# MAXIMAL CHAINS IN SOLVABLE GROUPS

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ABSTRACT

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by Armond E. Spencer

Some knowledge of the structure of a finite group can be derived from information about the structure of its maximal subgroups and the imbedding of these maximal subgroups. For example, two classical theorems in this vein, due to B. Huppert are: (1) If a finite group has all its maximal subgroups supersolvable, then the group is solvable. (2) If all the second maximal subgroups of a finite group are normal, then the group is supersolvable. Results of this type have been extended by, among others: W.E. Deskins, "On maximal subgroups," First Symposium in Pure Mathematics, American Mathematical Society (1959); Z. Janko, "Groups with invariant fourth maximal subgroups," Mathematische Zeitschrift, 82, (1963); J. Rose, "The influence on a finite group of its proper abnormal structure," Journal of the London Mathematical Society, 40, (1965).

The aim here is to extend some of the results of the above papers, under the added hypothesis of solvability. Although most of the definitions and some of the theorems can be stated in general for finite groups, the concern here is always with solvable groups.

Throughout let G denote a finite solvable group and define $h(G) = n$ if (1) Every upper chain in G of length n contains a proper subnormal entry; and (2) There exists at least one upper chain in G of length $(n-1)$ which contains no proper subnormal entry. (Note that $h(G) = 1$ if and only if G is nilpotent.) Using fairly classical techniques the following principal results are obtained.

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<u>Theorem</u>: If $h(G) \leq n$, then $\ell(G) \leq n$, where $\ell(G)$ denotes the nilpotent length of G, i.e. the length of the Fitting series for G.

<u>Theorem</u>: Let $\pi(G)$ denote the number of distinct prime divisors of the order of G. Suppose $\pi(G) = m$, then the following are true:

(1) If $h(G) \leq (m-1)$, then $h(G) = 1$, i.e. G is nilpotent.

(2) If $h(G) \leq (m+1)$, then G is a Sylow tower group for some ordering of the prime divisors of the order of G.

(3) If $h(G) = m \geq 2$, then G is a Sylow tower group by (2), moreover, the non-normal Sylow subgroups of G are cyclic, and the normal Sylow subgroups of G are cyclic or elementary abelian. This means that G is a split extension of a normal abelian Hall subgroup by a subgroup, all of whose Sylow subgroups are cyclic.

(4) These arithmetic limits are the best possible in the following sense: Given any integer $m \geq 2$,

(a) There exists a non-Sylow tower group G such that $h(G) \leq (m+2)$ and $\pi(G) = m$.

(b) There exists a non-nilpotent group G such that $h(G) \leq m$, and $\pi(G) = m$.

(c) There exists a group G, having at least one non-abelian Sylow subgroup, such that $h(G) \leq (m+1)$, and $\pi(G) = m$.

If $h(G) \leq 3$, and G is not nilpotent, then the structure of G is fairly simple. For example for groups having $h(G) = 2$, we have the following theorem.

<u>Theorem</u>: If $h(G) = 2$, then $G = PQ$, where P and Q are Sylow subgroups of G. P is a minimal normal subgroup of G. Q is cyclic, $Q_1$, the unique maximal subgroup of Q, is normal in G. In fact,

$Q_1 = Z(G)$, the center of G, and $Q_1 = \phi(G)$, the Frattini subgroup of G. $P = G'$, the derived group of G, and $Q/Q_1$ acts irreducibly on P.

Let $r(G)$ denote the minimal number of generators for G. The group just described has $h(G) = r(G) = 2$. This result can be extended.

<u>Theorem</u>: If G is not nilpotent, i.e. $h(G) \geq 2$, then $r(G) \leq h(G)$.

The function $h(G)$ in some sense measures the distribution of the subnormal subgroups of G in the lattice of all subgroups of G. Suppose $h(G) = n$. Then in G there exists an upper chain $G = G_o > G_1 > \ldots > G_{(n-1)} > G_n$, such that $G_n$ is the only proper subnormal entry in the chain. Clearly any conjugate of $G_n$ has this same imbedding property. More generally, any automorphic image of $G_n$ has this same imbedding property. We call H an <u>h-subgroup of G</u> if $h(G) = n$ and there exists an upper chain of length n from G to H such that H is the only proper subnormal entry in the chain. Natural questions at this point are: Are the h-subgroups conjugate, normal, or are they even of the same order? Partial answers can be given.

<u>Theorem</u>: If $h(G) \leq 3$, then the h-subgroup is unique.

<u>Theorem</u>: If $h(G) = n$, and there exists an upper chain $G = G_o > G_1 > G_2 > \ldots > G_{(n-1)} > G_n = H$, such that $h(G_i) = (n-i)$, $0 \leq i \leq (n-1)$, and H is an h-subgroup for each $G_i$, then H is normal in G.

Another measure of the distribution of the subnormal subgroups in the lattice of all subgroups was given by Deskins in the paper " A condition for the solvability of a finite group," <u>Illinois</u>

Journal of Mathematics, 2, (1961). There, the function $v(G)$, called the variance of G, is defined as: $v(G)$ is the maximum, taken over all upper chains, of the ratio $\mu(C)/\partial(C)$, where C denotes an upper chain, $\mu(C)$ denotes its length, and $\partial(C)$ denotes the number of subnormal entries in C. If C does not contain a subnormal entry, $\partial(C)$ is taken to be one. Clearly $h(G) \leq v(G)$, since the upper chain terminating in an h-subgroup has $h(G)$ entries and only one subnormal entry. Trivially, if $h(G) = 1$, then $v(G) = 1$. If $h(G) = 2$, then $v(G) = 2$. No example is known in which $h(G) \neq v(G)$.

A natural generalization of the functions $h(G)$ and $v(G)$ can be made by requiring the upper chains under consideration terminate above a particular subgroup of G. To be precise, we define $h(G:H) = n$ if (1) Every upper chain of length n whose terminal entry contains H contains a proper subnormal entry, and (2) There exists at least one upper chain of length $(n - 1)$ whose terminal entry contains H, and which contains no proper subnormal entry. A similar definition is made for $v(G:H)$. With this definition, $h(G)$ and $v(G)$ are simply $h(G: \langle 1 \rangle)$ and $v(G: \langle 1 \rangle)$ respectively. Of course, if H is normal in G, then $h(G:H)$ is simply $h(G/H)$. The aim is to use these definitions to generalize some of the earlier results, and also to gain some information about the placement of H in the lattice of subgroups. For example, if $v(G:H) = 1$, then the smallest normal subgroup of G containing H also contains the hypercommutator of G.

Finally, we have that $h(G) \leq 3$ implies solvability for G, and $h(G) \leq 4$ does not.

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# PREFACE

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# INTRODUCTION

An attempt to describe the structure of a finite group leads in a natural way to the consideration of two basic types of questions:
(1) Given the two group theoretic properties X and Y, if certain subgroups of G have property X then does G have property Y?
(2) Given a family of subgroups of G distinguished by some imbedding property, how is this family distributed in the lattice of all subgroups of G?

Examples of answers to the first type of question have been given by: B. Huppert [8] If all proper subgroups of a finite group G are supersolvable, then G is solvable; and J. Rose [10] If all the proper abnormal subgroups of G are nilpotent, then G has a normal Sylow subgroup P such that the factor group $G/P$ is nilpotent. Examples of answers to the second type of question have been given by: B. Huppert [8] If every second maximal subgroup of G is normal, then G is supersolvable; and W.E. Deskins [3] If the sublattice of subnormal subgroups constitutes a "fair portion" of the lattice of all subgroups, then G is solvable. Z. Janko [9] has described the structure of finite groups satisfying the property that every fourth maximal subgroup is invariant.

The aim here is to extend some of the results due to Janko, Huppert, and Rose, using some techniques suggested by Deskins. Such an attempt leads quite naturally to the definition of the function $h(G)$ which is the length of the longest upper chain in G having only its terminal entry subnormal in G. In a nilpotent group this length is one. In a non-abelian simple group, it is simply the length of the longest upper chain. In particular, the problem under consideration here is how the function h(G) restricts the structure of G if G is assumed to be finite and solvable.

For the sake of completeness, Chapter I contains some basic definitions and results pertaining to solvable groups, upper chains, and subnormal subgroups. In Chapter II the function h(G) is defined, and some structure theorems are proved, showing how h(G) relates to such things as the nilpotent length of G, the number of distinct prime divisors of the order of G, and the Sylow structure of G. Chapter III contains some generalizations of the function h(G), along with some results concerning the placement in the lattice of subgroups of a family of subgroups distinguished by their imbedding properties.

Although most of the definitions and some of the theorems can be formulated for non-solvable groups --and in some cases for infinite groups -- the groups considered herein are assumed to be finite and solvable unless otherwise stated. An index of notation is given in the appendix.

# CHAPTER I

## DEFINITIONS AND ELEMENTARY RESULTS

For the sake of completeness and easy reference this chapter contains some basic definitions and properties of solvable groups. Except for some minor observations, no pretense of originality is made, although some proofs are included where no immediate reference is avaliable.

Throughout let G denote a finite solvable group. A subgroup H of G is called maximal in G if there does not exist a subgroup K of G such that $G \gneq K \gneq H$. This idea extends inductively to the definition of an n-th maximal subgroup.

Definition 1.1: H is a first maximal subgroup of G if H is maximal in G. H is called an n-th maximal subgroup of G if H is maximal in an (n-1)-th maximal subgroup of G.

