 (a) there is a constant $k > 0$ and a $t_1 > t_o$ such that $D^{2j}y_\tau(t) \geq k\, D^{2j}y(t)$ for $t \geq t_1$. Thus

$$(2.10) \qquad w'(t) + N_1 k(t-T)^{2j} f(t,C) \leq 0, \quad t \geq t_1 .$$

An integration then yields

$$w(t) - w(t_1) + N_1 k \int_{t_1}^{t} (s-T)^{2j} f(s,C)\,ds \leq 0.$$

In view of the hypothesis, we must ultimately have $w(t) < 0$, which implies that $D^{n-1}y_1(t) < 0$ for large t. This contradicts the assumption that y is a B_j-solution.

Now suppose that n is even and $j \geq n/2$ or that n is odd and $j \geq (n+1)/2$. Let

$$w(t) = \frac{D^{n-1}y_1(t)}{D^{2j-n}y_1(t)} .$$

Then we see from (1.1) that

$$w'(t) + \frac{D^{n-1}y_1(t)\; D^{2j-n+1}y_1(t)}{[D^{2j-n}y_1(t)]^2} + \frac{y_\tau(t)}{D^{2j-n}y_1(t)} f(t,y_\tau(t)) = 0.$$

Since $D^{n-1}y_1(t)$ and $D^{2j-n+1}y_1(t)$ are positive for $t > T_1$, we have

$$(2.11) \qquad w'(t) + \frac{y_\tau(t)}{D^{2j-n}y_1(t)} f(t,y_\tau(t)) \leq 0, \quad t > T_1 .$$

From (2.6) we obtain

$$t^{2j}D^{2j-n}y_1(t) \leq 2^{2j}(2j-n)!\; M \prod_{j=1}^{n-1} [(2j-n)Mm^{-1} + j]\; y(t)$$

for $t > 2T_1$ so that for $t > 2T_1 + T = T_2$,

$$(t-\tau(t))^{2j}D^{2j-n}y_{1_\tau}(t) \le 2^{2j}(2j-n)!\ M \prod_{j=1}^{n-1}[(2j-n)Mm^{-1}+j]y_\tau(t).$$

Setting

$$N_2^{-1} = 2^{2j}(2j-n)!\ M \prod_{j=1}^{n-1}[(2j-n)Mm^{-1} + j]$$

and using the fact that $0 \le \tau(t) \le T$, we can rewrite this as

$$y_\tau(t) \ge N_2(t-T)^{2j}D^{2j-n}y_{1_\tau}(t), \quad t > T_2 .$$

Combining this with (2.11) we get

$$w'(t) + N_2(t-T)^{2j}\frac{D^{2j-n}y_{1_\tau}(t)}{D^{2j-n}y_1(t)} f(t,y_\tau(t)) \le 0, \quad t > T_2 .$$

Since $y'(t) > 0$ for $t > T_1$, there is a $t_o > T_2$ and a $C > 0$ such that $y_\tau(t) \ge C$ for $t \ge t_o$. Hence

$$w'(t) + N_2(t-T)^{2j}\frac{D^{2j-n}y_{1_\tau}(t)}{D^{2j-n}y_1(t)} f(t,C) \le 0, \quad t \ge t_o.$$

By Lemma 2.2 (b), there is a constant $K > 0$ and a $t_1 > t_o$ such that $D^{2j-n}y_{1_\tau}(t) \ge KD^{2j-n}y_1(t)$ for $t \ge t_1$. Thus, with N_1 and k replaced by N_2 and K, (2.10) holds as in the first part of the proof and we arrive at the same contradiction.

Finally, if n is odd and $j = \frac{n-1}{2}$, then letting

$$w(t) = \frac{D^{n-1}y_1(t)}{D^{n-1}y(t)} ,$$

we see from (1.1) that

$$w'(t) + \frac{D^{n-1}y_1(t)\ D^n y(t)}{[D^{n-1}y(t)]^2} + \frac{y_\tau(t)}{D^{n-1}y(t)} f(t,y_\tau(t)) = 0.$$

Since $D^{n-1}y_1(t)$ and $D^n y(t)$ are positive for $t > T_1$, one has

$$(2.12) \quad w'(t) + \frac{y_\tau(t)}{D^{n-1}y(t)} f(t, y_\tau(t)) \leq 0, \ t > T_1 .$$

From (2.8) we obtain

$$y_\tau(t) \geq N_3(t - T)^{n-1} D^{n-1} y_\tau(t), \ t > T_2$$

where $T_2 \geq 2T_1 + T$ and $N_3^{-1} = 2^{n-1} \prod_{j=1}^{n-1} (Mm^{-1} + j)$.

Since $y'(t) > 0$ for $t > T_1$, there is a $t_o > T_2$ and a $C > 0$ such that $y_\tau(t) \geq C$ for $t \geq t_o$. Moreover, by Lemma 2.2 (a), there is a constant $k > 0$ and a $t_1 > t_o$ such that $D^{n-1}y_\tau(t) \geq k D^{n-1}y(t)$ for $t \geq t_1$. Thus (2.10) holds once again (with N_1 replaced by N_3), and the conclusion follows as before.

Corollary 2.1. Suppose for all constants $C > 0$

$$\int^\infty f(t, C)\,dt = +\infty .$$

Then all solutions of (1.1) are oscillatory.

Corollary 2.2. Suppose $p(t) > 0$ and

$$\int^\infty p(t)\,dt = +\infty .$$

Then all solutions of the equation

$$(2.13) \quad D^n[r(t)D^n y(t)] + p(t)y_\tau^{2\gamma+1}(t) = 0, \ \gamma \geq 0$$

are oscillatory.

Remark 1. Under the hypothesis $0 < m \leq r(t) \leq M$, the only nonoscillatory solutions of (1.1) are of types $B_o, \ldots, B_{n-1}$.

If y is nonoscillatory of type A_j for some $j = 0,\dots,n-1$, then y is nonoscillatory of type B_k for some $k=0,\dots,n-1$. There are two cases: (i) If n is even and $j \le (n-2)/2$ or n is odd and $j \le (n-3)/2$, then $y, Dy,\dots,D^{2j+1}y$ are all eventually positive and $D^{2j+2}y$ is ultimately negative. This is precisely the case if y is a B_j-solution. Hence $j = k$; (ii) If n is even and $j \ge n/2$ or n is odd and $j \ge (n-1)/2$, then an A_j-solution is a B_k-solution for some $k = j,\dots,n-1$.

If $D^s r$, $s = 0,\dots,n-k$, are positive and $(-)^j D^{(n-k+j)} r>0$ for $j = 1,\dots,k-1$, then an A_k-solution is a B_k-solution. In other words, if $D^k r > 0$ for $k = 0,\dots,j$, $j \le n - 1$ and $(-)^k D^k r > 0$ for $k = j + 1,\dots,n-1$, then an A_j-solution is a B_j-solution.

<u>Remark 2</u>. It is clear that the condition

$$\text{(H)} \qquad \int^{\infty} \frac{dt}{r(t)} = +\infty$$

is sufficient to guarantee that all nonoscillatory solutions of (1.1) are of type B_j for some j.

In view of the previous remarks, we may state without proof the following results.

<u>Theorem 2.2</u>.

(a) Suppose n is even and $k \le (n-2)/2$ or n is odd and $k \le (n-3)/2$. If, for all constants $C > 0$,

$$\text{(2.14)} \qquad \int^{\infty} t^{2k} f(t,C)\,dt = +\infty;$$

then (1.1) has no solutions of type A_k.

(b) Suppose n is even and $k \ge n/2$ or n is odd and $k \ge (n-1)/2$. If $D^j r > 0$, $j = 0,\ldots,(n-k)$; $(-)^j D^j r > 0$ for $j = n-k+1,\ldots,n-1$ and if, for all constants $C > 0$, (2.14) holds, then (1.1) has no solutions of type A_k.

Theorem 2.3.

(a) Suppose n is even and $k \le (n-2)/2$ or n is odd and $k \le (n-3)/2$. If for some $k = 0,\ldots,n-1$

$$\int^{\infty} t^{2k}\, p(t)\,dt = +\infty \; ; \tag{2.15}$$

then (2.13) has no solutions of type A_k.

(b) Suppose n is even and $k \ge n/2$ or n is odd and $k \ge (n-1)/2$. If $D^j r > 0$, $j = 0,\ldots,(n-k)$; $(-)^j D^j r > 0$, $j = n-k+1,\ldots,n-1$; and if (2.15) holds, then (2.13) has no solutions of type A_k.

