CONSTRUCTION OF ANTOINE-TYPE MAPS BETWEEN CONTINUA IN EUCLIDEAN SPACE

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ABSTRACT

CONSTRUCTION OF ANTOINE - TYPE MAPS
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By

James Michael Sobota

In this paper we consider modifications of the general extension problem proposed by Antoine. We restrict our attention to continuous functions defined on E^n, taking one given continuum to another. In general we work with the following definition:

Definition 2.1 Two continua A and B in E^n are weakly equivalent if there exists a continuous $f: E^n \to E^n$ such that

1.) $f(A) = B$

2.) $f(E^n - A) = E^n - B$.

Theorem 2.4 Any non-separating continuum $A \subset E^3$ is weakly equivalent to a point.

Theorem 2.7 Let W be a wild arc in E^3 such that $W = A \cup B$ where A and B are tame arcs in E^3 and $A \cap B = p$. Then W is weakly equivalent to a tame arc.

Theorem 2.9 Let W be a wild arc in E^3 such that W lies on the boundary of a 3-cell in E^3. Then any tame arc A in E^3 is weakly equivalent to W.

Theorem 2.11 Any arc W which is the union of a finite number of tame arcs is weakly equivalent to any arc which lies on the

boundary of a 3-ball in E^3 or which lies on the boundary of a 2-disc in E^3.

In Chapter IV we generalize some of these results to higher dimensions. We have the following theorem:

<u>Theorem 4.2</u> For any $k > 0$, there exists a wild k-cell $W^k \subset S^{k+3}$ which is weakly equivalent to a tame k-cell $T^k \subset S^{k+3}$. In fact W^k goes homeomorphically onto T^k.

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CHAPTER I

INTRODUCTION

The problems we are concerned with in this paper arise from the problem Antoine discussed in [2]. He asked whether given a homeomorphism between compact sets in E^n, does the homeomorphism extend to E^n, or if not, does it extend to a homeomorphism on a neighborhood of the domain set. Antoine answered his question negatively via an example in E^3. The Antoine Necklace is a cantor set in E^3 lying on an arc and the arc has the property that a homeomorphism of this arc onto a segment in E^3 does not extend to a homeomorphism of E^3 to itself. Fox and Artin in [9] give more examples, among them examples of arcs in E^3 such that homeomorphisms of these arcs onto segments in E^3 extend only to tapered neighborhoods of the arcs (see also Persinger [14]). The extension problem for 2-spheres in E^3 was answered negatively by Alexander in [1], the Horned Sphere of Alexander being the example.

We form several generalizations and restrictions of Antoine's problem, we give solutions and partial solutions in the later chapters.

We first ask the general question: given pairs (X,A) and (Y,B), when can a map $f\colon A \to B$ (onto) be extended to an epi-map $f\colon X \to Y$ such that f takes $X - A$ onto $Y - B$.

We give examples showing simple cases where this need not happen.

Example 1.1 Consider the pairs (X,A) and (Y,B) where $X = \{0,1,2,3\}$, $A = \{0,1\}$, $Y = \{0,1,2,3,4\}$, $B = \{0,1\}$ and all spaces are discrete. Let $f: A \to B$ be defined by $f(x) = x$. f cannot extend to an epi-map $F: X \to Y$ such that $X - A$ goes onto $Y - B$. But this example is somewhat trivial due to the fact that the cardinality of Y is greater than that of X.

Example 1.2 Let $X = Y = Z^+$ (the positive integers), $A = \{3,4,5,\ldots\}$, $B = \{6,7,8,\ldots\}$. Let $f: A \to B$ be defined by $f(x) = x + 3$. Again f cannot extend since $Y - B$ has greater cardinality than $X - A$.

We next give an example where the trouble is not caused by cardinality differences. The following theorem is from Hempel [10].

Theorem 1.3 Let M be a closed, connected 2-manifold which is tamely embedded in E^3 and let f be a map of E^3 onto itself such that f restricted to M is a homeomorphism and $f(E^3 - M) = E^3 - f(M)$. Then $f(M)$ is tamely embedded in E^3.

Example 1.4 Let $X = Y = E^3$ and let $A = S^2$, $B = A^2$ the Alexander horned sphere. Let $f: A \to B$ be a homeomorphism. By Theorem 1.3, we know that if f extends to $F: X \to Y$ such that $F(X - A) = Y - B$ then B is tame. But we know A^2 is wild in E^3, hence no extension exists.

Example 1.5 Let $X = [0,\infty)$, $A = [a,b]$, $Y = L$ (L is the "long line", see Hocking and Young [11] p. 55), $B = [a,b]$, where $[a,b]$ is a nice interval in L. Let $f: A \to B$ be defined by $f(x) = x$. f cannot be extended to a map from X onto Y because Y is "too long".