It should be noted that H does not determine n. For example, consider $S_4$, the symmetric group on four letters. Let H be a Sylow 2-subgroup of $S_3$ in $S_4$. Then H is maximal in $S_3$, which is in turn maximal in $S_4$, hence H is second maximal in $S_4$. However, by the Sylow theorems, H is maximal in a subgroup of order four, which is second maximal in $S_4$, so that H is third maximal in $S_4$.

For each n-th maximal subgroup H of G there exists a sequence of subgroups $G = G_o > G_1 > G_2 > \ldots > G_n = H$, such that for each i $G_i$ is a maximal subgroup of $G_{(i-1)}$. Such a sequence is called an upper chain of length n from G to H. There is a relationship between n and the index of H in G.

Theorem 1.2: If H is a maximal subgroup of G, then $[G:H]$ is the power of a prime. (Note that G is assumed to be finite and solvable.)

Proof: Case 1: H is normal in G. In this case, since H is maximal, G/H has no non-trivial subgroups. However, G/H is solvable, hence is cyclic of prime order, and the theorem is proved.

Case 2: H non-normal in G: This case is handled by induction on the order of G. If H contains a non-trivial subgroup N with N normal in G, then by induction $[G/N:H/N] = p^{\alpha}$ for some prime p. However, $[G/N:H/N] = [G:H]$ and the theorem is proved. So suppose H does not contain a non-trivial subgroup normal in G, and let M be a minimal normal subgroup of G. $|M| = p^{\beta}$ for some prime p. H is maximal and does not contain M, thus G = MH. Since M is normal in G, $M \cap H$ is normal in H. Also, since M is abelian, $M \cap H$ is normal in M. Thus $M \cap H$ is normal in G, but by supposition H does not contain a non-trivial subgroup normal in G, thus $M \cap H = \langle 1 \rangle$. Then $[G:H] = |M|$ and the theorem is proved. ▯

To extend this theorem to n-th maximal subgroups, it is convenient to have some notation.

Definition 1.3: Let $\pi(G:H)$ denote the number of distinct prime divisors of $[G:H]$. $\pi(G:\langle 1 \rangle)$ will be denoted simply by $\pi(G)$.

With this notation, theorem 1.2 can be stated as: If H is a maximal subgroup of G, then $\pi(G:H) = 1$. This can be generalized as follows:

Theorem 1.4: If H is an n-th maximal subgroup of G, then $\pi(G:H) \leq n$.

Proof: The proof is by induction on n. Theorem 1.2 proves the theorem in the case $n = 1$. Suppose the theorem is true for $(n-1)$, i.e., if K is an $(n-1)$-th maximal subgroup of G, then $\pi(G:K) \leq (n-1)$. Let H be an n-th maximal subgroup of G. Then there exists an upper chain of length n from G to H. $G_{(n-1)}$, the $(n-1)$-th entry in the chain, is an $(n-1)$-th maximal subgroup of G, so by induction we have $\pi(G:G_{(n-1)}) \leq (n-1)$. By theorem 1.2 $\pi(G_{(n-1)}:H) = 1$. Since $[G:H] = [G:G_{(n-1)}][G_{(n-1)}:H]$, clearly $\pi(G:H) \leq \pi(G:G_{(n-1)}) + \pi(G_{(n-1)}:H) \leq (n-1) + 1 = n$. ▯

Some other properties of solvable groups which will be used to some extent have to do with Hall subgroups and Sylow systems.

<u>Definition 1.5</u>: A subgroup H of G is called a <u>Hall subgroup</u> of G if $(|H|, [G:H]) = 1$. An integer n is called a <u>permissible Hall divisor</u> of $|G|$ if there exists an integer m such that $mn = |G|$, and $(m,n) = 1$.

P. Hall [5] proved the following theorem.

<u>Theorem 1.6</u>: Let G be a finite solvable group, and n a permissible Hall divisor of $|G|$. Then the following hold:

(a) There exists a subgroup H of G of order n.

(b) Any two subgroups of G of order n are conjugate.

(c) If K is a subgroup of G such that the order of K divides n, then K is contained in a subgroup of order n.

Hall [6] also proved the existence of a collection $\{S_1, S_2 \ldots S_t\}$ of pairwise premutable Sylow subgroups of G such that the product of the $S_i$ is G. Such a collection is called a <u>Sylow system</u> for G. Let $D = \bigcap_{i=1}^{t} N(S_i)$. D is called the <u>Sylow system normalizer</u> for the system, or in short, a <u>System normalizer</u>. Several results due to Hall and Carter [2] can be collected into the following theorem.

<u>Theorem 1.7</u>: Let G be a finite solvable group, and D a system

normalizer in G. Then the following are true:

(a) D is nilpotent.

(b) All of the system normalizers of G are conjugate.

(c) D is not contained in any proper normal subgroup of G.

(d) The intersection of all the system normalizers of G contains the hypercenter of G.

D can be further characterized by an imbedding property. In order to do so, it is necessary to have a definition.

Definition 1.8: A subgroup H of G is called abnormal in G if the following two conditions hold:

(1) Every subgroup of G containing H is self-normalizing, i.e., if $G \geq K \geq H$, then $N(K) = K$.

(2) H is not contained in any two distinct conjugate subgroups of G.

Theorem 1.9: The system normalizer D can be characterized as follows: D is minimal with respect to the property that D can be joined to G by a chain of subgroups $D = G_o < G_1 < G_2 < \ldots < G_n = G$, such that for each i, $G_{(i-1)}$ is abnormal in $G_i$. The minimality refers to the fact that no proper subgroup of D has this property.

At this point it might be worth while pointing out some other properties of abnormal subgroups. It follows immediately from the definition that any subgroup containing an abnormal subgroup is itself abnormal. Also a maximal subgroup is either normal or abnormal. Standard theorems show that the normalizer of a Sylow subgroup is abnormal, and this can be extended to the same theorem about Hall subgroups.

<u>Theorem 1.10</u>: The normalizer of a Hall subgroup is abnormal.

<u>Proof</u>: Let H be a Hall subgroup of G and N = N(H) its normalizer. Let $K \geq N$, and $x \in N(K)$. Now $H \leq K$, so $H^x \leq K$. However, H is a Hall subgroup of K, so by theorem 1.6(b) H and $H^x$ are conjugate in K, i.e., there exists a $y \in K$ such that $H^x = H^y$. Then $xy^{-1}$ normalizes H so that $xy^{-1} \in K$. However, since $y \in K$, this implies that $x \in K$. Thus $N(K) = K$, and any subgroup containing N is self-normalizing. Now to show that N cannot belong to two distinct conjutates. Suppose $N \leq L \cap L^x$. Then H and $H^x$ are Hall subgroups of L and are thus conjugate in L, i.e., there exists a $y \in L$ such that $H^x = H^y$. But then $xy^{-1} \in N(H) \leq L$. However, $y \in L$, so $x \in L$ and $L = L^x$. ▯

Although for maximal subgroups the terms abnormal and non-normal are synonymous, in general this is not the case. For example, in $A_4$, the alternating group on four letters, the subgroups of order two are not normal, however, since they are in the four group which is normal, they are not abnormal. Probably the most natural imbedding property for subgroups is normality, however, for many purposes it is convenient to extend the relation of normality to the transitive relation of subnormality.

<u>Definition 1.11</u>: A subgroup H of G is <u>subnormal in G</u> (accessible in G, subinvariant in G) if there exists a chain $G=G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = H$, such that for each i, $G_i$ is normal in $G_{(i-1)}$. Such a chain is called a <u>subnormal chain</u>. The notation for H is subnormal in G is: $H \triangleleft \triangleleft G$.

Clearly this relation is transitive, and although the product of two subnormal subgroups need not be a group, the following theorem shows that the collection of subnormal subgroups forms a sublattice of the lattice of all subgroups of G.

Theorem 1.12: [11, p. 448] If H and K are subnormal in G, then $\langle H,K \rangle$ and $H \cap K$ are subnormal in G.

The theorem which states that the intersection of a normal subgroup N with a subgroup H is normal in H also extends to subnormal subgroups.

Theorem 1.13: If H is subnormal in G and K is a subgroup of G, then $H \cap K$ is subnormal in K.

Proof: Let $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = H$ be a subnormal chain from G to H. Let $K_i = K \cap G_i$. Then the chain $K = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_n = H \cap K$ is a subnormal chain from K to $H \cap K$. ▯

Well known theorems about Sylow subgroups yeild the fact that a subnormal Sylow subgroup is characteristic. The same is true for Hall subgroups.

Theorem 1.14: A subnormal Hall subgroup is characteristic.

Proof: It suffices to prove normality since by theorem 1.6(b) any two Hall subgroups of the same order are conjugate. To prove normality, we proceed by induction on the index of the Hall subgroup. Let H be a subnormal Hall subgroup of G. If [G:H] is a prime, then H is maximal, but since a maximal subnormal subgroup is normal, H is normal. So suppose [G:H] = m, and assume the induction hypothesis: If a group has a subnormal Hall subgroup of index less than m, then the subgroup is normal. Let $G_1$ be a maximal normal subgroup of G containing H. Since H is a Hall subgroup of G, H is a Hall subgroup of $G_1$. Also H is subnormal in $G_1$, so by the induction hypothesis, H is normal in $G_1$. By the remark at the beginning of the proof, H is characteristic in $G_1$. Hence H is characteristic in a normal subgroup of G, thus H is normal in G.▯

A subnormal, maximal subgroup is, of course, normal, so it is natural to ask if the same is true for second maximal, third maximal, and so forth. In $A_4$ the subgroups of order two are second maximal and subnormal, but not normal, so the answer to the question is no. However, Deskins [3, lemma 1] proved the following theorem.

