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TOPICS ON THE THEORY OF HOMOGENEOUS RANDOM FIELDS

By

Ahmad Reza Soltani

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ABSTRACT

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A.R. Soltani

A set of random variables $\xi(t) = \{\xi_x(t),\ x \in X,\ t \in E^n\}$; where E^n is the Cartesian product of E with itself n-times and X is any given set; is called a random field, E being the set of real numbers R or the set of integers Z. Let $\mathcal{E}$ denote the expected value. It is assumed that $\mathcal{E}\xi_x(t) = 0$, $\mathcal{E}|\xi_x(t)|^2 < \infty$ as elements of a Hilbert space H of random variables ξ, $\mathcal{E}|\xi|^2 < \infty$ with the scalar product $\mathcal{E}\xi\overline{\eta}$, $\xi, \eta \in H$; and that X is a linear space. It is also assumed that $\xi_x(t)$ is linear in x and continuous in t in the Hilbert space. H. A random field $\xi(t)$ is called homogeneous if $\mathcal{E}\xi_x(t)\overline{\xi}_y(s)$ depends only on $t-s$, for all $x, y \in X$. Let $M(t)$ be the closed linear span of $\xi_x(t)$, $x \in X$, in H and $M(S) = \bigvee_{t \in S} M(t)$ be the closed linear span of $M(t)$, $t \in S$, in H. The dimension of the field is defined to be the dimension of $M(0)$. A random field is called r-regular or regular if $\bigvee_{t \in E^n} M(s\colon\ |s-t| > r)^{\perp} = M(E^n)$ or $\bigcap_{r>0} M(s\colon\ |s| > r) = \{0\}$ respectively. A field is called minimal if it is r-regular, for $r \to 0$.

The following topics on the theory of homogeneous random fiels are disucssed in this thesis.

(i) Regularities, (ii) L-Markov and Markov properties, (iii) Interpolation .

In Chapter I we assume that X is a separable Hilbert space, and give necessary and sufficient conditions for an infinite dimensional continuous parameter, $t \in R^n$, random field to be regular. Wold-Cramér concordance theorem is established. In particular, we prove that the spectral measure of any regular field is absolutely continuous with respect to the Lebesgue measure.

Chapter II deals with $B(X, Y)$-valued homogeneous random fields, where $B(X, Y)$ is the set of bounded linear operators from the Banach space X to the Hilbert space Y. Effective conditions for a $B(X, Y)$-valued homogeneous continuous or discrete parameter random field to be regular, r-regular, minimal, L-Markov or Markov is given. This Chapter extends our results of Chapter I on regularity, and the recent work of Rozanov of minimality and Markov property on continuous parameter $(t \in R^n)$ Hilbert space-valued random fields to the $B(X, Y)$-valued random fields. New results for discrete parameter $(t \in Z^n)$ fields are also given.

In Chapter III the interpolation problem of a finite dimensional discrete parameter homogeneous random field is discussed. A recipe formula for the linear interpolator of the random field $M(t)$, $t \in Z^n$, under the assumption that the spectral density and its inverse are square integrable is obtained. This in the univariate case, extends the recent work of Salehi and the earlier work of Rozanov, where the boundedness of the spectral density was assumed.

To my wife Afsaneh and my son Sohrab

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INTRODUCTION

The main purpose in the thoery of homogeneous random fields is to study and analyze the behavior of a family of random variables $\{\xi_x(t),\ x \in X,\ t \in E^n\}$, where X is any given set and E^n stands for the Cartesian product of E with itself n times, E being the set of real numbers R or the set of integrs Z. The questions which are raised in this regard have drawn the attention of many mathematicians and probabilists, and some important results in this area are included in the work of Helson and Lowdenslager [7], Kotani [10], McKean [21], Molchan [27], [28], Kotani and Okabe [11], Pitt [32], [33] and Rozanov [37], [39]. Topics in this theory include extrapolation theory, interpolation theory, L-Markov and Markov properties, regularities and prediction on finite domain. In particular the concepts of L-Markov and Markov properties and regularities have been investigated by several authors in recent years, where satisfactory answers have been obtained, c.f. Kallianpur and Mandrekar [8], Makagon and Weron [15], Molchan [28], Pitt [33], Rozanov [39], [40], Salehi and Scheidt [44]. Each topic may be considered for the univariate fields, multivariate fields, Hilbert space-valued fields or Banach space-valued fields. In a general setting with the use of the Kolmogorov isomorphism, the study of homogeneous random fields reduces to investigate the behavior of a family of closed subspaces $H(t)$, $t \in E^n$ of a Hilbert space Y. Let $\bigvee_{t\in E^n} H(t)$ denote the span closure of $H(t)$, $t \in E^n$, in Y. It is assumed that the family of operators U_t, $t \in E^n$, defined on $\bigvee_{t\in E^n} H(t)$ onto $\bigvee_{t\in E^n} H(t)$ by $U_t\, H(s) = H(s+t)$ are unitary and strongly continuous. The dimension of the field is defined to be the dimension of $H(0)$. The problems of regularities can be formulated as obtaining spectral charactrization for the fields with the property that for a bounded domain $S_r = \{t \in E^n: \ |t| > r\}$, $\bigvee_{t\in E^n} U_t\, H(S)_r^{\perp} = \bigvee_{t\in E^n} H(t)$ or $\bigcap_{r>0} H(S_r) = \{0\}$, where $H(S) = \bigvee_{t\in S} H(t)$ and $\perp$ stands for orthogonal complement in $H(E^n)$. Such fields are called r-regular or regular respectively. A field is miminal if it is r-regular, for $r \to 0$.

This thesis consists of three chapters. Each chapter starts with an introduction which provides ancillary materials to the chapter and contains a complete description of the topics of the chapter and the historical background of the related problems. Here is a brief description of the content of this thesis. In Chapter I we discuss finite dimensional as well as infinite dimensional regular fields. Necessary and sufficient conditions in terms of the spectral density of the field are obtained. Wold-Cramér concordance theorem is established. In particular we prove that the spectral measure of a regular field is absolutely continuous with repsect to the Lebesgue measure. This chapter extends the work of Rozanov [37] and Pitt [32] in the univariate case to the infinite dimensional case.

Chapter II gives a complete spectral charactrization for a Banach space-valued field or more generally for a $B(X,Y)$-valued field to be r-regular, regular, minimal, L-Markov or Markov, where

$B(X, Y)$ is the class of bounded linear operators on a Banach space X into a Hilbert space Y. This chapter extends the work of Rozanov [39] on continuous parameter $(t \in R^n)$ Hilbert space-valued random fields to the $B(X, Y)$-valued random fields. New results on discrete fields $(t \in Z^n)$ in this regard are also obtained. The techniques in Rozanov's work [39], and the existence of a square root for a nonnegative operator-valued function from a Banach space into its dual, [13], [25] are used in this Chapter.

Chapter III deals with the interpolation problem of a finite dimensional homogeneous discrete parameter random field $H(t)$, $t \in Z^n$. We obtain a recipe formula for the linear interpolator of the random field $H(t)$, $t \in Z^n$. More precisely let $\{x_k\}$, $k = 1,\ldots,q$, be an orthonormal basis in $H(0)$. We assume that each x_k, $k = 1,\ldots,q$ does not belong to $(\bigvee_{t\neq 0} H(t))\vee(H(0)\backslash x_k)$, where $H(0)\backslash x_k = \vee\{x_\ell, \ell = 1,\ldots,q, \ell \neq k\}$ (we call such a field a field with imperfect interpolation, this terminalogy is being adapted from Dym and McKean [5]). Then we give a recipe formula for expressing the linear projection of x_k, $k \in T_k$ on $\vee\{x_\ell\colon\ x \notin T_\ell, \ell = 1,\ldots,q\}$ as an infinite series expansion, where it is assumed that all the elements of $x_\ell(t)$, $\ell = 1,\ldots,q$ are known except for the values $x_\ell(t)$, $t \in T_\ell$, $\ell = 1,\ldots,q$; T_ℓ, $\ell = 1,\ldots,q$ are finite domains in Z^n. This result constitutes an extension of the recent work of Salehi [42], where similar recipe formula is obtained for univariate fields under much stronger assumption. This problem was first studied by Rozanov in 1960 [35].

CHAPTER I

ON REGULARITY OF HOMOGENEOUS RANDOM FIELDS

Introduction. Consider a family of real or complex-valued random variables $\xi_x(t)$, over a probability space $(\Omega, \mathcal{B}, P)$ where the index x runs through a set X and t is a point in R^n; we call $\xi(t) = \{\xi_x(t), x \in X, t \in R^n\}$ a random field. Let E denote the expected value. We assume that $E\, \xi_x(t)$ is zero and the correlation function $E\, \xi_x(s)\, \overline{\xi_y(t)}$ is continuous and is invarriant with respect to simultaneous translation of s and t, for arbitary $x, y \in X$. In this case the random field $\xi(t)$ is called homogeneous in the wide sense.

Now let X to be a separable Hilbert space and let $M(t)$ be the closed linear span of the variables $\xi_x(t)$, $x \in X$, considered as elements of the Hilbert space $L^2(\Omega, \mathcal{B}, P)$ of random variables ξ, $E|\xi|^2 < \infty$ with scalar product $E\, \xi_1\overline{\xi}_2$. Since $M(t)$ contains the complete information about ξ at t, it is natural to call $M(t)$, $t \in R^n$ a homogenous random field. {As an example of $\xi_x(t)$ we can consider an X-valued Gaussian random process $\xi(t)$ and define $\xi_x(t)$ to be the inner product of $\xi(t)$ with x in X this family is called an X-valued Gaussian family}. Following the work [3], [17], [31], [38], [39], [43], [46], etc., we may assume that $\xi_x(t)$, $t \in R^n$, is linear in the variable x.

Let U_t, $t \in R^n$, be a continuous group of unitary operators defined by the relation $U_t \xi_x(s) = \xi_x(t + s)$ on the closed linear span $M(R^n)$ of all variables $\xi_x(t)$, $x \in X$, $t \in R^n$ in $L^2(\Omega, \mathcal{B}, P)$, evidently $M(t) = U_t M(0)$.

Here we assume that there is a spectral density $f(\lambda)$, a bounded linear positive operator-valued function of the variable $\lambda \in R^n$ acting on the Hilbert space X such that

$$E\xi_x(s)\overline{\xi_y(t)} = \int_{R^n} e^{i\lambda(s-t)}(f(\lambda)x,\ y)d\lambda \qquad s,\ t \in R^n;\ x,\ y \in X.$$

Note that $(f(\lambda)x,x)$ is Lebegue integrable which implies $f^{\frac{1}{2}}(\lambda)\ x \in L^2\ (R^n,\ X,\ d\lambda)$, where $f^{\frac{1}{2}}(\lambda)$ is the square root of $f(\lambda)$ and $L^2(R^n,\ X,\ d\lambda)$ consists of all X-valued Lebesgue measurable functions $x(\lambda)$ with square integrable norm $\|x(\lambda)\|$; the inner product between x and y in this space is given by $\int_{R^n} (x(\lambda),\ y(\lambda))d\lambda$.

Corresponding to $M(t)$ we will consider the unitary isomorphic field

$$H(t) = e^{i\lambda t}\ \overline{f^{\frac{1}{2}}(\lambda)X}\ , \qquad t \in R^n,$$

where closure is taken in $L^2(R^n,\ X,\ d\lambda)$.

For $S \subseteq R^n$, let $H(S) = \bigvee_{t \in S} H(t)$ be the closed linear span of the spaces $H(t)$, $t \in S$ in $L^2(R^n,\ X,\ d\lambda)$. Clearly $H(R^n) = \bigvee_{t \in R^n} H(t)$.

We denote by $\|f(\lambda)\|$ the operator norm of $f(\lambda)$. In this chapter frequently we require that $\|f(\lambda)\|$ is integrable, i.e.,

$$\int_{R^n} \|f(\lambda)\| d\lambda < \infty. \tag{1.0.1}$$

This condition is automatically satisfied for an X-valued Gaussian family.

1.0.2 <u>DEFINITION</u>. The field $H(t)$ is called r-regular (for fixed $r > 0$) if

$$\bigvee_{t \in R^n} e^{i\lambda t} H(s: \ |s| > r)^{\perp} = H(R^n),$$

is called minimal if it is r-regular, for $r \to 0$, and is called regular if

$$\bigcap_r H(s: \ |s| > r) = \{0\}$$

($\perp$ stands for the orthogonal complement in $H(R^n)$).

We observe that "regularity" is equivalent to

$$\bigvee_r H(s: \ |s| > r)^{\perp} = H(R^n).$$

Note that for any fixed r_0 and $|s| < r - r_0$,
$H(t: \ |t| > r) \subseteq H(t: \ |t-s| > r_0) = e^{i\lambda s} H(t: \ |t| > r_0)$. Therefore

$$\bigvee_r H(t: \ |t| > r)^{\perp} \supseteq \bigvee_{s \in R^n} e^{i\lambda s} H(t: \ |t| > r_0)^{\perp}. \tag{1.0.3}$$

This shows every r-regular field is regular and obviously every minimal field is r-regular. But an r-regular field need not be minimal. As Theorem 1.1.17 shows each regular field is r-regular for some r when f has a finite rank. This problem remains open when the rank of f is not finite.

When the dimension of the field, i.e., $N = \dim H(0)$ is one, necessary and sufficient conditons for a homogeneous random field to be regular is given by Rozanov [37] for the discrete parameter space, $t \in Z^n$, and by D. Pitt for $t \in R^n$ [32]. The substance of their work is that every regular field is r-regular for some r. As we already mentioned r-regular fields are regular.

In the case of finite dimentional homogeneous random field, Salehi and Scheidt [44] have obtained necessary and sufficient conditions for r-regularity. They also considered the problem of regularity and gave a set of sufficient conditions which amounts to the notion of r-regularity. A. Makagon and A. Weron [15] have the same results under slightly weaker assumptions.

The problem of minimality for infinite dimensional case has been analyzed by Yu. A. Rozanov [39], where satisfactory answer to this problem is obtained. Rozanov's definition of minimality is in terms of conjugate system, and is equivalent to ours (c.f. Theorem 2.2.14). What remains to be studied is the problem of regularity for infinite dimensional as well as finite dimensional fields which is the subject of this chapter.

This chapter consists of two sections. In Section 1 we will give necessary and sufficient conditions for regularity (Theorem 1.1.13). In Section 2 we give the Wold-Cramér concordance theorem for a homogeneous random field $H(t) = e^{i\lambda t}\, \overline{g^{\frac{1}{2}}(\lambda)X}$, where $g(\lambda)$ is a positive operator-valued function which is the density of $F(\lambda)$ (the spectral measure of $H(t)$) with respect to (w.r.t.) some positive σ-finite measure τ. Our Theorem 1.1.18 shows that the analogue of results of Rozanov ([37], p. 384) and Pitt ([32], p. 385) for regularity of scalar-valued case remains valid for the vector-valued case.

Before closing this discussion we point out that the concept of one-sided regularity for stationary processes indexed by the reals or integers was introduced in connection with the time domain analysis of such processes. This notion played an important role in the extrapolation theory of univariate, [4], [9], [48]; multivariate [7], [19], [36]; infinite dimensional processes [2], [17], [24], [38], where satisfactory analytic characterization in term of spectral density for one-sided regularity have been obtained (see the forthcoming article [43] for further references and information). The concept of regularity as discussed in the present chapter is connected with the study of multiparameter stationary processes, i.e., random fields over R^n. Its role to the problem of minimality and interpolation is similar to the role of one-sided regularity to the problem of extrapolation of stationary processes with real parameter.

1.1 Regularity. In this section we discuss the problem of regularity for a homogeneous random field, the main result being Theorem 1.1.13. X is a seprable Hilbert space, $f(\lambda)$ is a spectral density. The symbols $L^2(\mathbb{R}^n, X, d\lambda), H_t, H(R^n)$, etc. are the same as introduced in the earlier section. We discuss some of the known results as they relate to our work.

Let A be a subspace of the separable Hilbert space $L^2(R^n, X, d\lambda)$. Let a_n, $n \geq 1$, be an orthonormal basis in A. By $\bar{\bar{A}}(\lambda)$ a.e. λ is meant the closed subspace in the Hilbert space X generated by all values $a_n(\lambda)$, $n \geq 1$. The subspace $\bar{\bar{A}}(\lambda)$ a.e. λ is independent of the choice of the basis a_n, $n \geq 1$, (c.f. [ ]).

Note that $\overline{f^{\frac{1}{2}}(\lambda)X}$ (closure in X) is a subspace of X a.e. λ. Since $f^{\frac{1}{2}}$ is bounded it easily follows that

$$\overline{f^{\frac{1}{2}}(\lambda)X} = (\overline{\overline{f(\cdot)X}})(\lambda) \qquad \text{a.e. } \lambda. \tag{1.1.1}$$

The following lemmas (Lemma 1.1.2 and 1.1.3) are due to Rozanov [38], and are stated here for later use; Lemma 1.1.2 also can be found in Helson's book [6].

1.1.2 LEMMA. Let A be a seprable Hilbert space and let B be a subspace of $L^2(R^n, A, d\lambda)$ then the doubly invariant subspace $L = \bigvee_{t \in R^n} e^{i\lambda t} B$ of $L^2(R^n, A, d\lambda)$, consists of all measurable functions $a(\lambda) \in L^2(A, R^n, d\lambda)$ such that $a(\lambda) \in \bar{\bar{B}}(\lambda)$ a.e. λ.

An immidiate consequence of Lemma 1.1.2 is that $\bar{\bar{L}}(\lambda) = \bar{\bar{B}}(\lambda)$ a.e. λ.

1.1.3 LEMMA. Let $S \subseteq R^n$, then $H(S)^{\perp} = f^{-\frac{1}{2}} \mathcal{B}_S$. Where $f^{-\frac{1}{2}}(\lambda)$ is the inverse of the restriction of $f^{\frac{1}{2}}(\lambda)$ to $\overline{f^{\frac{1}{2}}(\lambda)X}$, and is defined from $f^{\frac{1}{2}}(\lambda)X$ onto $\overline{f^{\frac{1}{2}}(\lambda)X}$ with the properties that $f^{-\frac{1}{2}}(\lambda)f^{\frac{1}{2}}(\lambda)a(\lambda) = a(\lambda)$ for any $a(\lambda) \in \overline{f^{\frac{1}{2}}(\lambda)X}$ and $f^{\frac{1}{2}}(\lambda)f^{-\frac{1}{2}}(\lambda)a(\lambda) = a(\lambda)$ for any $a(\lambda) \in f^{\frac{1}{2}}(\lambda)X$; and $\mathcal{B}_S$ consists of all X-valued Lebesgue measurable functions $b(\lambda)$ with

(i) $b(\lambda) \in f^{\frac{1}{2}}(\lambda)X$ a.e. λ (ii) $f^{-\frac{1}{2}}(\lambda)b(\lambda) \in L^2(R^n, X, d\lambda)$ and (iii) $\int_{R^n} e^{-i\lambda t}(b(\lambda), x)d\lambda = 0$ for all $t \in S$, and $x \in X$. In addition if condition (1.0.1) holds, then $\tilde{b}(t)$, the Fourier transform of $b(\lambda)$, is a well-defined X-valued Lebsegue measurable function for $t \in R^n$. To see this note that

$$|\int e^{-i\lambda t}(b(\lambda),x)d\lambda| = |\int e^{-i\lambda t}(f^{\frac{1}{2}}(\lambda)f^{-\frac{1}{2}}(\lambda)b(\lambda), x)d\lambda|$$

$$= |\int e^{-i\lambda t}(f^{-\frac{1}{2}}(\lambda)b(\lambda), f^{\frac{1}{2}}(\lambda)x)d\lambda|$$

$$\leq \int |(f^{-\frac{1}{2}}(\lambda)b(\lambda), f^{\frac{1}{2}}(\lambda)x)|d\lambda \tag{1.1.4}$$

$$\leq \int \|f^{-\frac{1}{2}}(\lambda)b(\lambda)\|\|f^{\frac{1}{2}}(\lambda)x\|d\lambda$$

$$\leq \{\int\|f^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2d\lambda\}^{\frac{1}{2}}\{\int\|f^{\frac{1}{2}}(\lambda)x\|^2d\lambda\}^{\frac{1}{2}}$$

$$= \{\int\|f^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2d\lambda\}^{\frac{1}{2}}\{\int(f(\lambda)x, x)d\lambda\}^{\frac{1}{2}}$$

$$\leq \{\int\|f^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2d\lambda\}^{\frac{1}{2}}\{\int\|f(\lambda)\|d\lambda\}^{\frac{1}{2}}\|x\|.$$

Therefore $\int e^{-i\lambda t}(b(\lambda), x)d\lambda$ defines a bounded linear functional on X, and thus there exists an X-valued Lebesgue measurable function $\tilde{b}(t)$ such that for each $x \in X$

$$\int e^{-i\lambda t}(b(\lambda), x)d\lambda = (\tilde{b}(t), x).$$

Furthermore

$$\|\tilde{b}(t)\| \leq \{\int \|f^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2d\lambda\}^{\frac{1}{2}}\{\int \|f(\lambda)\|d\lambda\}^{\frac{1}{2}} \tag{1.1.5}$$

{The integrals above are taken over R^n, and in the future whenever the domain of the integration is missing, it is understood that the integral is taken over R^n}.

In this chapter we let $T = \{t \in R^n: \ |t| \leq r\}$ and $S = T^c = \{t \in R^n: \ |t| > r$.

1.1.6 LEMMA. $\bigvee_S H(S)^\perp$ consists of all functions $a(\lambda) \in L^2(R^n, X, d\lambda)$ such that

$$a(\lambda) \in \overline{\bigcup_S \overline{\overline{H(S)}}^\perp (\lambda)} \qquad \text{a.e. } \lambda.$$

Proof. For $S_2 \subset S_1$, $H(S_2) \subset H(S_1)$ and therefore $H(S_2)^\perp \supset H(S_1)^\perp$ which says $H(S_r)^\perp$ are increasing sequence of subspaces as $r \to \infty$. This permits us to consider only a countable number of $H(S)^\perp$s.

$\bigvee_S H(S)^\perp$ is a doubly invarriant subspace. To see this note that for each S and $t \in R^n$ there is a pair of S', S'' such that $e^{i\lambda t} H(S)^\perp$ is contained in $H(S')^\perp$ and contains $H(S'')^\perp$ therefore:

$$e^{i\lambda t} \bigvee_S H(S)^\perp = \bigvee_S e^{i\lambda t} H(S)^\perp = \bigvee_S H(S)^\perp .$$

Now by Lemma 1.1.2, $\bigvee_S H(S)$ consists of all functions $a(\lambda)$ in $L^2(R^n, X, d\lambda)$ such that $a(\lambda) \in \overline{\overline{(\bigvee_S H(S)^\perp)}}(\lambda)$ a.e. λ and the latter is a.e. λ equal to $\overline{\bigcup_S \overline{\overline{H(S)}}^\perp(\lambda)}$ which completes the proof.

1.1.7 LEMMA. $\bigvee_S H(S)^{\perp} = H(R^n)$ if and only if

$$\overline{\bigcup_S \overline{\overline{(f^{-\frac{1}{2}}B_S)}}(\lambda)} = \overline{f^{\frac{1}{2}}(\lambda)X} \quad \text{a.e. } \lambda$$

Proof. Suppose $\overline{\bigcup_S \overline{\overline{(f^{-\frac{1}{2}}B_S)}}(\lambda)} = \overline{f^{\frac{1}{2}}(\lambda)X}$ a.e. λ. Let $a(\lambda) \in H(R^n)$ with $a(\lambda) \perp \bigvee_S H(S)^{\perp}$. For any S by (1.0.3) we have $\bigvee_{S_r} H(S_r)^{\perp} \supseteq \bigvee_t e^{i\lambda t} H(S)^{\perp}$. Thus $a(\lambda) \perp \bigvee_t e^{i\lambda t} H(S)^{\perp}$ for each S. This means that $\int e^{i\lambda t}(a(\lambda),c(\lambda))d\lambda = 0$ for any S, $t \in R^n$ and $c(\lambda) \in H(S)^{\perp}$. This implies that $a(\lambda) \perp c(\lambda)$ a.e λ in X which is equivalent to $a(\lambda) \perp \overline{\overline{H(S)}}(\lambda)$ a.e. λ. But by Lemma 1.1.3 $H(S)^{\perp} = f^{-\frac{1}{2}}B_S$, therefore $a(\lambda) \perp \overline{\overline{(f^{-\frac{1}{2}}B_S)}}(\lambda)$ a.e. λ for any S. As already mentioned in the proof of Lemma 1.1.6 we may consider a countable number of $H(S)^{\perp}$. Therefore $a(\lambda) \perp \bigcup_S \overline{\overline{(f^{-\frac{1}{2}}B_S)}}(\lambda)$ a.e. λ, and hence using the assumption we get $a(\lambda) \perp \overline{f^{\frac{1}{2}}(\lambda)X}$ a.e. λ. But $a(\lambda) \in H(R^n)$ and by Lemma 1.1.2 $H(R^n)$ consists of all functions $d(\lambda) \in L^2(R^n, X, d\lambda)$ with $d(\lambda) \in \overline{f^{\frac{1}{2}}(\lambda)X}$ a.e. λ; so that $a(\lambda)$ has to be zero a.e. λ showing $\bigvee_S H(S)^{\perp} = H(R^n)$.

For the necessity note that $\bigvee_S H(S)^{\perp} = H(R^n)$ implies $\overline{\overline{(\bigvee_S H(S)^{\perp})}}(\lambda) = \overline{\overline{H(R^n)}}(\lambda) = \overline{f^{\frac{1}{2}}(\lambda)X}$ a.e. λ, where the second equality holds by Lemma 1.1.2. Now by Lemma 1.1.6 $\overline{\overline{(\bigvee_S H(S)^{\perp}}}(\lambda) = \overline{\bigcup_S \overline{\overline{H(S)}}^{\perp}(\lambda)}$ a.e. λ and by Lemma 1.1.3 $\overline{\overline{H(S)}}^{\perp}(\lambda) = \overline{\overline{(f^{-\frac{1}{2}}B_S)}}(\lambda)$ a.e., λ. Therefore $\overline{\bigcup_S \overline{\overline{(f^{-\frac{1}{2}}B_S)}}(\lambda)} = \overline{f^{\frac{1}{2}}(\lambda)X}$ a.e. λ. This completes the proof.

The following important Lemma under the condition that $f(\lambda)$ is nuclear (trace class) and its nuclear norm is integrable is due to Rozanov [39]. This Lemma is still true under our weaker assumption (1.0.1). Since Rozanov's proof is too condenced, and is based on the nuclearity of $f(\lambda)$ we will give the proof in detail below. We point out that the technique of our proof is the same as the one given by Rozanov.