We do know however that certain epi-maps extend to epi-maps preserving complements. If A and B are equivalently embedded in E^3 we can extend homeomorphisms. The following theorem and example also tell us more.

Theorem 1.6 Let $A \neq \emptyset$ be a proper open subset of X. There exists a pair (Y,B) such that any map $f: A \to B$ extends to an epi-map $F: X \to Y$ such that $X - A$ is mapped onto $Y - B$.

Proof: Let Y be Sierpinski Space, B the open point. If A is proper and closed we have the same result.

Example 1.7 The pair $(E^1, [0,1])$ has the property that if $[0,1]$ is mapped onto a non separating continuum $B \subset M^n$, an n-manifold, in such a way that the images of $\{0\}$ and $\{1\}$ are arcwise accessible from $M^n - B$, then the map extends to a map from E^1 onto M^n which preserves complements. The illustration of this fact is found in the proof of Theorem 2.9.

We might also ask whether a homeomorphism between arcs on spheres in E^n extends to a continuous function preserving complements. This is given a negative answer in Example 1.8.

A similar question for 2-spheres is found to have a negative answer in Example 1.4.

Example 1.8 Let T be a tame arc on S^2, a tame 2-sphere in E^3 and let W be a wild arc on the horned sphere of Alexander, A^2. Let $H: S^2 \to A^2$ be a homeomorphism extending the homeomorphism $h: T \to W$. If we assume H extends to $H: E^3 \to E^3$ such that $H(E^3 - S^2) = E^3 - A^2$, then by Theorem 1.3, A^2 is tame. We know this is false.

Another possible investigation is that of connected functions. A Connected function is a function (not necessarily continuous) which preserves connected sets. We could ask whether an epi-map $f: A \to B$ extends to a connected function. This is a weakening of the problem since a continuous function is connected but not conversely.

Example 1.9 A connected function which is not continuous.

Let $X = \{(x,y) : 0 \leq x \leq 1, -1 \leq y \leq 1\}$ and let

$$Y = \{(x,y) : 0 < x \leq 1, \quad y = \sin \tfrac{1}{x}\} \cup \{(0,y) : -1 \leq y \leq 1\}.$$

Define $f: X \to Y$ by

$$f(x,y) = \begin{cases} (x, \sin \frac{1}{x}), & x > 0 \\ (0,0), & x = 0 \end{cases}$$

f is obviously not continuous but it is connected.

In the remaining chapters we discuss more aspects of the extension problem. In Chapter III we discuss the extensions of proper maps. We give some "folklore results" and use these to obtain some answers to extension problems.

The main emphasis of Chapters II, IV, and V is the actual construction of maps. In Chapter II we ask whether there exists a map $f\colon E^3 \to E^3$ such that $f(A) = B$ and $f(E^3 - A) = E^3 - B$. We construct some maps f for some specified continua A and B. In the last part of Chapter II we weaken our requirements and construct maps between complements of non-separating continua in E^3. In Chapter IV we generalize Chapter II to higher dimensions, and construct a map from S^{k+3} onto S^{k+3} taking a wild k-cell onto a tame one.

Chapter V contains a different approach to the construction problems in Chapters II and IV. We construct functions which map wild arcs onto tame arcs but preserve complements only modulo a set of 3-dimensional Lebesgue measure zero.

In Chapter VI we consider some new definitions related to those of Chapter III and show some results dealing with more general point-set topology. We also consider some pathology in measure and dimension.

In this paper we will adopt the following definitions and notation. By E^n we will mean n-dimensional euclidean space with the usual topology, and S^n will denote the unit n-sphere.

$$S^n = \{(x_1, x_2, \ldots x_{n+1}) : x_1^2 + x_2^2 + \ldots x_{n+1}^2 = 1\} \subset E^{n+1}.$$

An <u>embedding</u> is a homeomorphism of a space A into a space B, and an <u>arc</u> is the homeomorphic image of the closed unit interval. A and B are equivalently embedded in S^n if there exists a homeomorphism of S^n onto itself taking A onto B. We adopt the definitions of wild and tame as found in Fox and Artin [9]. The piecewise linear topology discussed may be found in Hudson [12], and the dimension theory in Hurewicz and Wallman [13]. By $m_i(A)$ we mean the i-dimension Lebesgue measure of the set A, as can be found in Rudin [16].

Our goal in this paper is not necessarily the most general result possible, but rather constructive results which depend on the positioning of the continua A and B in E^n.

CHAPTER II

WEAK EQUIVALENCE BETWEEN CONTINUA

In this chapter we answer some modifications of Antoine's question in E^3. We concern ourselves primarily with the construction of continuous functions. We make the following definition.

<u>Definition 2.1</u> Two continua A and B in E^n are <u>weakly equivalent</u> if there exists a continuous $f: E^n \to E^n$ such that

1.) $f(A) = B$

2.) $f(E^n - A) = E^n - B$.