Theorem 1.15: If the subnormal subgroup H of a finite group G is contained in the non-normal maximal subgroup K of G, and no subgroup of K properly containing H is subnormal in G, then H is normal in G.

Note in particular that in this case $H = \mathrm{Core}_G K = \bigcap_{x \in G} K^x$. Moreover, under the hypothesis of the theorem, $H = \langle L \mid L \leq K, L \lhd\lhd G \rangle$.

In contrast to the maximal, subnormal subgroups, consider the minimal non-subnormal subgroups.

Theorem 1.16: If H is not subnormal in G, but every proper subgroup of H is subnormal in G, then H is cyclic of prime power order.

Proof: The proof consists of showing that H has only one maximal subgroup. Suppose H has two distinct maximal subgroups, $M_1$ and $M_2$. Then by theorem 1.12 $\langle M_1, M_2 \rangle$ is subnormal in G, but this is impossible since $\langle M_1, M_2 \rangle = H$. So H has a unique maximal subgroup M. Now any element outside M must generate H, so H is cyclic. If two distinct primes, p and q divide $|H|$, then H possesses subgroups of index p and q respectively, but this is impossible since H has only one maximal subgroup. So H is a p-group for some prime p. ▯

Given a group G that satisfies some group theoretic property X, it may be of some interest to know whether subgroups of G, factor groups of G, or extensions of G have property X. This notion leads to the definition of a formation as defined by Gaschütz [4].

Definition 1.17: A formation $\mathfrak{K}$ is a collection of finite solvable groups satisfying:

(1) $\langle 1 \rangle \in \mathfrak{K}$.

(2) If $G \in \mathfrak{K}$, and $N \lhd G$, then $G/N \in \mathfrak{K}$.

(3) If $N_1$ and $N_2$ are normal subgroups of G such that $G/N_i \in \mathfrak{K}$, $i = 1,2$, then $G/(N_1 \cap N_2) \in \mathfrak{K}$.

The collection consisting of the identity subgroup alone is a formation, as is the collection of all finite solvable groups. Some other non-trivial examples are:

(1) The set of all abelian groups

(2) The set of all nilpotent groups

(3) The set of all supersolvable groups

(4) The set of all p-groups for a fixed prime p.

In the above examples, all except the first have the further property that if $G/\Phi(G)$ belongs to the formation, then G also belongs. A formation which has this property is called saturated. Formally, Gaschütz defined a saturated formation in the following way:

Definition 1.18: A formation $\mathfrak{K}$ is saturated if given a group G which does not belong to $\mathfrak{K}$, if M is a minimal normal subgroup of G and $G/M \in \mathfrak{K}$, then M has a complement in G and all the complements to M are conjugate.

Gaschütz later proved that the conjugacy follows from the existence, and that the saturation property can be characterized as follows:

Theorem 1.19: A formation $\mathfrak{F}$ is saturated if and only if the following holds: If G is a finite solvable group such that $G/\phi(G)$ belongs to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.

Theorem 1.19, along with the definition, indicates that the saturated formation concept is very convenient in describing the structure of a finite solvable group. A particular saturated formation which will be important here is the collection of groups having Sylow towers. In a supersolvable group, the Sylow subgroup for the largest prime divisor of the order of the group is normal. Thus there exists a sequence of normal subgroups of G, $\langle 1\rangle = G_n \triangleleft G_{(n-1)} \triangleleft \ldots \ldots \triangleleft G_1 \triangleleft G_0 = G$, such that $G_{(i-1)}/G_i$ is isomorphic to the $p_i$-Sylow subgroup of G, where $p_n$ is the largest prime divisor, $p_{(n-1)}$ is the next largest, and so forth. Such a sequence is called a Sylow tower for G in the natural order of the primes, and G is called a Sylow tower group. More generally we have the following definition.

Definition 1.20: Let $p_1 < p_2 < p_3 < \ldots < p_n < \ldots$ be an arbitrary ordering of the primes. A group G is a Sylow tower group for this ordering if there exists a sequence of subgroups in G, $G = G_0 \triangleright G_{\alpha_1} \triangleright \ldots \ldots \triangleright G_{\alpha_n} \triangleright G_{\alpha_{(n+1)}} = 1$, such that $G_{\alpha_r}/G_{\alpha_{(r+1)}}$ is the normal $p_{\alpha_{(r+1)}}$ Sylow subgroup of $G/G_{\alpha_{(r+1)}}$, and as integers, $\alpha_1 < \alpha_2 < \alpha_3 \ldots$.

A more descriptive, and perhaps clearer, way to get this definition is to let "larger" and "smaller" refer to the given arbitrary ordering and repeat the description as given for the tower in a supersolvable group.

Note that a nilpotent group is a Sylow tower group for every ordering of the primes, and that subgroups and factor groups of a Sylow tower group are Sylow tower groups for the same ordering of the primes. The collection of all Sylow tower groups is not a formation, however, since the direct product of two Sylow tower groups need not be a Sylow tower group. Consider $G = S_3 \times A_4$. $S_3$ has a Sylow tower for the natural order $2 < 3$, and $A_4$ has a tower for the order $3 < 2$. However G does not have a Sylow tower for if it did, then the tower would be inherited by both subgroups. If we take the direct product of two groups having Sylow towers for the same ordering of the primes, then the direct product is again a Sylow tower group. More generally, a lemma due to Baer [1] gives the following theorem.

Theorem 1.21: Given a fixed ordering of the primes, the set of all groups having Sylow towers for this ordering is a saturated formation.

In a solvable group G the length of the derived series may be taken as some sort of a measure of how far G deviates from being abelian. In a similar manner, we can define the nilpotent length of G as a measure of the deviation from nilpotence.

Definition 1.22: The hypercommutator $D(G)$ of G is the intersection of all normal subgroups N of G such that $G/N$ is nilpotent.

By the saturated formation property of nilpotence, $D(G)$ is characterized by: $D(G)$ is minimal with respect to $G/D(G)$ is nilpotent. If G is nilpotent $D(G) = \langle 1 \rangle$, and since $G/G'$ is nilpotent, $D(G)$ is a proper subgroup of G as long as G is solvable. We can use this to define a lower nilpotent series for G.

Definition 1.23: The normal series $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \ldots \triangleright G_\ell = \langle 1 \rangle$, in which $G_i = D(G_{(i-1)})$, is called the lower nilpotent series for G. $\ell = \ell(G)$ is called the <u>nilpotent length</u> of G.

In a similar fashion, we define the upper nilpotent series, or <u>Fitting series for G</u>.

<u>Definition 1.24</u>: F(G), the <u>Fitting subgroup of G</u>, is the maximal, normal, nilpotent subgroup of G. (Note that since the product of normal nilpotent subgroups is normal and nilpotent, F(G) is well defined.) The <u>Fitting series for G</u> is the series $\langle 1 \rangle = F_0 \lhd F_1 \lhd \ldots \lhd F_r = G$, where $F_i/F_{(i-1)}$ is $F(G/F_{(i-1)})$.

The following theorem justifies the use of the term "nilpotent length" for the length of the lower nilpotent series.

<u>Theorem 1.25</u>: The lower nilpotent series and the Fitting series have the same length. Moreover, the length of any nilpotent series, i.e., a normal series with nilpotent factors, is greater than or equal to $\ell(G)$.

<u>Proof</u>: The proof is by induction on $|G|$; the theorem being trivially true when G is nilpotent. Consider the two series:

$$\langle 1 \rangle = F_0 \lhd F_1 \lhd F_2 \lhd \ldots \lhd F_r = G \qquad \text{(the Fitting series)}$$

$$\langle 1 \rangle = D_\ell \lhd D_{(\ell-1)} \lhd D_{(\ell-2)} \cdots \lhd D_1 \lhd D_0 = G \qquad \text{(the lower nilpotent series)}$$

Since $G/F_{(r-1)}$ is nilpotent, $D_1 \leq F_{(r-1)}$. Thus by the induction hypothesis $(\ell-1) \leq (r-1)$, so that $\ell \leq r$. On the other hand $D_{(\ell-1)}$ is nilpotent so $D_{(\ell-1)} \leq F_1$. Considering the length of the factor $G/F_1$ in the group $G/D_{(\ell-1)}$, we have again by induction, $(r-1) \leq (\ell-1)$. So $r \leq \ell$, and we have equality, $r = \ell$. The second remark concerning the length of any nilpotent series follows easily using the same type of argument.$\square$

As was remarked earlier, the set of all nilpotent groups constitutes a saturated formation. This can now be extended to the collection of all groups having a fixed bound on the nilpotent length.

Theorem 1.26: The set $\mathfrak{N}_n$ of all groups having nilpotent length less than or equal to n is a saturated formation.

Proof: The trivial group certainly belongs to $\mathfrak{N}_n$ for each n. The remainder of the proof follows from the following four lemmas.

Lemma 1.27: Let N be a normal subgroup of G, then $\ell(G/N) \leq \ell(G)$.

Proof: The image under the homomorphism of the lower nilpotent series is a nilpotent series, thus by theorem 1.25 $\ell(G/N) \leq \ell(G)$. ▯

Lemma 1.28: If H is a subgroup of G, then $\ell(H) \leq \ell(G)$.

