A REFORMULATION OF HARROD GROWTH THEORY

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JAY BERNARD SPECTOR

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This is to certify that the

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ABSTRACT

A REFORMULATION OF HARROD GROWTH THEORY

By

Jay Bernard Spector

In this dissertation, a nonmonetary dynamic model based on expectations of the firm is constructed. We show that this model is capable of explaining, from a general equilibruim point of view, many of the properties of the original Harrod growth model. We also show that the model is more general than the Harrod model, since the latter restricts initial conditions and expectations. Finally, we show how to generalize the model by incorporating in it inventory behavior and full employment constraints. We derive the result that a nonmonetary dynamic economy will always experience business cycles if expectations are "fully adaptive" and the rate of growth of labor is less than the "warranted" rate of the economy.

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A REFORMULATION OF HARROD GROWTH THEORY

By

Jay Bernard Spector

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who have waited so

and for

who made it possible

for

my parents

who have waited so long for this dissertation

and for

murray webster

who made it possible

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Finally I would like to express my gratitude to Murray Webster for all that he has done.

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CHAPTER 1

Section 1.1. Introduction

The purpose of this dissertation is to shed some light on the theory of growth first proposed by Roy Harrod, and to extend this theory to situations more complex than those discussed by Harrod.

The phrase "theory of growth first proposed by Roy Harrod" may mean different things to different people. In this dissertation, we shall define "Harrod growth theory" to be that nonmonetary theory of growth which assumes fixed capital-output ratio production functions and which can be formulated in terms of discrete period difference equations. We shall not discuss in this dissertation those nonmonetary fixed capital-output ratio models, first developed by Domar,[1] which are "continuous" and treated with the use of differential equations.

Harrod growth theory, as we define it, has had a long and varied development in the literature of economics. Harrod first proposed his basic model in 1939, as a dynamic version of the Keynesian model.[2] His conclusions with regard to the dynamic stability of the Keynesian economy were so startling that his model immediately became the source

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[1] E. Domar, "Capital Expansion, Rate of Growth, and Employment," Econometrica, (April, 1946) pp. 137-147; "Expansion and Employment" American Economic Review, (March 1947) pp. 34-55. Also Essays in the Theory of Growth (Oxford University Press, 1957).

[2] R. F. Harrod, "An Essay in Dynamic Theory," Economic Journal (March, 1939) pp. 14-33. Also Towards a Dynamic Economics (Macmillan, 1948).

of intense controversy among economists. The period from 1940 to 1955 saw a large number of articles written on the subject. After the mid-fifties, however, with the appearance of the "neo-classical" growth models[3] and the renewed interest in monetary theory, interest in Harrod growth waned. By the sixties, the subject had, to a certain extent, become outdated. At present, economists, no longer doing much research in the field, seem to have settled into two different schools of thought concerning the theory.

The first school would seem to believe that Harrod growth theory has been properly developed and may be studied as a useful first approximation to the nature of economic growth. The second school, on the other hand, would seem to believe that Harrod growth theory is fairly useless because of the simplistic assumptions it makes. Economists of this school have discarded the Harrod model in favor of other models which, they hope, are more "realistic" descriptions of the growth process.

This author, however, disagrees, in part, with both these schools of thought. With regard to the first school, the author believes that

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[3]The first neoclassical growth model was that of R. M., Solow, which appeared in the article "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics (Feb., 1956) pp. 65-94. This article was the first in a long line of articles on neoclassical growth models. The author must admit that even he was stunned by the vast numbers of articles on such growth models and the relative dearth of articles on Harrod type growth written after 1955 (and especially after 1960). In the Review of Economic Studies, for instance, the author estimates that there are probably about 30 or 40 articles on neoclassical growth theory from 1960 on, and only one article whose main focus is on Harrod growth theory. While the imbalance is not so heavy in other journals, there has been a drastic decline in the number of articles on Harrod growth in them also, during this period. Furthermore, most of the articles on Harrod growth in the sixties have concentrated on the "stability" of the Harrod system--only a small part of Harrod's original model. (See pages 17 and 18 in the text.)

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while "Harrod growth theory" may indeed by useful, it is not really a theory. A theory comprises a set of assumptions, a modus operandi, and a group of testable hypotheses which may be deduced logically from the methods (modus operandi) of the theory. A theory, whether "correct" or "incorrect," will, because of its logical deductive methods, be accepted as a theory by the professionals in the field.

Unfortunately, as we shall attempt to show in Section 2 of this chapter, Harrod growth theory is not altogether a logical theory. In places, its assumptions are vague or even contradictory. Many economists--among them Hicks,[4] Baumol,[5] Alexander,[6] Kaldor,[7] and Samuelson[8]--have studied the theory and noticed these contradictions. In trying to "rehabilitate" the theory however, these economists have arrived at conclusions which are sometimes radically different from those of Harrod. These differences manifest themselves in all aspects of the "theory": the magnitude of the warranted rate, the full

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[4] J. R. Hicks, "Mr. Harrod's Dynamic Theory," Economica (May 1949) pp. 109-123.

[5] W. J. Baumol, Economic Dynamics (Macmillan, 1970) Chpater 2 and Chapter 9, Section 2. Also W. J. Baumol, "Notes on Some Dynamic Models," Economic Journal (December 1948) pp. 506-521.

[6] S. S. Alexander, "Mr. Harrods Dynamic Theory," Economic Journal, (December 1950) pp. 724-739.

[7] N. Kaldor, and J. A. Mirlees, "A New Model of Economic Growth," Review of Economic Studies (June 1962) pp. 175-192. For a concise but lucid summary of the main points of this article, see R. G. D. Allen, Macroeconomic Theory (Macmillan, 1968) pp. 215-218.

[8] P. A. Samuelson, "Interaction Between the Multiplier Analysis and the Principle of Acceleration," Review of Economics and Statistics, (May 1939), pp. 75-78. Also see T. F. Dernburg and J. D. Dernburg, Macroeconomic Analysis, An Introduction to Comparative Statics and Dynamics (Addison Wesley, 1969) Chapter 8, Sections 2 and 3, pp. 132-149.

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utilization of capital, the stability of the growth path, etc. In view of these difficulties, therefore, it would seem premature at the present time to say that we should accept Harrod growth theory and deduce what we may from it. Rather, it would seem appropriate to reformulate the Harrod problem in such a way that a plausible and logical theory, acceptable to all economists from a theoretical point of view, can be constructed.

With regard to the second school of thought, this author must confess that he too believes that the assumptions of Harrod growth theory are "unrealistic." In Section 3 of this chapter, he shall attempt to specify the reasons for this assertion. However, "unreal" assumptions are no reason to discard a theory. According to many economists and scientists, a theory stands or falls, not on how reasonable its assumptions seem, but rather on how well it predicts.[9] Harrod growth theory, if properly formulated, may turn out to predict better than other growth models. Even if it does not, this author believes that there would be much to be gained from a reformulation of the present theory. For, before we can understand complex problems, it is necessary to understand simple ones. The Harrod problem, with its omission of monetary factors and its rigidly specified production function, is a simple problem for which we do not yet have a satisfactory

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[9] M. Friedman, Essays in Positive Economics, "The Methodology of Positive Economics," (University of Chicago Press, 1953) pp. 8-9. Some sample quotes: "viewed as a body of substantive hypotheses, theory is to be judged by its predictive power...only factual evidence can show whether it [a theory] is right or wrong... ." "The only relevant test of the validity of a hypothesis is comparison of its predictions with experience" etc.

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solution. In spite of this, other more complex growth models have borrowed some of the methods used in solving the Harrod problem, to obtain their conclusions. Needless to say, their conclusions seem hardly more satisfactory than those of Harrod models. It may be--and the author sincerely hopes that this will come to pass--that a better solution of the Harrod problem will lead to better solutions of neo-classical and other growth models.

In this dissertation, therefore, we will attempt to solve the Harrod growth problem in a manner that will be theoretically and logically acceptable to economists. In so doing, we shall attempt to fill a gap in the theoretical framework of economics today. Also, we shall briefly attempt to show why our model may be more empirically accurate than other Harrod-like models. Before proceeding with the mathematical apparatus necessary for our task, however, it seems appropriate to restate the Harrod problem. This chapter, therefore, will serve as a brief review of some of the more relevant theoretical and empirical literature on Harrod growth. In Chapter 2, 3, and 4, a new model will be presented and some of its simplest conclusions derived. In subsequent chapters, the model will be developed to incorporate different assumptions.

Section 1.2. The Harrod-Model; Theoretical Difficulties with Harrod-Like Models

In this section, we shall attempt to show some theoretical difficulties of the Harrod model and subsequent Harrod-like models.

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In order to do this, let us summarize the model as it was first perceived by Harrod.[10]

Basically, the Harrod model raised two important questions. First, does there exist in an economy characterized by a constant marginal (average) propensity to save and a constant capital-output ratio, a rate of growth which, under proper circumstances, will always be maintained by businessmen? Second, if this rate exists, what will happen to the above economy if the initial gorwth rate does not equal the "maintainable" rate?

Harrod answered the first question in the affirmative and called this maintainable rate of growth "the warranted rate of growth." Furthermore, he specified the warranted rate, henceforth to be designated w_0, as $\frac{s}{c}$, where s equals the marginal propensity to save and c equals the capital output ratio.

How did Harrod arrive at his conclusions? Essentially Harrod assumed that if businessmen had increased production (income) by a certain percentage w over the previous period's production, and if all goods had been exactly cleared by the market (with no excess supply or demand), they (businessmen) would wish to increase next's period's production by the <u>same</u> percentage w. Mathematically we can write this statement in the following manner. Define

$$U_t \equiv E_t - Y_t ,$$

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[10]The summary of the Harrod model given in the next 3 pages of the text closely follows Alexander *op. cit.*

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where E_t represents total (desired) expenditures on capital goods and consumption goods in period t, and Y_t represents total production (of capital goods and consumption goods) in period t. It is clear that U_t is nothing more than the excess demand for capital goods and consumption goods (excess supply if $E_t < 0$) in t. Then, according to Harrod, the rate of growth of production in period t, G_t, will be maintained in the next period if, and only if, $U_t = 0$. Or

$$G_{t+1} = G_t \quad \text{if, and only if,} \quad U_t = E_t - Y_t = 0 \ .$$

On the other hand, if $U_t \neq 0$, this period's rate of growth will not be continued. For businessmen produce goods to sell them. If expectations concerning the growth of sales have not been met--i.e., if businessmen have overproduced or underproduced--they will change their expectations for the next period and will not increase production again by w percent. Rather they will decrease the rate of growth of production for overproduction and increase it for underproduction. More concisely we can write

$$G_{t+1} = G_t + F(U_t)$$

where F( ) is any sign preserving function.

The magnitude of the warranted rate of growth can now easily be determined. Harrod assumed that saving in period t is a linear function of income in the preceeding period. Thus, $S_t = sY_{t-1}$ in any period t. He also assumed that the amount of investment goods desired by businessmen in period t is $I_t = c(Y_t - Y_{t-1})$ (where c is the ca

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is the capital output ratio.) Since $U_t \equiv E_t - Y_t = 0$ is the condition for warranted growth, it must be that

$$Y_t - sY_{t-1} + c(Y_t - Y_{t-1}) - Y_t = 0$$

or

$$sY_{t-1} = c(Y_t - Y_{t-1}) \qquad 1.2.1$$

along the warranted or "equilibrium" growth path. Equation 1.2.1 implies that

$$cY_t = (s+c)Y_{t-1}$$

or

$$Y_t = \left(1 + \frac{s}{c}\right)Y_{t-1} .$$

Thus, if $w = \frac{s}{c}$ were the rate of growth in period t, and if goods in this period had been exactly cleared by the market, businessmen would maintain this rate of growth for the next period, and their goods would again be exactly cleared by the market. In this fashion, growth would occur at a steady rate of $w = \frac{s}{c}$ through all periods.

What, however, if the rate of growth in period t had not been $\frac{s}{c}$, or if goods had not been exactly cleared in this period? Harrod stated that if the economy grew initially at a rate other than the warranted rate, or if goods had not been exactly cleared in some period, it would be impossible for the economy to return to equilibrium growth. Furthermore, he postulated that growth rates would diverge away from the warranted rate in the direction of initial divergence. Thus, if G_t were greater than w_0, it would continue to increase in subseque

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in subsequent periods away from w_0 as deficits in production occurred, and if it were less than w_0, it would continue to decrease away from w_0 as surpluses in production occurred. The model, therefore, gave rise to the first surprising conclusion that the more businessmen tried to produce to "catch up" with deficits in past production, the more they would fall behind, and vice versa. This conclusion promptly became known as the "knife edge" instability of the model. The slightest deviation from steady warranted growth would send the economy into either an inflationary spiral or a deep depression.

Almost immediately after Harrod published his theory of growth, various objections were raised. The first point which was seriously questioned was Harrod's choice of lags in his functions S_t and I_t, and, therefore, his choice of $w_0 = \frac{s}{c}$ as the warranted rate. The question was raised because Harrod simply postulated his lags without giving reasons for them--an approach which proved unsatisfactory to many. Hicks, for example, observed that Harrod's equation for the warranted rate "does at once look decidedly queer."[11] He proposed that, while the lagged saving function should be sY_{t-1}, that for investment should be $c(Y_{t-1} - Y_{t-2})$. Hicks' lags gave rise, however, to the perhaps stranger result that there might be two warranted rates of growth

$$G_{w1,2} = \frac{c+1-s \pm \sqrt{(c+1-s)^2 - 4c}}{2} - 1 ,$$

and then only if $(c+1-s)^2 - 4c > 0$.[12]

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[11] Hicks, op. cit., p. 110.

[12] These formulas are a direct consequence of Hicks' formulation of the lags. However, they are not given in Hicks article but rather on page 733 footnote 2 of Alexander's already cited article.

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pp. 203-209

Baumol thought it more reasonable to assume that $I_t = c(Y_t - Y_{t-1})$ and $S_t = sY_t$.[13] With these assumptions, the warranted rate of growth now becomes $w_0 = \frac{c}{s-c}$. This is derived as follows. Since the condition for clearing of markets is $E_t - Y_t = 0$, the equation for growth along the warranted path must be

$$Y_t - sY_t + c(Y_t - Y_{t-1}) - Y_t = 0$$

or

$$Y_t = \frac{c}{(c-s)} Y_{t-1} .$$

Thus, since $1 + w_0 = \frac{c}{c-s}$ along the warranted path,

$$1 - \frac{c}{(c-s)} = -w_0$$

and

$$w_0 = \frac{s}{(c-s)} .$$

A third model was proposed by Allen.[14] In this model,

$$S_t = sY_t$$

and

$$I_t = c(Y_{t+1} - Y_t) .$$

This lag in income for the investment function is actually not a lag at all but rather a "lead." The interpretation of this strange investment function is that businessmen can foresee the future, and

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[13] W. J. Baumol, Economic Dynamics, (Macmillan, 1970) pp. 157-158.

[14] R. G. D. Allen, Macroeconomic Theory, (Macmillan, 1968), pp. 203-209.

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buy capital goods now, in accordance with next period's perceived needs. Incredible as it may seem, this "lag" does give the Harrod warranted rate, $w_0 = \frac{s}{c}$. For if

$$U_t = E_t - Y_t = 0 ,$$

$$Y_t - sY_t + c(Y_{t+1}-Y_t) - Y_t = 0$$

and

$$cY_{t+1} = (c+s)Y_t .$$

Thus

$$w_0 = \frac{s}{c} .$$

(We shall learn later in Chapter 4 that such leads in the investment function are really not so strange--indeed they are the most plausible form of the investment function--when we introduce the concept of expectations into our analysis.)

Alexander introduced a new wrinkle to the problem by bringing up the possibility of lagging consumption and investment not by one period, but rather by p and $q+p$ periods respectively.[15] Then

$$C_t = aY_{t-p}$$

and

$$I_t = c(Y_{t-p}-Y_{t-p-q}) .$$

Under these assumptions, Alexander proved that a warranted rate of growth exists only if

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[15] S. S. Alexander, op. cit., pp. 734-737.

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$$\frac{q}{(q+p)}(a+c) > 1 .$$

Alexander's model was of course closely related to distributed lag models. These models, it will be remembered, make consumption and investment each period a weighted average of income in past periods. Thus, for these models

$$C_t = b_0 + b_1Y_{t-1} + b_2Y_{t-2} + b_3Y_{t-3} \cdots$$

and

$$I_t = c_1(Y_{t-1}-Y_{t-2}) + c_2(Y_{t-2}-Y_{t-3}) + c_3(Y_{t-3}-Y_{t-4}) + \cdots$$ [16]

(Using our market clearing equation $E_t - Y_t = 0$, we could determine the magnitude of the warranted rate of growth in such models.)

Finally, this author feels that a still different set of lags is reasonable. Following many economists, let us assume that $C_t = bY_{t-1}$. Also, let us assume, as does Harrod, that $I_t = c(Y_t-Y_{t-1})$. Our condition for clearing markets--$U_t = 0$--now implies that

$$U_t = bY_{t-1} + c(Y_t-Y_{t-1}) - Y_t = 0 ,$$

or

$$(c-1)Y_t = (c-b)Y_{t-1}$$

$$Y_t = \frac{c-b}{c-1}Y_{t-1} .$$

The warranted rate is now

$$w = \frac{c-b}{c-1} - 1 = \frac{1-b}{c-1} = \frac{s}{c-1} .$$

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[16] R. G. D. Allen, Mathematical Economics, (Macmillan, 1963) pp. 79-83, in particular, p. 80.

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For $c >>> 1$, this warranted rate is almost equal to the Harrod warranted rate.

The five models mentioned above are not the only possible ones.[17] Nevertheless, they are sufficient to indicate that, within the theoretical framework of the Harrod model, there is great ambiguity as to the magnitude of the warranted rate. Of course, this is not necessarily bad. Different assumptions in identical models may simply be different behavioral specifications of a problem. Unfortunately, however, the reasons for these different specifications seem to be entirely lacking. The various authors cited above have not attempted in their writings to give economic reasons for their choice of lags. Rather, they seem to have chosen their lags on an *à priori* basis with only the briefest of comments, if any, concerning the economic "reasonableness" of their assumptions. Therefore, while the models for determining the warranted rate of growth are certainly mathematically valid, their economic content is, at times, very obscure. The question of the magnitude of the warranted rate in Harrod-like models must, therefore, be said to be very much in doubt.

Related to the question of lags in the consumption and investment functions for determining the warranted rate of growth, is still another theoretical problem with the Harrod model. Harrod interpreted the warranted rate of growth as a rate of growth desired by businessmen if certain initial conditions had been met and markets were being cleared in each period. Essentially this is a demand oriented

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[17] R. G. D. Allen, *Macroeconomic Theory*, (Macmillan, 1968) pp. 225-228. Also Kaldor and Mirlees, *loc. cit*.

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assumption. It says nothing about supply considerations and whether, from a factor point of view, businessmen are happy with growth at the warranted rate. What, however, if the supply of capital is either insufficient or too plentiful for this period's production? Stated differently, what if $K_t \neq cY_t$, where K_t equals the amount of capital which businessmen have on hand each period.

If we assume those lags postulated by Harrod in the consumption and investment functions, it turns out that the amount of capital on hand (i.e., which businessmen have) is not a linear function of Y along the warranted path. Stated differently, in the model as Harrod formulated it, businessmen do not have the desired level of capital each period to produce optimally each period's output.[18] The proof is fairly simple. Suppose $K_t = cY_t$ for some period t. This, according to

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[18] In reading the literature on Harrod growth theory, the author was initially surprised to learn that in the simple Harrod (not Domar) model, capital was not fully employed along the warranted path. Indeed, Harrod himself made this mistake in his 1939 article. He said there on page 18 "If the value of the increment of stock of capital per unit of increment of output which naturally occurs, C_p, is equal to C, the amount of capital per unit increment of output required by technological and other conditions...then clearly the increase [in capital stock] which actually occurs is equal to the increase which is just fit by the circumstances." However, later, as a result of work by others mentioned below, Harrod realized his error. In his 1951 *Economic Journal* article, on page 273, he writes, "In my analysis [of 1939] I *assumed* [emphasis mine] that on the line of warranted" advance the existing condition of stocks and equipment was satisfactory. ... But if my postulate does not correctly depict the reaction of the representative entrepreneur, it may be necessary for stock and equipment instead of being satisfactory on the warranted line to be chronically deficient or redundant." Hicks points out that there exists a chronic deficiency on pages 117 to 120 of *Capital and Growth*, as does Allen on page 206 of *Macroeconomic Theory*. Both authors show that $K_t = cY_{t-1}$ in the Harrod model, not $K_t = cY_t$. Our proof of the "insufficiency" of capital is slightly different from the above two sources.

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Harrod, is the optimal level of capital in period t. Suppose now that the economy is growing in warranted fashion. We have seen that the condition $U_t \equiv E_t - Y_t = 0$ is equivalent to the condition $I_t = sY_{t-1}$. Thus, for warranted growth,

$$K_{t+1 \text{ on hand}} \equiv cY_t + sY_{t-1} .$$

But, with the lags postulated by Harrod, $w_0 = \frac{s}{c}$. Our formula for $K_{t+1 \text{ on hand}}$ now becomes

$$K_{t+1 \text{ on hand}} = cY_t + \frac{sY_t}{1+\frac{s}{c}} = \left(c + \frac{s}{1+\frac{s}{c}}\right) \frac{Y_{t+1}}{(1+\frac{s}{c})}$$

which does not equal cY_{t+1}. The surprising conclusion we have arrived at is that if capital, in the Harrod model, is initially optimal (fully employed), it will not be so in later periods, if growth occurs in warranted fashion. Indeed, assuming Harrod lags, we can show that, if $K_t = cY_{t-1}$ initially, then it will always equal cY_{t-1} along the warranted path. This implies that capital is always in deficit along the warranted path.[19]

A similar argument can be made for the Baumol version of the Harrod model. In this model $w_0 = \frac{s}{c-s}$. If we assume that

$$K_{t \text{ on hand}} = cY_t \quad \text{for some period } t ,$$

then since

$$I_t = sY_t ,$$

$$K_{t+1 \text{ on hand}} = cY_t + sY_t \quad \text{on the warranted path.}$$

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[19]See footnote 18.

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$$K_{t+1 \text{ on hand}} = \frac{(c+s)Y_{t+1}}{1 + \frac{s}{c-s}} = \frac{(c+s)(c-s)}{c} Y_{t+1} ,$$

which is less than cY_{t+1}. Thus, if capital is initially optimal in the Baumol version of the Harrod model, it will not be so in later periods, if growth occurs in warranted fashion.

It can be shown in the models considered above, with one exception, that capital is not fully employed along the warranted path. The only exception to this proposition is the model where $S_t = sY_t$ and $I_t = c(Y_{t+1}-Y_t)$. Allen shows, on page 204 of his book Macroeconomic Theory, that this model does have

$$K_{t+1 \text{ desired}} = cY_{t+1} = K_{t+1 \text{ on hand}} ,$$

along the warranted path.

In view of these results, we might ask the following question. Under the assumption of our models, might we not expect that businessmen would try to adjust for this discrepancy between desired capital and capital on hand along the warranted path, and spend more or less on investment than the amounts postulated for equilibrium growth by the various authors? In a sense, we are really asking whether the various representations of Harrod-like growth which we have studied so far are logical! For these models, while postulating logical behavior on the part of businessmen from a demand point of view, i.e., clearing of markets, for warranted growth to continue, do not also postulate logical behavior on the part of businessmen from a supply point of view, i.e., full employment of capital, for this growth to continue. As Allen states with reference to Harrod's version of the model, This raises a question of the interpretation of the model since it might appear that the flow conditions $I_t \quad sY_{t-1}$ involve a lag more appropriate to disequilibrium analysis than the present context.[20]

Hahn and Matthews have also pointed out that the Harrod models are "demand oriented" (as opposed to the supply oriented neoclassical growth models).[21] A theory of growth, however, should have logical bases from both a supply and a demand point of view. About the disturbing conclusion that warranted growth is not consistent in the period analysis with full employment of capital, all our authors are silent.

A third objection which has been raised against the Harrod model is whether his assumption of "knife edge" instability holds true. A careful reading of Harrod's original paper will show that Harrod originally postulated (not deduced) the instability of his model. It is not surprising, therefore, the question of stability was later raised by some economists. Unfortunately, no real unanimity of opinion with regard to this question has developed. Most economists have come to the conclusion that the model is indeed unstable, as Harrod first said. Alexander has given a rather elegant proof of this instability in his previously cited article.[22] On the other hand, Rose[23] and Jorgenson[24] have come to the conclusion that the Harrod

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[20]R. G. D. Allen, Macroeconomic Theory, p. 207.

[21]F. H. Hahn and R. C. O. Matthews, "The Theory of Economic Growth: A Survey," Economic Journal (December 1964) p. 789.

[22]S. S. Alexander, op. cit., p. 731.

[23]H. Rose, "On the Possibility of Warranted Growth," Economic Journal (June 1959) pp. 313-332.

[24]D. Jorgenson, "On Stability in the Sense of Harrod," Economica (August 1960) pp. 313-332.

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model may, if disturbed from equilibrium growth, return to it. They are able to show that Alexander's proof rests on an implicit assumption he does not state--namely, that deficits in production $(E_t > Y_t)$ occur only when the rate of growth is already higher than the warranted rate of growth, thus leading to further increases of the rate of growth away from the warranted rate, and that surpluses in production $(E_t < Y_t)$ occur only when the rate of growth is less than the warranted rate, thus leading to further decreases in the rate of growth. Alexander's assumptions seem to be best suited for initial conditions resulting from a shock in the system away from equilibrium growth. In a growth problem, however, the initial conditions do not have to be specified so narrowly as Harrod and Alexander have done. We may very well specify an initially high rate of growth occurring along with surpluses in production as part of our problem, and we shall do so later. Under these circumstances, a high rate of growth need not get higher. Thus, our only conclusion seems to be that in the model, as it stands today, there is a great deal of vagueness with regard to criteria for growth stability. Since stability criteria depend upon adjustment mechanisms by the firm, a specification of business behavior, which is more complete than that given by present models, would seem to be necessary for answering the question of whether the Harrod model is a "stable" growth model.[25]

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[25] See also F. H. Hahn and R. C. O. Matthews, op. cit., pp. 805-809, especially 805, for a discussion of this point.

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A fourth objection to Harrod's theory and subsequent analyses of it (and the most important objection from the author's point of view) is the complete lack of detail in the model on the nature of growth, if the warranted rate is deviated from. Harrod stated only that $G_{t+1} = G_t + F(U_t)$, without specifying at all the nature of $F(\ )$. As such, there is no way in the model that we can quantitatively state what the levels of income in each period will be if deviation from the warranted path takes place. Furthermore, all subsequent analyses of Harrod's model, which one might have expected would have tried to attack this problem, have focused their attention only on a determination of the warranted rate of growth. Generally, all these analyses have used the Samuelson-Hicks approach to growth--that $E_t = Y_t$ -- along the growth path. This equation states, however, that growth is occurring in the warranted or equilibrium fashion. It states nothing about growth off the equilibrium path.[26]

In particular, even the more complex growth models, the neoclassical models, have started with this assumption. Thus, the literature in this field in the last fifteen years has dealt only with growth under the Keynesian equilibrium assumption that supply equals demand for final goods and services in the economy. This orientation of growth theory, however, clearly implies a return to a comparative statics approach to growth. While neoclassical growth theory may be beneficial to economics in introducing more realistic production

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[26]The Phillips model (A. W. Phillips, "A Simple Model of Employment, Money, and Prices in a Growing Economy," Economica, November 1961), may appear to be an exception to this statement. However, even along the disequilibrium path in the Phillips model S_t is assumed to equal I_t.

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functions into growth theory, it (like warranted rate theories) in no way solves the problem of how an economy will grow under certain non-equilibrium initial conditions. We might even say, from this point of view, that Harrod models are better than neoclassical models, in that they, at least, ask, if they do not solve, the question of how growth will occur.

Finally, one last objection to Harrod models, as they stand today. In 1941, Lloyd Metzler, in a classic article, raised the question of how the desire by businessmen to hold inventories would affect the Keynesian system.[27] Since that time, very little work has been done in attempting to integrate this theory of inventories into a theory of growth. It is clear, however, that as income grows, businessmen may wish to increase their stock of inventories. Similarly, if income falls, they may wish to decrease their stock of inventories. It seems desirable that a theory of growth be able to incorporate within it a theory of inventories. At the present time, it is not clear how this may be done within the context of Harrod-like models.

In summary, it seems to this author that we can critize Harrod-like growth models on at least five points:

1. There is no clear cut specification of which lags in consumption and investment functions to use. Therefore, there is ambiguity with regard to the magnitude of the warranted rate of growth.

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[27] L. Metzler, "The Nature and Stability of Inventory Cycles," Review of Economics and Statistics (August 1941) pp. 113-129.

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2. The models with one exception do not imply full employment of capital along the warranted path, and do not indicate how, or whether, businessmen will try to compensate for this.

3. Whether or not, or under what circumstances, there is growth instability cannot be treated by these models. Additional assumptions or information, not specified in existing models, are needed to answer this question.

4. The models say very little about the path of income when growth rates deviate from the warranted path.

5. It is not clear how inventory behavior can be incorporated into these models.

Section 1.3. Empirical Difficulties of Harrod-Like Models

In the previous section, we attempted to show some theoretical difficulties of Harrod-like models. In this section, we shall attempt to indicate some of the empirical problems associated with these models. It will be convenient to divide these problems into two categories--those associated with the assumptions of the models, and those associated with the conclusions of the models.

We start with the former. The two fundamental assumptions of most Harrod-like growth models are the constancy of the rate of saving and the constancy of the capital-output ratio. The first assumption would appear to be a fairly good approximation to consumer behavior. Numerous economists have studied the consumption function, and one fact seems to emerge fairly clearly--namely, that in the "long run" consumption re

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consumption remains a constant function of disposable income.[28] Of course, in the "short run," there are slight variations and various economists, such as Friedman,[29] Dusenberry,[30] and Ando and Modigliani,[31] have attempted to explain these. Nevertheless, it seems safe to say that the assumption of a constant saving-income ratio is empirically adequate for the purposes of Harrod growth theory.

However, with regard to the second assumption, the constancy of the capital-output ratio, there is considerable more doubt. The Harrod investment function is essentially nothing more than the rigid accelerator first proposed by J. M. Clark.[32] The earliest studies concerning such investment functions were carried out by Tinbergen.[33] Tinbergen found that the degree of correlation between changes in output and investment was low. (Investment expenditures were only one

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[28] See for example S. Kuznets, "Proportion of Capital Formation," American Economic Review (May 1952) pp. 507-526. Also R. A. Goldsmith, A Study of Saving in the United States, (Princeton University Press, 1955).

[29] M. Friedman, A Theory of the Consumption Function, (Princeton University Press, 1957).

[30] J. S. Duesenberry, Income Saving and the Theory of Consumer Behavior, (Harvard University Press, 1949).

[31] A. Ando and F. Modigliani, "The Life Cycle Hypothesis of Saving: Aggregate Implications and Tests," American Economic Review (March 1963) pp. 55-82.

[32] J. M. Clark, "Business Acceleration and the Law of Demand: A Technical Factor in Economic Cycles," Journal of Political Economy (March 1917) pp. 217-235.

[33] J. Tinbergen, "Statistical Evidence on the Acceleration Principle," Economica (May 1938) pp. 164-176.

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half of what the accelerator principle predicted.) He found instead, that investment behavior could be better explained by profits than by acceleration. Following Tinbergen, other economists--among them Knox,[34] Tsiang,[35] and Kaldor[36]--came to the conclusion that the rigid accelerator was severely lacking as an explanation of investment behavior.

Beginning in the early fifties, however, one could begin to discern among economists, a shift in opinion back towards some form of acceleration principle as the chief explanation of investment behavior. Chenery[37] and Kocyck[38] suggested that if overcapacity, distributed lags, and expectations were incorporated into a theory of investment, the acceleration principle might be valid after all. Subsequent work tended to confirm this. Modigliani and Kisselgoff[39] in a study of the electric power industry confirmed the acceleration

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[34] A. Knox, "The Acceleration Principle and the Theory of Investment," Economica (August 1952) pp. 269-297.

[35] S. C. Tsiang, "Accelerator, Theory of the Firm, and the Business Cycle," Quarterly Journal of Economics (August 1951) pp. 325-341.

[36] N. Kaldor, "Mr. Hicks on the Trade Cycle," Economic Journal (December 1951) pp. 833-847, especially p. 837.

[37] H. B. Chenery, "Overcapacity and the Acceleration Principle," Econometrica (Jan. 1952) pp. 1-28.

[38] L. M. Kocyck, Distributed Lags and Investment Analysis, (North Holland Publishing Company, 1954).

[39] F. Modigliani and A. Kisselgoff, "Private Investment in the Electric Power Industry and the Acceleration Principle," Review of Economics and Statistics (November 1957) pp. 363-379.

principle, when account was taken of the "characteristics peculiar to the industry," (i.e., lags). Eisner,[40] in a study of eight different industries, found that the acceleration principle gave high correlation coefficients if expectations were taken into account. Jorgenson[41] also carried out work similar to Eisner's, as did Diamond[42] and Deleuuw.[43] By the end of the sixties, significant empirical work had been done to indicate that a "variable expectations, distributed lag" accelerator could accurately describe investment behavior. The names most frequently associated with such research were Chow,[44] Eisner,[45] Jorgenson and Stephenson,[46] and Jorgenson, Hunter, and Nadiri.[47]

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[40]R. Eisner, "Investment: Fact and Fancy," American Economic Review (May 1963) pp. 237-241. Also "A Distributed Lag Investment Function," Econometrica (January 1960) pp. 1-29. Also see footnote 43 below.

[41]J. J. Diamond, "Further Development of a Distributed Lag Investment Function," Econometrica (October 1962) pp. 788-800.

[42]D. Jorgenson, "Capital Theory and Investment Behavior," American Economic Review (May 1963) pp. 247-259.

[43]F. Deleuuw, "The Demand for Capital Goods by Manufacturing: A Study by Quarterly Time Series," Econometrica (July 1962) pp. 407-423.

[44]G. Chow, "Multiplier, Accelerator and Liquidity Preference in the Determination of National Income in the United States," Review of Economics and Statistics (February 1967) pp. 1-15.

[45]R. Eisner, op. cit. Also "A Permanent Income Theory of Investment: Some Empirical Explorations," American Economic Review (June 1967) pp. 363-390. Also see C. E. Ferguson, "On Theories of Acceleration and Growth," Quarterly Journal of Economics (February 1960) pp. 79-99.

[46]D. Jorgenson and J. Stephenson, "Investment Behavior in U.S. Manufacturing 47-60," Econometrica (April 1967) pp. 169-220.

[47]D. Jorgenson, J. Hunter, and M. Nadiri, "A Comparison of Alternative Econometric Models of Quarterly Investment Behavior," Econometrica (March 1970) pp. 187-212.

Recent empirical work, therefore, has tended to support the acceleration principle and thus vindicate, in part, the original formulation of the Harrod growth model. Nevertheless, in spite of this rehabilitation of the acceleration principle, one cannot jump to the conclusion that the investment functions of Harrod-like models are empirically satisfactory. In all the models considered in the previous section, the investment functions were devoid of "expectational" factors; in all but one case (page 12) they were devoid of "multiperiod lag" factors. Furthermore, it is not all clear how such factors (especially the expectational factors) can be meaningfully incorporated into present Harrod-like models. We are, therefore, forced to conclude that the investment assumptions of Harrod-like models are seriously deficient in an empirical sense.

The same is perhaps even more true of the conclusions of present Harrod-like models. Harrod-like models are "growth" models. They predict, for those models which are unstable that income will either explode or contract indefinitely, and for those models which are stable, that income will expand indefinitely at a constant warranted rate. They do not allow for cycles in any form. Growth over the last hundred years in advanced countries, however, has consisted of a long term upward trend interrupted at fairly constant intervals by business cycles. The Harrod-like models of the previous section cannot account for such growth patterns.

Of course, some would argue that the models of the previous section are the most elementary and naive Harrod-like models, and that other models, slightly more complex modifications of the Harrod model, are capa

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are capable of "explaining" long term upward growth interrupted by cycles. This argument may at first appear to be correct. Certainly the statement is valid; there are many such models. Generally these models fall into three categories: 1) those which rely on exogenous variables to produce cycles (wars, technological discoveries, sudden population surges, etc.); 2) those which rely on endogenous variables such as ratchet effect to produce cycles and; 3) those which rely upon external constraints to produce cycles. Some names associated with the first category are Frisch,[48] Kaldor,[49] Kalecki,[50] Hansen,[51] Schumpeter,[52] Tsiang,[53] and Adelman and Adelman.[54] The second category

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[48]R. Frisch, Propogation Problems and Impulse Problems in Dynamic Economics," Economic Essays in Honor of Gustav Cassel, George Allen, and Unwin Ltd. (1933).

[49]N. Kaldor, "The Relation of Economic Growth and Cyclical Fluctuations," Economic Journal (March 1954) pp. 53-71.

[50]M. Kalecki, "Trends and Business Cycles Reconsidered," Economic Journal (June 1968) pp. 263-276.

[51]A. H. Hansen, Business Cycles and National Income, (W. W. Norton and Company, 1951). Also see J. S. Duesenberry, Business Cycles and Economic Growth, pp. 36-38 for a summary of the Hansen model.

[52]J. Schumpeter, Business Cycles (McGraw Hill, 1939).

[53]S. C. Tsiang, loc. cit.

[54]I. Adelman and F. L. Adelman, "The Dynamic Properties of the Klein Goldberger Model," Econometrica (October 1959) pp. 596-625.

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consists of the models of Mathews,[55] Smithies,[56] Metzler,[57] Samuelson[58] Phillips,[59] and, to a certain extent, Duesenberry.[60] In the third category, belong such individuals as Hicks,[61] Goodwin,[62] Minsky,[63] and Niesser.[64]

However, in this author's opinion, there is one very serious difficulty with all these models. This is that in all the models, no matter what their differences, the only way an upward trend for growth is obtained is through the addition of an autonomous (or exogenous, or "trend") component to the investment function. This assumption may

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[55] R. C. O. Mathews, "The Saving Function and the Problem of Trend and Cycle," Review of Economic Studies, Vol 22 (1955) pp. 75-98.

[56] A. Smithies, "Economic Fluctuations and Growth," Econometrica (January 1957) pp. 1-52.

[57] L. Metzler, loc. cit.

[58] P. A. Samuelson, loc. cit. (This model is not so much a growth model as it is a cycle model. It has however, served as the basis for many cycle-trend models and may easily be made into such a model by the introduction of a trend component to investment; see below.)

[59] Phillips, loc. cit. (Same comment as in footnote 57.)

[60] J. S. Duesenberry, Business Cycles and Economic Growth (McGraw Hill, 1958).

[61] J. R. Hicks, loc. cit. Also A Contribution to the Theory of the Trade Cycle (Oxford University Press, 1950).

[62] R. Goodwin, "The Nonlinear Accelerator and the Persistence of Business Cycles," Econometrica (January 1951) pp. 1-17.

[63] H. Minsky, "A Linear Model of Cyclical Growth," Review of Economics and Statistics (May 1959) pp. 133-145.

[64] H. Niesser, "Critical Notes on the Acceleration Principle," Quarterly Journal of Economics (May 1954) pp. 253-274.