<u>Examples</u> Any two points are weakly equivalent. In fact, any two equivalently embedded continua are weakly equivalent. The solid flat torus in E^3 is weakly equivalent to an unknotted circle, however they are not equivalent.

In the remainder of this chapter we restrict our study to non-separating continua in E^3. Theorem 2.2 is a corollary to Theorem 2.4 but a simpler technique suffices so we include the proof.

<u>Theorem 2.2</u> Any tame arc A in E^3 is weakly equivalent to a point b' in E^3.

Proof: We construct the desired continuous function $f: E^3 \to E^3$ as follows. We first assume that A is a straight segment, for if not there exists an E^3 homeomorphism taking A onto a straight segment.

Let b be the midpoint of A. We enclose A in a decreasing sequence of closed cubes $\{C_i\}$ converging down to A. Similarly we enclose the point b' in a decreasing sequence of closed cubes $\{C_1'\}$ centered at and converging to b'. Consider the collection $\{E^3 - C_i\}$ of open sets which form an open covering of $E^3 - A$.

We construct a family $\{f_i\}$ of continuous functions, each f_i will be defined on $Cl(E^3 - C_i)$ and hence also defined on $E^3 - C_i$.

There exists $f_1 : Cl(E^3 - C_1) \to Cl(E^3 - C_1')$. We extend f_1 to $f_2 : Cl(E^3 - C_2) \to Cl(E^3 - C_2')$, so that $f_1 = f_2$ on the intersection of their domains. We construct f_2 as follows. For each $x \in Fr(C_1)$ we map the part of the straight line segment $\overline{bx}$ lying in $Cl(C_1 - C_2)$ onto the part of the segment $\overline{b'f_1(x)}$ lying in $Cl(C_1' - C_2')$. See Figure 1.

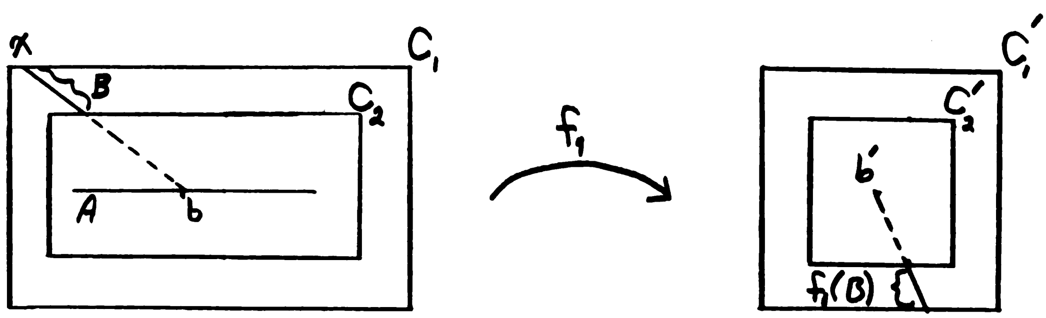

Figure 1.

We continue in this way, extending each

$f_i\colon \mathrm{Cl}(E^3 - C_i) \to \mathrm{Cl}(E^3 - C_i')$ to $f_{i+1}\colon \mathrm{Cl}(E^3 - C_{i+1}) \to \mathrm{Cl}(E^3 - C_{i+1}')$.

We have thus constructed a collection of maps $\{f_i\}$ with domains $\{E^3 - C_i\}$ such that the maps agree on the intersections of their domains. There exists a unique continuous $f\colon E^3 - A \to E^3 - b'$ which extends the f_i's. We extend f to a map defined on E^3 (we denote the extension by f also) by defining $f(A) = b'$. This gives the required function.

<u>Lemma 2.3</u> Let A be a non-separating continuum in E^3. Then there exists a decreasing sequence of compact 3-manifolds with connected boundary $\{M_i\}$ such that $\cap M_i = A$, $M_{i+1} \subset \mathrm{Int}\, M_i$.

<u>Proof</u>: We first triangulate E^3. Let M_1 be a second derived neighborhood of the set of simplicies meeting A in E^3. For each i let M_{i+1} be such a second derived neighborhood in M_i.

We claim that the $\{M_i\}$ just defined is the required collection of manifolds. By Theorem 2.11 in Hudson [12], the M_i are all regular neighborhoods, hence we have that each M_i is a compact 3-manifold, and since A does not separate, we assume each M_i has a connected boundary.

If we require the mesh of the subdivisions to go to zero, we have $A = \cap M_i$.

<u>Theorem 2.4</u> Any non-separating continuum $A \subset E^3$ is weakly equivalent to a point.