Proof: Let $G = D_0 \triangleright D_1 \triangleright D_2 \triangleright \dots \triangleright D_\ell = \langle 1 \rangle$ be the lower nilpotent series for G. Let $H_i = H \cap D_i$. Then the $H_i$ form a nilpotent series for H. ▯

Lemma 1.29: If $G = H \times K$, then $\ell(G) \leq \max \{\ell(H), \ell(K)\}$.

Proof: Let $\ell(H) = m$, $\ell(K) = n$, with $m \geq n$. Let $H = H_0 \triangleright H_1 \triangleright \dots \triangleright H_m = \langle 1 \rangle$ and $K = K_0 \triangleright K_1 \triangleright \dots \triangleright K_n$ be the lower nilpotent series for H and K respectively. Let $L_i = H_i \times K_i$ for $0 \leq i \leq n$, $L_i = H_i \times \langle 1 \rangle$ for $i > n$. Then the $L_i$ form a nilpotent series in G of length m. ▯

Lemma 1.30: If $N_i$ is a normal subgroup of G, i = 1,2, Then $G/N_1 \cap N_2$ is isomorphic to a subgroup of $G/N_1 \times G/N_2$.

Proof: Consider the map $\tau$ from G to $G/N_1 \times G/N_2$ defined by $\tau(x) = (xN_1, xN_2)$. $\tau$ is clearly a homomorphism, and $N_1 \cap N_2$ is clearly in the kernel of $\tau$. Suppose $x \in \ker(\tau)$. Then $xN_1 = N_1$ and $xN_2 = N_2$, so $x \in N_1 \cap N_2$; thus $\ker(\tau)$ is contained in $N_1 \cap N_2$, and equality holds. The image of $\tau$ is then isomorphic to $G/N_1 \cap N_2$. ▯

These four lemmas show that the set $\mathfrak{R}_n$ is a formation. It remains to show that the formation is saturated.

Lemma 1.31: $\ell(G/\phi(G)) = \ell(G)$.

Proof: It is only necessary to show $\ell(G) \leq \ell(G/\phi(G))$. Let F denote the Fitting subgroup F(G), and $\phi$ the Frattini subgroup $\phi(G)$. Then by [11, theorem 7.4.9] $F(G/\phi) = F/\phi$, so $\phi$ is proper in F, and the Fitting series for $G/\phi$ coincides with the Fitting series for G. ▯

It should be noted that theorem 1.26 can also be proved using the technique of local definition of a saturated formation as defined by Gaschütz.

# CHAPTER II

## THE FUNCTION h(G)

The theorems due to Huppert and Janko which were mentioned in the introduction have to do primarily with the occurence of normal subgroups in the lattice of all subgroups. If we weaken the condition that every k-th maximal subgroup of G is normal to the condition that in every upper chain of length k there must occur a subnormal subgroup, we get the definition of the function h(G).

Definition 2.1: h(G) = n if every upper chain of length n in G contains a proper subnormal entry, and there exists at least one upper chain in G of length (n-1) which contains no proper subnormal entry.

Theorem 2.2: h(G) = 1 if and only if G is nilpotent.

Proof: If h(G) = 1, then every upper chain of length one contains a subnormal entry. This means that every maximal subgroup is subnormal. However a subnormal maximal subgroup is normal, so every maximal subgroup is normal, and so G is nilpotent. Conversely, if G is nilpotent, every subgroup is subnormal so h(G) = 1.▯

From the definition we see that if h(G) = n, then there exists an upper chain in G of length n such that the only subnormal entry in the chain is the terminal entry. Such a chain will be called an h-chain for G, and the terminal entry will be called an h-subgroup. Of some interest later on is the question of the nature of an h-chain and h-subgroup, however at this point the primary interest is on how the function h(G) restricts the structure of G.

Theorem 2.2 can be extended to theorems on the structure of G when h(G) = n, and to do so it is convenient to first note how the function h(G) behaves on subgroups and factor groups.

Theorem 2.3: If H is a non-normal maximal subgroup of G, then $h(H) \leq h(G) - 1$.

Proof: Let $H = H_0 > H_1 > H_2 > H_3 > \dots > H_n$ be an h-chain for H. Then $H_n$ is the only entry in the chain subnormal in H. By theorem 1.13, $H_n$ is the only entry in the chain which is subnormal in G. Thus adjoining G to the chain we obtain an upper chain in G of length (n+1) with only the terminal entry subnormal, thus $h(G) \geq (n+1)$. ▯

Theorem 2.3 is only a slight modification of Lemma 2 [3].

Theorem 2.4: If N is a normal subgroup of G, then $h(G/N) \leq h(G)$.

Proof: Suppose $h(G/N) = n$ and let $G/N > G_1/N > G_2/N > \dots > G_n/N$ be an h-chain for G/N. Since subnormality is invariant under the homomorphism, the upper chain $G > G_1 > G_2 > \dots > G_n$ has $G_n$ as its only subnormal entry. Thus $h(G) \geq n$. ▯

We are now in a position to prove the first structure theorem, which gives an upper bound on the nilpotent length of G.

Theorem 2.5: Let $\ell(G)$ denote the nilpotent length of G. Then $\ell(G) \leq h(G)$.

Proof: The proof is by induction on h(G), the theorem being trivially true if h(G) = 1. Suppose the theorem is true for all groups K having $h(K) \leq (n-1)$ and is false for some group K having h(K) = n. Among such groups for which the theorem is false let G be one of minimal order. We will show that such a group G cannot exist. In short, we assume the theorem is true for all groups K for which $h(K) < n$, and for all groups K for which $h(K) = n$ and $|K| < |G|$. Let M be a minimal normal subgroup of G. By theorem 2.4, $h(G/M) \leq n$, so by the minimality of G, $\ell(G/M) \leq n$. If N is another minimal normal subgroup of G, by the same argument $\ell(G/N) \leq n$. $N \cap M$ is trivial, so by the saturated formation property this implies that $\ell(G) \leq n$, which is a contradiction. Thus no such N exists and M is the unique minimal normal subgroup of G. Also by the saturated formation property M has a complement L in G. Since M is the unique minimal normal subgroup of G and $L \cap M = \langle 1 \rangle$, L is non-normal. Since M is minimal, L is maximal, so by theorem 2.3, $h(L) \leq (n-1)$. By the induction hypothesis, $\ell(L) \leq (n-1)$. Thus there exists a nilpotent chain of length (n-1) from G to M. Since M is abelian, adjoining M to this chain, we obtain a nilpotent series in G of length n. This is a contradiction, so G does not exist. ▯

$S_3$, the symmetric group on three letters has the properties $h(S_3) = \ell(S_3) = 2$, showing that the arithmetic conditions in the theorem cannot be improved.

The converse of theorem 2.5 is false, in fact, we can find a group G such that h(G) is arbitrarly large and $\ell(G) = 2$.

<u>Theorem 2.6</u>: Given $n \geq 2$, there exists a group G such that $h(G) \geq n$ and $\ell(G) = 2$.

<u>**Proof**</u>: Let p be a prime of the form $p = 2^n k + 1$. A theorem of Dirichlet guarantees the existence of such a prime. Let P be a group of order p, and let G be the holomorph of P, i.e., the split extension of P by its automorphism group L. L is cyclic of order $2^n k$. L is maximal, and non-normal in G, hence L does not contain

a non-trivial subnormal subgroup of G. If L did contain a subgroup subnormal in G, then by theorem 1.15, L would contain a subgroup N normal in G. But then N would centralize P which is impossible, since L is the automorphism group of P. Since L is cyclic of order $2^n k$, there exists an upper chain in L of length n. Adjoining G to this chain, we obtain an upper chain in G of length (n+1) with no subnormal entries, thus $h(G) \geq (n+1)$. G/P and P are both cyclic so $\ell(G) = 2$. ▯

A natural question concerning such functions as h(G) is: How does this function behave on subgroups, factor groups, direct products, semi-direct products etc.? Partial answers have already been given: h is strictly decreasing on non-normal maximal subgroups; h is non-increasing on factor groups; and theorem 2.6 indicates that h(P) and h(L) do not restrict h(G) even if G is the semi-direct product of L and P. A similar, but more precise, statement can be made in the case of direct products.

<u>Theorem 2.7</u>: Let $G = H \times K$, where H is not nilpotent, i.e., $h(H) \geq 2$. Suppose the order of K is divisible by m not necessarily distinct primes, i.e., $|K| = \prod p_i^{\alpha_i}$, where $\sum \alpha_i = m$. Then $h(G) \geq h(H) + m$.