1.1.8 LEMMA. Suppose condition (1.0.1) is satisfied. Let G be any closed subspace of $H(S)^{\perp}$. Let $\{a_k(\lambda)\}$ be a complete orthonormal system in G. Define $b_k(\lambda) = f^{\frac{1}{2}}(\lambda)a_k(\lambda)$, then there exists a sub-maximal system of $b_k(\lambda)$, $k \geq 1$ denoted by $b_i^*(\lambda)$, $i = 1, 2, \ldots, M_G$ (M_G being finite or infinite) which are a.e., λ linearly independent in X. (Maximality here means that off a set of measure zero $b_k(\lambda) \in B(\lambda)$, where $B(\lambda)$ is the linear span of $b_i^*(\lambda)$, $i = 1, \ldots, M_G$ in X). Clearly $M_G \leq \dim X$.

Proof. Let $a(\lambda) \in G$. Then by Lemma 1.1.3, $a(\lambda) = f^{-\frac{1}{2}}(\lambda)b(\lambda)$, where $b(\lambda) \in \mathcal{B}_{\{s:\ |s| > r\}}$. Furthermore by (1.1.4) $(b(\lambda), x)$ is integrable and $\tilde{b}(t)$ is a well-defined X-valued function. Now $b(\lambda) \in \mathcal{B}_{\{s:\ |s| > r\}}$ implies that $\tilde{b}(t) = 0$ for $|t| > r$. Therefore $(b(\lambda), x) = \frac{1}{(2\pi)^n} \int_{|t| \leq r} e^{i\lambda t}(\tilde{b}(t), x)dt$. Now let

$$b(z)x = \frac{1}{(2\pi)^n} \int_{|t| \leq r} e^{izt}(\tilde{b}(t), x)dt, \text{ where } z \in \mathbb{C}^n \text{ and } z \cdot t = \sum_{i=1}^{n} z_i t_i.$$

$b(z)x$ is an entire analytic function defining on $\mathbb{C}^n$. In fact $b(z)x$ is analytic in each coordinate and the analiticity follows from Hartogs Theorem [30]. Also $(b(\lambda), x)$ is the boundry value of $b(z)x$ with $\operatorname{Re} z = \lambda$. For $|z| \leq \epsilon$ we have

$$|b(z)x| \leq \frac{1}{(2\pi)^n} \int_{|t|\leq r} e^{izt} |(\tilde{b}(t), x)| dt$$

$$\leq \frac{1}{(2\pi)^n} \int_{|t|\leq r} e^{|izt|} |(\tilde{b}(t), x)| dt$$

$$\leq \frac{1}{(2\pi)^n} e^{r\epsilon} \int_{|t|\leq r} |(\tilde{b}(t), x)| dt$$

$$\leq \frac{1}{(2\pi)^n} e^{r\epsilon} \{\int \|f(\lambda)\| d \cdot \int \|f^{-\frac{1}{2}}(\lambda) b(\lambda)\|^2 d\lambda\}^{\frac{1}{2}} \|x\| \int_{|t|\leq r} dt,$$

where the last inequality is by (1.1.4). Therefore with $C = \frac{1}{(2\pi)^n} \int_{|t|\leq r} dt$ we have

$$|b(z)x| \leq C\, e^{\epsilon r} \{\int \|f(\lambda)\| d\lambda \cdot \int \|f^{-\frac{1}{2}}(\lambda) b(\lambda)\|^2 d\lambda\}^{\frac{1}{2}} \|x\| \text{ for } |z| \leq \epsilon . \tag{1.1.9}$$

Similarly $\overline{(b(\lambda), x)}$ is the boundry value of the entire analytic function $\bar{b}(z)x$ defined by $\bar{b}(z)x = \int_{|t|\leq r} e^{izt} \overline{(\tilde{b}(t), x)} dt$, $\operatorname{Re} z = \lambda \cdot |\bar{b}(z)x|$ is also bounded for $|z| \leq \epsilon$ by the same bound occuring in (1.1.9). Also we have

$$|(b(x),x)| = \left| \frac{1}{(2\pi)^n} \int_{|t|\leq r} e^{i\lambda t} (\tilde{b}(t), x) \right| \leq \frac{1}{(2\pi)^n} \int_{|t|\leq r} |(\tilde{b}(t), x)| dt =$$

$$= \frac{1}{(2\pi)^n} \int_{|t|\leq r} \left| \int_{R^n} e^{-i\lambda t} (b(\lambda), x) d\lambda \right| dt \tag{1.1.10}$$

$$\leq \frac{1}{(2\pi)^n} \int_{|t|\leq r} \int_{R^n} |(b(\lambda), x)| d\lambda dt = \left(\frac{1}{(2\pi)^n} \int_{|t|\leq r} dt\right) \int_{R^n} |(b(\lambda), x)| d\lambda$$

$$= C\int_{R^n} |(b(\lambda),\ x)|d\lambda.$$

Now by using (1.1.10) and (1.1.4) we obtain that

$$|(b(\lambda),\ x)| \leq C \int_{R^n} |(b(\lambda),\ x)|d\lambda \leq C\ \{\int \|f(\lambda)\|d\lambda \cdot \int \|f^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2 d\lambda\}^{\frac{1}{2}}\|x\| .$$

therefore

$$\|b(\lambda)\| \leq C\ \{\int \|f(\lambda)\|d\lambda \cdot \int \|f^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2\}^{\frac{1}{2}}$$

(1.1.11)

$$= C\ \{\int \|f(\lambda)\|d\lambda \cdot \int \|a(\lambda)\|^2 d\lambda\}^{\frac{1}{2}} .$$

Now let $\{x_k\}$ be an orthonormal basis in X, and let $\{\alpha_k\}$ be a sequence of scalars with $\alpha_k \neq 0$ for all k and $\sum_k |\alpha_k|^2 < \infty$. We point out that for each λ, $b_i(\lambda) = f^{\frac{1}{2}}(\lambda)a_i(\lambda)$, $i = 1,\ldots,N$ are linearly independent in X if and only if the corresponding sequences $\{(b_i(\lambda),\ \alpha_k x_k)\}_{k=1}^{\infty}$, $i = 1,\ldots,N$ which can be regarded as rows of a matrix are linearly independent. The next step is to consider the matrix $\{(b_i(\lambda),\ \alpha_k x_k)\}_{k=1}^{\infty}$, $i = 1,\ldots,N$, and the Gram matrix of the sub-matrix consisting of the first m columns of the matrix $\{(b_i(\lambda),\ \alpha_k x_k)\}_k$, $i = 1,\ldots,N$; k finite or infinite. Suppose $d_{ij}^m(\lambda)$, $i,\ j = 1,\ldots,N$ are the entries of the Gram matrix. Then

$$d_{ij}^m(\lambda) = \sum_{k=1}^{m} (b_i(\lambda),\ \alpha_k x_k)(\overline{b_j(\lambda),\ \alpha_k x_k}) .$$

We observed earlier that $(b_i(\lambda),\ x)$, $(\overline{b_j(\lambda),\ x})$ are the boundry values of entire analytic functions $b_i(z)x$, $\overline{b}_j(z)x$ respectively. Therefore $d_{ij}^m(\lambda)$ is the boundry value of the entire analytic function $d_{ij}^m(z)$ and

$$|d^m_{ij}(z)| \le \sum_{k=1}^{m} |b_i(z)\alpha_k x_k| |\bar{b}_j(z)\alpha_k x_k|$$

$$\text{(1.1.12)} \qquad \le C^2 e^{2\epsilon r} \int \|f(\lambda)\| d\lambda \{\int \|f^{-\frac{1}{2}}(\lambda) b_i(\lambda)\|^2 d\lambda \cdot \int \|f^{-\frac{1}{2}}(\lambda) b_j(\lambda)\|^2\} \sum_{k=1}^{m} |\alpha_k|^2 \|x_k\|^2$$

$$= C^2 e^{2\epsilon r} \int \|f(\lambda)\| d\lambda \sum_{k=1}^{m} |\alpha_k|^2 \quad \text{for} \quad |z| \le \epsilon,$$

where the second inequality is by (1.1.9) and the equality is by the fact that $\|x_k\| = 1$ for all k and $f^{-\frac{1}{2}}(\lambda)b_i(\lambda) = a_i(\lambda)$, where $\{a_i(\lambda)\}$ is an orthonormal set in G with $\|a_i\|_{L^2(K)} = 1$.

Now since $\sum_k |\alpha_k|^2 < \infty$, (1.1.12) implies that each $d^m_{ij}(z)$, $i, j = 1,\dots,N$ converges uniformly on compact subsets of $\mathbb{C}^n$ to the entire analytic function $d_{ij}(z) = \sum_{k=1}^{\infty} (b_i(z)\alpha_k x_k)(\bar{b}_j(z)\alpha_k x_k)$ as $m \to \infty$ [30]. Define $D^m(\lambda)$ to be the determinant of the Gram matrix $\{d^m_{ij}(\lambda)\}$, $i, j = 1,\dots,N$, then $D^m(\lambda)$ are also the boundry values of entire analytic functions $D^m(z)$ for all m.

Clearly by uniformity in the argument given above, $\lim_{m\to\infty} D^m(z)$ is an entire analytic function. This implies that $\lim_{m\to\infty} D^m(\lambda)$ is the boundry value of an entire analytic function, and therefore it either vanishes identically or is different from zero, a.e. λ. In the latter case we agree to call the elements $b_i(\lambda)$, $i = 1,\dots,N$ a.e. λ linearly independent in X. The above procedure on N permits us to construct a sub-maximal system of $b_k(\lambda)$, say $b^*_{ki}(\lambda)$, $i = 1, 2,\dots,M_G$, which are a.e. λ linearly independent in X. Here maximality means that if $b_i(\lambda)$ is different from $b^*_{ki}(\lambda)$, $i = 1, 2,\dots,M_G$, then for each λ of a set of measure zero $b_i(\lambda), b_{k1}(\lambda),\dots,b_{kM}(\lambda)$ are linearly dependent, i.e., $b_i(\lambda) \in B(\lambda)$, a.e., λ. Obviously $M_G \leq \dim X$ and M_G could be finite or infinite. Proof of our lemma is complete.

A X-valued function $b(\lambda)$ in called weakly integrable if $\int (b(\lambda), x) d\lambda < \infty$. A weakly integrable X-valued function b is called S-exponential if $\int e^{-i\lambda t}(b(\lambda), x) d = 0$ for $t \in S$. When $S = (t \in R^n\colon\ |t| > r)$ we use the phrase r-exponential instead of S-exponential. Rozanov [39] and Pitt [32] have adopted this definition which reduces to the classical definition of exponential functions of type r on R by the help of the Paley-Wiener theorem, Dym and McKean [5], and exponential functions of type $T = S^c$ on R^n by the help of an n-dimensional version of the Paly-Wiener theorem, Stein [45]. An operator-valued function $\phi(\lambda)\colon\ X \to X$ is called r-exponential if for each $x \in X$, the X-valued function $\phi(\lambda)x$ is r-exponential.

The following Theorem gives necessary and sufficient conditions for regularity. Characterization for r-regularity could be deduced from Rozanov's work [39]. The extension of this criterion to the Banach space is given by our Theorem 2.2.14.

1.1.13 THEOREM. A homogeneous random field over a separable Hilbert space X, with spectral density $f(\lambda)$ satisfying the condition (1.0.1) is regular if and only if there exists a family of r-exponential operator-valued functions $\phi_r(\lambda)$, $r \to \infty$, such that

(i) Each $\phi_r(\lambda)$ in a Hilbert-Schmidt operator and $\phi_r(\lambda)X \subseteq f^{\frac{1}{2}}(\lambda)X$ a.e. λ.

(ii) $\overline{f^{-\frac{1}{2}}(\lambda)\phi_{r_1}(\lambda)X} \subseteq \overline{f^{-\frac{1}{2}}(\lambda)\phi_{r_2}(\lambda)X}$ a.e. λ with $r_1 < r_2$ and

$\overline{\bigcup_r \overline{f^{-\frac{1}{2}}(\lambda)\phi_r(x)X}} = \overline{f^{\frac{1}{2}}(\lambda)X}$ a.e. λ

(iii) For each r, $f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)$ is a Hilbert-Schmidt operator a.e. λ and $\int \| f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)\|_2^2 \, d\lambda < \infty$. Here $\| \; \|_2$ stands for the Hilbert-Schmidt norm.

Note: The condition (1.0.1) on $f(\lambda)$ is used only for the necessity part of the Theorem.

Proof. Sufficiency: Suppose there exists a family of r-exponential Hilbert-Schmidt operator-valued functions $\phi_r(\lambda)$ satisfying (i), (ii) and (iii). For each $x \in X$ define $\phi_r^x(\lambda) = \phi_r(\lambda)x$. The fact that ϕ_r is r-exponential satisfying (i) and (iii) implies $\phi_r^x(\lambda)$ belongs to $\mathcal{B}_{S_r}$ where $S_r = (t \in R^n\colon \; |t| \geq r)$. Thus $\phi_r X \subseteq \mathcal{B}_{S_r}$, where $\phi_r X$ and $\mathcal{B}_{S_r}$ are well defined classes of X-valued function and the inclusion is pointwise. Now $\phi_r X \subseteq \mathcal{B}_{S_r}$ implies $f^{-\frac{1}{2}}\phi_r X \subseteq f^{-\frac{1}{2}}\mathcal{B}_{S_r}$, where the inclusion is in $L^2(R^n, X, d\lambda)$. This implies that $(\overline{\overline{f^{-\frac{1}{2}}\phi_r X}})(\lambda) \subseteq (\overline{\overline{f^{-\frac{1}{2}}\mathcal{B}_{S_r}}})(\lambda)$ a.e. λ. Since $f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)$ is bounded linear operator a.e. λ we have $(\overline{\overline{f^{-\frac{1}{2}}\phi_r X}})(\lambda) = \overline{f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)X}$ a.e. λ, therefore $\overline{f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)X} \subseteq (\overline{\overline{f^{-\frac{1}{2}}\mathcal{B}_{S_r}}})(\lambda)$ a.e. λ. This means that $\overline{\bigcup_r \overline{f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)X}} \subseteq \overline{\bigcup_{S_r} (\overline{\overline{f^{-\frac{1}{2}}\mathcal{B}_{S_r}}})(\lambda)}$ a.e. λ. Using (ii) we get $\overline{f^{\frac{1}{2}}(\lambda)X} \subseteq \overline{\bigcup_S (\overline{\overline{f^{-\frac{1}{2}} \; \mathcal{B}_S}})(\lambda)}$ a.e. λ, and since the converse of this inclusion is always true we obtain $\overline{f^{\frac{1}{2}}(\lambda)X} = \overline{\bigcup_S (\overline{\overline{f^{-\frac{1}{2}}\mathcal{B}_S}})(\lambda)}$ a.e. λ. Thus by

Lemma 1.1.7 we have $H(R^n) = \bigvee_S H(S)^\perp$ which says the process is regular.

Necessity: $\{H(S_r)^\perp: S_r = (t: |t| > r)\}$ is an increasing sequence w.r.t. r, i.e., $H(S_{r_1})^\perp \subseteq H(S_{r_2})^\perp$ for $S_{r_1} \supset S_{r_2}$.

Let $\{a_{r,k}(\lambda)\}_k$ be an orthonormal basis in $H(S_r)^\perp$. Consider the sequence $b_{r,k}(\lambda) = f^{\frac{1}{2}}(\lambda)a_{r,k}(\lambda)$, $k = 1,\ldots$.

By Lemma 1.1.8 for each $H(S_r)^\perp$ there exists a maximal system $\{b^*_{r,i}(\lambda)\}$ $i = 1,\ldots,M_r$ (M_r being finite or infinite) which a.e. λ are linearly independent in X, and such that if $B_r(\lambda)$ denotes the linear span of $b^*_{r,i}(\lambda)$, $i = 1,\ldots,M_r$, then each $b_{r,k}$ belongs to $B_r(\lambda)$ a.e. λ. Each $a(\lambda) \in H(S_r)^\perp$ can be approximated by $a_{r,k}(\lambda)$ in $L^2(R^n, X, d\lambda)$, i.e., $\sum_{k=1}^{N} \alpha_k a_{r,k}(\lambda) \to a(\lambda)$ in $L^2(R^n, X, d\lambda)$, therefore there exists a subsequence of $\{\sum_{k=1}^{N} \alpha_k a_{r,k}(\lambda)\}_N$ which converges to $a(\lambda)$ pointwise in X. But $\sum_{k=1}^{N} \alpha_k a_{r,k}(\lambda) \in f^{-\frac{1}{2}}(\lambda)B_r(\lambda)$ a.e. λ. Thus $a(\lambda) \in \overline{f^{-\frac{1}{2}}(\lambda)B_r(\lambda)}$ a.e. λ. This clearly shows that

$$\overline{\overline{H(S_r)^\perp}}(\lambda) \subseteq \overline{f^{-\frac{1}{2}}(\lambda)B_r(\lambda)} \text{ a e. } \lambda, \tag{1.1.14}$$

$$\overline{f^{-\frac{1}{2}}(\lambda)B_{r_r}(\lambda)} \subseteq \overline{f^{-\frac{1}{2}}(\lambda)B_{r_2}(\lambda)} \text{ a.e. } \lambda. \tag{1.1.15}$$

Now based on $b^*_{r,i}(\lambda)$, $i = 1,\ldots,M_r$; we define the operator valued function $\phi_r(\lambda)x_k = \mu_k b^*_{r,k}(\lambda)$, where $\{x_k\}_k$ in an orthonormal basis in X and $\mu_k > 0$ for $k = 1,\ldots,M_r$ with $\sum_{k=1}^{M_r} \mu_k^2 < \infty$; and $\mu_k = 0$ for superplus x_k' s.

We extend each $\phi_r(\lambda)$ to a Hilbert-Schmidt operator-valued function on X such that $\phi_r(\lambda) \subset f^{\frac{1}{2}}(\lambda)X$ a.e. λ and $f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)$ is a Hilbert-Schmidt operator a.e. λ with $\int \|f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)\|_2^2\, d\lambda < \infty$. Furthermore each $\phi_r(\lambda)$ is r-exponential, i.e., $\int e^{-i\lambda t}(\phi(\lambda)x,y)d\lambda = 0$ for $x,y \in X$, $t \in S_r$ (see p. 11 of [39] and Chapter II p. 49 for extension to the Banach space).

By the way that $\phi_r(\lambda)$ is constructed we have $\overline{f^{-\frac{1}{2}}(\lambda)B_r(\lambda)} = \overline{f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)X}$ a.e. λ. Thus by (1.1.15) $\overline{f^{-\frac{1}{2}}(\lambda)\phi_{r_1}(\lambda)X} \subseteq \overline{f^{-\frac{1}{2}}(\lambda)\phi_{r_2}(\lambda)X}$. Furthermore $(\overline{\overline{f^{-\frac{1}{2}}B_{S_r}}}(\lambda) = \overline{\overline{H(S_r)}}^{\perp}(\lambda) \subseteq \overline{f^{-\frac{1}{2}}(\lambda)B_r(\lambda)} = \overline{f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)X}$ a.e. λ. The first equality is by Lemma 1.1.3 and the inclusion is by (1.1.14). Thus

$$\overline{\bigcup_r (\overline{\overline{f^{-\frac{1}{2}}B_{S_r}}})(\lambda)} \subseteq \overline{\bigcup_r \overline{f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)X}} \quad \text{a.e. } \lambda.$$

But by Lemma 1.1.7, regularity implies that

$$\overline{\bigcup_r (\overline{\overline{f^{-\frac{1}{2}}B_{S_r}}})(\lambda)} = \overline{f^{\frac{1}{2}}(\lambda)X} \quad \text{a.e. } \lambda.$$

Therefore we get $\overline{f^{\frac{1}{2}}(\lambda)X} = \overline{\bigcup_r \overline{f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)X}}$ a.e. λ and the proof is complete.

1.1.16 <u>COROLLARY</u>. Let $f(\lambda)$, the spectral density of a regular homogeneous random field satisfy the condition (1.0.1). Then $f(\lambda)$ has constant rank a.e. λ. This constant value is called the rank of f.

Proof. From the construction of $\phi_r(\lambda)$ in the proof of Theorem 1.1.13 we see that rank of $\phi_r(\lambda)X$ is constant a.e. $\lambda = M_r$. By 1.1.13 (i), $\phi_r(\lambda)X \subseteq f^{\frac{1}{2}}(\lambda)X$ a.e. λ. But $f^{-\frac{1}{2}}(\lambda)$ is 1-1 on $f^{\frac{1}{2}}(\lambda)X$ onto $\overline{f^{\frac{1}{2}}(\lambda)X}$. Thus rank of $f^{-\frac{1}{2}}(\lambda)\phi_r(\lambda)X = M_r$ a.e. λ. The result follows by 1.1.13 (ii).

As a Corollary to our Theorem 1.1.13 we get the following interesting result.

1.1.17 THEOREM. Let $H(t)$, $t \in R^n$, be a regular homogeneous random field with spectral density $f(\lambda)$ satisfying (1.0.1). If f is of finite rank then the process is r-regular for some $r > 0$.

Proof. The construction procedure for $\phi_r(\lambda)$ shows that each $\phi_r(\lambda)X$ has a constant dimension M_r a.e. λ. Also by Corollary 1.1.16, $f^{\frac{1}{2}}(\lambda)X$ has a constant dimension a.e. λ. Further more $\overline{f^{-\frac{1}{2}}(\lambda)\phi_{r_1}(\lambda)X} \subseteq \overline{f^{-\frac{1}{2}}(\lambda)\phi_{r_2}(\lambda)X}$ for $r_1 < r_2$. Therefore by Theorem 1.1.13 (ii), there exists r_0 such that $\overline{f^{-\frac{1}{2}}(\lambda)\phi_{r_0}(\lambda)X} = \overline{f^{\frac{1}{2}}(\lambda)X}$ a.e. λ. and this is equivalent to say that the process is r_0-regular, see [39] page 12. This also follows from our Theorem 2.2.14. This completes the proof.

We remark that for the univariate case the class $\mathcal{B}_S$, which was introduced in Lemma 1.1.3 coincides with the class of all functions φ of exponential type $T = S^c$ with $\int_{R^n} \frac{|\varphi|^2}{f} d\lambda < \infty$. For additional information on functions of exponential type see [5] and [45]. In the case that $t \in Z^n$, the elements of $\mathcal{B}_S$ are polynomials of the form $\sum_{k\in T} a_k e^{ik\cdot\theta}$ with $\int_{T^n} \frac{|\sum_{k\in T} a_k e^{ik\cdot\theta}|^2}{f} d\sigma < \infty$, where T^n is the n-dimensional torus and, $d\sigma$ in the Lebesques measure on T^n. The following Theorem which is an immidiate consequence of our Theorem 1.1.17 extends the results of Rozanov [37] and Pitt [32] to q-variate processes over R^n and, completes the work of Salehi and Scheidt [44] and Makagon and Weron [15] in interpolation theory.

1.1.18. THEOREM. Let $\xi(t)$ be a q-variate homogeneous random field with $t \in R^n$ and spectral distribution $F(\lambda)$. Then $\xi(t)$ is regular if and only if $F(\lambda)$ is absolutly continuous w.r.t. The Lebesgue measure and there exists a nonzero r-exponential matrix-valued function $\varphi(\lambda)$ such that

(i) rang $\varphi(\lambda)$ = rang $F'(\lambda)$ a e. λ, and

(ii) $\int \varphi^* F'^{-1} \varphi \, d\lambda$ exists and $\neq 0$.

Proof. The fact that for a regular field, $F(\lambda)$ is absolutely continuous will be proved in section 2. When the dimension of the field in finite, condition (1.0.1) automatically holds. From the proof of Theorem 1.1.17 follows that in the finite dimensional case, the spectral set of conditions for regularity given in Theorem 1.1.13 reduces to the present set of conditions (i) - (ii), and the proof is complete.

2. <u>The Wold-Cramér concordance</u>. Suppose the homogeneous random field $\xi(t)$ has a spectral distribution $F(\lambda)$ which is absolutly continuous w.r.t. a σ-finit positive measure τ, i.e., there exists a weakly integrable bounded linear positive operator-valued function $g(\lambda)$ with

$$E\xi_x(t)\overline{\xi_y(s)} = \int e^{i\lambda(t-s)}(g(\lambda)x,y)d\tau, \quad x,y \in X.$$

The bounded operator-valued function $g(\lambda)$ is called the density of $F(\lambda)$ w.r.t. τ, i.e., $g(\lambda) = \frac{dF(\lambda)}{d\tau}$.

The homogeneous random field $M(t)$ can be represented within a unitary isomorphism as

$$H(t) = e^{i\lambda t}\overline{g^{\frac{1}{2}}(\lambda)X} \tag{1.2.1}$$

where the closure is taken in $L^2(R^n, X, d\tau)$.

Similar to Lemma 1.1.3 $H(S)^{\perp}$ admits the following representation.

1.2.2 <u>LEMMA</u>. $H(S)^{\perp} = g^{-\frac{1}{2}} \mathcal{B}_S^g$ where $g^{-\frac{1}{2}}(\lambda)$ is the inverse of the restriction of $g^{\frac{1}{2}}(\lambda)$ to $\overline{g^{\frac{1}{2}}(\lambda)X}$ and $\mathcal{B}_S^g$ consists of all X-valued measurable function $b(\lambda)$ with (i) $b(\lambda) \in g^{\frac{1}{2}}(\lambda)X$ a.e. τ (ii) $g^{-\frac{1}{2}}(\lambda)b(\lambda) \in L^2(R^n, X, d\tau)$, (iii) $\int e^{-i\lambda t}(b(\lambda), x)d\tau = 0$, for all $x \in X$ and $t \in S$.

The proof is similar to the proof of Lemma 1.1.3, and therefore it is only sketched.

Proof. Take $a(\lambda) \in H(S)^{\perp}$, then $\int e^{-i\lambda t}(a(\lambda), g^{\frac{1}{2}}(\lambda)x)d\tau = 0$ for all $t \in S$ or $\int e^{-i\lambda t}(g^{\frac{1}{2}}(\lambda)a(\lambda), x)d\tau = 0$, $t \in S$. Let $b(\lambda) = g^{\frac{1}{2}}(\lambda)a(\lambda)$, then (i) and (iii) are obvious, and

$g^{-\frac{1}{2}}(\lambda)b(\lambda) = a(\lambda) \in L^2(R^n, X, d\tau)$ which implies (ii). The proof of the converse follows by reversing the order of discussion given above.