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appear to be innocuous. But it carries profound and disturbing implications. To illustrate, let us consider the Hicks model. In his model of growth with cycles, Hicks simply assumes that the investment function has, in addition to its accelerator component, a component which grows according to the function $I_t = I_o(1+r)^t$. The economic basis for the introduction of this component of investment is never made clear by Hicks. Furthermore, Hicks never indicates whether these "autonomous" investment goods will be used--a fact which has been pointed out by Kaldor[65] and Robertson.[66] Of course, the mathematical reason for this component of investment is perfectly clear. Without it, a long term upward growth trend could not be obtained in the model. Income would either converge to a zero equilibrium level or oscillate around such a level (with oscillations of ever increasing amplitude). From an economic point of view, therefore, the assumption of an autonomous component of investment, which Hicks and all other economists, who purport to explain trend and cycle growth, use, is most obscure. This assumption seems to be introduced on an *à priori* basis to achieve a desired conclusion, and many economists have pointed out that it is "illogical." In view of these facts, we may repeat our earlier point that the empirical validity of the "conclusions" of Harrod-like models is very much in doubt.

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[65] N. Kaldor, *loc. cit.*

[66] D. H. Robertson, "Thoughts on Meeting Some Important Persons," *Quarterly Journal of Economics* (May 1954) pp. 181-190, especially p. 183.

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Section 1.4. A New Approach to the Problem

In view of all the theoretical and empirical objections which can be raised against "Harrod growth theory," as it presently stands, it seems appropriate to "go back" and see if we can solve the Harrod problem in a more logical and empirically satisfying fashion.

This author believes that the only way we can understand the solution to this problem is through a model which places the firm in the central position as the determiner of economic growth. Furthermore, he believes that such a model should have several important characteristics. It should emphasize the use of microvariables much more than present growth models; it should quantitatively incorporate expectations of firms into its dynamic structure; and, perhaps most important, because the firm rarely operates in a setting where goods are always cleared, it should be formulated in such a way that growth need not occur in equilibrium fashion (i.e., with $S_t = I_t$).

We now turn to the task of building such a model.

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CHAPTER 2

Section 1. The Basic Model

We shall begin our reformulation of the Harrod growth problem with a simple model in which there are only two types of goods--capital goods and consumption goods--and two types of factors--capital and labor.

We start by defining a "period." A period is a certain interval of time in which firms and individuals make and carry out economic decisions. A period is not specified as being of any prescribed length of time; it is not a "year" or a "quarter." Rather, it is described by two important characteristics: first, that if any businessman decides at the beginning of a period to produce a good, be it a capital or consumption good, that good can be produced and brought to market no sooner or later than the end of the period; second, that if a decision has been made at some time during a period; it cannot be changed for the rest of that period. The types of decisions which firms make during a period are, from a product-market point of view, how many goods to produce and what price to charge for them and, from a factor market point of view, how many factors to hire or buy at the prices the owners of the factors are charging.[1] The

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[1]The phrase "at the prices the owners of the factors are charging" may strike some readers as being a bit unusual. For what it implies is that all owners of factors of production-laborers, lenders, capitalists--set the prices for their factors or services. Clearly, in the real

decisions which consumers (i.e., the household sector) make are: 1) how many consumer goods to buy at the prices which businessmen have set; 2) how to allocate savings and other financial assets between cash and bonds and what rate of interest to demand for funds which are loaned; 3) how many units of labor services to offer to businessmen and what price to charge for these services.[2]

Each period will be divided into two parts. The first part of the period will be called the "production phase." The second part, which we have referred to above as "the end of the period," will be called the "market phase."

The first two behavioral assumptions of our model concern prices. We shall assume that in each period, firms and individuals set prices which are constant over time. Also we shall assume that firms and

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world, this is not what happens quite often. Banks (borrowers) usually set the rate of interest at which they borrow. Firms, as buyers of labor, usually tell workers what their wages will be. Nevertheless, for the purposes of this dissertation, it seems not entirely unreasonable to make the assumption that all factors set the prices for their services. First, this assumption introduces a symmetry into our model. Just as firms, as sellers of goods, set the prices on their goods, so do factors, as sellers of their goods or services, set their prices. Secondly, and perhaps more importantly, this assumption clearly demonstrates the dynamic properties of our model. For in a dynamic model it must be that prices and all other variables are endogenously determined. Prices do not just simply drop out of the air. They are set by profit maximizing or utility maximizing individuals. Even a firm which is in equilibrium in perfect competition does not "take" the equilibrium price. He sets price at this value because it is his most rational profit maximizing value. Letting factors set their prices is, therefore, one very clear, if somewhat unusual, way of demonstrating microeconomic profit maximization in a dynamic setting.

[2]Notice again in 2 and 3 the assumption, stated earlier, that factors set their prices for their services, and see footnote 1 above.

and individuals set prices which are constant over time. Finally, we shall assume that firms and individuals selling identical goods (or services) set identical prices. The above assumptions make price determination exogenous to our model and for a work which, as we stated in Chapter 1, aspires to be microeconomically oriented, this is a serious problem. Nevertheless, these assumptions have been made by virtually all authors in the field of nonmonetary growth economics. For a full discussion of the implications of these assumptions the reader is referred to Hicks' Capital and Growth[3] and to Baumol's Economic Dynamics.[4] At this point, let us simply say that our price assumptions will make our model a "nonmonetary" one. Also, let us note that they will simplify our growth equations in later chapters enormously.

Having stated that prices are constant, we must now ask how firms will make output (production) decisions. In order to answer this question we must make some further assumptions. We shall assume that output decisions are made by businessmen at the beginning of the period--or more precisely at the beginning of the production phase of the period. At that time, firms know what price they will charge for their goods and also individuals have announced to firms what wages they wish and how many hours they are willing to offer at these wages. Also we shall assume that at the beginning of the production phase of period t, each firm has a fixed nonchangeable amount of

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[3] J. R. Hicks, Capital and Growth (Oxford University Press, 1965), Chapter 7, "The Fixprice Method."

[4] W. J. Baumol, Economic Dynamics (Macmillan, 1970), Chapter 8, "Period Analysis."

capital on hand. This fixed amount of capital on hand shall be designated as $K_{t \text{ on hand}}$. We assume that this capital is owned outright by the firm, that it does not depreciate, and that how much capital is presently on hand is determined by how much capital the firm bought in the "market phase" of the last period. (See pp. 41-42 below.)

We also assume that each firm has two ways of producing goods. The first is with capital in a certain ratio to labor, and the second is with just labor. The first method is said to have a fixed capital output ratio, designated by c. Both methods of production shall be described by production functions which are homogeneous of degree 1. The first may be described by the equation

$$Y_t = \frac{K_t}{c} \quad (\text{or} \quad K_t = cY_t)$$

where K_t is the stock of capital on hand in period t; the second by the equation

$$Y_t = Al_t$$

where l_t is the amount of labor used in production method 2. These assumptions are sometimes said to embody the linear programming model.[5]

The first method of production shall be assumed to be the preferred method of production because less labor has to be used in this method. (We assume that capital is costless to operate once it has been bought.)[6] However, both methods will be assumed to be profitable

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[5] R. G. D. Allen, Macroeconomic Theory, pp. 37-40, 209-211.

[6] We assume also that there is no imputed cost of capital.

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at the given wage of labor. Because of the latter assumption, the <u>criteria for profit maximization will be production at the maximum level of expected sales</u>. We prove this as follows. Suppose the firms expect some maximum level of sales in period t, $E_{t\ exp}$, where $E_{t\ exp}$ is in dollar terms. (We shall see how firms determine the magnitude of expected sales in our discussion of the market phase (p. 42). For the moment, let us simply say that such a quantity does exist in the minds of businessmen.) In our model, expected profits in period t may now be written

$$\Pi_{exp} = P_t Q_{1t} + P_t Q_{2t} - \alpha Q_{1t} - \beta Q_{2t} ,$$

where

Q_{1t} = the amount of goods produced by method 1

Q_{2t} = the amount of goods produced by method 2

α = the cost per unit by method 1

β = the cost per unit by method 2

Also

$$P_t(Q_{1t} + Q_{2t}) \leq E_{t\ exp} ,$$

and

$$Q_{1t} \leq \frac{K_t}{c} \text{ on hand} .$$

The solution to this profit maximization problem is obviously to produce at the maximal possible level ($E_{t\ exp}$) and to use method 1, the cheaper of these two methods, ($\alpha < \beta$), as much as possible. Therefore, given the wage of labor, the amount of capital on hand, and the expected value of expenditures in t, businessmen can now <u>explicitly</u> determine their desired production levels.

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The reader will note that the above model implies that businessmen produce enough goods to satisfy expected sales. Implicit in this model, however, is the assumption that in producing to meet sales, businessmen disregard the inventories which they have on hand at the beginning of the production phase. Or stated differently, we are assuming in the above model that businessmen desire to maximize profits, at the same time maintaining whatever level of inventories they have on hand at the beginning of the period. This type of behavior will be defined to be "passive inventory adjustment behavior" and will be treated in Chapter 3, Section 1, with regard to the Keynesian model, and in Chapters 4, 5, and 6, with regard to the Harrod model.

What, however, if businessmen wish to maintain some level of inventories as they maximize profits? In this case, the inequality

$$P_t(Q_{1t} + Q_{2t}) \leq E_{t\ exp}$$

in the above model should be replaced by the inequality

$$P_t(Q_{1t} + Q_{2t}) \leq E_{t\ exp} + (Inv_{t\ desired} - Inv_{t\ on\ hand}),$$

if

$$E_{t\ exp} + (Inv_{t\ desired} - Inv_{t\ on\ hand}) > 0,$$

and

$$P_t(Q_{1t} + Q_{2t}) = 0,$$

if

$$E_{t\ exp} + (Inv_{t\ desired} - Inv_{t\ on\ hand}) < 0.$$

The solution to this linear programming problem is almost exactly the same as before. If production takes place it will be at a level given by $E_{t\ exp} + (Inv_{t\ desired} - Inv_{t\ on\ hand})$ and will be carried out as much as possible by method 1 "the capital intensive method." This type of profit maximization, in which the firm desires to maintain some level of inventories will be known as nonpassive inventory adjustment and will be treated in Chapters 7, 8, 9, and also in Chapter 10, Section 1.

We can now summarize all the economic activities which occur at the beginning of the production phase. First, laborers set the wage rate at some constant level; second, firms attempt to hire a certain amount of labor at this wage rate with the intent of producing a certain level of output. This output is consistent with profit maximization, the capital "constraint," and inventory adjustment behavior. We may state, therefore, that the only transaction which occurs at the beginning of the production phase is the hiring but not the paying of labor.

Finally one last point. We have seen that at the wage rate set by workers a certain quantity of labor will be offered by workers and a certain quantity of labor will be demanded by firms. What if these quantities are not equal? It is clear that if the quantity of labor demanded is less than or equal to the quantity supplied, businessmen will be able to produce as much as they please. In this case, there will be no labor constraint. We shall study this case in Chapter 3 to 9. However, it may be that in some period, the quantity of labor demanded will be greater than that supplied. In this case, the "availability" of labor will serve as a "constraint" on the economy and prevent firms from producing what they wish to. This case of "labor constraint" will be discussed in Chapter 10.

We now describe the end of the production phase. At the end of the production phase, all goods which were started at the beginning of the production phase have been completed. Businessmen now pay the factors of production--that is, they pay labor, make interest payments on any outstanding bonds, and distribute to themselves "profits." The first payment--to labor--is clear. The last two "payments" however, require some explanation. With regard to interest payments, we shall assume that firms have at the beginning of the production phase a certain amount of debt. This debt shall be in the form of consoles. These consoles are the result of borrowings in the market phase of previous periods and will be explained in our discussion of the market phase (p. 43). For the moment, we simply say that firms find it necessary to make interest payments each period on these consoles and that they pay their interest at the end of the production phase. With regard to profits, we shall assume that the "profits" which businessmen distribute to themselves equal the profits they expect to make. For example, if businessmen have produced \$100 worth of goods expecting to sell this amount, and if their total costs for labor and interest amount to \$80, they distribute \$20, their expected profits, to themselves. This assumption is equivalent to the assumption that there is no business saving in the economy and has been made by Metzler, among others, in his classic article on Inventory Cycles.[7]

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[7] L. Metzler, op. cit., p. 115.

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We now realize that our assumptions concerning payments to the factors of production imply that the total income at the end of the production phase equals the dollar value of production.

The production phase is therefore that part of the period in which businessmen take resources, capital and labor, and transform these resources into finished goods. The dollar value of these goods produced in period t will be called, interchangeably, either production or income in period t, and will be designated by Y_t.

We can now proceed to the market phase of our period. In so doing, we will explain several things which we had to leave hanging in our discussion of the production phase--namely, how much capital businessmen possess at the beginning of period t, what level of debt they have at this time, and what level of sales they expect during this period.

The first thing which occurs during the market phase of our period is that businessmen make known to buyers the prices of the consumption and capital goods which they have on hand. (We assume that firms which produce goods also sell them.) As stated above, these prices are determined at the beginning of the production phase. It is only their assignment, or announcement, which takes place at the beginning of the market phase.

We view the assignment of prices as taking place in the following manner. Firms place outside their stores signs indicating their prices. These prices are instantaneously made known to all individuals and businessmen in society--in other words, there is perfect knowledge of prices, with no time lag, in the economy. Also, by our first

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by our first characteristic of a period, these prices cannot be changed during the remainder of the market phase. Thus, if during the market phase sales go more slowly or quickly than at first anticipated, businessmen cannot change their prices.

Once prices become available to consumers and businessmen, these two groups can rationally make decisions concerning expenditures. First, consumers can decide how many dollars to spend on consumption goods, given their existing financial assets. Second, they can decide how to allocate their remaining financial assets after consumption between cash and bonds. (Since in our model the owner of a good or asset fixes its prices in accordance with information from previous periods, the consumer fixes the rate of interest, expecting to sell a certain amount of money in exchange for bonds. At the risk of boring the reader, let us point out that if sales of money go differently than expected, consumers are not allowed to change the rate of interest.) Third, businessmen can decide how much money to borrow and how many capital goods to buy.

Our first assumption in the market phase is that the only determinant of consumption in the market phase is income. We note that this assumption implies that there is no wealth effect with respect to all forms of wealth other than income. Furthermore, we will assume that all individuals, no matter what their income level, desire to consume the same proportion of their income. If we now assume, as we have already done, that production in a period equals income in that period, and that there is no problem in spending this period's income on this period's consumption goods (since the market phase of our period comes after the production phase), we may write

$$C_t = bY_t,$$

where b is the marginal (average) propensity to consume.[8,9]

With some additional assumptions we can also determine the level of business investment in the market phase of any period. Suppose businessmen can see what total expenditures on capital and consumption goods, in the market phase of previous periods, were. If we assume that businessmen have some expectation concerning future expenditures which is based on past realized expenditures, then in the market phase they will form an expectation for what expenditures will be in the next period. For example, if in t-1, expenditures were $100 and businessmen expect expenditures to increase each

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[8]The assumption that wealth effects play no part in determining consumption expenditures allows us, if we wish, to make less restrictive assumptions with respect to price and wage fluctuations. Instead of assuming constancy of prices, as we have done, we can simply assume that all prices and wages change proportionally. If this assumption is made there are no relative price effects. Consequently, since income is paid out in nominal terms, and since wealth effects are assumed away, there is no change in the formula for C_t.

[9]The formula $C_t = bY_t$ may present one difficulty to the astute reader. It may be noted that, since interest payments are fixed costs, expected profits may be negative. If this is the case, profiteers would have negative income and assuming their consumption is zero, C_t would be greater than bY_t. To obviate this difficulty, simply assume that when expected profits are negative interest payments are not equal to the rate of interest times the amount of bonds outstanding, but rather equal income minus wages. In this case profits are zero, total income equals total production, and C_t still equals bY_t. Notice that this assumption also makes our model more realistic in that firms, in practice, sometimes do default on interest payments.

period by 10%, then in the market phase of t, businessmen will expect expenditures in period $t+1$ to be $100\ (1.1)^2 = 121$. How do businessmen now prepare to produce this level of goods?

In order to answer this question, we must make one further assumption. Suppose we assume that all bonds in the economy are consoles with a rate of interest r_0. Then if businessmen borrow to finance capital expenditures, the cost of capital in the production of one unit by method 1 is r_0cP_0, where P_0 is the price of capital. The "capital intensive" method of production, therefore, has a total cost per unit per period of $r_0cP_0 + w_{10}L_0$, where L_0 is the amount of labor necessary to produce a unit by this method. The cost of producing one unit by the second method, the "labor intensive method" is simply $w_{10}L_1$, where L_1 is the amount of labor necessary to produce a unit by this method. If we assume now that

$$w_{10}L_1 > w_{10}L_0 + r_0cP_0 ,$$

the capital intensive method--as seen in the market phase-- is clearly preferred.

Total expenditures on capital are now easy to determine. Since the capital method of production is the cheaper of the two methods (even when the cost of capital is taken into consideration), then if 1) capital can be used immediately, 2) expectations are positive, and 3) sales in future periods are expected to grow at the same rate as in the next period, businessmen will attempt to buy

$$c(E_{t+1\ \mathrm{exp}}) - K_{t\ \mathrm{on\ hand}}$$

capital goods, in the market phase of period t. Furthermore, if businessmen succeed in buying this amount of capital they will have, at the beginning of the production phase of period t+1, $c(E_{t+1\ exp})$ capital on hand. (It may be, however, that capital production and capital inventories will not be sufficient to let desired capital expenditures be achieved. This case will be considered in Chapter 9.)

Finally, with some further assumptions we can explain how businessmen form their expectations concerning expenditures at the beginning of the production phase. (Notice that on page 34 above, we simply stated that there exists some expected level of expenditures in the period.) We shall assume that businessmen at the beginning of the production phase can observe the expenditures which occurred in last period's and previous periods' market phase. Given that businessmen have some expectations for the future they can use these previous observed results to guess what (maximum) expenditures will be this period. For example, if expenditures in the market phase of t were $100 and businessmen expect expenditures to increase by 10% each period, expected expenditures, for period t+1, as seen at the beginning of the production phase of t+1, will be $110. Notice that expenditures expected at this time do not have to equal expenditures expected in the market phase of t. Note also that expenditures do not have to be equal to the amount of goods produced. If expenditures are less than production inventories will pile up; if greater, inventories will be depleted. We shall assume in Chapters 3 to 8 that expenditures can always be satisfied through production or inventories. In Chapter 9 we shall remove this assumption.

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There is now only one remaining question. How do businessmen obtain the money to pay for capital expenditures and also to pay for next periods expected production. The answer to this question is fairly simple. We shall assume that at the beginning of the market phase businessmen have some "leftover" money. Also we shall assume they expect to take in money during the market phase from sales. If more money is now needed to finance capital expenditures and future production it will be borrowed from consumers who we shall assume are willing at the beginning of the market phase to sell businessmen money at a rate of interest r_0. (This explains why businessmen have a certain level of bonds outstanding at the beginning of the market phase. See page 37.)

We shall make two further assumptions concerning this borrowing: first, that businessmen will always borrow enough money from consumers (so that they never run short in the next period), and, second, that consumers always have enough money to lend to firms to satisfy their desires. The latter is equivalent to saying that there is an "exogenous force which expands the money supply and makes it plentiful." A more precise reason for this assumption is the following.

We shall see in later chapters that, under certain circumstances, income may explode in a Harrod sense. If prices are assumed to be constant, there may be eventually not enough money to support high levels of income. More rigorously, it must be in our model that the velocity of money is always less than one when defined with respect to a period. For if money circulates but once in a period, as it doe

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as it does in our model, then those dollars which circulate will have velocity of one (when defined with respect to the period) and those which do not will have a velocity equal to zero. The velocity of the total money supply V_{total} is given by the formula

$$M_0 V_{total} = M_{circulating} \cdot 1 + M_{noncirculating} \cdot 0$$

$$V_{total} < 1 \ .$$

Therefore, if M_0 is the total amount of money in the economy and v the velocity of money, then the identify

$$Mv = \sum_{i=1}^{N} P_i Q_i$$

implies that there is an upper limit to the amount of goods which can be produced in a period, which depends upon prices and the amount of money in the economy. Thus, if under our assumptions of constant prices, we do not wish the amount of money in the economy to limit the amount of production--<u>if we do not wish the availability of money to serve as a constraint on our system</u>--we will have to posit an exogenous force which expands the money supply and makes it "sufficiently plentiful." This assumption, it may be noted, is very similar to our previous assumption that the demand for money by businessmen is always less than the amount which consumers have available for loanable funds.

Our model is now complete. All our variables are endogenously determined by our behavioral assumptions. At this point however, it may be desirable to summarize concisely the main points of our model. These are:

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1) At the beginning of the period K_t, $E_{t\ exp}$, $E_{t+1\ exp}$ (and $E_{t+2\ exp...}$), Inv_t, and $Inv_{t\ desired}$ are predetermined. Production in t, Y_t, $= E_{t\ exp} + (Inv_{t\ desired} - Inv_{t\ on\ hand})$.

2) In the market phase

$$E_t = C_t + I_t$$

$$C_t = bY_t$$

$$I_t = c(E_{t+1\ exp}) - K_t$$

3) At the end of the period inventories have accumulated or de-accumulated over desired levels, if expected sales are different from actual sales.

Section 2.2. Applications of the Model; the "Plan" of the Dissertation

The assumptions listed above are now sufficient to describe Harrod-like economies. In the remainder of this dissertation, we shall make our assumptions more specific to the description of such economies.

In Chapter 4, we shall consider a Harrod-like model where expectations are constant, businessmen disregard their inventory positions, and the supply of labor and inventories is infinite. Such a model will be called the simple unconstrained Harrod model. In Chapters 5 and 6, we shall again consider unconstrained Harrod-like models. However, in these chapters, the forms of the expectation will be different from that in Chapter 4. In Chapter 5, the expectation will be

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will be "additive," in Chapter 6, the expectation will be "adaptive." We shall observe in all these unconstrained models that our patterns of growth are very similar to the Harrod pattern of growth. Income will either explode or contract in these models, with capital and production continually in surplus or deficit, or else it will expand in Harrod warranted fashion.

In Chapters 7, 8, and 9, we shall indicate how inventory adjustment behavior can be incorporated in our model. At the same time (in Chapter 9), we shall show how the assumption of infinite inventories can be removed from our models. The inventory adjustment behavior which we shall discuss in these chapters will closely follow that first proposed by Metzler[10] and thus we will attempt to do for the unconstrained Harrod model what Metzler did for the unconstrained Keynesian model.

Finally, in Chapter 10, we shall remove our last assumption, namely that of infinite supplies of labor. In so doing, we shall show that our Harrod-like models, instead of being growth models are growth and cycle models. We shall prove in this chapter that, under the proper circumstances, a Harrod-like economy will always "turn down" when it hits a full employment ceiling.

Before proceeding with the above models, however, let us relax the assumption concerning optimal investment criteria made above and consider a model, in which, in any period, businessmen desire to spend only $I_t = I_0$ on capital goods. Together with our other

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[10] L. Metzler, loc. cit.

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assumptions, this assumption makes up the Keynesian model. Of course, we realize that, within the context of our model, this assumption seems very "irrational." Nonetheless, because the crude Keynesian model has been so important in the development of economics, let us digress from the main task of this dissertation to investigate its consequences as a dynamic theory.

Section 3

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CHAPTER 3

Section 3.1. Keynesian Models with Passive Inventory Behavior

We start off our discussion of the dynamic Keynesian model by assuming, as before, that $C_t = bY_t$. In addition, we assume that $I_t = I_o$. Total expenditures in the market phase of any period t are, therefore, $E_t = C_t + I_t = bY_t + I_o$.

The question now arises as to what production (income) will be in period $t+1$. If businessmen have simple expectations concerning sales in $t+1$-i.e., if they believe that sales in $t+1$ will be the same as sales in t- and if businessmen do not care what levels of inventories they maintain, then, at the beginning of the production phase of $t+1$, they will decide to produce exactly what was sold in period t. Thus,

$$Y_{t+1} = bY_t + I_o \qquad 3.1.1$$

Equation 3.1.1 is a very simple first order difference equation whose solution can be written as

$$Y_t = (Y_o - \frac{I_o}{1-b})(b)^t + \frac{I_o}{1-b}, \qquad 3.1.2$$

where Y_o is the initial level of income. If $0<b<1$, it is clear that the Keynesian model, regardless of initial income, converges to a stable equilibrium when expectations are simple.

The fact that income does converge to an equilibrium level under simple expectations also allows us to place a slightly more charitable explanation (

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explanation (other than irrationality) on why business demand for investment may be at a constant level each period. Let us assume in the Keynesian model that capital depreciates. (We shall make the opposite assumption in the Harrod model). Let us also assume that businessmen desire to add capital only to the extent that it depreciates; in other words, that they desire to maintain a fixed level of capital, no matter what income is. If $I_t = I_o$ is the amount by which capital depreciates each period, then, at least in equilibrium, investment behavior will be rational if businessmen already have the optimal level of capital for that income level.

In considering Keynesian-like models, however, we do not have to restrict ourselves to models with simple expectations. Suppose, for instance, that businessmen believe that sales will always expand by some percent w over sales in the previous period. Then if production is solely for sales purposes (no inventory adjustment behavior), we may write

$$Y_{t+1} = (1 + w)(bY_t + I_o). \qquad 3.1.3$$

The solution to this equation is

$$Y_t = I_o \frac{(1 + w)}{1 - (1 + w)b} + ((1 + w)b)^t (Y_o - \frac{I_o(1 + w)}{1 - (1+w)b}), \qquad 3.1.4$$

if $(1 + w)b \neq 1$, and

$$Y_t = Y_o + (1 + w)I_o t, \qquad 3.1.5$$

if $(1 + w)b = 1$.

If $(1 + w)b < 1$, our solution can be described exactly as before. A stable equilibrium is reached. The only difference in our problem is

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problem is that, whereas before in equilibrium, sales equalled production, now sales in equilibrium are <u>less</u> than production. The proof is as follows.

$$E_{equilibrium} = bY_{equilibrium} + I_o = \frac{b(1+w)I_0}{1-(1+w)b} + I_0$$

$$= \frac{I_o}{1-(1+w)b},$$

and

$$Y_{equilibrium} = \frac{(1+w)I_0}{1-(1+w)b}.$$

For positive w, it is clear that equilibrium income, or production, exceeds sales. Thus, $Y_e \neq C_e + I_e$. Surprising as it may seem there is nothing wrong with this analysis. For we are assuming in this model that businessmen do not mind the increase in their inventory position which occurs, and that they do not change their expectations, even though these expectations are continually disappointed.

For $(1+w)b \geq 1$, equations 3.1.4 and 3.1.5 indicate that the Keynesian model is no longer stable. Both solutions explode to a value of income equal to infinity. Furthermore, in both these instances, sales eventually must become <u>less</u> than production. For when $(1+w)b = 1$,

$$E_t = b(Y_o + I_o(1+w)t) + I_o \rightarrow bI_o(1+w)t$$

and

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When $b(1+w) > 1$,

$$E_t \to b((1+w)b)^t \frac{(Y_o - I_o(1+w))}{1 -((1+w)b)}$$

and

$$Y_t \to ((1+w)b)^t \frac{(Y_o - I_o(1+w))}{1 -((1+w)b))} .$$

These somewhat unusual results--that sales in the Keynesian model are less than production either in equilibrium or explosion--point out a flaw in this model which we shall discuss at the end of this chapter. They also point out, however, a fundamental difference between our approach and that of other economists in dynamizing the Keynesian model.

Many economists in Keynesian dynamics start out by postulating the relationship

$$Y_t = C_t + I_t$$

to determine growth. They then plug in some lagged consumption function, usually $C_t = bY_{t-1}$, to get a growth equation

$$Y_t = bY_{t-1} + I_0 . \qquad 3.1.1.a$$

Since the above equation is the same as one of our previous equations--equation 3.1.1.a--it might appear that the $Y_t = C_t + I_t$ approach to growth is similar to ours. This, however, is not at all true. First, equation 3.1.1.a implies a very particular, and unusual, type of expectation--i.e. an expectation which is always self-fulfilling. Stated differently, an implicit assumption of equation 3.1.1.a is that businessmen can somehow predict expenditures in the next period and produce accordingly. Our approach, however, does not specify expectations so narrowl

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so narrowly. It allows for a wide variety of expectations which may or may not be self fulfilling. Second, equation 3.1.1.a always implies equilibrium growth - growth in which markets are cleared. Our approach shows that growth is fundamentally a disequilibrium process. In our first model, with simple expectations, production equalled expenditures only in equilibrium; in our second model, with constant non-simple expectation, production equalled expenditures for only one nonequilibrium value of income. Thus while the $Y_t = C_t + I_t$ approach to growth may appear, in one case, to be similar to our approach, it is fundamentally different in that it specifies business expectations much more narrowly than ours and allows only equilibrium, or market-clearing, growth.

We can now turn to other Keynesian models with non-simple expectations. First, to use a model suggested by Metzler, let us assume that businessmen expect sales in period t+1 to equal sales in period t plus a constant, A, times the difference in sales in periods t and t-1. Mathematically, let

$$E_{t+1 \text{ expected}} = (bY_t + I_o) + A((bY_t + I_o) - (bY_{t-1} + I_o)). \qquad 3.1.6$$

Then

$$Y_{t+1} = (b + Ab)Y_t - (bA)Y_{t-1} + I_o. \qquad 3.1.7$$

For $bA < 1$, it can be shown that the solution to 3.1.7 is stable. Furthermore, the equilibrium level of income is

$$Y_e = \frac{I_o}{1-b},$$

as before. In this equilibrium, sales will equal production. In general, however, sales will not equal production while the system is moving toward equil

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toward equilibrium. Finally, for $bA > 1$, the model is unstable, and income either diverges to infinity or contracts to zero (minus infinity). It can also be shown that when income diverges, sales will eventually become less than production each period, and that when income contracts, sales will always become greater. These are indeed most unusual results, and we shall discuss their implications at the end of the chapter.

Finally, we consider a slightly more difficult Keynesian growth model. Suppose we assume that businessmen believe that sales in any period will grow by the same percentage that they did in the preceding two periods. Then, if we assume passive inventory behavior,

$$E_{t\ \text{expected}} = (bY_{t-1} + I_o)\,\frac{(bY_{t-1} + I_o)}{(bY_{t-2} + I_o)} \qquad 3.1.8$$

and

$$Y_t = (bY_{t-1} + I_o)\,\frac{(bY_{t-1} + I_o)}{(bY_{t-2} + I_o)}. \qquad 3.1.9$$

These equations are nonlinear difference equations and we shall not attempt to solve them here. However, a few numerical examples will reveal that these equations have exactly the same properties as equations 3.1.6 and 3.1.7.

In general, it should be clear by now that we can incorporate any type of expectation into our noninventory adjustment Keynesian model. It should also be clear that these models either give an equilibrium solution or a solution which either diverges to infinity or contracts to zero. Of course, for nonlinear or extremely complicated expectations, it may be impossible to solve the difference equatio

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equations we write. This, however, seems secondary. For our chief purpose in this section is to show how to make macroparameters meaningfully dependent on microdecisions of firms within the framework of the dynamic Keynesian model (See chapter 1, Section 4). Having succeeded in this purpose, we will now be able to do the same for the Harrod model.

Section 3.2. Keynesian Inventory Adjustment Models

In Section 3 of this chapter we shall indicate why the Keynesian model is in many respects unsatisfactory as a dynamic model, and we shall try to show why a more realistic specification of the investment function, as in the Harrod model, would be desirable for a dynamic theory. Before proceding to do this, however, let us indicate how inventory adjustment behavior, which we shall study at greater length in Chapters 7, 8, and 9, can be incorporated into the Keynesian model.

Suppose, therefore, that we consider a model in which businessmen try to compensate for the gains or losses in inventories which have occurred only in the last period. It is clear that the gains or losses in each period t may be written as

$$(bY_t + I_o) - Y_t.$$

Therefore, by the assumptions of our model, we can write that

$$Y_{t+1} = E_{t\ \text{expected}} + (bY_t + I_o) - Y_t.$$

If we now consider models with the four different types of expectations of the previous section, we can write the following equations

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$$Y_{t+1} = bY_t + I_o + (bY_t + I_o) - Y_t \qquad 3.2.1$$

$$Y_{t+1} = (1+w)(bY_t + I_o) + (bY_t + I_o) - Y_t \qquad 3.2.2$$

$$Y_{t+1} = bY_t + I_o + A((bY_t + I_o) - (bY_{t-1} + I_o)) + (bY_t + I_o) - Y_t \qquad 3.2.3$$

$$Y_{t+1} = \frac{(bY_t + I_o)^2}{(bY_{t-1} + I_o)} + (bY_t + I_o) - Y_t \qquad 3.2.4$$

The first of the above equations is the simple expectations case; the second, the case in which businessmen expect sales to increase by some constant percentage each period; the third, the case where sales are expected to be last period's sales plus a constant times the difference in the two previous period's sales; and the fourth, the case where sales are expected to grow by the same percentage they grew in the preceding period.

Of course, other types of inventory behavior can be assumed. Following Metzler, let us assume that businessmen desire either to maintain a fixed level of inventories each period or a supply of inventories in each period proportional to expected sales in that period. For the first case, we can write the inventory adjustment term in each period $t+1$ as

$$(bY_t + I_o) - E_{t \text{ expected}} ,$$

and for the second, as

$$k(E_{t+1 \text{ expected}}) - k(E_{t \text{ expected}}) + (bY_t + I_o) - E_{t \text{ expected}} .$$

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For the fixed level inventory adjustment case, our equations of growth now become

$$Y_{t+1} = bY_t + I_o + (bY_t + I_o) - (bY_{t-1} + I_o) \qquad 3.2.5$$

$$Y_{t+1} = (1 + w)(bY_t + I_o) + (bY_t + I_o) - (1+w)(bY_{t-1} + I_o) \qquad 3.2.6$$

$$Y_{t+1} = bY_t + I_o + A(bY_t + I_o) - (bY_{t-1} + I_o) + bY_t + I_o \qquad 3.2.7$$

$$- ((bY_{t-1} + I_o) + A((bY_{t-1} + I_o) - (bY_{t-2} + I_o)))$$

$$Y_{t+1} = \frac{(bY_t + I_o)^2}{(bY_{t-1} + I_o)} + (bY_t + I_o) \qquad 3.2.8$$

$$- \frac{(bY_{t-1} + I_o)^2}{(bY_{t-2} + I_o)} .$$

and for the proportional inventory adjustment case, they become

$$Y_{t+1} = bY_t + I_o + k(bY_t + I_o) - k(bY_{t-1} + I_o) \qquad 3.2.9$$

$$+ (bY_t + I_o) - (bY_{t-1} + I_o)$$

$$Y_{t+1} = (1+w)(bY_t + I_o) + k((1+w)bY_t + I_o)-(1+w)(bY_{t-1} + I_o))$$

$$+ (bY_t + I_o) - (1+w)(bY_{t-1} + I_o) \qquad 3.2.10$$

$$Y_{t+1} = bY_t + I_o + A((bY_t + I_o)-(bY_{t-1} + I_o)) + k((bY_t + I_o)$$

3.2.11

$$+ A((bY_t + I_o)-(bY_{t-1} + I_o))) - k((bY_{t-1} + I_o)$$

$$+ A((bY_{t-1} + I_o)-(bY_{t-2} + I_o))) + (bY_t + I_o)$$

$$- ((bY_{t-1} + I_o) + A((bY_{t-1} + I_o) - bY_{t-2} + I_o)))$$

$$Y_{t+1} = \frac{(bY_t + I_o)^2}{(bY_{t-1} + I_o)} + \frac{k(bY_t + I_o)^2}{(bY_{t-1} + I_o)} - \frac{k(bY_{t-1} + I_o)^2}{bY_{t-2} + I_o} \qquad 3.2.12$$

$$+ \ (bY_t + I_o) - \frac{(bY_{t-1} + I_o)^2}{(bY_{t-2} + I_o)}$$

We shall not discuss here the solutions to the above equations. Some of these equations, it will be noted, have already been derived by Metzler. We therefore refer the reader to Metzler's article for a fuller discussion of the solution to these equations. In general, however, for all these equations, including those not obtained by Metzler, we can make the following statements. First there exists an equilibrium solution to all these equations. This equilibrium value is $Y_e = \frac{I_o}{1-b}$ for all but equations 3.2.2, 3.2.6, 3.2.10, and is $Y_e = \frac{I_o(1+w)}{1-(1+w)b}$ for these. Second, this equilibrium may not be stable. If it is not, income will either diverge to plus infinity or contract to zero. If the former obtains, sales are eventually <u>less</u> than production each period; if the latter, sales are eventually <u>greater</u> than production each period. Finally, these characteristics of the solutions are exactly the same as those in the passive inventory adjustment models. We can therefore say that introducing inventory adjustment behavior into our Keynesian models does not alter the nature of Keynesian growth in any fundamental way.

Section 3.3. Difficulties with Dynamic Keynesian Models

It should now be apparent from the previous sections of this chapter that the dynamic Keynesian model suffers from serious difficulties.

One difficulty lies in the assumption that $I_t = I_o$ each period. The first thing that we can say about this assumption is that it makes one of our behavioral assumptions exogenous to our model. As such, our model is "incomplete " from a theoretical point of view. More importantly, however, this assumption implies, to a certain extent, irrationality on the part of businessmen. For it seems hard to imagine that as income changes, businessmen will insist on buying the same amount of capital goods each period. Only under one set of circumstances will this behavior be at all "rational." This is when income is already in equilibrium, businessmen have the desired level of capital, and capital depreciates at a rate of $I_t = I_o$ each period. Needless to say, this set of circumstances is most restrictive and not at all consistent with a "dynamic" model.

There are also difficulties associated with the interpretation of the solutions of the dynamic Keynesian model. First, there is the problem that this model may, under certain circumstances, have a stable solution. Clearly a stable solution is not consistent with a model which tries to explain economic growth. Second, there is the problem that when the model is unstable, we obtain the counterintuitive result that production exceeds sales for divergence, and is less than sales for contraction. It would seem much more reasonable that shortages would develop as income grew and that surpluses would occur as income contracted.

CHAPTER 4

Section 4.1. The Fundamental Equation for Constant Expectations Passive Inventory Adjustment, Harrod-like Growth

In view of the difficulties associated with the Keynesian model, let us now abandon our irrational Keynesian assumption that $I_t = I_o$. In doing so, we shall arrive at a model which obviates many of the Keynesian difficulties, and gives results very similar to those of Harrod's original model.[1]

We shall make four assumptions in addition to those already made in Chapter 2. These are: (1) that there are initially an infinite number of inventories in the economy; (2) that there is an infinite labor force; (3) that businessmen always believe that sales will increase each period by some constant percent w; (4) that capital does not depreciate. In later chapters, we shall relax all these assumptions except for the last. This assumption is made only for mathematical simplicity.

We may begin by recalling our assumption concerning capital in Section 2.1. According to this assumption, businessmen desire to have on hand in any period t a stock of capital proportional to

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[1]In the remainder of this dissertation, we shall rely heavily on the use of difference equations. For a discussion of such equations, the reader is referred to S. Goldberg, Introduction to Difference Equations (Wiley, 1958), and H. Levy and F. Lassman, Finite Difference Equations, (Macmillan, 1961).

expected sales in that period. This assumption may be written mathematically as

$$K_{t\ desired} = c(sales_{expected\ in\ t})$$

Obviously the only way that businessmen can attain this level of capital, if they do not already have it, is through purchases of capital in the market phase of period t-1. In the market phase of period t-1, however, sales in t are not known. Furthermore, even sales in the market phase of period t-1 are not yet known since investment spending occurs simultaneously with other purchases. Thus, in the market phase of period t-1, the last seem sales are those in the market phase of period t-2. At that time, sales were $E_{t-2} = bY_{t-2} + I_{t-2}$.