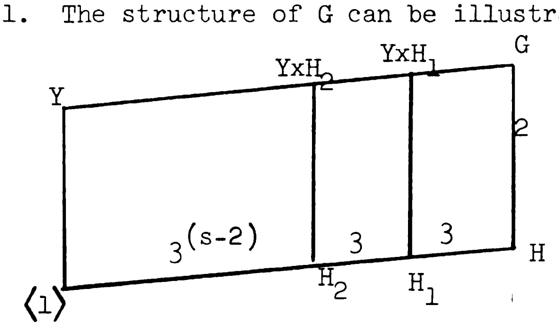
<u>Proof</u>: Let $h(H) = r$, and $H = H_0 > H_1 > \ldots > H_r$ be an h-chain for H. Let $K = K_0 > K_1 > K_2 > \ldots > K_m = \langle 1 \rangle$ be a composition series for K. Consider the chain $G = H \times K > H_1 \times K > H_1 \times K_1 > H_1 \times K_2 > H_1 \times K_3 > \ldots$ $\ldots > H_1 \times K_m = H_1 > H_2 > H_3 > \ldots > H_r$. If one of these subgroups is sub-normal in G, then its projection on H is subnormal in H. However, these projections are precisely $H_1, H_2, H_3, \ldots, H_r$, and of these, only $H_r$ is subnormal in H. Thus only the last entry in the

chain is subnormal in G. The chain is of length (r+m), so we have $h(G) \geq (r+m)$. ▯

Note that although the set of groups for which $h(G) = 1$ is a saturated formation the same is not true for the groups satisfying $h(G) \leq n$, if $n > 1$.

We have shown that h(G) is not restricted by the values of the function h on direct factors and semi-direct factors. Consider the same question for subgroups in general. Is h(G) restricted by h(K) as K ranges over all subgroups of G? The answer is no, as is shown in the following example.

Let H be a cyclic group of order $3^s$. $H = \langle x \rangle$. Let y be an element of order two such that $y^{-1}xy = x^{(3^{(s-1)}+1)}$. Since $3^s$ and $(3^{(s-1)}+1)$ are relatively prime, $y^{-1}xy$ generates H. Let $G = \langle x,y \rangle$ subject to the above relation. Then $|G| = 2\cdot 3^s$. G is the semi-direct product of H and $Y = \langle y \rangle$. Let $H_1 = \langle x^3 \rangle$. Then we have $y^{-1}x^3y = x^{3(3^{(s-1)}+1)} = x^{3^s+3} = x^3$, so Y centralizes $H_1$. Now $N_G(Y) = Y \times H_1$, therefore since Y isn't subnormal in G, no proper subgroup of G containing Y is subnormal in G. The only maximal subgroups of G are H and the conjugates of $Y \times H_1$, so all proper subgroups of G are nilpotent. Consider an upper chain from G to Y. Since Y is not subnormal, this chain contains no subnormal entry. Therefore $h(G) = s+1$, and yet for every proper subgroup K of G, $h(K) = 1$. The structure of G can be illustrated by a diagram.



In contrast to the example, which shows that an upper bound is not immediately avaliable, except of course, for the obvious bound given by the length of a composition series, a lower bound for h(G) is avaliable.

Theorem 2.8: Let $\pi(G)$ denote the number of distinct prime divisors of $|G|$. Then if $h(G) < \pi(G)$, $h(G) = 1$, i.e., G is nilpotent.

Proof: Suppose the theorem is false and let G be a counter-example. Let $h(G) = n$, then $\pi(G) \geq (n+1)$. Let P be a non-normal Sylow subgroup of G, and consider an upper chain from G to P which passes through N(P). N(P) is abnormal, so no entry in the chain above N(P) is subnormal in G. However, since P is not subnormal, no entry below N(P) can be subnormal. Since $\pi(G:P) = n$, by theorem 1.4 P is at least an n-th maximal subgroup of G, so the length of the chain is at least n. $h(G) = n$, so the chain must contain a subnormal entry. This is a contradiction, so no such P exists. Therefore all the Sylow subgroups of G are normal, and so G is nilpotent. ▯

Theorem 2.8 is the best possible in the following sense.

Theorem 2.9: Given n there exists a group G satisfying $h(G) = \pi(G) = n$.

Proof: For $n = 1$, any p-group satisfies the condition. For $n = 2$, consider $S_3$. $h(S_3) = \pi(S_3) = 2$. For $n > 2$, let $Z_n$ be a cyclic group whose order is the product of (n-2) distinct primes greater than 3. Let $G_n = S_3 x Z_n$. By theorem 2.7, $h(G_n) \geq n$, but since $|G_n|$ is the product of n distinct primes, $h(G_n) = n$. ▯

The groups generated in theorem 2.9 are all supersolvable, however supersolvability is not a consequence of the condition that $h(G) = \pi(G)$, as is seen in $A_4$. In $A_4$ the only non-subnormal subgroups are the Sylow 3-subgroups, so $h(G) = 2$, $\pi(G) = 2$, and $A_4$ is not supersolvable. The structure of $A_4$ does suggest, however, what is true in this case.

Theorem 2.10: If $h(G) = \pi(G) \geq 2$, then:

(1) G is a Sylow tower group for some ordering of the primes.

(2) The non-normal Sylow subgroups of G are cyclic.

(3) The normal Sylow subgroups of G are cyclic or elementary abelian.

(4) The theorem is the best possible in the following sense: Given $n \geq 3$, there exists a group G such that $h(G) = \pi(G) + 1 = n$, and G has at least one non-abelian Sylow subgroup.

Part of the proof of theorem 2.10 follows from a slightly more general theorem.

Theorem 2.11: If $h(G) \leq \pi(G) + 1$, then G is a Sylow tower group for some ordering of the primes. This is the best possible in the sense that given $n \geq 4$, there exists a non-Sylow tower group G such that $h(G) = n = \pi(G) + 2$.

Proof of theorem 2.11: The proof is by induction on $h(G)$, the theorem being trivially true if $h(G) = 1$. So suppose the theorem is true for all groups K such that $h(K) \leq (n-1)$, and is false for some group K for which $h(K) = n$. Among such groups, let G be one of minimal order. We show that G cannot exist.

G must satisfy the following conditions:

(1) Every non-normal maximal subgroup of G is a Sylow tower group.

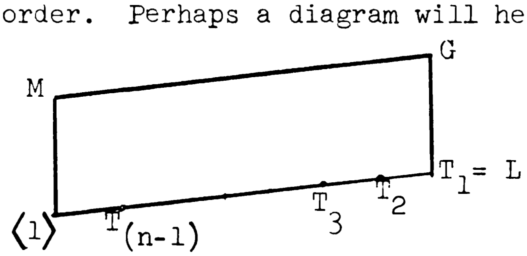
Suppose H is a non-normal maximal subgroup of G. Then by theorem 2.3, $h(H) \leq (n-1)$. By theorem 1.2, $\pi(H) \geq \pi(G) - 1$, so by the induction hypothesis, H is a Sylow tower group.

(2) G does not contain a normal Sylow subgroup.

Suppose P is a normal Sylow subgroup of G. Then P satisfies the properties: P is normal, P is a Hall subgroup of G, P is a Sylow tower group. Choose $K \geq P$, so that K is maximal with respect to these three properties. Then K satisfies: K is normal in G, K is a Hall subgroup of G, K is a Sylow tower group, and no subgroup of G properly containing K has these properties. Since $K \geq P$, K is non-trivial, and since G is not a Sylow tower group K is proper in G. As a normal Hall subgroup of G, K has a complement L in G. L is a Hall subgroup of G, so by theorem 1.1, N(L) is abnormal in G. If $N(L) \neq G$, then N(L) is contained in an abnormal maximal subgroup of G, which by (1) is a Sylow tower group. But then L is a subgroup of a Sylow tower group, so L is a Sylow tower group. But L is isomorphic to G/K, so that G/K is a Sylow tower group. Then since K is a Hall subgroup, G is a Sylow tower group, which is a contradiction. Therefore $N(L) = G$, and $G = K \times L$. Suppose $\pi(K) = r$. Then $\pi(L) = \pi(G) - r$. If $h(L) \leq \pi(G) - r + 1$, then L is a Sylow tower group by the induction hypothesis. But L is not a Sylow tower group, so $h(L) \geq \pi(G) - r + 2$. Then by theorem 2.7 we have that $h(G) \geq h(L) + m$, where m is the length of a composition series for K. Clearly $m \geq \pi(K) = r$, thus $h(G) \geq h(L) + m \geq (\pi(G) - r + 2) + r = \pi(G) + 2$. This is a contradiction to the hypothesis of the theorem, so no such K exists, and G does not possess a normal Sylow subgroup.

(3) G has a unique minimal normal subgroup M, and moreover, G/M is supersolvable.

Let M be a minimal normal subgroup of G. By (2), M is not a Sylow subgroup of G, so $\pi(G/M) = \pi(G)$. By theorem 2.4, $h(G/M) \leq h(G)$, so by the minimality of G, G/M is a Sylow tower group. The set of groups which are Sylow tower for the same ordering as the ordering in G/M constitute a saturated formation by theorem 1.21, so by the definition of saturated formation, M has a complement L in G. L is isomorphic to G/M, so L is a Sylow tower group. Let $L = T_1 \triangleright T_2 \triangleright T_3 \triangleright \ldots \triangleright T_{(n-1)} \triangleright \ldots \triangleright \langle 1 \rangle$ be a Sylow tower for L. We refine this series and adjoin G to obtain the upper chain: $G = T_0 > T_1 \triangleright \ldots \triangleright T_2 \triangleright \ldots \triangleright T_3 \triangleright \ldots \quad \triangleright T_{(n-1)} \triangleright \ldots \triangleright \langle 1 \rangle$. Since $h(G) = n$, some one of the first n entries in this chain is subnormal in G. Therefore a subnormal entry must occur at or above $T_{(n-1)}$, unless $[T_1 : T_{(n-1)}]$ is square free. So if $[T_1 : T_{(n-1)}]$ is not square free, $T_{(n-1)}$ is subnormal in G. Now $T_{(n-1)}$ is a Sylow tower group, so contains a normal Sylow subgroup S. If S is a Sylow subgroup of G, then S is a subnormal Sylow subgroup of G, and is thus normal in G. This contradicts (2), so S is not a Sylow subgroup of G. Then SM is a normal Sylow subgroup of G, which is again a contradiction, so $[T_1 : T_{(n-1)}]$ is square free. Now $T_{(n-1)}$ is an (n-1)-th maximal subgroup of G, is not subnormal, and in the $T_i$ chain, no entry above $T_{(n-1)}$ is subnormal in G. Since $h(G) = n$, every maximal subgroup of $T_{(n-1)}$ is subnormal in G, so by theorem 1.16, $T_{(n-1)}$ is cyclic of prime power order. Perhaps a diagram will help illustrate the structure of G.