Let $F = F_a + F_s$ and $\tau = \tau_a + \tau_s$ be the Cramér-Lebesgue decomposition w.r.t. λ. Clearly $F(A) = \int_A \frac{dF_a}{d\lambda} d\lambda + F_s(A)$. Now F is absolutly continusous w.r.t. τ, therefore

$F(A) = \int_A \frac{dF}{d\lambda} d\tau = \int_A \frac{dF}{d\tau} \cdot \frac{d\tau_a}{d\lambda} d\lambda + \int_A \frac{dF}{d\tau} d\tau_s$. Therefore

$\int_A \frac{dF}{d\lambda} \cdot \frac{d\tau_a}{d\lambda} d\lambda - \int_A \frac{dF_a}{d\lambda} d\lambda = \int_A \frac{dF}{d\tau} d\tau_s - F_s(A)$ which implies

$$\frac{dF_a}{d\lambda} = g(\lambda) \frac{d\tau_a}{d\lambda} \quad \text{a.e. } \lambda. \tag{1.2.3}$$

Because $g(\lambda)$ is a.e. λ a bounded operator-valued function, (1.2.3) defines $\frac{dF_a}{d\lambda}$ as a bounded operator-valued function a.e. λ. We denote $\frac{dF_a}{d\lambda}$ by the usual notation $f(\lambda)$. In summary this discussion shows that $f(\lambda)$ is a weakly integrable positive operator-valued function. With this preparation we state the following lemma.

1.2.4 <u>LEMMA</u>. Let $H_\xi(t)$ be a homogeneous random field over a separable Hilbert space X, with spectral distribution $F(\lambda)$ admitting the spectral representation (1.2.1). Let $H_\eta(t)$ be the random field corresponding to the absolutly continuous part of $F(\lambda)$. Then $H_\xi(S)^\perp$ is isomorphic to $H_\eta(S)^\perp$, where S is the complement of a bounded set T.

Proof. Let $H_\eta(t) = e^{i\lambda t}\,\overline{f^{\frac{1}{2}}(\lambda)X}$, where $f = \frac{dF_a}{d\lambda}$ as above. Then $H_\eta(S)^\perp = f^{-\frac{1}{2}}\,\mathcal{B}_S^f$ and $H_\xi(S)^\perp = g^{-\frac{1}{2}}\,\mathcal{B}_S^g$ where $\mathcal{B}_S^f$ and $\mathcal{B}_S^g$ were described in Lemmas 1.1.3 and 1.2.3. We establish a correspondence between $H_\eta(S)^\perp$ and $H_\xi(S)^\perp$ as follows. Take $b(\lambda)$ in $\mathcal{B}_S^g$, then $\int e^{-i\lambda t}(b(\lambda), x)d\tau = 0$ for all $x \in X$ and $t \in S$. This implies that $(b(\lambda), x)d\tau$ is absolutly continuous w.r.t The Lebesgue measure with density $[(b(\lambda), x)d\tau]/d\lambda = (b(\lambda), x)\,\frac{d\tau_a}{d\lambda}$ for all $x \in X$. Therefore $b(\lambda)$ can be taken to be zero on the singular part of τ and $\int e^{-i\lambda t}(b(\lambda), x)d\tau = \int e^{-i\lambda t}(b(\lambda), x)d\tau_a$. Now let $d(\lambda) = \frac{d\tau_a}{d\lambda}\,b(\lambda)$. Then $d(\lambda)$ is a X-valued Lebesgue measurable function satisfying (i) $d(\lambda) \in f^{\frac{1}{2}}(\lambda)X$ a.e.λ by (1.2.3),

$$\text{(ii)}\qquad \int e^{-i\lambda t}(d(\lambda), x)d\lambda = \int e^{-i\lambda t}\left(\frac{d\tau_a}{d\lambda}\,b(\lambda), x\right)d\lambda$$

$$= \int e^{-i\lambda t}(b(\lambda), x)d\tau_a = \int e^{-i\lambda t}(b(\lambda), x)d\tau = 0$$

for all $x \in X$, $t \in S$ and

$$\text{(iii)}\quad \int \|f^{-\frac{1}{2}}(\lambda)d(\lambda)\|^2 d\lambda = \int \|g^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2\,\frac{d\tau_a}{d\lambda}\,d\lambda$$

$$= \int \|g^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2 d\tau_a = \int \|g^{-\frac{1}{2}}(\lambda)b(\lambda)\|^2 d\tau$$

therefore $d(\lambda) \in \mathcal{B}_S^f$. Let $f^{-\frac{1}{2}}(\lambda)d(\lambda)$ in $H_\eta(S)^\perp$ correspond to $g^{-\frac{1}{2}}(\lambda)b(\lambda)$ in $H_\xi(S)^\perp$. Computations in (iii) shows that this map is norm preserving. Furthermore for any $d(\lambda)$ in $\mathcal{B}_S^f$ define $b(\lambda)$ through $b\,\frac{d\tau_a}{d\lambda} = d$. The calculation above shows that this map is also onto. Hence the proof is complete.

1.2.5 <u>DEFFINITION</u>. A random field $H(t)$, $t \in R^n$, is said to be singular if $\bigcap_r H(S_r) = H(R^n)$.

1.2.6 <u>COROLLARY</u>. Let $H_\xi(t)$ be a homogeneous random field as in Lemma 1.2.4. Then $H_\xi(t)$ is singular if and only if $\mathcal{B}_S^f \equiv 0$ for all S.

Proof. Suppose $\mathcal{B}_S^f \equiv 0$, then $f^{-\frac{1}{2}} \mathcal{B}_S^f \equiv 0$ which says $H_\eta(S)^\perp \equiv 0$. Therefore by Lemma 1.2.4 $H_\xi(S)^\perp = 0$ which is equivalent to say that $H_\xi(t)$ is singular. Now assuming the process is singular, then $H_\xi(S)^\perp \equiv 0$ and by Lemma 1.2.4 $H_\eta(S)^\perp \equiv 0$ or $f^{-\frac{1}{2}} \mathcal{B}_S^f \equiv 0$ and since for any $b(\lambda) \in \mathcal{B}_S^f$ we have $b(\lambda) = f^{\frac{1}{2}}(\lambda) f^{-\frac{1}{2}}(\lambda) b(\lambda)$ we get $\mathcal{B}_S^f \equiv 0$ and this completes the proof.

1.2.7 <u>LEMMA</u>. Let $H_\xi(t)$, $t \in R^n$, be a homogeneous random field with spectral distribution $F(\lambda)$ satisfying (1.2.1). Let $F_a(\lambda)$ and $F_s(\lambda)$ denote the absolutely continuous part and the singular part of $F(\lambda)$ w.r.t. the Lebesgue measure. Then $\xi_x(t) = \eta_x(t) \oplus \zeta_x(t)$, $x \in X$, $t \in R^n$; $H_\eta(R^n) \perp H_\zeta(R^n)$, where η and ζ have spectral measures F_a and F_s respectively. Moreover $H_\xi(S) = H_\eta(S) \oplus H_\zeta(S)$ for any S the complement of a bounded set $T \subseteq R^n$, and $H_\xi(S)^\perp = H_\eta^\perp(S)$.

Proof. We note that $d\tau = d\tau_a + d\tau_s$, where τ_a and τ_s are absolutely continuous part and singular part of the measure τ w.r.t. the Lebesgue measure. Therefore

$$g(\lambda)d\tau = g(\lambda)d\tau_a + g(\lambda)d\tau_s . \tag{1.2.8}$$

Now let $H_\eta(t) = \overline{e^{i\lambda t} 1_{A^c} g^{\frac{1}{2}}(\lambda) X}$ and $H_\zeta(t) = \overline{e^{i\lambda t} 1_A g^{\frac{1}{2}}(\lambda) X}$, where A is the support of τ_s and closure is taken in $L^2(R^n, X, d\tau)$.

Hence $\xi_x(t) = \eta_x(t) \oplus \zeta_x(t)$, $x \in X$, $t \in R^n$. The subspaces $H_\eta(R^n)$ and $H_\zeta(R^n)$ are perpendicular to each other in $L^2(R^n, X, d\tau)$, because

$$\int (e^{i\lambda t} 1_{A^c} g^{\frac{1}{2}}(\lambda)x, e^{i\lambda s} 1_{A^c} g^{\frac{1}{2}}(\lambda)y)d\tau = \int e^{i\lambda(t-s)}(g^{\frac{1}{2}}(\lambda)x, g^{\frac{1}{2}}(\lambda)y) 1_A 1_{A^c} d\tau = 0.$$

We also note that by (1.2.3) $H_\eta(t)$ admits the spectral representation $H_\eta(t) = \overline{e^{i\lambda t} f^{\frac{1}{2}}(\lambda)X}$ with closure in $L^2(R^n, X, d\lambda)$. Therefore by Lemma 1.2.4 $H_\xi(S)^\perp = H_\eta(S)^\perp$. Furthermore $H_\eta(S) = H_\eta(R^n) \ominus H_\eta(S)^\perp \subseteq H_\xi(R^n) \ominus H_\xi(S)^\perp = H_\xi(S)$ this along with $\xi_x(t) = \eta_x(t) + \xi_x(t)$ show that $H_\zeta(S) \subseteq H_\xi(S)$ which completes the proof.

The well-known Wold decomposition theorem for the random field $H_\xi(t)$, $t \in R^n$, states that $\xi(t)$ can be uniquely decomposed into $\xi_x(t) = \eta_x(t) \oplus \xi_x(t)$, $x \in X$, $t \in R^n$; $H_\eta(R^n) \perp H_\xi(R^n)$, where $H_\eta(t)$ is a regular and $H_\zeta(t)$ is a singular field, and that they are subordinated to $H_\xi(t)$. The following result establishes the so-called Wold-Concordance Theorem which extends the work of Rozanov [37], Pitt [32], Makagon and Weron [15], Salehi and Scheidt [44] to the infinite dimensional case.

1.2.9 THEOREM. (Wold-Crámer Concordance) let $H_\xi(t)$, $t \in R^n$ be a homogeneous random field with spectral representation given by (1.2.1). Let $F_a(\lambda)$ and $F_s(\lambda)$ denote absolutely continuous part and singular part of $F(\lambda)$ w r t. the Lebesgue measure respectively. Suppose the density of $F_a(\lambda)$ satisfies the conditions (i), (ii) and (iii) of Theorem 1.1.7. Then $\xi(t)$, $t \in R^n$, can be uniquely decomposed in the form $\xi_x(t) =$

$\eta_x(t) \oplus \zeta_x(t)$, $x \in X$, $t \in R^n$, where the field $H_\eta(t)$ is regular, $H_\zeta(t)$ is singular and $H_\eta(R^n) \perp H_\zeta(R^n)$. Moreover for any set S the complement of a bounded set T, $H_\xi(S) = H_\eta(S) \oplus H_\zeta(S)$ and $F_\eta(\lambda) = F_a(\lambda)$ and $F_\zeta = F_s$.

Proof. By Lemma 1.2.7 $\xi_x(t)$ can be written as $\xi_x(t) = \eta_x(t) \oplus \zeta_x(t)$, $x \in X$, $t \in R^n$, $H_\eta(R^n) \perp H_\zeta(R^n)$; $H_\eta(t)$, $H_\zeta(t)$ have spectral measures F_a and F_s respectively and $H_\eta(t)$, $H_\zeta(t)$ are subordinated to $H_\xi(t)$. Since the density of F_a satisfies the conditions (i), (ii) and (iii) of the Theorem 1.1.13, by Theorem 1.1.7 $H_\eta(t)$ is regular Furthermore by Lemma 1.2.6 $H_\zeta(t)$ is singular, therefore the proof is complete by the uniqueness of the Wold decomposition.

1.2.10 <u>THEOREM</u>. The spectral measure of a regular homogeneous random field $H(t)$, $t \in R^n$, with spectral representation given by (1.2.1) is absolutely continuous with respect to the Lebesgue measure.

Proof. Let $H_\eta(t)$ and $H_\zeta(t)$ be the components of $H_\xi(t)$ in the decomposition of $H_\xi(t)$ in Lemma 1.2.7. Now $H_\xi(t)$ is regular which implies $\bigvee_S H_\xi(S)^\perp = H_\xi(R^n)$. But by Lemma 1.2.7 $H_\xi(S)^\perp = H_\eta(S)^\perp$. Therefore $\bigvee_S H_\eta(S)^\perp = H_\xi(R^n)$. Also by the construction of $H_\eta(t)$ we have $H_\eta(R^n) \subseteq H_\xi(R^n)$. Hence $H_\eta(R^n) \supseteq \bigvee_S H_\eta(S)^\perp = H_\xi(R^n)$, which implies $H_\eta(R^n) = H_\xi(R^n)$, $H_\eta(R^n)$ is regular and $H_\zeta(R^n) \equiv 0$. Thus $H_\zeta(t) \equiv 0$ which by Lemma 1.2.7 implies F_s the singular part of the spectral measure F is zero and this completes the proof.

1.2.11 <u>COROLLARY</u>. Suppose F the spectral measure of a regular homogeneous random field is of trace class (this is satisfied for the finite dimensional case). Then F is absolutely continuous w.r.t. Lebesgue measure.

Proof. Spectral representation (1.2.1) is automathically valid in this case. Now apply Theorem 1.2.10.

CHAPTER II

ON B(X, Y)-VALUED HOMOGENEOUS RANDOM FIELDS

<u>INTRODUCTION</u>. Let X be a complex-Banach space and $(\Omega, \mathcal{B}, P)$ be some probability space. Let ξ be a random variable of second order over $(\Omega, \mathcal{B}, P)$ taking values in the Banach space X, i.e., ξ: $\Omega \to X$, $x^*(\xi(\omega)) \in L^2(\Omega, \mathcal{B}, P)$ for all $x^* \in X^*$, where X^* is the space of bounded conjugate linear functionls on X, namely $X^* = \{x^*: x^* = \overline{x}', x' \in X'\}$, where X' stands for the space of bounded linear functionals on X. Let us define the linear operator T on X^* into $L^2(\Omega, \mathcal{B}, P)$ by $Tx^* = x^*(\ (\omega))$. Then by closed graph theorem T belongs to $B(X^*, Y)$, where $Y = L^2(\Omega, \mathcal{B}, P)$ and $B(X^*, Y)$ denotes the space of bounded linear operators from X^* to Y. This association induces a $B(X^*, L^2(\Omega, \mathcal{B}, P))$-valued process for each X-valued process. Therefore in general, we may consider $B(X, Y)$-valued stochastic processes, where X and Y are arbitary complex Banach and Hilbert spaces respectively.

Let ξ_t, $t \in R^n$ = the n-dimensional euclidean space, be a $B(X, Y)$-valued stochastic process, i.e., for each $t \in R^n$, $\xi_t \in B(X, Y)$. ξ_t is called homogeneous if $(\xi_t x, \xi_s y)_Y$ depends only on $t-s$ for all $x,y \in X$; $t, s \in R^n$; it is called continuous if $(\xi_t x, \xi_s y)_Y$ is continuous in t and s, where $(\ ,\ )_Y$ stands for the inner product in the Hilbert space Y. In this work we assume that the

the processes are always continuous. Corresponding to ξ_t define a closed subspace in Y spanned by the elements $\xi_t x$, $x \in X$. This closed subspace is denoted by $M(t)$. Since the strong closure of the family $\{\xi_t A\colon\ A \in B(X, X)\}$ is the same as $B(X, M(t))$(c.f. [22] p. 9, [31] p. 335), as far as terminology is concerned we will make no distinction between ξ_t and $M(t)$, $t \in R^n$, and we call them $B(X, Y)$-valued homogeneous random fields.

For any $S \subseteq R^n$, define $M(S) = \bigvee_{t \in S} M(t)$, the span closure in Y of the subspaces $M(t)$, $t \in S$. Clearly $M(R^n) = \bigvee_{t \in R^n} M(t)$.

2.0.1 DEFINITION. A $B(X, Y)$-valued homogeneous random field ξ_t, $t \in R^n$, is called r-regular $(r > 0)$ if

$$\bigvee_{t \in R^n} U_t\, M(t\colon\ |t| > r)^{\perp} = M(R^n),$$

where U_t is the continuous unitary shift operator defined on $M(R^n)$ onto $M(R^n)$ by $U_t \xi_s x = \xi_{t+s} x$ for all $x \in X$, and $t, s \in R^n$ ($\perp$ stands for orthogonal complement in $M(R^n)$).

2.0.2 DEFINITION. A $B(W, Y)$-valued homogeneous random field η_t, $t \in R^n$, is called an r-conjugate field to the random field ξ_t, $t \in R^n$ if

$$M_\eta(0) \subseteq M_\xi(t\colon\ |t| > r)^{\perp},$$

$$M_\eta(R^n) = M_\xi(R^n)$$

$$\dim M_\eta(0) \leq \dim M_\xi(0),$$

where W is a complex Banach space and $r > 0$.

2.0.3 DEFINITION. A $B(X, Y)$-valued homogeneous random field ξ_t, $t \in R^n$, is called minimal if it is r-regular, as $r \to 0$.

NOTE. The notion of minimality was first introduced by Rozanov [39] in terms of the conjugate system. Indeed he called a random field minimal, if for each r, $r \to 0$, the field has an r-conjugate field. Clearly this concept of minimality implies ours. We will prove that under the assumption (2.2.1) these two concepts of minimality are equivalent.

2.0.4 DEFINITION. A $B(X, Y)$-valued homogeneous random field ξ_t, $t \in R^n$, is called regular if

$$\bigcap_{r>0} \mathcal{M}(t: \ |t| > r) = \{0\}.$$

This concept of regularity is equivalent to

$$\bigvee_{r>0} \mathcal{M}(t: \ |t| > r)^{\perp} = \mathcal{M}(R^n).$$

This chapter consists of four sections. Section 1 consists of spectral representation of a $B(X, Y)$-valued homogeneous random field and some ancillary results for later use. In Section 2 necessary and sufficient conditions in terms of the spectral density for r-regularity, minimality or regularity is given respectively. It will be shown that every r-regular field has an r-conjugate field (Theorem 2.2.14) (evidently a random field with an r-conjugate field is r-regular). In Theorem 2.2.14, we will also give necessary and sufficient conditions for a homogeneous random field to admit an r-conjugate field. This result is the extension of the work of Rozanov [39] to the $B(X, Y)$-valued random fields (Rozanov obtained necessary and sufficient conditons for a Hilbert space valued homogeneous continuous parameter, $t \in R^n$, random field to admit an r-conjugate field (c.f. [39], 1976)). As a corollary to our work we derive necessary and sufficient conditions for minimality of a discrete parmaeter, $t \in Z^n$, random fields which independently has been obtained by Makagon [14]. We will also show that the conjugate field is a Hilbert space-valued field acting on Y with certain spectral representation. The work of regularity problem extends our result in Chapter 1 to B(X, Y)-valued fields.

In Section 3 the concept of complete minimality for the Banach space case will be discussed, and sufficient conditions for a minimal field to be completely minimal will be given. The key to this result is Theorem 2.3.8, which says that under certain conditions on spectral density, each r-regular fields admits a Hilbert space-valued spectral representation with a nuclear (trace class) density. The notion of complete minimality was introduced by Rozanov in 1976 [39] for Hilbert space-valued continuous parameter homogeneous random fields, and plays important role in analyzing Markov property. The role of conjugate field, which is an extension of biorthagonality (c.f. Masani [18], Nadkarni [29]) in Markov property was first observed by Kallianpur and Mandrekar [8]. We also introduce such a notion for discrete parameter random fields. Similar results as the continuous case are obtained for the discrete case. In particular we prove that every finite dimensional discrete parameter homogeneous minimal random field satisfies certain geometrical property (c.f. Theorem 2.3.18).

Section 4 discusses the L-Markov and Markov properties for the Banach space case. L-Markov property for discrete fields was first introduced by Rozanov (c.f. [37] 1967). Recent works in this topic are [1], [8], [10], [28], [40]. Necessary and sufficient conditions for a completely minimal field to be L-Markov or Markov are given. This also is an extension to the work in [39] to the $B(X, Y)$-valued homogeneous fields. Similar results for the discrete case are obtained. In summary this chapter extends Rozanov's work [39] and the work in Chapter I to $B(X, Y)$-valued continuous parameter random fields with new results on discrete parameter random fields. The techniques that we use are similar to the ones employed by Rozanov [39]. The existence of a square root for a $B^+(X, X^*)$-valued function (c.f. Miamee-Salehi [25] and Masani [20] is crucial in carrying out our work. Throughout this chapter the measurablity of any X^*-valued function b is understood to be in weak* sense, i.e., for each $x \in X$, $b(\lambda)x$ is a complex-valued measurable function. For such functions the integrals are taken in the sense of Pettis.

2.1 <u>Spectral Representation and Preliminaries</u>. Let ξ_t, $t \in R^n$, be a $B(X, Y)$-valued homogeneous random field. It is known that there exists a unique $B^+(X, X^*)$-valued measure F such that

$$(\xi_t x, \xi_s y) = \int_{R^n} e^{i\lambda(t-s)} (F(d\lambda)x)y \ , \quad x, y \in X,$$

where $B^+(X, X^*)$ stands for bounded linear positive operators on X to X^* [ ]. $F(\lambda)$ is called the spectral measure of the process ξ_t. When the derivative of F with respect to the Lebesgue measure exists we say that the process has a density. In this chapter we assume that the process ξ_t has a density $f(\lambda)$, a unique measurable $B^+(X, X^*)$-valued function with

$$(\xi_t x,\ \xi_s y) = \int_{R^n} e^{i\lambda(t-s)}\ (f(\lambda)x)y\ d\lambda \quad x,y \in X. \tag{2.1.1}$$

We also assume that the Banach space X is separable.

An interesting factorization result which was proved by A.G. Miamee and H. Salehi [25] is the following: for a separable Banach space X, any weakly integrable $B^+(X, X^*)$-valued function $f(\lambda)$ has a square root. More precisely there exists a separable Hilbert space K ($\dim K \leq \dim X$) and a measurable $B(X, K)$-valued function $Q(\lambda)$ such that

$$f(\lambda) = Q^*(\lambda)Q(\lambda) \quad \text{a.e. } \lambda, \tag{2.1.2}$$

where $Q^*(\lambda)\colon\ K \to X^*$ is the adjoint of $Q(\lambda)$ defined by

$$(Q^*(\lambda)Q(\lambda)x)y = (Q(\lambda)x,\ Q(\lambda)y),\ x,\ y \in X \tag{2.1.3}$$

<u>NOTE</u>. As we mentioned, the factorization result given above under the assumption that the Banach space X is separable was proved by A.G. Miamee and H. Salehi. This assumption was later relaxed to the separability of $F(R^n)X$ by A. Makagon [13], [14], where F is the spectral measure of the field. Since such a factorization is essential in our work, we assume that $F(R^n)X$ is separable. The justification for the separability assumption and the study of the Banach space-valued fields is discussed in [25] p. 548. We will continue with developing the spectral representation of a $B(X, Y)$-valued homogeneous random field.

From (2.1.2) and (2.1.3) we have

$$(f(\lambda)x)y = (Q^*(\lambda)Q(\lambda)x)y = (Q(\lambda)x,\ Q(\lambda)y),\ x,\ y \in X \tag{2.1.4}$$

Let $f(\lambda)$ be the density of the process ξ_t. Then (2.1.1) and (2.1.4) give the following:

$$(\xi_t x, \xi_s y) = \int_{R^n} e^{i\lambda(t-s)} (Q(\lambda)x, Q(\lambda)y)_K d\lambda$$

(2.1.5)

$$= \int_{R^n} (e^{i\lambda t} Q(\lambda)x, e^{i\lambda s} Q(\lambda)y)_K d\lambda$$

The above identity defines the so called Kolmogorov isomorphism map between the time and spectral domains. To be more precise let $H(t)$ be the span closure of $e^{i\lambda t} Q(\lambda)x$, $x \in X$ in $L^2(K) = L^2(R^n, K, d\lambda)$, where $L^2(K)$ consists of all measureable K-valued functions $x(\lambda)$ with square integrable norm $\|x(\ )\|$, with the inner product defined by $(x(\lambda), y(\lambda)) = \int_{R^n} (x(\lambda), y(\lambda))_K d\lambda$. Then by (2.1.5) $M(t) \subseteq Y$ is isomorphic to $H(t) \subseteq L^2(K)$. Evidently from (2.1.5) we have

(2.1.6) $$H(t) = e^{i\lambda t}\, \overline{Q(\lambda)X} ,$$

where closure is taken in $L^2(K)$. Define $H(S) = \bigvee_{t \in S} H(t)$, the span closure in $L^2(K)$ of $H(t)$, $t \in S \subseteq R^n$. Evidently $H(S)$ and $M(S)$ are isomorphic. Therefore we may consider $H(t)$ instead of $M(t)$. Definitions 2.0.1, 2.0.2, 2.0.3, and 2.0.4 can be defined for $H(t)$, $t \in R^n$ in a similar way.

<u>NOTATIONS</u>. Let A be a subset of $L^2(K)$, then A is separable and has an orthonormal basis. By $\overline{\overline{A}}(\lambda)$, we mean the span closure of the elements of the orthonormal basis of A in X (see Section 1.1). It is clear that $\overline{\overline{A}}(\lambda)$ is defined, a.e., λ and is independent of the choice of the orthormal basis. Let $Q(\lambda)$ be the square root

of $f(\lambda)$. Then $\overline{Q(\lambda)X}$ is a well defined closed subspace of K a.e. λ, and since $Q(\lambda)$ is bounded and linear we have $\overline{Q(\lambda)X} = \overline{(Q\,X)}(\lambda)$ a.e. λ (c.f. Section 1.1). We also introduce the following notations which are used heavily through this Chapter.

$$K(\lambda) = \overline{Q(\lambda)X} \quad \text{and} \quad X^*(\lambda) = Q(\lambda)(\overline{Q(\lambda)X}) \quad \text{a.e. } \lambda. \tag{2.1.7}$$

The following elementary lemma is essential.