Let us assume, as stated above, that businessmen believe that sales will increase at a rate w every period, where w is greater than -1.[2] Then in period t-1 (market phase or production phase), businessmen believe that sales in period t will be

$$Sales_{t\ expected} = (1+w)^2 E_{t-2} \ . \qquad 4.1.1$$

The desired level of capital in period t is, therefore,

$$K_{t\ desired} = c(1+w)^2 E_{t-2} \ .$$

[2] In the remainder of this dissertation, we shall sometimes speak of w as a percent, even though it is actually a number. When we do this we understand w percent to mean 100 w percent.

The same analysis was also carried out by businessmen in period t-2. Assuming sufficient production <u>or inventories</u> of capital goods in this period to allow $K_{t-1 \text{ desired}}$ to be achieved (assumption 1 above), we have

$$K_{t-1 \text{ on hand}} = c(1+w)^2 E_{t-3} . \qquad 4.1.1a$$

In view of equations 4.1.1 and 4.1.1a, we may, therefore, conclude that

$$I_{t-1} = c(1+w)^2 (E_{t-2} - E_{t-3}) . \qquad 4.1.1b$$

Finally, we know that

$$Y_t = (1+w)E_{t-1}, \qquad 4.1.2$$

for all periods (again by assumption 3 made above). Plugging 4.1.2 into 4.1.1b, we obtain

$$I_{t-1} = c(1+w)(Y_{t-1} - Y_{t-2}) . \qquad 4.1.1c$$

Also, since $E_t = bY_t + I_t$

$$Y_t = (1+w)(bY_{t-1} + I_{t-1}) \qquad 4.1.2a$$

(Equations 4.1.1c and 4.1.2a may be interpreted in the following manner. The model we are considering in this chapter is a passive inventory adjustment model. Production in any period is undertaken only to meet expected sales which is exactly what equation 4.1.2a states. Furthermore, since expected sales in a period equal production in that period, we can write investment not only as a function of past sales but also as a function of past and present income, as in equation 4.1.1c.)

We now have two equations (4.1.1c and 4.1.2a) in two unknowns. To solve these equations, let us substitute equation 4.1.1c into 4.1.2a to obtain

$$Y_t = (1+w)(bY_{t-1}+c(1+w)Y_{t-1}-c(1+w)Y_{t-2})$$

or

$$Y_t = (b(1+w)+c(1+w)^2)\ Y_{t-1}-c(1+w)^2Y_{t-2}\quad . \qquad 4.1.3$$

Equation 4.1.3 will henceforth be known as the "fundamental" equation of simple Harrod-like growth. In a sense, however, a better name for this equation might be "the fundamental equation of simple <u>unconstrained</u> Harrod-like growth." For in our model, businessmen are able to produce exactly what they wish, by our assumptions of "dual production methods" and infinite supplies of labor. Also, by our assumption of infinite inventories, they are able to obtain exactly that level of capital which they feel they will need in the next period. Thus, there are no "constraints" on the desires of businessmen in our model.

We now proceed to discuss some of the implications of the fundamental equation.

Section 4.2 The Warranted Rate of Growth; Consequences of the Model when the Expected Growth Rate Equals the Warranted Rate

The first question that we ask with regard to equation 4.1.3 is whether it is possible for sales in our model to grow continually at the rate expected by businessmen. Stated differently, we are aski

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asking whether it is possible for E_t to be equal to $(1+w)E_{t-1}$ in all periods, where w is the percentage rate of growth expected by businessmen.

The answer to our question is that such expectation fulfilling growth is possible, but only if businessmen feel that sales will grow by $w = \frac{1-b}{c}$ percent each period. The proof is as follows.

Since the equation $Y = (1+w)E_{t-1}$ implies that E_t is a linear combination of Y_{t+1}, equation 4.1.3 can be rewritten as

$$E_t = (c(1+w)^2+b(1+w))E_{t-1} - c(1+w)^2E_{t-2} \text{ .} \qquad 4.1.3a$$

The characteristic equation of 4.1.3a is

$$\lambda^2 - (b(1+w)+c(1+w)^2)\lambda + c(1+w)^2 = 0 \text{ .} \qquad 4.2.1$$

If one of the roots of 4.2.1 is now equal to $(1+w)$, and the other root has a zero coefficient in front of it (due to initial conditions), sales will grow each period at the percentage rate w expected by businessmen. Therefore, let $\lambda = (1+w)$ and plug this into 4.2.1 to obtain

$$(1+w)^2 - ((1+w)b+c(1+w)^2(1+w)+c(1+w)^2 = 0$$

or

$$(1-b) + c(1+w) - c = 0$$

$$w = \frac{1-b}{c} \text{ .}$$

When $w = \frac{1-b}{c}$, the other root of the characteristic equation 4.1.3a must be $\lambda_2 = c(1+w) = c(1 + \frac{1-b}{c}) = (c+1-b)$. This is easy

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easy to show if we realize that $\lambda_1\lambda_2 = c(1+w)^2$. As such the general solution of equation 4.1.3, when $(1+w) = 1+\frac{1-b}{c}$, may be written

$$Y_t = A_o\left(1+\frac{1-b}{c}\right)^t + B_o c\left(1+\frac{1-b}{c}\right)^t , \qquad 4.2.2$$

where A_o and B_o depend upon initial conditions. If the initial conditions are specified in such a way that $Y_1 = Y_o(1+\frac{1-b}{c})$, then the two equations

$$Y_o = A_o + B_o \qquad 4.2.3$$

$$Y_1 = Y_o\left(1+\frac{1-b}{c}\right) = A_o\left(1+\frac{1-b}{c}\right) + B_o c\left(1+\frac{1-b}{c}\right) \qquad 4.2.4$$

imply that $A_o = Y_o$ and $B_o = 0$. Our solution now becomes

$$Y_t = Y_o\left(1+\frac{1-b}{c}\right)^t . \qquad 4.2.5$$

The solution represented by equation 4.2.5 clearly represents the Harrod warranted growth solution. For it states that when the expected rate of growth is $\frac{1-b}{c}$ percent, income, and sales, will grow at this rate. Furthermore, this solution has two nice properties. First, markets are always cleared along the warranted path; and second, capital is always fully employed.

The proof of these two statements is fairly easy. We know that sales in any period along this path equal

$$\text{Sales}_t = bY_t + I_t = bY_o(\ \ )^t + c(\ \ )(Y_o(\ \ )^t - Y_o(\ \ )^{t-1})$$

where $(\ \ ) = 1+\frac{1-b}{c}$. Then

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$$\text{Sales}_t = Y_o(\quad)^t(b+c(\quad)-c)$$

$$= Y_o(\quad)^t(b+c+1-b-c)$$

$$= Y_o(\quad)^t$$

$$= \text{production in } t .$$

Similarly,

$$K_{t+1 \text{ on hand}} = c(1+w)Y_t = cY_o(\quad)^t(\quad)$$

$$= cY_o(\quad)^{t+1}$$

$$= cY_{t+1} = K_{t+1 \text{ desired}} .$$

Thus, warranted growth, if realized, is growth in a succession of Keynesian equilibrium states, where $Y_t = C_t + I_t$ and capital is fully employed. The first of these properties was of course postulated by Harrod and served as the basis for his warranted path. The second, however, was not a property of the Harrod model, as we have seen in Chapter 1, and we have observed that, because of this, Harrod's model was slightly illogical from a supply point of view. Consequently, we may say that our model is at least a little better than Harrod's model in that it implies consistency by businessmen along the warranted path from both a demand and <u>supply</u> point of view.

Finally, we can understand why the only model in Chapter 1 which did imply full employment of capital along the warranted path was that in which

$$I_t = c(Y_{t+1}-Y_t) . \qquad 4.2.6$$

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Since, in our model, $Y_{t+1} = (1+w)Y_t$ and $Y_t = (1+w)Y_{t-1}$ when expectations are fulfilled, equation 4.2.6 becomes

$$I_t = c(1+w)(Y_t - Y_{t-1})$$

along the warranted path, which is identical to our equation 4.1.1c.

We now consider what happens to income and sales when $w = \frac{1-b}{c}$ but $Y_1 \neq (1+\frac{1-b}{c})Y_o$. In these circumstances, if we assume that $c > 1$, and $b < 1$, as is usually done, $c(1+\frac{1-b}{c}) > 1+\frac{1-b}{c}$, and the second root of the characteristic equation dominates our solution as $t \to \infty$. The nature of the growth path may be determined by looking at the sign of the coefficient in front of this root.

In particular, if $Y_1 < \left(1+\frac{1-b}{c}\right) Y_o$, B_o will be negative, and if $Y_1 > \left(1+\frac{1-b}{c}\right) Y_o$, B_o will be positive. To prove this, let

$$A_o + B_o = Y_o \qquad 4.2.6$$

and

$$A_o\left(1+\frac{1-b}{c}\right) + B_o c\left(1+\frac{1-b}{c}\right) = Y_1 . \qquad 4.2.7$$

Multiplying 4.2.6 by $1+\frac{1-b}{c}$, we have

$$A_o(\quad) + B_o(\quad) = Y_o(\quad), \qquad 4.2.8$$

where $(\quad) = 1+\frac{1-b}{c}$. Subtracting 4.2.8 from 4.2.7, we have

$$B_o(\quad)(c-1) = Y_1 - Y_o(\quad) .$$

Since $(c-1) > 0$, B_o will be negative when $Y_1 - Y_o(\quad) < 0$, and positive when $Y_1 > Y_o(\quad)$.

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We now have a general solution to 4.1.3, when $w = \frac{1-b}{c}$.

$$Y_t = Y_o(\ \ )^t \qquad \text{when } Y_1 = Y_o(\ \ ) \qquad 4.2.9$$

$$Y_t = A_o(\ \ )^t + B_o(c(\ \ ))^t, \ B_o > 0, \ \text{when } Y_1 > Y_o(\ \ ) \qquad 4.2.10$$

$$Y_t = A_o(\ \ )^t + B_o(c(\ \ ))^t, \ B_o < 0, \ \text{when } Y_1 < Y_o(\ \ ) \qquad 4.2.11$$

The solution 4.2.10 explodes to $+\infty$ and the solution 4.2.11 contracts to $-\infty$.

Only one small problem remains. Equation 4.2.11 implies that after some t, income is negative. However, in the real world, as well as in our model, income can never be negative. This equation also implies that after some t, $I_t = c(1+w)(Y_t-Y_{t-1})$, becomes negative. Since there is no depreciation in our model, this too makes no sense. In order to get around these problems, let us assume that if, for some t, Y_t is less than Y_{t-1}, $I_t = 0$ and not $c(1+w)(Y_t-Y_{t-1})$. Our equation for growth now becomes not 4.1.3 but rather

$$Y_t = b\left(1+\frac{1-b}{c}\right)Y_{t-1} \qquad 4.2.12$$

which has a solution

$$Y_t = Y_o\left[b(1+\frac{1-b}{c})\right]^t . \qquad 4.2.13$$

Since $b(1+\frac{1-b}{c}) < 1$, for $c > 1$ and $b < 1$, this solution goes to zero as $t \to \infty$. Equations 4.2.9, 4.2.10, 4.2.11, and 4.2.13 now specify growth completely.

These solutions also demonstrate the knife edge instability of the Harrod model. For in specifying our initial conditions, we stated that

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that if Y_1 exactly equalled $Y_o(1+\frac{1-b}{c})$, income would grow at a constant warranted rate. If, however, Y_t is <u>ever so slightly greater</u> that $Y_o(1+\frac{1-b}{c})$, the economy will diverge to infinity. In each period, the rate of growth will increase until finally as $t \to \infty$, it becomes $c(1+\frac{1-b}{c}) - 1$. Similarly if $Y_1 < (1+\frac{1-b}{c})\, Y_o$ <u>by any amount, no matter how small</u>, the solution will eventually be "cut off," and income will contract to zero. Moreover, when income explodes, sales will exceed production and capital will be insufficient. When income contracts, production will exceed sales and capital will be overly-sufficient.

The proofs of the last two statements are fairly easy. E_t satisfies the same difference equation as Y_t. Its solution can be written as

$$E_t = A'_o\left(1+\frac{1-b}{c}\right)^t + B'_o c\left(1+\frac{1-b}{c}\right)^t$$

For B'_o positive (divergence),

$$E_t > \left(1+\frac{1-b}{c}\right) E_{t-1} .$$

But

$$Y_t = \left(1+\frac{1-b}{c}\right) E_{t-1} .$$

Therefore,

$$E_t > Y_t .$$

For B'_o negative (contraction),

$$E_t < \left(1+\frac{1-b}{c}\right) E_{t-1} = Y_t$$

Similarly, the amount of capital desired by businessmen in the market phase of t, and on hand in $t+1$, is

$$K_{t+1 \text{ desired}} = K_{t+1 \text{ on hand}} = c\left(1+\frac{1-b}{c}\right) Y_t .$$

But

$$K_{t+1 \text{ needed}} = c\left(1+\frac{1-b}{c}\right) E_t .$$

Therefore,

$$K_{t+1 \text{ needed}} > K_{t+1 \text{ on hand}}$$

for divergence, and

$$K_{t+1 \text{ needed}} < K_{t+1 \text{ on hand}}$$

for contraction.

Thus, when income diverges, sales always exceed production and capital always proves insufficient. Similarly, when income contracts, sales are always less than production each period, and capital is always in excess. We are confronted with Harrod's original premise that for "upward" divergence from the warranted path, businessmen will find themselves producing too little, with an "inadequate" supply of capital, and that for "downward" divergence (contraction), businessmen will find themselves producing too much, with an excess of capital. Furthermore, for "upward" divergence, we may prove that the deficits in production and capital become greater as $t \to \infty$. In this case, the difference between sales and production is

$$E_t - \left(1+\frac{1-b}{c}\right) E_{t-1} ,$$

which for large t, behaves as

$$c\left(1+\frac{1-b}{c}\right) - \left(c\left(1+\frac{1-b}{c}\right)\right)^t\left(1+\frac{1-b}{c}\right)$$

or

$$(c-1)(c)^{t-1}\left(1+\frac{1-b}{c}\right)^t \quad .$$

The latter expression clearly increases as t increases. Similarly since $K_{t+1 \text{ needed}} - K_{t+1 \text{ on hand}}$ equals

$$c\left(1+\frac{1-b)}{c}\right)E_t - c\left(1+\frac{1-b}{c}\right)Y_t \ ,$$

the difference between needed and actual capital each period also becomes greater as t increases. For downward divergence, it is clear that the amount of capital over-sufficiency gets greater and greater each period since capital is fixed and income is decreasing. However, because production is becoming smaller and smaller, the excess of production over sales will decrease as t increases.

For $w = \frac{1-b}{c}$, our model is now virtually identical to Harrod's. First, we have a warranted path. Along this warranted path, markets are cleared as Harrod stated and capital is fully employed. Second, if the system diverges at any time from this warranted path, it can never return to it. Income will either expand faster and faster, with underproduction and insufficiency of capital each period, or contract with overproduction and underemployment of capital. The only difference between our model, under these circumstances, and Harrod's, is that Harrod in his discussion of the knife edge talked about inflationary and deflationary price pressures, whereas we describe pressures in terms of inventory accumulation or decumulation.

Section 4.3 Growth When $w \neq \frac{1-b}{c}$

In the previous section we investigated growth according to equation 4.1.3 when $w = \frac{1-b}{c}$, and saw how similar, under this condition, growth in our model is to that postulated by Harrod. Unfortunately, there is nothing in our theory, or in the real world, which states that the expected rate of growth has to equal the warranted rate of the economy. Indeed, one of the most realistic types of expectations by businessmen might be simple expectations, where $w = 0$. Let us, therefore, consider what happens to our model when $w \neq \frac{1-b}{c}$.

We will divide our types of expectations into three classes.

1) $(1+w) < \frac{2c^{\frac{1}{2}}-b}{c}$

2) $\frac{2c^{\frac{1}{2}}-b}{c} < (1+w) < 1+\frac{1-b}{c}$

3) $(1+w) > 1+\frac{1-b}{c}$

Each of these "classes" of expectations, as we shall see, has different properties. It will turn out to be fairly easy to categorize the type of growth our model displays, by stating into which class our expected growth rate falls.

For the first class of expectations, we can state that, while income may first increase or decrease, it <u>must</u> eventually contract to zero. The proof of this statement is simple. The characteristic equation of our model is

$$\lambda^2 - (b(1+w)+c(1+w)^2)\lambda + c(1+w)^2 = 0 \cdot \qquad 4.1.3a$$

The roots of this equation are

$$\lambda_{1,2} = b(1+w)+c(1+w)^2 \pm \frac{\sqrt{(b(1+w)+c(1+w)^2)^2-4c(1+w)^2}}{2}$$

$$\lambda_{1,2} = (1+w)\ \frac{(b+c(1+w)) \pm \sqrt{(b+c(1+w))^2-4c}}{2} \ . \qquad 4.3.1$$

If

$$(b+c(1+w))^2 - 4c < 0 \ ,$$

or, equivalently, if

$$(1+w) < \frac{2c^{\frac{1}{2}}-b}{c} \qquad 4.3.2$$

(class 1 above), both roots of the characteristic equation 4.1.3a will be complex. The solution to the difference equation can now be written in the form

$$Y_t = R^t(A_o \cos \theta t + B_o \sin \theta t) \qquad 4.3.3$$

where R is the modulus of the complex roots $\lambda_{1,2} = h+ij$, and $\theta = \tan^{-1}(\frac{h}{j})$. A_o and B_o depend on initial conditions. Equation 4.3.3 exhibits oscillatory behavior, and after a while, no matter what the signs or magnitudes of A_o and B_o, income must begin to decrease. When this occurs, we realize that I_t would be negative. Therefore, we cut off our solution and let $I_t = 0$ at this point. Our growth equation now becomes

$$Y_t = (1+w)(bY_{t-1}+0) \ . \qquad 4.3.4$$

The condition $(1+w) < \frac{2c^{\frac{1}{2}}-b}{c}$, however, implies that $(1+w) < 1+\frac{1-b}{c}$ since

$$\frac{2c^{\frac{1}{2}}-b}{c} < 1+\frac{1-b}{c}$$

is equivalent to

$$2c^{\frac{1}{2}} - b < (1+c-b)$$

$$0 < 1 - 2c^{\frac{1}{2}} + c$$

$$0 < (1-c^{\frac{1}{2}})^2 ,$$

which is obviously true. Since we have already seen on page 68 that $b(1+\frac{1-b}{c}) < 1$, it also must be that $b(1+w) < 1$, for $(1+w) < \frac{2c^{\frac{1}{2}}-b}{c}$, and income, according to equation 4.3.4, must contract to zero.

Thus, we arrive at the very important conclusion that when $(1+w) < \frac{2c^{\frac{1}{2}}-b}{c}$, income will eventually decrease to zero, no matter how high the initial rate of growth. This last statement is not in accord with Harrod's postulate that if initially the rate of growth is higher than the warranted rate, the economy will explode with ever increasing growth rates. However, the result we have obtained is perfectly plausible. In an economy in which expectations are always "low," there will be great difficulty in getting production to expand very quickly. This in turn will cause investment to be "small," which will reinforce continued slowness of growth.

The next class of expectations, which we wish to deal with, is that where $(1+w)$ is less than $1+\frac{1-b}{c}$ but greater than $\frac{2c^{\frac{1}{2}}-b}{c}$. In order to facilitate the discussion of the solution to equation 4.1.3 under this assumption, let us first prove the following extremely useful theorem.

Theorem 4.3.1: If $\frac{2c^{\frac{1}{2}}-b}{c} < (1+w) < 1+\frac{1-b}{c}$ the two roots of the characteristic equation 4.1.3a are real and greater than $1+\frac{1-b}{c}$

Proof: First, because $(1+w) > \frac{2c^{\frac{1}{2}}-b}{c}$, the two roots cannot be complex. Second, because the two roots are given by 4.3.1, it must be that

$$\lambda_{min} = (1+w)\ \frac{(b+c(1+w) - \sqrt{(b+c(1+w)^2-4c)}}{2}.$$

If we can prove that $\lambda_{min} > 1+\frac{1-b}{c}$, it must also follow that λ_{max} is greater than $1+\frac{1-b}{c}$.. Therefore, suppose that

$$\lambda_{min} = (1+w)\ \frac{(b+c(1+w) - \sqrt{(b+c(1+w))^2-4c}}{2} > 1+\frac{1-b}{c}\ .$$

Letting $w_o = \frac{1-b}{c}$, we have

$$(1+w)(b+c(1+w))-2(1+w_o) > (1+w)\ \sqrt{(b+c(1+w))^2-4c}\ . \qquad 4.3.5$$

Both sides of 4.3.5 are positive since

$$\begin{aligned}(1+w)(b+c(1+w))-2(1+w_o) &> \frac{2c^{\frac{1}{2}}-b}{c}\,(2c^{\frac{1}{2}}) - 2(1+w_o)\\ &= 4-\frac{2b}{c^{\frac{1}{2}}} - 2 - \frac{2(1-b)}{c}\\ &= 2-\frac{2b}{c^{\frac{1}{2}}} - \frac{2}{c} + \frac{2b}{c}\\ &> 2-\frac{2}{c}\\ &> 0\ .\end{aligned}$$

Consequently, we may square both sides of 4.3.5 to obtain

$$(1+w)^2(b+c(1+w))^2-4(1+w_o)(1+w)(b+c(1+w))+4(1+w_o)^2 >$$

$$(1+w)^2(b+c(1+w))^2-4c(1+w)^2\ ,$$

or

$$4(1+w_o)^2-4(1+w_o)(1+w)(b+c(1+w))+4c(1+w)^2 > 0$$

$$(1+w_o)^2+c(1+w)^2-(1+w_o)(1+w)b-c(1+w)^2(1+w_o) > 0 \qquad 4.3.6$$

If we can now prove that for $w < w_o$, equation 4.3.6 is valid, we have proved our theorem. Accordingly, let us consider the left hand side of 4.3.6. For $w < w_o$, this expression is greater than

$$(1+w_o)^2 + c(1+w)^2 - (1+w_o)^2 b - c(1+w)^2(1+w_o) . \qquad 4.3.7$$

Simplifying 4.3.7 we have

$$(1-b)(1+w_o)^2 - w_o c(1+w)^2$$

$$(1-b)(1+w_o)^2 - \frac{1-b}{c} c(1+w)^2$$

$$(1-b)((1+w_o)^2 - (1+w)^2)$$

which is certainly greater than 0, for $w < w_o$. Thus, when $w < w_o$, the left hand side of 4.3.6 is greater than 0, and our theorem is proved.

We can now discuss the nature of growth when $\frac{2c^{\frac{1}{2}}-b}{c} < (1+w) < 1+\frac{1-b}{c}$. Let the solution to our problem be written in the form

$$Y_t = A_o(\lambda_1)^t + B_o(\lambda_2)^t ,$$

where A_o and B_o depend on initial conditions, and λ_1 and λ_2 are greater than $1+\frac{1-b}{c}$. Let $\lambda_1 < \lambda_2$. If $Y_1 > \lambda_1 Y_o$, then income will diverge to infinity; if $Y_1 < \lambda_1 Y_o$, income will contract. The proof is as follows. Income will expand or contract according to the sign of B_o. For B_o positive, it will expand, and, for B_o negative, it will contract. We know that

$$A_o + B_o = Y_o \qquad 4.3.8$$

and

$$A_o\lambda_1 + B_o\lambda_2 = Y_1 \ . \qquad 4.3.9$$

Multiply 4.3.8 by λ_1 to get

$$A_o\lambda_1 + B_o\lambda_1 = Y_o\lambda_1 \ . \qquad 4.3.8a$$

If we subtract 4.3.8a from 4.3.9, we have

$$B_o(\lambda_2-\lambda_1) = Y_1 - Y_o\lambda_1 \ .$$

For $Y_1 > Y_o\lambda_1$, B_o is positive and income expands; for $Y_1 < \lambda_1 Y_o$, B_o is negative and income contracts. We may now draw the following important conclusions. When $(1+w)$ is "low" (less than the warranted rate), but greater than $\frac{2c^{\frac{1}{2}}-b}{c}$, income may or may not explode. Whether or not explosion or contraction occurs depends upon the magnitude of our expectations and initial conditions. In particular, no matter what w is, the initial growth rate must be higher than the warranted rate of growth $w_o = \frac{1-b}{c}$, for explosion. Finally, if the initial rate of growth is less than $(\lambda_{min}-1)$, income will initially expand but eventually contract. When the latter occurs, we cut off our solution and let $I_t = 0$. The economy will now slowly but steadily contract to zero income level since $b(1+w) < 1$.

A simple numerical example may help to clarify the preceding statements. Suppose $w = .2$, $b = .5$, and $c = 2$. Since $w_o = .25$ and $w < w_o$, our equation for the growth of income becomes

$$Y_t - (3.48)Y_{t-1} + (2.88)Y_{t-2} = 0$$

The characteristic roots of this equation are $\lambda_1 = 1.35$ and $\lambda_2 = 2.13$. Thus, by our previous discussion, unless $Y_1 > 1.35Y_o$, income must eventually converge to zero. If $Y_1 > 1.35Y_o$, however, the economy will explode. Because $(\lambda_2)^t >>> (\lambda_1)^t$ for large t, income will eventually increase at a rate of growth equal to (λ_2-1).

Finally, it should be pointed out that if in any period, the growth rate exceeds the expected rate of growth in sales, sales will exceed production and needed capital will exceed actual capital on hand. This will occur whether the economy ultimately diverges or contracts. The proof is very easy.

$$E_t = A_o'(\lambda_1)^t + B_o'(\lambda_2)^t .$$

Suppose, for some period t, we can write

$$E_t = (1+g)E_{t-1} ,$$

where g is greater than w. Then $Y_t = (1+w)E_{t-1}$ and $Y_t < E_t$. Furthermore,

$$\begin{aligned} K_{t \text{ needed}} &= cY_t \\ &= c(1+w)E_{t-1} \\ &> c(1+w)Y_{t-1} , \end{aligned}$$

which equals $K_{t \text{ on hand}}$, when $E_{t-1} > Y_{t-1}$. Similarly, it may be shown that when the growth rate is less than the expected rate of growth, production exceeds sales and capital on hand exceeds needed capital. Thus, off the warranted path, sales will never equal production, and capital will never be optimal.

The above conclusions differ in several respects from Harrod's conclusions. For they state that even if the initial rate of growth is higher than the warranted rate, the economy need not diverge. Furthermore, even if sales exceed production and capital is insufficient in a certain period, it is not necessarily true that this state of affairs will continue. After a while, due to low expectations, production may become greater than sales each period, and capital may be over-utilized. Of course, our results do not disagree completely with those of Harrod. Harrod noted that off the warranted path, "sales will never equal production and capital will never be optimal" just as we have. Furthermore, when $w < w_o$, our model predicts that if the initial rate of growth is less than the warranted rate, income will contract, and never return to the warranted path. Both of these conclusions are similar to Harrod's.

Finally, a curious result which differs from one of Harrod's conclusions. Harrod stated that income could grow at a constant rate only if that rate was the warranted rate. It is clear, however, that one of the coefficients in the solution for Y_t

$$Y_t = A_o(\lambda_1)^t + B_o(\lambda_2)^t$$

may vanish under the proper initial conditions. In this circumstance, we will get a constant rate of growth not equal to the warranted rate. In each period, sales will be greater than production (inventories will be drawn down) and capital will be insufficient, since the expected rate of growth is less than the actual rate of growth, which equals either (λ_1-1) or (λ_2-1). The reason that this strange result obtains is that in our model businessmen are assumed not to change their expectations, even though these expectations are not realized. In this case, it just fortuitously happens that income expands, but not at increasing rates. In a model in which expectations could change, this result would not obtain. We shall prove the last statement in Chapter 6.

We now turn to the case where $w > \frac{1-b}{c}$. It is convenient to discuss separately two situations, one where $\frac{1-b}{c} < w < \frac{1}{b} - 1$, and the other where $w > \frac{1}{b} - 1$. (It is clear that $\frac{1}{b} - 1 > \frac{1-b}{c}$ since $\frac{1}{b} - 1 - \frac{1-b}{c} = \frac{1-b}{c} - \frac{1-b}{c} > 0$.) Before we do so, however, let us state and prove a theorem very similar to Theorem 4.3.1.

<u>Theorem 4.3.2</u>: If for equation 4.1.3 $w > \frac{1-b}{c}$, then one root of the characteristic equation 4.1.3a is greater than $1+\frac{1-b}{c}$, and the other is less than $1+\frac{1-b}{c}$.

<u>Proof</u>: Define $\frac{1-b}{c}$ to be w_o. Then

$$\lambda_{max} = (1+w)\ \frac{(b+c(1+w) + \sqrt{(b+c(1+w))^2-4c)}}{2}$$

$$>(1+w_o)\ \frac{(b+c(1+w_o) + \sqrt{(b+c(1+w_o)^2-4c)}}{2}$$

$$>(1+w_o)\ \frac{b+c(1+\frac{1-b}{c}) + \sqrt{(b+c(1+\frac{1-b}{c}))^2-4c}}{2}$$

$$>(1+w_o)\ \frac{c+1 + \sqrt{(c-1)^2}}{2}$$

$$>(1+w_o)c\ ,$$

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which for $c > 1$, proves the first part of our theorem. The proof of the second part of our theorem follows very closely the proof of Theorem 4.3.1. Since we are trying to determine circumstances under which $\lambda_{min} < (1+w_o)$, we reverse the inequality in 4.3.6 to obtain

$$(1+w_o)^2 + c(1+w)^2 - (1+w)(1+w_o)b - c(1+w)^2(1+w_o) < 0 . \qquad 4.3.10$$

We now ask when the left hand side of 4.3.10 is less than 0. Assume that $(1+w) > (1+w_o)$. It is clear that under this assumption, the left hand side of 4.3.10 is less than

$$(1+w_o)^2 + c(1+w)^2 - (1+w_o)^2 b - c(1+w)^2(1+w_o) ,$$

which equals

$$(1-b)(1+w_o)^2 - c(1+w)^2 w_o$$

$$(1-b)((1+w_o)^2 - (1+w)^2) ,$$

which is less than zero. Equation 4.3.10 is satisfied and the theorem is proved.

Thus for $\frac{1-b}{c} < w < \frac{1}{b} - 1$, our solution may be written in the form

$$Y_t = A_o(\lambda_1)^t + B_o(\lambda_2)^t , \qquad 4.3.11$$

where λ_1 is less than $1+\frac{1-b}{c}$, λ_2 is greater than $1+\frac{1-b}{c}$ and A_o and B_o depend on initial conditions. Furthermore, except for the magnitudes of the roots, this case is exactly similar to that when $\frac{2c^{\frac{1}{2}}-b}{c} < (1+w) < (1+w_0)$. Income will either expand or contract, with the concomitant deficits or surpluses in production. The only difference in the two cases is that the economy is more likely to expand when $w > w_o$ than when $w < w_o$. This result obtains because the condition for explosion is $Y_1 > Y_o(\lambda_1)$, and λ_1 is smaller for $w > w_o$ than it is for $w < w_o$.

Finally, we consider the case where $w > w_o$ and $(w) > \frac{1}{b} - 1$. We can rewrite the latter condition as $(b)(1+w) > 1$. In this case, it is easy to see that there must always be explosive growth. For the lowest initial condition that we may have in equation 4.1.3 is $Y_1 = Y_o$. (Otherwise investment would be less than zero and equation 4.1.3 would not be valid.) If we can now show that $\lambda_{min} < 1$, B_o must always be less than zero, and income must always explode. The latter statement, however, must be true, since

$$\lambda_{min} = (1+w)\left[\frac{b+c(1+w) - \sqrt{(b+c(1+w)^2-4c}}{2}\right] < 1$$

implies that

$$(1+w)\left[b+c(1+w) - \sqrt{(b+c(1+w)^2-4c)}\right] < 2$$

or

$$(1+w)(b+c(1+w))- 2 < (1+w) \sqrt{(b+c(1+w))^2-4c} .$$

Squaring, we have

$$(1+w)^2(b+c(1+w))^2-4(1+w)(b+c(1+w))+4 < (1+w)^2(b+c(1+w))^2-4c(1+w)^2$$

or

$$4 + 4c(1+w)^2 - 4(1+w)b - 4c(1+w)^2 < 0$$

$$1 - (1+w)b < 0 .$$

The latter implies that λ_{min} of equation 4.1.3 will be less than 1 when $b(1+w) > 1$.

Thus, when $b(1+w) > 1$, one of the roots of our characteristic equation 4.1.3a must be less than one, and, due to the constraints imposed on the initial conditions, the economy must diverge. Stated differently, business expectations concerning sales are so high that even if production exceeds sales and capital is underutilized by a great deal in some period, businessmen will "keep faith" and increase production. Income will eventually explode, with insufficiency of sales and capital, and expectations will be realized.

We may now compare the conclusions of our "high expectations" with those of the Harrod model. First, to point out similar conclusions, both models state that if the initial rate of growth is greater than the warranted rate, the economy will explode. Second, they both state that when this occurs, sales and capital will eventually become insufficient each period. Third, they both predict that the warranted path can never be achieved. On the other hand, there are dissimilar conclusions. Harrod stated that if initially the rate of growth were less than the warranted rate, income would contract. This is clearly false in our model. The condition for explosion in our model is $Y_1 > Y_o(\lambda_1)$. Since λ_1 in the case of high expectations is less than $1+w_o$, it is entirely conceivable that the economy will explode even when the initial rate of growth is less than the warranted rate. Indeed, when expectations are very high $(b(1+w) > 1)$, the economy will explode no matter how low the initial rate of growth. Thus, in our model, sales and capital may be overly sufficient in some period without causing contraction. Finally, our model, as opposed to Harrod's, predicts that the economy may grow at a constant nonwarranted rate of growth. This occurs when A_o or B_o is zero in expression 4.3.11. If A_o is zero, the interpretation is the same as in the low expectations case. (See page 79 paragraph 3.) If B_o is zero and A_o positive, production and capital will be overly sufficient each period but income will still continue to grow at a constant rate each period. This strange result obtains because of the constant "optimistic" expectations which businessmen have. Again, we shall see in Chapter 6 that when expectations are "adaptive," this type of growth will not be possible.

Section 4.4 Review and Comment

We can now summarize the results we have obtained when expectations with regard to sales are constant and businessmen disregard their inventory positions.

First, there exists a "warranted growth path" in the economy. Along this path, the economy will grow at a constant rate $w_o = \frac{1-b}{c}$. Capital will be fully employed and sales will equal production.

Second, the slightest disturbance of growth from this path will cause the economy to veer off into a state of excess demand (depletion of inventories) or deep depression.

Third, if sales are expected to grow at a rate less than the warranted rate, expected and actual growth pattern will not be the same. For "very" low expectations, the economy will always contract. For higher expectations, it may contract or explode. In general, unless the initial growth rate is "much" higher than the warranted rate, the economy will fall into deep depression.

Fourth, if sales are expected to grow at a rate higher than the warranted rate, expected and actual growth patterns will not be the same. For "very" high expectations, the economy will always explode. For lower expectations, it may explode or contract. In general, unless the initial growth rate is "much" lower than the warranted rate, the economy will be plagued by continual excess demand.

We have now concluded our discussion of the simplest Harrod-like economy. Before proceding to discussions of more complicated (and realistic) models, it seems appropriate at this point to briefly discuss three points.

The first point concerns the appropriateness of our model as a dynamic model. In Chapter 3, we investigated the Keynesian model and found it seriously wanting in dynamic terms. The reasons, to repeat ourselves briefly, were that: 1) the assumption $I_t = I_o$ made one of our variables exogenously determined; 2) the model, under certain circumstances, reached a stable equilibrium; and 3) when explosion occurred in the model, sales became less than production, and when contraction occurred, sales became greater than production.

Our Harrod-like model, however, obviates all of these difficulties. First, our model endogenously determines the level of investment each period. Second, it is impossible for our economy to reach an equilibrium (other than zero income). Third, our model predicts that when explosion occurs, sales will eventually exceed production, and that when the economy contracts sales will eventually be less than production. The latter conclusions are in accord with the observations that when a real economy expands "shortages" develop and that when it contracts "surpluses" pile up. Thus, the "Harrod" model seems far more appropriate as a growth model than the Keynesian model.

Our second point is of historical interest. We have seen that our model, while amazingly similar to Harrod's, does arrive at conclusions concerning whether an economy will explode or contract, somewhat different from Harrod's. Realizing that it is extremely difficult to read another person's mind (even when that person's thoughts are in print), the author would like to suggest that the reason for this is the Harrod envisioned (non-mathematically) a special case of our model.

Harrod, like ourselves, seems to think of growth as occurring because of business expectations. This is perhaps made most clear in his discussion of the warranted rate of growth in pages 81 and 82 of Towards a Dynamic Economics.[3] Furthermore, he states as we do that there are three possible rates of expectations: $w = \frac{1-b}{c}$, $w < \frac{1-b}{c}$, and $w > \frac{1-b}{c}$. He states, however, that if w is greater than $\frac{1-b}{c}$, "their experience will tend to drive them further from it" (the warranted rate). Similarly, if w is less than the warranted rate. We have seen that these statements in general are not true. In one special case, though, they are correct. Suppose that initially $Y_1 = (1+w)Y_o$, where w is the coefficient of

[3]Some sample quotes, "The decision by each entrepreneur to continue producing at the rate... ." I define G_w as that over all rate in which they carry on a similar advance... ." "Some may be dissatisfied and have to adjust upwards or downwards... ."

expectations. If $w < w_o$, income <u>will</u> contract (and businessmen might be tempted to decrease even more their expected rate of growth causing further contraction), and if $w > w_o$ the opposite will occur. <u>All of Harrod's statements concerning the instability of his model would be correct, if the initial rate of growth were equal to the expected rate of growth. Unfortunately, we cannot make this assumption.</u>

For the initial conditions in our problem, as in any dynamic problem, are arbitrary. Thus, even if expectations are high, initial changes in income might be low, and vice versa. Initial conditions being inconsistent with expectations could be the result of miscounting by businessmen or the result of some exogenous force (the government) which, for one period, increases or decreases total expenditures in the economy. In short, there is, in general, no way we can relate the initial conditions of our problem to expectations, and we must, therefore, consider all possible combinations of these factors.

Our third point concerns a statement we made in Section 1.3 with regard to the empirical accuracy of Harrod-like models. At that time, we indicated that the most appropriate description of investment behavior might be a multilag accelerator function. Unfortunately, in our model, so far, the investment function has only a single current lag, when expressed with respect to income, and only a single one period lag, when expressed with respect to expenditures. The question now remains as to how to incorporate multiperiod accelerator lags into our model.