We have that the Sylow subgroups of L are cyclic, so L is supersolvable. Thus we have shown that the quotient group G/N is supersolvable, where N is any minimal normal subgroup. If G has two distinct minimal normal subgroups, then by the saturated formation property of supersolvability, G is supersolvable, and therefore is a Sylow tower group. This is a contradiction, thus G has a unique minimal normal subgroup.

(4) Using the same notation as in (3), L is of square free order.

We already have that $[L:T_{(n-1)}]$ is square free, so it only remains to show that $T_{(n-1)}$ is of prime order. If $T_{(n-1)}$ is not of prime order, then $T_{(n-1)}$ contains a subgroup subnormal in G. But then L contains a subnormal subgroup, and therefore contains a normal subgroup of G. However, M is the unique minimal normal subgroup of G, so this is impossible. Therefore $T_{(n-1)}$ is of prime order, and so L is of square free order.

(5) $h(G) = 3$.

Let P denote the Sylow p-subgroup of G, where $|M| = p^{\alpha}$. Then since $|L|$ is square free, $|P| = p^{\alpha+1}$, and M is a maximal subgroup of P. P is not cyclic, since G is not supersolvable, and P is not subnormal, so there exists a subgroup R of P such that R is a maximal subgroup of P and R is not subnormal. Now $\pi(G:P) = n-2$, so R is the (n-1)-th entry in an upper chain from G through P to R. No entry in this chain is subnormal in G, so by theorem 1.16, R is cyclic. Now $RM = P$, and $R \cap M$ is maximal in R and in M. Since R is cyclic and M is elementary abelian, $R \cap M$ is both cyclic and elementary abelian. Therefore $|R \cap M| = 1$ or p, so $|M| = p$ or $p^2$.

Let q denote the largest prime divisor of $|G|$, and Q the Sylow q-subgroup of G. Then Q is a normal subgroup of L, since L is supersolvable. However, Q is not normal in G, so $L = N(Q)$. By the Sylow theorems, q divides $[G:L] - 1$. However $[G:L] = |M| = p$ or $p^2$. So we have $q|(p-1)$ or $q|(p^2-1)$. Since q is the largest prime divisor of the order of G, the first possibility cannot hold. Hence $q|(p^2-1)$, and since $q > p$, $q|(p + 1)$ and so $q = (p + 1)$. Hence $q = 3$, and $p = 2$, and since q is the largest prime divisor of $|G|$, $\pi(G) = 2$. By hypothesis, $h(G) = \pi(G) + 1$, so $h(G) = 3$.

(6) The final contradiction

From (5) we see that $|G| = 24$. Let $G = G_0 > G_1 > G_2 > G_3$ be an upper chain where $|G_1| = 8$. Then since $h(G) = 3$, and $G_1$ is nilpotent, $G_3$ is subnormal in G, i.e., all subgroups of order 2 in G are subnormal. ( Note that this means that G is not $S_4$.) Finally, consider Q, the Sylow 3-subgroup of G. By the Sylow theorems, $[G:N(Q)] = 1$ or 4. Since Q is not normal in G, $[G:N(Q)] = 4$, and $|N(Q)| = 6$. The Sylow 2-subgroup of N(Q) is subnormal in G, so is subnormal, hence normal, in N(Q). Therefore N(Q) is abelian. Hence Q is in the center of its normalizer, so by Burnside's theorem, Q has a normal complement. However, this complement of Q is of order eight, which means that G is a Sylow tower group. This final contradiction shows that G does not exist, proving the theorem. ▯

To show that this theorem is the best possible in the sense that the arithmetic conditions cannot be relaxed, we begin with $S_4$. In $S_4$ there exists a non-subnormal subgroup of order two. Therefore the upper chain, $S_4 = G_0 > G_1 > G_2 > G_3 > \langle 1 \rangle$, where $G_3$ is a non-subnormal subgroup of order two and $G_1$ is a Sylow 2-subgroup, has $\langle 1 \rangle$ as its only subnormal entry. Therefore $h(S_4) = 4$, since the longest upper chain in $S_4$ is of length four. $\pi(S_4) = 2$, and $S_4$ is not a Sylow tower group. Now given $n > 4$, let $Z_n$ be a cyclic group whose order is $(n - 4)$ distinct primes greater than 3. Let $G_n = S_4 \times Z_n$. Then using the same argument as in the proof of theorem 2.9, we have: $h(G_n) = (n + 4)$, $\pi(G_n) = (n + 2)$, and since $S_4$ is a subgroup of $G_n$, $G_n$ is not a Sylow tower group. ▯

Now to return to the proof of theorem 2.10. Part (1), concerning the existence of a Sylow tower in G, follows from theorem 2.11, so it only remains to show that G has the proper types of Sylow subgroups. Let $h(G) = \pi(G) \geq 2$. Suppose P is a non-normal Sylow subgroup of G. Consider an upper chain from G to P, which passes through N(P). Since N(P) is abnormal and P is not subnormal, no entry in the chain is subnormal in G. However, since $\pi(G:P) = (n - 1)$, the length of the chain is $(n - 1)$. Since $h(G) = n$, every maximal subgroup of P is subnormal in G, so by theorem 1.16, P is cyclic. Since there are $(n - 1)$ entries in the chain, and $(n - 1)$ distinct primes involved, each entry in the chain is a Sylow complement in its predecessor. Then the proof of theorem 2.10 follows from the following lemma.

<u>Lemma 2.12</u>: If S is a Sylow complement for the Sylow subgroup R of T, and S is maximal in T, then R is elementary abelian.

<u>Proof</u>: If S is normal in T, then since S is maximal, $[T:S] = r$, a prime, and $|R| = r$, so R is elementary abelian. If S is not normal in T, let $K = \mathrm{Core}_T(S)$, i.e., the largest subgroup of S which is normal in T. Then RK/K is the unique minimal normal subgroup of T/K. Therefore RK/K is elementary abelian, however since $(|R|, |K|) = 1$,

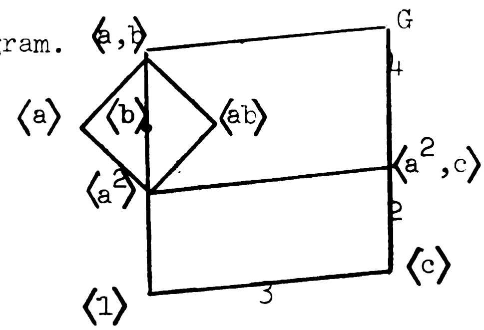
RK/K and R are isomorphic, and R is elementary abelian. ▯

Applying this lemma to each entry in the chain, we have that all the Sylow subgroups of G, except P, are elementary abelian. ▯

Actually, we have proved much more than theorem 2.10 states. We have shown that all except one of the Sylow subgroups are elementary abelian, so if there are at least 2 distinct non-normal Sylow subgroups, for distinct primes, that is, then all of the non-normal Sylow subgroups are of prime order. This follows since they must be cyclic and elementary abelian. Also note that if K is the product of all the normal Sylow subgroups of G, then K is abelian, and G/K has cyclic Sylow subgroups, therefore is supersolvable. Therefore the nilpotent length of G is no more than three.

The fact that theorem 2.10 is the best possible in the sense of the arithmetic conditions, follows from an example.

Let $G = \langle a, b, c \rangle$, with $a^4 = c^3 = 1$, $b^2 = a^2$, $ba = a^3b$, and $c^2ac = b$. G is the extension of the Quaternion group of order eight by an automorphism of order three which permutes the subgroups of order four. A skeleton for the structure of G is given in the following diagram.




Since $\langle a,b \rangle$ is normal in G, an h-chain for G is obtained by taking the chain $G > \langle a^2, c \rangle > \langle c \rangle > \langle 1 \rangle$. Thus $h(G) = 3$, $\pi(G) = 2$, and the Sylow 2-subgroup is not abelian. Using the same technique as in theorem 2.9, this example can be extended to arbitrary n.

Theorems 2.8, 2.9, 2.10, and 2.11 are related and can be summed up as follows:

Suppose $\pi(G) = m \geq 2$. Then:

(1) If $h(G) \leq (m + 1)$, G is a Sylow tower group for some ordering of the primes.

(2) If $h(G) = m$, the non-normal Sylow subgroups of G are cyclic, and the normal Sylow subgroups are cyclic or elementary abelian. If for two distinct primes G has non-normal Sylow subgroups, then G has an abelian, normal Hall subgroup such that the quotient group is of square free order.

(3) If $h(G) \leq (m - 1)$, then $h(G) = 1$, i.e., G is nilpotent.

(4) In each case the given inequality is the best possible.