2.1.8 LEMMA.

(a) For any $a(\lambda) \in \overline{Q(\lambda)X}$ we have $(Q(\lambda)x,\ a(\lambda))_K = \overline{(Q^*(\lambda)a(\lambda))(x)}$

(b) There exists a linear operator $Q^{*-1}(\lambda)\colon\ X^*(\lambda) \to K(\lambda)$ with the following properties:

(i) $Q^{*-1}(\lambda)Q^*(\lambda)a(\lambda) = a(\lambda)$ for any $a(\lambda) \in K(\lambda)$

(ii) $Q^*(\lambda)Q^{*-1}(\lambda)b(\lambda) = b(\lambda)$ for all $b(\lambda) \in X^*(\lambda)$

Proof. (a) Let $a(\lambda) \in K(\lambda)$, then there exists a sequence $\{Q(\lambda)y_n,\ y_n \in X\}$ such that $Q(\lambda)y_n$ tends to $a(\lambda)$ in K norm. This obviously implies

$$\begin{aligned}
(Q(\lambda)x,\ a(\lambda)) &= \lim_n\ (Q(\lambda)x,\ Q(\lambda)y_n) \\
&= \lim_n\ (Q^*(\lambda)Q(\lambda)x)(y_n) \qquad \text{by (2.1.3)} \\
&= \lim_n\ \overline{(Q^*(\lambda)Q(\lambda)y_n)}(x).
\end{aligned}$$

But $Q(\lambda)y_n$ tends to $a(\lambda)$ implies that $Q^*(\lambda)Q(\lambda)y_n$ tends to $Q^*(\lambda)a(\lambda)$ in X^* norm, because $Q^*(\lambda)$ is bounded. Now strong convergence in X^* implies weak convergence, therefore

$\lim_n \overline{(Q^*(\lambda)Q(\lambda)y_n)}(x) = \overline{(Q^*(\lambda)a(\lambda))}(x)$ and the proof is finished.

(b) All we need is to show that $Q^*(\lambda)$: $K(\lambda) \to X^*(\lambda)$ is one-to-one. Let $Q^*(\lambda)a(\lambda) = Q^*(\lambda)b(\lambda)$ for $a(\lambda), b(\lambda) \in K(\lambda)$. This implies that for any $x \in X$, $Q^*(\lambda)a(\lambda)x = Q^*(\lambda)b(\lambda)x$. This by part (a) is equivalent to $(a(\lambda), Q(\lambda)x) = (b(\lambda), Q(\lambda)x)$ for all $x \in X$, or $(a(\lambda)-b(\lambda), Q(\lambda)x) = 0$ for all $x \in X$, which implies $a(\lambda)-b(\lambda) \perp K(\lambda)$. But $a(\lambda)-b(\lambda) \in K(\lambda)$. Therefore for $a(\lambda) = b(\lambda)$ which completes the proof.

2.2 <u>Regularities</u>. The main purpose of this Section is to obtain necessary and sufficent conditions for a $B(X, Y)$-valued homogeneous field $H(t)$, $t \in R^n$, to be r-regular, minimal or regular. We will show in this section that every r-regular field has an r-conjugate field. The main results are Theorems 2.2.14, 2.2.19 and 2.2.22. Theorems 2.2.14(b) and 2.2.22 extend the work of Rozanov (c.f. [39] p. 12) and the work in Chapter 1 to $B(X, Y)$-valued homogeneous random fields respectively. Throughout this Section we assume that the random field $H(t)$, $t \in R^n$, admits the spectral representation (2.1.6) and its spectral density $f(\lambda)$ satisfies the following condition

$$(2.2.1) \qquad \int_{R^n} \|f(\lambda)\| d\lambda < \infty .$$

We start with the following lemma.

2.2.2 <u>LEMMA</u>. For any X^*-valued measurable function $b(\lambda)$ satisfying

(i) $b(\lambda) \in X^*(\lambda)$, a.e., λ and (ii) $\int_{R^n} \|Q^{*^{-1}} b(\lambda)\|^2 d\lambda < \infty$,

the function $\tilde{b}(t)$, $t \in R^n$ defined by $\tilde{b}(t)x = \int_{R^n} e^{-i\lambda t} b(\lambda)x d\lambda$ is an X^*-valued measurable function and

$$\|\tilde{b}(t)\| \leq (\int \|f(\lambda)\| d\lambda \cdot \int \|Q^{*^{-1}}(\lambda)b(\lambda)\|^2 d\lambda)^{\frac{1}{2}}. \tag{2.2.3}$$

When the domain of integration is not specified, it is understood that the integration is taken over R^n.

Proof. All we need is to show that $\overline{\int e^{-i\lambda t} b(\lambda)x d\lambda}$ defines a bounded linear functional on X. The linearity of $\overline{\int e^{-i\lambda t} b(\lambda)x d\lambda}$ is obvious once we have shown that $\int |b(\lambda)x| d\lambda < \infty$. By using (i), (ii) and 2.1.8(a) we have

$\overline{b(\lambda)}x = (Q^*(\lambda)Q^{*^{-1}}(\lambda)b(\lambda))(x) = (Q(\lambda)x, Q^{*^{-1}}(\lambda)b(\lambda))$. Therefore

$$\begin{aligned} |\int e^{-i\lambda t}\overline{b(\lambda)x} d\lambda| &\leq \int |(Q(\lambda)x,\ Q^{*^{-1}}(\lambda)b(\lambda))|\, d\lambda \\ &\leq \int \|Q(\lambda)x\|\ \|Q^{*^{-1}}(\lambda)b(\lambda)\| d\lambda \\ &\leq (\int \|Q(\lambda)x\|^2)^{\frac{1}{2}} \cdot (\int \|Q^{*^{-1}}(\lambda)b(\lambda)\|^2 d\lambda)^{\frac{1}{2}}. \end{aligned}$$

But $\|Q(\lambda)x\|^2 = (Q(\lambda)x, Q(\lambda)x) = (Q^*(\lambda)Q(\lambda)x)(x) = (f(\lambda)x)(x) \leq \|f(\lambda)\|\|x\|^2$.
Thus

$$|\int e^{i\lambda t}\ \overline{b}(\lambda)x d\lambda| \leq \int |\overline{b}(\lambda)x| d\lambda \leq (\int \|f(\lambda)\| d\lambda \cdot \int \|Q^{*^{-1}}(\lambda)b(\lambda)\|^2)^{\frac{1}{2}} \|x\| \tag{2.2.4}$$

This shows that $e^{i\lambda t}\ b(\lambda)x d\lambda$ defines a bounded linear functional on X satisfying (2.2.3). The proof is complete.

2.2.5 DEFINITION. A X^*-valued measurable function $b(\lambda)$ is called weakly integrable if $\int|b(\lambda)x|d\lambda < \infty$ for any $x \in X$, and is called S-exponential if in addition

$$\int e^{-i\lambda t}\, b(\lambda)x d\lambda = 0 \quad \text{for all} \quad x \in X \quad \text{and} \quad t \in S,$$

where S is a subset of R^n (see the discussion on S-exponential functions following Lemma 1.1.8).

Similar to Chapter 1, let B_S denote the class of all S-exponential functions $b(\lambda)$ with

(i) $b(\lambda) \in X^*(\lambda)$, a.e. λ, and (ii) $\int\|Q^{*^{-1}}(\lambda)b(\lambda)\|^2 d\lambda < \infty$.

With this notation the following important lemma characterizes the orthogonal complement of $H(S)$ in $H(R^n)$ which will be denoted by $H(S)^{\perp}$.

2.2.6 LEMMA. $H(S)^{\perp} = Q^{*^{-1}} B_S$

Proof. Let $a(\lambda) \in H(S)^{\perp}$. Then $a(\lambda) \perp H(S)$ which is equivalent to $\int(a(\lambda), e^{i\lambda t}\, Q(\lambda)x)_K\, d\lambda = 0$ for all $t \in S$ and $x \in X$. But by lemma 2.1.8(a) $(a(\lambda), e^{i\lambda t}\, Q(\lambda)x)_K = e^{-i\lambda t}\,(a(\lambda), Q(\lambda)x)_K = e^{-i\lambda t}(Q^*(\lambda)a(\lambda))(x)$. (Here we note that by Lemma 1.1.2 $H(R^n)$ consists of all K-valued functions $a(\lambda)$ which take values in $K(\lambda)$, a.e., λ. Therefore $a(\lambda) \in H(S)^{\perp}$ satisfies the requirement of Lemma 2.1.8 which already used in the above identity). Therefore $\int e^{-i\lambda t}\,(Q^*(\lambda)a(\lambda))(x)d\lambda = 0$ for all $t \in S$ and $x \in X$. This implies that $b(\lambda) = Q^*(\lambda)a(\lambda)$ is an S-exponential function. $b(\lambda)$ obviously takes values in $X^*(\lambda)$ a.e. λ. But $Q^{*^{-1}}(\lambda)b(\lambda) = a(\lambda)$ which belongs to $L^2(K)$ i.e. $\int\|Q^{*^{-1}}(\lambda)b(\lambda)\|^2 d\lambda < \infty$. Thus $b(\lambda) \in B_S$ which finishes one part of the proof, i.e., any function $a(\lambda) \in H(S)^{\perp}$ is of the form $a(\lambda) = Q^{*^{-1}}(\lambda)b(\lambda)$ with $b(\lambda) \in \mathcal{B}_S$. The proof of the remaining follows by just reversing the argument given above.

The following lemma plays an important role in developing the main results of this section. As it was mentioned earlier in Chapter I, under the assumption that $f(\lambda)$ is nuclear, namely of trace class, and its nuclear norm is integrable the lemma below was proved by Rozanov [39] for the Hilbert space-valued homogeneous random fields. In Chapter I we relaxed the nuclearity of f and the integrability of its nuclear norm to the integrability of $\|f(\lambda)\|$ (c.f. Lemma 1.1.8). Although with the help of Lemmas 2.2.2 and 2.2.6 the proof for the Banach space case can be carried out similar to the one given in Lemma 1.1.8, nevertheless we give the proof in detail for the convenience of the readers. In summary we extend below Lemma 1.1.8 to the $B(X, Y)$-valued homogeneous random fields under the assumption (2.2.1).

2.2.7 <u>LEMMA</u>. Let G be any closed subspace of $H(s\colon |s| > \epsilon)^{\perp}$ and let $\{a_k(\lambda)\}$ be a complete orthonormal system in G. Also let $b_k(\lambda) = Q^*(\lambda)a_k(\lambda)$. Then there exists a sub-maximal system of $b_k(\lambda)$'s denoted by $b_{k_i}(\lambda)$, $i = 1, 2, \ldots, M_G$ (M_G being finite or infinite) which are a.e. λ linearly independent in X^* (maximality means that off a set of measure zero $b_k(\lambda) \in B(\lambda)$ for all k, where $B(\lambda)$ is the linear span of $b_{k_i}(\lambda)$, $i = 1, 2, \ldots, M_G$ in X^*). Clearly $M_G \leq \dim X^*(\lambda) \leq \dim X^*$, a.e. λ.

Proof. Let $a(\lambda) \in G$. Then by Lemma 2.2.6 $a(\lambda) = Q^{*^{-1}}(\lambda)b(\lambda)$, where $b(\lambda) \in \mathcal{B}_{\{s\colon |s| > \epsilon\}}$. Furthermore by the proof of Lemma 2.2.2 $b(\lambda)x$

is integrable and $\tilde{b}(t)$ is a well defined X^*-valued function.

Now $b(\lambda) \in \mathcal{B}_{\{s:\ |s| > \epsilon\}}$ implies that $\tilde{b}(t) = 0$ for $|t| > \epsilon$.

Therefore $b(\lambda)\ x = \frac{1}{(2\pi)^n} \int_{|t| \le \epsilon} e^{i\lambda t}\, \tilde{b}(t)x dt$. Now let

$b(z)x = \frac{1}{(2\pi)^n} \int_{|t| \le \epsilon} e^{izt}\, \tilde{b}(t)xdt$, where $z \in ¢^n$ and $zt = \sum_{i=1}^{n} z_i t_i$.

$b(z)x$ is an entire analytic function defined on $¢^n$. In fact $b(z)x$ is analytic in each coordinate and the analiticity follows from Hartogs Theorem [30]. Finally $b(\lambda)x$ is the boundry value of $b(z)x$ with $\operatorname{Re} z = \lambda$.

For $|z| \le r$ we have

$$
\begin{aligned}
|b(z)x| &\le \frac{1}{(2\pi)^n} \int_{|t| \le \epsilon} |e^{izt}|\ |\tilde{b}(t)x| dt \\
&\le \frac{1}{(2\pi)^n} \int_{|t| \le \epsilon} e^{|izt|}\ |\tilde{b}(t)x| dt \\
&\le \frac{1}{(2\pi)^n} e^{\epsilon r} \int_{|t| \le \epsilon} |\tilde{b}(t)x| dt \\
&\le \frac{1}{(2\pi)^n} e^{\epsilon r} \left(\int \|f(\lambda)\| d\lambda \cdot \int \|Q^{*^{-1}}(\lambda) b(\lambda)\|^2 d\lambda\right)^{\frac{1}{2}} \|x\| \int_{|t| \le \epsilon} dt,
\end{aligned}
$$

where the last inegualıty is by (2.2.3). Therefore with

$C = \frac{1}{(2\pi)^n} \int_{|t| \le \epsilon} dt$ we have

$$
(2.2.8) \qquad |b(z)x| \le C\, e^{\epsilon r} \left(\int \|f(\lambda)\| d\lambda \cdot \int \|Q^{*^{-1}}(\lambda) b(\lambda)\|^2 d\lambda\right)^{\frac{1}{2}} \|x\| \quad \text{for} \quad |z| \le r
$$

Similarly $\overline{b(\lambda)x}$ is the boundry value of the entire analytic function $\bar{b}(z)x$ defined by $\bar{b}(z)x = \int_{|t| \le \epsilon} e^{izt}\, \overline{\tilde{b}(t)}, x\ dt$, $\operatorname{Re} z = \lambda$. $|\bar{b}(z)x|$ is also bounded for $|z| \le r$ by the same bound occurring in (2.2.8).

Also we have

$$|b(\lambda)x| = \left|\frac{1}{(2\pi)^n}\int_{|t|\le\epsilon} e^{i\lambda t}\,\tilde{b}(t)x\,dt\right| \le \frac{1}{(2\pi)^n}\int_{|t|\le\epsilon} |\tilde{b}(t)x|\,dt$$

$$= \frac{1}{(2\pi)^n}\int_{|t|\le\epsilon}\left|\int_{R^n} e^{-i\lambda t} b(\lambda)x\,d\lambda\right|dt$$

(2.2.9)

$$\le \frac{1}{(2\pi)^n}\int_{|t|\le\epsilon}\int_{R^n} |b(\lambda)x|\,d\lambda\,dt = \left(\int_{R^n} |b(\lambda)x|\,d\lambda\right)\frac{1}{(2\pi)^n}\int_{|t|\le\epsilon} dt$$

$$= C\int_{R^n} |b(\lambda)x|\,d\lambda.$$

Now by using (2.2.9) and (2.2.4) we obtain that

$$|b(\lambda)x| \le C\int_{R^n} |b(\lambda)x|\,d\lambda \le C\left(\int\|f(\lambda)\|d\lambda\cdot\int\|Q^{*^{-1}}(\lambda)b(\lambda)\|^2\right)^{\frac{1}{2}}\|x\| .$$

Therefore

$$\|b(\lambda)\| \le C\left(\int\|f(\lambda)\|d\lambda\cdot\int\|Q^{*^{-1}}(\lambda)b(\lambda)\|^2\right)^{\frac{1}{2}}$$

(2.2.10)

$$= C\left(\int\|f(\lambda)\|d\lambda\cdot\int\|a(\lambda)\|^2 d\lambda\right)^{\frac{1}{2}} .$$

Now let $\{x_k\}$ be a dense linear set in X with $\|x_k\| = 1$ for all k, and let $\{\alpha_k\}_k$ be a sequence of scalars with $\alpha_k \neq 0$ for all k, $\sum_k |a_k|^2 < \infty$. Each $b_i(\lambda) = Q^*(\lambda)a_i(\lambda)$ takes values in X^*. For each λ and each $x \in X$, $b_i(\lambda)x$ is uniquely determined by the sequence $\{b_i(\lambda)\alpha_k x_k\}_{k=1}^{\infty}$. In fact for each λ, $b_i(\lambda)x$ is a conjugate of a bounded linear functional on X and each bounded linear functional is uniquely determined by its values on a dense linear subset.

Also for each λ, $b_i(\lambda)$, $i = 1,\dots,N$ are linearly independent in X^* if and only if the corresponding sequences $\{b_i(\lambda)\alpha_k x_k\}_{k=1}^{\infty}$, $i = 1,\ldots,N$ which can be regarded as rows of a matrix are linearly independent. To see this let us assume that $b_i(\lambda)$, $i = 1,\ldots,N$ are linearly independent in X^* and there exists β_i, $i = 1,\ldots,N$ such that

$$\beta_1\{b_1(\lambda)\alpha_k x_k\}_k + \beta_2\{b_2(\lambda)\alpha_k x_k\}_k + \ldots + \beta_N\{b_N(\lambda)\alpha_k x_k\}_k = 0 \text{ for all } k.$$

This implies that $\beta_1\, b_1(\lambda)\alpha_k x_k + \beta_2\, b_2(\lambda)\alpha_k x_k + \ldots + \beta_N b_n(\lambda)\alpha_k x_k = 0$ for all k or

$$\{\beta_1\, b_1(\lambda) + \beta_2 b_2(\lambda) + \ldots + \beta_N b_N(\lambda)\}\alpha_k x_k = 0 \text{ for all } k. \text{ But}$$

$\beta_1 b_1(\lambda) + \beta_2 b_2(\lambda) + \ldots + \beta_N b_N(\lambda)$ is a conjugate of a bounded linear functional and by the argument above it has to be the zero element in X^*, i.e. $\beta_1 b_1(\lambda) + \beta_2 b_2(\lambda) + \ldots + \beta_N b_N(\lambda) = 0$. But $b_i(\lambda)$, $i = 1,\ldots,N$ are linearly independent therefore β_i, $i = 1,\ldots,N$ have to be zero. The proof of the other part is stright forward. The next step is to consider the matrix $\{b_i(\lambda)\alpha_k\ x_k\}_k$, $i = 1,\ldots,N$ (k being finite or infinite) and look at the Gram matrix of the sub-matrix consisting of the first m columns of the matrix $\{b_i(\lambda)\alpha_k x_k\}_k$, $i = 1,\ldots,N$; k finite or infinite. Suppose $d^m_{ij}(\lambda)$, $i, j = 1,\ldots,N$ are the entires of the Garm matrix. Then

$$d^m_{ij}(\lambda) = \sum_{k=1}^{m} (b_i(\lambda)\alpha_k x_k)(\overline{b_j(\lambda)\alpha_k x_k}).$$

we observed earlier that $b_i(\lambda)x$ and $\overline{b_j(\lambda)x}$ are the boundry values of entire analytic functions $b_i(z)x$ and $\overline{b}_j(z)x$ respectively. Therefore $d^m_{ij}(\lambda)$ is the boundry value of the entire analytic function

$d^m_{ij}(z)$ and

$$|d^m_{ij}(z)| \leq \sum_{k=1}^{m} |(b_i(z)\alpha_k x_k)| |\bar{b}_j(z)\alpha_k x_k|$$

$$(2.2.11) \quad \leq C^2 e^{2\epsilon r} \int \|f(\lambda))\|d\lambda \sum_{k=1}^{m} (\int \|Q^{*^{-1}}(\lambda)b_i(\lambda)\|^2 d\lambda)^{\frac{1}{2}} (\int \|Q^{*^{-1}}(\lambda)b_j(\lambda)\|^2 d\lambda)^{\frac{1}{2}} |\alpha_k|^2 \|x_k\|^2$$

$$= C^2 e^{2\epsilon r} \int \|f(\lambda)\|d\lambda \sum_{k=1}^{m} |\alpha_k|^2 \quad \text{for} \quad |z| \leq r,$$

where the second inequality is by (2.2.8) and the equality is by the fact that $\|x_k\| = 1$ for all k and $Q^{*^{-1}}(\lambda)b_i(\lambda) = a_i(\lambda)$, where $\{a_i(\lambda)\}$ is an orthonormal set in G with $\|a_i\|_{L^2(K)} = 1$.

Now since $\sum_k |\alpha_k|^2 < \infty$, (2.2.11) implies that each $d^m_{ij}(z)$, $i, j = 1,\ldots,N$ converges uniformly on compact subsets of $\mathbb{C}^n$ to the entire analytic function $d_{ij}(z) = \sum_{k=1}^{\infty} (b_i(z)\alpha_k x_k)(\bar{b}_i(z)\alpha_k x_k)$ as $m \to \infty$ [30].

Define $D^m(\lambda)$ to be the determinant of the Gram matrix $\{d^m_{ij}(\lambda)\}$ $i, j = 1, 2,\ldots,N$, then $D^m(\lambda)$ are also the boundry values of entire analytic functions $D^m(z)$ for all m. Clearly by uniformity of the argument given above, $\lim_{m\to\infty} D^m(z)$ is an entire analytic functions.

This implies that $\lim_{m\to\infty} D^m(\lambda)$ is the boundry value of an entire analytic function, and therefore it either vanishes identically or is different from zero a.e. λ. In the latter case we agree to call the elements $b_i(\lambda)$, $i = 1,\ldots,N$ a.e. λ linearly independent in X^*. The above procedure on N permits us to construct a sub-maximal system of $b_k(\lambda)$, say $b_{k_i}(\lambda)$, $i = 1, 2,\ldots,M_G$, which are a.e. λ linearly independent in X^*. Here maximality means that if $b_j(\lambda)$ is different from $b_{k_i}(\lambda)$, $i = 1, 2,\ldots,M_G$, then for each λ off a set of measure zero. $b_j(\lambda), b_{k1}(\lambda), b_{k2}(\lambda), \ldots, b_{kM}(\lambda)$ are linearly dependent, i.e., $b_j(\lambda) \in B(\lambda)$ a.e. λ. Obviously $M_G \leq \dim X^*(\lambda) \leq \dim X^*$ and M_G could be finite or infienite. The proof of our lemma is complete.

We recall from the introduction that a homogeneous field $H_r^*(t) = e^{i\lambda t} H_r^*(0)$ is called an r-conjugate field of the field $H(t)$ if (i) $H_r^*(0) \subset H(s: |s| > r)^{\perp}$, (ii) $H_r^*(R^n) = H(R^n)$ and $\dim H_r^*(0) \leq \dim H(0) \leq \dim K$.

The following lemma is an immediate consequence of lemma 1.1.2.

2.2.12 LEMMA. Let A be any subspece of $H(R^n)$, then

$$\bigvee_t e^{i\lambda t} A = H(R^n) \text{ if and only if } \overline{A}(\lambda) = K(\lambda) \text{ a.e. } \lambda$$

Proof: By Lemma 1.1.2 $\bigvee_t e^{i\lambda t} A = H(R^n)$ if and only if $\overline{\overline{A}}(\lambda) = \overline{\overline{H(R^n)}}(\lambda)$ a.e. λ. But by the same lemma $H(R^n) = \bigvee_t e^{i\lambda t} \overline{Q(\lambda)X}$ implies that $\overline{\overline{H(R^n)}}(\lambda) = K(\lambda)$ a.e. λ. and proof is finished.

2.2.13 COROLLARY. Let $H_r^*(t)$ be an r-conjugate field of $H(t)$, then

$$\overline{\overline{H_r^*(0)}}(\lambda) = K(\lambda) \text{ a.e. } \lambda.$$

Proof: Use Lemma 2.2.12 and the fact that $H(R^n) = H_r^*(R^n) = \bigvee_t e^{i\lambda t} H_r^*(t)$.

The following theorem gives necessary and sufficient conditions for a Banach space valued homogeneous random field $H(t)$, $t \in R^n$, to be r-regular. It also shows that the Defenitions 2.0.1 and 2.0.2 are equivalent, in particular, every r-regular field has an r-conjugate field. Part (b) of the theorem given below extends the work of Rozanov (c.f. [39] p. 12) to the $B(X, Y)$-valued homogeneous fields.

2.2.14 THEOREM. Let $H(t)$, $t \in R^n$, be a $B(X, Y)$-valued homogeneous random field with spectral density $f(\lambda)$ satisfying (2.2.1). Then

(a) the field $H(t)$, $t \in R^n$, is r-regular if and only if there exists an r-exponential $B(K,X^*)$-valued function $\varphi_r(\lambda)$ such that

(i) $\varphi_r(\lambda)K \subseteq X^*(\lambda)$ a.e.λ (c.f. (2.1.7) for $X^*(\lambda)$),

(ii) $\overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\, K} = K(\lambda)$ a.e.λ (c.f. (2.1.7) for $K(\lambda)$),

(iii) $g_r(\lambda) = [Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^*[Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]$ is a nuclear function in K i.e. $Q^{*^{-1}}(\lambda)\varphi(\lambda)$ is a.e. λ a Hilbert-Schmidt operator on K to K and $\int_{R^n} \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\|_2^2 d\lambda < \infty$, where $\|\ \|_2$ stands for the Hilbert-Schmidt operator norm.

(b) Each $\varphi_r(\lambda)$ satisfying (i), (ii) and (iii) defines a certain r-conjugate field for $H(t)$, namely

$$H_r^*(t) = e^{i\lambda t}\, \overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K}, \quad t \in R^n, \tag{2.2.15}$$

with spectral density $g_r(\lambda)$. Furthermore corresponding to each r-conjugate field there exists an r-exponential $B(K, X^*)$-valued function $\varphi_r(\lambda)$ satisfying conditions (i), (ii), (iii) given above, and each r-conjugate field admits the spectral representation (2.2.15) with the help of the corresponding r-exponential function.

NOTE. We point out that the conjugate field is a K-valued field with density $g_r(\lambda)$ defined on K to K.

Proof. Suppose the field $H(t)$, $t \in R^n$ is r-regular, i.e. $\bigvee_t e^{i\lambda t} H(s\colon\ |s| > r)^\perp = H(R^n)$.