To answer this question, let us assume that firms have different stores in different locations. Assume that "expenditures data" from different locations "come in" at different times. In particular, assume that each firm has three different stores A, B, C, which report their sales with no lag, with a lag of one period, and with a lag of two periods respectively. Suppose, moreover that store A always does α percent of all business in a period, store B does β percent, and store C does γ percent, $(\alpha + \beta + \gamma = 1)$. It is clear that at the beginning of the production phase of period t, firms will expect sales to be

$$E_{t\ exp} = (1+w)\alpha E_{t-1} + (1+w)^2\beta E_{t-2} + (1+w)^3\gamma E_{t-3}$$

and that

$$Y_t = (1+w)\alpha E_{t-1} + (1+w)^2\beta E_{t-2} + (1+w)^3\gamma E_{t-3} . \qquad 4.4.1$$

Similarly in the market phase of period t, investment expenditures will be given by the formula

$$I_t = c(1+w)^2\alpha(E_{t-1}-E_{t-2})+c(1+w)^3\beta(E_{t-2}-E_{t-3})+c(1+w)^4\gamma(E_{t-3}-E_{t-4}). \qquad 4.4.2$$

Solving equations 4.1.1 and 4.1.2 for E_t, we obtain

$$\begin{aligned} E_t &= b\alpha(1+w)E_{t-1} + b\beta(1+w)^2E_{t-2} + b\gamma(1+w)^3E_{t-3} \\ &\quad + c(1+w)^2\alpha(E_{t-1}-E_{t-2}) + c(1+w)^3\beta(E_{t-2}-E_{t-3}) \qquad 4.4.3 \\ &\quad + c(1+w)^4\gamma(E_{t-3}-E_{t-4}) . \end{aligned}$$

Equation 4.4.3 is clearly not the same difference equation as equation 4.1.3. But the type of growth it describes is very similar to that of equation 4.1.3. In particular, there exists, for equation 4.4.3, a warranted rate of growth and this warranted rate is $\frac{1-b}{c}$. To prove this, let us plug $E_t = (1+w)E_{t-1}$ into equation 4.4.3 to obtain

$$(1+w)^4 = b\alpha(1+w)(1+w)^3 + b\beta(1+w)^2(1+w)^2 + b\gamma(1+w)^3(1+w)$$
$$+ c(1+w)^2\alpha((1+w)^3-(1+w)^2) + C(1+w)^3\beta((1+w)^2-(1+w))$$
$$+ c(1+w)^4\gamma((1+w)-1) .$$

Since $\alpha + \beta + \gamma = 1$, we have

$$(1+w)^4 = b(1+w)^4 + c(1+w)^5 - c(1+w)^4$$

or

$$(1+w) = 1 + \frac{1-b}{c} .$$

The above model is only a three period lag model. Clearly, however, our methods can be extended to any type of distributed lag criteria. Of course, such models are "harder" to work and, from a theoretical point of view, they are cumbersome. Therefore, we shall not use them in the rest of this dissertation. This does not deny, however, our ability to use them for empirical purposes, which is what we set out to show.

CHAPTER 5

Section 5.1. The Fundamental Equation for Simple Additive Expectations Harrod-like Growth

The model we have discussed in the last chapter is "unrealistic" in the sense that expectations are constant and exogenous to the system. We might expect that in the real world businessmen, upon seeing that expectations are not fulfilled would change them. Accordingly, we might wish to investigate expectations which are "endogenous" to the model.

One particular expectation which appears to be endogenous is that where businessmen feel that sales in period $t+1$ will equal sales in period t plus the change in sales between periods t and $t-1$. A model based on this type of expectation is one in which businessmen do not anticipate growth at a certain percentage rate, but rather feel that the change in sales in the next period (and all periods thereafter) will be the same absolutely as that in the last period. The reasonableness of such a model may be open to question. If sales grew from \$100 to \$200 between periods t and $t-1$, a 100% increase, it might seem strange to assume that businessmen will expect sales to grow only from \$200 to, \$300 - a 50% increase, between periods t and $t+1$. Nonetheless, this type of expectation does seem to satisfy our condition for endogenousness and various economists such as Metzler have investigated it in other contexts. We shall, therefore, attempt to find out whether such a model gives us typical Harrod-like growth.

Mathematically, we begin by writing

$$E_{t+1\ \text{expected}} = (bY_t + I_t) + ((bY_t + I_t) - (bY_{t-1} + I_{t-1})) \quad ,$$

which expresses businessmen's expectations concerning sales. Because we assume that businessmen produce only to meet expected sales (i.e., they do not try to adjust their inventories), we may also write that

$$Y_{t+1} = (bY_t + I_t) + ((bY_t + I_t) - (bY_{t-1} + I_{t-1})) \ . \qquad 5.1.1$$

The question now arises as to how businessmen plan capital expenditures in the market phase of period t, given this same expectation. The answer is fairly simple.

$$K_{t+1\ \text{desired}} = c(\text{Sales}_{\text{expected in } t+1})$$

$$= c(\text{Sales}_{\text{expected in } t} + (\text{Sales}_{\text{exp in } t} - \text{Sales}_{t-1})) \ .$$

(Again, since we are in the market phase of period t, businessmen <u>cannot</u> see sales in this period, and, hence, make investment decisions on the basis of expected sales.) Expected sales in t, in the absence of inventory adjustments, are Y_t. Accordingly,

$$K_{t+1\ \text{desired}} = c(Y_t + Y_t - (bY_{t-1} + I_{t-1})) \qquad 5.1.2$$

$$= c(2Y_t - (bY_{t-1} + I_{t-1})) \ .$$

In period t-1, however, this same analysis was carried out by businessmen. Therefore, assuming businessmen were able to obtain all the capital goods which they desired,

$$K_{t\ desired} = K_{t\ on\ hand} = c(2Y_{t-1} - (bY_{t-2} + I_{t-2}))\ , \qquad 5.1.2a$$

and

$$I_t = K_{t+1\ desired} - K_{t\ on\ hand}$$

$$= c(2Y_t - (bY_{t-1} + I_{t-1})) - c(2Y_{t-1} - (bY_{t-2} + I_{t-2}))\ .$$

We now have two difference equations in two unknowns.

$$Y_t = 2bY_{t-1} - bY_{t-2} + 2I_{t-1} - I_{t-2} \qquad 5.1.3$$

$$I_t = c((2Y_t - 2Y_{t-1}) - (bY_{t-1} + I_{t-1}) + (bY_{t-2} + I_{t-2})) \qquad 5.1.4$$

We would now like to write this system of equations as a single linear difference equation in one of the unknowns Y_t. From a conceptual point of view, the simplest procedure for doing this is to introduce the E operator defined as $EY_t = Y_{t+1}$. We may now rewrite equations 5.1.3 and 5.1.4 as

$$(E^2 - 2bE + b)Y_{t-2} = (2E-1)I_{t-2} \qquad 5.1.3a$$

and

$$(2cE^2 - (2c + bc)E + bc)Y_{t-2} = E^2 + cE - E)I_{t-2}\ . \qquad 5.1.4a$$

Multiplying 5.1.3a by $(E^2 + cE - c)$, we get

$$(E^2 + cE - c)(E^2 - 2bE + b)Y_{t-2} = (2E-1)(E^2 + cE - c)I_{t-2}\ ,$$

and, similarly, multiplying 5.1.4a by (2E-1), we have

$$(2E-1)(2cE^{2} - (2c + 2b)E + cb)Y_{t-2} = (2E-1)(E^{2} + cE - E)I_{t-2} \ .$$

Equating the left hand sides of the above expressions, we obtain

$$(E^{2}+cE-c)(E^{2}-2bE+b)Y_{t-2} = (2E-1)(2cE^{2}-(2c+2b)E+cb)Y_{t-2}$$

The latter implies that

$$(E^{4}-2bE^{3}+bE^{2}+cE^{3}-2bE^{2}+bcE-cE^{2}+2bcE-2b)Y_{t-2}$$

$$= (2cE^{3}-2(2c+2b)E^{2}+2cbE-2cE^{2}+(2c+2b)E-2b)Y_{t-2}$$

or

$$(E^{4}+E^{3}(-2b+c-4c)+E^{2}(b-2bc-c+2(2b+2c)+2c)$$

$$+E(2bc+bc-2bc-2c-cb))Y_{t-2} = 0 \ .$$

Upon simplifying, we obtain,

$$(E^{4}+E^{3}(-2b-3c)+E^{2}(5c+b)-2cE)Y_{t-2} = 0$$

or

$$Y_{t+3}-(2b+3c)Y_{t+2}+(5c+b)Y_{t+1}-2cY_{t} = 0 \ . \qquad 5.1.5$$

Another and simpler way of deriving equation 5.1.5 is the following. From equation 5.1.1

$$Y_{t} = E_{t-1} + E_{t-1} - E_{t-2} = 2E_{t-1} - E_{t-2} \ .$$

From equation 5.1.2a

$$I_t = c(2Y_t - E_{t-1} - 2Y_{t-1} + E_{t-2})$$

$$= c(2(2E_{t-1} - E_{t-2}) - E_{t-1} - 2(2E_{t-2} - E_{t-3}) + E_{t-2})$$

$$= c(3E_{t-1} - 5E_{t-2} + 2E_{t-3}) \quad .$$

Now

$$E_t = bY_t + I_t = b(2E_{t-1} - E_{t-2}) + c(3E_{t-1} - 5E_{t-2} + 2E_{t-3})$$

or

$$E_t = (2b + 3c)E_{t-1} - (5c + b)E_{t-2} + 2cE_{t-3} \quad .$$

Since $Y_t = 2E_{t-1} - E_{t-2}$ implies that Y_t is a linear combination of lagged values of E_t, it must be that Y_t satisfies the same difference equation as E_t. Therefore,

$$Y_{t+3} = (2b + 3c)Y_{t+2} - (5c + b)Y_{t+1} + 2cY_t \quad ,$$

which is the same as above. We have used the first method of deriving this equation because it is conceptually the simplest and perhaps most clearly shows the economics of our problem. From now on, however, we shall try to use the second and easier method in deriving our growth equations.

Section 5.2. The Nature of Growth under Simple Additive Expectations

Equation 5.1.5 is a third order linear difference equation. Since its characteristic equation admits the possibility of three different roots, three different initial values of income will be necessary to specify exactly the path of income. Therefore, if we were given the values of b, c, and the three initial conditions, it would be easy to solve equation 5.1.5 and determine how our economy grows. Even if b, c, and the initial conditions are not given explicitly, however, it is still possible to say a great deal about the possible types of growth which occur as a result of equation 5.1.5. In order to do this, let us prove the following two extremely important theorems.

Theorem 5.2.1: The characteristic equation of 5.1.5,

$$(\lambda)^3 - (3c + 2b)(\lambda)^2 + (5c + b)\lambda - 2c = 0, \qquad 5.1.5a$$

always has one real root between 0 and 1. For b and c "low", the other two roots will be complex conjugates whose modulus is greater than 1. As b and/or c increase, the other two roots, at some point will become real, equal, and greater than 1. As b and c increase still further, one of these roots increases to $+\infty$, and the other decreases to 0.

Theorem 5.2.2: If in every period, we increase the right hand side of equation 5.1.5 (or any nth order linear difference equation) by some positive amount, the solution to 5.1.5 will become greater than before, and for given initial conditions will be more likely to explode.

Proof of Theorem 5.2.1: First, we prove that equation 5.1.5 always has a real root between 0 and 1. If we substitute the value $\lambda = 1$ into the left hand side of 5.1.5a, we obtain

$$1 - (2b + 3c) + (5c + b) - 2c = 1 - b > 0.$$

If we do the same with $\lambda = 0$, we get

$$-2c < 0 .$$

Since the value of the left hand side of equation 5.1.5a is greater than 0 for $\lambda = 1$, and less than 0 for $\lambda = 0$, there must always be 1 or 3 roots of 5.1.5a between 0 and 1. But there cannot be 3. For, if there were, $\lambda_1\lambda_2\lambda_3$ would be < 1, which is impossible, since $\lambda_1\lambda_2\lambda_3 = 2c > 1$.

Let us now define the left hand side of 5.1.5a as the characteristic function of our problem. For $\lambda < 0$, this function is always less than 0. It also crosses the λ axis between 0 and 1, and goes to ∞ as λ goes to ∞. The graph of this function must therefore look like (1), (2), or (3) below.

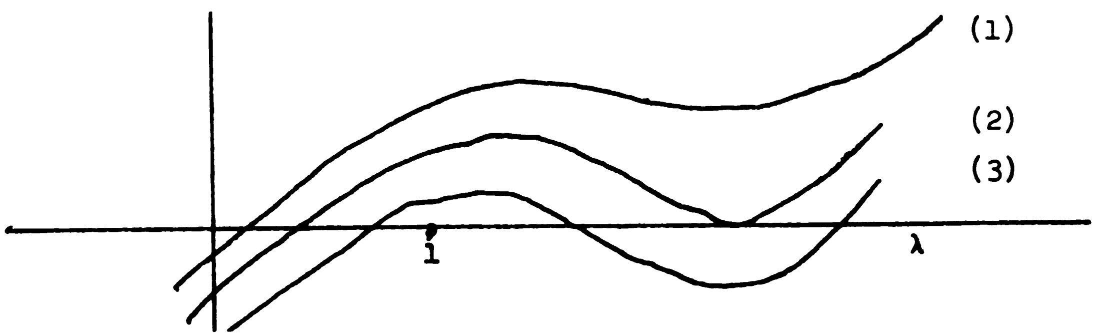

Figure 1. The Characteristic Graph of Equation 5.1.5

We now wish to investigate how the graph of this characteristic function changes as we increase or decrease b and c. Suppose we change b by Δb. Our new characteristic function is

$$\lambda^3 - (2(b + \Delta b) + 3c)\lambda^2 + (5c + b + \Delta b)\lambda - 2c \quad . \qquad 5.2.1$$

Figure 2. Effects of Positive Δb, Δc, on the Characteristic Graph of Equation 5.1.5

If we subtract our old characteristic function from 5.2.1, we obtain

$$-2\Delta b(\lambda)^2 + \Delta b\lambda \quad . \qquad 5.2.2$$

Similarly, if we change c by Δc, and subtract the old characteristic function from the new, we obtain

$$\Delta c(-3(\lambda)^2 + 5(\lambda) - 2),$$

which equals

$$\Delta c(-3(\lambda)^2 + 3\lambda - 3\lambda + 5\lambda - 2 + 2\lambda - 2\lambda) \qquad 5.2.3$$

$$\Delta c(-3(\lambda)^2 + 3\lambda - 2 + 2\lambda)$$

$$\Delta c(-3(\lambda)(\lambda - 1) + 2(\lambda - 1)) \ .$$

Expressions 5.2.2 and 5.2.3 are both negative for $\lambda > 1$ and Δc, Δb positive. Thus, as we increase b and c, the characteristic function for $\lambda > 1$ decreases. Graphically, this implies that

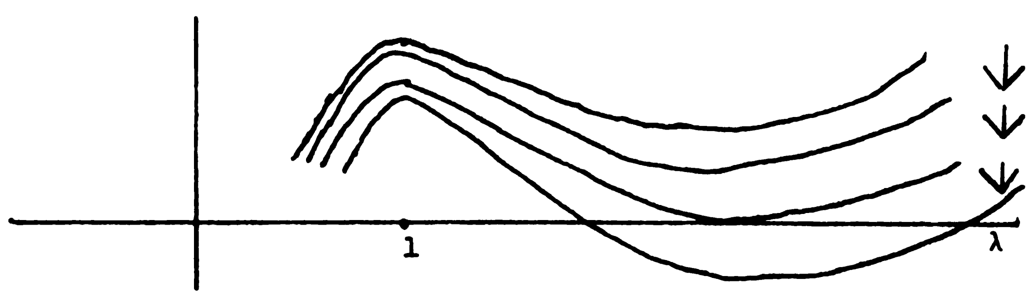

Figure 2.

As b and c increase, the graph of the characteristic function moves down. At some values of b and c, the graph becomes tangent to the axis. When this happens, the roots are real and equal (also greater than 1). As we increase b and c still further, the graph intersects the λ axis in two places, and two distinct real roots exist. It is clear that the upper and lower roots are increasing and decreasing respectively. This completes the proof.

Proof of Theorem 5.2.2: Suppose that we have a set of initial conditions Y_2, Y_1, Y_o, which explodes for equation 5.1.5. Increase the right hand side of equation 5.1.5 by some positive amount. Then $Y_3' > Y_3$. By assumption, Y_3, Y_2, Y_1 explodes. Certainly, therefore, Y_3', Y_2, Y_1 will also explode for equation 5.1.5. Now continue on in this fashion in every period. The new solution is greater than the old solution and is bound to explode, if the old one explodes. Furthermore, if we have initial conditions which do not explode, it may be that, with the addition of our positive terms, these initial conditions will give a solution which does explode. Thus, the likelihood of explosion is greater if we increase the right hand side of 5.1.5 every period by some positive amount.

We now have all the necessary mathematical apparatus for a discussion of growth, according to equation 5.1.5. First, as in the Harrod model, the only possible equilibrium value of income is zero. Furthermore, when b and c are "low", income will contract to this value no matter what the initial conditions of our problem.

The latter statement obtains because for "low" b and c, our solution may be written in the form

$$Y_t = R^t(A_o\cos\theta t + B_o\sin t\theta) + C_o(\lambda_1)^t \quad , \qquad 5.2.4$$

where $|R| > 1$, $\lambda_1 < 1$, and A_o, B_o, and C_o depend on initial conditions. (This statement is, of course, a direct consequence of

Theorem 5.2.1.) Sooner or later this oscillatory solution must cause income to decline. This, in turn, causes investment to be negative, according to equation 5.1.3. We, therefore, cut off our solution at this point by letting $I_t = 0$. Our equation of growth becomes

$$Y_t = bY_{t-1} + (bY_{t-1} - bY_{t-2}) \quad , \qquad 5.2.5$$

and income converges to zero, since equation 5.2.5 has a stable equilibrium at this point.

The fact that income cannot explode, when b and c are low, in equation 5.1.5, is of course very similar to the fact that income cannot explode, when w is low, in equation 4.1.3. In both cases, moreover, when income contracts, production exceeds expenditures and capital is overly sufficient each period. The latter statement can be proved by noting that for equation 5.1.5, $E_t = bY_t$ in the "cut off" region. Since $b < 1$, $E_t < Y_t$. Similarly, if Y_t is declining, and $K_{t \text{ on hand}}$ is fixed, K_t must be overly sufficient. Thus, equation 5.1.5, for low values of b and c, has exactly the same type of growth as equation 4.1.3 for low values of w.

As b and c become higher, the roots of equation 5.1.5 become positive. The solution to equation 5.1.5 may now be written as

$$Y_t = A_o(\lambda_1)^t + B_o(\lambda_2)^t + C_o(\lambda_3)^t \quad , \qquad 5.2.6$$

where λ_1 is less than 1, λ_2 and λ_3 are greater than 1, and A_o, B_o, and C_o depend on initial conditions. This solution will explode or contract depending on initial conditions. In general, we can say that as b and c increase, the chances for explosion, under given initial conditions, become greater. For if we increase b and/or c, in equation 5.1.5, we are either adding a term

$$\Delta b(2Y_{t+2} - Y_{t+1}) ,$$

or a term

$$\Delta c(3Y_{t+2} - 5Y_{t+1} + 2Y_t) = \Delta c(2I_{t-1} - I_{t-2}) ,$$

or both, to equation 5.1.5. But so long as equation 5.1.5 is valid, it must be that $Y_{t+3}(=(2Y_{t+2} - Y_{t-1}))$ and I_{t+3} produced $(=(2I_{t-1} - I_{t-2}))$, are positive. Therefore, when we increase b and/or c, we add a positive term to equation 5.1.5 in each period. By Theorem 5.2.2, income is more likely to explode. Also, the solution, given by 5.2.6, will explode faster, as b and/or c are increased. This follows either by Theorem 5.2.1 which states that the highest root of 5.1.5a increases as b and/or c increase, or by Theorem 5.2.2.

Finally, if income does explode, then sales will exceed production and capital will be insufficient each period. This statement may be proved as follows. Consider the equation

$$Y_t = 2E_{t-1} - E_{t-2} \quad .$$

Let λ_3 be the highest root of 5.1.5. Since E_t also satisfies 5.1.5, we have that when explosion occurs,

$$E_t \to C_o'(\lambda_3)^t = C_o'(\lambda_3)^{t-2}(\lambda_3)^2$$

and

$$Y_t \to 2C_o'(\lambda_3)^{t-1} - C_o'(\lambda_3)^{t-2} = C_o'(\lambda_3)^{t-2}(2\lambda_3 - 1)$$

But $(\lambda_3)^2 > 2(\lambda_3) - 1$, which proves that eventually sales must exceed production. Similarly, capital must become insufficient each period. For from equation 5.1.2 we have

$$K_{t+1 \text{ desired}} = c(2Y_t - E_{t-1}) = c(2(2E_{t-1} - E_{t-2}) - E_{t-2})$$

$$= c(3E_{t-1} - 2E_{t-2}) \ .$$

But

$$K_{t+1 \text{ needed}} = c(Y_{t+1}) = c(2E_t - E_{t-1}) \ .$$

Let λ_3 be the highest root of equation 5.1.5a. Then

$$K_{t+1 \text{ desired}} = K_{t+1 \text{ on hand}} \to c(3C_o'(\lambda_3)^{t-1} - 2C_o'(\lambda_3)^{t-2}$$

$$\to cC_o'(\lambda_3)^{t-2}(3\lambda_3 - 2)$$

and

$$K_{t+1 \text{ needed}} \to c(2C_o'(\lambda_3)^{t} - C_o'(\lambda_3)^{t-1}$$

$$\to cC_o'(\lambda_3)^{t-2}(2(\lambda_3)^2 - \lambda_3) \ .$$

Since

$$2(\lambda_3)^2 - \lambda_3 > 3(\lambda_3) - 2 \ ,$$

it must be that

$$K_{t+1 \text{ needed}} > K_{t+1 \text{ on hand}}$$

as $t \to \infty$.

We can now see that the two models described by equations 4.1.3 and 5.1.5 are in many respects identical. Both of these models are unstable; the only "equilibrium" which can be attained is at zero income. For "low" values of the parameters, the models are never able to diverge. For "higher" values of the parameters, they may contract or diverge in Harrod-like fashion, depending on the initial conditions. For divergence, production and capital will be insufficient in both models; for contraction, production and capital will be more than desired. In only one respect do the two models differ. Whereas in the multiplicative model of equation 4.1.3, it is possible for income to expand along a warranted path, where sales and capital are always as expected, in the additive model of equation 5.1.5, no such expansion is possible.

For if there were the possibility of warranted growth in the additive model, the solution to equation 5.1.5 would have to be of the form,

$$Y_t = A_o + B_o t \qquad \text{or} \qquad E_t = A_o' + B_o' t \quad , \qquad 5.2.7$$

where A_o and B_o depend on initial conditions. Clearly, only with a solution of this form would sales increase by a constant amount each period, in accordance with our expectation 5.1.1. But a solution of the form 5.2.7 implies a double root of 1 as a solution to the characteristic equation of 5.1.5. 1, however, cannot be a solution or double solution of equation 5.1.5 since, as we have seen, $\lambda = 1$ implies that the left hand side of 5.1.5a equals $1-b$, which for $b < 1$ does not equal zero.

Section 5.3. A General Additive Model; Properties of this Model

We have seen in the preceding two sections that the simple additive model, while displaying the instability of Harrod's model, does not give a warranted path along which expectations are realized. As such, it might appear that the form of the expectation function (multiplicative versus additive) is critical in getting a Harrod-like equilibrium growth solution. Fortunately, this is not so. By changing our model of the previous section slightly, we can make it more general and also arrive at an equilibrium growth solution.

Suppose we change the assumptions of the simple additive model by saying that businessmen feel that sales in some period $t+1$ will equal sales in the preceding period t, plus a constant A times the change in sales of the preceding two periods, t and $t-1$. (For $A = 1$, we have the simple additive model of the previous section.) With our general additive expectation, we may now write our equations of growth. First, we have

$$Y_t = E_{t-1} + A(E_{t-1} - E_{t-2}) \quad . \qquad 5.3.1$$

Also, proceding as before, we have

$$K_{t+1 \text{ desired}} = c(Y_t + A(Y_t - E_{t-1})) \qquad 5.3.2$$

$$= c((E_{t-1} + A(E_{t-1} - E_{t-2}))$$

$$+ A(E_{t-1} + A(E_{t-1} - E_{t-2}) - E_{t-1})$$

$$= c(E_{t-1} + AE_{t-1} - AE_{t-2} + AE_{t-1} + A^2E_{t-1}$$

$$- A^2E_{t-2} - AE_{t-1})$$

$$= c(E_{t-1}(1 + A + A^2) - E_{t-2}(A + A^2)). \qquad 5.3.3$$

The above implies that

$$I_t = c((E_{t-1}(1+A+A^2)-E_{t-2}(A+A^2))-c(E_{t-2}(1+A+A^2)-E_{t-3}(A+A^2))$$

$$= cE_{t-1}(1+A+A^2)-cE_{t-2}(A+A^2+1+A+A^2)+cE_{t-3}(A+A^2)$$

$$= cE_{t-1}(1+A+A^2)-cE_{t-2}(1+2A+2A^2)+cE_{t-3}(A+A^2) \ . \qquad 5.3.4$$

Now Y_t is a linear combination of lagged values of E_t, and, hence, satisfies the same difference equation as E_t. Let us therefore find the equation of growth for E_t, and having done this, simply plug in Y_t. To do this, let us start with the identity

$$E_t = bY_t + I_t \ . \qquad 5.3.5$$

By plugging 5.3.1 and 5.3.4 into 5.3.5, we have

$$E_t = b(E_{t-1}+A(E_{t-1}-E_{t-2})+cE_{t-1}(1+A+A^2)-cE_{t-2}(1+2A+2A^2)$$

$$+cE_{t-3}(A+A^2),$$

or

$$E_t = (b+c(1+A+A^2)+bA)E_{t-1}-(bA+c(1+2A+2A^2)E_{t-2}+c(A+A^2)E_{t-3} \ . \qquad 5.3.6$$

(The reader should note that equation 5.3.6 reduces to equation 5.1.5 when $A = 1$.) Since Y_t satisfies the same difference equation as E_t, we also have

$$Y_t = (b+c(1+A+A^2)+bA)Y_{t-1}-(bA+c(1+2A+2A^2Y_{t-2}+c(A+A^2)Y_{t-3} \ . \qquad 5.3.6a$$

We now wish to ask the following question. Is there a solution to equation 5.3.6 such that $E_t = AE_{t-1}$? For if there is, then

$$Y_t = E_{t-1} + A(E_{t-1} - E_{t-2})$$

$$= E_{t-1} + AE_{t-1} - AE_{t-2}$$

$$= AE_{t-1}$$

$$= E_t$$

and markets are cleared. Also,

$$K_{t+1 \text{ desired}} = K_{t+1 \text{ on hand}} = cE_{t-1}(1+A+A^2) - E_{t-1}(1+A)$$

$$= cE_{t-1}(A^2),$$

and

$$K_{t+1 \text{ needed}} = cY_{t+1}$$

$$= c(E_t + A(E_t - E_{t-1})$$

$$= c(AE_{t-1} + AE_t - AE_{t-1})$$

$$= c(AE_t)$$

$$= cA^2E_{t-1} .$$

Thus,

$$K_{t+1 \text{ needed}} = K_{t+1 \text{ on hand}} ,$$

which implies full employment of capital.

The answer to our question is yes. We may also show that $E_t = AE_{t-1}$ only when $A = 1 + \frac{1-b}{c}$, $E_2 = AE_1$, and $E_1 = AE_o$. The proof is very simple. If A is one of the roots of 5.3.6 then,

$$A^3 = (b+c(1+A+A^2)+bA)A^2-(bA+c(1+2A+2A^2))A+c(A+A^2)$$

$$A^3 = bA^2+cA^2+cA^3+cA^4+bA^3-bA^2-cA-2cA^2-2cA^3+cA+cA^2$$

$$A^3 = (b+c-b-2c+c)A^2+(c+b-2c)A^3+cA^4-(-c+c)A ,$$

and

$$A^4c = A^3(1-b+c)$$

$$A = 1 + \frac{1-b}{c} . \qquad 5.3.7$$

If $A = 1 + \frac{1-b}{c}$, our solution may now be written in the form

$$E_t = A_o(\lambda_1)^t + B_o(\lambda_2)^t + C_o(1 + \frac{1-b}{c})^t .$$

If, furthermore, $E_2 = (1 + \frac{1-b}{c})E_1$ and $E_1 = (1 + \frac{1-b}{c})E_o$, we may write

$$A_o + B_o + C_o = E_o , \qquad 5.3.8$$

$$A_o(\lambda_1)+B_o(\lambda_2)+C_o(1 + \frac{1-b}{c}) = E_o(1 + \frac{1-b}{c}) , \qquad 5.3.9$$

$$A_o(\lambda_1)^2+B_o(\lambda_2)^2+C_o(1 + \frac{1-b}{c})^2 = E_o(1 + \frac{1-b}{c})^2 . \qquad 5.3.10$$

The solution to equations 5.3.8, 5.3.9, and 5.3.10 can easily be shown to be $A_o = 0$, $B_o = 0$ and $C_o = E_o$. Thus, when $A = 1 + \frac{1-b}{c}$ $E_1 = (1 + \frac{1-b}{c})E_o$ and $E_2 = E_1(1 + \frac{1-b}{c})$, our solution to equation 5.3.6 may be written in the form

$$E_t = E_o(1 + \frac{1-b}{c})^t \quad . \qquad 5.3.11$$

As shown above, this is a growth solution along which markets are cleared and capital is fully employed.

We have now derived the fundamental equation for general additive expectations growth, and seen that a general additive model allows, under the right conditions, a warranted growth solution. The rest of this section will be devoted to showing that in addition to this property, the general additive model possesses practically all the other properties of the constant multiplicative expectations model of Chapter 4. We begin as usual with several mathematical theorems.

Theorem 5.3.1: The characteristic equation of 4.1.3

$$\lambda^2 - (b(1+w)+c(1+w)^2)\ \lambda+c(1+w)^2 = 0 \qquad 4.1.3a$$

either has two positive roots or two complex roots whose modulus is greater than 1. For b, c, and w "low", the roots will be complex. As b, c, and/or w increase, the roots will at some point become real, equal, and greater than 1. As we increase b still further, one root will increase to $+\infty$, and the other will decrease towards zero. As we increase c and/or w one root will increase to $+\infty$, and the other will either decrease, or be between 0 and 1.

Theorem 5.3.2: The characteristic equation of 5.3.6

$$\lambda^3-(b+c(1+A+A^2)+bA)\lambda^2-(bA+c(1+2A+2A^2))\lambda+c(A+A^2) = 0 \qquad 5.3.6a$$

always has one real root between 0 and 1. For b, c, and A low the other two roots will be complex conjugates or real and less than 1. As b, c and/or A increase, the other two roots will at some point become real, equal, and greater than 1. As b, c, and/or A increase still further, one of these roots increases to infinity and the other decreases to 1.

<u>Proof of Theorem 5.3.1</u>: Define the left hand side of equation 4.1.3a to be the characteristic function of equation 4.1.3. For $\lambda < 0$, this function is greater than 0. Since this function goes to infinity as λ goes to infinity, the graph of this function must look like

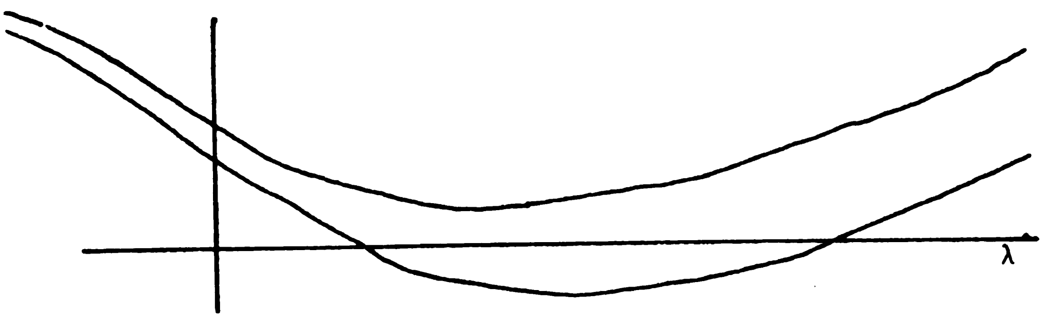

Figure 3. The Characteristic Graph of Equation 4.1.3

Consider now what happens if we change b by Δb. Our "new" characteristic function is

$$\lambda^2-(b+\Delta b)(1+w)+c(1+w)^2)\lambda+c(1+w)^2 = 0 \; . \qquad 5.3.12$$

If we subtract our old characteristic function (equation 4.1.3a) from our new characteristic function, we obtain

$$-\Delta b(1+w) \; . \qquad 5.3.13$$

If we carry out the same procedure for c and w, we obtain

Figure 4. Effects of Positive Δb, Δc, Δw, on the Characteristic Graph of Equation 4.1.3

$$-\Delta c(1+w)^2\lambda+\Delta c(1+w)^2 \text{ ,} \qquad 5.3.14$$

and

$$-\Delta wb\lambda-\Delta w2c(1+w)\lambda+\Delta w(2c(1+w))-c(\Delta w)^2\lambda+c(\Delta w)^2 \text{ ,} \qquad 5.3.15$$

respectively. For $\lambda > 1$, and Δb, Δc, Δw positive, all these terms are negative. Hence, in the region $\lambda > 1$, the characteristic graph of 5.3.6a must move as shown below, when we increase b, c, or w

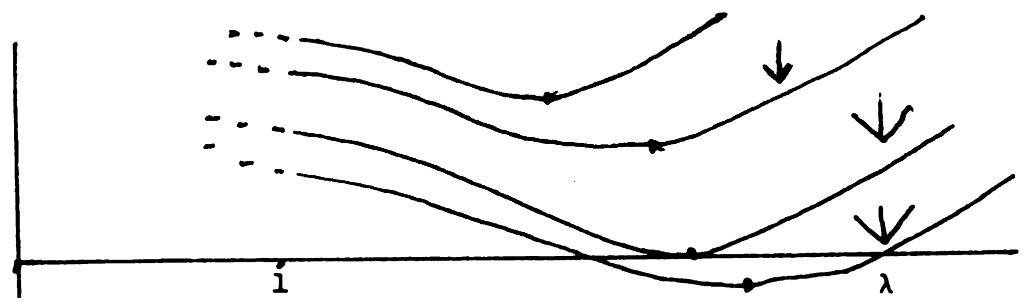

Figure 4.

As we change b, c, or w, the roots change from complex to real equal and greater than 1. If we increase b still further, the graph must decrease and intersect the λ axis in two places. Thus, the upper root increases and the lower root decreases for positive changes in b. The same is true of changes in c and w, except when $b(1+w) > 1$. In this case, our lower root is less than 1 (see pages 82 and 83), and the lower root need not decrease for positive changes in c and/or w. The root, however, will always remain between 0 and 1.

Proof of Theorem 5.3.2: First, it is clear that there exists a root of 5.3.6a between 0 and 1, since the values $\lambda = 0$ and

Figure 5. Effects of Positive Δb, Δc, ΔA, on the Characteristic Graph of Equation 5.3.6

and $\lambda = 1$ plugged into the left hand side of 5.3.6a give $(1-b)$, a positive quantity, and $-c(A+A^2)$ a negative quantity, respectively. Furthermore, if we now increase b, c, and/or A, by Δb, Δc, ΔA respectively, we obtain as the differences between the old and the new characteristic functions,

$$-\Delta b(1+A)\lambda^2-\Delta bA\lambda \quad , \qquad 5.3.16$$

$$-\Delta c(1+A+A^2)\lambda^2+\Delta c(1+2A+2A^2)\lambda-\Delta c(A+A^2)$$

$$= -\Delta c(\lambda^2-\lambda)-\Delta c(A+A^2)(\lambda^2-2\lambda+1) \quad , \qquad 5.3.17$$

$$-\Delta A(b\lambda^2)+\Delta A(b\lambda)-\Delta(A+A^2)(\lambda^2-2\lambda+1) \quad . \qquad 5.3.18$$

All of these terms are negative for $\lambda > 1$ and positive changes. Hence, the characteristic graph must move in the $\lambda > 1$ region as,

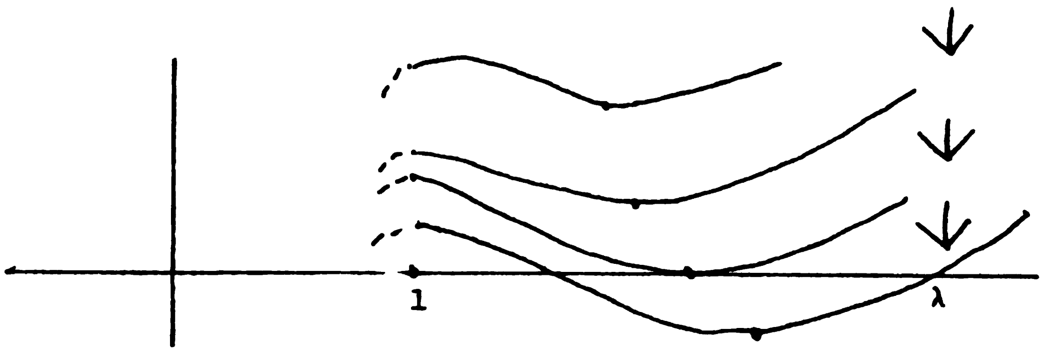

Figure 5.

For "low" b, c and/or w the two roots will be complex. As we increase b, c, and/or w, however, the two roots will at some point become real equal, and greater than 1. Further increases in these parameters will cause the upper root to increase and the lower root to decrease. The lower root however, cannot decrease below one, since there cannot be two roots between 0 and 1, when the values $\lambda = 1$ and $\lambda = 0$, plugged into the characteristic function give positive and negative results respectively.

With the help of Theorems 5.3.1, 5.3.2, 5.2.2, which applies to <u>any</u> nth order linear difference equation, and some of our previous results, we are now able to describe and compare the types of growth possible under equations 4.1.3 and 5.3.6. We shall show that the two models described by these equations are identical in many respects. In particular, we shall see that changing the form of the expectation from multiplicative to additive does not change the Harrod-like nature of growth. Our conclusions may be listed as follows:

1) Both models are "dynamic" in that they do not reach an equilibrium other than 0. This equilibrium is primarily the result of the fact that income cannot be physically less than 0; the "equilibrium" can therefore, be looked upon as a constraint.

2) Both models give warranted growth when the coefficient of expectations is $(1 + \frac{1-b}{c})$ and the initial conditions are "correct". Along this warranted path, markets are cleared and capital is just sufficient.

3) When the values of the parameters of our models are "low", income must contract under all circumstances. This occurs because when the values of the parameters of our models are low, the solutions of equations 4.1.3 and 5.3.6 can be written respectively as

$$Y_t = R^t(A_o\cos\theta t + B_o\sin\theta t) \ ,$$

where A_o and B_o depend upon initial conditions, and either

$$Y_t = R^t(A_o\cos\theta t + B_o\sin\theta t) + C_o(\lambda_1)^t \ ,$$

or

$$Y_t = A_o(\lambda_1)^t + B_o(\lambda_2)^t + C_o(\lambda_3)^t ,$$

where A_o, B_o, and C_o depend on initial conditions and λ_1, λ_2, λ_3 are less than zero. Both solutions must eventually cause income to decline, and thus investment will become zero. When this occurs, we cut off our solution and write

$$Y_t = b(1+w)Y_{t-1}$$

and

$$Y_t = (b+bA)Y_{t-1} - bAY_{t-2}$$

for equations 4.1.3 and 5.3.6. For low values of b, w, and A, both models will clearly contract. Furthermore, they do so in Harrod-like fashion with production exceeding sales and capital overly sufficient. The latter two results obtain, since $E_t = bY_t < Y_t$ and K_t is fixed.

4) As the parameters of both our models increase, the economies described by equation 4.1.3 and 5.3.6 may explode under suitable initial conditions. This is clearly true, since the roots of our equations become real and greater than 1 as we increase b, c, w, and/or A. Furthermore, as we increase these parameters, both models, under very general circumstances, are more likely to explode.