Letting $\gamma = 0$ and $r(t) \equiv 1$ in (2.13), we obtain results for the equation

$$D^{2n} y(t) + p(t) y_{\tau}(t) = 0$$

analogous to Bradley's results [1] for the linear second order equation

$$y''(t) + p(t) y_{\tau}(t) = 0.$$

An improvement can be made easily in the nonoscillation criteria of Theorem 2.1. We state this as

Theorem 2.4. Equation (1.1) has no solution of type B_j if, for all constants $C > 0$,

$$\int^{\infty} t^{2j} f(t, Ct^{2j})\,dt = +\infty. \tag{2.16}$$

<u>Proof</u>: Suppose $y(t)$ is a solution of type B_j, where n is even and $j \le (n-2)/2$ or n is odd and $j \le (n-3)/2$. Since $D^{2j+1}y(t) > 0$ for $t > T_1$, there is a $t_1^* \ge t_o > T_2$ and a $C^* > 0$ such that $D^{2j}y_T(t) \ge C_1^*$ for $t \ge t_1^*$. Thus

$$y_T(t) \ge N_1 C_1^* (t-T)^{2j}, \quad t \ge t_1^*$$

so that (2.10) becomes

$$w'(t) + N_1 k \ (t-T)^{2j} f(t, N_1 C_1^* (t-T)^{2j}) \le 0$$

for $t \ge t_1$, where we may assume that $t_1 \ge t_1^*$. Letting $C = N_1 C_1^*$ and integrating from t_1 to t, we must ultimately have $w(t) < 0$ which implies that $D^{n-1}y_1 < 0$ for large t, which is absurd.

In a similar manner we can modify the second and third parts of the proof of Theorem 2.1 by observing that in the second part $D^{2j-n+1}y_1(t) > 0$ for $t > T_1$ and in the third part $D^n y(t) > 0$ for $t > T$. Hence there are constants C_2^*, C_3^*, t_2^*, t_3^* for which $D^{2j-n}y_1(t) \ge C_2^*$ if $t \ge t_2^*$ and $D^{n-1}y(t) \ge C_3^*$ if $t \ge t_3^*$, respectively. Thus we have

$$y_T(t) \ge N_2 C_2^* \ (t-T)^{2j}, \ t \ge t_2^* \ \text{ or}$$

$$y_T(t) \ge N_3 C_3^* \ (t-T)^{2j}, \ t \ge t_3^*$$

so that (2.10) becomes

$$w'(t) + N_i K(t-T)^{2j} \ f(t, N_i C_i^* (t-T)^{2j}) \le 0, \quad t \ge t_i^*$$

where $i = 2,3$; taking $C = N_i C_i^*$ and using the hypothesis if follows that in each case an integration from t_1 to t

implies that $w(t) < 0$ for large t which contradicts the fact that y is a B_j-solution.

We may restate Theorem 2.4 as

<u>Theorem 2.4</u>'

(a) Suppose that n is even and $j \le (n-2)/2$ or n is odd and $j \le (n-3)/2$. If for all constants $C > 0$ (2.16) holds; then either (1.1) is oscillatory or for sufficiently large t, $y\, D^{2j}y < 0$.

(b) Suppose that n is even and $j \ge n/2$ or n is odd and $j \ge (n-1)/2$. If for all constants $C > 0$ (2.16) holds; then either (1.1) is oscillatory or for sufficiently large t, $yD^{2j-n}y_1 < 0$.

For $j = n-1$ and $r(t) \equiv 1$, (b) reduces to the alternative that either (1.1) is oscillatory or $yD^{2n-2}y < 0$ which is essentially Theorem 3.1 of Ladas [6].

Chapter 3. The Asymptotic Character of Certain Solutions

In this chapter some special results are given on the asymptotic behavior of solutions of the equation

$$(3.1) \qquad D^{2n}y(t) + f(t,y_\tau(t))y_\tau(t) = 0,$$

where $f(t,u)$ satisfies the three conditions in section one.

<u>Lemma 3.1</u>. Let $y(t)$ be a solution of (3.1) which is eventually positive. Then

$$\lim_{t\to\infty} (2n-1)!t^{-(2n-1)}\, y(t) = \lim_{t\to\infty} D^{2n-1}y(t).$$

<u>Proof</u>: Suppose that $y(t)$ is a solution of (3.1) which is eventually positive. Then there is a $T_1 > 0$ such that $y(t)$ is positive for $t > T_1$. Hence $y_\tau(t) > 0$ for $t - \tau(t) > T_1$, i.e., for $t > T_1 + T = T^*$. Then by Taylor's Theorem with Remainder, for $t > T^*$

$$(3.2) \qquad (2n-1)!\, R(t) = (2n-1)!\, y(t) - \int_{T^*}^{t} (t-s)^{2n-1} D^{2n}y(s)ds$$

$$= (2n-1)!\, y(t) + \int_{T^*}^{t} (t-s)^{2n-1} y_\tau(s)f(s,y_\tau(s))\,ds,$$

where

$$R(t) = \sum_{k=0}^{2n-1} \frac{1}{k!}\, D^k y(T^*)\,(t-T^*)^k .$$

Since $y_\tau(s)$ and hence $-D^{2n}y(s)$ are positive for $s > T^*$, condition three together with $(t - T^*) > (t - s) > 0$ imply that

$$(2n-1)!\,R(t) \le (2n-1)!\,y(t) + (t-T^*)^{2n-1}[D^{2n-1}y(T^*) - D^{2n-1}y(t)].$$

Dividing this by $(t-T^*)^{2n-1}$ and noting that

$$\lim_{t\to\infty} (2n-1)!\ (t-T^*)^{2n-1} R(t) = D^{2n-1}\ y(T^*),$$

it follows upon passage to the limit that

$$(3.3)\quad \lim_{t\to\infty} D^{2n-1}y(t) \le \underline{\lim}_{t\to\infty} (2n-1)!\,(t-T^*)^{-(2n-1)}y(t).$$

We remark that this inequality could also be obtained directly from Lemma 2.1 if y is assumed to be a solution of type B_{n-1}.

To prove the reverse inequality, we choose an η for which $T^* < \eta < t$. By restricting s to lie in the interval $[T^*,\eta]$, $(t-s)^{2n-1} \ge (t-\eta)^{2n-1}$ and

$$(2n-1)!\ R(t) \ge (2n-1)!\,y(t) + (t-\eta)^{2n-1}\int_{T^*}^{\eta} y_T(s) f(s,y_T(s))ds$$

$$= (2n-1)!\,y(t) + (t-\eta)^{2n-1}[D^{2n-1}y(T^*) - D^{2n-1}y(\eta)].$$

Multiplying this by $(t-T^*)^{-(2n-1)}$, keeping η fixed and letting $t\to\infty$ through a sequence of points for which $(t-T^*)^{-(2n-1)}y(t)$ tends to its upper limit, we have

$$D^{2n-1}y(T^*) \ge \overline{\lim}_{t\to\infty} (2n-1)!\,(t-T^*)^{-(2n-1)}y(t) + D^{2n-1}y(T^*) - D^{2n-1}y(\eta)$$

from which it follows that

$$\overline{\lim}_{t\to\infty} (2n-1)!\ (t-T^*)^{-(2n-1)}y(t) \le D^{2n-1}y(\eta).$$

Since η is arbitrary and $\lim_{t\to\infty} D^{2n-1}y(t)$ exists, (3.4) $$\overline{\lim_{t\to\infty}}\ (2n-1)!\ (t-T^*)^{-(2n-1)}y(t) \le \lim_{t\to\infty} D^{2n-1}y(t).$$

By combining (3.3) and (3.4) we obtain the desired result.