Put $H_r(0) = H(s\colon\ |s| > r)$. Let $\{a_k(\lambda)\}$ be an orthonormal basis in $H_r(0)^\perp$. Also let $b_k(\lambda) = Q^*(\lambda)a_k(\lambda)$. By Lemma 2.2.7

there exists a sub-maximal system $b_{ki}(\lambda)$, $i = 1, 2, \ldots, M_r$ (M_r is finite or infinite) such that $b_{ki}(\lambda)$, $i = 1, 2, \ldots, M_r$ are a.e. λ linearly independent in X^*. Take $a(\lambda) \in H_r(0)^\perp$, then there exists a sequence of the form $\sum_{k=1}^{n} \alpha_k a_k(\lambda)$ which converges to $a(\lambda)$ in $L^2(K)$, and therefore it has a subsequence which converges to $a(\lambda)$ a.e. λ. But by Lemma 2.2.7 each $b_k(\lambda) \in B(\lambda)$ a.e. λ, where $B(\lambda)$ is the linear span of $b_{ki}(\lambda)$, $i = 1, 2, \ldots, M_r$, and therefore $a_k(\lambda) \in Q^{*^{-1}}(\lambda)B(\lambda)$ a.e. λ. Thus we obtain that $a(\lambda) \in \overline{Q^{*^{-1}}(\lambda)B(\lambda)}$ a.e. λ. This implies that $\overline{\overline{H_r(0)^\perp}}(\lambda) \subseteq \overline{Q^{*^{-1}}(\lambda)B(\lambda)}$ a.e. λ. But $Q^{*^{-1}}(\lambda)B(\lambda)$ is the linear span of $a_{ki}(\lambda)$, $i = 1, 2, \ldots, M_r$ therefore $\overline{Q^{*^{-1}}(\lambda)B(\lambda)} \subseteq \overline{\overline{H_r(0)^\perp}}(\lambda)$ a.e. λ. Therefore

$$\overline{\overline{H_r(0)^\perp}}(\lambda) = \overline{\overline{Q^{*^{-1}}(\lambda)B(\lambda)}}, \text{ a.e., } \lambda. \tag{2.2.15}$$

But $H(t)$, $t \in R^n$, is r-regular, i.e., $\bigvee_t e^{i\lambda t} H_r(0)^\perp = H(R^n)$, and this by Lemma 2.2.12 is equivalent to

$$\overline{\overline{H_r(0)}}^\perp(\lambda) = K(\lambda) \quad \text{a.e. } \lambda. \tag{2.2.16}$$

Now from (2.2.15) and (2.2.16) we obtain that

$$\overline{Q^{*^{-1}}(\lambda)B(\lambda)} = K(\lambda) \quad \text{a.e. } \lambda. \tag{2.2.17}$$

Also note that $X^*(\lambda) = Q^*(\lambda)K(\lambda)$ which implies $\dim \overline{X^*(\lambda)} \leq \dim K(\lambda)$. But $\dim B(\lambda) = M_r$ and (2.2.17) implies $\dim K(\lambda) \leq \dim B(\lambda) = M_r$. Also according to Lemma 2.2.7 we have $M_r \leq \dim X^*(\lambda)$, therefore by putting all these together we obtain

$$\dim \overline{X^*(\lambda)} \leq \dim K(\lambda) \leq M_r \leq \dim X^*(\lambda) \quad \text{a.e. } \lambda.$$

This shows that the $\dim \overline{X^*(\lambda)} = \dim K(\lambda) = M_r$, a.e., λ, which says $f(\lambda) = Q^*(\lambda)Q(\lambda)$ has a.e.λ constant rank.

The $B(K,X^*)$-valued function $\varphi_r(\lambda)$ can be constructed in the following way, define $\varphi_r(\lambda)y_k = \mu_k b_k(\lambda)$, $k = 1, 2, \ldots, M_r$, where $\{y_k\}$ is an orthogonal basis in K and μ_k's are scalars subject to $\sum_k \mu_k^2 < \infty$ and $\mu_k = 0$ for superplus y_k with $k > M_r$. Since

$$\sum_k \|\varphi_r(\lambda)y_k\|^2 = \sum_k \mu_k^2\|b_k(\lambda)\|^2 \leq C \sum_k \mu_k^2 \left(\int\|f(\lambda)\|d\lambda \cdot \int\|a_k(\lambda)\|^2 d\lambda\right)$$
$$= C \int\|f(\lambda)\|d\lambda \sum_k \mu_k^2$$

(The inequality is by (2.2.10)), $\varphi_r(\lambda)$ can be extended continuously to an operator over the whole K. We denote this extension also by $\varphi_r(\lambda)$. $\varphi_r(\lambda)$ satisfies the required properties, i.e.

(i) $\varphi_r(\lambda)K \subseteq \overline{X^*(\lambda)}$, (ii) $\overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K} = K(\lambda)$ a.e.λ,

(ii) $\int e^{-i\lambda t} (\varphi_r(\lambda)y)x \, d\lambda = 0$ for $|t| > r$, $y \in K$, $x \in X$ and

(iv) $g(\lambda) = [Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^* [Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]$ is nuclear a.e.λ with $\int \operatorname{trace} g(\lambda)d\lambda = \int\sum_k (g(\lambda)y_k, y_k)d\lambda < \infty$.

For (i) note that

$$\int_{R^n} \sum_k \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k\|_K^2 d\lambda = \int_{R^n} \sum_k \|Q^{*^{-1}}(\lambda)\mu_k b_k\|^2 d\lambda = \int_{R^n} \sum_k \mu_k^2\|a_k(\lambda)\|^2 d\lambda$$

$$= \sum_k \mu_k^2 \int_{R^n} \|a_k(\lambda)\|_K^2 d\lambda = \sum_k \mu_k^2 < \infty, \tag{2.2.18}$$

which implies $\sum_k \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k\|^2 < \infty$ a.e. λ, i.e., $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)$ can be extended to a Hilbert-Schmidt operator on K to K dentoed by $\psi_r(\lambda)$. Obviously $\psi_r(\lambda)y_k = Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k \in K(\lambda)$ which implies

$\psi(\lambda)K \subseteq K(\lambda)$, a.e. λ. But since $Q^*(\lambda) \in B(K, X^*)$, a.e. λ, we may consider the composition of $Q^*(\lambda)$ and $\psi_r(\lambda)$ denoted by $\tilde{\varphi}_r(\lambda)$, i.e.,

$$\tilde{\varphi}_r(\lambda)y = Q^*(\lambda)\psi_r(\lambda)y \quad \text{a.e. } \lambda$$

clearly $\tilde{\varphi}_r(\lambda)$ is a $B(K, X^*)$-valued function satisfying (i) and $Q^{*^{-1}}(\lambda)\tilde{\varphi}_r(\lambda)$ is, a.e. λ, Hilber-Schmidt operator-valued function on K to K. But $\tilde{\varphi}_r(\lambda)y_k = \varphi_r(\lambda)y_k$ for all k, a.e., λ, and since $\tilde{\varphi}_r$, φ_r are bounded linear operators we obtain that $\tilde{\varphi}_r(\lambda)y = \varphi_r(\lambda)y$ for any $y \in K$ a.e. λ which gives (i). (iv) follows from (2.2.18). For (ii) note $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\, K = Q^{*^{-1}}(\lambda)B(\lambda)$ (this is by the way that $\varphi_r(\lambda)$ was constructed). From this and (2.2.17) we obtain that $\overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\, K} = K(\lambda)$, a.e., λ.

For (iii) note that

$$\begin{aligned}\int e^{-i\lambda t}(\varphi_r(\lambda)y)x\, d\lambda &= \int e^{-i\lambda t}(Q^*(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y)x\, d\lambda \\ &= \int e^{-i\lambda t}(Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y,\ Q(\lambda)x)_K d\lambda \\ &= \int (Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y,\ e^{i\lambda t}Q(\lambda)x)_K d\lambda\ .\end{aligned}$$

Therefore (iii) is equivalent to show that $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \in H_r(0)^{\perp} = H(t\colon\ |t| > r)^{\perp}$. In fact we will show that $Q^{*^{-1}}(\lambda)\varphi_r(\lambda) \in H_r^*(0)$, where $H_r^*(0) = V\{a_k(\lambda),\ k = 1, 2, \ldots, M_r\}$ in $L^2(K)$. Let $\{y_k\}$ be an orthonormal basis in K, for $y \in K$ we have $y \sim \sum_{k=1}^{n} \alpha_k y_k$, i.e., $\|y - \sum_{k=1}^{n} \alpha_k y_k\| \to 0$ as $n \to \infty$, where α_k, $k = 1, 2, \ldots$ are scalars. Also let $\{a_k(\lambda)\}$ be an orthonormal basis in $H_r^*(0) \subseteq H_r(0)^{\perp}$, then

$$\|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y - \sum_{k=1}^{n} \mu_k\alpha_k a_k(\lambda)\|^2_{L^2(K)} = \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y - \sum_{k=1}^{n} \mu_k\alpha_k Q^{*^{-1}}(\lambda) b_k(\lambda)\|^2_{L^2(K)}$$

$$= \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y - Q^{*^{-1}} \sum_{k=1}^{n} \alpha_k\mu_k b_k(\lambda)\|^2_{L^2(K)}$$

$$= \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y - Q^{*^{-1}}(\lambda) \sum_{k=1}^{n} \alpha_k\varphi_r(\lambda)y_k\|^2_{L^2(K)}$$

$$= \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y - Q^{*^{-1}}(\lambda)\varphi_r(\lambda) \sum_{k=1}^{n} \alpha_k y_k\|^2_{L^2(K)}$$

$$= \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)(y - \sum_{k=1}^{n} \alpha_k y_k)\|^2_{L^2(K)}$$

$$= \int \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)(y - \sum_{k=1}^{n} \alpha_k y_k)\|^2_K \, d\lambda$$

$$\leq \|y - \sum_{k=1}^{n} \alpha_k y_k\|^2_K \int_{R^n} \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\|^2_K \, d\lambda$$

$$\leq \|y - \sum_{k=1}^{n} \alpha_k y_k\|^2_K \int_{R^n} \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\|^2_2 \, d\lambda$$

$$\leq C\, \|y - \sum_{k=1}^{n} \alpha_k y_k\|^2_K \quad \text{by} \quad (2.2.18).$$

Therefore $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y$ can be approximated by the members of $H_r^*(0)$ in $L^2(K)$, and since $H_r^*(0)$ is a closed subspace, therefore $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \in H_r^*(0)$. This gives (iii) and also shows that $\overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K} \subseteq H_r^*(0)$, where closure is taken in $L^2(K)$. But $a_k(\lambda) = Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\mu_k^{-1}y_k$ which implies that $H_r^*(0) \subseteq \overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K}$

as closed subspaces of $L^2(K)$. Therefore $H_r^*(t) = e^{i\lambda t} H_r^*(0) = e^{i\lambda t}\, \overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K}$, $t \in R^n$, which is (2.2.15). The above spectral representation says that $H_r^*(t)$ has a density $g(\lambda) = [Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^*[Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]$ with the desired nuclear property. Finally since $\overline{\overline{H^*}}(0)(\lambda) = V\{a_k(\lambda),\ k = 1, 2, \dots, M_r\}$ in K we have $\overline{\overline{H_r^*}}(0)(\lambda) = \overline{Q^{*^{-1}}(\lambda)B(\lambda)}$ a.e. λ. But $\overline{Q^{*^{-1}}(\lambda)B(\lambda)} = \overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K} = K(\lambda)$ a.e. λ. Therefore $\overline{\overline{H_r^*}}(0)(\lambda) = K(\lambda)$a.e. λ. Now Lemma 2.2.12 implies that $\bigvee_t e^{i\lambda t} H_r^*(0) = H(R^n)$. Also note, $\dim H_r^*(0) = M_r \le \dim K(\lambda)$. Therefore $H_r^*(t)$ is an r-conjugate field of $H(t)$. The proof of one part is now complete.

For the proof of the other part, suppose there exists an r-exponential $B(K, X^*)$-valued function $\varphi_r(\lambda)$ satisfying 2.2.14 (i), (ii) and (iii). Define $H_r^*(t) = e^{i\lambda t}\overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K}$, then $H_r^*(t)$ is an r-conjugate field of $H(t)$ i.e. (a) $H_r^*(0) \subseteq H(s:\ |s| > r)^{\perp}$ (b) $\bigvee_{t\in R^n} H_r^*(t) = H(R^n)$ and (c) $\dim H_r^*(0) \le H(0)$.

For (a) note that $\int (Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y,\ e^{i\lambda t}Q(\lambda)x)d\lambda$

$$= \int e^{-i\lambda t}(Q^*(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y)(x)d\lambda$$

$$= \int e^{-i\lambda t}(\varphi_r(\lambda)y)(x)d\lambda = 0 \quad \text{for} \quad |t| > r,\ x \in X,\ y \in K$$

Therefore $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \perp e^{i\lambda t}Q(\lambda)x$ in $L^2(K)$ for $|t| > r,\ x \in X,\ y \in K$

This implies that $H_r^*(0) \subset H(s:\ |s| > r)^{\perp}$.

(b) follows from 2.2.14 (ii) and Lemma 2.2.12. (c) also follows form 2.2.14 (ii). Now note that (b) also says that

$\bigvee_{t\in R^n} e^{i\lambda t} H(s:\ |s| > r)^{\perp} = H(R^n)$ which is equivalent to say that the process $H(t)$ is r-regular (this argument also shows that every field admitting an r-conjugate field is r-regular). The proof of part (a) is now complete.

(b) We already in the sufficiency part of the part (a) observed that each $\varphi_r(\lambda)$ defines a certain r-conjugate field for $H(t)$, which admits the spectral representation (2.2.15). Now suppose that $H_r^*(t)$ is an arbitary r-conjugate field to the random field $H(t)$, $t \in R^n$. Let $\{a_k(\lambda)\}$ be an orthonormal basis in $H_r^*(0)$. Put $b_k(\lambda) = Q^*(\lambda)a_k(\lambda)$. Define the operator-valued function $\varphi(\lambda)$ as $\varphi(\lambda)y_k = \mu_k b_k(\lambda)$ where $\{y_k\}$ is an orthonormal basis in K and μ_k's are scalars subject to $\sum_k \mu_k^2 < \infty$ and $\mu_k = 0$ for surplus y_k. Similar to the necessity part of part (a) one can show that $\varphi_r(\lambda)$ can be extended to an r-exponential $B(K, X^*)$-valued function satisfying the properties 2.2.14 (i), (ii), (iii) (c.f. p. 49). Evidently $H_r(t)$ admits the spectral representation (2.2.15) with the help of this corresponding r-exponential function φ. The proof of the Theorem is now complete.

We recall from Definition 2.0.3 that the field $H(t)$ is minimal if it is r-regular, for $r \to 0$. By using Theorem 2.2.14 we arrive at the following Theorem which gives necessary and sufficient condtions for a Banach space valued homogeneous random field $H(t)$, $t \in R^n$ to be minimal.

2.2.19 THEOREM. A $B(X, Y)$-valued homogeneous random field $H(t)$, $t \in R^n$ with spectral density $f(\lambda)$ satisfying (2.2.1) is minimal if and only if there exists a system of r-exponential $B(K, X^*)$-valued functions $\{\varphi_r(\lambda), r \to 0\}$ such that each $\varphi_r(\lambda)$ satisfies 2.2.14 (i),

(ii) and (iii). Furthermore $H_r^*(t) = e^{i\lambda t}\,\overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K}$, $r \to 0$ is a corresponding conjugate system to the field $H(t)$.

Let us for an instant take $t \in Z^n$, where Z is the set of all integers then Definition 2.0.3 for minimality is equivalent to

$$(2.2.20) \qquad \bigvee_{s \in Z^n} H(t:\ t \neq s)^{\perp} = \bigvee_{s \in Z^n} e^{is\lambda} H(t:\ t \neq 0)^{\perp} = H(Z^n).$$

As a corollary to Theorem 2.2.19 we obtain the following Theorem which gives necessary and sufficient conditions for a discrete parameter random field $H(t)$, $t \in Z^n$, to be minimal (c.f. Makagon [14]; Miamee and Salehi [26]).

2.2.21 <u>THEOREM</u>. A $B(X, Y)$-valued homogeneous random field $H(t)$, $t \in Z^n$, with spectral density $f(\lambda)$ satisfying (2.2.1) is minimal if and only if there exists a constant subspace $B \subset X^*(\lambda)$, a.e. λ, such that

$$\overline{Q^{*^{-1}}(\lambda)B} = K(\lambda) \quad \text{a.e. } \lambda \quad \text{and} \quad \int \|Q^{*^{-1}}(\lambda)y\|^2 d\lambda < \infty \quad \text{for any } y \in B.$$

Indeed there exists a constant $B(K, X^*)$-valued function φ, which satisfies 2.2.14 (i), (ii), (iii).

Proof. Note that r-exponential $B(K, X^*)$-valued functions $\varphi_r(\lambda)$ in Theorem 2.2.19 are identical to a constatn $B(K, X^*)$-valued function, say φ, a.e. λ for $r < 1$.

Take $B = \varphi K$, then B is a constant subspace of $X^*(\lambda)$, a.e. λ and by 2.2.14 (ii), (iii) it satisfies the required properties. The proof is complete.

We recall from Definiton 2.0.4 that the field $H(t)$, $t \in R^n$ is called regular if

$$\bigcap_{r>0} H(t:\ |t| > r) = \{0\}\ .$$

The following Theorem is an extension to our work in Chapter 1, Theorem 1.1.13. It gives necessary and sufficient conditions in terms of the spectral density of a $B(X, Y)$-valued homogeneous random field $H(t)$, $t \in R^n$ to be regular. The proof is similar to the proof of Theorem 1.1.13, and is only sketched.

2.2.22 <u>THEOREM</u>. A $B(X, Y)$-valued homogeneous random field $H(t)$, $t \in R^n$, with spectral density $f(\lambda)$ satisfying (2.2.1) is regular if and only if there exists a family of r-exponential $B(K, X^*)$-valued functions $\varphi_r(\lambda)$, $r \to \infty$, such that

(i) $\varphi_r(\lambda)K \subseteq X^*(\lambda)$ a.e. λ.

(ii) $\overline{Q^{*^{-1}}(\lambda)\varphi_{r_1}(\lambda)K} \subseteq \overline{Q^{*^{-1}}(\lambda)\varphi_{r_2}(\lambda)K}$ a.e. with $r_1 < r_2$ and

$$\overline{\overline{\bigcup_r Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K}} = K(\lambda) \text{ a.e. } \lambda.$$

(iii) For each r, $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)$ is a Hilbert-Schmidt operator a.e. λ from K to K and $\int \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\|_2^2 d\lambda < \infty$.

Proof. Proof is similar to the one give for Theorem 1.1.13. Indeed since $\bigvee_{r>0} H(s: |s| > r)^{\perp}$ is a doubly invariant subspace, by Lemma 2.2.12, regularity is equivalent to $\overline{\overline{\bigcup_{r>0} H(s: |s| > r)^{\perp}}}(\lambda) = K(\lambda)$ a.e. λ. The latter with the help of Lemma 2.2.6 is equivalent to

$$\overline{\overline{\bigcup_{r>0} (Q^{*^{-1}} B)(\lambda)_{(s:\ |s| > r)}}} = K(\lambda) \text{ a.e. } \lambda \text{ (c.f. Lemmas 1.1.6, 1.1.7).}$$

Now with a similar technique as one given in the proof of Theorem 1.1.13 one can show that the latter condition is equivalent to the set of conditions given above. Proof is complete.

2.3 Completely Minimal Fields. Necessary and sufficient conditons for a $B(X, Y)$-valued homogeneous random field $H(t)$, $t \in R^n$, to be minimal was obtained in Section 2, Theorem 2.2.19. In this Section we will consider a sub-calss of the class of the minimal fields, namely completely minimal fields, which plays an important role in charactrizing the L-Markov and Markov properties in terms of the spectral density. Such a charatrization is the subject of Section 4.

The notion of complete minimality for the Hilbert space-valued random fields was introduced by Rozanov [39], where sufficient conditions for a minimal field to be completely minimal were obtained in that work. The main attempt in this section is to extend his result to the $B(X, Y)$-valued random fields. The key to this extension is Theorem 2.3.8 which says that every $B(X, Y)$-valued r-regular field with spectral representation (2.1.6) admits a spectral representation in the form $H(t) = e^{i\lambda t}\, \overline{h^{\frac{1}{2}}(\lambda)K}$, where $h(\lambda)\colon\ K \to K$ is a.e. λ nuclear with integrable nuclear norm. We will also introduce the notion of complete minimality for discrete parameter random fields. Necessary and sufficient conditions for a discrete parameter random field to be completely minimal is given (c.f. Theorem 2.3.17). In particular we prove that every minimal field satifies the geometric property (2.3.4). First we introduce some notations.

NOTATIONS. Let $S \subseteq R^n$ be a bounded open region and T the complement of S, $T = S^c$. S^ϵ denotes an ϵ-neighborhood of $\overline{S}$ and $S^{-\epsilon}$ denotes the complement of the closure of T^ϵ. Also ∂S denotes the boundary of S and $\partial^\epsilon S$ denotes the ϵ-neighborhood of the ∂S.

2.3.1 DEFINITION. With the same notations as Section 2, we call a minimal field completely minimal if there exists a conjugate system $H_r^*(t)$, $r \to 0$, or by Theorem 2.2.19, equivalently. A corresponding system of r-exponential functions $\varphi_r(\lambda)$, $r \to 0$, for which

$$H(T)^{\perp} \subseteq \bigvee_{\substack{r<\delta \\ \text{Supp } u \subseteq S}} \tilde{u}(\lambda) Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K \subseteq H(T^{-\delta})^{\perp}, \text{ for any, } \delta > 0, \tag{2.3.2}$$

where T could be a bounded or unbounded region in R^n with $S = T^c$ and $\tilde{u}(\lambda)$ is a Lebesgue integrable scalar-valued function which is also square integrable with respect to $\|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y\|^2$, for all $y \in K$, with Fourier transform $u(t)$, $t \in R^n$ (supp=support).

NOTE. We will show in Lemma 2.3.5 that the second inclusion in (2.3.2) is always true, and in the case that S is bounded $\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \in H_r^*(S)$. Furthermore we always have $H_r^*(S) \subseteq H(T^{-\delta})^{\perp}$ for $r < \delta$. Therefore (2.3.2) reduces to

$$H(T)^{\perp} \subseteq \bigvee_{r<\delta} H_r^*(S) \subseteq H(T^{-\delta})^{\perp}, \text{ for any } \delta > 0. \tag{2.3.3}$$

(2.3.3) gives a better picture of the notion of complete minimality. In fact in the discrete case, $t \in Z^n$, minimality is equivalent to (2.2.20) and the conjugate system $H_r^*(t) = H(s: |s-t| > r)^{\perp}$, $r \to 0$, reduces to the conjugate field $H^*(t) = H(s: s \neq t)^{\perp}$. In this case since $H(T) \subseteq \bigcap_{t \in S} H(s: s \neq t)$, we obtain that always $H(T)^{\perp} \supseteq \bigvee_{t \in S} H(s: s \neq t)^{\perp} = H^*(S)$. Now if (2.3.3) is satisfied, it implies that $H(T)^{\perp} \subseteq H^*(S)$. Therefore complete minimality in the sense of (2.3.3) for the discrete parameter random field is equivalent to

$$H(T)^{\perp} = H^{*}(S)$$

(2.3.4) or

$$H(T) = \bigcap_{t \in S} H(s: \ s \neq t),$$

where S is a bounded domain in Z^n with complementary domain T. (2.3.3), for a bounded domain, and in general (2.3.2) are reasonable substituted for (2.3.4) in continuous parameter case.

2.3.5 <u>LEMMA</u>. Let $H(t)$, $t \in R^n$, be a $B(X, Y)$-valued homogeneous minimal random field. Then the following statements (a), (b) and (c) are satisfied.

(a) Let $\tilde{u}(\lambda)$ be an integrable scalar-valued function, square integrable with respect to the weight $\|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y\|^2$, $y \in K$ with supp $u \subseteq S$, then $\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \in H(T^{-r})^{\perp}$, where $T = S^c$, $T^{-r} = (\overline{S^r})^c$ and $u(t)$, $t \in R^n$, is the Fourier transform of $\tilde{u}(\lambda)$.

(b) For a bounded domain S and $\tilde{u}(\lambda)$ as in (a) $\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \in H_r^{*}(S)$

(c) Always $H_r^{*}(S) \subseteq H(T^{-r})^{\perp} \subseteq H(T^{-\delta})^{\perp}$ for $r < \delta$.

Proof. (a) Note that in order to show $\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \in H(T^{-r})^{\perp}$ it suffices to show $\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \perp H(T^{-r})$ or equivalently show that

$$\int_{R^n} (\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y, \ e^{i\lambda t}Q(\lambda)y')d\lambda = 0 \quad \text{for all} \quad t \in T^{-r} \quad \text{and} \quad y, y' \in K.$$

But by Lemma 2.1.8 (a) we have

$$\int_{R^n} (\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y,\ e^{i\lambda t}Q(\lambda)y')d\lambda = \int_{R^n} e^{-i\lambda t}\,\tilde{u}(\lambda)(Q^*(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y)(y')d\lambda$$

$$= \int_{R^n} e^{-i\lambda t}\,\tilde{u}(\lambda)(\varphi_r(\lambda)y)(y')d\lambda .$$

Now let $u_r(s)$ be the Fourier transform of the scalar-valued r-exponential function $(\varphi_r(\lambda)y)(y')$. Then $u_r(s) = 0$ for $|s| > r$. By using Plancherel identity we obtain $\int_{R^n} e^{-i\lambda t}\,\tilde{u}(\lambda)(\varphi_r(\lambda)y)(y')d\lambda =$ $\frac{1}{(2\pi)^n}\int_{R^n} u(s)u_r(s + t)ds$. But $u(s) = 0$ for $s \in T$ and $u_r(s + t) = 0$ for $|s + t| > r$. Therefore for $t \in T^{-r}$ $u_r(s + t) = 0$ for $s \in S$. Thus whenever $t \in T^{-r}$ $u(s)u_r(s + t) = 0$ on R^n, and this finishes the proof of part (a).

(b) Note that $\tilde{u}(\lambda) = \frac{1}{(2\pi)^n}\int_{R^n} e^{i\lambda t}\,u(t)dt = \lim \sum_k e^{i\lambda t_k}\,u(t_k)\Delta_k$.

The Riemann sums $\sum_k e^{i\lambda t_k}u(t_k)\Delta_k$ are bounded. In fact supp u $\in$ S which implies $\sum_k \Delta_k \leq$ volum of S, and $u(t)$ is a bounded function. Therefore $|\sum_k e^{i\lambda t_k}u(t_k)\Delta_k| \leq C$ (volum of S). But

$$\|\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y - \sum_{k=1}^{n} u(t_k)\Delta_k\, e^{i\lambda t_k}\, Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y\|^2$$

$$\leq 2\ |\tilde{u}(\lambda)|^2\|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y\|^2 + 2\ C^2\ (\text{volum of } S)^2\ \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y\|^2 .$$

The bound given above is independent of n and is integrable. Therefore by the bounded convergence theorem $\sum_k^n u(t_k)\Delta_k\, e^{i\lambda t_k}\, Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y$ converges to $\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y$ in $L^2(R^n, X)$.

But since $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)$ is the square root of the density of the r-conjugate field $H_r^*(t)$,

$$H_r^*(t) = \overline{e^{i\lambda t}\, Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K,\ e^{i\lambda t_k} Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y} \in H_r^*(S) \text{ for } t_k \in S.$$

For $t_k \notin S$, $u(t_k)\Delta_k\, e^{i\lambda t_k} Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y = 0$, therefore

$\sum_k u(t_k)\Delta_k\, e^{i\lambda t_k} Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y$ belongs to $H_r^*(S)$ and this implies that

$\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y \in H_r^*(S)$ which gives (b).