In light of the above situation, we are led naturally to the question of what structure follows from the hypothesis $h(G) \leq \pi(G)+2$, and so forth. As has been noted, $h(S_4) = 4$, and $\pi(S_4) = 2$. $S_4$ is the group of smallest order that is not a Sylow tower group, so is the smallest example of the class of groups Baer called critical. Baer defined: G is a critical group if G is solvable; G is not a Sylow tower group; and every subgroup and factor group of G is a Sylow tower group. We might expect then that the hypothesis that $h(G) \leq \pi(G) + 2$ would imply that G is critical. This, however, is not true. If we take $G = S_4 \times Z$, where Z is cyclic of order five, then $h(G) = 5$, $\pi(G) = 3$, and G is not critical since G contains the non-Sylow tower group $S_4$.

Certainly some more information about the structure of G is available, and perhaps some indication of what is true can be obtained by looking at the structure of groups having $h(G)$ very small. Of course, if $h(G) = 1$, $G$ is nilpotent, and no further information is available, or necessary, if we are concerned with the placement of the subnormal subgroups in the lattice of all subgroups, since all the subgroups are subnormal. Suppose that $h(G) = 2$, then we have the following structure.

Theorem 2.13: Suppose $h(G) = 2$. Then $G = PQ$; $P$ and $Q$ are Sylow subgroups of $G$; $P$ is a minimal normal subgroup of $G$; $Q$ is cyclic; $Q_1$, the maximal subgroup of $Q$, is normal in $G$, in fact, $Q_1 = \phi(G) = Z(G)$; $Q/Q_1$ acts irreducibly on $P$.

Proof: By theorem 2.8, if $\pi(G) > 2$, then $G$ is nilpotent, but by hypothesis $h(G) = 2$, so we have $\pi(G) = 2$. By theorem 2.10, $G$ contains a normal Sylow subgroup $P$, and also by theorem 2.10, the non-normal Sylow subgroup $Q$ is cyclic. By theorem 1.16, every maximal non-normal subgroup of $G$ is cyclic of prime power order. $N(Q)$ is abnormal, and is proper in $G$, therefore $N(Q)$ is cyclic of prime power order, i.e., $N(Q) = Q$, and $Q$ is a maximal subgroup of $G$. Since $Q$ is maximal, $P$ is a minimal normal subgroup. Let $Q_1$ denote the maximal subgroup of $Q$. $Q_1$ is subnormal in $G$, so by theorem 1.15, $Q_1$ is normal in $G$. Therefore $Q_1$ centralizes $P$, and is centralized by $Q$, so $Q_1 \leq Z(G)$. Since $Q$ is maximal, $Q$ does not centralize any subgroup of $P$, thus $Q_1 = Z(G)$. The maximal subgroups of $G$ are simply $Q_1P$ and the conjugates of $Q$, so $Q_1 = \phi(G)$. The irreducible action of $Q/Q_1$ on $P$ stems from the fact that $Q$ is maximal. ▯

It should be noted that from a theorem due to Rose [10, theorem 1] the fact that $h(G) = 2$ implies solvability for $G$, thus theorem 2.13 is true for finite groups in general.

Also note that such groups as described in theorem 2.13 do exist. Both $S_3$ and $A_4$, for example, satisfy $h(S_3) = h(A_4) = 2$. The groups G satisfying $h(G) = 2$ have the property that G is generated by two elements. This can be extended to a more general theorem.

<u>Theorem 2.14</u>: Let $r(G)$ denote the minimal number of generators for G, i.e., there exists a set of r elements which generates G, but no set of $(r - 1)$ elements generates G. Then if $h(G) \geq 2$, $r(G) \leq h(G)$.

<u>Proof</u>: The condition $h(G) \geq 2$ is certainly necessary, since we can find elementary abelian groups K with $r(K)$ arbitrarily large. The proof of theorem 2.14 follows from a slightly more general theorem about the function $r(G)$.

<u>Theorem 2.15</u>: If a group H has a cyclic k-th maximal subgroup, then $r(H) \leq (k + 1)$.

<u>Proof</u>: The proof is by induction on k. If $k = 1$, H has a cyclic maximal subgroup M. By the maximality of M, if $x \notin M$, $\langle x, M \rangle = H$. So if y generates M, x and y generate H, and the theorem is true. So suppose the theorem is true for all integers less than k, and let $H = H_0 > H_1 > \ldots > H_k$ be an upper chain from H to the cyclic k-th maximal subgroup $H_k$. $H_k$ is a cyclic $(k-1)$-th maximal subgroup of $H_1$, so by the induction hypothesis, $r(H_1) \leq k$. If $\Lambda$ is a set of k generators for $H_1$, and $x \notin H_1$, then $\Lambda \cup \{x\}$ will generate H. Therefore $r(H) \leq k + 1$. ▯

Now theorem 2.14 follows, since the next to last entry in an h-chain for G is cyclic and $(h(G) - 1)$-th maximal. ▯

# CHAPTER III

## SOME EXTENSIONS AND GENERALIZATIONS

The results of chapter II suggest several new questions concerning the function $h(G)$. For example, we have seen that the h-chain for a non-nilpotent group terminates in a subnormal cyclic p-group. In the case $h(G) = 2$ or $3$, this terminal member is normal in G. Whether or not this holds in general is not known. Partial answers can be given, however.

Definition 3.1: H is called an h-subgroup of G if H is the terminal entry in an h-chain for G.

Theorem 3.2: If H is an h-subgroup of G, and $\sigma$ is an automorphism of G, then $H^{\sigma}$ is an h-subgroup of G.

Proof: Since normality is preserved under the automorphism, the h-chain terminating in H is transformed into an h-chain terminating in $H^{\sigma}$. ▯

It has been pointed out that the situation can occur where $h(G)$ is large, and every proper subgroup of G is nilpotent. If this does not happen, then we can obtain a result concerning the normality of an h-subgroup.

Theorem 3.3: If $h(G) = n$, and in an h-chain from G to H, $G = G_0 > G_1 > G_2 > \ldots > G_n = H$, $h(G_i) = (n-i)$, and H is an h-subgroup for each $G_i$, then H is normal in G.

Proof: The proof is by induction on $h(G)$. If $h(G) = 1$, then G is nilpotent, H is maximal, and therefore is normal. Suppose the theorem is true for all groups K for which $h(K) \leq (n-1)$, and is false for some group K for which $h(K) = n$. Among such groups K for which $h(K) = n$, and the theorem is false, let G be one of minimal order. We prove the theorem by showing that such a group G cannot exist. Let $G = G_0 > G_1 > G_2 > \ldots > G_{(n-1)} > G_n = H$ be an h-chain for G. Since $h(G_1) = (n-1)$ and H is an h-subgroup for $G_1$, by the induction hypothesis, H is normal in $G_1$. Since for each i, $0 \leq i < n$ $h(G_i) = (n-i)$, we may assume that each $G_i$ is non-normal in $G_{(i-1)}$. Then $G_{(n-1)}$ has the property that it can be joined to G through a chain, each entry abnormal in its predecessor, and as well, every proper subgroup of $G_{(n-1)}$ is subnormal in G. Therefore by theorem 1.9, $G_{(n-1)}$ is a system normalizer of G. By the same argument, $G_{(n-1)}$ is a system normalizer of $G_1$. Now H is the core of $G_{(n-1)}$ in $G_1$, therefore H contains the hypercenter of $G_1$. H does not contain a non-trivial subgroup N normal in G, for if N is normal in G, by the minimality of G, we have H/N is normal in G/N and so H is normal in G. Thus the core of $G_{(n-1)}$ in G is trivial, so since $G_{(n-1)}$ contains the hypercenter of G, the hypercenter of G is trivial. Let $x \in H$, such that x is of prime order and belongs to $Z(G_1)$. Let $M = \langle x^g \mid g \in G \rangle$. M is a minimal normal subgroup of G. Consider a cheif series for G which passes through M. Since the hypercenter of G is trivial, M is not a central cheif factor of G. Therefore by [7, theorem 6.1], $G_{(n-1)} \cap M = \langle 1 \rangle$. However, $x \in G_{(n-1)} \cap M$, and we have a contradiction, so G does not exist. ▯

In the proof of theorem 2.10 it is clear that if $h(G) = \pi(G)$, then the h-subgroup is normal in G. Whether this is true in general is not known. From the proof of theorem 2.13, it is clear that if h(G) = 2, then the h-subgroup is unique. If h(G) = 3, it is still true that the h-subgroup is normal, but this requires a proof. Before proving this theorem, it is convenient to prove a theorem concerning subnormal p-groups.

Theorem 3.4: If P is a cyclic Sylow subgroup of G, and $P_1$, the maximal subgroup of P, is subnormal in G, then $P_1$ is normal in G.

Proof: The proof is by induction on $|G|$, the theorem being trivially true if G is a p-group. So suppose the theorem is true for all groups of order less than $|G|$, and let $G = G_0 \triangleright G_1 \triangleright \cdots$ $\cdots \triangleright G_n = P_1$ be a subnormal chain from G to $P_1$. $P_1$ is subnormal in $G_1$, and the Sylow p-subgroup of $G_1$ is cyclic, so by the induction hypothesis $P_1$ is normal in $G_1$. If $P_1$ is a Sylow subgroup of $G_1$, then $P_1$ is actually characteristic in $G_1$, therefore is normal in G. So suppose $P_1$ is not a Sylow subgroup of $G_1$. Then $G_1$ contains P. If P is normal in $G_1$, then as before, P is normal in G, and since $P_1$ is characteristic in P, $P_1$ is normal in G. So we may assume P is not normal in $G_1$. Then $\bigcap_{x \in G_1} P^x = P_1$, however since P is a Sylow subgroup of G, and $G_1$ is normal in G, $\bigcap_{x \in G} P^x = \bigcap_{x \in G_1} P^x$, hence $P_1$ is normal in G. ▯

Theorem 3.5: If h(G) = 3, and H is an h-subgroup of G, then H is normal in G.