The proof of the latter statement is fairly simple. Increasing b, c, and/or w in equation 4.1.3 is equivalent to adding the terms

$$\Delta b(1+w)Y_{t-1},$$

$$\Delta c(1+w)^2(Y_{t-1} - Y_{t-2}),$$

and

$$\Delta w(bY_{t-1}) + \Delta(1+w)^2(Y_{t-1} - Y_{t-2})$$

to the right hand side of equation 4.1.3. For Y_t and $I_t = c(1+w)^2 (Y_{t-1} - Y_{t-2})$ positive, all these terms are positive. Thus, by Theorem 5.2.2, the multiplicative model is always more likely to explode as we increase b, c, and/or w. Similarly, increasing b, c, and/or A in equation 5.3.6 is equivalent to increasing the right hand side of this equation by

$$\Delta bE_{t-1}+\Delta bA(E_{t-1} - E_{t-2}) = \Delta bY_t,$$

$$\Delta c((1+A+A^2)E_{t-1}-(1+2A+2A^2)E_{t-2}+(A+A^2)E_{t-3}) = I_t\Delta c ,$$

and

$$\Delta Ab(E_{t-1}-E_{t-2})+\Delta(A+A^2)c(E_{t-1}-2E_{t-2}+E_{t-3}) .$$

The first two of these terms will be positive whenever Y_t and I_t are positive. Thus, increasing b and/or c always increases the likelihood of explosion. The third term, however, will always be positive only when $(E_{t-1}-E_{t-2}) > (E_{t-2}-E_{t-3})$. Thus decreasing the coefficient of expectations in equation 5.3.6 may, under certain conditions, increase the likelihood of explosion!

This last result may at first seem very strange. It has however a simple explanation, which is a direct consequence of our additive expectation. According to equation 5.3.4, I_t may be written as

$$I_t = c(E_{t-1}-E_{t-2})+c(A+A^2)((E_{t-1}-E_{t-2})-(E_{t-2}-E_{t-3})) .$$

For $(E_{t-1}-E_{t-2})<(E_{t-2}-E_{t-3})$, I_t decreases as A increases. It may even become 0 for large enough A. The reason for this is that, in our additive model, expectations are based on the difference in the previous two period's sales. In period t-1 businessmen expected an increase in sales based on the term $E_{t-2}-E_{t-3}$, and invested accordingly. In period t, however, they realize that these expectations were not fulfilled. Since $(E_{t-1}-E_{t-2})<(E_{t-2}-E_{t-3})$, they are forced to the conclusion that they have overinvested. They therefore, cut back on investment in this period. The greater their past (and present expectations), the more they are forced to cut back. This decrease in investment spending may offset the other increase in next period's production caused by the increased expectation, and thus, decrease the likelihood of explosion.

We may, therefore, repeat our earlier statement that increasing the parameters of our models will, for given initial conditions, usually increase the likelihood of explosion. This will always occur in both models for increases in b and c. It will also occur in the multiplicative model, for increases in the coefficient of expectations, but need not occur for similar increases in the additive model, when $(E_{t-1}-E_{t-2})<(E_{t-2}-E_{t-3})$.

5) When expectations are very high, it will be impossible for both models to contract as long as $Y_t > Y_{t-1}$. We have already seen this for the multiplicative model (pages 82 and 83). Suppose now that $I_t = 0$ in the additive model. This is the lowest possible value of I_t in this model, and by Theorem 5.2.2, the least likely to cause explosion. Then,

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$$Y_t = bY_t + bA(Y_t - Y_{t-1}) \ . \qquad 5.3.19$$

It is clear that, depending on initial conditions, there always exists a high enough value of A, such that income explodes according to equation 5.3.19. Thus, in both models, if expectations are high enough, income must explode.

6) In both the additive and multiplicative models, if income explodes, capital and production must eventually become insufficient. We have already proved this assertion for the constant multiplicative model and for the special $(A = 1)$ additive model. Let us now prove it for the general additive model.

First we know that

$$Y_t = E_{t-1} + A(E_{t-1} - E_{t-2}),$$

which for explosion, goes to

$$A_o(\lambda_3)^{t-1} + AA_o((\lambda_3)^{t-1} - (\lambda_3)^{t-2})$$

$$A_o(\lambda_3)^{t-2}(\lambda_3(1+A) - A),$$

as $t \to \infty$, where A_o is a constant and λ_3 is the highest root of 5.3.6. But E_t approaches

$$A_o(\lambda_3)^t = A_o(\lambda_3)^{t-2}(\lambda_3)^2 \ .$$

We wish to show that

$$(\lambda_3)^2 > (\lambda_3(1+A) - A)$$

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$$(\lambda_3)^2 - (\lambda_3)(1+A) + A > 0$$

$$(\lambda_3 - A)(\lambda_3 - 1) > 0 .$$

Clearly $\lambda_3 > 1$. Also, $\lambda_3 > A$, since if $\lambda_3 < A$, $\lambda_2 < A$, and $\lambda_1 < 1$,

$$\lambda_1\lambda_2\lambda_3 < A^2 .$$

But

$$\lambda_1\lambda_2\lambda_3 = c(A+A^2) > A^2 .$$

Thus, the inequality holds and sales exceed production as $t \to \infty$.

We also know that

$$K_{t+1 \text{ desired}} = K_{t+1 \text{ on hand}} = cE_{t-1}(1+A+A^2)-cE_{t-2}(A+A^2)$$

$$c(A_o(\lambda_3)^{t-1}(1+A+A^2)-A_o(\lambda_3)^{t-2}(A+A^2)$$

$$cA_o(\lambda_3)^{t-2}(\lambda_3(1+A+A^2)-(A+A^2)) .$$

$$K_{t+1 \text{ needed}} = cY_{t+1} = c(E_t + A(E_t - E_{t-1}))$$

$$= cA_o((\lambda_3)^t + A((\lambda_3)^t - (\lambda_3)^{t-1}))$$

$$= cA_o(\lambda_3)^{t-2}((\lambda_3)^2 + A((\lambda_3)^2 - (\lambda_3)) .$$

We wish to show that

$$(\lambda_3)^2 + A((\lambda_3)^2 - (\lambda_3)) > (\lambda_3)(1+A+A^2) - (A+A^2) .$$

This equation is equivalent to

$$(1+A)(\lambda_3)^2 - (1+2A+A^2)\lambda_3 + (A+A^2) > 0$$

or

$$(\lambda_3)^2 - (1+A)\lambda_3 + A > 0$$

$$(\lambda_3 - A)(\lambda_3 - 1) > 0 ,$$

which, as shown above, is true. Therefore, capital on hand in period $t+1$ must eventually become insufficient when income explodes.

7) We saw in the constant multiplicative model that there exist situations in which a constant rate of growth can be maintained, even though expectations are not fulfilled. The same is true in the additive model. For

$$Y_t = A_o(\lambda_1)^t + B_o(\lambda_2)^t + C_o(\lambda_3)^t,$$

where $\lambda_1 < 1$ and λ_2, $\lambda_3 < 1$ in this model. If A_o, and either B_o or C_o are zero, because of initial conditions, constant rate growth will occur. Since λ_2 or $\lambda_3 \neq A$, except when $A = 1 + \frac{1-b}{c}$, expectations need not be fulfilled with such growth.

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CHAPTER 6

Section 6.1 Types of Expectations

We can now see that the two models considered in the previous chapters are almost identical in their properties. This would tend to suggest that any type of expectation, when plugged into our two phase model, would also give results similar to those of our previous models. Unfortunately, in order to verify this, we would have to consider an infinite number of expectations. A better approach might be to try to classify expectations and see if we can say anything about how the class of expectations affects our model.

In order to do this, it seems desirable to look more carefully at the "types" of expectations which economists have considered in recent years. Turnovsky has summarized these "types" very nicely in his article on "Stochastic Stability of Short Run Market Equilibria."[1] In this article, he indicates that there are five types of expectations: 1) static, 2) extrapolative, 3) adaptive, 4) weighted and 5) rational.

The first two are the simplest. Static expectations are expectations which do not change over time. They may be written

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[1] S. J. Turnovsky, "Stochastic Stability of Short Run Market Equilibria under Variations in Supply," Quarterly Journal of Economics (November 1968) pp. 666-681, especially 668-671.

in the form $w_t = w_{t-1}$ or $w_t = w_o$, a constant. Extrapolative expectations are expectations of the form

$$E_{t\ exp} = E_{t-1} + (E_{t-1} - E_{t-2}) \ .$$

These expectations were first considered by Metzler;[2] they have also been considered by Goodwin[3] and others. Since we have considered both these types of expectations already in Chapters 4 and 5 respectively, we shall not consider them further.

Of the remaining types, adaptive expectations are expectations in which the previous period's forecast is changed by an amount proportional to the most recently observed forecast error. In the terminology of this dissertation, we may write this expectation in the form

$$(1+w)_t = (1+w)_{t-1} + f\left(\frac{E_{t-1}}{E_{t-2}} - (1+w)_{t-1}\right)^{4}$$

where f( ) is sign preserving. These expectations were first considered in the late fifties, the names most frequently associated

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[2] L. Metzler, loc. cit.

[3] R. M. Goodwin, "Dynamical Coupling with Especial Reference to Markets Having Production Lags," Econometrica (July 1947) pp. 181-203, especially pp. 191-193.

[4] The reader may find this equation slightly unusual in that it is the expected rate of growth of sales, rather than sales, which is the variable being considered here. Thus, the expectation itself is subject to adaptive expectations.

with such expectations being Friedman,[5] Cagan,[6] Nerlove,[7] and Nerlove and Arrow.[8] Essentially, they imply that businessmen will somehow try to strike a balance between last period's realized and expected changes in sales. Thus, if $f(\ ) = .2(\ )$, $\frac{E_{t-1}}{E_{t-2}} = 1.2$, and $w_{t-1} = .1$, the expected rate of growth in period t would be $w_t = .12$.

Weighted expectations are expectations in which

$$w_t = \theta \frac{E_{t-1}}{E_{t-2}} + (1-\theta) \frac{E_t}{E_{t-1}} ,$$

where, $0 < \theta < 1$. Such expectations have been considered by Lovell,[9] Devletoglou,[10] and Gruenberg and Modigliani,[11] among others. If $\theta = 1$, weighted expectations reduce to a special case of adaptive expectations, where $f(\ ) = 1(\ )$. If $\theta = 0$, they are perfect

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[5] M. Friedman, A Theory of the Consumption Function, (Princeton University, 1957).

[6] P. Cagan, "The Monetary Dynamics of Hyper-Inflation," in Studies in the Quantity Theory of Money, ed. M. Friedman (University of Chicago Press, 1956).

[7] M. Nerlove, "Adaptive Expectations and Cobweb Phenomena," Quarterly Journal of Economics (May 1958) pp. 227-240. Also "Reply to Mills" Quarterly Journal of Economics (May 1961) pp. 335-338.

[8] M. Nerlove and K. J. Arrow, "A Note on Expectations and Stability," Econometrica (April 1958), pp. 297-305.

[9] M. Lovell, "Manufacturers' Inventories, Sales Expectations, and the Acceleration Principle," Econometrica (July 1961) pp. 293-314.

[10] E. A. Devletoglou, "Correct Public Prediction and the Stability of Equilibrium," Journal of Political Economy (April 1961) pp. 142-161.

[11] E. Gruenberg and F. Modigliani, "The Predictability of Social Events," Journal of Political Economy (December 1956) pp. 465-478.

predictors. From the latter, it is clear that weighted expectations imply some knowledge on the part of firms concerning the future.

Finally rational expectations are expectations whose mean over time equals the mean of the observable they are trying to predict. Stated differently, if businessmen have rational expectations, they will be correct at least on the average in guesses of future growth rates. Rational expectations were first formulated by Muth[12] and Mills.[13]

The question now remains as to which of the last three types of expectations we should use. It is clear that all three types of expectations are suitable for investigation. Nevertheless, we shall consider only one of these types, adaptive expectations, in our growth models. The reason for this choice is that the other two types of expectations either imply some knowledge of the future (weighted expectations) or constrain the expectation to have some desirable property (rational expectations).

In deciding to use adaptive expectations exclusively in our model, we are fully aware of the objections which Muth and Mills have raised against such expectations. We are fully aware that adaptive expectations are autoregressive and make predictions "which are incorrect in a simple and systematic way."[14] We shall note this

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[12] J. F. Muth, "Rational Expectations and the Theory of Price Movements," Econometrica (July 1961), pp. 315-335.

[13] E. S. Mills, Price, Output, and Inventory Policy (Wiley, 1962). Also "The Use of Adaptive Expectations in Stability Analysis: Comment," Quarterly Journal of Economics (May 1961) pp. 330-335.

[14] E. S. Mills, "The Use of Adaptive Expectations in Stability Analysis: Comment," Quarterly Journal of Economics (May 1961) p. 333.

in Section 2. On the other hand, adaptive expectations do not say anything about the future, can be expressed quite simply in mathematical language and do make correct predictions, as we shall see, along the warranted path. Also, we shall see that the irrationality which Muth and Mills were concerned about is precisely the thing which gives us the Harrod instability conditions.

Finally, one more point. Since there are many forms of adaptive expectations, we shall illustrate their use with one case which is a little simpler mathematically than others. This is the case where the adaptive expectation can be written in the form

$$(1+w)_t = (1+w)_{t-1} + 1 \left[\frac{E_{t-1}}{E_{t-2}} - (1+w)_{t-1}\right]$$

or

$$(1+w)_t = \frac{E_{t-1}}{E_{t-2}} .$$

Strictly speaking, we should call this case the case of "instantaneously" adaptive expectations, since equation 6.1.2 states that last period's realized change in sales will be immediately translated into this period's expected change in sales. For simplicity, however, we shall usually refer to equation 6.1.2 as the case of adaptive expectations.

Section 6.2. The Fundamental Equation for Adaptive Expectations

We start by assuming, as stated above, that $w_t = \frac{E_{t-1}}{E_{t-2}}$. With this expectation, we may write

$$Y_t = (1+w_t)(bY_{t-1}+I_{t-1}) = (bY_{t-1}+I_{t-1}) \frac{bY_{t-1}+I_{t-1}}{bY_{t-2}+I_{t-2}} .$$

In addition, we know that in the market phase of period t businessmen feel that sales will increase by

$$\frac{bY_{t-1}+I_{t-1}}{bY_{t-2}+I_{t-2}} - 1 \%$$

over this period's sales, which are expected to be

$$(bY_{t-1}+I_{t-1}) \frac{bY_{t-1}+I_{t-1}}{bY_{t-2}+I_{t-2}} .$$

We may now write

$$K_{t+1 \text{ desired}} = c(bY_{t-1}+I_{t-1}) \frac{bY_{t-1}+I_{t-1}}{bY_{t-2}+I_{t-2}} \frac{bY_{t-1}+I_{t-1}}{bY_{t-2}+I_{t-2}} ,$$

or

$$K_{t+1 \text{ desired}} = c \frac{(bY_{t-1}+I_{t-1})^3}{(bY_{t-2}+I_{t-2})^2} .$$

Similarly, however,

$$K_{t \text{ desired}} = K_{t \text{ on hand}} = c \frac{(bY_{t-2}+I_{t-2})^3}{(bY_{t-3}+I_{t-3})^2} .$$

Thus,

$$I_t = c \frac{(bY_{t-1}+I_{t-1})^3}{(bY_{t-2}+I_{t-2})^2} - c \frac{(bY_{t-2}+I_{t-2})^3}{(bY_{t-3}+I_{t-3})^2} . \qquad 6.1.4$$

We now have two equations 6.1.3 and 6.1.4 in two unknowns. In order to solve these equations it seems much simpler to transform them to expenditures variables and solve for E_t. If we do this, we obtain

$$Y_t = \frac{(E_{t-1})^2}{E_{t-2}} \qquad 6.1.3a$$

and

$$I_t = c \frac{(E_{t-1})^3}{(E_{t-2})^2} - c \frac{(E_{t-2})^3}{(E_{t-3})^2} . \qquad 6.1.4a$$

Making use of the identity $E_t = bY_t + I_t$, we obtain as our equation of growth when expectations are adaptive

$$E_t = b\,\frac{(E_{t-1})^2}{E_{t-2}} + c\,\frac{E(_{t-1})^3}{(E_{t-2})^2} - c\,\frac{(E_{t-2})^3}{(E_{t-3})^2}\ . \qquad 6.1.5$$

(We can also derive equation 6.1.5 in a more intuitive or economic fashion, which may prove insightful. This derivation will also help us a little later. We realize that, in general,

$$K_{t+1\ \text{desired}} = c(1+w)^2E_{t-1}$$

and

$$I_t = c(1+w_t)^2E_{t-1} - c(1+w_{t-1})^2E_{t-2}\ .$$

Thus, making use of the identity $E_t = bY_t + I_t$, and the equation $Y_t = (1+w_t)E_{t-1}$, we have,

$$E_t = b(1+w_t)E_{t-1} + c(1+w_t)^2E_{t-1} - c(1+w_{t-1})^2E_{t-2}\ .$$

The above equation is of exactly the same form as equation 4.1.3. Now, however, w_t is adaptive instead of constant, and since $w_t = \frac{E_{t-1}}{E_{t-2}}$, we obtain

$$E_t = b\,\frac{(E_{t-1})^2}{E_{t-2}} + c\,\frac{(E_{t-1})^3}{(E_{t-2})^2} - c\,\frac{(E_{t-2})^3}{(E_{t-3})^2}\ .) \qquad 6.1.5$$

Equation 6.1.5 is a third order non-linear difference euqation, and it might seem impossible to say anything about it in view of our current knowledge of nonlinear equations. Fortunately, however, this is not so. We can show that, if initial conditions are "correct," constant warranted growth, with clearing of markets and full employment of capital, may occur under equation 6.1.5. Furthermore, we can show that if initial conditions are not suitable for warranted growth, then income may either diverge in Harrod-like fashion, contract in Harrod-like fashion, or return towards the warranted path without ever quite achieving it. We should note that the last statement does not include the possiblity of the constant nonwarranted growth discussed in Chapters 4 and 5.

The first of the above statements is the simplest to prove, and so let us begin here. The existence of a warranted path depends, of course, on whether there exists a solution to equation 6.1.5 such that $\frac{E_t}{E_{t-1}} = w_o$, for all t. For if this were the case, we could write $E_t = A_o(w_o)^t$, and since

$$Y_t = w_o E_{t-1} = w_o A_o (w_o)^{t-1} = A_o (w_o)^t = E_t \ ,$$

sales would always equal production along this path. Similarly, since

$$K_{t+1 \text{ desired}} = K_{t+1 \text{ on hand}} = c\,\frac{(E_{t-1})^3}{(E_{t-2})^2}$$

$$= \frac{cA_o(w_o)^{3t-3}}{(w_o)^{2t-4}} = c(w_o)^{t+1}A_o$$

$$= c(E_{t+1}) = c(Y_{t+1}) = K_{t+1 \text{ needed}} \ ,$$

capital will always be just sufficient along this path. Let us, therefore, plus the equation $E_t = A_o(w_o)^t$ into 6.1.5, to obtain

$$(w_o)^t = \frac{b((w_o)^{t-1})^2}{(w_o)^{t-2}} + c\,\frac{((w_o)^{t-1})^3}{((w_0)^{t-2})^2} - c\,\frac{((w_o)^{t-2})^3}{((w_o)^{t-3})^2}$$

or

$$(w_o)^t = b(w_o)^{2t-2-t+2} + c(w_o)^{3t-3-2t+4} - c(w_o)^{3t-6-2t+6}$$

$$(w_o)^t = b(w_o)^t + c(w_o)^{t+1} - c(w_o)^t$$

$$1 = b + cw_o - c$$

$$w_o = 1 + \frac{1-b}{c} . \qquad 6.2.1$$

We may now conclude that if the economy grows at a constant rate, this rate must be $\frac{1-b}{c}$. We may also conclude that, only if $E_2 = (1+\frac{1-b}{c})\, E_1$, and $E_1 = (1+\frac{1-b}{c})\, E_o$, will such growth be possible. Thus, the warranted path is most unstable.

If initial conditions are not suitable for warranted growth, there are several possible types of growth. In order to discuss these possible types of growth, let us prove the following theorem.

Theorem 6.2.1: If in equation 6.1.5, the initial conditions are such that

$$\frac{E_{t-1}}{E_{t-2}} > \frac{E_{t-2}}{E_{t-3}} \text{ and } \frac{E_{t-1}}{E_{t-2}} > 1 + \frac{1-b}{c} ,$$

the economy will explode. Similarly, if

$$\frac{E_{t-1}}{E_{t-2}} < \frac{E_{t-2}}{E_{t-3}} \text{ and } \frac{E_{t-1}}{E_{t-2}} < 1 + \frac{1-b}{c} ,$$

the economy will contract.

Proof: First we show that if in equation 4.1.3 $E_1 = (1+w)E_o$ and $(1+w) > 1 + \frac{1-b}{c}$, then $E_2 > (1+w)E_1$. This follows since in equation 4.1.3

$$E_2 = (b(1+w)+c(1+w)^2(1+w)E_o - c(1+w)^2E_o$$

$$= (b(1+w)+c(1+w)^2-c(1+w))(1+w)E_o$$

$$= (b+c(1+w)-c)(1+w)E_1$$

$$> 1(1+w)E_1 ,$$

since $(b+c(1+w)-c) > 1$ for $(1+w) > 1+\frac{1-b}{c}$. Now equation 6.1.5 can be written in the form

$$E_t = (b(1+w_t)+c(1+w_t)^2)E_{t-1} - c(1+w_{t-1})^2E_{t-2} , \qquad 6.1.5a$$

as shown above. This implies, if $(1+w_t) > (1+w_{t-1})$, that

$$E_t > ((b(1+w_t)+c(1+w_t)^2)E_{t-1} - c(1+w_t)^2E_{t-2} ,$$

and if $(1+w_t) > 1+\frac{1-b}{c}$, that

$$E_t > (1+w_t)E_{t-1} ,$$

in equation 6.1.5. By induction, $\frac{E_t}{E_{t-1}} > \frac{E_{t-1}}{E_{t-2}}$, for all t, and the solution to equation 6.1.5 will have to diverge. The proof of the second part of the theorem is exactly analogous to that of the first part and rests upon showing that for equation 4.1.3, when $E_1 < (1+w)E_o$, and $(1+w) < 1+\frac{1-b}{c}$, $E_2 < (1+w)E_1$. This follows since

$$E_2 = (b(1+w)+c(1+w)^2)E_1 - c(1+w)^2E_o$$

$$= (b(1+w)+c(1+w)^2)E_o(1+w_o)E_o - c(1+w)^2E_o$$

$$= (b+c(1+w)-c)(1+w)E_1$$

$$< (1+w)E_1$$

for $(1+w) < 1 + \frac{1-b}{c}$.

We can now see that if $\frac{E_t}{E_{t-1}} > \frac{E_{t-1}}{E_{t-2}}$ and $\frac{E_t}{E_{t-1}} > 1 + \frac{1-b}{c}$, the economy explodes. Similarly, if $\frac{E_t}{E_{t-1}} < \frac{E_{t-1}}{E_{t-2}}$, the economy contracts. What, however, if

1) $\frac{E_t}{E_{t-1}} > \frac{E_{t-1}}{E_{t-2}}$ but $\frac{E_t}{E_{t-1}} < 1 + \frac{1-b}{c}$

2) $\frac{E_t}{E_{t-1}} < \frac{E_{t-1}}{E_{t-2}}$ but $\frac{E_t}{E_{t-1}} > 1 + \frac{1-b}{c}$.

In neither of these cases does Theorem 6.2.1 hold.

In order to answer this question, let us consider case 1) above. In this case one of three things may occur. First, after a finite number of periods, $\frac{E_t}{E_{t-1}}$ may become greater than $1 + \frac{1-b}{c}$. If this happens, the economy explodes by Theorem 6.2.1. Second, $\frac{E_t}{E_{t-1}}$, before reaching the value $1 + \frac{1-b}{c}$, may "turn around" and become less than $\frac{E_{t-1}}{E_{t-2}}$. If this happens, the economy contracts, by Theorem 6.2.1. Third, the system may eventually approach but never "quite" achieve the warranted path. We now prove this last statement.

Let us proceed as follows. Suppose $\frac{E_t}{E_{t-1}}$ is given and is less than $1 + \frac{1-b}{c}$. There exists a whole range of values of $\frac{E_{t-1}}{E_{t-2}}$ such that $1 + \frac{1-b}{c} > \frac{E_{t+1}}{E_t} > \frac{E_t}{E_{t-1}}$. Furthermore, it is obvious that each value of $\frac{E_{t+1}}{E_t}$ corresponds to one and only one value of $\frac{E_{t-1}}{E_{t-2}}$. Of all these values of $\frac{E_{t+1}}{E_t}$, which do not yield either immediate contraction or explosion, only some lead to the further conclusion $1+\frac{1-b}{c} > \frac{E_{t+2}}{E_{t+1}} > \frac{E_{t+1}}{E_t}$. Each value of these $\frac{E_{t+2}}{E_{t+1}}$ corresponds to one value of $\frac{E_{t-1}}{E_{t-2}}$. Continuing in this fashion, period by period, we can restrict the range values of $\frac{E_{t-1}}{E_{t-2}}$ which may give rise to growth approaching the warranted path as much as we please. Therefore, we can see that the range of values of $\frac{E_{t-1}}{E_{t-2}}$ which cause neither contraction or explosion, must approach an interval of length zero. This implies that, in the limit, there exists one and only one value of $\frac{E_{t-1}}{E_{t-2}}$ which lets income "approach" the warranted path. We may conclude that the "approach to warranted growth," just like warranted growth, is most unstable. Finally, to state what now must be obvious, the warranted path can never be achieved. For if it were, then $\frac{E_t}{E_{t-1}}$ would have to equal $1+\frac{1-b}{c}$ at some point, while $\frac{E_{t-1}}{E_{t-2}}$ would be less, giving explosion.

From the above, we may also prove that when $\frac{E_t}{E_{t-1}} > \frac{E_{t-1}}{E_{t-2}}$ and $\frac{E_t}{E_{t-1}} < 1+\frac{1-b}{c}$, the economy cannot achieve or approach a constant nonwarranted path. If a constant nonwarranted path were achieved at a rate less than $1+\frac{1-b}{c}$, expenditures would grow at this rate for two periods and then by Theorem 6.2.1, they would have to contract. Similarly, expenditures cannot grow continually at a rate almost equal to some rate below $\frac{1-b}{c}$. For if they did, then $w_t = w_o + \varepsilon$, and $w_{t-1} = w_o + \varepsilon'$ for all t, where w_o is less than $\frac{1-b}{c}$, and ε and ε' can be made as small as we like by letting $t \to \infty$. Equation 6.1.5 could now be written

$$E_t = (b(1+w_o)+c(1+w_o)^2)E_{t-1} + (b\varepsilon+c2\varepsilon+c\varepsilon^2)E_{t-1}$$
$$- (c(1+w_o)^2E_{t-2} - (c2\varepsilon'+c\varepsilon'^2)E_{t-2} . \qquad 6.2.2$$

This in turn may be written as

$$E_t = (b(1+w_o)+c(1+w_o)^2)E_{t-1} - c(1+w_o)^2 \frac{E_{t-1}}{1+w_o+\varepsilon} + \cdots .$$

or

$$E_t = (b(1+w_o)+c(1+w_o)^2)E_{t-1} - c(1+w_o)E_{t-1} + \varepsilon''E_{t-1}$$
$$< (1+w_o-A+\varepsilon'')E_{t-1}$$

where A corresponds to w_o and is finite. This implies that eventually, as $t \to \infty$ and the $\varepsilon \to 0$, expenditures must "turn around" and grow at a rate less than w_o. By theorem 6.2.1, this will cause contraction.

We have so far discussed only case 1 above. In order to discuss case 2, we simply reverse the signs of the inequalities in all the previous proofs. We may then show that the growth of income is exactly the same as before. Income may either explode, contract or approach the warranted path. If the latter obtains, the approach is very unstable. Also, income may not grow at a rate equal to or approaching a nonwarranted rate.

Having discussed all possible types of growth according to equation 6.1.5, we may now state that such growth, be it contraction, explosion or warranted growth, occurs in Harrod-like fashion. Stated differently, we can prove that for warranted growth, sales and capital are just sufficient each period, that for contraction, they must eventually become overly sufficient, and that for explosion, they must eventually be insufficient. We have already proved the first of these statements. To prove the last, let

$$\frac{E_t}{E_{t-1}} > \frac{E_{t-1}}{E_{t-2}} .$$

Since

$$Y_t = E_{t-1}\frac{E_{t-1}}{E_{t-2}} \quad \text{and} \quad E_t = E_{t-1}\frac{E_t}{E_{t-1}} ,$$

$$E_t > Y_t .$$

Also

$$K_{t+1} \text{ desired} = K_{t+1} \text{ on hand} = c(E_{t-1})\left(\frac{E_{t-1}}{E_{t-2}}\right)^2 ,$$

and

$$K_{t+1} \text{ needed} = c(E_t)\frac{E_t}{E_{t-1}} .$$

Therefore

$$K_{t+1} \text{ needed} > K_{t+1} \text{ on hand} .$$

For contraction, just reverse the signs of the inequalities to get the opposite conclusions.

Section 6.3. Summary and Comment on the Adaptive Model

We can now summarize the conclusions of the adaptive model.

First, there exists a warranted path in the adaptive model in which markets are fully cleared and capital fully employed each period. The rate of warranted growth will be $\frac{1-b}{c}$ as in our two previous models, and our warranted path will be extremely unstable.

Second, if growth is initially not along the warranted path, the economy can never achieve completely this path. Depending on initial conditions, the economy will either contract, explode, or approach the warranted path. If the economy explodes or contracts, it will do so in Harrod-like fashion, with either oversufficiency or undersufficiency of capital and sales.

Third, for a given value of $\frac{E_{t-1}}{E_{t-2}}$, there is only one value of $\frac{E_{t-2}}{E_{t-3}}$ which allows the economy to return towards the warranted path. Because of this, this type of growth, like warranted growth, is very unstable.

The above summary concludes our discussion of the adaptive expectations model. Before continuing with other models, let us make four brief comments on this model.

Our first comment concerns a "policy implication" of the adaptive model. We have noted in our previous models that the warranted path could never be achieved if the coefficients of expectations in these models were different from $1 + \frac{1-b}{c}$. This implies that only in a model where the coefficient of expectations was equal to $1 + \frac{1-b}{c}$, could a government possibly hope to restore an economy which had deviated from warranted growth back to a state where it would continue on this path period after period without further government intervention. In general, however, because we cannot expect from our constant multiplicative and additive models that w and A will equal $\frac{1-b}{c}$, the implications of these two models are that capitalism will not be self-equilibrating to full employment (of capital) growth, and that the only way this happy state can be approached is through continual government manipulation of the economy.

In a model with adaptive expectations, however, this last conclusion is considerably weakened. It is true that an economy in which expectations are adaptive, if left to its own devices, will almost certainly suffer deep recession or continued and increasing excess demand. If the government were lucky, however, it might be able, through expenditures or taxation, to alter aggregate demand so that the "right" set of initial conditions is obtained under which the economy, on its own, can return towards the warranted path. In these circumstances, if no exogenous force disturbed the economy, or if businessmen did not make "mistakes," the economy after the proper government action, could experience on its own, actual warranted growth. Thus, if expectations are adaptive, the prospects for full employment growth, with a minimum of government intervention, are considerably more sanguine than they are if expectations are constant.

Our second comment concerns the adaptive model, as compared with the Harrod model. At the end of Chapter 4, we made a number of statements concerning the relation between Harrod's original model and the constant multiplicative model. We pointed out that there were two important differences between our model and Harrod's--namely that Harrod did not consider the most general type of initial conditions, and also did not consider constant but non-warranted growth.

As far as the latter point is concerned, it should now be clear that if expectations are adaptive, constant non-warranted growth is impossible. What this means from an economic point of view is that in an adaptive model businessmen, upon seeing that expectations are not fulfilled, will change these expectations in the direction of realized changes in sales. Since the adaptive model is probably more "realistic" than the constant models of the previous chapters, the latter objection to Harrod's model is considerably less. However, it should still be remembered that constant non-warranted growth is possible for constant expectations models, and there is nothing in our theory to say that such expectations can be ruled out. As such, we may fault Harrod for dealing with only one particular type of expectation.

Our first objection to Harrod's original model, though, does remain valid. As we stated in Chapter 4, Harrod believed that if the growth rate at any time were different from $\frac{1-b}{c}$, it would continue to diverge away from this value. In an adaptive model, this would also be the case if $\frac{E_{t-1}}{E_{t-2}}$ were equal to $1 + \frac{1-b}{c}$ but $\frac{E_t}{E_{t-1}}$ were not. Under these circumstances, income would diverge in the direction of the disturbance. Nevertheless, it may very well be the case that, even though $\frac{E_{t-1}}{E_{t-2}}$ and $\frac{E_t}{E_{t-1}}$ are greater than $1 + \frac{1-b}{c}$, the economy will contract. This will occur if $\frac{E_{t-1}}{E_{t-2}}$ is significantly greater than $\frac{E_t}{E_{t-1}}$. In this case, businessmen would realize, in view of the low follow up growth rate in period t, that they had overinvested in the previous period and would buy significantly fewer investment goods in the next period. This might, in spite of high expectations, cause the growth rate to slow and income to eventually contract. Similarly, even though growth rates in our model might actually be less than $\frac{1-b}{c}$, the economy might still explode, if the second period growth rate were significantly higher than the first period's. Under these circumstances, businessmen, realizing that they had bought too little capital in view of this period's large gain in sales, would increase their capital expenditures. This would make income more likely to explode. These examples, therefore, show that Harrod's assertion that "small" growth rates cause contraction and "large" growth rates cause explosion is incorrect. They also show why Harrod could not envision income returning towards the warranted path.

Our third comment concerns the "irrationality" of our expectation. We noted, in Section 2, that the Harrod model with adaptive expectations was most unstable. We also noted that away from the warranted path businessmen will always make incorrect guesses as to the future and that these guesses will get worse over time. This property of the Harrod model is clearly the result of the type of expectation we have used in this chapter. For if we chose a weighted expectation or better still a rational expectation as our expectation, we would clearly not obtain our instability characteristics. Thus, we may conclude that rational expectations (and to a certain extent, weighted expectations) are inconsistent with the Harrod like properties of our model.

Finally, our last comment concerns a point, made earlier in Chapter 1, concerning the empirical validity of the Harrod investment function. At that time, we stated that empirical research had indicated that the investment function should have, in addition to a lag factor, some variable expectations factor. In Chapter 4, we indicated how to build models with lags. Now it should be clear that we can also build models with variable expectations in the investment function. For the simple investment function of this chapter

$$I_t = c\left(\frac{E_{t-1}}{E_{t-2}}\right)^2 E_{t-1} - c\left(\frac{E_{t-2}}{E_{t-3}}\right)^2 E_{t-2}$$

can be written

$$I_t = c\left(\frac{E_{t-1}}{E_{t-2}}\right) Y_t - c\left(\frac{E_{t-2}}{E_{t-3}}\right) Y_{t-1} .$$

The latter is not a simple rigid accelerator function but rather a variable expectations accelerator in which the coefficients in front of the output variables depend on expectations. Thus, our model is capable of explaining many of Eisner's empirical conclusions. It is also clear that, by postulating lags in sales information, as in Section 4 of Chapter 4, we can build the variable expectations multi-period lag functions, which other economists, such as Chow and Jorgenson, have used to explain investment behavior.

CHAPTER 7

Section 7.1. Introduction to Inventory Adjustment Models[1]

In our expectation models so far, we have assumed two things which may appear to the reader to be most unrealistic.

First, we have assumed that there exists in our economies an infinite supply of labor, or at the very least, a supply of labor growing at a rate faster than the most explosive rate of growth of income in these economies. The necessity of this assumption is clear. We have seen that if the economy explodes, capital, after some period of time, becomes insufficient relative to desired levels of production. Yet in spite of this, we have assumed that production proceeds just as fast as businessmen desire. The only way this can occur is if businessmen use the "other," labor intensive, method of production. Furthermore, we have seen in all our models that when the economy explodes, the gap between desired and actual capital on hand becomes greater over time. This implies that during an explosive phase, more and more labor will have to be forthcoming. Thus, to meet businessmen's needs, the labor supply will either have to be infinite or else it will have to grow at a rate faster than the fastest rate of explosion in the economy (i.e., the highest root of the characteristic

[1] In this chapter, and the next, any reference to Metzler will be with regard to his already cited article on the "Nature and Stability of Inventory Cycles."

equation). Neither assumption is satisfactory from a realistic point of view.

Second, and in a similar vein, we have assumed in the above models that an infinite supply of inventories is initially on hand in businessmen's warehouses. We must assume this, since we are postulating in our models that even though expenditures exceed production in the explosive phase of the economy, consumers and businessmen still obtain their desired goods. For example, with regard to investors, we stated that $K_{t+1 \text{ desired}} = c(1+w)^2 E_{t-1}$. We then deduced that $I_t = c(1+w)^2(E_{t-1} - E_{t-2})$. The latter, however, makes sense only if we can assume that $K_{t \text{ desired}}$ was achieved in period t, or stated differently, if businessmen were able to obtain all the investment goods they desired in $t-1$. Likewise, with regard to consumers. Here, we have assumed that desired consumption in period t is always equal to bY_t. However, if consumers in period t could not obtain bY_t consumption goods (since expenditures exceeded production), we would clearly have to increase postulated consumption in period $t+1$, to account for the "frustrated" demand in period t. (Mathematically, as we shall see later, C_t would become $(bY_t + C_{t-1} - (1+w)Y_{t-2})$. It is now clear that if only a finite amount of inventories were on hand initially, they would soon become exhausted in the explosive phase of our economy, since in this phase the gap between expenditures and production becomes greater each period. The equations describing consumption and investment in the explosive phase of our previous models would not be correct, and our models would be inadequate descriptions of explosive growth. Thus, it is necessary to assume infinite initial inventories.

These two problems--the assumptions of infinite labor supplies and infinite inventories will occupy our attention for the remainder of this dissertation. In the next three chapters we shall discuss the second of these problems. We shall begin in chapters 7 and 8 by showing how to incorporate inventory adjustment behavior into our constant multiplicative model. We shall postulate two types of inventory adjustment as in the Keynesian model - inventory behavior, in which businessmen wish to maintain each period a fixed level of inventories, and inventory behavior, in which businessmen try to maintain in each period a level of inventories proportional to expected sales in that period. With the results of chapters 7 and 8, we shall, in chapter 9, be able to drop the assumption of infinite inventories. At all times, however, it will be understood in these three chapters, that even though capital is insufficient relative to desired production, desired production is achieved through the labor intensive method of production. We reserve until chapter 10 a discussion of our second problem--the assumption of infinite supplies of labor.

Section 7.2. The Fundamental Equation of Growth for Fixed Level Inventory Adjustment Behavior

In order to ultimately drop the assumption of infinite initial inventories, it is necessary to show how businessmen adjust their inventory positions in the constant multiplicative model. It will be remembered that, earlier, we have assumed that inventory changes do not affect production plans at all. This is known as passive inventory adjustment behavior. Clearly, however, a more realistic model would assume that if expenditures exceeded production in some period, businessmen would try to make up for lost inventories in the next period, and hence would produce more in this period. Similarly, a more realistic model would assume that if, in some period t, production exceeded sales, businessmen would decrease production in the next period $t+1$, in order to get rid of inventories which had involuntarily accrued. In the rest of this chapter, therefore, we consider a more realistic model where businessmen desire to maintain a fixed level of inventories each period.

In considering this model, we must ask how a firm will plan production on the basis of past information with regard to sales and production. As usual, we realize that if, in any given period, businessmen expect sales to increase by w percent over last period's sales, then production for sales purposes in the next period will be

$$Y_t = (1+w)(bY_{t-1} + I_{t-1}) \; . \qquad 7.2.1$$

We now realize, however, that, in general, inventories on hand will be different from desired inventories. In that case, we must add or subtract a term to the above equation to get Y_t total. The question now arises as to what the magnitude of this term is, and whether it can be expressed in terms of past variables. Metzler has shown that if businessmen are attempting to maintain a fixed level of inventories, the amount by which they fail in each period to attain this level, is sales in that period minus expected sales in that period. In our model, this means that

$$(bY_{t-1} + I_{t-1}) - (1+w)(bY_{t-2} + I_{t-2}) \qquad 7.2.2$$

is the inventory discrepancy term.