(c) Note that $H_r^*(t) \subseteq H(s: |s-t| > r)^{\perp}$. Therefore $H_r^*(t) \perp H(T^{-r})$ with $t \in S$ which implies $H_r^*(S) \subseteq H(T^{-r})^{\perp}$. The second inclusion is equivalent to $H(T^{-\delta}) \subseteq H(T^{-r})$ with $r < \delta$ and the second inclusion in (c) is always satisfied.

2.3.6 <u>LEMMA.</u> Suppose $H_r^*(t)$ is an r-conjugate field of $H(t)$, i.e., $H_r^*(t)$ is a homogeneous random field with $H_r^*(R^n = H(R^n)$, $H_r^*(0) \subseteq H(s: |s| > r)^{\perp}$ and $\dim H^*(0) \leq \dim H(0)$. Then $H(t)$, $t \in R^n$, is an r-conjugate field of $H_r^*(t)$, $t \in R^n$.

Proof. All we need is to show $H(0) \subseteq H_r^*(s: |s| > r)^{\perp}$. For this it is enough to show that $H(0) \perp H_r^*(t)$ for $|t| > r$ since $H_r^*(R^n) = H(R^n)$. Let fix t_0 with $|t_0| > r$. Then since $H_r^*(t)$ is an r-conjugate field of $H(t)$ we have $H_r^*(t_0) \subseteq H(s: |s-t_0| > r)^{\perp}$ which says $H_r^*(t_0) \perp H(s)$ for all s with $|s-t_0| > r$. But $|t_0| > r$, therefore $H_r^*(t_0) \perp H(0)$.

2.3.7 <u>COROLLARY</u>. $H_r^*(t)$, an r-conjugate field of the field $H(t)$ is r-regular .

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Proof. It is an immediate consequence of the Lemma 2.3.6 and the fact that every field with an r-conjugate field is r-regular.

According to Theorem 2.2.14, a conjugate field of every homogeneous random field with spectral density satisfying (2.2.1) has a nuclear density on K to K. But by Lemma 2.3.6 if $H^*(t)$ is a conjugate field of $H(t)$, then $H(t)$ itself is a conjugate field of $H^*(t)$. Therefore we may apply Theorem 2.2.14 to $H^*(t)$. Consequently we obtain a nuclear density on K for $H(t)$, i.e., $H(t) = e^{i\lambda t}\overline{h^{\frac{1}{2}}(\lambda)K}$, where $h(\lambda)$ is a $B(K, K)$-valued function; $h(\lambda)$ is nuclear a.e. λ. This result enables us to establish sufficient conditions for complete minimality. It is by itself an interesting result, saying that every $B(X, Y)$-valued homogeneous r-regular random field admits the spectral representation (2.3.9) given below. Here is the detail.

2.3.8 <u>THEOREM</u>. Let $H(t) = e^{i\lambda t}\ \overline{Q(\lambda)X}$, $t \in R^n$, be a spectral representation of an r-regular $B(X, Y)$-valued homogeneous random field with spectral density $f(\lambda) \in B^+(X, X^*)$ satisfying (2.2.1). Then there exists a separable Hilbert space K ($\dim K \leq \dim X$) and a $B^+(K, K)$-valued nuclear function $h(\lambda)$ with Legesgue integrable nuclear norm such that

$$H(t) = e^{i\lambda t}\ \overline{h^{\frac{1}{2}}(\lambda)K}. \tag{2.3.9}$$

Indeed there exists a $B(K, X^*)$-valued r-exponential function $\varphi_r(\lambda)$ and a $B(K, K)$-valued r-exponential function $\psi_r(\lambda)$, such that

$$h^{\frac{1}{2}}(\lambda) = [Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^{-1}\psi_r(\lambda). \tag{2.3.10}$$

Analogous to 2.2.14 (i), (ii), (iii) the following properties are easily read off.

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(i) $\psi_r(\lambda)K \subseteq \overline{[Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]K}$ a.e. λ

(ii) $\overline{h^{\frac{1}{2}}(\lambda)K} = \overline{Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K} = K(\lambda)$ a.e. λ.

(iii) $h(\lambda) = ([Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^{-1}\psi(\lambda))^*([Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^{-1}\psi_r(\lambda))$ is a nuclear operator a.e. λ, and its nuclear norm is integrable.

Proof. As we mentioned above the Theorem follows by applying Theorem 2.2.14 to the $H^*(t)$ with the help of Lemma 2.3.6. But let us give some details for the construction of the r-exponential $B(K, K)$-valued function $\psi_r(\lambda)$. The method of constructing $\psi_r(\lambda)$ is similar to the one given for $\varphi_r(\lambda)$ in Theorem 2.2.14. All we need is to replace $H^*(0)$, $Q^*(\lambda)$ and $Q^{*^{-1}}(\lambda)$ by $H(0)$, $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)$ and $[Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^{-1}$ respectively. Indeed let $\{a_k(\lambda)\}$ be an orthonormal basis in $H(0) = \overline{Q(\lambda)X}$, closure in $L^2(K)$. Define $b_k(\lambda) = Q^{*^{-1}}(\lambda)\varphi_r(\lambda)a_k(\lambda)$. Then define $\psi_r(\lambda)y_k = \mu_k Q^{*^{-1}}(\lambda)\varphi_r(\lambda)a_k(\lambda)$, where $\{y_k\}$ is an orthonormal basis in K and μ_k, $k = 1, 2, \ldots$ are scalars subject to $\sum_k |\mu_k|^2 < \infty$, where $\mu_k = 0$ only for surplus y_k's. Properties 2.3.8 (i), (ii), (iii) and the fact that $\psi_r(\lambda)$ is an r-exponential $B(K, K)$-valued function with $\psi_r(\lambda)$ being of Hilbert-Schmidt type, a.e. , can be carried out in a similar way as 2.2.14 (i), (ii), (iii).

We are now in a position to investigate conditons under which a minimal field is completely minimal.

2.3.11 <u>THEOREM.</u> A $B(X, Y)$-valued homogeneous minimal random field $H(t)$, $t \in R^n$, with density $f(\lambda)$ is completely minimal if

(i) $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)h^{\frac{1}{2}}(\lambda) = [Q^{*^{-1}}(\lambda)\varphi_r(\lambda)][Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^{-1}\psi_r(\lambda) = \psi_r(\lambda) \to I$,

as $r \to 0$ in the sense of strong convergence in $K(\lambda)$.

(ii) $\|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\|_2 \cdot \|h^{\frac{1}{2}}(\lambda)\| \leq C$

Proof. By Lemma 2.3.5 all we need is to show the first inclusion in (2.3.2), i.e.,

$$H(T)^{\perp} \subseteq \bigvee_{\substack{r<\delta \\ \text{supp } \mu \subseteq S}} \tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K.$$

The inclusion follows by proving the following steps.

Step 1: Every function $a(\lambda)$ of the space $H(R^n)$ can be approximated by functions $\psi_r(\lambda)a(\lambda)$ as $r \to 0$ in $L^2(R^n, K)$.

Step 2: For each r and $a(\lambda) \in H(T)^{\perp}$, $\psi_r(\lambda)a(\lambda)$ can be approximated by $\sum_k \tilde{u}_k(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k$, where $\{y_k\}$ is an orthonormal basis in K and $\tilde{u}_k(\lambda) = (h^{\frac{1}{2}}(\lambda)a(\lambda), y_k)$.

Proof of step 1. Any function $a(\lambda) \in H(R^n)$ takes values in $K(\lambda)$ a.e. λ and 2.3.11 (i) implies pointwise convergence, i.e., $\psi_r(\lambda)a(\lambda) \to a(\lambda)$ as $r \to 0$ a.e., λ in K norm.

$$\text{But } \|\psi_r(\lambda)a(\lambda) - a(\lambda)\|_K^2 = \|(\psi_r(\lambda) - I)a(\lambda)\|^2$$

$$\leq 4\,\{\|\psi_r(\lambda)\|^2 + \|I\|^2\}\,\|a(\lambda)\|^2$$

$$\leq 4\,\{C^2 + 1\}\,\|a(\lambda)\|^2 \qquad \text{by 2.3.11 (ii).}$$

Now since $a(\lambda) \in H(R^n)$, $4\{C^2 + 1\}\,\|a(\lambda)\|^2$ is integrable function. Therefore by bounded convergence theorem

$$\int_{R^n} \|\psi_r(\lambda)a(\lambda) - a(\lambda)\|^2 d\lambda \to 0 \quad \text{as} \quad r \to 0.$$

Proof of step 2. Let $a(\lambda) \in H(T)^{\perp}$, then by using spectral representation (2.3.9) we get

$$\int_{R^n} (a(\lambda), e^{i\lambda t} h^{\frac{1}{2}}(\lambda) y) d\lambda = 0 \quad \text{for all} \quad t \in T.$$

This implies that the function $u(t)$, the Fourier transform of the function $\tilde{u}(\lambda) = (h^{\frac{1}{2}}(\lambda) a(\lambda), y)$, has support inside $S = T^c$.

But $h^{\frac{1}{2}}(\lambda) a(\lambda) \in K(\lambda) \subset K$. Thus

$$h^{\frac{1}{2}}(\lambda) a(\lambda) = \sum_k (h^{\frac{1}{2}}(\lambda) a(\lambda), y_k) y_k, \quad \text{which implies}$$

$$Q^{*^{-1}}(\lambda) \varphi_r(\lambda) h^{\frac{1}{2}}(\lambda) a(\lambda) = \sum_k (h^{\frac{1}{2}}(\lambda) a(\lambda), y_k) Q^{*^{-1}}(\lambda) \varphi_r(\lambda) y_k$$

$$= \sum_k \tilde{u}_k(\lambda) Q^{*^{-1}}(\lambda) \varphi_r(\lambda) y_k$$

Each $\tilde{u}_k(\lambda) Q^{*^{-1}}(\lambda) \varphi_r(\lambda) y_k \in L^2(K)$, because

$$\begin{aligned}
\sum_k \|\tilde{u}_k(\lambda) Q^{*^{-1}}(\lambda) \varphi_r(\lambda) y_k\|^2 &\le \sum_k |\tilde{u}_k(\lambda)|^2 \|Q^{*^{-1}}(\lambda) \varphi_r(\lambda)\|^2 \|y_k\|^2 \\
&= \|Q^{*^{-1}}(\lambda) \varphi_r(\lambda)\|^2 \sum_k |\tilde{u}_k(\lambda)|^2 \\
&\le \|Q^{*^{-1}}(\lambda) \varphi_r(\lambda)\|_2^2 \, \|h^{\frac{1}{2}}(\lambda) a(\lambda)\|^2 \\
&\le \|Q^{*^{-1}}(\lambda) \varphi_r(\lambda)\|_2^2 \, \|h^{\frac{1}{2}}(\lambda)\|^2 \, \|a(\lambda)\|^2 \\
&\le C^2 \, \|a(\lambda)\|^2 ,
\end{aligned}$$

which implies $\sum_k \|\tilde{u}_k(\lambda) Q^{*^{-1}}(\lambda) \varphi_r(\lambda) y_k\|^2$ is an integrable function and consequently $\int \|\tilde{u}_k(\lambda) Q^{*^{-1}}(\lambda) \varphi_r(\lambda) y_k\|^2 d\lambda < \infty$.

Now in order to use the bounded convergence theorem what is left is to show $\left\|\sum_{k=1}^{n} \tilde{u}_k(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k\right\|^2$ is uniformly bounded by an integrable function. But

$$\begin{aligned}
\left\|\sum_{k=1}^{n} \tilde{u}_k(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k\right\|^2 &\leq \left(\sum_{k=1}^{n} |\tilde{u}_k(\lambda)| \, \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k\|\right)^2 \\
&\leq \sum_{k=1}^{n} |\tilde{u}_k(\lambda)|^2 \cdot \sum_{k=1}^{n} \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k\|^2 \\
&\leq \sum_{k=1}^{\infty} |\tilde{u}_k(\lambda)|^2 \cdot \sum_{k=1} \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k\|^2 \\
&= \|h^{\frac{1}{2}}(\lambda)a(\lambda)\|^2 \cdot \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\|_2^2 \\
&\leq \|h^{\frac{1}{2}}(\lambda)\|^2 \|Q^{*^{-1}}(\lambda)\varphi_r(\lambda)\|_2^2 \, \|a(\lambda)\|^2 \\
&\leq C^2 \, \|a(\lambda)\|^2 \; ,
\end{aligned}$$

and evidently $\int_{R^n} \|a(\lambda)\|^2 d\lambda < \infty$.

Therefore $\sum_k \tilde{u}_k(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y_k$ converges to $Q^{*^{-1}}(\lambda)\varphi_r(\lambda)h^{\frac{1}{2}}(\lambda)a(\lambda) = \psi_r(\lambda)a(\lambda)$ in $L^2(K)$ and the proof of the theorem is complete.

NOTE 1. In case of Hilbert space, $h(\lambda) = f(\lambda)$ and the functions φ, ψ are related by $\psi(\lambda) = f^{-\frac{1}{2}}(\lambda)\varphi(\lambda)f^{\frac{1}{2}}(\lambda)$, $\varphi(\lambda) = f^{\frac{1}{2}}(\lambda)\psi(\lambda)f^{-\frac{1}{2}}(\lambda)$, a.e. λ.

NOTE 2. A conjugate system with the properties 2.3.11 (i) and (ii) for finite dimensional random fields with certain properties on the spectral density (including some growth conditions on f^{-1}) always exists [10], [28], [33].

EXAMPLE. Let us give a simple example to the Theorem 2.3.11. Let $\varphi_r^*(\lambda)$ be r-exponential scalar-valued function corresponding to the density $f_1(\lambda)$ of a univariate field satisfying 2.3.11 (i). Note 2.3.11 (ii) for the univariate case is trival. Now let $X = L^p([0,1], d\lambda), \frac{1}{p} + \frac{1}{q} = 1$ $2 \leq p < \infty$ then $X^* = L^q([0,1], d\lambda)$. Define the density function $f(\lambda)$ to be $f(\lambda) = f_1(\lambda)I$ where I is the identity on X into X^*. K, $K(\lambda)$, $X^*(\lambda)$ are $L^2([0,1], d\lambda)$, $f_1^{\frac{1}{2}}(\lambda)L^p$, $f_1(\lambda)L^p$ respectively. The operator-valued functions $Q(\lambda)$, $Q^*(\lambda)$, $Q^{*^{-1}}(\lambda)$, $\varphi_r(\lambda)$, $\psi_r(\lambda)$ are $f_1^{\frac{1}{2}}(\lambda)I_1$, $f^{\frac{1}{2}}(\lambda)I_2$, $f^{-\frac{1}{2}}(\lambda)I_3$, $\varphi_r(\lambda)I_2$, $\varphi_r(\lambda)I_4$ respectively where I_1: $L^p \to L^2$, I_2: $L^2 \to L^p$, I_3: $L^2 \subset L^q \to L^2$ and I_4: $L^2 \to L^2$ are identity operators.

Let us now consider the discrete case, i.e. $t \in Z^n$. We start with the following lemma.

2.3.12 <u>LEMMA</u>. Let T be any domain in Z^n with complementary domain S. Then for a minimal field $H(t)$, $t \in Z^n$,

$$\bigvee_{\text{supp } \mu \subseteq S} \tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi K \subseteq H(T)^{\perp}, \tag{2.3.13}$$

where φ is a constant $B(K, X^*)$-valued function given in the proof of the Theorem 2.2.21 and $\tilde{u}(\lambda)$ is a Lebesgue integrable scalar-valued function which is also square integrable w.r.t. the weight $\|Q^{*^{-1}}(\lambda)\varphi y\|^2$, for $y \in K$ $\cdot u(t)$ is the Fourier transform of $\tilde{u}(\lambda)$.

Proof. It is sufficient to show $\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi y \perp H(t)$, $t \in K$. This is equivalent to $\int_{T^n} (\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi y, e^{i\lambda t}Q(\lambda)x)d\lambda = 0$ for all $t \in T$, $x \in X$, $y \in K$ or $\int_{T^n} e^{-i\lambda t}\, \tilde{u}(\lambda)(\varphi y)x\, d\lambda = 0$ for all $t \in T$

$x \in X$, $y \in K$. But $\int_{T^n} e^{-i\lambda t}\,\tilde{u}(\lambda)(\varphi y)x\,d\lambda = (\varphi y)x \int_{T^n} e^{-i\lambda t}\,\tilde{u}(\lambda)d\lambda =$ $(\varphi y)x\; u(t) = 0$ for $t \in T$. Proof of the Lemma is complete.

In view of this lemma it is appropriate to make the following definition.

2.3.14 <u>DEFINITION</u>. A $B(X, Y)$-valued homogeneous discrete parameter minimal random field $H(t)$, $t \in Z^n$, is called completely minimal if equality occurs in (2.3.13), i.e.,

$$\bigvee_{\text{supp } u \subseteq S} \tilde{u}(\lambda) Q^{*^{-1}}(\lambda)\varphi K = H(T)^{\perp}. \tag{2.3.15}$$

2.3.16 <u>REMARK</u>. When S is bounded $\tilde{u}(\lambda) = \sum_{t \in S} u(t)e^{i\lambda t}$ is a polynomial, and since $Q^{*^{-1}}(\lambda)\varphi$ is the density of the conjugate field $H^{*}(t) = H(s\colon\; s \neq t)^{\perp}$, i.e., $H^{*}(t) = e^{i\lambda t}\, Q^{*^{-1}}(\lambda)\varphi K$, we obtain that $\bigvee_{\text{supp } u \subseteq S} \tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi K = H^{*}(S)$. Therefore in the case that S is bounded (2.3.15) is equivalent to $H^{*}(S) = H(T)^{\perp}$, which is equivalent to the geometrical property (2.3.4).

The following Theorem is a Corollary to Theorem 2.3.11, and it gives sufficient conditions for a $B(X, Y)$-valued discrete parameter minimal field to be completely minimal.

2.3.17 <u>THEOREM</u>. A $B(X, Y)$-valued homogeneous discrete parameter minimal field is completely minimal if the constant $B(K, K)$-valued function ψ, given in Theorem 2.3.8, satisfies the conditons (i) ψ is identity on $K(\lambda)$ and (ii) $\|\,\|g^{\frac{1}{2}}(\lambda)\|\,\|_2\, \|\,\|g^{-\frac{1}{2}}(\lambda)\psi\|\,\| < C$, where $g^{\frac{1}{2}} = Q^{*^{-1}}(\lambda)\varphi$, and g is the density of the conjugate field $H^{*}(t) = H(s\colon\; s \neq t)^{\perp}$.

NOTE. When the dimension of a field $H(t)$, $t \in Z^n$ is finite the condition 2.3.17 (i) as we show in the proof of Theorem 2.3.18 given below is not needed and a sufficient condition for a minimal field to be completely minimal is that $\|f(\lambda)\| \, \|f^{-1}(\lambda)\| < C$, a.e., λ, where f^{-1} is the generalized inverse of f. In the following we show that every finite dimensional minimal random field $H(t)$, $t \in Z^n$ satisfies the geometric property (2.3.4) or by Remark 2.3.16 equivalently satisfies (2.3.15) for bounded domains S with complementary domains T. This interesting result is the content of the next Theorem.

2.3.18 THEOREM. Let $H(t)$, $t \in Z^n$, be a finite dimensional homogeneous discrete parameter minimal random field. Then for any bounded domain S in Z^n with complementary domain T,

$$H(T) = \bigcap_{s \in S} H(t: \; t \neq s)$$

NOTE. Theorem 2.3.18 says that every discrete parameter minimal field is completely minimal in the weak sense (restricted to bounded sets).

Proof. By Remark 2.3.16 Theorem follows by showing that (2.3.15) holds in this case. We recall from Theorem 2.2.21 that the field $H(t)$, $t \in Z^n$, is minimal if and only if there exists a constant $B(X, X)$-valued function φ such that

(i) $\varphi X \subseteq f^{\frac{1}{2}}(\lambda)X$, a.e. λ, (ii) $f^{-\frac{1}{2}}(\lambda)\varphi X = f^{\frac{1}{2}}(\lambda)X$ a.e. λ,

(iii) $\int_{T^n} \|f^{-\frac{1}{2}}(\lambda)\varphi\|_2^2 d\lambda < \infty$. These conditons are equivalent to (i) rang $f = \varphi X$ is constant, a.e. , (ii) $\int_{T^n} \|f^{-1}\| d\lambda < \infty$, where

$f^{-1} = (f^{-\frac{1}{2}})^* f^{-\frac{1}{2}}$ and X is a finite dimensional Hilbert space (c.f. [15]). Now assume the field $H(t)$, $t \in Z^n$, is minimal. Define the operator P on X onto φX to be the projection onto φX. Clearly $PX = \varphi X$. Also since any $a(\lambda) \in H(Z^n)$ takes values in $f^{\frac{1}{2}}(\lambda)X$, a.e. λ, we have $f^{-\frac{1}{2}}(\lambda)P\, f^{\frac{1}{2}}(\lambda)a(\lambda) = a(\lambda)$, a.e. λ. Take $a(\lambda) \in H(T)^{\perp}$, then $u_x(t) = \int e^{-i\lambda t}\, (f^{\frac{1}{2}}(\lambda)a(\lambda), x) = 0$ for $t \in T$, $x \in X$. Therefore for any $x \in X$ $\tilde{u}_x(\lambda) = (f^{\frac{1}{2}}(\lambda)a(\lambda), x)$ is Lebesgue integrable with supp $u_x(t) \subseteq S$. But S is bounded, therefore $\tilde{u}_x(\lambda)$ is bounded. This implies that $\tilde{u}_x(\lambda)$ is square integrable w.r.t. the weight $\|f^{-\frac{1}{2}}(\lambda)\varphi y\|^2$, $y \in X$. Now let $\{x_k\}_{k=1}^n$ be an orthonormal basis in X. Then

$$f^{\frac{1}{2}}(\lambda)a(\lambda) = \sum_{k=1}^{n} (f^{\frac{1}{2}}(\lambda)a(\lambda), x_k)x_k$$

which implies that

$$a(\lambda) = f^{-\frac{1}{2}}(\lambda)P\, f^{\frac{1}{2}}(\lambda)a(\lambda) = \sum_{k=1}^{n} (f^{\frac{1}{2}}(\lambda)a(\lambda), x_k)f^{-\frac{1}{2}}(\lambda)Px_k, \text{ a.e. } \lambda.$$

Put $\tilde{u}_k(\lambda) = (f^{\frac{1}{2}}(\lambda)a(\lambda), x_k)$ and note that $Px_k \in \varphi X$. Then the expression given above implies that $a(\lambda) \in \bigvee_{\text{supp } \mu \subseteq S} \tilde{u}(\lambda)f^{-\frac{1}{2}}(\lambda)\varphi X$. Therefore $H(T)^{\perp} \subseteq \bigvee_{\text{supp } \mu \subseteq S} \tilde{u}(\lambda)f^{-\frac{1}{2}}(\lambda)\varphi X$. Proof is complete by applying Lemma 2.3.12.

2.4 Markov Minimal Fields. In this section we will discuss Markov and L-Markov properties of a homogeneous random field. Necessary and sufficient conditions for a completely minimal $B(X, Y)$-valued field to be Markov or L-Markov will be given. The result extends the work of Rozanov [39] to the $B(X, Y)$-valued fields. Similar new results for discrete case are also obtained. With the same notations as in Section 3 (c.f. Notations preceding Definition 2.3.1), in this Section we shall consider bounded open regions $S \subseteq R^n$ for which

$$(2.4.1) \qquad \overline{(S^{\epsilon})^{-\epsilon}} = \overline{(S^{-\epsilon})^{\epsilon}} = \overline{S}$$

for sufficiently small $\epsilon > 0$ (For instance, this property is possessed by regions with finite piecewise smooth boundries).

2.4.2 DEFINITION. We call a field $H(t)$, $t \in R^n$, Markov if for any bounded domain S (satisfying (2.4.1)) with complementary domain T,

$$(2.4.3) \qquad P(S)H(T) \subseteq H(\partial^{\epsilon}S)$$

for sufficiently small $\epsilon > 0$, where $P(S)$ stands for projection onto $H(S)$.

The following results are due to Rozanov [39], which can be observed also from the work of Kallianpur and Mandrekar [8] and Mandrekar [16].

(a) Markov property (2.4.3) implies

$$(2.4.4) \qquad P(S^{\delta})H(T^{\delta}) \subseteq H(\partial^{\epsilon}S) \quad \epsilon > \delta$$

(b) (2.4.4) implies that

$$(2.4.5) \qquad P_+(S)H_+(T) = H_+(\partial S),$$

where $H_+(S) = \bigcap_{\epsilon>0} H(S^{\epsilon})$ and $P_+(S)$ is projection onto $H_+(S)$;

(c) (2.4.5) implies that

$$(2.4.6) \qquad H_+(S)^{\perp} \perp H_+(T)^{\perp} \quad ;$$

(d) (2.4.6) holds if and only if

(2.4.7) $$H(S^{\delta})^{\perp} \perp H(S^{\delta})^{\perp} \qquad \text{for any } \delta > 0 \;;$$

(e) if the following condition

(2.4.8) $$H(S_1) \cap H(T_2) \subseteq H((S_1 \cap T_2)^{\epsilon})$$

holds for sufficiently small $\epsilon > 0$ and bounded domains S_1, S_2 with $S_2 \subset S_1$, i.e., T_1 and S_2 are disjoint, then (2.4.3) through (2.4.7) are equivalent.

Let us now also consider L-Markov fields. L-Markov property can be described as follows: Let L be a sufficiently good neighborhood of zero, namely such that the regions $S_L = \overline{S} + L = \{s + t: \; s \in \overline{S}, t \in L\}$ satisfies (2.4.1). We denote $\partial_L S = \partial S + L$ the thickened boundry of S and by $\partial_L^{\epsilon} S$ its ϵ-neighborhood.

2.4.9 <u>DEFINITION</u>. A homogeneous field $H(t)$, $t \in R^n$ is called L-Markov if for every bounded domain S satisfying (2.4.1),

(2.4.10) $$P(S_L)H(T_L) \subseteq H(\partial_L^{\epsilon} S) \;,$$

for sufficiently small $\epsilon > 0$.

In analogy with the case of Markov property, assuming (2.4.8) we have (c.f. Rozanov [39]) the following equivalent properties for the determination of L-Markov property.

(2.4.11)
$$\begin{aligned} H(t) \text{ is L-Markov} &\Leftrightarrow P(S_L^{\delta})H(T_L^{\delta}) \subseteq H(\partial_L^{\epsilon} S) \quad \epsilon > \delta \\ &\Leftrightarrow P_+(S_L)H_+(T_L) = H_+(\;_L S) \\ &\Leftrightarrow H_+(S_L) \perp H_+(T_L)^{\perp} \end{aligned}$$

$$\Leftrightarrow \; H(S_L^{\delta})^{\perp} \perp H(T_L^{\delta})^{\perp}, \; \delta > 0$$

2.4.12 <u>LEMMA</u>. The property (2.4.8) is satisfied for completely minimal fields.