<u>Theorem 3.1</u>. Equation (3.1) has a solution $y(t) > 0$ satisfying

(3.5) $$y(t) \sim kt^{2n-1}, \quad 0 < k$$

if, and only if, for all $C > 0$

(3.6) $$\int^{\infty} t^{2n-1} f(t, Ct^{2n-1})dt < \infty .$$

<u>Proof</u>: First suppose that (3.6) holds. Choose $T_o > 0$ sufficiently large so that

$$\int_{T_o}^{\infty} t^{2n-1} f(t, Ct^{2n-1})dt \le (2n-1)! - \frac{1}{2} .$$

Now consider the solution $y(t) = y(t,T_o)$ of (3.1) subject to: $D^k y(T_o) = 0$, $k = 0,1,\ldots,n-$; $D^k y_1(T_o) = 0$, $k = 0,\ldots,n-2$; provided $n \ge 2$; $D^{n-1}y_1(T_o) = 1$ and $y(t) = 0$ for $T_o - T \le t \le T_o$; $y(t,T_o)$ is positive on some open interval whose left-hand endpoint is T_o. Let $t = T_1$ be the first zero of $y(t,T_o)$ in (T_o,∞). By Taylor's Theorem with Remainder

(3.7) $$(t-T_o)^{2n-1} = (2n-1)!y(t,T_o) + \int_{T_o}^{t} (t-s)^{2n-1} y_\tau(s) f(s, y_\tau(s)).$$

Since $y(s) \ge 0$ for $T_o - T \le s \le T_1$, $y_\tau(s) \ge 0$ for $T_o - T \le s - \tau(s) \le T_1$, i.e. for $s \ge T_o - T + \tau(s)$ and hence $y_\tau(s) \ge 0$ for $s > T_o$. A similar argument shows that $y_\tau(s) \ge 0$ for $s < T_1$. Thus

(3.8) $$(2n-1)!y(t) = (2n-1)!y(t,T_o) \le (t-T_o)^{2n-1}, \quad T_o < t < T_1$$

Moreover, letting $t = T_1$ in (3.7)

$$(T_1 - T_o)^{2n-1} = \int_{T_o}^{T_1} (T_1 - s)^{2n-1} y_\tau(s) f(s, y_\tau(s)) ds$$

$$\le (T_1 - T_o)^{2n-1} \int_{T_o}^{T_1} y_\tau(s) f(s, y_\tau(s)) ds.$$

By condition (iii) and (3.8),

$$(2n-1)! y_\tau(s) f(s, y_\tau(s)) \le (s - \sigma(s))^{2n-1} f(s, (2n-1)!^{-1} (s - \sigma(s))^{2n-1})$$

$$\le s^{2n-1} f(s, Cs^{2n-1}),$$

where $C = (2n-1)!^{-1}$ and $\sigma(s) = \tau(s) + T_o$.

Substituting this in the previous inequality, we obtain

$$(2n-1)! \int_{T_o}^{T_1} s^{2n-1} f(s, Cs^{2n-1}) ds \le \int_{T_o}^{\infty} s^{2n-1} f(s, Cs^{2n-1}) ds .$$

This contradicts the initial choice of T_o and demonstrates the existence of a positive nonoscillatory solution $y(t)$. The first half of Theorem 3.1 then follows from Lemma 3.1.

To prove the second assertion, suppose that (3.1) has a positive solution $y(t)$ satisfying (3.5). By Lemma 3.1, $\lim_{t\to\infty} D^{2n-1} y(t) = (2n-1)!k$, so that

$$(3.9) \quad \int_{T_1}^{\infty} y_\tau(s) f(s, y_\tau(s)) ds = - \int_{T_1}^{\infty} D^{2n} y \, ds = D^{2n-1} y(T_1) - (2n-1)!k.$$

The hypothesis (3.5) ensures that for $\epsilon > 0$ given there is a $T^* > T_1$ for which $y(t) > (k - \epsilon) t^{2n-1}$ provided $t \ge T^*$. Hence, $y_\tau(t) \ge (k - \epsilon)(t - T)^{2n-1}$. By (ii) we have $f(s, y_\tau(s)) \ge f(s, (k - \epsilon)(s - T)^{2n-1})$. Since (3.9) is valid with T_1 replaced by T^*,

$$D^{2n-1}y(T^*) - (2n-1)!k \geq (k-\epsilon)\int_{T^*}^{t} (s-T)^{2n-1} f(s, (k-\epsilon)(s-T)^{2n-1})\,ds.$$

For $s - T \geq \frac{1}{2}s$, i.e. for $s \geq 2T$, we have

$$\int^{\infty} s^{2n-1} f(s, Cs^{2n-1})\,ds \leq 2^{2n-1} \int^{\infty} (s-T)^{2n-1} f(s, 2^{2n-1}C(s-T)^{2n-1})\,ds$$

$$\leq N_1 ,$$

where

$$N_1 = 2^{2n-1}(k-\epsilon)^{-1}[D^{2n-1}\, y(T^*) - (2n-1)!k],$$

$$C = 2^{1-2n}\,(k-\epsilon),$$

and the lower endpoint of integration is not less than $\max(T^*, 2T)$. This proves the theorem.

We remark that in the case $n = 2$ and $\tau(t) \equiv 0$, these results reduce to those of Leighton and Nehari [7] in the linear case and to those of Wong [11] in the nonlinear case.

Chapter 4. An Application of Lyapunov's Direct Method

In this chapter we use Lyapunov's second method to obtain nonoscillation criteria for the equation (1.1). We consider the equivalent system:

$$y_k(t) = D^k y(t), \quad k = 0,\ldots,n-1;$$

(4.1) $$z_k(t) = D^k[r(t)D^{n-1}y(t)], \; k = 0,\ldots,n-1; \quad \text{and}$$

$$Dz_{n-1}(t) = -f(t,y_\tau(t))y_\tau(t).$$

To simplify notation we shall let $\eta = (y_0,\ldots,y_{n-1})$ and $\zeta = (z_0,\ldots,z_{n-1})$. For the variables (t,η,ζ) we also define

$$R^1 = R = (-\infty,\infty);$$

$$R_T = [T,\infty), \; T \geq 0;$$

$$R^* = (0,\infty);$$

$$R_* = (-\infty,0);$$

$$R^{p*} = R^* \times R^* \times \cdots \times R^*, \; p \text{ times};$$

$$R_{p*} = R_* \times R_* \times \cdots \times R_*, \; p \text{ times};$$

$$R^*_* = R^* \times R_*;$$

$$R_{T,j} = R_T \times R^{(2j+1)*} \times (R^*_*)^{n-1-j} \times R;$$

$$S_{T,j} = R_T \times R_{(2j+1)*} \times (R_*^*)^{n-1-j} \times R.$$

In the following a scalar function v of the variables t,η,ζ will be called a Lyapunov function for (4.1) if it is

continuous in (t,η,ζ) in the domain of definition and is locally Lipschitzian in (η,ζ). Following Yoshizawa [12], we define

$$(4.2)\quad \dot{v}_{(1)}(t,\eta,\zeta) = \overline{\lim_{h\to 0^+}}\, \frac{1}{h}\{v(t+h,\eta(t+h),\zeta(t+h)) - v(t,\eta,\zeta)\}$$

Theorem 4.1. Suppose that there exist two continuous functions $V(t,\eta,\zeta)$ and $W(t,\eta,\zeta)$ which are defined on $R_{T,j}$ and $S_{T,j}$ respectively for some fixed T. Assume further that $V(t,\eta,\zeta)$ satisfy:

(i) Both $V(t,\eta,\zeta)$ and $W(t,\eta,\zeta)$ tend to infinity as $t\to\infty$ uniformly for (η,ζ) in $R^{(2j+1)*} \times (R^*_*)^{n-1-j} \times R$ or $R_{(2j+1)*} \times (R_*^*)^{n-1-j} \times R$, respectively;

(ii) $\dot{V}_{(1)}(t,\eta,\zeta) \le 0$ for all sufficiently large t, where (η,ζ) is a solution of (4.1) which for large t lies in the region $R^{(2j+1)*} \times (R^*_*)^{n-1-j} \times R$; and

(iii) $\dot{W}_{(1)}(t,\eta,\zeta) \le 0$ for all sufficiently large t, where (η,ζ) is a solution of (4.1) which for large t lies in the region $R_{(2j+1)*} \times (R_*^*)^{n-1-j} \times R$.

Then (1.1) has no solutions of type B_j.

Proof: Let $y(t)$ be a solution of (1.1) of type B_j. Since $y(t)$ and $y_1(t)$ are positive for large t, there is a positive T_o for which $(\eta(t),\zeta(t))$ lies in $R^{(2j+1)*} \times R$ for $t \ge T_o$. By (ii), for t sufficiently large, i.e., for $t \ge T_1 \ge T_o$,

$$V(t,\eta(t),\zeta(t)) < V(T_1,\eta(T_1),\zeta(T_1)).$$

On the other hand, condition (i) implies that there is a $T_2 > T_1$ for which

$$V(t,\eta(t),\zeta(t)) > V(T_1,\eta(T_1),\zeta(T_1))$$

for $t \geq T_2$, which is a contradiction.

By letting $y(t)$ be a negative solution of (1.1) of type B_j and considering $W(t,\eta(t),\zeta(t))$, we obtain an analogous contradiction.