Proof: By Lemma 2.3 we enclose A in a sequence of compact 3-manifolds with connected boundary $\{M_i\}$ such that $M_{k+1} \subset \operatorname{Int} M_k$. We enclose the point b in a sequence of closed 3-balls $\{B_k\}$ centered at and converging to b.

Let $N_k = \mathrm{Cl}(M_k - M_{k+1})$ which is a compact 3-manifold with two boundary components, ∂M_k and ∂M_{k+1}. Let $A_k = \mathrm{Cl}(B_k - B_{k+1})$, a closed anular region with boundary components ∂B_k and ∂B_{k+1}. We map N_k onto A_k as follows.

Remove open 2-discs U_k and U_{k+1} from ∂M_k and ∂M_{k+1} respectively, and remove open 2-discs V_k and V_{k+1} from ∂B_k and ∂B_{k+1}. Construct tubes from ∂U_k to ∂U_{k+1} and from ∂V_k to ∂V_{k+1}. Let C_k be the closed 3-cell bounded by U_k, U_{k+1} and part of the constructed tube. Let C_k' be the closed 3-cell bounded by V_k, V_{k+1} and part of the constructed tube.

There exists a homeomorphism g_k from C_k onto C_k' and hence a homeomorphism of ∂U_k onto ∂V_k. Similarly we have a homeomorphism $g_{k+1} : \partial U_{k+1} \to \partial V_{k+1}$. Since g_k is defined on a closed subset of $\partial M_k - U_k$ into $\partial B_k - V_k$, a 2-cell, we can extend g_k by Tietze's Extension Theorem to: $G_k : (\partial M_k - U_k) \to (\partial B_k - V_k)$, similarly g_{k+1} extends to G_{k+1}. These maps give rise to a map f_k from $(\partial M_k - U_k) \cup (\partial M_{k+1} - U_{k+1}) \cup \{$2-dimensional boundary of $C_k - (U_k \cup U_{k+1})\}$ into a 3-cell. Again by Tietze's Extension Theorem, f_k extends to $F_k : N_k \to A_k$.

If the U_k and V_k are chosen properly for each k, we have a family of open sets $\{E^3 - M_k\}$ covering $E^3 - A$ and a family $\{F_k\}$ of maps such that $F_i = F_j$ on $N_i \cap N_j$. Thus there is a unique extension $F : E^3 - A \to E^3 - b$. We again extend to a map on E^3 by setting $F(A) = b$. The result is the required map.

The following corollary is a special case of Theorem 2.4. We are restricting our study to arcs in the next several results.

Corollary 2.5 Any arc A in E^3 is weakly equivalent to a point.

We now extend to show certain arcs are weakly equivalent to others. We begin by giving an example of a wild arc which is weakly equivalent to a tame arc in E^3.

Example 2.6 Consider the wild arc W in Figure 2. A and B are tame arcs intersecting at p. We can map W onto B by shrinking A to p. This can be done by a map defined on E^3 which preserves complements.

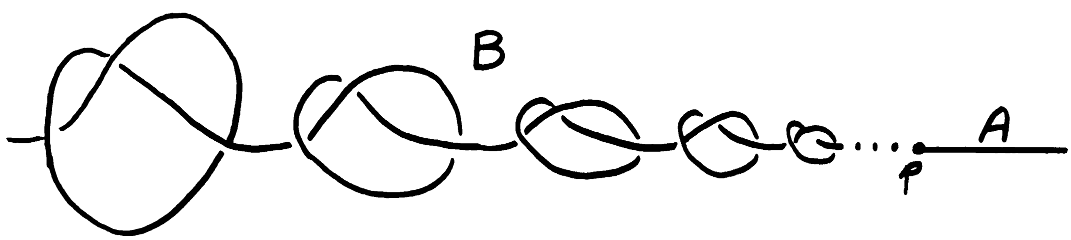

Figure 2

This example generalizes to Theorem 2.7.

Theorem 2.7 Let W be a wild arc in E^3 such that $W = A \cup B$ where A and B are tame arcs in E^3 and $A \cap B = p$. Then W is weakly equivalent to a tame arc.

Proof: Since A is tame in E^3, there exists a homeomorphism $h: E^3 \to E^3$ such that $h(A)$ is a straight segment in E^3. Since B is tame, $h(B)$ is also tame and $h(A) \cap h(B) = h(p)$. By Example 3.6 we can shrink $h(A)$ to $h(p)$.

Corollary 2.8 Let W be a wild arc in E^3 where $W = A_1 \cup A_2 \cup \cdots \cup A_n$ where A_i is a tame arc in E^3 and $A_i \cap A_{i+1} = p_i$. Then W is weakly equivalent to a tame arc.

Proof: Apply the proof of Theorem 2.7 n times.

We now give some constructive theorems to show that tame arcs are weakly equivalent to certain wild arcs. Hence we get a weak equivalence between certain types of wild arcs.