Proof. Let S_1 and S_2 be bounded domains with $T_1 \cap S_2 = \emptyset$. Then $(T_1 \cup S_2)^{-\epsilon} = T_1^{-\epsilon} \cup S_2^{-\epsilon}$. From (2.3.2) we have

$$H((S_1 \cap T_2)^{\epsilon})^{\perp} \subseteq \bigvee_{\substack{\alpha<\delta \\ \operatorname{supp} \mu \subseteq (T_1 \cup S_2)^{-\epsilon}}} \tilde{u}(\lambda) Q^{*^{-1}}(\lambda)\varphi_{\alpha}(\lambda)K \;,$$

where $u(t)$ is the Fourier transform of $\tilde{u}(\lambda)$ and can be represented as a sum $u(t) = u_1(t) + u_2(t)$, where

$$u_1(t) = \begin{cases} u(t) & , \; t \in T^{-\epsilon} \\ 0 & , \; t \notin T_1^{-\epsilon} \end{cases} \quad \text{and } u_2(t) = \begin{cases} u(t), & t \in S_2^{-\epsilon} \\ 0 \;, & t \notin S_2^{-\epsilon} \end{cases}$$

But $u_2(t)$ is a bounded function with bounded support $S_2^{-\epsilon}$, therefore its Fourier transform $\tilde{u}_2(\lambda)$ is a well defined bounded function. Thus by Lemma 2.3.5(b) $\tilde{u}_2(\lambda)Q^{*^{-1}}(\lambda)\varphi_{\alpha}(\lambda)y$ belongs to $H(T_2^{\epsilon-\delta})^{\perp}$ for $\alpha < \delta$ while the function $\tilde{u}_1(\lambda) = \tilde{u}(\lambda) - \tilde{u}_2(\lambda)$ is Lebesgue integrable and is square integrable with respect to the weight $\|Q^{*^{-1}}(\lambda)\varphi_{\alpha}(\lambda)y\|^2$ with $\operatorname{supp} u_1 \subseteq T_1^{-\epsilon}$. Therefore by Lemma 2.3.5(a)

$$\tilde{u}_1(\lambda)Q^{*^{-1}}(\lambda)\varphi_{\alpha}(\lambda)y \in H(S_1^{\epsilon-\alpha})^{\perp} \subseteq H(S_1^{\epsilon-\delta})^{\perp} \quad \text{for} \quad \alpha < \delta \,.$$

Now by taking $\delta = \epsilon$ we obtain

$$\tilde{u}(\lambda)Q^{*^{-1}}(\lambda)\varphi_{\alpha}(\lambda)y = \tilde{u}_1(\lambda)Q^{*^{-1}}(\lambda)\varphi_{\alpha}(\lambda)y + \tilde{u}_2(\lambda)Q^{*^{-1}}(\lambda)\varphi_{\alpha}(\lambda)y \in H(T_2)^{\perp} \vee H(S_1)^{\perp}.$$

Therefore

$$H((S_1 \cap T_2)^{\epsilon})^{\perp} \subseteq H(T_2)^{\perp} \vee H(S_1)^{\perp} ,$$

which is equivalent to (3.1.6) and the proof of lemma is complete.

2.4.13 REMARK. Before we state the main result of this Section, by using Lemma 2.2.6, we observe that the orthogonality of $H(S_L^{\delta})^{\perp}$ and $H(T_L^{\delta})^{\perp}$, $\delta > 0$ which gives the L-Markov property for completely minimal fields is equivalent to

$$\int_{R^n} (Q^{*^{-1}}(\lambda)b_1(\lambda),\ Q^{*^{-1}}(\lambda)b_2(\lambda))d\lambda = 0,\ b_1 \in \mathcal{B}_{S_L^{\delta}},\ b_2 \in \mathcal{B}_{T_L^{\delta}} \tag{2.4.14}$$

for an arbitary bounded domain S with complementry domain T. If by convention (c.f. Rozanov [38] and the note given below) we speak of the generalized Fourier transform $\tilde{f}^{-1}(t)$ as a linear operator from X^* to X defined by

$$\int_{R^n} (Q^{*^{-1}}(\lambda)b_1(\lambda),\ Q^{*^{-1}}(\lambda)b_2(\lambda))d\lambda = \int_{R^n\setminus S_L^{\delta}} \int_{R^n\setminus T_L^{\delta}} \tilde{b}_2(t)(\tilde{f}^{-1}(s-t)\tilde{b}_1(s))dsdt,$$

then (2.4.14) is equivalent to say that the $\operatorname{supp} \tilde{f}^{-1}$ lies in the domain $L - L = \{t-s,\ t,s \in L\}$,

$$\operatorname{supp} \tilde{f}^{-1} \subseteq L - L .$$

NOTE. When complete minimality is assumed the following Theorems: Theorem 2.4.15 and Theorem 2.4.18, give necessary and sufficient conditions for L-Markov and Markov properties respectively. The conditions are in terms of the conjugate fields. However it would be interesting to obtain effective L-Markov and Markov conditions in terms of the f^{-1}, the inverse of the spectral density $f(\lambda)$. For the univariate fields over R such conditions for Markov property is obtained by N. Levinson and H.P. McKean [12]. Indeed they proved that under some condition on the growth of f^{-1}, the univariate homogeneous field $H(t)$, $t \in R$, is Markov if and only if f^{-1} is an entire function of minimal exponential type. This result was extended to the univarite homogeneous reandom fields with parameter t in R^n, by Kotani [10]. We remark that under the condition that f^{-1} has a polynomial growth, the Markov and L-Markov properties for the univariate case, $t \in R^n$, also were studied by G. M. Molchan [28] and L. D. Pitt [33]. Kotani introduced the concept of generalized Fourier transform for a sufficiently large class of functions. Kotani showed that $\tilde{f}^{-1}$ the generalized Fourier transform of f^{-1} exists under some growth condition on f^{-1}, and is defined as an "ultradistribution". We note that when complete minimality is assumed and $\tilde{f}^{-1}$ exists, Markov property is equivalent to supp $\tilde{f}^{-1} = \{0\}$. The characterization of Markov property in terms of $\tilde{f}^{-1}$ for the multivariate random fields with parmeter t in R^n is discussed by Rozanov [39], p. 16. In view of these observations, the last paragraph of Remark 2.4.13 is in need of further scrutiny. To our knowledge, conditions for the existence of $\tilde{f}^{-1}$ have not been studied for the infinite dimensional case, and deserves serious investiaation.

The following Theorems are the main results of this section which extends the work of Rozanov [39] to the Banach space case.

2.4.15 <u>THEOREM</u>. A $B(X, Y)$-valued homogeneous completely minimal field $H(t)$, $t \in R^n$, is L-Markov if and only if for each $r > 0$, the Fourier transform $\tilde{g}_r(t)$ of the density of conjugate field $H_r^*(t)$

$$g_r(\lambda) = [Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^*[Q^{*^{-1}}(\lambda)\varphi_r(\lambda)],$$

satisfies the condition

$$\text{(2.4.16)} \qquad \operatorname{supp} \tilde{g}_r \subseteq L^r - L^r, \; r > 0$$

Proof. Because of the Lemma 2.4.12 it is sufficient to establish that (2.4.16) is equivalent to the last condition of (2.4.11), namely

$$\text{(2.4.17)} \qquad H(S_L^\delta)^\perp \perp H(T_L^\delta)^\perp, \quad \delta > 0.$$

Let us assume that (2.4.17) is satisfied. By Lemma 2.3.5(c) $H_r^*((S_L^{\delta+r})^c) \subseteq H(S_L^\delta)^\perp$ and $H_r^*((T_L^{\delta+r})^c) \subseteq H(T_L^\delta)^\perp$. Therefore (2.3.16) implies that $H_r^*((T_L^{\delta+r})^c) \perp H_r^*((S_L^{\delta+r})^c)$. This is equivalent to

$$\int e^{-i\lambda(s-t)}(Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y,\, Q^{*^{-1}}(\lambda)\varphi_r(\lambda)x)d\lambda = 0 \quad \text{for} \quad s \in (S_L^{\delta+r})^c,\; t \in (T_L^{\delta+r})^c,$$

$$x, y \in K.$$

Now let ν be outside the closed region $\overline{L^r - L^r}$. Then there exist a $\delta > 0$ such that $\nu = s-t$ with $s \in (S_L^{\delta+r})^c$, $t \in (T_L^{\delta+r})^c$, which implies

$$\int e^{-i\lambda\nu}(g_r(\lambda)x,\, y)d\lambda = 0.$$

This implies (2.4.16).

Now suppose (2.4.16) is satisfied. Since the field $H(t)$ is completely minimal according to (2.3.2) we have

$$H(T_L^\delta)^\perp \subseteq \bigvee_{r<\epsilon} \tilde{u}_1(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K,\; H(S_L^\delta)^\perp \subseteq \bigvee_{r<\epsilon} \tilde{u}_2(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K\;,$$

where $u_1(t) = 0$ for $t \in T_L^\delta$ and $u_2(t) = 0$ for $t \in S_L^\delta$.

$u_1(t)$ has compact support and by (2.4.16) $\tilde{g}_r(t)$ also has compact support, $u_1(\lambda)$ and $(g_r(\lambda)x, y)$ are bounded functions, therefore

$$\int_{R^n} (\tilde{u}_1(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)x, \tilde{u}_2(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)y)d\lambda$$

$$= \int_{R^n} \tilde{u}_1(\lambda)(g_r(\lambda)x, y)\,\overline{\tilde{u}_2}(\lambda)d\lambda$$

$$= \int_{R^n} \overline{u}_2(t) \int_{R^n} u_1(s)(\tilde{g}_r(s-t)x, y)ds\, dt = 0$$

because for $t \in S_L^\delta$, $u_2(t) = 0$ and for $t \notin S_L^\delta$, $\int_{R^n} u_1(s)(\tilde{g}_r(s-t)x,y)dsdt = 0$. The latter follows from the fact that by (2.4.16) $(\tilde{g}_r(s-t)x, y)$ is only different from zero on $\{s:\ s-t \in \overline{L^\epsilon - L^\epsilon}\}$ with $r < \epsilon$, but for $t \notin S_L^\delta$ we have $\{s:\ s-t \in \overline{L^\epsilon - L^\epsilon}\} \subset T_L^{2\delta-\epsilon} \subseteq T_L^\delta$ and by using the fact that $u_1(t) = 0$ for $t \in T_L^\delta$ we obtain $u_1(t)(\tilde{g}_\alpha(s-t)x, y) = 0$ for $t \notin S_L^\delta$, $s \in R^n$. Proof is complete by noting that the elements of $H(S_L^\delta)^\perp$ and $H(T_L^\delta)^\perp$ can be approximated by the elements of $\bigvee_{u_1} \tilde{u}_1(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K$ and $\bigvee_{u_2} \tilde{u}_2(\lambda)Q^{*^{-1}}(\lambda)\varphi_r(\lambda)K$ as $r \to 0$ respectively.

Markov property is equivalent to the L-Markov property with respect to all arbitary small neighborhoods L. The following theorem is a stright consequence of the Theorem above.

2.4.18 THEOREM. A $B(X, Y)$-valued homogeneous completely minimal field is Markov if and only if for each $r > 0$, the spectral density

$$g_r(\lambda) = [Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]^*[Q^{*^{-1}}(\lambda)\varphi_r(\lambda)]$$

of the conjugate field $\overset{*}{H}_r(t)$ is a 2r-exponential function, i.e.,

$$\int_{R^n} e^{-i\lambda t}\, g_r(\lambda)d\lambda = 0 \quad , \quad |t| > 2r \quad .$$

Remark 2.4.13 and the fact that for completely minimal fields (2.4.16) and (2.4.17) are equivalent imply that if the generalized Fourier transform exists, a completely minimal field is L-Markov or Markov if supp $\tilde{f}^{-1} \subseteq L - L$ or supp $\tilde{f}^{-1} = \{0\}$ respectively. As we mentioned in the note preceding the Theorem 2.4.15, the existence of the generalized Fourier transform has not recieved satisfactory attention for the Hilbert space case as well as the Banach space case, and is need of further study.

Let us now consider the L-Markov property for discrete parameter random fields, i.e., $t \in Z^n$. We start by introducing the following notations.

NOTATIONS. Let L be a fixed finite neighborhood of zero in Z^n. We are assuming that $0 \in L$. For any bounded domain $S \subseteq Z^n$, define $S^L = S + L = \{s + \ell, s \in S, \ell \in L\}$. Also by L-boundry of S we mean $\partial^L S = S^L \backslash S$. Note that $\partial^L S \subset T$, where T is the complementary domain of S in Z^n.

2.4.19 DEFINITION. A discrete parameter random field $H(t)$, $t \in Z^n$, is called L-Markov if

$$(2.4.20) \qquad P(T)H(S) \subseteq H(\partial^L S),$$

where S is a bounded domain in Z^n with complementary domain T and $P(T)$ stands for projection onto T.

2.4.21 LEMMA. (a) Under the assumption

$$(2.4.22) \qquad H(S^L) \cap H(T) = H(\partial^L S),$$

the following condition is equivalent to (2.4.20)

$$P(T)H(S) = H(S^L) \cap H(T) \tag{2.4.23}$$

(b) (2.4.23) is always equivalent to

$$H(S^L)^\perp \perp H(T)^\perp \tag{2.4.24}$$

Proof. (a) (2.4.20) implies (2.4.23) from the following expression

$$H(S^L) \cap H(T) \subseteq P(T)H(S^L) = P(T)H(S) \subseteq H(\partial^L S) = H(S^L) \cap H(T).$$

It is clear that (2.4.23) implies (2.4.20).

(b) Note that for two closed subspaces A and B, since $P(A)B$ is a splitting subspace [40], we always have $A \vee B \ominus A = B \ominus P(A)B$. Assuming (2.4.23) with the help of expression given above we obtain that

$$\begin{aligned} H(T)^\perp = H(Z^n) \ominus H(T) &= H(T) \vee H(S) \ominus H(T) \\ &= H(S) \ominus P(T)H(S) \\ &= H(S) \ominus H(S^L) \cap H(T) \\ &\subseteq H(S^L) \ominus H(S^L) \cap H(T) \subseteq H(S^L). \end{aligned}$$

Therefore $H(T)^\perp \perp H(S^L)^\perp$. Now suppose (2.4.24) holds. Then $H(S^L) = H(T)^\perp \oplus H(T) \cap H(S^L)$, which implies that $P(T)H(S^L) = H(T) \cap H(S^L)$ or $P(T)H(S) = H(T) \cap H(S^L)$, which is (2.4.23). Proof of the Lemma is complete.

The following Lemma is similar to Lemma 2.4.12 with a similar proof.

2.4.25 LEMMA. Let $H(t)$, $t \in Z^n$ be a $B(X, Y)$-valued homogeneous completely minimal random field. Then

$$H(S_1) \cap H(T_2) = H(S_1 \cap T_2) \ , \tag{2.4.26}$$

where S_1 and S_2 are bounded domains in Z^n with complementary domains T_1 and T_2 respectively satisfying $T_1 \cap S_2 = \emptyset$.

Let $S_1 = S^L$ and $S_2 = S$ in (2.4.26). Then we obtain that (2.4.22) is satisfied for discrete parameter completely minimal fields.

The following Theorem gives necessary and sufficient condition for a Banach space-valued completely minimal field to be L-Markov. This Theorem in the univariate case implies Rozanov's Theorem (c.f. [37] Theorem 3).

2.4.25 THEOREM. A $B(X, Y)$-valued homogeneous discrete parameter completely minimal random field $H(t)$, $t \in Z^n$, is L-Markov if and only if

$$\text{supp } \tilde{g} \subseteq L,$$

where $\tilde{g}$ is the Fourier transform of the density $g(\lambda) = [Q^{*^{-1}} \varphi]^* [Q^{*^{-1}} \varphi]$; of the conjugate field $H^*(t) = H(s: \ s \neq t)^{\perp}$.

Proof. Proof is similar to the proof of Theorem 2.4.18.

2.4.26 COROLLARY. Let $H(t)$, $t \in Z^n$, be a finiate dimensional homogeneous minimal random field. Suppose the density of the field, $f(\lambda)$, satisfies $\|f\| \, \|f^{-1}\| < c$, where f^{-1} is the generalized inverse. The $H(t)$, $t \in Z^n$, is L-Markov if and only if

$$f^{-1}(\lambda) = A(I - \sum_{t \in L \setminus 0} A(t) e^{i\lambda t}) \tag{2.4.27}$$

where $A(t)$, $t \in L$ are matrices with $A = A(0)$, $\|A\| > 0$.

Proof. This follows from Theorem given above and the note to the Theorem 2.3.17. Also note that the point zero always has to be in the supp $\tilde{f}^{-1}$. Proof is complete.

2.4.28 <u>COROLLARY</u>. Let $H(t)$, $t \in Z^n$, be a finite dimensional homogeneous minimal random field. If the spectral density $f(\lambda)$ is bounded i.e., $\|f(\lambda)\| < C$ a.e. λ, then $H(t)$, $t \in Z^n$ is L-Markov if and only if (2.4.27) holds.

Proof. With a similar proof to the one given for Lemma 2.4.21 (b), L-Markov property (2.4.20) implies (2.4.24), and the latter implies that supp $\tilde{f}^{-1} \subseteq L$, which implies that (2.4.27) holds. For the other part suppose (2.4.27) holds. This implies $\|f^{-1}\| < c$ a.e. λ. But $\|f\| < c$ a.e. λ. Therefore $\|f\| \, \|f^{-1}\| < c$ a.e. λ. Now apply Corollary 2.4.26. Proof is complete.

CHAPTER III

A RECIPE FORMULA FOR THE LINEAR INTERPOLATOR

Introduction. A set of reals or complex-valued random variables $\xi_x(t)$ over a probability space $(\Omega, \mathcal{B}, P)$ depending on a parameter t in Z^n = the Cartesian product of Z (set of integers) with itself n-times, where the index x runs through a set X, is called a (discrete parameter) random field. Let E denote the expected value. We assume that $E\,\xi_x(t) = 0$ and $E|\xi_x(t)|^2 < \infty$ as elements of a Hilbert space $H = L^2(\Omega, \mathcal{B}, P)$ of random variable ξ, $E|\xi|^2 < \infty$, with the scalar product $E\,\xi\bar{\eta}$, $\xi, \eta \in H$.

In this chapter we assume that X is a finite dimensional Hilbert space and for each $t \in Z^n$, $\xi_x(t)$ is linear in the variable x. As such, one can express $\xi_x(t)$ in the form

$$\xi_x(t) = \xi(t)x;\ x \in X,\ t \in Z^n,$$

where for each t, $\xi(t)$ is a bounded linear operator on X into the Hilbert space H. In view of the relation above, the operator-valued $\xi(t)$, $t \in Z^n$, is also called a random field. These two versions of a random field will enterchangbly be used throughout this chapter. A random field $\xi(t)$ is called homogeneous if $E\,\xi_x(t)\overline{\xi_y(s)}$ depends only on $t-s$ for all $x, y \in X$. Let x_k, $k = 1,\ldots,q$ be an orthonormal basis in X, then we will refer to $\xi_k(t) = \xi_{k_x}(t)$, $k = 1,\ldots,q$, as the kth component of the homogeneous random field $\xi(t)$. We will also call $\xi(t)$, a q-variate homogeneous random field denoted by $\xi(t) = \{\xi_k(t)\}_{1 \le k \le q}$ (c.f. Notations 3.1).

The main purpose of this Chapter is to derive a recipe formula for the linear interpolator in the time domain which seems to have potential applications. To be more precise let $\xi(t) = \{\xi_k(t)\}_{1 \le k \le q}$, $t \in Z^n$, be a q-variate homogeneous random field and let T_k, $k = 1,\dots,q$, be finite subsets of Z^n. Let us assume that all the values $\xi_k(t)$, $k = 1,\dots,q$, are known except for the values $\xi_k(t)$, $t \in T_k$, $k = 1,\dots,q$. The problem which is the subject of this Chapter is to express the linear projection of the unknown component $\xi_k(t_0)$, $t_0 \in T_k$, in terms of the known components $\xi_\ell(t)$, $t \notin T_\ell$, $\ell = 1,\dots,q$, as an infinite series expansion. In a recent paper [42] H. Salehi derived an explicit representation for the linear interpolator of a univariate random field under the boundedness assumption on the spectral density of the field. Earlier works in this area were carried out by A.M. Yaglom [48] and Yu. A. Rozanov [35]. Our intention in this Chapter is to carry out an algorithm for the linear interpolator of a q-variate random field under a more relaxed assumption on the spectral density. This algorithm reduces to the one given by Salehi in [42] for the univariate case. We replace the boundedness condition on the spectral density by its square norm interability.

3.1. Notations and Priliminaries. A matrix a consisting of elements a_{kj} $(k = 1,\dots n,\ j = 1,\dots,m)$ have n rows and m columns, will be denoted by $\{a_{kj}\}_{1 \le k \le n}^{1 \le j \le m}$ or simply $\{a_{kj}\}_n^m$. A matrix a consisting of a single row of elements $a_1,\ldots,a_m$, will be called a row vector denoted by $\{a_j\}^{1\le j\le m}$ or $\{a_j\}^m$; and by $\{b_i\}_{1\le i\le n}$ or $\{b_i\}_n$ we mean a matrix b consisting of a single column of elements $b_1,\ldots,b_n$, such a matrix is called a column vector. As usual the product of two matrices $\{a_{kj}\}_m^n$ and $b = \{b_{j\ell}\}_n^{n'}$ is a matrix $c = \{c_{k\ell}\}_m^{n'}$ where $c_{k\ell} = \sum_{j=1}^{n} a_{kj}b_{j\ell}$. Further, the adjoint of a matrix $a = \{a_{kj}\}_n^m$ is defined to be the matrix $a^* = \{\overline{a}_{jk}\}_m^n$, where $\overline{a}_{jk}$ is the complex conjugate of a_{jk}. Also the Euclidean norm, $\|a\|$, of a matrix $a = \{a_{kj}\}_n^m$ is defined to be

$$\|a\| = \left(\sum_{k=1}^{n}\sum_{j=1}^{m} |a_{kj}|^2\right)^{\frac{1}{2}} . \tag{3.1.1}$$

It is known that $\xi(t)$ has the spectral representation $\xi(t) = \int e^{i\lambda t}\Phi(d\lambda)$, where $\Phi = \{\Phi_k\}_{1\le k\le q}$, is the random spectral measure associated with the multi-dimensional homogeneous random field [19], [34], [36] (whenever the domain of integration is missing, it is understood that the integration is over T_n, the n-dimensional torus, which is the Cartesian product of the unit circle T with itself n times). Define the measures $F_{k\ell}(\Delta) = E(\Phi_k(\Delta)\overline{\Phi_\ell(\Delta)})$, $k, \ell = 1,\ldots,q$, then the square matrix $\{F_{k\ell}\}_q^q$ is called the spectral measure of the process $\xi(t)$. In the case that all the elements $F_{k\ell}$ are absolutely continuous with respect (w.r.t.) to the Lebesgue measure λ, $f(\lambda) = \{f_{k\ell}(\lambda)\}_q^q$ with $f_{k\ell}(\lambda) = \frac{dF_{k\ell}}{d\lambda}(\lambda)$ is called the spectral density of the q-variate homogeneous random field $\xi(t)$. In this work we assume that the process has a density $f(\lambda)$.

The Hilbert space $L^2(F)$, which plays an important role in the analysis of a homogeneous random field, consists of all vector-valued functions $\varphi(\lambda) = \{\varphi_k(\lambda)\}^q$ with $\int[\varphi(\lambda)f(\lambda)\varphi^*(\lambda)\,d\lambda = \int \sum_{k,\ell=1}^{q} \varphi_k(\lambda)f_{k\ell}(\lambda)\overline{\varphi}_\ell(\lambda)d\lambda < \infty$. The inner product between φ and ψ in $L^2(F)$ is given by $<\varphi,\psi> = \int \varphi(\lambda)f(\lambda)\psi^*(\lambda)d\lambda$. Let H_ξ be the span clousure of the random variable $\xi_k(t)$, $t \in Z^n$ and $k = 1,\ldots,q$, in H, i.e.,

$$H_\xi = V\ \{\xi_k(t),\ t \in Z^n,\ k = 1,\ldots,q\}\ .$$

Then there is an isometry between H_ξ and $L^2(F)$ which to any $h \in H_\xi$ corresponds a unique $\varphi = \{\varphi_k\}^q$ in $L^2(F)$ with $h = \int\varphi(\lambda)\Phi(d\lambda)$. For the definition of the integral $\int\varphi(\lambda)\Phi(d\lambda)$ and the isometry in a general setting see [17], [34]. In our case the integral $\int\varphi\Phi(d\lambda)$ reduces to $\sum_{k=1}^{q} \int\varphi_k(\lambda)\Phi_K(\lambda)$. For addional information on q-variate homogeneous processes see [19].

Let $N(T)$ be the closed subspace of H_ξ spanned by the differences $\xi_k(t) - \hat{\xi}_k(t)$, $t \in T_k$, $k = 1,\ldots,q$, where $\hat{\xi}_k(t)$ is the projection of $\xi_k(t)$, $t \in T_k$, onto the $V\{\xi_k(t),\ t \notin T_k,\ k = 1,\ldots,q\}$, and let $\Delta(T)$ be the closed subspace of $L^2(F)$ corresponding to $N(T)$ under the isometry map between $L^2(F)$ and H_ξ. Let $B(T)$ be the space of vector-valued functions $b(\lambda) = \{b_k(\lambda)\}^q$ whose components $b_k(\lambda)$ are trigonometric polynomials of the form

$$b_k(\lambda) = \sum_{t\in T_k} a_k(t)e^{i\lambda t},$$

satisfying (i) $b^*(\lambda) \in$ Range $f(\lambda)$ a.e.λ; (ii) the integral $\int b(\lambda)f^{-1}(\lambda)b^*(\lambda)d\lambda < \infty$, where $f^{-1}(\lambda)$ is the inverse of the restriction of $f(\lambda)$ to the Range $f(\lambda)$.

The following lemma is a special case of Lemma 1.1.3 or 2.2.6. It is also given in [38] and [44].

3.1.2 LEMMA. $\Delta(T) = \mathcal{B}(T)f^{-1}$, meaning that for any $\varphi \in \Delta(T)$, there exists a unique $b(\lambda)$ in $\mathcal{B}(T)$ such that $\varphi(\lambda) = b(\lambda)f^{-1}(\lambda)$.