<u>Lemma 4.1</u>. For $(t,\eta(t),\zeta(t)) \in R_{T,n-1}$ assume that there exists a Lyapunov function $v(t,\eta(t),\zeta(t))$ satisfying:

(i) $z_{n-1}v(t,\eta,\zeta) > 0$;

(ii) $\dot{v}_{(1)}(t,\eta,\zeta) \leq -\lambda(t)$, where $\lambda(t)$ is a continuous function defined on R_T such that

$$\underline{\lim}_{t\to\infty} \int_T^t \lambda(s)ds \geq 0 \tag{4.3}$$

for $T \geq T^*$ sufficiently large.

Moreover, suppose there exist a T_1 and a function $w(t,\eta,\zeta)$ which for (t,η,ζ) in the region $R_{T_1} \times R^{(2n-1)*} \times R_*$, is a Lyapunov function satisfying:

(iii) $z_{n-1} \leq w(t,\eta,\zeta) \leq b(z_{n-1})$;

where $b(u)$ is a continuous function, $b(0) = 0$ and $b(u) < 0$ for $u \neq 0$; and

(iv) $\dot{w}_{(1)}(t,\eta(t),\zeta(t)) \leq -\rho(t)w(t,\eta(t),\zeta(t))$,

where $\rho(t) \geq 0$ is a continuous function such that

$$\int^\infty \exp\{-\int_T^t \rho(s)ds\}dt = +\infty . \tag{4.4}$$

If $(\eta(t),\zeta(t))$ is a solution of (4.1) which lies in the region $R^{(2n-1)*} \times R$ for sufficiently large values of t, then $z_{n-1}(t) \geq 0$ for large t.

Proof: Suppose there is a sequence $\langle t_k \rangle$ for which $t_k \to \infty$ as $k \to \infty$ and $z_{n-1}(t_k) < 0$. Assume $t_k \geq T^*$ and that t_k is sufficiently large so that by (4.3),

$$\varliminf_{t\to\infty} \int_{t_k}^{t} \lambda(s)\,ds \geq 0, \quad t \geq t_k$$

and $y_0(t),\ldots,y_{n-1}(t)$, $z_0(t),\ldots,z_{n-2}(t)$ are positive, were we assume $n \geq 2$. For the case $n = 1$, see Yoshizawa [2]. Consider the function $v(t,\eta(t),\zeta(t))$ for $t \geq t_k$.

$$(4.5) \quad v(t,\eta(t),\zeta(t)) \leq v(t_k,\eta(t_k),\zeta(t_k)) + \int_{t_k}^{t} \dot{v}_{(1)}(s,\eta(s),\zeta(s))\,ds$$

$$\leq v(t_k,\eta(t_k),\zeta(t_k)) - \int_{t_k}^{t} \lambda(s)\,ds.$$

Since $z_{n-1}(t_k) < 0$, $v(t_k,\eta(t_k),\zeta(t_k)) < 0$, so there is a $T_1 \geq t_k$ for which

$$\int_{t_k}^{t} \lambda(s)\,ds \geq \frac{1}{2}\, v(t_k,\eta(t_k),\zeta(t_k)) ,$$

which implies that for $t \geq T_1$,

$$v(t,\eta(t),\zeta(t)) \leq \frac{1}{2}\, v(t_k,\eta(t_k),\zeta(t_k)) < 0.$$

By (i), $z_{n-1}(t) < 0$ for $t > T_1$. By (iii), there is a $T_2 > T_1$ and a Lyapunov function $w(t,\eta(t),\zeta(t))$ defined on $R_{T_2} \times R^{(2n-1)*} \times R_*$. For this $w(t,\eta(t),\zeta(t))$ we have by (iv)

$$z_{n-1}(t) \le w(t,\eta(t),\zeta(t)) \le w(T_2,\eta(T_2),\zeta(T_2))\exp[-\int_{T_2}^{t}\rho(s)ds] ,$$

where $t \ge T_2 > T_1$. By (iii),

$$z_{n-1}(t) \le b(z_{n-1}(T_2))\exp[-\int_{T_2}^{t}\rho(s)ds] .$$

Substituting this into the above expression, one gets

$$z_{n-1}(u) = [z_{n-2}(u)]' \le b(z_{n-1}(T_2))\exp[-\int_{T_2}^{u}\rho(s)ds] .$$

Integrating from T_2 to t, we arrive at

$$z_{n-2}(t) \le z_{n-2}(T_2) + b(z_{n-1}(T_2)) \int_{T_2}^{t} \exp[-\int_{T_2}^{t}\rho(s)ds]dt.$$

Letting $t\to\infty$ and using (4.4), it follows that $z_{n-2}(t) < 0$ for sufficiently large t, which is a contradiction.

By the same argument we can prove the following lemma.

Lemma 4.2. For $(t,\eta(t),\zeta(t)) \in S_{T^*,n-1}$, assume that there exists a Lyapunov function $v(t,\eta(t),\zeta(t))$ satisfying:

(i) $z_{n-1}\, v(t,\eta,\zeta) > 0$; and

(ii) $\dot{v}_{(1)}(t,\eta,\zeta) \le -\lambda(t)$, where $\lambda(t)$ is a continuous function defined on R_{T^*} and for large T,

$$\varliminf_{t\to\infty} \int_{T}^{t} \lambda(s)ds \ge 0 .$$

Moreover, assume that there exists a T_1 and a function $w(t,\eta,\zeta)$ which for (t,η,ζ) in the region $R_{T_1} \times R_{(2n-1)^*} \times R^*$ is a Lyapunov function satisfying

(iii) $-z_{n-1} \le w(t,\eta,\zeta) \le b(z_{n-1})$, where $b(u)$ is a continuous function, $b(0) = 0$ and $b(u) < 0$ for $u \neq 0$; and

(iv) $\dot{w}_{(1)}(t,\eta(t),\zeta(t)) \leq -\rho(t)w(t,\eta(t),\zeta(t))$,

where $\rho(t) \geq 0$ is a continuous function for which

$$\int^{\infty} \exp\,[-\int_{T_1}^{t} \rho(s)\,ds\,]dt = +\infty \ .$$

If $(\eta(t),\zeta(t))$ is a solution of (4.1) which lies in the region $R_{(2n-1)*} \times R$ for sufficiently large values of t, then $z_{n-1}(t) \leq 0$ for large t.

<u>Remark 1</u>: Since $0 < m \leq r(t) \leq M$, condition (4.4) is equivalent to

$$(4.6) \qquad \int^{\infty} \frac{1}{r(t)} \{\exp[-\int_{T_1}^{t} \rho(s)\,ds\,]\}dt.$$

To see this we merely note that

$$M\int^{t} \frac{1}{r(u)} \exp[-\int_{T}^{u} \rho(s)\,ds\,]du \geq \int^{t} \exp[-\int_{T}^{u} \rho(s)\,ds\,]du \geq m\int^{t} \frac{1}{r(u)} \exp[-\int_{T}^{u} \rho(s)\,ds\,]du.$$

In the case $n = 1$, we have $v(t,\eta,\zeta) = v(t,y,y')$ since $y = z$ and $z_1 = y'$. Condition (4.6) arises naturally in the proof of Lemma 4.1.

<u>Remark 2</u>: Suppose we let $\rho(t) \equiv 0$ in each of the two lemmas. Condition (4.4) is then trivially valid, and the alternative condition (4.6) reduces to

$$(4.7) \qquad \int^{\infty} \frac{dt}{r(t)} = +\infty \ .$$

Thus, we may replace condition (iv) by $\dot{w}_{(1)}(t,\eta,\zeta) \leq 0$ and obtain two easy corollaries whose statements are left to the reader.

Remark 3: Let $r(t) \equiv 1$ and $f(t,y_\tau(t))$ be nonnegative. As already noted, solutions of type B_1 are solutions of type A_1. The lemma asserts that a solution $y(t)$ for which $D^k y(t) > 0$, $k = 0,1,\ldots,2n-2$ must satisfy $D^{2n-1}y(t) > 0$, i.e., $y(t)$ must be a solution of type A_{n-1}. But this is obvious from Kiguradze's lemma.

Theorem 4.2. Suppose there are continuous functions $a(t)$, $b(t)$, $\alpha(z_{n-2})$ and $\beta(z_{n-2})$ satisfying:

a) For large T,

$$\varliminf_{t\to\infty} \int_T^t a(s)ds \geq 0, \quad \varliminf_{t\to\infty} \int_T^t b(s)ds \geq 0;$$

b) $z_{n-2}\alpha(z_{n-2}) > 0$, $\alpha'(z_{n-2}) = \frac{d}{dz_{n-2}} [\alpha(z_{n-2})] \geq 0$,

where y_k $(k = 0,\ldots,n-1)$ and $z_k (k = 0,\ldots,n-2)$ are nonnegative for large t,

$$z_{n-2}\,\beta(z_{n-2}) > 0, \quad \beta'(z_{n-2}) = \frac{d}{dz_{n-2}} [\beta(z_{n-2})] \geq 0,$$

where y_k $(k = 0,\ldots,n-1)$ and $z_k (k = 0,\ldots,n-2)$ are nonpositive for large t; and

c) $a(t)\,\alpha(z_{n-2}) \leq f(t,y_\tau(t))y_\tau(t)$ for large t, $y \geq 0$,

$b(t)\,\beta(z_{n-2}) \geq f(t,y_\tau(t))y_\tau(t)$ for large t, $y \leq 0$.