Theorem 2.9 Let W be a wild arc in E^3 such that W lies on the boundary of a 3-cell in E^3. Then any tame arc A in E^3 is weakly equivalent to W.

Proof: We define the map $f: E^3 \to E^3$, such that $f(A) = W$ and $f(E^3 - A) = E^3 - W$, by the composition of three maps.

We first assume A lies on the x-axis in E^3. We define $f_1 : E^3 \to E^3_+ = \{(x,y,z) : z \geq 0\}$ by

$$f_1(x,y,z) = \begin{cases} (x,y,z) & z \geq 0 \\ (x,y,-z) & z < 0 . \end{cases}$$

We note that $f_1(A) = A$.

Let $X = B^3 \cup L$ where B^3 is a closed 3-ball and L is a ray with end point $\{0\}$ in ∂B^3. We define $f_2 \colon E^3_+ \to X$ noting that f_2 can be chosen so that $f_2(A) \subset \partial B^3 - \{0\}$, preserving complements.

Assume $W \subset C^3$ where C^3 is a closed 3-cell in E^3. We construct $f_3 \colon X \to E^3$ using a collection of maps. There is a homeomorphism $h_o : B^3 \cup [0,1] \to C^3 \cup K$ where K is an arc meeting C^3 at only one end point and $h_o(A) = W$. We now map the "tail" of L onto the remainder of E^3 by repeated applications of the Hahn-Mazurkiewicz theorem, Hocking and Young [11] p. 129. We write $E^3 - C^3 = \cup F_i$ where each F_i is compact, connected and locally connected and $F_i \subset \operatorname{Int} F_{i+1}$. We extend h_o to h_1 by defining:

$h_1 = h_o$ on $B^3 \cup [0,1]$,

h_1 maps $[3/2, 2]$ onto F_1 and

h_1 maps $[1, 3/2]$ onto the path from $h_o(1)$ to $h_1(3/2)$.

We extend h_1 to h_2 by defining:

$h_2 = h_1$ on $B^3 \cup [0,2]$,

h_2 maps $[5/2, 3]$ onto F_2 and

h_2 maps $[2, 5/2]$ onto the path from $h_1(2)$ to $h_2(5/2)$.

We continue extending each h_{k-1} to h_k by defining:

$h_k = h_{k-1}$ on $B^3 \cup [0,k]$,

h_k maps $[\frac{2k+1}{2}, k+1]$ onto F_k and
h_k maps $[k, \frac{2k+1}{2}]$ onto the path from $h_{k-1}(k)$ to $h_k(\frac{2k+1}{2})$.
This gives a map $f_3 \colon X \to E^3$.

Finally we define $f = f_3 f_2 f_1$.

<u>Theorem 2.10</u> Let W be a wild arc in E^3 such that W lies on the boundary of a 2-disc. Then any tame arc $A \subset E^3$ is weakly equivalent to W.

<u>Proof</u>: The same technique used in the proof of Theorem 2.9 suffices here.

We now combine Corollary 2.8 with Theorems 2.9 and 2.10 to show certain wild arcs are weakly equivalent.

<u>Theorem 2.11</u> Any arc W which is the union of a finite number of tame arcs is weakly equivalent to any arc which lies on the boundary of a 3-ball in E^3 or which lies on the boundary of a 2-disc in E^3.

We define two continua A and B in E^n to be <u>very weakly equivalent</u> if there exists a map f from $E^n - A$ onto $E^n - B$. In [6], M. Brown defined the concept of cellularity in a manifold. A set A in an n-manifold is <u>cellular</u> if there is a decreasing sequence of n-cells $\{C_i\}$ such that $C_{i+1} \subset \operatorname{Int} C_i$ and $A = \bigcap C_i$. A set is <u>pointlike</u> it it has the same complement as a point. It is known that a set is pointlike if and only if it is cellular.

<u>Remark</u> Any two cellular sets in E^n are very weakly equivalent. The converse is false as seen in Example 2.13. We do however have the following.

<u>Theorem 2.12</u> Any non-separating continua A in E^3 is very weakly equivalent to any cellular set B in E^3.

<u>Proof</u>: We mimic the proof of Theorem 2.4. Enclose A in a decreasing sequence of 3-manifolds and enclose B in a decreasing sequence of 3-cells. We construct the map exactly as in the proof of Theorem 2.4 except we omit the last extension.

<u>Example 2.13</u> There is a non-cellular arc A E^3 which is very weakly equivalent to a cellular arc B. We use Example 1.1 in Fox and Artin [9]. The complement of this arc is not simply connected hence the arc is not cellular. But by Theorem 2.12 the arc is very weakly equivalent to a segment.