The reason for this statement should be clear. Suppose that Y_t has been produced in period t. If businessmen are attempting to maintain a fixed level of inventories, then they have produced enough in period t to (1) meet expected sales, and (2) achieve the level of inventories desired. Consequently, in the production phase of t, they believe that they are "even" as far as inventories are concerned. Unfortunately, in the market phase of t, sales will not necessarily equal expected sales. If sales exceed expected sales, businessmen are now "down" in inventories by precisely this excess, and if sales are less than expected sales, businessmen will have more inventories on hand than they desire by exactly this amount. We are now able to write for total production in period t

$$Y_t = (1+w)(bY_{t-1} + I_{t-1}) + bY_{t-1} + I_{t-1} - (1+w)(bY_{t-2} + I_{t-2}). \qquad 7.2.3$$

We also have to determine, however, what investment in period t is. It turns out that in the fixed level adjustment model, I_t is precisely what it was before in the constant model without inventory adjustment--namely

$$I_t = c(1+w)^2((bY_{t-1} + I_{t-1}) - (bY_{t-2} + I_{t-2})). \qquad 7.2.4$$

The reason for this result is that in the market phase of t, businessmen feel that they will be "even" as far as inventories are concerned. They do not realize that expected sales may not equal actual sales. As such, they desire capital only for expected sales and

$$K_{t+1 \text{ desired}} = c(1+w)^2(bY_{t-1} + I_{t-1})$$

Since businessmen went through the same type of analysis in the previous period t-1, we obtain the formula for investment in t by subtracting the equivalent expression for $K_{t\ desired}$ from equation 7.2.5. In doing so, we obtain equation 7.2.4.

We now have two difference equations in two unknowns. To solve for Y_t, we may rewrite equations 7.2.3 and 7.2.4 as

$$Y_t-((1+w)b+b)Y_{t-1}+Y_{t-2}(1+w)b=(1+w+1)I_{t-1}-(1+w)I_{t-2} ,$$

and

$$c(1+w)^2bY_{t-1}-c(1+w)^2bY_{t-2}=I_t-c(1+w)^2I_{t-1}+I_{t-2}c(1+w)^2.$$

Using E operators, we can further rewrite these equations as

$$((E^2-(1+w)b+b)E+(1+w)b)Y_{t-2}=((1+w+1)E-(1+w))I_{t-2} , \qquad 7.2.3a$$

and

$$(c(1+w)^2Eb-c(1+w)^2b)Y_{t-2}=(E^2-c(1+w)^2E+c(1+w)^2)I_{t-2}. \qquad 7.2.4a$$

Multiplying 7.2.3a by $(E^2-c(1+w)^2E+c(1+w)^2)$ and 7.2.4a by $((1+w+1)E-(1+w))$, we have

$$(E^2-(1+w)b+b)E+(1+w)b)(E^2-c(1+w)^2E+c(1+w)^2Y_{t-2}$$
$$= (cb(1+w)^2E-cb(1+w)^2)((1+w+1)E-(1+w))Y_{t-2} . \qquad 7.2.5$$

Define X to be $(1+w+1)E-(1+w)$ and Y to be $(c(1+w)^2E-c(1+w)^2)$. Then

$$(E^2-bX)(E^2-Y)Y_{t-2} = (bY)(X) ,$$

which implies that

$$(E^4-(bX+Y)E^2)Y_{t-2} = 0$$

or

$$(E^4-(b(1+w)+b)E^3+b(1+w)E^2-c(1+w)^2E^3+c(1+w)^2E^2)Y_{t-2} = 0 \qquad 7.2.6$$

or

$$Y_{t+2}-(b(1+w)+b+c(1+w)^2)Y_{t+1}+(c(1+w)^2+b(1+w))Y_{t-2} = 0 \quad . \qquad 7.2.7$$

An easier way to derive this equation, at least from a mathematical point of view, is to realize that

$$E_t = bY_t + I_t$$

and that

$$Y_t = (1+w)E_{t-1} + E_{t-1} - (1+w)E_{t-2} \qquad 7.2.3b$$

$$I_t = c(1+w)^2E_{t-1} - c(1+w)^3E_{t-2} \quad . \qquad 7.2.4b$$

Then

$$E_t = (b(1+w)+b+c(1+w)^2)E_{t-1} - (1+w)b+c(1+w)^2)E_{t-2} \, . \qquad 7.2.7a$$

Since Y_t is a linear combination of lagged values of E_t, it follows that

$$Y_t = (b(1+w)+b+c(1+w)^2)Y_{t-1} - ((1+w)b+c(1+w)^2Y_{t-2} \quad , \qquad 7.2.7$$

which is the result we obtained above.

Section 7.3. Implications of the Fixed Level Inventory Adjustment Model

Our equation for the growth of income, when businessmen believe sales will increase by w percent each period and when they try to maintain a fixed level of inventories each period, is now given in the form of equation 7.2.7. It remains to discuss the characteristics of such growth and, in particular, to compare it with growth under the passive adjustment of Chapter 4, Section 1.

Let us start by asking whether there exists a warranted rate of growth in the fixed level model, and if so, what its value is. As usual, whether or not a warranted path exists, depends on whether there exists an expectation w such that one of the roots of the characteristic equation of 7.2.7 equals $(1+w)$. Accordingly, let us plug $(1+w)$ into the characteristic equation of 7.2.7 to get

$$(1+w)^2 - (b(1+w)+b+c(1+w)^2)(1+w)+c(1+w)^2 + b(1+w) = 0 \quad .$$

Simplifying, we have

$$(1+w)^2 - b(1+w)^2 - b(1+w) + c(1+w)^3 + c(1+w)^2 + b(1+w) = 0$$

$$(1-b) - c(1+w) + c = 0$$

or

$$1+w = 1 + \frac{1-b}{c} \quad . \qquad 7.3.1$$

Therefore, if $1+w = 1 + \frac{1-b}{c}$ one of the roots of 7.2.7 will equal $1+w$ and if initial conditions are such that $Y_1 = (1 + \frac{1-b}{c})Y_0$ growth will occur along the Harrod warranted path. Furthermore, it is easy to show that along this path all markets are cleared and capital is just fully employed. For if

$$E_t = (1+w_0)E_{t-1} \qquad \text{or} \qquad E_t = A_0(1 + \frac{1-b}{c})^t \quad ,$$

then

$$Y_t = (1+w)E_{t-1} + E_{t-1} - (1+w)E_{t-2} \quad 7.3.2$$

$$= (1+w)(1+w)^{t-1}A_0 + (A_0(1+w)^{t-1} - (1+w)(1+w)^{t-2})$$

$$= E_t \quad ,$$

which shows that markets, in each period, are being cleared exactly. Furthermore, under these circumstances,

$$K_{t+1 \text{ desired}} = c(1+w)^2 E_{t-1} \text{ by } 7.2.5 \quad 7.3.3$$

$$= cE_t(1+w)$$

$$= c(Y_{t+1})$$

$$= K_{t+1 \text{ needed}} \quad ,$$

which shows that capital is just sufficient in each period.

This conclusion should not really be very surprising. For suppose, in a passive inventory model, that growth were occurring along the warranted path. Then markets would be cleared exactly in each period. Also, if inventories on hand were already at some desired fixed level, they would remain that way, and even if there were a desire by businessmen for inventory adjustment, it would never have to be implemented. Consequently, we should have expected from the first that the warranted rates of growth are the same in the passive and fixed level inventory adjustment models, and that, under "equilibrium" conditions, growth should occur in exactly the same fashion in both models.

What now if $1+w = 1 + \frac{1-b}{c}$ in the fixed level model but initial conditions are not suitable for warranted growth? If we compare equations 7.2.7 and 4.1.3, we can see that the coefficient in front of Y_{t-2} is higher in equation 7.2.7 than it is in equation 4.1.3. But the lower root when $1+w = 1 + \frac{1-b}{c}$ is the same in both models. This implies that the upper root of equation 7.2.7, $\lambda_{high,fixed}$, is always greater than the upper root of equation 4.1.3, $\lambda_{high,passive}$, for $1+w = 1 + \frac{1-b}{c}$.

The economic consequences of this are now fairly simple to interpret. Our analysis states that if the warranted rate is deviated from in an upward direction, income will diverge "up" faster in the fixed level model than in the passive model. Similarly, if the deviation is in the downward direction, income will decrease faster in the fixed level model than in the passive model. The economic reason for this should be fairly clear. What is happening in the fixed level model is that if the rate of increase in income decreases from the warranted rate, the rate of increase in expenditures decreases even faster. (Or very simply, sales become less than production.) Therefore, businessmen, in addition to decreasing future growth in production proportionally to sales, also try to get rid of some of the inventories which have involuntarily piled up. They do so by producing less in the next period than they would have in a passive model, believing that part of this period's expected sales can be supplied through the unwanted inventories. But this reduces the rate of increase in sales (aggregate demand) even more than in the passive model, since $C_t = bY_t$, and so all attempts to get rid of unwanted inventories fail and businessmen find that more and more inventories pile up each period. A similar argument can be made when the growth rate deviates from the warranted rate in an upward fashion to show that inventories will always be depleted in each period, no matter how hard businessmen try to maintain them.

We now consider the case where $w \neq \frac{1-b}{c}$. This, of course, is the more likely case, since we do not expect businessmen to hit upon the warranted rate as the expected rate. Before attempting to discuss this situation, it will be very helpful to prove the following theorems.

Theorem 7.3.1 : If $(1+w) < (1 + \frac{1-b}{c})$, then the smaller root of 7.2.7 is less than the smaller root of 4.1.3 and the higher root of 7.2.7 is greater than the higher root of 4.1.3.

Theorem 7.3.2 : If $(1+w) > (1 + \frac{1-b}{c})$, then the smaller root of 7.2.7 is greater than the smaller root of 4.1.3 !! and the higher root of 7.2.7 is (still) greater than the higher root of 4.1.3.

Proof of Theorems 7.3.1 and 7.3.2

Consider the two equations

$$(\lambda_L)^2 - ((1+w)^2c + b(1+w))(\lambda_L) + c(1+w)^2 = 0$$

$$(\lambda_L')^2 - ((1+w)^2c + b + b(1+w))(\lambda_L') + c(1+w)^2 + b(1+w) = 0,$$

where λ_L is the lower root of 4.1.3, and λ_L' is the lower root of 7.2.7. Suppose we plug λ_L into the second of the above equations. Then we obtain

$$\lambda_L^2 - ((1+w)^2c + b(1+w) + b)\lambda_L + c(1+w)^2 + b(1+w) . \qquad 7.3.4$$

We wish to ask whether this expression is greater or less than zero. Clearly

$$(\lambda_L)^2 - ((1+w)^2 c + b(1+w))\ \lambda_L + c(1+w)^2 = 0$$

and hence

$$- b(\lambda_L) + b(1+w)$$

is all that remains of our expression 7.3.4. However, if $1+w < (1 + \frac{1-b}{c})$, we know that $\lambda_L > (1+w)$, by Theorem 4.3.1. Therefore, this expression is less than zero. Now when λ_L' is zero, the expression

$$(\lambda_L')^2 - ((1+w)^2 c + b(1+w) + b)\ \lambda_L' + c(1+w)^2 + b(1+w)$$

is positive. Consequently, when $1+w < (1 + \frac{1-b}{c})$, there _must_ be a root for equation 7.2.7 which is less than the lower root of 4.1.3. This proves the first half of Theorem 7.3.1.

Similarly, if $1+w > (1 + \frac{1-b}{c})$, $- b\ \lambda_L + b(1+w) > 0$, since λ_L is $< (1+w)$ by Theorem 4.3.2. Therefore, the quantity $- b\ \lambda_L + b(1+w)$ is greater than zero and, since $\lambda_L' = 0$ implies expression 7.3.4 is also greater than zero, we can infer that there is no root between zero and λ_L, for equation 7.2.7. This proves the first half of Theorem 7.3.2.

Proving that $\lambda_{H7.2.7} > \lambda_{H4.1.3}$ is now easy. Upon plugging $\lambda_H = \infty$ into the characteristic equation of 7.2.7, we get $+ \infty$. Put $\lambda_{H4.1.3}$ into 7.3.4, to get

$$\lambda_H^2 - ((1+w)^2 + b(1+w) + b)\ \lambda_H + c(1+w)^2 + b(1+w) \quad ,$$

which equals

$$- b \lambda_H + b(1+w).$$

But λ_H is greater than $(1+w)$ for all w. Therefore, this quantity is less than zero, and equation 7.2.7 has a root between λ_H and $+ \infty$. This completes our proof.

We are now able to interpret equation 7.2.7. When $1+w < (1 + \frac{1-b}{c})$, the chances for explosion are, as in the passive model, only slight. The initial conditions will have to have the property that $Y_1 > (1 + \frac{1-b}{c})Y_0$. However, now in the fixed level model, the chances for explosion are better than in the passive model. The reason mathematically, of course, is that $\lambda_{L7.2.7}$ is lower than $\lambda_{L4.1.3}$ for $(1+w) < 1 + \frac{1-b}{c}$. Economically, we interpret this mathematical conclusion as follows. If the initial changes in expenditures (or income) is high and greater than the expected value of changes in expenditures, then inventories will be depleted. Then, in the fixed level model, businessmen will increase production more than in the passive model, so as to make up for lost inventories. This, in turn, will cause still greater increases in expenditures in the next period. Thus, for a given set of initial conditions, when $1+w < 1 + \frac{1-b}{c}$, a model which might have contracted under passive inventory adjustment assumptions, would initially be more explosive and might even eventually explode under fixed level inventory adjustment. Similarly, if $1+w > 1 + \frac{1-b}{c}$ and initially $Y_1 < (1 + \frac{1-b}{c})Y_o$, an economy which might ordinarily explode in the passive model would not necessarily explode in the fixed level model. For $Y_1 < (1 + \frac{1-b}{c})Y_o$ implies that expenditures in period 1 had been less than expected expenditures in this period. Therefore, inventories have piled up, and, because of this, there is a tendency on the part of businessmen to get rid of these excess inventories, by producing less in the next period than they would have in a passive model. This will cause income to be less explosive initially in the fixed level model and may even cause the system to eventually contract, in spite of the high expectations.

In both models, however, whatever may happen initially, income will eventually increase or decrease at a faster rate in the fixed level model than in the passive model. Mathematically, of course, this is because $\lambda_{H7.2.7} > \lambda_{H4.1.3}$. Economically, this occurs in the explosive case because as sales start to exceed production, inventories are drawn down. Businessmen will now try to produce more to compensate for this depletion of inventories and, thus, when sales exceed production, the rate of growth in production will be due to two positive effects instead of one as in the passive model. Stated differently, even if income in the passive model is initially increasing faster than in the fixed level model, eventually, if income explodes in the fixed level model, it must catch up with and surpass income in the passive model. Similarly, if sales start to fall below expected sales and inventories pile up, businessmen will try to get rid of the excess inventories. They will therefore increase production less than in a passive model, and, eventually, income in the fixed level model will have to fall below that in the passive model.

Finally, we should point out that just as in the passive model there is the possibility of constant growth at a rate other than the warranted rate, so there is a possibility of such growth in our constant expectation fixed level model. Clearly, this will occur if one of the coefficients in the solution $Y_t = A_0(\lambda_1)^t + B_0(\lambda_2)^t$ becomes zero. If $1+w < 1 + \frac{1-b}{c}$, both λ_1 and λ_2 are greater than $1+w$ and in this case growth will occur with $E_t > Y_t$ and inventories continually being depleted. Similarly, if $1+w > 1 + \frac{1-b}{c}$, growth will take place as above if $A_0 = 0$; but if $B_0 = 0$, growth will occur with $E_t < Y_t$ and inventories always piling up! These two types of solutions will of course not be permitted in a model with adaptive expectations.

CHAPTER 8

Section 8.1. The Fundamental Equation for Proportional Inventory Adjustment Growth

We now change the assumption made in the previous chapter that businessmen wish to maintain a fixed level of inventories in all periods. This assumption, while perfectly valid as a means of describing business behavior, can perhaps be improved upon, if we assume that businessmen wish to maintain in each period an amount of inventories which is proportional to expected sales in that period.

The rationale for the latter assumption is as follows. Let us assume that businessmen desire to maintain inventories for only one purpose - namely, to satisfy customers in case actual sales exceed expected sales (and realized production) in a certain period. If a businessman believes that there is a possibility of expected sales going astray, it seems reasonable to assume that he will feel the magnitude of the deviation from actual sales will be greater, the greater actual (and expected) sales are. Thus, if a company expects to sell one million units in period t, as opposed to one thousand units, it might wish to keep on hand ten thousand units of inventories, in the first case, but only ten units, in the second case. (The reason that it will not always keep on hand a huge number of inventories is, of course, that there is a cost to inventory maintenance.) In this chapter, we shall assume that the relation between desired inventories and expected sales is a simple linear one. This will keep our difference equations linear and, hence, explicitly solvable.

Accordingly, let us assume that businessmen desire to maintain a level of inventories in each period equal to k(sales expected in $t+1$), where $k > 0$. It may seem reasonable to restrict k to be between 0 and 1, since we might question whether firms would keep inventories on hand greater than expected sales. We shall see, however, that there is no fundamental difference in the solution for income whether $k < 1$ or $k > 1$. Therefore, we shall simply let k be > 0.

Since sales expected in $t+1$ are $(1+w)(bY_t + I_t)$, we would expect that production in period $t+1$ would be

$$Y_{t+1} = (1+w)(bY_t + I_t) \text{ for sales purposes} , \qquad 8.1.1$$

and

$$Y_{t+1} = k(1+w)(bY_t + I_t) - k(1+w)(bY_{t-1} + I_{t-1}) \qquad 8.1.2$$
$$\text{for inventory purposes} .$$

The latter formula arises as follows. Since desired inventories in t are $k(1+w)(bY_{t-1} + I_{t-1})$, we would expect this to be achieved inventories in t. Therefore, production in $t+1$, to achieve $k(1+w)(bY_t + I_t)$ inventories, would have to be given by equation 8.1.2. Unfortunately, this is not entirely correct. While it is true that at the beginning of period t inventories on hand were $k(1+w)(bY_{t-1} + I_{t-1})$, inventories may have accumulated or decumulated during the market phase of t. The exact amount by which this occurred would be the difference between realized and expected sales. Therefore, we should correct our earlier equation to read

$$Y_{t+1\ \text{inventory}} = k(1+w)(bY_t + I_t) - k(1+w)(bY_{t-1} + I_{t-1}) + (bY_t + I_t) - (1+w)(bY_{t-1} + I_{t-1}) \ ,$$

and

$$Y_{t+1\ \text{total}} = (1+w)(bY_t + I_t) + k(1+w)(bY_t + I_t) - k(1+w)(bY_{t-1} + I_{t-1}) + (bY_t + I_t) - (1+w)(bY_{t-1} + I_{t-1}) \ . \qquad 8.1.3$$

We must now determine the formula for I_t. It is clear that just as before, $K_{t+1\ \text{desired}} = c(1+w)^2(bY_t + I_t)$ for sales purposes. In addition, however, there must be another component of desired capital to account for inventory production in $t+1$. (The businessman wishes to produce inventories in the cheapest possible manner. Thus, he will also use the capital intensive process for the production of these goods.) The businessman believes that inventories on hand at the end of period t will be $k(1+w)(bY_{t-1} + I_{t-1})$, since he expects his guess of sales in t to be realized. But he also believes that sales in $t+1$ will be $(1+w)^2(bY_{t-1} + I_{t-1})$. As such, he desires to add another $k((1+w)^2(bY_{t-1} + I_{t-1}) - (1+w)(bY_{t-1} + I_{t-1}))$ to his already existing stock of inventories. In order to do this in the optimal fashion, he will need an additional

$$ck(1+w)^2(bY_{t-1} + I_{t-1}) - ck(1+w)(bY_{t-1} + I_{t-1}) \qquad 8.1.4$$

of capital. Consequently our total desired capital in $t+1$ is

$$K_{t+1 \text{ desired}} = c(1+w)^2(bY_{t-1} + I_{t-1}) + ck(1+w)^2(bY_{t-1} + I_{t-1}) - ck(1+w)(bY_{t-1} + I_{t-1}) \quad . \qquad 8.1.5$$

Investment in period t is therfore given by

$$I_t = c(1+k)(1+w)^2(bY_{t-1} + I_{t-1}) - ck(1+w)(bY_{t-1} + I_{t-1}) - c(1+k)(1+w)^2(bY_{t-2} + I_{t-2}) + ck(1+w)(bY_{t-2} + I_{t-2}). \qquad 8.1.6$$

We now have two equations in two unknowns:

$$Y_{t+1} = (1+w)(bY_t + I_t) + k(1+w)(bY_t + I_t) - k(1+w)(bY_{t-1} + I_{t-1}) + (bY_t + I_t) - (1+w)(bY_{t-1} + I_{t-1}), \qquad 8.1.3$$

and

$$I_t = c(1+k)(1+w)^2(bY_{t-1} + I_{t-1}) - ck(1+w)(bY_{t-1} + I_{t-1}) - c(1+k)(1+w)^2(bY_{t-2} + I_{t-2}) + ck(1+w)(bY_{t-2} + I_{t-2}) \quad . \qquad 8.1.6$$

(The reader may note that the proportional model, with $k = 0$, reduces to the fixed level model, where the level of inventories desired each period is zero. This should not be too surprising, since no mention was made in the fixed model of the absolute level of inventories desired. That was an initial condition in this model, which, once given, had no bearing whatsoever on the growth pattern. Because, as we shall see shortly, the introduction of a finite value of k causes a somewhat different pattern of growth from that in the fixed level model, it is appropriate to discuss both models separately.)

As before, we can now proceed to solve equations 8.1.3 and 8.1.6 with the use of E operators. We have seen, however, that this process is especially cumbersome. Let us, therefore, introduce the expenditures transformation directly and write our equations in terms of E_t. Doing this, we obtain as our two equations for Y_t and I_t :

$$Y_t = ((1+w)(1+k)+1)E_{t-1} - ((1+w)(1+k)E_{t-2} \; ; \qquad 8.1.3a$$

$$I_t = c(1+k)(1+w)^2E_{t-1} - ck(1+w)E_{t-1} - c(1+k)(1+w)^2E_{t-2} \qquad 8.1.6a$$

$$+ ck(1+w)E_{t-2} \; .$$

Making use of the identity $E_t = bY_t + I_t$, we have

$$E = (b((1+w)(1+k)+1)) + c(1+k) \; (1+w)^2 - ck(1+w))E_{t-1}$$

$$- (b(1+w)(1+k) + c(1+k)(1+w)^2 - ck(1+w))E_{t-2} \; .$$

Since Y_t is a linear combination of the lagged E_t, it must satisfy the same difference equation as E_t. Therefore,

$$Y_t = (b((1+w)(1+k)+1) + c(1+k)(1+w)^2 - ck(1+w))Y_{t-1} \qquad 8.1.7$$

$$- (c(1+k)(1+w)^2 - ck(1+w) + b(1+w)(1+k))Y_{t-2} \; .$$

The use of E operators to solve equations 8.1.3 and 8.1.6 can be shown to give the same result. Finally, we should note that what we said on page 155 about the equality of the fixed level and proportional adjustment nodel when $k = 0$ is true, since for $k = 0$, equations 8.1.7 and 7.2.7 are identical.

Section 8.2. Properties of the Fundamental Equation for Constant Expectations, Proportional Inventory Adjustment, Growth

We now wish to investigate the properties of equation 8.1.7. As usual, we begin by asking whether there exists an expectation w such that if businessmen believe that sales will increase by w percent each period, their expectations, under the proper initial conditions, will be fulfilled. Or stated differently, does there exist in the proportional model a warranted rate of growth?

If there exists such an expectation, then one of the roots of the characteristic equation of 8.1.7 will have to equal $(1+w)$. Therefore plug $(1+w)$ into the characteristic equation of 8.1.7 to get

$$(1+w)^2 - c(1+w)^3 - b(1+w)^2 - bk(1+w)^2 - b(1+w) + ck(1+w)^2 \qquad 8.2.1$$

$$- ck(1+w)^3 + c(1+w)^2 + ck(1+w)^2 - ck(1+w) - b(1+w)(1+k) = 0 .$$

Equation 8.2.1 implies that

$$(1+w) - c(1+w)^2 - b(1+w) - bk(1+w) - b + ck(1+w)$$

$$- ck(1+w)^2 + c(1+w) + ck(1+w) - ck + b + bk = 0 ,$$

or somewhat differently

$$(1+w)^2(c)(1+k) + (1+w)(- 1 + b + bk - ck - ck - c) \qquad 8.2.1a$$

$$+ (ck - bk) = 0 .$$

Solving for $(1+w)$, we get

$$1+w = \frac{(c(k+1) + ck + 1 - b - bk) \pm \sqrt{(\qquad)^2 - 4c(1+k)(ck-bk)}}{2c(1+k)} \qquad 8.2.2$$

It now appears that we have two warranted rates of growth. Fortunately, one of these warranted rates, w_{lower}, turns out to be negative, and so we may exclude it from consideration. The proof is fairly easy. If $w_L < 0$, then $1+w < 1$. We wish to show that one of the roots of equation 8.2.2 lies between 0 and 1. Plug in the value 0 (for 1+w) into equation 8.2.1a. This gives ck-bk, which is greater than 0. Now plug in for 1+w the value 1. This gives

$$c(1+k) + (-1 + bk + b - 2ck - c) + ck - bk$$

which equals $-1 + b$, and which, in turn, is less than 0. Consequently, since one value of 1+w is always between 0 and 1, we may exclude this expectation from consideration as a possible warranted rate of growth expectation.

The other warranted rate expectation is, however, permissible. We shall now show that the warranted rate of growth for equation 8.1.7 is less than $(1 + \frac{1-b}{c})$. The simplest way of showing this might appear to be to ask whether the top root of equation 8.2.1a is less than $(1 + \frac{1-b}{c})$. By squaring both sides of equation 8.2.2, we can then check the assertion. The author has done this and verified the above statement. The proof, however, is very cumbersome and tedious. An easier way of proceding is as follows. We know that at $(1+w) = 1$, the value of the left hand side of the characteristic equation 8.2.1a is negative. Now plug the value $1+w = (1 + \frac{1-b}{c})$ into the left hand side of equation 8.2.1a. If this implies that the left hand side of equation 8.2.1a is greater than 0, then our warranted rate of growth is less than $(1 + \frac{1-b}{c})$. Doing this, we have

$$((1 + 2(\frac{1-b}{c}) + (\frac{1-b}{c})^2)c(1+k) + (1 + \frac{1-b}{c})(- 1 + b + bk - 2ck - c)$$

$$+ ck - bk)$$

for the left hand side of equation 8.2.1a. This equals

$$(- 1 + b) + 2(\frac{1-b}{c})(1+k)c + c(1+k)((\frac{1-b}{c})^2$$

$$+ (\frac{1-b}{c})(- 1 + b + bk - 2ck - c) .$$

Expanding, we have

$$2(1-b) + 2k(1-b) + c(1+k)(\frac{1-b}{c})^2 - (\frac{1-b}{c})^2 + bk(\frac{1-b}{c})$$

$$- 2k(1-b) - 2(1-b) ,$$

or

$$k(\frac{1-b}{c})^2 + \frac{(1-b)bk}{c} ,$$

which is a positive quantity. Consequently, the permissible warranted rate of growth is between 1 and $(1 + \frac{1-b}{c})$. (Notice that when $k = 0$, the value $1 + \frac{1-b}{c}$ plugged into equation 8.2.1a gives 0, thus implying that in a fixed level model, with a desired fixed level of 0, the warranted rate of growth is $(1 + \frac{1-b}{c}))$.

An interesting point obtains when k becomes very high, i.e., when $k \to \infty$. Then

$$(1+w_{warranted}) \to \frac{((2ck - bk) + \sqrt{(\quad)^2 - 4ck(ck - bk)})}{2ck}$$

$$\to \frac{((2c - b) + \sqrt{(2c - b)^2 - 4c(c - b)})}{2c}$$

$$\rightarrow \frac{((2c - b) + \sqrt{(c + c - b)^2 - 4c(c - b)})}{2c}$$

$$\rightarrow \frac{((2c - b) + \sqrt{(c - (c - b))^2})}{2c}$$

$$\rightarrow \frac{(2c - b + c - c + b)}{2c}$$

$$\rightarrow 1$$

Thus, for $k \rightarrow \infty$, the warranted rate of growth approaches 1.

We may now ask what is the economic meaning of the fact that the warranted rate of growth is less than $(1 + \frac{1-b}{c})$. In our passive inventory model, we found that, when expectations were less than $(\frac{1-b}{c})$, and initial conditions met expectations, income contracted. Here, however, we find that income may continue to expand. Why? Clearly, what is happening is that, in the absence of inventory adjustments, the growth in income will not be sufficient to expand sales enough at the "low" rate of expectation, for explosion. With proportional inventory adjustments, however, sales will grow more, due to the fact that producing inventories generates income, and this, in turn, generates sales. Thus, a lower rate of expectations for sales is needed in the proportional inventory adjustment model than in the passive model for a warranted rate growth path.

We can also demonstrate that, along the warranted path, expectations will be fulfilled, capital will be fully employed, and production will equal expected sales plus desired inventory adjustments. The proof is as follows:

If

$$E_t = (1+w)E_{t-1}$$

and initial conditions are correct, then,

$$Y_t = ((1+w)(1+k)+1)E_{t-1} - (1+w)(1+k)E_{t-2}$$

$$= (1+k)E_t + E_{t-1} - (1+k)E_{t-1} \quad \text{along the warranted path}$$

$$= E_t + k(E_t - E_{t-1}) \quad ,$$

which is exactly what businessmen expected. Therefore, expectations for clearing of goods and inventory accumulation are realized along the warranted path.

Similarly,

$$K_{t+1 \text{ desired}} = c(1+w)^2(1+k)E_{t-1} - ck(1+w)E_{t-1}$$

$$= c(1+k)E_{t+1} - ckE_t \quad \text{along the warranted path}$$

$$= c(E_{t+1} + kE_{t+1} - kE_t) \text{ along the warranted path}$$

$$= cY_{t+1} \quad \text{along the warranted path}$$

$$= K_{t+1 \text{ needed}} \quad \text{along the warranted path.}$$

Therefore, capital in our model is just sufficient along the warranted path.

With information concerning the warranted rate now given, it becomes clear that if expectations equal the "warranted" expectation, then, depending on initial conditions, there will either be contraction, explosion, or equilibrium growth. What however, if expectations are not so fortuitously chosen? With the help of several familiar looking theorems, we shall be able to describe these situations.

<u>Theorem 8.2.1</u> : If $w < w_{warr}$, both roots of 8.1.7 are greater than $1+w_{warr}$. If $w > w_{warr}$, one of the roots of 8.1.7 is less then $1+w_{warr}$ and one root greater than $1+w_{warr}$.

<u>Proof</u>: The characteristic equation of 8.1.7 is

$$\lambda^2 - ((b)(1+w)1+k)+1) + c(1+k)(1+w)^2 - ck(1+w))\lambda \qquad 8.1.7a$$

$$+ b(1+w)(1+k) + c(1+k)(1+w)^2 - ck(1+w).$$

We first prove that the two roots of this characteristic equation are greater than 1. If $\lambda = 0$ then the left hand side of 8.1.7a is clearly greater than zero; if $\lambda = 1$ the left hand side equals 1-b which again is greater than zero. The two roots of 8.1.7a are therefore greater than one and the characteristic graph looks as

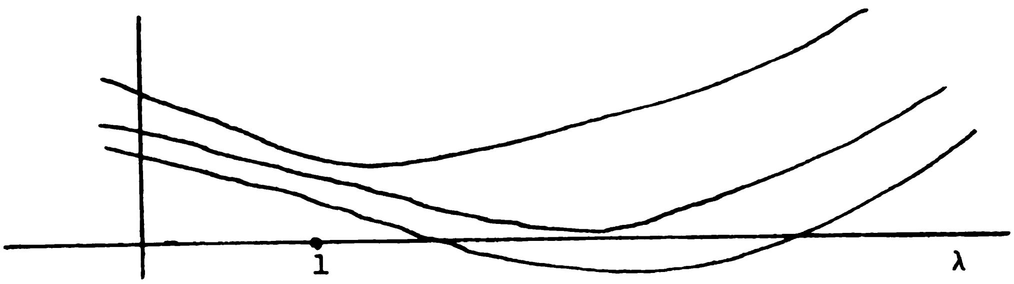

Figure 6. The Characteristic Graph of Equation 8.1.7

Now if we increase w, the coefficient of λ in the left hand side of 8.1.7a clearly increases by some positive amount. This amount is also equal to the increase in the third term of 8.1.7a. However, since $\lambda > 1$, this implies that the characteristic function decreases as w increases and increases as w decreases. Graphically, we have

Figure 7. Effects of Positive Δb, Δc, Δw, on the Characteristic Graph of Equation 8.1.7

Figure 8. The Characteristic Graph of Equation 8.1.7, when $\lambda = (1 + w_{warranted})$

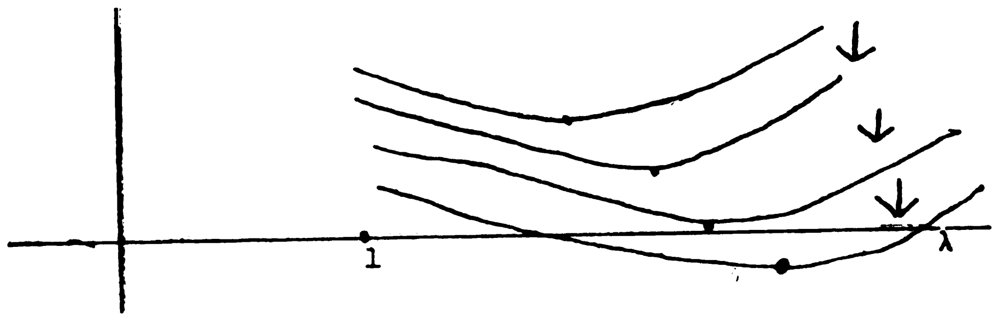

Figure 7.

Since for $w = w_{warranted}$, the characteristic graph looks as follows,

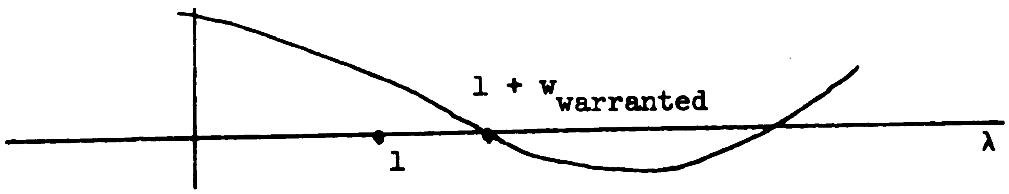

Figure 8.

an increase in w clearly causes the lower root to decrease below $(1+w_{warr})$, the upper root increasing, and a decrease in w causes the lower root to increase above $1+w_{warr}$, the upper root decreasing.

Theorem 8.2.2 : If $1+w < 1+w_{warr}$, the lower root of equation 8.1.7 is lower than that of equation 4.1.3. If $1+w > 1+w_{warr}$, it is impossible to say whether the lower root of 4.1.3 is higher or lower than that of 8.1.7, except when $b(1+w) > 1$, in which case the lower root of 4.1.3 is lower. In general, the higher $1+w$, the more likely the lower root of equation 4.1.3 will be less than that of equation 8.1.7.

Proof:

$$\lambda_{1L}^2 - c(1+w)^2 + b(1+w))\ \lambda_{1L} + c(1+w)^2 = 0 \quad , \qquad \begin{matrix} 8.2.6 \\ (4.1.3) \end{matrix}$$

where λ_{1L} is the lower root of equation 4.1.3. Consider

$$\lambda_{1L}^2 - (c(1+w)^2 + b(1+w) + bk(1+w) + b - ck(1+w) + ck(1+w)^2)\lambda_{1L}$$

$$+ (c(1+w)^2 + b(1+w) + bk(1+w) - ck(1+w) + ck(1+w)^2). \qquad 8.2.7$$

If this is less than zero, then there exists a root of equation 8.1.7 between 1 and λ_{1L}. If it is greater than 0, there is no such root. Subtract equation 8.2.6 from expression 8.2.7 to get

$$- \lambda_{1L}(bk)(1+w) + b - ck(1+w) + ck(1+w)^2) \qquad 8.2.8$$

$$+ (bk)(1+w) + b(1+w) - ck(1+w) + ck(1+w)^2) \quad .$$

We know that for $1+w > (1 + \frac{1-b}{c})$, $\lambda_{1L} > 1+w$. Then expression 8.2.8 is less than zero. Therefore, for $1+w < 1+w_{warr}$, the lower root of equation 8.1.7 is less than that of equation 4.1.3. Also, for $1+w > 1+w_{warr}$, λ_{1L} starts to decrease towards 1. For $\lambda_{1L} > 1$, it is impossible to say whether expression 8.2.8 is less or greater than zero. However, clearly, as $1+w$ increases and $\lambda_{1L} \to 1$, expression 8.2.8 is increasing. Therefore, as $\lambda_{1L} \to 1$, the chances are better that expression 8.2.8 becomes positive and the lower root of 4.1.3 is less than that of 8.1.7. For $b(1+w) > 1$, $\lambda_{1L} < 1$, and expression 8.2.8 is clearly greater than zero. Under these circumstances, $\lambda_{1L} < \lambda_{L8.1.7}$.

Theorem 8.2.3 : The upper root of the inventory equation 8.1.7 is always greater than that of equation 4.1.3.

Proof:

$$\lambda_{1H}^2 - (b(1+w) + c(1+w)^2)\,\lambda_{1H} + c(1+w)^2 = 0 \quad ,$$

4.1.3 where λ_{1H} is the upper root of 4.1.3. Plug λ_{1H} into 8.1.7, to get the expression

$$\lambda_{1H}^2 - (c(1+w)^2 + b(1+w) + bk(1+w) + b - ck(1+w) + ck(1+w)^2 L\ \lambda_{1H}$$
$$+ (c(1+w)^2 + b(1+w) + bk(1+w) - ck(1+w) + ck(1+w)^2) \ . \qquad 8.2.10$$

Now subtract 4.1.3 from 8.2.10 to get the expression

$$- \lambda_{1H}(bk(1+w) + b - ck(1+w) + ck(1+w)^2) + (ck(1+w)^2 \qquad 8.2.11$$
$$- ck(1+w) + b(1+w)(1+k)) \ .$$

Will 8.2.11 be negative? For $\lambda_{1H} > 1+w$ this is certainly true, and, therefore, because the value of infinity plugged into the characteristic equation of 8.1.7 gives a positive result, this implies that there exists a root of 8.1.7 between ∞ and λ_{1H}.

Theorem 8.2.4 : The upper root of 8.1.7 is greater than the upper root of 7.2.7. The lower root of 8.1.7 is lower than the lower root of 7.2.7.

Proof: Let $\lambda_{L7.2.7}$ be plugged into 8.1.7. Subtract 7.2.7 from this expression to get

$$(- b(1+w) - ck(1+w)^2 - ck(1+w))\ \lambda_{L7.2.7}$$

$$- (- b(1+w) - ck(1+w)^2 - ck(1+w)) .$$

For $\lambda_{L7.2.7} > 1$, this expression is negative. Since in equation 8.1.7, $\lambda = 1$ implies the left hand side is positive, there exists a root of equation 8.1.7 between 1 and $\lambda_{L7.2.7}$. The proof is exactly the same for showing that there exists a root between $\lambda_{H\,7.2.7}$ and $+\infty$ in equation 8.1.7.