If $(\eta(t),\zeta(t))$ is a solution of (4.1) which for large t lies in the region $R^{(2n-1)^*} \times R$; then $z_{n-1}(t) \geq 0$ for large t. If $(\eta(t),\zeta(t))$ is a solution of (4.1) which for large t lies in the region $R_{(2n-1)*} \times R$; then $z_{n-1}(t) \leq 0$ for large t.

Proof: Let $\lambda(t) = a(t)$ or $b(t)$, $\rho(t) \equiv 0$ and define $v(t,\eta,\zeta)$ and $w(t,\eta,\zeta)$ by

$$v(t,\eta(t),\zeta(t)) = \frac{z_{n-1}(t)}{\alpha[z_{n-2}(t)]}$$

and

$$w(t,\eta(t),\zeta(t)) = z_{n-1}(t) + \alpha[z_{n-2}(t)] \int_T^t a(s)ds.$$

Conditions (i), (ii), and (iii) of Lemma 4.1 hold. In particular,

(i) $$z_{n-1}\ v(t,\eta,\zeta) = \frac{z_{n-1}^2(t)}{\alpha[z_{n-2}(t)]} > 0 \quad \text{since} \quad z_{n-2} > 0.$$

(ii)$_1$ $$\dot{v}_{(1)}(t,\eta,\zeta) = \frac{1}{\alpha^2(z_{n-2})} \{\alpha z'_{n-1} - z_{n-1}^2\ \alpha'(z_{n-2})\ \}$$

$$\leq \frac{z'_{n-1}}{\alpha(z_{n-2})} \quad ,$$

by (b). Using (1.1)

$$\dot{v}_{(1)}(t,\eta(t),\zeta(t)) \leq \frac{-f(t,y_T(t))y_T(t)}{\alpha[z_{n-2}(t)]} \leq -\ a(t).$$

(ii)$_2$ $$\varliminf_{t\to\infty} \int_T^t \lambda(s)ds = \varliminf_{t\to\infty} \int_T^t a(s)ds \geq 0,$$

for large t by (a).

(iii) $$z_{n-1} \leq w(t,\eta,\zeta) \leq z_{n-1} + \alpha(z_{n-2}) \int_T^t a(s)ds,$$

since $z_{n-2} \geq 0$.

Also

$$\alpha(z_{n-2}) \int_T^t a(s)ds \leq \int_T^t \alpha(z_{n-2})\ a(s)ds \leq \int_T^t f(s,y_T(s))ds$$

i.e., $w(t,\eta(t),\zeta(t)) \leq z_{n-1}(t) + [z_{n-1}(T) - z_{n-1}(t)] = z_{n-1}(T)$.

So we may let the function $b(u)$ of Lemma 4.1 to be the constant function $b(u) = - z_{n-1}(T)$.

(iv) $\dot{w}_{(1)}(t,\eta,\zeta) = z_{n-1}'(t) + a(t)\alpha(z_{n-2}(t)) + [\int_T^t a(s)ds]\alpha(z_{n-2}(t))z_{n-1}(t) < 0$

for large t by (a) provided z_{n-1} is assumed negative.

Moreover, suppose we let

$$v(t,\eta(t),\zeta(t)) = \frac{z_{n-1}(t)}{\beta[z_{n-2}(t)]} \quad \text{and}$$

$$w(t,\eta(t),\zeta(t)) = - z_{n-1}(t) - \beta[z_{n-2}(t)] \int_T^t b(s)ds.$$

Similar routine computations show that v and w satisfy the three conditions of Lemma 4.2.

<u>Theorem 4.3</u>. Suppose that, in addition to the hypotheses of Theorem 4.2,

$$\int^\infty a(s)ds = \int^\infty b(s)ds = +\infty .$$

Then (1.1) has no solutions of type B_{n-1}.

<u>Proof</u>: Suppose we define

$$V(t,\eta(t),\zeta(t)) = \begin{cases} \dfrac{z_{n-1}(t)}{\alpha[z_{n-2}(t)]} + \displaystyle\int^t a(s)ds , & y \geq 0 \\ \displaystyle\int_0^t a(s)ds , & y < 0 \end{cases}$$

$$W(t,\eta(t),\zeta(t)) = \begin{cases} \dfrac{z_{n-1}(t)}{\beta[z_{n-2}(t)]} + \displaystyle\int_0^t b(s)ds , & y < 0 \\ \displaystyle\int_0^t b(s)ds , & y \geq 0 . \end{cases}$$

Assume that $y(t)$ is a solution of (1.1) of type B_{n-1}. Then for large t, y_k $(k = 0,\ldots,n-1)$ and z_k $(k=0,\ldots,n-1)$ are positive

$$V(t,\eta(t),\zeta(t)) \geq \int_0^t a(s)ds \quad \text{and}$$

$$W(t,\eta(t),\zeta(t)) \geq \int_0^t b(s)ds \ .$$

Thus V and W both tend to infinity as $t \to \infty$ uniformly. Next,

$$\dot{V}_{(1)}(t,\eta(t),\zeta(t)) = \left[\frac{z_{n-1}}{\alpha(z_{n-2})}\right]' + a(t) \leq -a(t) + a(t) = 0 \ ;$$

$$\dot{W}_{(1)}(t,\eta(t),\zeta(t)) = \left[\frac{z_{n-1}}{\beta(z_{n-2})}\right]' + b(t) \leq -b(t) + b(t) = 0 \ ,$$

Hence V and W satisfy the three conditions of Theorem 4.1, and the proof is complete.

<u>Theorem 4.4</u>. Suppose that there are continuous functions $a(t)$, $b(t)$, $\alpha(y)$ and $\beta(y)$ satisfying:

a) $\int^\infty a(s)ds = \int^\infty b(s)ds$;

b) $y\alpha(y) > 0$, $\alpha'(y) \geq 0$, where y and y' are nonnegative for large t;

$y\beta(y) > 0$, $\beta'(y) \geq 0$, where y and y' are nonpositive for large t; and

c) $a(t)\ \alpha(y) \leq f(t,y_\tau(t))y_\tau(t)$,

$b(t)\ \beta(y) \geq f(t,y_\tau(t))y_\tau(t)$.

Then (1.1) has no solutions of types $B_0,\ldots,B_{n-1}$.

Proof:

Let

$$V(t,\eta(t),\zeta(t)) = \begin{cases} \dfrac{z_{n-1}(t)}{\alpha(y(t))} + \displaystyle\int_0^t a(s)ds , & y \geq 0 \\ \displaystyle\int_0^t a(s)ds , & y < 0 \end{cases}$$

$$W(t,\eta(t),\zeta(t)) = \begin{cases} \displaystyle\int_0^t b(s)ds , & y > 0 \\ \dfrac{z_{n-1}(t)}{\beta(y(t))} + \displaystyle\int_0^t b(s)ds , & y \leq 0 . \end{cases}$$

V and W will then satisfy the three conditions of Theorem 4.1. The details are omitted.

We observe that Theorem 4.4 is only one of a sequence of similar results. Suppose that n is even and $j \leq (n-2)/2$ or n is odd and $j \leq (n-3)/2$. Then we may replace y by y_{2j} in conditions (b) and (c) and require $D^k y$, $k = 0,1,\ldots,$ $2j+1$ to have the usual signs. Similarly, if n is odd and $j = \frac{n-1}{2}$, we may replace y by z in (b) and (c). Finally, if n is even and $j \geq n/2$ or n is odd and $j \geq (n+1)/2$, we replace y by z_{2j} in (b) and (c). In each case we conclude that (1.1) has no solutions of type $B_j,\ldots,B_{n-1}$.

Theorem 4.5. Let $p(t) > 0$

If $$\int^\infty t^{2j}\, p(t)dt = +\infty ;$$

then there are no solutions of

(4.8) $$D^n[r(t)D^n y(t)] + p(t)y_\tau(t) = 0$$

of type B_j.