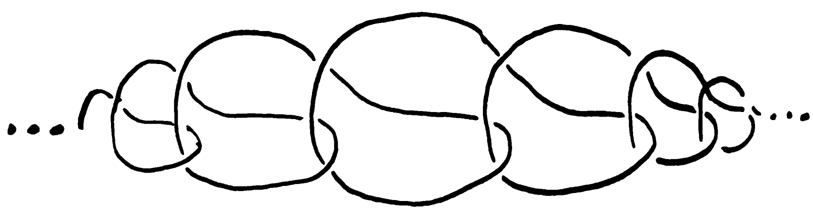

Figure 3

We may extend Theorem 2.12 to non-separating continua in E^n using techniques similar to those used in the proof of Theorem 2.9. We first prove the more general result.

<u>Theorem 2.14</u> Any two open connected n-manifolds are continuous images of each other.

<u>Proof</u>: Let M_1 and M_2 be open connected n-manifolds and let L be a locally flat ray in M_1 such that L is closed in M_1. Enclose L in a product neighborhood $E^{n-1} \times [0,1)$ as in Figure 4.

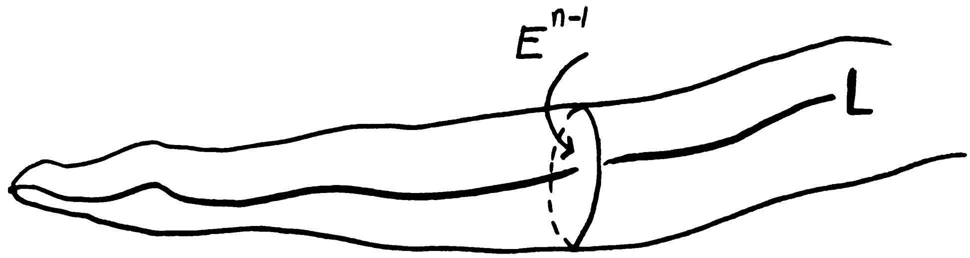

Figure 4

Let $M_1' = Cl(E^{n-1} \times [0,1))$. There is a copy of E^{n-1} in M_1' and M_1' retracts onto this copy of E^{n-1}. There is $f_1: M_1' \to E^{n-1}$. f_1 extends to an epi-map $f_2: M_1 \to E^{n-1} \times [0,1)$. We also have a retraction $f_3: E^{n-1} \times [0,1) \to L$.

Construct a map $f_4 : L \to M_2$ using the Hahn-Mazurkiewicz Theorem as in the proof of Theorem 2.9.

Define $f = f_4\ f_3\ f_2 : M_1 \to M_2$.

<u>Theorem 2.15</u> Any two non-separating continua in E^n are very weakly equivalent.

<u>Proof</u>: The complements of non-separating continua in E^n are open connected n-manifolds.

CHAPTER III

PROPER EQUIVALENCE OF SETS

Let $f: X \to Y$ be a continuous surjection. f is called a proper map if for each compact $A \subset Y$, $f^{-1}(A)$ is compact in X. We can modify the general Antoine type problem in terms of proper maps. Given sets in E^n when does there exist proper maps between them and when do these maps extend to proper maps on E^n? We first state some known results on proper maps.

Lemma 3.1 Let M_1 and M_2 be connected n-manifolds and let $f: M_1 \to M_2$ be a proper map. Then f extends to a map from the one point compactification of M_1 onto the one point compactification of M_2.

Lemma 3.2 If M_1 and M_2 are connected n-manifolds and $f: M_1 \to M_2$ is a proper map, then M_1 has at least as many ends as M_2, $|e(M_1)| \geq |e(M_2)|$.

Lemma 3.3 If $F: M_1 \to M_2$ is a proper map, f extends to a map from the Freudenthal compactification of M_1 to that of M_2.

Definition 3.4 Let M_1 and M_2 be connected n-manifolds. M_1 and M_2 are properly equivalent if there exists proper maps $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_1$.

<u>Theorem 3.5</u> Let A be a convergent sequence of points in S^n and let B be a cantor set in S^n. $S^n - A$ and $S^n - B$ cannot be properly equivalent.

<u>Proof</u>: If $f: S^n - A \to S^n - B$ were proper, then by Lemma 3.2 $|e(S^n - A)| \geq |e(S^n - B)|$. But $S^n - B$ has an uncountable number of ends and $S^n - A$ has only countably many.

The above method of proof shows that if A and B are compact and B has more components than A then $S^n - A$ is not properly equivalent to $S^n - B$.

<u>Theorem 3.6</u> If M_1 and M_2 are properly equivalent, then they have the same number of ends.

The following theorem was proved by Fort in [8]. We give a much simpler proof here.

<u>Theorem 3.7</u> Let C_1 and C_2 be cantor sets in E^n. If $E^n - C_1$ is homeomorphic to $E^n - C_2$, then C_1 and C_2 are equivalently embedded.