The following theorem gives a necessary and sufficient condition for $\xi_k(t_0) - \hat{\xi}_k(t_0)$ to be different form zero for each k, $1 \leq k \leq q$, i.e., the interpolation is imperfect (see McKean [5]). This theorem was originally proved by Kolmogorov for the univariate case [9]. Extensions to multivariate case were carried out by Rozanov [36] and Masani [18].

3.1.3 THEOREM. In order to have imperfect interpolation for the q-variate homogeneous random field, it is necessary and sufficient that f is non-singular a.e.λ, and that

$$\int \|f^{-1}\| d\lambda < \infty. \tag{3.1.4}$$

It should be noted that the notion of "imperfect interpolation" introduced here is equivalent to the concept of "minimality" given by Rozanov [36] and "minimality of full rank" given by Masani [18].

The following theorem gives a recipe for $\hat{\xi}_k(t_0)$, $t_0 \in T_k$, the projection of the $\xi_k(t_0)$, $t_0 \in T_k$, onto the $V\{\xi_j(t): t \notin T_j, j = 1,\dots,q\}$, as a spectral integral.

3.1.5 THEOREM. Let $\xi(t) = \{\xi_k(t)\}_{1\le k\le q}$, $t \in Z^n$, be a q-variate homogeneous Gaussian random field which has imperfect interpolation. Also let $\hat{\xi}_k(t_0)$, $t_0 \in T_k$, be the projection of $\xi_k(t_0)$, $t_0 \in T_k$, (a fixed but arbitrary component of $\xi(t_0)$) onto $V\{\xi_k(t)$, $t \notin T_k$, $k = 1,\ldots,q\}$, where T_k, $k = 1,\ldots,q$ are finite subsets of Z^n. Then

$$\hat{\xi}_k(t_0) = \int \hat{\varphi}_k(\lambda)\Phi(d\lambda). \tag{3.1.6}$$

$\hat{\varphi}_k(\lambda) = \{\hat{\varphi}_{kj}(\lambda)\}^q$ has the form

$$\hat{\varphi}_k(\lambda) = e^{i\lambda t_0}\delta_k - b_k(\lambda)f^{-1}(\lambda), \tag{3.1.7}$$

where $\delta_k = \{\delta_{kj}\}^q$ with $\delta_{kj} = 0$ for $j \neq k$ and $\delta_{kk} = 1$;

$$\begin{aligned} &b_k(\lambda) = \{b_{kj}(\lambda)\}^q \quad \text{with} \\ &b_{kj}(\lambda) = \sum_{t\in T_j} a_{kj}(t)e^{i\lambda t}, \quad j = 1,\ldots,q\,; \end{aligned} \tag{3.1.8}$$

and the coefficients $\{a_{kj}(t):\ t \in T_j,\ j = 1,\ldots,q\}$ can be obtained from the following system of equations:

$$\begin{aligned} &\sum_{j=1}^{q}\sum_{s\in T_j} \hat{p}_{j\ell}(s-t)a_{kj}(s) = 0 \quad \text{for} \quad t \in T_\ell,\ \ell \neq k, \\ &\sum_{j=1}^{q}\sum_{s\in T_j} \hat{p}_{jk}(s-t)a_{kj}(s) = 1 \quad \text{for} \quad t = t_0 \quad \text{and} \\ &\sum_{j=1}^{q}\sum_{s\in T_j} \hat{p}_{jk}(s-t)a_{kj}(s) = 0 \quad \text{for} \quad t \in T_k\backslash t_0. \end{aligned} \tag{3.1.9}$$

where $f^{-1}(\lambda) = \{p_{j\ell}(\lambda)\}_q^q$ and $\hat{p}_{j\ell}(t)$, $t \in Z^n$, is the Fourier coefficient of $p_{j\ell}$ at t, namely, $\hat{p}_{j\ell}(t) = \int e^{i\lambda t} p_{j\ell}(\lambda) d\lambda$, $\lambda t = \sum_{j=1}^{n} \lambda_j t_j$.

Note. We point out that this Theorem can be found in [36] p. 101 where the third equation in (3.1.9) is missing. For completness we will give the proof below.

Proof. By the isomorphism between the time domain H_ξ and spectral domain $L^2(F)$, we have $\xi_k(t_0) - \hat{\xi}_k(t_0) = \int h_k \Phi(d\lambda)$ where $h_k \in \Delta(T)$.

Therefore by LEMMA 3.1.2 $h_k(\lambda) = b_k(\lambda) f^{-1}(\lambda)$, where $b_k(\lambda) = \{b_{kj}(\lambda)\}^q$ with $b_{kj}(\lambda) = \sum_{t \in T_j} a_{kj}(\lambda) e^{it\lambda}$, $j = 1,\ldots,q$. This implies that the corresponding image of $\hat{\xi}_k(t_0)$ in $L^2(F)$ is $e^{i\lambda t_0}\delta_k - b_k(\lambda) f^{-1}(\lambda)$ which is orthogonal to $\Delta(T)$, i.e., $\langle e^{i\lambda t_0}\delta_k - b_k(\lambda) f^{-1}(\lambda), g(\lambda)\rangle = 0$ for any $g(\lambda) \in \Delta(T)$. Now take $g(\lambda) = e^{i\lambda t}\delta_\ell f^{-1}(\lambda)$, $t \in T_\ell$, which in view of (3.1.5) is in $\Delta(T)$. The orthogonality can be expressed as

$$\int (e^{i\lambda t_0}\delta_k - b_k(\lambda) f^{-1}(\lambda)) f(\lambda) (e^{i\lambda t}\delta_\ell f^{-1}(\lambda))^* d\lambda = 0 \quad \text{for all}$$

$$t \in T_\ell, \ \ell = 1,\ldots,q.$$

or

$$\int e^{i\lambda t_0}\delta_k f(\lambda) [e^{i\lambda t}\delta_\ell f^{-1}(\lambda)]^* d\lambda - \int b_k(\lambda)(e^{i\lambda t}\delta_\ell f^{-1}(\lambda))^* d\lambda = 0, \tag{3.1.10}$$

$$t \in T_\ell, \ \ell = 1,\ldots,q.$$

But $f(\lambda) = \{f_{ij}(\lambda)\}_q^q$ and $f^{-1}(\lambda) = \{p_{mn}(\lambda)\}_q^q$, therefore $\delta_k f(\lambda) = \{f_{ki}\}^q$ and $\delta_\ell f^{-1}(\lambda) = \{p_{\ell n}\}^q$. Thus

$$b_k(\lambda)\ (e^{i\lambda t}\delta_\ell f^{-1}(\lambda))^* = e^{-i\lambda t} \sum_{j=1}^{q} b_{kj}(\lambda)\overline{p}_{\ell j}(\lambda) \quad \text{and}$$

$$e^{i\lambda t_0}\delta_k f(\lambda)[e^{i\lambda t}\delta_\ell f^{-1}(\lambda)]^* = e^{i\lambda(t_0-t)} \sum_{j=1}^{q} f_{kj}(\lambda)\ \overline{p}_{\ j}(\lambda)$$

Substituting the above quantities in (3.1.10) we obtain

$$(3.1.11)\quad \int e^{i\lambda(t_0-t)} \sum_{j=1}^{q} f_{kj}(\lambda)\overline{p}_{\ell j}(\lambda)d\lambda - \int e^{-i\lambda t} \sum_{j=1}^{q} b_{kj}(\lambda)\overline{p}_{\ell j}(\lambda)d\lambda = 0,$$

$$t \in T_\ell,\ \ell = 1,\ldots,q.$$

But since $ff^{-1} = I$ we have

$$(3.1.12)\qquad \sum_{j=1}^{q} f_{kj}\ p_{j\ell} = 1 \quad \text{for} \quad k = \ell$$

$$= 0 \quad \text{for} \quad k \neq \ell.$$

Now let $k \neq \ell$, then with the help of (3.1.8), (3.1.12) and the fact that f^{-1} is self-adjoint, (3.1.9) becomes

$$\sum_{j=1}^{q} \sum_{s\in T_j} a_{kj}(s) \int e^{-i\lambda t}\ e^{i\lambda s}\ p_{j\ell} = 0\ ,\ t \in T_\ell,\ \ell \neq k$$

or

$$(3.1.13)\qquad \sum_{j=1}^{q} \sum_{s\in T_j} a_{kj}(s)\ \hat{p}_{j\ell}(s-t) = 0\ ,\ t \in T_\ell,\ \ell \neq k,$$

which is the first equation in (3.1.9)

For $k = \ell$ and $t \in T_k \backslash t_0$ similar to (3.1.13) we have

$\sum_{j=1}^{q} \sum_{s\in T_j} a_{kj}(s)\hat{p}_{jk}(s-t) = 0$, and for $k = \ell$, $t = t_0$ we have

$1 - \sum_{j=1}^{q} \sum_{s \in T_j} a_{kj}(s)\hat{p}_{jk}(s-t) = 0$, which gives the last two equations in (3.1.9). As in [42] we can easily show that the system of equation (3.1.9) has a unique solution. This completes the proof.

3.2 A Recipe Formula. In order to derive a recipe for $\hat{\xi}_k(t_0)$, $t_0 \in T_k$, as a series expansion involving the known values $\xi_\ell(t)$, $t \notin T_\ell$, $1 \leq \ell \leq q$, we are forced to impose certain additional condition on the spectral density. We have already mentioned that under the boundedness of f and the square integrability of f^{-1} Rozanov [35] and Salehi [42] have obtained such expansions. In this section we remove the restrictive boundedness assumption on f and replace it by a more relaxed condition, namely the square intergrability of f.

We now state the main result of this Chapter.

3.2.1 THEOREM. Suppose that the spectral density $f(\lambda)$ of a q-variate homogeneous random field $\xi(t)$, $t \in Z^n$, satisfies

(i) $\int \|f(\lambda)\|^2 d\lambda < \infty$ and (ii) $\int \|f^{-1}(\lambda)\|^2 d\lambda < \infty$.

Then the following statements hold:

(a) bf_m^{-1} as well as $(bf^{-1})_m$ converge to bf^{-1} in $L^2(F)$ as $m \to \infty$ for any polynomial $b(\lambda)$ and in particular for the polynomial b_k given in (3.1.8), where for any $g \in L^2(d\ )$, g_m is defined by

$$g_m = \{g_{ij}(m,\lambda)\}_{1 \leq i \leq q}^{1 \leq j \leq q} \quad \text{with} \quad g_{ij}(m,\lambda) = \sum_{t \in F_m} \hat{g}_{ij}(t) e^{-i\lambda t},$$

$\hat{g}_{ij}(t) = \int e^{i\lambda t} g_{ij}(\lambda)d\lambda$ and $F_m = \{t \in Z^n: \|t\| \le m\}$, $m = 1,2,\ldots$

(b) The random variable $\hat{\xi}_k(t_0)$, $t_0 \in T_k$, giving the best linear interpolator of $\xi_k(t_0)$ based on $\xi_i(\ell)$, $\ell \notin T_i$, $1 \le i \le q$, can be obtained from the following formula:

$$\hat{\xi}_k(t_0) = \sum_{i=1}^{q} \sum_{s \notin T_i} \beta_i(s)\xi_i(s), \tag{3.2.2}$$

where

$$\beta_i(s) = -\sum_{j=1}^{q} \sum_{t \in T_j} \hat{p}_{ji}(t-s)a_{kj}(t),\ s \notin T_i,$$

and the coefficients $a_{kj}(t)$, $t \in T_j$, $1 \le j \le q$ are obtained from the system of equations (3.1.9).

The convergence in (3.2.2) is understood to be in the space H_ξ.

Proof. (a) We give the proof for $b_k f_m^{-1}$ wit the polynomial $b_k(\lambda)$ given in (3.1.8). The proof can be carried out similarly for the remaining cases. Note that $b_k(\lambda)f^{-1}(\lambda)$ is in $L^2(F)$. We recall that $b_k(\lambda)f_m^{-1}(\lambda) = \{\sum_{i=1}^{q} b_{ki}(\lambda)p_{ij}(m,\lambda)\}^{1\le j\le q}$ and

$b_k(\lambda)f^{-1}(\lambda) = \{\sum_{i=1}^{q} b_{ki}(\lambda)p_{ij}(\lambda)\}^{1\le j\le q}$. Thus

$$b_k(\lambda)f^{-1}(\lambda) - b_k(\lambda)f_m^{-1}(\lambda) = \{\sum_{i=1}^{q} b_{ki}(\lambda)(p_{ij}(\lambda) - p_{ij}(m,\lambda))\}^{1\le j\le q} .$$

Therefore

$$(b_k(\lambda)f^{-1}(\lambda) - b_k(\lambda)f_m^{-1}(\lambda))f(\lambda)(b_k(\lambda)f^{-1}(\lambda) - b_k(\lambda)f_m^{-1}(\lambda))^* =$$

$$\{\sum_{r=1}^{q} b_{kr}(\lambda)(p_{r\ell}(\lambda)-p_{r\ell}(m,\lambda)\}^{1\leq\ell\leq q}\{\sum_{j=1}^{q}\sum_{i=1}^{q} f_{\ell j}\,\overline{b_{ki}}(\lambda)(\overline{p_{ij}}(\lambda) - \overline{p_{ij}}(m,\lambda))\ {}_{1\leq\ell\leq q}$$

$$= \sum_{\ell=1}^{q}\sum_{r=1}^{q}\sum_{j=1}^{q}\sum_{i=1}^{q} b_{kr}(\lambda)\overline{b_{ki}(\lambda)}f_{\ell j}(\lambda)(p_{r\ell}(\lambda) - p_{r\ell}(m,\lambda))(\overline{p_{ij}}(\lambda) - \overline{p_{ij}}(m,\lambda)).$$

Let

$$A(\lambda) = \sum_{\ell=1}^{q}\sum_{r=1}^{q}\sum_{j=1}^{q}\sum_{i=1}^{q} b_{kr}(\lambda)\overline{b_{ki}(\lambda)}f_{\ell j}(\lambda)p_{r\ell}(\lambda)\overline{p_{ij}}(\lambda),$$

$$B_m(\lambda) = \sum_{\ell=1}^{q}\sum_{r=1}^{q}\sum_{j=1}^{q}\sum_{i=1}^{q} b_{kr}(\lambda)\overline{b_{ki}(\lambda)}f_{\ell j}(\lambda)p_{r\ell}(\lambda)\overline{p_{ij}}(m,\lambda),$$

$$C_m(\lambda) = \sum_{\ell=1}^{q}\sum_{r=1}^{q}\sum_{j=1}^{q}\sum_{i=1}^{q} b_{kr}(\lambda)\overline{b_{ki}(\lambda)}f_{\ell j}(\lambda)p_{r\ell}(m\ \lambda)\overline{p_{ij}}(\lambda) \quad \text{and}$$

$$D_m(\lambda) = \sum_{\ell=1}^{q}\sum_{r=1}^{q}\sum_{j=1}^{q}\sum_{i=1}^{q} b_{kr}(\lambda)\overline{b_{ki}(\lambda)}f_{\ell j}(\lambda)p_{r\ell}(m,\lambda)\overline{p_{ij}}(m,\lambda) \quad \text{be the}$$

four terms of the expansion of the expression above. Therefore

$$\int(b_k(\lambda)f^{-1}(\lambda)-b_k(\lambda)f_m^{-1}(\lambda))f(\lambda)(b_k(\lambda)f^{-1}(\lambda)-b_k(\lambda)f_m^{-1}(\lambda))^* d\lambda =$$

(3.2.3)

$$\int A\ (\lambda)d\lambda - \int B_m(\lambda)d\lambda - \int C_m(\lambda)d\lambda + \int D_m(\lambda)d\lambda$$

Now

$$A(\lambda) = \sum_{r=1}^{q}\sum_{i=1}^{q}\sum_{j=1}^{q} b_{kr}(\lambda)\overline{b_{ki}}(\lambda)\ \{\sum_{\ell=1}^{q} f_{\ell j}(\lambda)p_{r\ell}(\lambda)\}\overline{p_{ij}}(\lambda). \quad \text{But}$$

$f^{-1}f = I$ giving $\sum_{\ell=1}^{q} f_{\ell j}p_{r\ell} = 1$ for $r = j$ and $\sum_{\ell=1}^{q} f_{\ell j}p_{r\ell} = 0$

for $r \neq j$ a.e. λ.

Therefore $A(\lambda) = \sum_{j=1}^{q} \sum_{i=1}^{q} b_{kj}(\lambda)\overline{b_{ki}}(\lambda)\overline{p}_{ij}(\lambda)$ a.e. λ.

thus $\int A(\lambda)d\lambda = \sum_{j=1}^{q} \sum_{i=1}^{q} \int b_{kj}(\lambda)\overline{b_{ki}}(\lambda)\overline{p}_{ij}(\lambda)d\lambda$

but $b_{ki}(\lambda) = \sum_{t\in T_i} a_{ki}(t)e^{it\lambda}$, Therefore we get

$$\int A(\lambda)d\lambda = \sum_{j=1}^{q} \sum_{i=1}^{q} \sum_{t\in T_i} \sum_{t\in T_j} a_{kj}(t)\overline{a_{ki}(s)} \int e^{i(t-s)\lambda}\,\overline{p}_{ij}(\lambda)d\lambda$$

$$= \sum_{j=1}^{q} \sum_{i=1}^{q} \sum_{t\in T_i} \sum_{t\in T_j} a_{kj}(t)\overline{a_{ki}}(s)\hat{\overline{p}}_{ij}(s-t)\,.$$

Similarly

$$B_m(\lambda) = \sum_{r=1}^{q} \sum_{j=1}^{q} \sum_{i=1}^{q} b_{kr}(\lambda)\overline{b_{ki}}(\lambda)\{\sum_{\ell=1}^{q} f_{\ell j}(\lambda)p_{r\ell}(\lambda)\}\overline{p}_{ij}(m,\lambda)$$

$$= \sum_{j=1}^{q} \sum_{i=1}^{q} b_{kj}(\lambda)\overline{b_{ki}}(\lambda)\overline{p}_{ij}(m,\lambda)$$

$$= \sum_{j=1}^{q} \sum_{i=1}^{q} \sum_{t\in T_j} \sum_{s\in T_i} \sum_{\ell\in F_m} a_{kj}(t)\overline{a_{ki}}(s)\hat{\overline{p}}_{ij}(\ell)e^{i\lambda(t-s+\ell)}$$

Therefore

$$\int B_m(\lambda)d\lambda = \sum_{j=1}^{q} \sum_{i=1}^{q} \sum_{t\in T_j} \sum_{s\in T_i} \sum_{\ell\in F_m} a_{kj}(t)\overline{a_{ki}}(s)\overline{\hat{p}}_{ij}(\ell) \int e^{i\lambda(t-s+\ell)}d\lambda$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{t\in T_j} \sum_{s\in T_i} a_{kj}(t)\overline{a_{ki}}(s)\hat{\overline{p}}_{ij}(t-s) \quad \text{for } m \text{ large enough.}$$

Also

$$C_m(\lambda) = \sum_{\ell=1}^{q} \sum_{r=1}^{q} \sum_{i=1}^{q} b_{kr}(\lambda)\overline{b}_{ki}(\lambda)p_{r\ell}(m,\lambda) \sum_{j=1}^{q} f_{\ell j}(\lambda)\overline{p_{ij}}(\lambda)$$

$$= \sum_{r=1}^{q} \sum_{i=1}^{q} b_{kr}(\lambda)\overline{b}_{ki}(\lambda)p_{rj}(m,\lambda) \text{ , because } f^{-1} \text{ is self adjoint.}$$

Therefore

$$\int C_m(\lambda)d\lambda = \sum_{r=1}^{q} \sum_{i=1}^{q} \sum_{t\in T_r} \sum_{s\in T_i} \sum_{j\in F_m} a_{kr}(t)\overline{a_{ki}}(s)\hat{p}_{ri}(j) \int e^{i\lambda(t-s-j)}d\lambda$$

$$= \sum_{r=1}^{q} \sum_{i=1}^{q} \sum_{t\in T_r} \sum_{s\in T_i} a_{kr}(t)\overline{a_{ki}}(s)\hat{p}_{ri}(t-s)$$

$$= \sum_{r=1}^{q} \sum_{i=1}^{q} \sum_{t\in T_r} \sum_{s\in T_i} a_{kr}(t)\overline{a_{ki}}(s)\hat{\hat{p}}_{ir}(t-s) \quad \text{for sufficiently}$$

large m.

Finally

$$D_m(\lambda) = \sum_{\ell,r,j,i=1}^{q} \sum_{T_r,T_i} \sum_{x\in F_m} \sum_{y\in F_m} a_{kr}(t)\overline{a_{ki}}(s)\hat{p}_{r\ell}(x)\overline{\hat{p}}_{ij}(y)e^{i\lambda(t-s-x+y)}f_{\ell j}(\lambda)$$

Let $$A_{mn} = \sum_{\ell,r,j,i=1}^{q} \sum_{T_r,T_j} \sum_{x\in F_n} \sum_{y\in F_m} a_{kr}(t)\overline{a_{ki}}(s)\hat{p}_{r\ell}(x)\overline{\hat{p}}_{ij}(y)[e^{i\ (t-s-x)}f_{\ell j}(\lambda)\hat{j}(y).$$

We show below that A_{mn} as, $n \to \infty$, converges uniformly in m and the same is true for A_{mn}, as $m \to \infty$. This will allow us to write

$$\lim_{m} \int D_m(\lambda)d\lambda = \lim_{m\to\infty} \lim_{n\to\infty} A_{mn} \text{ ,}$$

because

$$\int D_m(\lambda)d\lambda = \sum_{\ell,r,j,i} \sum_{T_r,T_i} \sum_{x\in F_m} \sum_{y\in F_m} a_{kr}(t)\overline{a_{ki}}(s)\hat{p}_{r\ell}(x)\overline{\hat{p}_{ij}}(y)$$

$$[e^{i\lambda(t-s-x)}f_{\ell j}(x)\hat{]}(y) = A_{mm}$$

Now for any (i,ℓ,j), $1 \leq i,\ell,j \leq q$, since $f_{\ell j}$ and p_{ij} are in $L^2(\lambda)$ we have $\sum_{y\in F_m} \overline{\hat{p}_{ij}}(y)[e^{i\lambda(t-s-x)}f_{\ell j}(\lambda)\hat{]}(y) \to \langle p_{ij}, e^{i\lambda(t-s-x)}f_{\ell j}\rangle$

uniformly in x, where $\langle\cdot\rangle$ stands for inner product in $L^2(d\lambda)$, the convergence is clear and the uniformity follows from the Cauchy-Schawrtz inequality and the fact that $\|f_{\ell j}\|_2 = \|e^{i\lambda(r-s-x)} f_{\ell j}\|_2$ for all x.

But $\langle p_{ij}, e^{i\lambda(t-s-x)}f_{\ell j}\rangle = \int e^{-i\lambda(t-s-x)}p_{ij}(\lambda)\overline{f_{\ell j}}(\lambda)d\lambda$. This implies that $\sum_{j=1}^{q} \sum_{y\in F_m} \overline{\hat{p}_{ij}}(y)[e^{i\lambda(t-s-x)}f_{\ell j}(\lambda)\hat{]}(y)$ converges uniformly in

x to

$$\sum_{j=1}^{q} \int e^{-i\lambda(t-s-x)}p_{ij}(\lambda)\overline{f_{\ell j}}(\lambda)d\lambda = \int e^{-i\lambda(t-s-x)} \sum_{j=1}^{q} p_{ij}(\lambda)f_{j\ell}(\lambda)d\lambda,$$

which is equal to zero when $i \neq \ell$ and is equal to

$\int e^{-i\lambda(t-s-x)}d\lambda$ for $i = \ell$; and the latter is zero for $x \neq t-s$

Therefore

$$\int D_m(\lambda)d\lambda \to \sum_{r=1}^{q}\sum_{i=1}^{q}\sum_{t\in T_r}\sum_{s\in T_i} a_{kr}(t)\overline{a_{ki}}(s)\hat{p}_{ri}(t-s)$$

$$= \sum_{r=1}^{q}\sum_{i=1}^{q}\sum_{t\in T_r}\sum_{s\in T_i} a_{kr}(t)a_{ki}(s)\bar{\hat{p}}_{ir}(t-s)$$

thus $\int A_m(\lambda)d\lambda$, $\int B_m(\lambda)d\lambda$, $\int C_m(\lambda)d\lambda$ and $\int D_m(\lambda)d\lambda$ converge to the same limit and the proof is complete by (3.2.3).

(b) Recall from Theorem 3.1.5 that $\hat{\xi}_k(t_0) = \int\varphi_k(\lambda)\Phi(d\lambda)$, where

$$\varphi_k(\lambda) = e^{i\lambda t}\delta_k - b_k(\lambda)f^{-1}(\lambda).$$

But by part (a) $(b_k f^{-1})_m = \{\sum_{s\in F_m}\sum_{t\in T_i}\sum_{i=1}^{q} a_{ki}(t)\dot{p}_{ij}(t+s)e^{-i\lambda s}\}^{1\le j\le q}$

converges to $b_k f^{-1}$ in $L^2(F)$. Now let

$$\beta_j(s) = -\sum_{i=1}^{q}\sum_{t\in T_i}\hat{p}_{ij}(t-s)a_{ki}(t),\ j = 1,\dots,q.$$

Then from (3.1.9) we have

$$e^{i\lambda t_0}\delta_k - (b_k(\lambda)f^{-1}(\lambda))_m = \{e^{i\lambda t_0}\delta_{kj} - \sum_{s\in F_m}\sum_{i=1}^{q}\sum_{t\in T_i} a_{ki}(t)\hat{p}_{ij}(t-s)e^{i\lambda s}\}^{1\le j\le q}$$

$$= \{\sum_{s\in F_m\setminus T_j}\beta_j(s)e^{i\lambda s}\}^{1\le j\le q}.$$

Since $e^{i\lambda t_0}\delta_k - (b_k f^{-1})_m$ converges to φ_k in $L^2(F)$, by using the isomorphism betwen the time domain and spectral domain we obtain that

$$\hat{\xi}_k(t_0) = \sum_{j=1}^{q}\sum_{s\notin T_j}\beta_j(s)e^{i\lambda s},$$

which completes the proof.

REMARK. In this chapter our main Theorem 3.2.1 is derived under the assumption that $f(\lambda)$ is nonsingular a.e.λ, and that f^{-1} is in $L^2(\lambda)$. The problem of obtaining a recipe for the case when f may be less than full rank remains open (the case of singular f has been considered in mathematical literature in connection with regularity of homogenous random field (c.f. [15])).

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