<u>Proof</u>: Let $y(t)$ be a solution of type B_j. For equation (4.8),

$$f(t,y_\tau(t))y_\tau(t) = p(t)y_\tau(t) \geq \mu(t-T)^{2j}\begin{cases} y_{2j}(t-T) \\ z(t-T) \\ z_{2j}(t-T) \end{cases},$$

depending on whether (i) n is even and $j \leq (n-2)/2$ or n is odd and $j \leq (n-3)/2$; (ii) n is odd and $j = (n-1)/2$; (iii) n is even and $j \geq n/2$ or n is odd and $j \geq (n+1)/2$. We note that μ is a known constant (determined in section two) once case (i), (ii) or (iii) is prescribed. We let

$$\lambda(t) = a(t) = b(t) = t^{2j}\, p(t) .$$

With the choices of V and W prescribed by Theorem 4.4 and the remarks following it, it follows that (4.8) has no solutions of type B_j.

Chapter 5. A More General Delay Differential Equation

Throughout this section vectors in R^{2n} will be denoted by lower case Greek letters and scalars by lower case Latin letters. To facilitate the discussion we shall also adopt the following notation:

$$\mu = (x_1, x_2, \cdots, x_{2n}) \;;$$

$$\mu_k = \left(\frac{x_1}{x_k}, \frac{x_2}{x_k}, \cdots, \frac{x_{2n}}{x_k}\right) \;;$$

$$\tau_i = (\tau_{i,1}(t), \tau_{i,2}(t), \cdots, \tau_{i,2n}(t)) \;;$$

$$t\epsilon_{2n} = (t, t, \cdots, t) \;, \quad 2n \text{ times}; \quad \text{and}$$

$$\mu^* = (y_o, \cdots, y_{n-1}, z_o, \cdots, z_{n-1}) \;.$$

For the vector $\sigma = (s_1, \ldots, s_{2n})$ we shall form the composites:

$$\mu(\sigma) = [x_1(s_1(t)), \ldots, x_{2n}(s_{2n}(t))]; \quad \text{and}$$

$$\mu^*(\sigma) = [y_o(s_1(t)), \ldots, y_{n-1}(s_n(t)), z_o(s_{n+1}(t)), \ldots, z_{n-1}(s_{2n}(t))].$$

The purpose of this section is to present conditions for the nonexistence of certain types of nonoscillatory solutions of the even order delay equation:

$$(5.1) \quad D^n[r(t)D^n y](t) + \sum_{i=1}^{N} f_i(t) F_i[\mu^*(t\epsilon_{2n} - \tau_i(t))] = 0,$$

where $0 < m \le r(t) \le M$ and the delays $\tau_{i,k}(t)$ satisfy $0 \le \tau_{i,k}(t) \le T$. It will be assumed throughout that:

(i) $f_i(t) \geq 0$; $f_i(t)$ and $F_i(\mu)$ are continuous functions of the variables t and μ respectively;

(ii) $\operatorname{sgn} F_i(\mu) = \operatorname{sgn} x_1$,

$$F_i(-\mu) = -F_i(\mu); \text{ and}$$

(iii) $F_1(\mu) \neq 0$ if $\mu \neq 0$.

Lemmas 2.1 and 2.2 are still valid for equation (5.1), as are certain analogues of the theorems in sections two through four. However, somewhat different hypotheses will be considered here.

Theorem 5.1. Suppose there is an index j $(1 \leq j \leq N)$ and some $q \geq 0$ which for all $\mu \in R_{2n}$ and for all $c \in R$ satisfy:

$$(5.2) \qquad F_j(c\mu) \geq c^{2q+1}F_j(\mu) \quad \text{and}$$

$$(5.3) \qquad \int^{\infty} t^{2k} f_j(t)\,dt = +\infty$$

for some integer $k = 0,1,\ldots,n-1$. Then (5.1) has no solutions of type B_k.

Proof: Let $y(t)$ be a solution of type B_k. First suppose that n is even and $k \leq (n-2)/2$ or that n is odd and $k \leq (n-3)/2$. Define

$$w_k(t) = [D^{2k}y(t)]^{-1}D^{n-1}z_o(t).$$

Then we see from (5.1) that

$$(5.4) \quad w_k'(t) = [D^{2k}y(t)]^{-1}D^n z_o(t) - [D^{2k}y(t)]^{-2}D^{n-1}z_o(t)D^{2k+1}y(t).$$

There is a $T_1 > 0$ such that $D^s y(t) > 0$ $(s = 0, \ldots, 2k+1)$ for $t > T_1$. Beginning with $D^{2k+2} y(t)$ the various derivatives of y and z alternate in sign. Hence $y_{\tau_{i,1}}(t) > 0$ for $t - \tau_{i,1}(t) > T_1$, i.e., for $t > T_1 + T$. Thus, for $t > T_1$

$$w_k'(t) < [D^{2k} y(t)]^{-1} D^n z_o(t) = -\sum_{i=1}^{N} [D^{2k} y(t)]^{-1} f_i(t) F_i[\mu^*(t\epsilon_{2n} - \tau_i(t))].$$

For $t > T_1 + T$,

$$(5.5) \quad w_k'(t) < -f_j(t)[D^{2k} y(t)]^{-1} F_j[\mu^*(t\epsilon_{2n} - \tau_j(t))].$$

Since $y'(t)$ is positive on (T_1, ∞), $y(t)$ is an increasing function for $t > T_1$. Thus, for $t > T_1 + T$

$$(5.6) \qquad y_{\tau_{j,1}}(t) \geq y_{\tau_{j,1}}(T_1 + T),$$

which implies that

$$-[y_{\tau_{j,1}}(t)]^{2q} \leq -[y_{\tau_{j,1}}(T_1 + T)]^{2q}.$$

Using this and (5.2), one gets

$$w_k'(t) \leq -f_j(t)[D^{2k} y(t)]^{-1} [y_{\tau_{j,1}}(t)]^{2q+1} F_j[\mu_1^*(t\epsilon_{2n} - \tau_j(t))]$$

$$(5.7) \qquad \leq -f_j(t)[D^{2k} y(t))]^{-1} y_{\tau_{j,1}}(t)[y_{\tau_{j,1}}(T_1+T)]^{2q} F_j[\mu_1^*(t\epsilon_{2n} - \tau_j)].$$

By (iii), $F_j[\mu_1^*(t\epsilon_{2n} - \tau_j(t))]$ is positive for $t > T_1 + T$ and does not tend to zero as $t \to \infty$ because of (i). Thus there is a $k_{j,1} > 0$ and a $T_* \geq T_1 + T$ for which $F_j[\mu_1^*(t\epsilon_{2n} - \tau_j(t))] \geq k_{j,1} > 0$ if $t > T_2$. Moreover, by Lemma 2.2 there is a constant $k_{j,2} > 0$ and a $T_3 \geq T_2$ for which $y_{\tau_{j,1}}(t) \geq k_{j,2}\, y(t)$. Hence, for $t > T_3$,

(5.8) $w_k'(t) \leq -k_{j,1}k_{j,2}[y_{\tau_{j,1}}(T_1+T)]^{2q} y(t)[D^{2k}y(t)]^{-1} f_j(t)$.

By Lemma 2.1, $t^{2k}D^{2k}y(t) \leq 2^{2k}(2k+1)!\; y(t)$ for $t \geq 2T_1$. For $T^* = \max(2T_1, T_3)$, we have

(5.9) $$w_k' \leq -\, k_1 t^{2k} f_j(t)$$

where $k_1 = k_{j,1}k_{j,2}[y_{\tau_{j,1}}(T_1 + T)]^{2q}\, 2^{-2k}(2k+1)!^{-1}$.

Integrating (5.9) from T^* to ∞,

$$\lim_{t\to\infty} w_k(t) - w_k(T^*) \leq -\, K_1 \int_{T^*}^{\infty} t^{2k} f_j(t)\,dt = -\,\infty$$

Noting that $0 \leq \lim_{t\to\infty} w_k(t)$, it follows that $w_k(T^*) = \infty$, which is absurd since $D^{n-1}z_o(T^*) > 0$ and $D^{2k}y(T^*) \neq 0$.