<u>Proof</u>: Let $h: E^n - C_1 \to E^n - C_2$ be a homeomorphism. Then h is proper and by Lemma 3.3 h extends to $H : E^n \to E^n$ which is a homeomorphism.

<u>Theorem 3.8</u> Let A and B be cellular continua in S^n. Then $S^n - A$ and $S^n - B$ are properly equivalent.

CHAPTER IV

WEAK EQUIVALENCE BETWEEN k-CELLS IN S^n

Chapter IV extends some results of Chapter II to higher dimensions. Using Theorem 2.4 with E^3 replaced by S^3, we construct a map defined on S^4 which takes a wild arc to a tame arc and preserves complements. We use an easy induction to show a wild k-cell in S^{k+3} is weakly equivalent to a tame k-cell. We lower the dimension to $k+2$ using Corollary 2.8 with E^3 replaced by S^3.

<u>Theorem 4.1</u> There exists a wild arc $W \subset S^4$ which is weakly equivalent to a tame arc $T \subset S^4$.

<u>Proof</u>: A wild arc A in S^3 is weakly equivalent to a point in S^3 by Corollary 2.5. Thus there exists $f: S^3 \to S^3$ such that $f(A) = b$, $f(S^3 - A) = S^3 - b$, and $\pi_1(S^3 - A) \neq 1$. We form the quotient space S^3/A by the natural map $p : S^3 \to S^3/A$. The map $f\,p^{-1} : S^3/A \to S^3$ is epi, carries the point $p(A)$ to b in S^3 and preserves complements. Bing shows in [4] that the suspension of S^3/A is topologically S^4. The suspension of $p(A)$ is a wild arc $W \subset S^4$ and the suspension of b is a tame arc $T \subset S^4$. Using the product structure of the suspension we can extend fp^{-1} to a map $F : S^4 \to S^4$. By construction $F(W) = T$ and $F(S^4 - W) = S^4 - T$.

Theorem 4.2 For any $k > 0$, there exists a wild k-cell $W^k \subset S^{k+3}$ which is weakly equivalent to a tame k-cell $T^k \subset S^{k+3}$. In fact W^k goes homeomorphically onto T^k.

Proof: We induct on k. Theorem 4.1 bases the induction. The product structure of the suspension gives the inductive step.

Theorem 4.3 For $k > 0$, there exists a wild k-cell $W^k \subset S^{k+2}$ which is weakly equivalent to a tame k-cell.

Proof: We again induct with Corollary 2.8 used as a base for the induction.

CHAPTER V

A MAP ON E^3 PRESERVING COMPLEMENTS a.e.

In this chapter we weaken condition 2.) of Definition 3.1 so that the map need only preserve complements modulo a set of 3-dimensional Lebesgue measure zero. We use the concept of decomposition space to construct the desired function for certain wild arcs.

The concept of decomposition space comes from R. L. Moore, L. Vietoris, and P. Alexandroff. The following definition is from Whyburn [17].

Definition 5.1 Let X be a separable metric space, a decomposition G of X is a representation

$$X = \cup A_\alpha \ , \ A_\alpha \in G$$

where the A_α's are closed and disjoint. The decomposition space X' of G is formed by taking elements of G as points of X', and open sets in X' are those sets of elements of G whose union is open in X.

In [5] Bing gives an example of decomposition of E^3 whose resulting decomposition space is again E^3. As elements of G he chooses two linked circles g_0 and g_1 which bound discs perpendicular to each other and intersecting along a common radius. He also chooses a parameterized family of figure-eight's, g_t, $0 \leq t \leq 1$, where the loops of g_t have radii t and $1 - t$. The other elements of G are points of E^3.

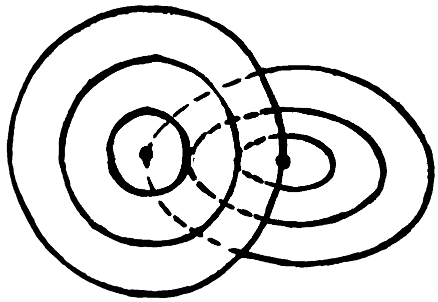

Figure 5

Persinger [15] has exhibited a family of wild arcs in E^3 which are equivalent to arcs embedded in a 3-book in E^3. If we assume one of these arcs in a 3-book is embedded in the union of the discs in Bing's example, when the decomposition space is obtained we will have mapped a wild arc in E^3 onto a segment in E^3 via a map defined on all of E^3. This map is seen to preserve the complement of the wild arc modulo a set of 3-dimensional Lebesgue measure zero.

CHAPTER VI

SOME POINT-SET RESULTS AND PATHOLOGY

This chapter contains more definitions resulting from the study of the Antoine problem. We use these new definitions to briefly develop some basic point-set properties.