With the above theorems, we can now compare proportional inventory adjustment growth to that under passive inventory adjustment, and also fixed level inventory adjustment.

First, it is clear from Theorem 8.2.4 that growth is more explosive under the proportional assumptions than it is under the fixed level assumptions. Furthermore, income is more likely to explode in the former than the latter. The reason for both these assertions is Theorem 8.2.4. This Theorem tells us that the higher root of 8.1.7 is higher than that of 7.2.7, implying greater growth eventually, no matter what the initial conditions, and that the lower root of 8.1.7 is lower than that of 7.2.7, implying that under given initial conditions, the former equation is more likely to explode.

However, this conclusion could also have been arrived at without recourse to Theorem 8.2.4, if we had reasoned in a more intuitive fashion. The only two differences between the fixed level and proportional models are that, in the latter, there is an additional positive term for inventory production in any period, and an additional term, also positive, for investment expenditures. (These terms will be positive for $Y_1 > Y_0$, which is our "valid" initial condition. For $Y_1 < Y_0$, these terms will be negative. However, under these circumstances, our equations no longer apply, since there is a negative component to investment expenditures.) Clearly, a model exactly the same as another, except for the fact that it has more positive forces causing income growth, will explode faster in an upward direction than the other. Also, it is clear that, for given initial conditions, this model is more likely to explode than the other. Indeed, if the two models start out with the same initial conditions, then, as long as income is increasing, income in the proportional model will be higher than that in the fixed level model. Only if income starts to decline will the proportional model ever have income less than that in the fixed model. In this case, as income falls, businessmen will decrease their inventories proportionally to the fall in income, thus reducing income more than in a fixed level model under similar circumstances, and eventually causing income to fall below that in a fixed level model.

It also is clear that income will explode much more rapidly, at least eventually, in the proportional model than in the passive model. Again, this mathematically follows from Theorem 8.2.3, which states that the higher root of 8.1.7 is greater than the higher root of 4.1.3. Economically, this means that as income increases, businessmen, if expectations for sales are exceeded, will attempt to do two things - (1) increase inventories on hand and (2) replace depleted inventories - in the proportional model, that they would not do in the passive model. The opposite conclusion is true in the opposite direction. As a result of income decreasing, businessmen will try to reduce desired inventories and also get rid of unwanted inventories which have inadvertantly piled up -- in the proportional model --

but will do neither in the passive model. This will cause income to fall even further in the proportional model than in the passive model, and this in turn will cause future consumption (expenditures) to be less in this model. In either direction income will change more rapidly in the proportional model than in the passive model.

Finally, as far as the likelihood of explosion or contraction is concerned, Theorem 8.2.2 tells us that the proportional model is more likely to explode for small w (w less than w_{warr}) than the passive model. This is due to the fact that if, for given initial conditions, a passive model contracts, it might still be that, initially, realized gains in sales are greater than expected gains in sales. In a proportional model, if these circumstances obtained, businessmen would try to replenish inventories which have been accidentally depleted, as well as increase inventories "proportionally" to the growth in income. These two effects might give the system just enough thrust to explode. On the other hand, for large w (w greater than w_{warr}), we cannot say whether the passive model or proportional model is more likely to explode, for given initial conditions. For now there are two effects in the proportional model -- each of which has a different sign. On the one hand, there is the positive proportional inventory effect; on the other hand, there may be, if expectations are higher than initial changes in sales, an accidental pile-up of unwanted inventories (as in the fixed level model). This effect will cause businessmen to reduce income in the future, as they try to satisfy demand, by depleting their unwanted inventories. Since the latter effect might be greater than the former, this might cause income to decrease relative to the passive model, thus decreasing the chances for explosion relative to the passive model.

CHAPTER 9

Section 9.1. Inventory Models with Depleted Inventories; the Passive Adjustment Case

So far, in all the models we have discussed, we have assumed there exists a sufficient amount of inventories to satisfy each period's consumption and investment demand.

In the passive inventory model of Chapter 4, for instance, we have assumed that $I_t = c(1+w)(Y_t - Y_{t-1})$. This implies that capital on hand in t was $c(1+w)Y_{t-1}$, and the latter implies that businessmen had no trouble in obtaining this level of capital. Likewise, in the inventory models of Chapters 7 and 8, it was assumed that there was never any problem in obtaining desired capital levels. As far as consumers are concerned, we have always posited $C_t = bY_t$, as consumption in each period. This, of course, implies that there is no unsatisfied demand from previous periods--i.e., that consumers had no problem in obtaining in previous periods those consumer goods which they had demanded.

What now if due to a lack of inventories, demand in a certain period cannot be satisfied for either capital or consumption goods? Is it possible to write equations for the growth of income, and to discuss the nature of this growth? The answer to this question is yes, and the problem can be handled in two ways.

Let us start, as usual, with the passive adjustment model. We have seen in the passive model that

$$C_t = bY_t \qquad 9.1.1$$

and

$$I_t = c(1+w)(Y_t - Y_{t-1}) \quad , \qquad 9.1.2$$

if demand has been satisfied in period t-1. What are the respective equations for consumption and investment, however, if demand has not been satisfied? The answer to this question is that they are the same if we introduce the concept of negative inventories into the passive model!

Suppose we change the passive model slightly by saying that, in this model, businessmen do not wish to let demand go unsatisfied, even though goods are not sufficient to satisfy this demand. Under these assumptions, the only way businessmen can do this is to promise to supply in the future what they cannot supply in this period. In making this assumption we say that businessmen are willing to incur negative inventories. Since consumers are now promised the goods which they demand, there is no sense in demanding them again, and, thus, in the next period they will demand only $C_{t+1} = bY_{t+1}$ of consumption goods. Similarly, if investors have been promised capital goods which they have not yet obtained, they will demand only $I_{t+1} = c(1+w)(Y_{t+1} - Y_t)$ of investment goods in any period. Thus, the expenditures functions in the passive adjustment model will remain the same as before (equations 4.1.1 and $C_t = bY_t$), if we allow businessmen to accumulate negative inventories.

However, equation 4.1.2 of the passive model does not. Businessmen have promised to deliver in the next period what they cannot deliver in this period. The only way they can do this, they will feel, is to increase production in the next period over next period's expected sales. Therefore, they will produce in period t+1

$$Y_t = (1+w)(bY_t + I_t) + \underline{bY_t + I_t - (1+w)(bY_{t-1} + I_{t-1})}\ , \qquad 9.1.3$$

where the underlined term is positive and represents those goods which they had to promise in t since inventories did not exist. We notice that equations 9.1.1, 9.1.2, and 9.1.3 are exactly the same as those in the fixed level adjustment model. We can, therfore, say that this new "passive" model we have assumed is essentially one in which businessmen do not care about maintaining inventories when inventories are positive, but do try to maintain inventories at a level of zero when inventories are negative. Also, since we have seen that the equation

$$Y_t = (c(1+w)^2 + b(1+w) + b)Y_{t-1} - (c(1+w)^2 + b(1+w))Y_{t-2} \qquad 7.2.7$$

is more explosive than our fundamental equation of simple growth 4.1.3, we clearly can say that, when inventories have been depleted, the growth of income will accelerate more than in a passive model with positive inventories. Thus, the stock of inventories will become even more depleted and negative inventories (promises to supply goods in future periods) will grow more and more.

The perceptive reader may have noted that we have actually used an artifice to arrive at these conclusions. For there can be no getting around the fact that we have changed our passive model

fundamentally with the introduction of negative inventories, by making a model, which is passive when inventories are positive, into a non-passive, zero fixed level maintenance, model, when inventories are negative. Our model, while it may be perfectly valid in the real world, does make inventory assumptions which are slightly "inconsistent" in the two areas of positive and negative inventories.

In reality, the only reason we have done this is to show parallels between the negative inventory passive model of this section and the negative inventory fixed level and proportional models of the next section. The latter two models are models in which the introduction of the concept of negative inventories is in no way inconsistent with our previous assumptions. In view of the "inconsistency" which exists in the passive model, however, it seems desirable to see if there is another way to describe passive inventory adjustment growth when inventories have become depleted.

Let us assume, therefore, that businessmen do not care about satisfying the "excess" demand of a certain period but still do try to anticipate future sales. This is of course a "completely" passive model. We can, under these assumptions, write

$$Y_t = (1+w)(C_{t-1} + I_{t-1}) \quad . \qquad 9.1.4$$

However, C_t and I_t will now be different from their former values. In particular, for consumers, we now realize that C_t is no longer bY_t but rather

$$C_t = bY_t \underline{+ C_{t-1} - (1+w)C_{t-2}} \; (C_{t-1} - (1+w)C_{t-2} > 0) \, . \qquad 9.1.5$$

The underlined term represents the unsatisfied consumer demand of the previous period and is equal to total consumption desired in period t-1 minus anticipated consumption production in this period, namely $(1+w)C_{t-2}$. For investors, we also realize that investment expenditures in period t are no longer given by

$$I_t = c(1+w)^2((C_{t-1}+ I_{t-1}) - (C_{t-2} + I_{t-2})) ,$$

but rather must be written as

$$I_t = c(1+w)^2((C_{t-1}+ I_{t-1}) - (C_{t-2} + I_{t-2})) \qquad 9.1.6$$

$$\underline{+ I_{t-1} - (1+w)I_{t-2}} \quad .$$

Again, our interpretation is the same. In addition to the usual part of the demand for capital goods, there is also an unsatisfied demand part - $(I_{t-1} - (1+w)I_{t-2})$ - where $(I_{t-1} - (1+w)I_{t-2})$ is greater than zero. This part of demand represents the excess of total investment demand in t-1 over total investment goods produced in the same period, and is added to business demand for capital in period t.

We now have three equations in three unknowns:

$$Y_t = (1+w)(C_{t-1} + I_{t-1}) = (1+w)E_{t-1} \quad , \qquad 9.1.4$$

$$C_t = bY_t + C_{t-1} - (1+w)C_{t-2} \quad , \qquad 9.1.5$$

$$I_t = c(1+w)^2(E_{t-1} - E_{t-2}) + I_{t-1} - (1+w)I_{t-2} \qquad 9.1.6$$

To solve, let us write

$$E_t = C_t + I_t \qquad 9.1.7$$

$$= bY_t + C_{t-1} - (1+w)C_{t-2} + c(1+w)^2(E_{t-1} - E_{t-2})$$

$$+ I_{t-1} - I_{t-2}(1+w)$$

$$= b(1+w)E_{t-1} + c(1+w)^2(E_{t-1} - E_{t-2}) + E_{t-1} - (1+w)E_{t-2} .$$

Upon rewriting equation 9.1.7, we have

$$E_t = (b(1+w) + c(1+w)^2 + 1)E_{t-1} - ((c(1+w)^2 + (1+w))E_{t-2} . \qquad 9.1.8$$

Also, since $Y_t = (1+w)E_{t-1}$,

$$Y_t = (b(1+w) + c(1+w)^2 + 1)Y_{t-1} - (c(1+w)^2 + (1+w)))Y_{t-2} . \qquad 9.1.8a$$

(Needless to say, these equations for E_t and Y_t could also have been derived with the use of E operators. However, we shall omit the proof.)

We now have two different models which describe inventory adjustment growth when inventories have become depleted. It is of interest to ask which of these two models is more likely to explode, under given initial conditions, and which has the greater speed of explosion.

To answer the latter question, we must investigate the "higher" root of each equation. Clearly, the model which has the higher "higher" root has the greater speed of explosion.

We now prove that the higher root of equation 9.1.8 is greater than that of equation 7.2.7, and that the "completely" passive model has a greater speed of explosion than the negative inventory passive model. The proof is as follows:

We know that

$$\lambda_H^2 - (c(1+w)^2 + b(1+w) + b)\ \lambda_H + c(1+w)^2 + b(1+w) = 0 . \qquad 7.2.7a$$

(λ_H is the top root of 7.2.7a)

Plug λ_H into the left hand side of equation 9.1.8 to get

$$\lambda_H^2 - ((c(1+w)^2 + b(1+w) + 1))\ \lambda_H + c(1+w)^2 + (1+w) .$$

Subtracting zero or equation 7.2.7a from this expression, we have

$$- \lambda_H(1-b) + (1-b)(1+w) ,$$

or

$$- (1-b)(\lambda_H - (1+w)) , \qquad 9.1.9$$

which for $\lambda_H > (1+w)$ is negative. But since the left hand side of 9.1.9 is negative, and since $+\infty$ when plugged into the left hand side of equation 9.1.8 gives a positive (plus infinity) value, there must exist a root of 9.1.8 between λ_H and $+\infty$. Thus, the top root of 9.1.8 is greater than that of 7.2.7, and the completely passive model adjusts faster than the negative inventory model.

We now wish to investigate which of the models is more likely to explode. To do this let us investigate the lower roots of the two equations. Let us repeat the process in the above proof only this time with λ_L of equation 7.2.7. We now obtain

$$- (1-b)(\lambda_L - (1+w)) . \qquad 9.1.10$$

If this expression is less than zero, then since the value $\lambda_L = 1$ plugged into the left hand side of equation 9.1.8 gives $(1+w)(1-b)$ a positive result, the lower root of 9.1.8 will be less than that of 7.2.7. For $1+w < 1 + \frac{1-b}{c}$, the completely passive model will be more likely to explode by Theorem 7.2.1 which states that $\lambda_L > (1+w)$ for equation 7.2.7. For $1+w > 1 + \frac{1-b}{c}$, however, $\lambda_L < 1+w$. It would thus appear that, for these values of w, the negative inventory model is more likely to explode. This, however, is incorrect. For when $1+w > 1+ \frac{1-b}{c}$, it must be that $E_1 > (1+w)E_o$. Otherwise, equations 9.1.8 and 7.2.7 would not be valid (see conditions in equations 9.1.3, 9.1.5, and 9.1.6). Since both models, for $1+w > 1+ \frac{1-b}{c}$, have roots less than $1 + \frac{1-b}{c}$, both models are guaranteed to explode when $1+w > 1 + \frac{1-b}{c}$.

We can now see that the completely passive model accelerates more quickly than the negative inventory model. There is, however, a much cleaner proof than the above, and since we shall use an exactly similar proof in the next section, let us introduce it here. We know that we can rewrite equation 9.1.8 as

$$E = (b(1+w) + c(1+w)^2 + b)E_{t-1} - c(1+w)^2 + b(1+w))E_{t-2}$$

$$\underline{+ ((1-b)(E_{t-1} - (1+w)E_{t-2})} \, .$$

The nonunderlined part of this equation is nothing more than equation 7.2.7. For $E_{t-1} > (1+w)E_{t-2}$, the underlined part of this equation is always greater than zero. Therefore, since the completely passive model is the same as the negative inventory model, except for an always positive term in its difference equation, it must always explode faster and be more likely to explode than the negative inventory passive model. The latter results from Theorem 5.2.2.

Mathematically, we have now shown that a negative inventory model is less explosive than a completely passive model, when inventories are depleted. We may ask, however, why we might expect this result from an economic point of view. The reason is rather subtle. In the negative inventory model, businessmen increased output immediately to satisfy demand. As a result of this, demand in the next period was less than it would have been in a completely passive model. Or stated differently, there was a once and for all effect on production, and businessmen felt that the deficit in the period would not continue. Accordingly, they did not multiply their production plans by $(1+w)$ times the deficit. In the completely passive model, however, the lack of satisfaction by consumers and investors causes demand to be raised. Because there is increased demand for goods and services, and because businessmen cannot separate one demand from another, they believe that this demand will continue forever. As such, they make plans to produce this level of income and in the following period, they buy even more capital to do so. This, in turn, will make for larger future incomes. Hence, we can expect that income will grow faster in a completely passive model than in a passive negative inventory model. Finally, just to repeat a point, in both models the growth of income will be more likely to explode and have a greater speed of explosion, when inventories have been depleted; it is simply that in the "completely" passive model the likelihood and speed of explosion are greater.

Section 9.2. Depleted Proportional and Fixed Level Inventory Adjustment Models

We now turn to the case of inventory depletion, when businessmen are attempting to maintain a level of inventories in each period which is either proportional to expected sales in that period or at some fixed level.

Again, we start with the negative inventory model of the previous section. We saw in this section that the concept of negative inventories in a passive model implied a change in businessmen's behavior in this model when inventories became depleted. For initially in this model, when inventories were positive, there was completely passive behavior by businessmen; when inventories became depleted, however, the concept of negative inventories implied a zero fixed level type of adjustment. Thus, business behavior, under the assumption of negative inventories, was "assymetrical" between the positive and negative regions of inventory levels.

Fortunately, no such "assymetry" exists when the concept of negative inventories is introduced into the proportional and fixed level models of Chapters 7 and 8. If businessmen are attempting to maintain some level of inventories, whether fixed or proportional, we can still assume that they wish to do this, even if they have at present run out of inventories. Stated differently, the desire to maintain some positive level of inventories should in no way be affected by the fact that businessmen have run out of inventories. Thus, the expression for inventory adjustment in nonpassive models is exactly the same when businessmen have promised to customers to supply goods that they do not presently have as when they still have some positive level of inventories. We can therfore still write

$$Y_{t \text{ for inventories}} = k(1+w)(bY_{t-1} + I_{t-1}) - k(1+w)(bY_{t-2} + I_{t-2})$$

$$+ (bY_{t-1} + I_{t-1}) - (1+k)(1+w)(bY_{t-2} + I_{t-2}) .$$

The only difference between the two cases is that now when inventories are depleted

$$(bY_{t-1} + I_{t-1}) - (1+w)(1+k)(bY_{t-2} + I_{t-2}) > 0 , \qquad 9.2.1$$

whereas formerly the reverse was true--namely,

$$(bY_{t-1} + I_{t-1}) - (1+w)(1+k)(bY_{t-2} + I_{t-2}) < 0 . \qquad 9.2.2$$

(What this means, of course, is that, in the present instance, misjudged expenditures have exceeded production of goods for all purposes by more than production for the sake of inventories so that inventory positions are now negative. Formerly, however, while expenditures may have been misjudged, they were not misjudged so heavily as to make inventories negative.) Nonetheless, as far as our equation is concerned, this difference is of no consequence. Also, since goods have been promised to consumers and investors, we still have the same consumption and investment functions as before when inventories were positive. Therefore,

$$C_t = bY_t$$

$$I_t = (c(1+w)^2(1+k)E_{t-1} - c(1+w)kE_{t-1}) - (\qquad {}_{t-2} \qquad).$$

We can now easily see that the equations for the growth of income are exactly the same as before--namely, equations 8.1.6 and $C_t = bY_t$.

Running out of inventories in the fixed level and proportional adjustment models, in no way whatsoever, changes the growth pattern of the economy when we make the assumption that businessmen are willing to incur negative inventories. In particular, the system will neither accelerate more nor be more likely to explode in the region of negative inventories than in the region of positive inventories. This means that, in the non-passive models, the initial level of inventories is completely irrelevant as far as our problem is concerned. This result clearly differs from the result we obtained for the passive model, where we found that running out of inventories causes the system to grow faster.

The model we have just described is a perfectly valid one. However, there may be some who, as we have said above, do not like the concept of negative inventories. These people would argue that even in a proportional or fixed level adjustment model, businessmen will not attempt to "take" customers' orders when inventories become depleted, but rather will simply say that no more goods exist. Since the latter assumption is not necessarily less realistic than our negative inventory assumption, let us attempt to build fixed level and proportional models very similar to the completely passive model of the previous section, where unsatisfied demand from previous periods carries over.

We start first with the fixed level model. Businessmen, as usual, will produce in this model

$$Y_t = (1+w)(C_{t-1} + I_{t-1}) \quad \text{for sales purposes} \quad .$$

Also, since inventories in any period are now assumed to be zero, $Y_t = C_o + I_o$ for inventory purposes, where C_o and I_o are the desired levels of consumption and investment inventories. Therefore,

$$Y_t = (1+w)(c_{t-1} + I_{t-1}) + C_o + I_o \quad . \qquad 9.2.3$$

C_t is now given exactly as before in the "completely" passive model, except that total production of consumption goods in period t, instead of being just $(1+w)C_{t-2}$, is now $(1+w)C_{t-2} + C_o$. Therefore,

$$C_t = bY_t + C_{t-1} - ((1+w)C_{t-2} + \underline{C_o}) \; . \qquad 9.2.4$$

Similarly,

$$I_t = c(1+w)^2(bY_{t-1} + I_{t-1}) - c(1+w)^2(bY_{t-2} + I_{t-2}) \qquad 9.2.5$$

$$+ I_{t-1} - ((1+w)I_{t-2} + \underline{I_o}) \; .$$

(The underlined terms are the only difference in the investment and consumption functions of our present model and those of the completely passive model of the previous section.)

We now have three equations in three unknowns. To solve, transform the equations to expenditures variables to get

$$Y_t = (1+w)E_{t-1} + C_o + I_o \; , \qquad 9.2.3a$$

$$C_t = bY_t + C_{t-1} - (1+w)C_{t-2} + C_o \; , \qquad 9.2.4a$$

$$I_t = c(1+w)^2(E_{t-1} - E_{t-2}) + I_{t-1} - ((1+w)I_{t-2} + I_o) \, . \qquad 9.2.5a$$

Using the identity $E_t = bY_t + I_t$, we have

$$E_t = b(1+w)E_{t-1} + b(C_o + I_o) + c(1+w)^2 E_{t-1} - E_{t-2}) + E_{t-1}$$

$$- (1+w)E_{t-2} - (C_o + I_o),$$

or

$$E_t = (c(1+w)^2 + b(1+w) + 1)E_{t-1} - (c(1+w)^2 + (1+w))E_{t-2}$$

$$- (1-b)(C_o + I_o), \qquad 9.2.6$$

To obtain the equivalent equation for Y_t, we make use of equation 9.2.3 above. Plugging the expression for Y_t into equation 9.2.6 gives

$$\frac{(Y_{t+1} - C_o + I_o)}{(1+w)} = \frac{(c(1+w)^2 + b(1+w) + 1)(Y_t - (C_o + I_o))}{(1+w)}$$

$$- \frac{(c(1+w)^2 + 1 + w)(Y_{t-1} - C_o + I_o)}{(1+w)} - (1-b)(C_o + I_o)$$

or,

$$Y_{t+1} - (C_o + I_o) = (c(1+w)^2 + b(1+w) + 1)Y_t - (c(1+w)^2$$

$$+ (1+w))Y_{t-1} + ((1+w)(b-1) + 1)(-C_o - I_o)$$

$$- (1+w)(1-b)(C_o + I_o)$$

or,

$$Y_{t+1} = (c(1+w)^2 + b(1+w)+1)Y_t - (c(1+w)^2 + (1+w))Y_{t-1} + 0 \qquad 9.2.7$$

Equation 9.2.7 is exactly the same as equation 9.1.8a. Clearly, this is as it should be if $C_o + I_o = 0$. If $C_o + I_o \neq 0$, however, our equation implies that these values will be reflected only in the initial conditions to our problem. Again, as in Chapter 7, the values of C_o and I_o are completely irrelevant to the problem of the nature of growth. Finally, since we have already discussed the characteristics of growth under equation 9.1.8a , there is no need to do so here.

Let us now discuss proportional inventory adjustment, when inventories have become depleted. In this case, we have

$$Y_t = (1+w)(C_{t-1} + I_{t-1}) + k(1+w)(C_{t-1} + I_{t-1}) - 0. \qquad 9.2.8$$

The 0 arises because we are assuming that businessmen have no physical inventories on hand and, also, have not promised any either.

The expression for C_t is the same as before in the fixed level depleted model, except that now total production in t-1 of consumption goods is $(1+w)(1+k)C_{t-2}$, whereas formerly it was $C_o + (1+w)C_{t-2}$. Therefore,

$$C_t = bY_t + C_{t-1} - (1+w)(1+k)C_{t-2} \qquad 9.2.9$$

Similarly, the expression for I_t is

$$\begin{aligned} I_t = {} & c(1+w)^2(1+k)(C_{t-1}+I_{t-1})-ck(1+w)(C_{t-1}+I_{t-1}) \qquad 9.2.10 \\ & -c(1+w)^2(1+k)(C_{t-2}+I_{t-2})+ck(1+w)(C_{t-2}+I_{t-2}) \\ & +I_{t-1}-(1+w)(1+k)I_{t-2} \ . \end{aligned}$$

It should be clear that these equations are valid only if

$$(C_{t-1}+I_{t-1}) > (1+w)(1+k)(C_{t-2}+I_{t-2}) = (1+w)(1+k)E_{t-2} \ .$$

Rewriting these equations in expenditures variables, we have

$$E_t = C_t + I_t - bY_t + c(1+w)^2(1+k)(E_{t-1}-E_{t-2})-ck(E_{t-1}-E_{t-2})+E_{t-1}$$

$$- E_{t-2}(1+w)(1+k)$$

$$E_t = b(1+w)(1+k)E_{t-1}+c(1+w)^2(1+k)(E_{t-1}-E_{t-2})-ck(E_{t-1}-E_{t-2})+E_{t-1}$$

or,

$$E_t = (c(1+w)^2(1+k)-ck(1+w)+b(1+w)(1+k)+1)E_{t-1} \qquad 9.2.11$$

$$-((c(1+w)^2(1+k)-ck(1+w)+(1+w)(1+k))E_{t-2} \ .$$

Likewise, since $Y_t = (1+w)(1+k)E_{t-1}$ we have

$$Y_t = (c(1+w)^2(1+k)-ck(1+w)+b(1+w)(1+k)+1)Y_{t-1} \qquad 9.2.12$$

$$-(c(1+w)^2(1+k)-ck(1+w)+(1+w)(1+k)Y_{t-2} \ .$$

Equation 9.2.11 is the same as equation 8.1.7 which is the negative inventory proportional model equation except for two facts. First, the b term in the coefficient of Y_{t-1} in 8.1.7 has become a 1, and second, the $b(1+w)(1+k)$ in the coefficient of Y_{t-2} in 8.1.7 has become a $(1+w)(1+k)$.

We may now ask which of the two proportional depleted inventory models is more explosive and which is more likely to explode. The answer to this question is the same as in the passive model of the previous section--namely, that the passive proportional model is more explosive and likely to explode than the negative inventory proportional model. Again, the reason is very simple and exactly similar to the argument on pages 177 and 178.

Equation 9.2.11 can be rewritten as

$$E_t = (b(1+w)(1+k)+c(1+w)^2(1+k)+b+(1-b)-ck(1+w))E_{t-1}$$

$$-(b)(1+w)(1+k)+c(1+w)^2(1+k)+(1-b(1+w)(1+k)-ck(1+w)E_{t-2} .$$

or,

$$E_t = \text{equation } 8.1.7 + (1-b)E_{t-1} - (1-b)(1+w)(1+k)E_{t-2} \qquad 9.2.12$$

But equation 9.2.1 is valid only if

$$E_t > (1+w)(1+k)E_{t-1} \quad \text{for all} \quad t ,$$

since otherwise inventories in each period would be positive. Therefore, because we have added to our negative inventory model terms which are always positive by the nature of our problem, it must be that the new model, the passive proportional model, is more explosive and likely to explode than the negative inventory model (see pages 177 and 178).

We now have all the information necessary to describe growth for all levels of inventories. Our conclusions may be summarized as follows:

1) For inventory adjustment in which businessmen act passively when inventories are positive and incur negative inventories when inventories are depleted, there is an increase in the speed and likelihood of explosion when inventories become depleted.

2) For inventory adjustment in which businessmen act passively when inventories are positive and do not wish to incur negative inventories when inventories are depleted (the completely passive model),

there is an increase in the speed and likelihood of explosion, when inventories become depleted. In this case, the speed and likelihood of explosion are even greater than in the above case.

3) For proportional and fixed level models in which we allow negative inventories, the economy continues to act in exactly the same fashion in the region of negative inventories as it did in the region of positive inventories.

4) In both fixed level and proportional models, if negative inventories are not allowed (the passive fixed level or passive proportional models), there is an increase in the speed and likelihood of explosion when inventories become depleted.

CHAPTER 10

Section 10.1. Generalizing the Non-Passive Inventory Adjustment Models to Non-Constant Expectations

In the last three chapters, we have discussed Harrod-like growth models, under assumptions of non-passive inventory adjustment and constant multiplicative expectations. Because we have not considered expectations other than those of the constant multiplicative type, it would seem appropriate at this point to discuss more general types of expectations - namely, additive and adaptive expectations - and their relation to the non-passive models.

The author, however, must admit that he sees very little value in doing this at all! It seems intuitively obvious that introducing additive and adaptive expectations should in no way cause changes in the patterns of growth in the non-passive models, which are different from the types of changes already discussed in Chapters 5 and 6 with respect to the simple Harrod-like model. On the other hand, a great deal of tedious and cumbersome work will be involved - in particular, for the additive model. Nonetheless, in the spirit of "verifying" mathematically all our conclusions, we shall now briefly indicate why the changes in growth patterns as a result of introducing adaptive expectations (the more important of the two expectations) into our non-passive models are exactly the same as those caused by introducing adaptive expectations into the passive model.

Let us, therefore, ask what are the equations for the growth of income when expectations, themselves, change - in particular, when businessmen expect sales in period t to increase over sales in period $t-1$ by the same percentage that they increased between periods $t-1$ and $t-2$.

The latter condition can be represented mathematically, by writing $W_t = \frac{E_{t-1}}{E_{t-2}}$. If, for simplicity, we consider a proportional negative inventory model, our equation of growth when expressed in expenditures terms becomes

$$E_t = (b((1+w_t)(1+k)+1)+c(1+k)(1+w_t)^2-c(k)(1+w_t))E_{t-1} \quad 10.1.1$$

$$+(-b(1+k)(1+w_{t-1})-c(1+k)(1+w_{t-1})^2+ck(1+w_{t-1}))E_{t-2} .$$

Equation 10.1.1 is of course simply equation 8.1.7 with the appropriate w_t substituted for w. (This is exactly the same procedure we used in Chapter 6 on page 124.) Substituting $w_t = \frac{E_{t-1}}{E_{t-2}}$ into equation 10.1.1, we have

$$E_t = (b(\left(\frac{E_{t-1}}{E_{t-2}}\right)(1+k)+1)+c(1+k)\left(\frac{E_{t-1}}{E_{t-2}}\right)^2-ck\left(\frac{E_{t-1}}{E_{t-2}}\right))E_{t-1} \quad 10.1.2$$

$$-(b\left(\frac{E_{t-2}}{E_{t-3}}\right)(1+k)+c(1+k)\left(\frac{E_{t-2}}{E_{t-3}}\right)^2-ck\left(\frac{E_{t-2}}{E_{t-3}}\right))E_{t-2} .$$

The solution to 10.1.2 looks terribly formidable. Nonetheless, just as in Chapter 6, we can now very easily show, with the aid of Theorem 10.1.1 below, that only three types of growth are possible - explosion, contraction, and constant growth at the warranted rate - and we can easily state under what conditions each of these results.

<u>Theorem 10.1.1</u>: If $\frac{E_{t-1}}{E_{t-2}} > (1+w_{warr})$ and $\frac{E_{t-1}}{E_{t-2}} > \frac{E_{t-2}}{E_{t-3}}$, the economy explodes. If $\frac{E_{t-1}}{E_{t-2}} < (1+w_{warr})$ and $\frac{E_{t-1}}{E_{t-2}} < \frac{E_{t-2}}{E_{t-3}}$, the economy contracts to zero income.

<u>Proof</u>: Proving Theorem 10.1.1 is equivalent to proving for equation 8.1.7 that $E_2 > (1+w)E_1$, if $E_1 = (1+w)E_o$ and $w > w_{warr}$, and that $E_2 < (1+w)E_1$, if $E_1 = (1+w)E_o$ and $w < w_{warr}$. (See Theorem 6.2.1.) Therefore, letting $E_1 = (1+w)E_o$ and substituting E_1 into equation 8.1.7, we obtain

$$\begin{aligned} E_2 &= (b((1+w)(1+k)+1)+c(1+w)^2(1+k)-ck(1+w)(1+k))E_o(1+w) \\ &\quad -(b(1+k)+c(1+k)(1+w)-ck))(1+w)E_o \\ &= (1+w)E_o(b(1+w)(1+k)+b+c(1+k)(1+w)^2-ck(1+w)-b(1+k) \\ &\quad -c(1+k)(1+w)+ck) \end{aligned} \qquad 10.1.3$$

The expression in brackets for $1+w = 1+w_{warr}$ is equal to $(1+w_{warr})$. This can easily be seen, by writing the expression in brackets as

$$(c(1+k)(1+w)^2+(1+w)(b+bk-ck-c(1+k))+(b-b-bk+ck)) , \qquad 10.1.4$$

which is exactly the same as the left hand side of expression 8.2.1a except for the term $(1+w)(-1)$. Therefore, for $(1+w) = (1+w_{warr})$, the term in brackets equals $(1+w)$ and $E_2 = (1+w)^2E_o$, or $E_2 = (1+w)E_1$. (Incidentally, this proves that in an adaptive inventory model, the same warranted rate of growth exists as in a constant multiplicative model.) Furthermore, if we change w by Δw in expression 10.1.4, we have

$$\Delta(\quad) = c(1+k)((1+w+\Delta w)^2-(1+w)^2)+(1+w+\Delta w-(1+w))(b+bk-ck-c(1+k)$$

$$+(b-b-bk+ck)-(b-b-bk+ck)$$

$$= c(1+k)(2\Delta w(1+w)+(\Delta w)^2)+\Delta w(b+bk-ck-c(1+k))$$

$$= \Delta w(2c(1+k)(1+w)+(b+bk-2ck-c))+(\Delta w)^2 c(1+k)$$

$$= \Delta w(2c(1+k)w+b+bk+c)+(\Delta w)^2 c(1+k)$$

$$> \Delta w .$$

This proves our initial assertion and hence the theorem.

Theorem 10.1.1 now enables us to state all that we want to know about the types of growth under adaptive expectations, without explicitly solving equation 10.1.2. First, there are three types of growth: explosion at ever increasing rates; contraction to zero income; and constant growth at the warranted rate, given by equation 8.2.2. Constant growth at a rate other than the warranted rate is now impossible in an adaptive model. For if sales grew in two consecutive periods at a rate $1+w > 1+w_{warr}$, then by the first half of Theorem 10.1.2 explosion will occur; and if $1+w < 1+w_{warr}$, contraction would occur. However, just as before in Chapter 6, growth continually approaching the warranted rate of growth, and eventually for large t becoming virtually identical to warranted growth, is possible, but only if the initial conditions are rigidly specified (i.e., there exists only one set of initial conditions which allows this). This path of growth, like the warranted path, is very unstable.

We can now see that introducing adaptive expectations into inventory models changes growth from the constant expectations model in exactly the same way that introducing adaptive expectations into passive models does. In particular, adaptive expectations disallow constant rate non-warranted growth, and allow the possibility of a return to the warranted path. Furthermore, since eventually expectations are being constantly disappointed, capital production and inventory accumulation will again be insufficient, when explosion occurs, and overly sufficient, when contraction occurs. For warranted growth, expectations will be just what businessmen hoped they would be.

Section 10.2. Cycles within Long Term Growth

So far in this dissertation, we have shown how to construct perfectly general inventory models under two types of expectations - constant multiplicative expectations and adaptive expectations. In these respective models, we have seen that three types of growth are possible, and that which type of growth occurs depends on the magnitude of expectations and initial conditions. The analysis that we have used, however, does lead to one conclusion that is at variance with growth as it actually occurs in the real world. In all our models, there is no possibility of cycles within the long term growth trend.

In the constant multiplicative model, for instance, growth rates for nonwarranted growth either (1) first increase and then decrease, leading to contraction of income, or else (2) continually increase approaching some highest rate. In the latter case, there is clearly no decrease in the rate of growth, let alone a decrease in income in any period, and in the former case, while there is one turning point of a cycle, there is no second turning point to complete the cycle. Similarly, in the adaptive model, the rate of growth for nonwarranted growth can either first decrease and then increase continually, first increase and then decrease continually, or increase or decrease continually. In all cases, we can never once get a full business cycle. This conclusion is, of course, also true in all models for warranted growth.

Such noncyclical growth, however, does not agree with the historical growth patterns which have emerged over the last hundred years in the United States and Western Europe. During this period, there has been a long term growth trend, with frequent downturns not only in the rate of growth but also in the level of income (i.e., business cycles). The question now arises as to whether it is possible to incorporate in our models assumptions which will lead to business cycles within a long term growth trend.

In order to answer this question, it may be helpful to recall our earlier statements, in Chapter 1, with respect to trend and cycle growth. At that time, we pointed out that three different types of assumptions could account for the above mentioned growth patterns. These assumptions were: 1) trend growth with stochastic disturbances, 2) trend growth with endogenous perturbations, 3) trend growth with external constraints. We would now like to ask how these assumptions can be fitted into our model and whether they are empirically reasonable explanations of trend and cycle growth.

The first two are the simplest and we shall begin here. Suppose we consider the passive adjustment model of Chapter 4 where

$$Y_t = (b(1+w)+c(1+w)^2)Y_{t-1}+c(1+w)^2Y_{t-2} .$$

Suppose also that we use as a proxy for stochastic expenditures disturbance the expenditures function

$$E_t = A^t\sin(\theta t).$$

Then the equation for Y_t now becomes

$$Y_t = (b(1+w)+c(1+w)^2Y_{t-1}+c(1+w)^2Y_{t-2}+A^t\sin\theta t) . \qquad 10.2.1$$

The solution to this equation is

$$Y_t = A(\lambda_1)^t+B_2(\lambda_2)^t+C(A^t)\sin\theta t , \qquad 10.2.2$$

where C does not depend on initial conditions. If we now restrict A to be less than λ_2, we will obtain growth with cycles. Unfortunately, however, it will not be trend growth, but rather growth at ever increasing rates. Only as $t \to \infty$, will a trend emerge, and then the rate of growth will be $(\lambda_2-1)\%$, which for reasonable values of b, c, and w will be well over 100%. Thus the only way to obtain reasonable trend and cycle growth is to posit an additional exogenous expenditures term $E_t = E_o(1+r)^t$, where r is some reasonable trend estimate of growth. In this case the formula for Y_t becomes

$$Y_t = A_1(\lambda_1)^t+B_2(\lambda_2)^t+C(A)^t\sin\theta t+D(1+r)^t . \qquad 10.2.3$$

If $B_o = 0$ and $A < (1+r)$, we get the desired conclusion.

If we consider a model with no exogenous disturbances, we can also get trend and cycle growth in a similar manner. Simply assume that there exists a trend component to expenditures and that the solution to the homogeneous part of equation 4.1.3 is oscillatory. Assume also that the value (1+r) in the trend component function, $E_o(1+r)^t$ is greater than R, where R is the modulus of the complex roots in the oscillatory homogeneous solution. Then

$$Y_t = R^t(A_o\cos(\theta t)+B_o\sin\theta t) + C(1+r)^t, \qquad 10.2.4$$

which gives us the desired conclusion.

The models given above, therefore, do tend to give us the "correct" growth patterns. Unfortunately, these models seem to be singularly barren. For, first, they require that the parameters of our problem be such that the magnitude of the oscillations be less than the magnitude of the trend rate of growth. Second, they are not necessarily stable growth patterns. In the case of exogenous disturbances, the slightest additional disturbance will cause B_o to become positive or negative, thus causing a non-trend pattern to occur. Third, they imply that the major causes of growth and cycles are exogenous. The question therefore remains as to whether cycles can be explained endogenously.