Now suppose n is even and $k \geq n/2$ or n is odd and $k \geq (n+1)/2$. Define

$$w_k(t) = [D^{2k-n}z_o(t)]^{-1} D^{n-1}z_o(t).$$

Equation (5.4) becomes:

$$w_k'(t) = [D^{2k-n}z_o(t)]^{-1}D^n z_o(t) - [D^{2k-n}z_o(t)]^{-2} D^{n-1}z_o(t)\, D^{2k-n+1}z_o(t).$$

Equations (5.5), (5.7) and (5.8) remain valid with $D^{2k}y(t)$ replaced by $D^{2k-n}z_o(t)$; (5.6) remains unchanged. By Lemma 2.1, we have for $t \geq T^*$

$$t^{2k}D^{2k-n}z_o(t) \leq 2^{2k}(2k-n)!\; M \prod_{j=1}^{n-1} [(2k-n)Mm^{-1} + j]\, y(t).$$

Thus, for $t \geq T^*$

$$w_k'(t) \leq -k_2\; t^{2k} f_j(t),$$

where

$$k_2 = k_{j,1}k_{j,2}[y_{\tau_{j,1}}(T_1+T)^{2q_2-2k}(2k-n)!^{-1}M^{-1}\prod_{i=1}^{n-1}[(2k-n)Mm^{-1}+i]^{-1}.$$

For the case that n is odd and $k = (n-1)/2$, we define:

$$w_k(t) = [D^{2k}y(t)]^{-1} D^{n-1}z_o(t).$$

The only change in the proof is that by Lemma 2.1,

$$t^{2k}D^{2k}y(t) \le 2^{2k} M \prod_{i=1}^{n-1}(Mm^{-1}+i).$$

The rest of the arguments proceed as before and the theorem is proved.

The following results are obtained easily upon considering more carefully the proof of Theorem 5.1.

Corollary 5.1. Under the hypotheses of Theorem 5.1, equation (5.1) has no solutions of type B_s $(s = k,\ldots,n-1)$.

Corollary 5.2. Suppose, in addition to (i), (ii), (iii) and (5.2), there is some integer $k = 0,1,\ldots,2n-2$ for which

$$\int^{\infty} t^k f_j(t)dt = +\infty .$$

Then (5.1) has no solutions of type B_s $(s = [\frac{k+1}{2}],\ldots,n-1)$.

Corollary 5.3. Suppose, in addition to (i), (ii) and (5.2), there is some integer $k = 1,\ldots,2n-1$ and some $j = 1,\ldots,N$ for which $F_j(\mu) \neq 0$ if $x_k \neq 0$ and

$$\int^{\infty} f_j(t)dt = +\infty .$$

Then (5.1) has no solutions of type B_s, $s = [\frac{k}{2}],\ldots,n-1$.

Denote by I_1 the set of indices i $(1 \le i \le N)$ for which $F_i(\mu)$ is nondecreasing with respect to x_j for each j $(1 \le j \le 2n)$. Let $I_{2,k}$ denote the set of indices i $(1 \le i \le N)$ for which $x_k^{-1} F_{i,k}(x_k)$ is nonincreasing with respect to x_k, where $F_{i,k}(x_k)$ is the function obtained from $F_i(\mu)$ by setting $x_j = 0$ for all $j \neq k$. Finally let $I_{1,k} = I_1 \cap I_{2,k}$. In terms of these notions we may give a different type of nonoscillation criterion.

<u>Theorem 5.2</u>. Let (5.1) satisfy, in addition to (i), (ii) and (iii), the following conditions:

(iv) $I_{1,k} \neq \emptyset$ for some $k = 1,\dots,2n-1$;

(v) there is a nonnegative function $\Phi(t)$ such that for all $c \ge 1$,

$$\int^{\infty} \{\Phi(t) \sum_{i \in I_{1,k}} c^{-1} t^{k-2n} f_i(t) F_{i,k}(t^{2n-k}) - P_k(t)(\Phi'(t))^2\} dt = +\infty ,$$

where

$$P_k^{-1}(t) = 4N_k(t-T)^{2n-k-1}\Phi(t) \quad \text{and}$$

$$N_k^{-1} = \begin{cases} 2^{2n-k-1}\, n\,!\, M \prod_{j=1}^{n-k-1} (nMm^{-1} + j), & 1 \le k \le n \\ 2^{2n-\bar{k}}\,(2n-\bar{k})\,!, & k \ge n+1,\ \bar{k} = k+1. \end{cases}$$

Then (5.1) has no solutions of type B_{n-1}.

<u>Proof</u>: Suppose $y(t)$ is a solution of type B_{n-1}. If $1 \le k \le n$, let

$$w(t) = -\Phi(t)D^{n-1}z_o(t)/D^{k-1}y(t-T)$$

Using previous notation, $z(t) < 0$ for $t > T_1 + T$. A simple computation shows that

$$w'(t) = -\Phi(t)D^n z_o(t)/D^{k-1}y(t-T) - \Phi'(t)D^{n-1}z_o(t)/D^{k-1}y(t)$$
$$+ \Phi(t)D^{n-1}z_o(t)D^k y(t-T)/[D^{k-1}y(t-T)]^2 .$$

We note that since $D^n z_o(t) < 0$,

$$D^k y(t-T) \geq N_k(t-T)^{2n-k-1}D^{n-1}z_o(t-T) \geq N_k(t-T)^{2n-k-1}D^{n-1}z_o(t),$$

where

$$N_k^{-1} = 2^{2n-\bar{k}}(n-1)!\, M n \prod_{j=1}^{n-k-1}(nMm^{-1} + j) .$$

Substituting for $D^n z_o(t)$ from equation (5.1) and using (i), (ii) and (iv), one finds

$$(5.10)\quad w'(t) \geq \Phi(t)\sum_{i\in I_{1,k}} f_i(t)[D^{k-1}y(t-T)]^{-1}F_{i,k}(D^{k-1}y(t-T)) + \Omega(t),$$

where

$$\Omega(t) = N_k(t-T)^{2n-k-1}\Phi^{-1}(t)(z(t))^2 + \Phi^{-1}(t)\Phi'(t)z(t).$$

Completing the square as suggested by the last two terms, we have

(5.11)

$$\Omega(t) = N_k\Phi^{-1}(t)(t-T)^{2n-k-1}[z^2(t)+4\Phi(t)\Phi'(t)P_k(t)z(t)]$$
$$= N_k\Phi^{-1}(t)(t-T)^{2n-k-1}\{[z(t)+2\Phi(t)\Phi'(t)P_k(t)]^2-4\Phi^2(t)(\Phi'(t))^2P_k^2(t)\}$$
$$\geq 4N_k\,\Phi(t)(t-T)^{2n-k-1}P_k^2(t)(\Phi'(t))^2$$
$$= P_k(t)(\Phi'(t))^2 .$$

Moreover, since $D^n z_o(t) < 0$, there is a constant $c_1 \geq 1$ and a $t_1 \geq T_1 + T$ for which

$$(5.12) \qquad D^{k-1}y(t) \leq c_1 t^{2n-k}, \quad t \geq t_1 .$$

We then have the chain of inequalities

$$(5.13) \qquad \begin{aligned} [D^{k-1}y(t-T)]^{-1}F_{i,k}(D^{k-1}y(t-T)) &\geq c_1^{-1}t^{k-2n}F_{i,k}(c_1 t^{2n-k}) \\ &\geq c_1^{-1}t^{k-2n}F_{i,k}(t^{2n-k}), \end{aligned}$$

where the first inequality follows because of (5.12) and the fact that $i \in I_{2,k}$ and the second because $i \in I_1$ and $c_1 \geq 1$.

Combining (5.10), (5.11) and (5.13), we have

$$(5.14) \quad w'(t) \geq \Phi(t) \sum_{i \in I_{1,k}} c_1^{-1}t^{k-2n}f_i(t)F_{i,k}(t^{2n-k}- P_k(t)(\Phi'(t))^2 .$$

Integrating this from t_1 to t and using (v), it follows that $w(t)$ is positive for sufficiently large t, which is a contradiction since $w(t) < 0$ for $t > T_1 + T$.