Definition 6.1 A function $f: X \to Y$ is $|\Delta|$-piece continuous if there exists an index set Δ and a decomposition $X = \cup X_\alpha$, $\alpha \in \Delta$ such that f restricted to X_α is continuous for each α.

Example 6.2 Any compact metric space X is the 2-piece continuous image of the closed unit interval I. Let $I = C \cup (I - C)$ where C is a cantor set. There is a continuous surjection $f_1 : C \to X$. Define $f_2 : I - C \to X$ to be a constant function.

Theorem 6.3 Let W be an arc in E^3 and A a tame arc. Then there exists a 2-piece continuous $f: E^3 \to E^3$ such that $f_1(W) = A$ and $f_2(E^3 - W) = E^3 - A$.

We may use the above definition to formulate the following question. Given a topological space X and an integer k, what can be said about X if every $f : X \to Y$ is k-piece continuous?

Theorem 6.4 If every $f : X \to Y$ is 1-piece continuous, then X is discrete.

Proof: Choose Y to be a discrete space with the same cardinality as X and let $f : X \to Y$ be a bijection. Since f is 1-piece continuous, the pre-images of open sets, hence the pre-images of points, are open. Thus points are open in X and X is discrete.

Theorem 6.5 If every $f : X \to Y$ is 2-piece continuous, then $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, X_1 and X_2 are discrete but X need not be. We may conclude however that every point x of X has basic neighborhoods of the form $x \cup U$ where $x \in X_i$, $U \subset X_j$, $i \neq j$.

Proof: Again we choose Y discrete having the same cardinality as X and $f : X \to Y$ a bijection. Since f is 2-piece continuous, $X = X_1 \cup X_2$ where f restricted to X_i is continuous. Each X_i must be discrete and we may choose them disjoint. X need not be discrete. If we choose Y to be a 2-point space, each X_i is closed.

The proof of Theorem 6.5 generalizes to n-piece continuous. We may state this as follows:

Theorem 6.6 If every $f : X \to Y$ is n-piece continuous, then $X = X_1 \cup X_2 \cup \cdots \cup X_n$, $X_i \cap X_j = \emptyset$ if $i \neq j$, and each X_i is discrete.

We also consider some examples in euclidean space dealing with pathology in measure and dimension. We ask what relation, if any, exists between $m_i(A)$ and $\dim(A)$ where A is a compact set in E^n. We also mention some known facts about what continuous functions and homeomorphisms do to measure and dimension.

It is well known that continuous functions do not preserve either measure or dimension. A constant function can decrease both measure and dimension, and space filling curves will increase both. It follows from the inductive definition of dimension that homeomorphisms preserve dimension; but in the case of measure this is not true.

<u>Example 6.7</u> We give an example of a homeomorphism which does not preserve measure. The homeomorphism given will also extend to the whole space. Let C be the usual cantor set in E^1, $C \subset [0,1]$. We know $M_1(C) = 0 = \dim(C)$. But there is a homeomorphism $h : C - C'$ where C' is a "fat" cantor set such that $m_1(C') = \rho$, $0 < \rho < 1$, and $\dim(C') = 0$. Since cantor sets are equivalently embedded in E^1, the homeomorphism extends. This same idea can be used in higher dimensional euclidean spaces.

We next give an example of a wild set W in E^3 having $m_3(W) = \dim(W) = 0$, yet the projection $\pi(W)$ into the x-y plane has $\dim \pi(W) = 1$.

Example 6.8 Consider "Antoine's Necklace" a wild cantor set in E^3. We shall construct Antoine's Necklace in such a way that the 3-dim measure of W is zero. Let the first solid torus T, in the construction have volume V. By considering a torus in T whose cross section is concentric with that of T and has a radius $1/\sqrt{2}$ times that of T, we obtain a torus whose volume is $\frac{1}{2} V$ (by a theorem of Pappus). In this new torus we construct the 4 linked tori prescribed in the first stage of the construction of W. The total volume of these four tori is less than $\frac{1}{2} V$. We continue the construction by constructing in each T_{ij} four linked tori whose total volume is less than $\frac{1}{2}$ of the volume of T_{ij}. In the limit we will have a wild cantor set with the added property that the measure is zero.

When we project W into the x-y plane we have an object of dimension 1. At each stage of the construction the projection is a continua. And stage by stage these continua are nested. Hence in the limit the intersection is a continua. Therefore since $\pi(W)$ is connected and has more than one point, it has dimension at least one.

If $\dim \pi(W) = 2$, then by Theorem IV.3 Hurewicz and Wallman, $\pi(W)$ contains a disc. Say this disc has radius ϵ. If we look at the pre-image of this disc, it lies in one of the solid tori but cannot lie in all of them since their cross sections go to zero. Therefore $\dim \pi(W) = 1$.

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