To answer this, let us investigate how our dynamic models are changed by the introduction of a labor force growing at a rate less than the maximum possible rate of explosion. (This, it will be remembered, is the third approach we discussed in Chapter 1.) As a first example, let us consider a passive model with infinite initial inventories. Suppose that for some parameters w, c, b, and some initial conditions Y_1, Y_o, we get explosive growth. Suppose, also, that, in some period t, labor becomes fully employed, and that, in subsequent periods, labor grows in such a way that income can increase at most by L_o percent each period. In the latter case, we can write that after this period t,

$$E_t = c(1+w)^2(E_{t-1}-E_{t-2})+bA_o(1+L_o)^t , \qquad 10.2.5$$

where

$$(1+w)E_t > A_o(1+L_o)^{t+1} . \qquad 10.2.6$$

The derivation of equation 10.2.5 is quite simple. We start with equation 4.1.1b which states that

$$I_t = c(1+w)^2(E_{t-1}-E_{t-2}) .$$

Since, by assumption, there is a limitation on the rate of growth of income after some period t, Y_t does not equal $(1+w)E_{t-1}$, but rather equals $A_o(1+L_o)^t$, after this period. When this occurs, we may write

$$C_t = bY_t = bA_o(1+L_o)^t ,$$

where A_o is the value of income in the period when Y_t no longer equals $(1+w)E_{t-1}$. Using the identity $E_t = bY_t+I_t$, we now obtain equation 10.2.1.

The only question now remaining is whether equation 10.2.1 will cause contraction when the labor "ceiling" is reached. The answer to this question is fairly simple. Equation 10.2.1 is exactly like equation 4.1.3 except for the fact that we have inserted the term $bA_o(1+L_o)^t$, which is always less than $b(1+w)E_{t-1}$. By

Theorem 5.2.2, this will decrease the speed and likelihood of explosion in our passive model. But it will not necessarily cause an economy, which would have exploded under equation 4.1.3, to contract. Contraction of income according to equation 10.2.1 will depend as in any difference equation upon the exact parameters and initial conditions of our problem. Labor ceilings, therefore, may or may not cause contraction.

Two numerical examples will illustrate the above assertion. First, suppose that $w = .2$, $c = 2$, $b = .5$. Let $Y_1 = 1.4Y_o$. From the information given on page 78, it is clear that in the absence of any labor constraints, this model explodes. Now let $L_o = .5$. Suppose that in period one, labor is fully employed. The solution to equation 10.2.5 may now be written

$$E_t = |\sqrt{2.88}|^t (A_o \cos\theta t + B_o \sin\theta t) + C_o (1.5)^t$$

Since $|1.5^t| < |\sqrt{2.88}|^t$, equation 10.2.1 must eventually contract. When this happens, equation 10.2.1 will not be valid, and the initial conditions for the unconstrained problem will also lead to contraction. The imposition of a labor constraint in our problem, thus, decreases the speed of explosion by enough to cause contraction.

On the other hand, suppose that $c = 10$, $w = 0$, $b = .5$, and $L_o = 0$! $L_o = 0$, of course, implies a very severe labor constraint. Also, suppose that $Y_o = 1$ and $Y_1 = 2$. It is clear that for these initial conditions and parameters, equation 4.1.3 explodes. Now let labor be fully employed in period two. Then after period two

$$E_t = c(1+w)^2 E_{t-1} - c(1+w)^2 E_{t-2} + (1)(1)^t \ .$$

We may now solve the above equation. The initial conditions $Y_o = 1$, $Y_1 = 2$ imply that $E_o = 2$ and $E_1 = bY_1+I_1 = (\frac{1}{2})2+10(2-1) = 11$. Furthermore, $E_3 = (\frac{1}{2})2+10(11)-20 = 91$. Since the roots of equation 10.2.1a are $\lambda_1 = 1.15$, $\lambda_2 = 8.75$, our solution is obtained by solving

$$11 = A_o(1.15)^1+B_o(8.75)^1+1$$

$$91 = A_o(1.15)^2+B_o(8.75)^2+1,$$

and is given as

$$E_t \approx -.2(1.15)^t+1.17(8.75)^t+1 \qquad t>2.$$

It is clear that E_t explodes and that $2 < (1)E_t$ for all t after 2. Thus, the most stringent labor condition does not cause a downturn in this problem. Furthermore, if we let labor grow at greater rates, the same conclusions would obtain. This follows, since increasing the maximal rate of income growth increases the right hand side of equation 10.2.1, which implies, by Theorem 5.2.2, greater speed and likelihood of explosion. Graphically, therefore, we may depict the rate of growth of income (production) as

Figure 9. Growth of Income; Labor Constrained Explosion

At this point, the reader may have noted a slightly unusual property of the above explosive model. In our example, with $L_o = 0$, income was stationary. Yet the demand for capital goods continually increased! The astute reader may believe that this is totally irrational, and that something is wrong "economically" with our model. Actually, however this conclusion is perfectly sensible! Consider an economy with three industries - one consumption goods industry, A, and two capital goods industries, B, and C. Industry A buys its capital from industry B. Industry B buys its capital from industry C, and industry C buys its capital from B. When income becomes stationary, industry A stops buying capital, but industries B and C do not stop buying capital from one another. Furthermore, they will not do so in subsequent periods if each period they disregard past experience and expect that next period enough labor will be forthcoming to meet anticipated expenditures. Our conclusions are, therefore, true in our passive model, if we assume that expectations with regard to labor are not "adaptive."

What, however, if businessmen realize that available labor is growing at a finite rate? In this case, businessmen will buy capital not in accordance with expenditure "needs" but rather, if labor is fully employed, in accordance with available labor. As a special case, let us assume that businessmen know how much income growth is possible with the labor constraint in the economy. In this case

$$I_t = c\,(Y_{t+1\ \text{expected}} - Y_t) \qquad 10.2.7$$

$$= c\,((1+L_o)Y_t - Y_t)\ ,$$

where L_o is the assumed known growth of labor in the economy.

Therefore ,

$$I_t = c(L_o)A_o(1+L_o)^t .$$

Our equation of growth of expenditures now becomes

$$E_t = (b+c(L_o))A_o(L_o)^t .$$

The above equation is valid, of course, only if

$$Y_t < (1+w)E_{t-1} .$$

The question now remains as to whether income will contract if businessmen's expectations are adaptive in the above manner. The answer is clearly no if

$$A_o(1+L_o)^t<(1+w)(b+c(L_o))A_o(1+L_o)^{t-1} \qquad 10.2.8$$

or

$$(1+L_o)<(1+w)(b+c(L_o)) , \qquad 10.2.9$$

for the same reason that equation 4.1.3 cannot contract if $b(1+w) > 1$. Indeed if $L_o = 0$, (if population growth is stationary) the above condition does reduce to $b(1+w) > 1$. Thus, even if expectations are adaptive with respect to labor growth, a very severe constraint will not necessarily cause contraction. Of course, however, if the above equation is not valid, as is probably the case, a downturn will occur.

Finally, let us consider a model in which expectations are adaptive with respect to both sales and capital needs. In this case,

$$Y_t = E_{t-1}\left(\frac{E_{t-1}}{E_{t-2}}\right) \quad .$$

Plugging $E_t = (b+c(L_o-1))A_oL^t$ into the above equation,we obtain

$$Y_t = \left(\frac{b+c(L_o-1)}{b+c(L_o-1)}\right)\left(\frac{A_oL_o{}^t}{A_oL_o{}^{t-1}}\right)(b+c(L_o-1)A_oL_o{}^{t-1} \qquad 10.2.10$$

$$= (b+c(L_o-1)A_oL^t \ .$$

Clearly the only way such growth can continue is if

$$(b+c(L_o-1))A_oL^t > A_oL_o{}^t \ , \qquad 10.2.11$$

where the latter is constrained income. But this implies that

$$(b+c(L_o-1) > 1 \qquad 10.2.12$$

or

$$L_o > 1 + \frac{1-b}{c} \ .$$

Thus, in a model with expectations adaptive to both sales and capital needs, the only way growth can continue to crawl along the ceiling is if the rate of growth of the labor force is greater than the warranted rate!!

We may now conclude that our model is capable of inducing at least a downturn in income if our parameters are suitable. Since cycles and downward turning points do seem to occur in the real world, this will impose some empirical constraints on the values of our parameters. Nevertheless, considering the last of the above models as the most realistic, such constraints (i.e., $L_o < w_{warranted}$) do not seem at all unreasonable.

Section 10.3. Cycles within Long Term Growth, Continued."Capital Constraints"

In the last section, we investigated the possibility of a downturn occurring because of a labor constraint in the economy. In this section, we consider the possibility of a downturn occurring because of a capital constraint.

In order to do this, we must change the model of this dissertation to allow the possibility of some capital constraint. For in the model as it stands so far, capital can never serve as a constraint, since there are two ways of producing goods and one of these ways requires only labor. Let us, therefore, assume a more rigid production than the one we have considered - one in which there exists only one method of producing goods in the economy. This method shall be that which uses capital and labor in some fixed proportion to output. We now ask how this capital constraint in production will affect the growth equation and also whether it will cause an expanding economy to eventually contract.

To answer these questions, let us realize that

(1) $Y_{t+1} = (1+w)E_t$ if $K_{t \text{ on hand}} = c(1+w)Y_t > c(1+w)E_t$,

$Y_{t+1} = Y_t + \frac{I_t}{c}$ if $K_{t \text{ on hand}} = c(1+w)Y_t < c(1+w)E_t$.

(2) $I_t = c(1+w)(Y_t - Y_{t-1})$ if $K_{t \text{ on hand}} > cY_t = K_{\text{needed}}$,

$I_t = c(1+w)^2 E_{t-1} - cY_t$ if capital is fully utilized in the preceding period.

(3) $E_t = bY_t + I_t$.

It is clear from the above statements that if capital is not being fully utilized, our equations of growth remain exactly as before. The capital "constraint" in no way interferes with growth in our economy. If $(1+w)E_t > Y_t + \frac{I_t}{c}$, however, capital is "inadequate." Our equations of growth now become

$$Y_{t+1} = Y_t + \frac{I_t}{c} \qquad 10.3.1$$

$$I_t = c(1+w)^2 E_{t-1} - cY_t \qquad 10.3.2$$

$$E_t = bY_t + I_t \ . \qquad 10.3.3$$

Simplifying, we have

$$I_t = c(1+w)^2(bY_{t-1} + I_{t-1}) - cY_t \qquad 10.3.2a$$

$$Y_{t+1} = Y_t + \frac{I_t}{c} \ .$$

Solving for Y_t, we obtain

$$(Y_{t+1} - Y_t)c = c(1+w)^2 bY_{t-1} + c(1+w)^2 c(Y_t - Y_{t-1})$$

or

$$Y_{t+1} = c(1+w)^2 Y_t + (b(1+w)^2 - c(1+w)^2 Y_{t-1} \ . \qquad 10.3.4$$

It can easily be shown that the warranted rate of growth for equation 10.3.4 is $1+w = 1 + \frac{1-b}{c}$, a fact which should be intuitively obvious since the non-constrained model of Chapter 4 has capital "just" fully employed along the warranted path. We may also show that equation 10.4.4 is less explosive than equation 4.1.3. This follows by rewriting equation 10.1.4 as

$$Y_{t+1} = (c(1+w)^2+b(1+w))Y_t-c(1+w)^2Y_{t-1}+(b(1+w)^2Y_{t-1}$$
$$-b(1+w)Y_t) \qquad 10.4.4a$$

and observing that this equation holds <u>only</u> if

$$cY_{t+1} = c(1+w)^2E_{t-1}$$
$$= c(1+w)(1+w)E_{t-1}$$
$$> c(1+w)Y_t \ .$$

Since the bracketed term in equation 10.4.4a is always negative under these circumstances, our conclusion follows from Theorem 5.2.2. Finally, we assert that while the capital constraint may slow down and <u>possibly</u> cause an expanding unconstrained economy to contract, this is not always the case. For equation A1.4.4 expansion or contraction will depend on whether

$$Y_t > \lambda_{LA1.4.4}Y_o \ . \qquad 10.4.5$$

Furthermore, if we run out of capital goods inventories in this model the right hand side of equation A1.4.4 will be changed by the addition of a term X, which represents the capital goods deficiency, and the subtraction of a term $\frac{bX}{c}$, which represents the consumption decrease. Running out of capital goods inventories will therefore add to the right hand side of the equation and make our model even more explosive. This is precisely analogous to our conclusions in Chapter 9. <u>Thus, we may assert that within our model there is no such thing as a capital constraint. Capital shortages and rigidities can cause a slow-down in the rate of growth but they cannot force the economy to turn down.</u>

Finally, two points of historical interest. In 1949, John Hicks proposed a "mathematical" model which purported to show how cycles occurred within a long term trend growth.[1] Hicks' basic model, as the reader is well aware, postulated an autonomous component of investment which continuously grows as $I_t = I_o(1+r)^t$. This component of investment, according to Hicks, was the result of "the natural growth of the economy (productivity and perhaps population)."[2] Because of this component, Hicks was able to achieve in his model, a non-zero floor for investment. This in turn prevented the Hicksian economy from contracting to a zero income level. Furthermore, as a result of the introduction of a "maximum level which Y(income) cannot exceed," Hicks believed that the rate of growth would have eventually to slow down, and that this slowdown, by reducing the "induced" component of investment, would always cause the economy to turn down. Consequently, Hicks believed that he had successfully constructed a "nonlinear" model in which business cycles occurred within a long term growth pattern.

There are, of course, several difficulties with Hicks' model. Hicks' assumption of a component of investment which does not depend on changes in income or expected sales is extremely hard to swallow. Also, his treatment of the downward turning point, or ceiling, is extremely obscure. Quite apart from these difficulties, however, is the simple fact that Hicks' assertion of a necessary downturn is theoretically incorrect. For if we add an autonomous component of

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[1] J. R. Hicks, loc. cit.

[2] J. R. Hicks, op. cit., p. 112

investment to the right hand side of equation 10.2.1, this increases the likelihood of explosion. Those economies which are sure to explode with the labor constraint (i.e., $L_o > 1 + \frac{1-b}{c}$) will also explode if we assume, in addition to an induced component of investment, an autonomous component of investment. Therefore, Hicks' theoretical conclusion that a capitalist economy must always experience business cycles as it reaches a resource ceiling is incorrect.

Yet the reason for Hicks' error is perfectly understandable. It lies in the fact that he considered a special case of our model. In his famous trade cycle theory article, Hicks, like Harrod, considered growth as occurring along a warranted path where $S_t = I_t$. From our work in this dissertation, we realize that if we disturb such a path, divergence takes place. In particular, if we disturb it in a downward direction by imposing a labor constraint, or "ceiling," as Hicks calls it, we obtain an immediate downturn in the rate of growth. Hicks' conclusions are therefore correct but only within the context of his special assumption that growth occurs in equilibrium fashion.

Similarly, in 1951 Robert Goodwin considered a model in which a capital constraint generated cycles.[3] As we have shown, our model arrives at the opposite conclusion. The reason for the difference between our model and Goodwin's is that, in Goodwin's model, the capacity of the capital goods industry is assumed to be fixed. Stated differently, in Goodwin's model the capital goods industry does not buy capital goods for itself. In our model, no such assumption

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[3] R. Goodwin, loc. cit.

is made. Thus, the only constraint is a labor constraint. Dernburg and Dernburg put the matter nicely when they said

> The case for a real resource ceiling rests on the assumption that the supply of factors of production cannot indefinitely be expanded as rapidly as the cyclical growth for the demand for these factors. Since the capital stock grows endogenously as the system expands and at a rate that tends to maintain a constant capital output ratio, a resource ceiling resulting from growing relative capital shortage is impossible. Thus, if business cycles are brought to a halt by collision with a ceiling, this ceiling must result from the full employment of the labor force. The essential point to note is that capital equipment holds a unique place as a factor of production because it expands endogenously as output itself expands. The same is not true of the labor-force, and it is for this reason that the real-resource ceiling hypothesis may be a plausible one.[4]

Section 10.4. Price Fluctuations and the Need for a Monetary Growth Theory

In the last section, we explained why an exogenous labor constraint may cause an explosive economy to turn down. This will cause part of a business cycle. The question remains, however, as to how income will turn up once it has started to decline.

One way of answering this question is to assume that, in the real world, capital depreciates at a rate faster than the rate of decline in income. If this is so, then eventually, so long as income remains positive, capital must become insufficient and businessmen must begin to spend more on capital. This will lead to increases in consumption expenditures and to an upturn.

However, it is not necessarily true that depreciation rates will be so high. In particular, in our model, we have assumed <u>no</u>

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[4] T. F. Dernburg and J. D. Dernburg, <u>op</u>. <u>cit</u>., p. 164

depreciation. How then can we explain the upturn?

The upturn can be explained if we relax still another of the assumptions made in Chapter 2. This assumption is the constancy of prices in the economy, or as Hicks calls it, the "fixprice" assumption. If prices are allowed to fluctuate, then, during a contraction, we can expect that goods prices as well as interest rates will fall. In the absence of money illusion, these two facts will give rise to wealth effects. C_t is given by the formula $C_t = bY_t + f(r_t, p_t)$, and as p_t and r_t fall, C_t increases. (For those who do not prefer this formalism, we can argue that consumption out of present income becomes larger and larger as prices and interest rates fall - i.e., b increases.) As a result of these wealth effects, as income continues to fall, $b(1+w)$ will have to become greater than 1. This will cause an upturn, as we have already seen in Chapter 4.

The process will now continue as before until the economy hits a new ceiling. This time, however, due to labor growth the ceiling will be at a higher level. Thus, graphically, income expands or contracts in the following manner

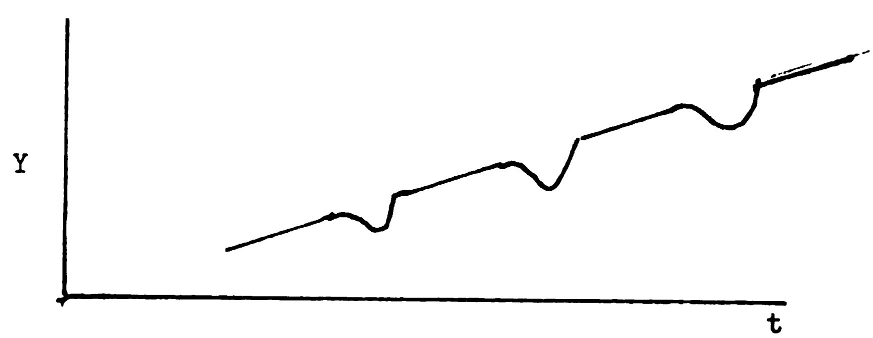

Figure 10. Trend and Cycle Growth

In arriving at the happy conclusion that price effects can give us business cycles within a long-term growth trend, however, we are beginning to overstep our ground. For, under no circumstances, in this dissertation, have we said anything about monetary factors in our growth models. While the above arguments concerning wealth effects and the upturn may seem perfectly plausible, they are not a scientific proof. They do not rest upon the firm ground of a theoretical quantitative model. In this dissertation, we have not shown how to incorporate price changes into our model. We do not know how price changes will effect businessmen's behavior, and, in turn, we do not know how businessmen will change prices as the result of past experience. Furthermore, if we incorporate a discussion of price changes in the down part of our model, we should also be able to incorporate them into the up part of our model. In short, we should be able to ask most generally how price increases or decreases - as shortages or surpluses develop - will affect the growth of income.

Questions concerning price changes and their effects on the growth of income, however, will not be answered in this dissertation. This is not to belittle their importance; it is the author's opinion that they are of paramount importance to the future of economics. It is simply that a discussion of these questions, if successful, would take at least as much time as that already spent on this dissertation. We shall, therefore, leave to the future, or to others, this undertaking.

CHAPTER 11

Section 11.1. Summary and Conclusions

In the preceding chapters of this dissertation, we have built a number of dynamic models which purport to describe growth patterns in economies characterized by various types of expectations and inventory behavior. It may seem to some, however, perhaps because of the "apparent" mathematical complexity of our models, that these chapters have been little more than a series of quantitative exercises in dynamic model building. It seems appropriate, therefore, in this concluding chapter, to try to put some of our ideas into perspective and to show why we feel they are important in the context of growth economics today. In order to do this, let us briefly summarize some of the more important points of each of the preceding chapters.

In Chapter 2, we have tried to show how to build very simple but perfectly general models to describe dynamic economies. These general models are simply dynamic extensions of current static models. They make use of expectations in a very simple straightforward way, and they state that growth can occur in an economy only as the result of specific actions undertaken by businessmen. These actions are the result of decisions based on past levels of expenditures and expectations as to future levels of expenditures. Finally, these models, even though price variables are excluded, are general equilibrium models, suitable for all types of behavioral assumptions.

It may be observed that in the rest of this dissertation we consider only one type of model, namely, the two-phase model where we divide a period into two parts - a production part and a market part. This two-phase model is introduced only because of its relative mathematical simplicity and in no way denies the possibility of working with other models with slightly different lags and assumptions. It does turn out, however, that if we employ this two-phase model, we arrive at some rather elegant and simple mathematical conclusions and are more easily able to compare our theories with those of Harrod and others.

In Chapter 3, we digress from the main point of the dissertation to show how the model we have discussed in Chapter 2 can give a perfectly general dynamic Keynesian model. Stated differently, we show that under the unsatisfactory assumption that $I_t = I_o$, our model is equivalent to the Keynesian model. We also show that the Keynesian model will be stable if we assume that businessmen believe that sales in each period will be the same as they were in the last period. However, if expectations are not of this form, then we cannot necessarily say this. Furthermore, if we bring inventory behavior into our model, there is no assurance that the model will be stable even with simple expectations.

Many of the equations in this chapter may appear very familiar. In particular, our Keynesian inventory models give equations which in a number of instances are identical to those previously derived by Metzler. However, for all the various types of Keynesian models which we consider, our interpretations of the equations are fundamentally different from those of previous authors in that we do not start by assuming that gross national product equals $C_t + I_t + G_t$. Because of this fundamental difference in interpretation, we are able to show that the Keynesian model is quite unsatisfactory as a dynamic model, in that it leads either to a stable equilibrium solution or an explosive growth pattern in which production exceeds sales.

In Chapter 4 we begin the main body of the work of this dissertation. In Section 1 of this chapter, we derive a perfectly general equation to describe a dynamic economy under simple Harrod-like assumptions. Our equation of growth turns out to be

$$Y_t = ((1+w)b+c(1+w)^2)Y_{t-1}-c(1+w)^2Y_{t-2}$$

and is given on page 63. In deriving this equation, we have simply said that it is businessmen, who, by their desires, decide how much growth there will be in the economy. Furthermore, they decide this on the basis of two factors and two factors alone - namely, sales in the previous period and expectations concerning the future. In the rest of this chapter, we discuss the characteristics of growth in the economy as a result of this equation and the assumptions it embodies. We show the following:

1) There exists in the economy a warranted rate of growth which is given by the value $\frac{1-b}{c}$, the rate first derived by Harrod. If this rate is the one at which the economy starts to grow and if expectations are correct, then this rate will always be maintained. Furthermore, when the economy grows at this rate, capital is fully

employed and markets are always cleared. The latter conclusion is very much different from that in the Harrod model since we have seen that one of the major flaws in the Harrod model is that capital is not fully employed along the warranted path.

2) When the economy initially starts out off the warranted path, three things can happen. First, there can be explosion with capital always insufficient for production (in which case we assume that production occurs by the labor intensive method of production). Second, there can be contraction to zero income, with capital always in too great supply. And third, there can be constant growth at a nonwarranted rate with capital either in abundance or deficit. In each case, if capital is too abundant, then the supply of goods exceeds the demand for goods and inventories pile up in businessmen's warehouses. If capital is insufficient, the reverse happens and inventories are drawn down. Needless to say, these types of growth except for the last are identical to Harrod's. However, they differ in one important respect. Harrod believed that, if initially the rate of growth were less than the warranted rate, the economy would have to contract and that, if initially the rate of growth were higher than the warranted rate, the economy would have to explode. We show that this is not necessarily so and that, whether or not contraction or explosion occurs, depends entirely on the values of expectations which businessmen have - i.e., by how much businessmen feel sales will grow in the next period. Indeed, when expectations are very low - for instance when businessmen feel that sales will stay the same each period - it may be impossible for the economy to

explode, no matter how high the initial rate of growth. Similarly, when expectations are very high, it may be impossible for the economy to contract. Finally, as far as the third type of growth is concerned, this is clearly in direct opposition to Harrod's model. We do show later however, that when expectations change, this type of growth is impossible.

3) For the inelastic expectations that we use in this chapter, i.e., for expectations which have the property that businessmen always believe that sales will increase by some constant percent no matter what has happened in the past, the model is grossly unstable with respect to the warranted path. The slightest deviation from the warranted path causes the economy to explode or contract to zero income. Furthermore, if the economy starts off initially growing at a rate other than the warranted rate, it can never achieve the warranted rate. These properties are the famed "knife edge" properties of the Harrod model.

In Chapter 5 we change the form of the expectation used in Chapter 4. We now let expectations be such that businessmen believe that sales will increase by some constant times the difference in sales in the preceding two periods. We call this the additive model. Mathematically, instead of the expectation function

$$\text{Sales}_{t+1\ \text{expected}} = (1+w)(\text{Sales}_t) \quad w\text{, constant}$$

where w is the coefficient of expectations used in Chapter 4, we let

$$\text{Sales}_{t+1\ \text{expected}} = \text{Sales}_t + A(\text{Sales}_t - \text{Sales}_{t-1})$$

where A is some constant. We then show that this equation has exactly all the properties that the previous model did, including the fact that there exists warranted rate growth only if the coefficient of expectations in this model is $A = 1 + \frac{1-b}{c}$. In this chapter, we also digress to show that in general for both the models which we have discussed, the economies are more likely to explode for high values of the marginal propensity to consume, and capital output ratio.

So far in Chapters 4 and 5, we have assumed that expectations are constant - or inelastic. Stated differently, we have assumed that businessmen always expect sales to increase by some constant percentage over last period's sales or by some constant times the difference in the preceding two periods' sales. In Chapter 6, we change this assumption by letting businessmen change their expectations in the direction of realized changes in sales, when their previous expectations have been disappointed. In particular, we let businessmen believe that sales will increase in any period by the same percentage that they increased by in the preceding two periods. Mathematically, we let

$$\text{Sales}_{t+1 \text{ expected}} = \text{Sales}_t \left(\frac{\text{Sales}_t}{\text{Sales}_{t-1}}\right)$$

The equation for the growth of income turns out to be nonlinear when expectations are "adaptive". Nevertheless, by making use of some properties of our model in Chapter 4, we are able to characterize all the possible types of growth. We show that warranted growth is still possible at the identical rate as before. Constant nonwarranted growth, however, is no longer possible. For nonwarranted growth, we are able to show that the economy may either contract to a zero income level or explode, exactly as before, or may return under the proper initial conditions towards warranted rate growth. In other words, Harrod's conclusion that once a system has been disturbed from equilibrium growth it can never return to such growth is incorrect. However, in fairness to Harrod, it is also true that both paths of warranted growth are very unstable. Thus, the slightest disturbance of the economy while it is travelling along either of these paths will cause contraction or explosion.

The model we have considered in Chapter 6 is probably the model most similar to Harrod's original model since Harrod also thought of an economy where expectations themselves change. Again, we see that the conclusions we obtain are in many respects similar to those of Harrod. However, Harrod was mistaken in saying that if the growth rate is at any time below the warranted rate, the economy will contract, and if above, will explode. Again, we are able to show that, if the system is growing at a rate higher than the warranted rate, this does not necessarily imply contraction, and vice versa, if it is growing at a rate lower than the warranted rate.

Chapter 6 completes our work with different forms of expectations. Our work, however, is still in a sense incomplete. For in Chapters 4, 5, and 6 one assumption has been made that is most unrealistic - namely, that businessmen do not care about their inventory holdings. (This type of behavior may be called, following Metzler, passive inventory behavior.) However, it is important to consider inventories in our models not just because they may be important in the "real world" but more importantly because the losses or gains in inventories are the signals by which businessmen know whether or not their expectations are being realized. The next three chapters, therefore, try to show how the economy expands when businessmen do consider how many inventories they have on hand. The work is, of course, similar to Metzler's famous work on inventory cycles, and we attempt to improve our version of the Harrod model in much the same way that Metzler improved the Keynesian model. As an aside, it is perhaps interesting to note that we are easily able to incorporate such inventory behavior into our model. It is an encouraging sign that our methods are capable of being extended to more complex situations, without having to introduce unrealistic or complicated qualifications.

Let us now be more specific. In Chapter 7, we consider a model in which businessmen desire to maintain a fixed level of inventories in all periods. We are then able to show that there exists a warranted rate of growth with the same value as before. The types of possible growth also turn out to be the same as before. Thus, nothing drastic happens to our model when we introduce such inventory behavior. However, we do compare the likelihood and speed of explosion in models with such inventory behavior and those without. We show that the speed with an economy adjusts is greater for the inventory case than it is for the noninventory case. We also show that the likelihood of explosion is greater for the inventory model, when initially expectations were disappointed by being too low, and less when expectations were disappointed in the upward direction.

In Chapter 8, we consider a model where businessmen try to maintain some constant proportion of expected sales as inventories. We then show that the warranted rate of growth is not $\frac{1-b}{c}$ but rather some number less than this and greater than zero, whose value decreases as the k value of the proportion increases. Aside from this one fact, we obtain the exact same types of growth patterns as we obtained in the fixed level and passive models. However, we do compare the speed of adjustment and likelihood of explosion for all three models. We are able to show that the speed of adjustment in the proportional model is greater than that in the other two models. Also we are able to prove that the proportional model is always more likely to explode than the fixed level model and in most circumstances, but not all, will be more likely to explode than the passive model.

Chapter 9 discusses a point not explicitly mentioned in Chapters 7 and 8. In these chapters, we have implicitly assumed that inventories on hand were always positive. In Chapter 9, we show that this assumption is unnecessary, if we define something called a negative inventory. A negative inventory is an obligation to deliver a certain good to a customer in the future. The negative inventory has resulted, because in the past inventories have been depleted and the only way consumer demand could be satisfied was for businessmen to promise to supply these goods in the future. If we accept this type of inventory behavior as a valid representation of businessmen's behavior, then it quickly becomes apparent that our growth equations are in no way affected by the depletion of inventories. Thus, there is no need to change our models in any way. On the other hand, if we are not willing to posit such negative inventory behavior on the part of businessmen, and if we consider a model where businessmen simply tell customers that they do not have any more goods when their inventories run out, our models and equations of growth are changed. Under these circumstances we are able to show that the likelihood of explosion and speed of adjustment are greater for models in which businessmen do not wish to incur negative inventories.

In Chapter 10 we show that all our models for inventory behavior can be written with changing expectations, instead of constant expectations, and that the consequences are exactly like those discussed in Chapter 6. We also show that cycles can be induced in our models by labor constraints and wealth effects. Finally we show the limitations of our models and explain why it would be desirable to try to build similar monetary growth models.

It should now be clear from this summary chapter that the purpose of this dissertation is not simply to "reformulate Harrod growth theory," as we suggested in Chapter 1. In reality, our purpose has been not simply to change Harrod's theories slightly, but rather to build, upon Harrod's assumptions of rigid production functions and monetary neutrality, a new dynamic theory, based upon expectations, which can qualify as a true dynamic theory. In so doing, it is true that we have obtained many of Harrod's results as special cases of our more general model. The main point of this work, however, has been to fill what we consider to be a serious theoretical gap in the literature of economics today - namely, the lack of a coherent dynamic model.

Finally, it will no doubt be observed by some that these models - or exercises - are "unrealistic" and cannot be relied upon as predictors of economic growth. These observations may, of course, be correct, since we have neglected many factors in our analyses. The unreality of these models is, however, not the point. What is of primary importance in this dissertation is that we are able to build rational and logical dynamic models based on very simple macro- and micro-economic ideas. In short, it is not so much the "ends" of this dissertation, which are important, as the "means." For if we have been successful in our attempts to build coherent general equilibrium macrodynamic models, we will have done three things. First, we will have clarified several points not quite understood in present dynamic models. Second, we will have put nonmonetary economic dynamics on a more solid theoretical foundation. And third, we will have pointed a direction in which economists in the future can turn in their efforts to build more "monetary" macrodynamic models.

BIBLIOGRAPHY

BIBLIOGRAPHY

Adelman, I. and Adelman, F. L. "The Dynamic Properties of the Klein Goldberger Model," Econometrica (October, 1959).

Alexander, S. S. "Mr. Harrod's Dynamic Theory," Economic Journal (December, 1950).

Allen, R. G. D. Mathematical Economics (Macmillan, 1963).

________. Macroeconomic Theory (Macmillan, 1968).

Ando, A. and Modigliani, F. "The Life Cycle Hypothesis of Saving: Aggregate Implication and Tests," American Economic Review (March, 1963).

Baumol, W. J. "Notes on Some Dynamic Models," Economic Journal (December, 1948).

________. Economic Dynamics (Macmillan, 1970).

Cagan, P. "The Monetary Dynamics of Hyper-Inflation" in Studies in the Quantity Theory of Money (University of Chicago Press, 1956).

Chenery, H. B. "Overcapacity and the Acceleration Principle," Econometrica (January, 1952).

Chow, G. "Multiplier, Accelerator and Liquidity Preference in the Determination of National Income in the United States," Review of Economics and Statistics (February, 1967).

Clark, J. M. "Business Acceleration and the Law of Demand: A Technical Factor in Economic Cycles," Journal of Political Economy (March, 1917).

Deleuuw, F. "The Demand for Capital Goods by Manufacturing: A Study by Quarterly Time Series," Econometrica (July, 1962).

Dernburg, T. F. and Dernburg, J. D. Macroeconomic Analysis: An Introduction to Comparative Statics and Dynamics (Addison Wesley, 1969).

Devletoglou, E. A. "Correct Public Prediction and the Stability of Equilibrium," Journal of Political Economy (April, 1961).

Diamond, J. J. "Further Developments of a Distributed Lag Investment Function," Econometrica (October, 1962).

Domar, E. "Capital Expansion, Rate of Growth and Employment," Econometrica (April, 1946).

________. "Expansion and Unemployment," American Economic Review (March, 1947).

________. Essays in the Theory of Growth (Oxford University Press, 1957).

Duesenberry, J. S. Income, Saving, and the Theory of Consumer Behavior (Harvard University Press, 1949).

Eisner, R. "A Distributed Lag Investment Function," Econometrica (October, 1962).

________. "Investment: Fact and Fancy," American Economic Review (May, 1963).

________. "A Permanent Income Theory of Investment: Some Empirical Explorations," American Economic Review (June, 1967).

Ferguson, C. E. "On Theories of Acceleration and Growth," Quarterly Journal ot Economics (February, 1960).

Friedman, M. Essays in Positive Economics (University of Chicago Press, 1953).

________. A Theory of the Consumption Function (Princeton University Press, 1957).

Frisch, R. "Propagation Problems and Impulse Problems in Dynamic Economics," Economic Essays in Honor of Gustav Cassel (George Allen, and Unwin Ltd., 1933).

Goldberg, S. Introduction to Difference Equations (John Wiley and Sons Inc., 1958).

Goldsmith, R. A. A Study of Saving in the United States (Princeton University Press, 1955).

Goodwin, R. M. "Dynamical Coupling with Especial Reference to Markets Having Production Lags," Econometrica (July, 1947).

________. "The Nonlinear Accelerator and the Persistence of Business Cycles," Econometrica (January, 1951).

Gruenberg, E. and Modigliani, F. "The Predictability of Social Events," Journal of Political Economy (December, 1954).

Hahn, F. and Matthews, R. C. O. "The Theory of Economic Growth: A Survey," Economic Journal (December, 1964).

Hansen, A. H. Business Cycles and National Income (W. W. Norton, 1951).

Harrod, R. F. "An Essay in Dynamic Theory," Economic Journal (March, 1939).

________. Towards a Dynamic Economics (Macmillan, 1948).

________. "Second Essay in Dynamic Theory," Economic Journal (June, 1960).

Hicks, J. R. "Mr. Harrod's Dynamic Theory," Economica (May, 1949).

________. A Contribution to the Theory of the Trade Cycle (Oxford University Press, 1950).

________. Capital and Growth (Oxford University Press, 1965).

Jorgenson, D. "On Stability in the Sense of Harrod," Economica (August, 1960).

________. "Capital Theory and Investment Behavior," American Economic Review (May, 1963).

________. and Stephenson, J. "Investment Behavior in U. S. Manufacturing 47-60," Econometrica (April, 1967).

________. Hunter, J., and Nadiri, M. "A Comparison of Alternative Econometric Models of Quarterly Investment Behavior," Econometrica (March, 1970).

Kaldor, N. "Mr. Hicks on the Trade Cycle," Economic Journal (December, 1951).

________. "The Relation of Economic Growth and Cyclical Fluctuations," Economic Journal (March, 1954).

________. and Mirlees, J. A. "A New Model of Economic Growth," Review of Economic Studies (June, 1962).

Kalecki, M. "Trends and Business Cycles Reconsidered," Economic Journal (June, 1968).

Kisselgoff, A. and Modigliani, F. "Private Investment in the Electric Power Industry and the Acceleration Principle," Review of Economics and Statistics (November, 1957).

Knox, A. "The Acceleration Principle and the Theory of Investment," Economica (August, 1952).

Kocyck, L. M. Distributed Lags and Investment Analysis (North Holland, 1954).

Kuznets, S. "Proportion of Capital Formation," American Economic Review (May, 1952).

Levy, H. and Lassman, F. Finite Difference Equations (Macmillan, 1960).

Lovell, M. "Manufacturers, Sales Expectations, and the Acceleration Principle," Econometrica (July, 1961).

Matthews, R. C. O. "The Saving Function and the Problem of Trend and Cycle," Review of Economic Studies (Volume 22, 1955).

Metzler, L. "The Nature and Stability of Inventory Cycles," Review of Economics and Statistics (August, 1941).

Mills, E. S. "The Use of Adaptive Expectations in Stability Analysis: Comment," Quarterly Journal of Economics (May, 1961).

________. Price, Output, and Inventory Policy (John Wiley and Sons Inc., 1962).

Minsky, H. "A Linear Model of Cyclical Growth," Review of Economics and Statistics (May, 1959).

Muth, J. F. "Rational Expectations and the Theory of Price Movements," Econometrica (July, 1961).

Nerlove, M. "Adaptive Expectations and Cobweb Phenomena," Quarterly Journal of Economics (May, 1958).

________. "Reply to Mills," Quarterly Journal of Economics (May, 1961).

________. and Arrow, K. J. "A Note on Expectations and Stability," Econometrica (April, 1958).

Niesser, H. "Critical Notes on the Acceleration Principle," Quarterly Journal of Economics (May, 1954).

Phillips, A. W. "A Simple Model of Employment, Money and Prices in a Growing Economy," Economica (November, 1961).

Robertson, D. H. "Thoughts on Meeting Some Important Persons," Quarterly Journal of Economics (May, 1954).

Rose, H. "On the Possibility of Warranted Growth," Economic Journal (June, 1959).

Samuelson, P. A. "Interaction Between the Multiplier Analysis and the Principle of Acceleration," Review of Economics and Statistics (May, 1934).

Schumpeter, J. Business Cycles (McGraw Hill, 1939).

Smithies, A. "Economic Fluctuations and Growth," Econometrica (January, 1957).

Solow, R. M. "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics (February, 1956).

Tinbergen, J. "Statistical Evidence on the Acceleration Principle," Economica (May, 1938).

Tsiang, S. C. "Accelerator, Theory of the Firm, and the Business Cycle," Quarterly Journal of Economics (August, 1951).

Turnovsky, S. J. "Stochastic Stability of Short Run Market Equilibria Under Variations in Supply," Quarterly Journal of Economics (November, 1968).