Now suppose $k \geq n + 1$, then $k - 1 \geq n$, and we let

$$w(t) = - \Phi(t) D^{n-1}z_o(t)/D^{k-n-1}z_o(t-T) .$$

Since y is of type B_{n-1}, we have $w(t) < 0$ for $t > T_1 + T$.

$$\begin{aligned} w'(t) = - \Phi(t) D^n z_o(t)/D^{k-n-1}z_o(t-T) - \Phi'(t) D^{n-1}z_o(t) D^{k-n-1}z_o(t-T) \\ + \Phi(t) D^{n-1}z_o(t) D^{k-n}z_o(t-T)/[D^{k-n-1}z_o(t-T)]^2 . \end{aligned}$$

Substituting for $D^n z_o(t)$ from (5.1) and using (i), (ii) and (iv), we obtain

$$w'(t) \geq \Phi(t) \sum_{i\in I_{1,k}} f_i(t)[D^{k-n-1}z_o(t-T)]^{-1}F_{i,k}(D^{k-n-1}z_o(t-T))$$

$$+ \Omega(t) ,$$

where $\Omega(t) \geq -[4N_k^*(t-T)^{2n-k-1}\Phi(t)]^{-1}(\Phi^1(t))^2$ as in (5.11) with $N_k^* = [2^{2n-\bar{k}}(2n-\bar{k})!]^{-1}$(5.12) then becomes

$$D^{k-n-1}z_o(t) \leq c_1^* t^{2n-k} , t \geq t_1^* ,$$

where $c_1^* \geq$ and $t_1^* \geq T_1 + T$. The inequalities of (5.13) now become

$$[D^{k-n-1}z_o(t-T)]^{-1}F_{i,k}(D^{k-n-1}z_o t-T)) \geq (c_1^*)^{-1}t^{k-2n}F_{i,k}(c_1^*t^{2n-k})$$

$$\geq (c_1^*)^{-1}t^{k-2n}F_{i,k}(t^{2n-k}).$$

Thus (5.14) remains valid with c_1 replaced by c_1^* and N_k replaced by N_k^*. An integration from t_1^* to t results in the same contradiction as before.

<u>Remark</u>: By requiring, instead of (ii), that $F_i(c\mu) = cF_i(\mu)$, we may assume in the proof of Theorem 5.2 that $c_1 = 1$ (or $c_1^* = 1$) and take $c = 1/t$ to obtain as a trivial corollary integral criteria independent of the parameter c and thus easier to apply for a specific verification.

<u>Corollary 5.4</u>. Let (5.1) satisfy, in addition to (i), (iii) and (iv), the following conditions

(ii)′ $F_i(c\mu) = cF_i(\mu)$, $i \in I_{1,k}$; and

(v) there is a nonnegative function $\Phi(t)$ for which

$$\int^{\infty}\{\Phi(t)\sum_{i\in I_{1,k}} f_i(t)F_{i,k}(1)-[4N_k(t-T)^{2n-k-1}\Phi(t)]^{-1}(\Phi'(t))^2\}dt=+\infty$$

Then (5.1) has no solutions of type B_{n-1}.

If we denote by $I_{2,k}^{+}$ the set of indices i $(1 \le i \le N)$ for which $x_k^{-1}F_{i,k}(x_k)$ is nondecreasing with respect to x_k for $1 \le k \le 2n-1$ and let $I_{1,k}^{+} = I_1 \cap I_{2,k}^{+}$, we may modify the estimates of Theorem 5.2 and state the following result.

<u>Theorem 5.3</u>. Let (5.1) satisfy, in addition to (i), (ii) and (iii), the following conditions:

(iv) $I_{1,k}^{+} \neq \emptyset$ for some $k = 1,\ldots,2n-1$;

(v) there is a nonnegative function $\Phi(t)$ such that for all $c \ge 1$ and for all $d > 0$,

$$\int^{\infty}\{\Phi(t)\sum_{i\in I_{1,k}^{+}} c^{-1}t^{k-2n}f_i(t)F_{i,k}(dt^{2n-k-1})-P_k(t)(\Phi'(t))^2\}dt=+\infty.$$

Then (5.1) has no solutions of type B_{n-1}.

<u>Proof</u>: Suppose $y(t)$ is a solution of (5.1) of type B_{n-1}. If $1 \le k \le n$, let

$$w(t) = -\Phi(t)D^{n-1}z_o(t)/D^{k-1}y(t-T).$$

As in Theorem 5.2, (5.10) and (5.11) imply that

$$w'(t) \ge \Phi(t)\sum_{i\in I_{1,k}^{+}} f_i(t)[D^{k-1}y(t-T)]^{-1}F_{i,k}(D^{k-1}y(t-T)) - P_k(t)\Phi^{-1}(t)(\Phi'(t))^2,$$

where $P_k^{-1}(t) = 4N_k(t-T)^{2n-k-1}\Phi$. As before, there is a $c_2 \ge \max(N_1,\ldots,N_n) = N$ and a $t_2 \ge T_1 + T$ such that

$$D^{n-2}z_o(t) \le c_2(t-T), \quad t \ge t_2 .$$

Since $i \in I^+_{1,k}$, we have the following chain of inequalities

$$[D^{k-1}y(t-T)]^{-1}F_{i,k}(D^{k-1}y(t-T)) \ge \frac{F_{i,k}[N^{-1}(t-T)^{2n-k-1}D^{n-2}z_o(t-T)]}{N^{-1}(t-T)^{2n-k-1}D^{n-2}z_o(t-T)}$$

$$(5.16) \qquad \ge \frac{F_{i,k}[N^{-1}(t-T)^{2n-k-1}D^{n-2}z_o(t_2-T)]}{c_2N^{-1}(t-T)^{2n-k}}$$

$$= c^{-1}(t-T)^{k-2n}F_{i,k}[d(t-T)^{2n-k-1}],$$

where $c = c_2N^{-1} \ge 1$ and $d = N^{-1}D^{n-2}z_o(t_2-T)$.
For $n + 1 \le k \le 2n-1$, let

$$w(t) = -\Phi(t)D^{n-1}z_o(t)/D^{k-n-1}z_o(t-T)$$

Since (5.16) holds with $D^{k-1}y(t-T)$ replaced by $D^{k-n-1}z_o(t-T)$, the result now follows upon integration from t_2 to t as in Theorem 5.2.

<u>Remark</u>: Since the criteria in Theorem 5.3 depends on two parameters c and d, it seems difficult to apply. Moreover, the discrepancy in the power of t in the $F_{i,k}$ term gives rise to a weaker test for oscillation. Application of Theorem 5.3 to the equation

$$(5.17) \qquad [r(t)y''(t)]'' + p(t)y_\tau(t) = 0$$

shows that a proper choice of $\Phi(t)$ results in criteria which agrees to a large extent with previous results. Here $2n = 4$, $f_1(t) = p(t)$ and $F_1(\mu) = x_1$. Conditions (i), (ii) and (iii) are clearly valid; $x_1^{-1}F_{1,1}(x_1) = 1$ which is trivially nondecreasing. Letting $\Phi(t) = t^{3-\delta}$, $0 < \delta \le 2$, (v) becomes

$$\int^{\infty} [ct^{2-\delta} p(t) - \frac{(3-\delta)^2}{2t^{1+\delta}}]dt = +\infty .$$

For $\delta > 0$, this is equivalent to

$$(5.18) \qquad \int^{\infty} t^{2-\delta}\, p(t)dt = +\infty .$$

Thus, if (5.18) holds for any $0 < \delta \le 2$, there are no solutions of (5.17) of type B_1.

<u>Theorem 5.4</u>. Let (5.1) satisfy, in addition to (i), (ii) and (iii) the following conditions:

(iv) $I^+_{1,k} \neq \emptyset$ for some $k = 1,\dots,2n-2$;

(v) there is a nonnegative function $\Phi(t)$ which for all $c > 0$ satisfies

$$\int^{\infty} \{\Phi(t) \sum_{i \in I^+_{1,k}} c^{-1} t^{k+1-2n} F_{i,k}(ct^{2n-k-1}) - P_k(t)(\Phi'(t))^2\}dt = +\infty.$$

Then (5.1) has no solutions of type B_{n-1}.

<u>Proof</u>: It is sufficient to note that

$$[D^{k-1}y(t-T)]^{-1} F_{i,k}(D^{k-1}y(t-T)) \ge \frac{F_{i,k}[N^{-1}(t-T)^{2n-k-1} D^{n-2} z_o(t_2-T)]}{N^{-1}(t-T)^{2n-k-1} D^{n-2} z_o(t_2-T)}$$

$$= c_o^{-1}(t-T)^{k+1-2n} F_{i,k}[c_o(t-T)^{2n-k-1}]$$

$$\ge c^{-1} t^{k+1-2n} F_{i,k}(ct^{2n-k-1}),$$

where $c = 2^{k+1-2n} c_o = 2^{k+1-2n} N^{-1} D^{n-2} z_o(t_2-T) > 0$.

Remark: Theorem 5.4 corrects the inadequacy of Theorem 5.3. If we consider the equation (5.17) again and apply Theorem 5.4 with $\Phi(t) = t^2$, (v) becomes

$$\int^{\infty} [t^2 p(t) - \frac{1}{4N_k t^2}]dt = + \infty ,$$

which is equivalent to

$$\int^{\infty} t^2 p(t) dt = + \infty . \tag{5.19}$$

Hence (5.19) implies the nonexistence of solutions of (5.17) of type B_{n-1}. This criterion was already established in section two for the simpler equation (1.1).

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