Proof: The proof is done in two cases, depending on $\pi(G)$. Let $G = G_0 > G_1 > G_2 > G_3 = H$ be an h-chain for G.

Case 1: $\pi(G) = 3$.

In this case, $G_2$ is a Sylow subgroup of G. In any case $G_2$ is a cyclic p-group, so by theorem 3.4, H is normal in G.

Case 2: $\pi(G) = 2$.

If $G_2$ is a Sylow subgroup of G, then as in case 1, H is normal in G. So we may assume that $G_2$ is not a Sylow subgroup. $G_2$ is a cyclic p-group, so if $G_2$ is normal in $G_1$, then H is normal in $G_1$. If $G_2$ is not normal in $G_1$, by theorem 1.15, H is normal in $G_1$. So in all cases H is normal in $G_1$, so suppose $N_G(H) = G_1$. We will show that this assumption leads to a contradiction. Let $|G_2| = p^{\alpha}$.

Case 2.1: $[G: G_1] = q^{\beta}$.

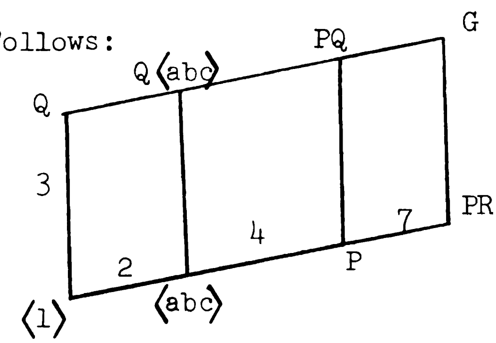
By theorem 3.4, we see that $G_2$ is not a Sylow subgroup of G, thus $G_1$ is a Sylow p-Sylow subgroup of G. Let L be a normal maximal subgroup containing H. $L \not\geq G_1$, thus $[G:L] = p$. In the subnormal chain $G \triangleright L \triangleright \ldots T \triangleright H$, T normalizes H, thus $T < G_1$. However, since $[G_1: H] = p^2$, T is the Sylow p-subgroup of L, so T is normal in L. Thus L contains the normal Sylow subgroup Q of G (G is a Sylow tower group) and L normalizes T. Thus T and Q centralize each other, so Q centralizes H, so $Q \leq N(H) = G_1$. This is a contradiction, so case 2.1 cannot hold.

Case 2.2: $[G:G_1] = p^{\nu}$.

In this case, $G_2$ is a Sylow subgroup of $G_1$. Since $h(G) = 3$, G is a Sylow tower group. $G_1$ contains the normal Sylow subgroup Q of G, but then G/Q is a p-group, and $G_1$ is normal in G. This is a contradiction, showing that case 2.2 cannot hold, and the proof is complete. ▯

In consideration of the question of the normality of the h-subgroup, we are led to an attempt to extend results similar to theorem 1.15, in the form: If H is a second maximal subgroup, and H is subnormal in G, then if H is contained in a non-normal maximal subgroup of G, H is normal in G. The natural attempt to generalize this would be: If H can be joined to G through a chain such that each entry is maximal and non-normal in the entry immediately above, and if H is subnormal in G, H is normal in G. This conjecture is false, as can be seen in the following example:

Let $G = \langle a,b,c,x,y \rangle$ subject to: $\langle a,b,c \rangle$ is an elementary abelian group of order eight. $x^3 = y^7 = 1$. $a^x = b$, $b^x = c$, and $c^x = a$. $a^y = b$, $b^y = c$, and $c^y = ab$. $xyx^{-1} = y^2$. Let P, Q, and R be the two, three, and seven Sylow subgroups, respectively. Then a diagram showing showing the skeleton of the lattice of subgroups of G is as follows:




$|G| = 168$, and the Sylow 2-subgroup of G is normal. $N(R) = QR$, and $N(Q) = Q\langle abc \rangle$. Consider the chain: $G > PQ > Q\langle abc \rangle > \langle abc \rangle$. Since $Q\langle abc \rangle = N(Q)$ is abnormal, PQ is abnormal. Since P is normal, $\langle abc \rangle$ is subnormal in G, however since R is transitive on the elements of order two, $\langle abc \rangle$ is not normal in G. Note that the h-chain for G is the chain: $G > PQ > Q\langle abc \rangle > Q > \langle 1 \rangle$.

The function h(G) can be viewed as a measure of how far down in the lattice of subgroups one can go without encountering a subnormal subgroup. Another measure of the distribution of subnormal subgroups in the lattice of all subgroups was given by Deskins [3]. He defined the function v(G), called the <u>variance of G</u>, as follows:

Definition 3.6: $v(G)$ is the maximum, taken over all upper chains, of the ratio $\mu(C)/\partial(C)$, where C denotes an upper chain in G, $\mu(C)$ denotes the length of C, and $\partial(C)$ denotes the number of subnormal entries in C, if there are any. If not, then $\partial(C)$ is taken to be one.

Deskins proved that if $v(G) < 4$, then G is solvable. It is clear from the defintion that $v(G) = 1$ if and only if G is nilpotent, in which case $h(G) = 1$. In the proof of theorem 2.13, it is clear that if $h(G) = 2$, then $v(G) = 2$. Whether equality holds in general has not yet been answered. Part of the answer is easy.

Theorem 3.7: In any finite group G ( not necessarily solvable), $h(G) \trianglelefteq v(G)$.

Proof: The h-chain is one of the upper chains to be considered in taking the maximum. ▯

Thus far we have been considering the distribution of subnormal subgroups in G by considering all upper chains. Suppose the upper chains under consideration are only those which pass through or terminate above a specified subgroup of G. Then the distribution might tell something about how this subgroup is contained in the lattice.

Definition 3.8: Let H be a subgroup of G. Define $h(G:H) = n$ if (1) Every upper chain of length n from G to K, where $K \geq H$, contains a proper subnormal entry, and (2) There exists at least one upper chain of length (n-1) from G to L, with $L \geq H$, containing no proper subnormal entry.

In a similar fashion, we define $v(G:H)$. With this definition, $h(G)$ and $v(G)$ are simply $h(G:\langle 1 \rangle)$, and $v(G:\langle 1 \rangle)$ respectively.

If H is a normal subgroup of G, then h(G:H) is simply h(G/H), and all the theorems of chapter II can be applied to G/H, but this does not yeild any essentially new information.

We know that v(G) = 1 if and only if G is nilpotent, so it is natural to consider the class of subgroups K of G satisfying the property v(G:K) = 1. Since D(G), the hypercommutator of G, has the property G/D(G) is nilpotent, D(G) belongs to this class. Maximal subgroups also belong, and since subnormality is invariant under automorphisms, the class is characteristic. One might suspect that D(G) is characterized as minimal in this class. However, in $A_4$, the subgroups of order two have the property that they are contained only in subnormal subgroups, and yet $D(A_4)$ is of order four. In this case the class of subgroups K having v(G:K) = 1 is simply the class of all non-trivial subgroups. We can further restrict our attention to members of the class which are subnormal in G. This subclass, in the case of $A_4$, is contained in the hypercommutator. This property is not generally true as can be seen in the following example:

Let $G = \langle a,b,c,x \rangle$ . $\langle a,b,c \rangle$ is an elementary abelian group of order eight. $X = \langle x \rangle$ is of order three. $a^x = b$, $b^x = ab$, $c^x = bc$. Then $D(G) = \langle a,b \rangle$ . Let $K = \langle a , c \rangle$ . Then v(G:K) = 1, yet K neither contains nor is contained in D(G).

A few remarks concerning the behavior of the function v(G:H) can be made.

Theorem 3.9: If $G \geq K \geq H$, then $v(G:K) \leq v(G:H)$.

Proof: The upper chain in which v(G:K) is assumed is also to be considered in evaluating v(G:H). ▯

Theorem 3.10: If H is a subgroup of G such that $v(G:H) = 1$, then $\langle H^x \mid x \in G \rangle$ contains the hypercommutator of G.

Proof: Let $K = \langle H^x \mid x \in G \rangle$ . Then K is normal in G, and by theorem 3.9, $v(G:K) = 1$. Therefore G/K is nilpotent, and so K contains D(G). ▯

Theorem 3.10 can be extended in the following manner.

Theorem 3.11: If H is a subgroup of G such that $h(G:H) \leq n$, then $\langle H^x \mid x \in G \rangle$ contains $D_n$, the nth term of the lower nilpotent series.

Proof: As in theorem 3.9, if $G \geq K \geq H$, then $h(G:K) \leq h(G:H)$. Let $K = \langle H^x \mid x \in G \rangle$ . Then $h(G/K) = h(G:K) \leq n$. By theorem 2.5, $\ell(G/K) \leq n$, therefore $K \geq D_n(G)$. ▯

Two other methods of generalizing the functions h(G) and v(G) give rise to what might be called the lower variance of a group, and the full variance of a group. For the lower variance, we essentially replace upper chains beginning with G by lower chains beginning with $\langle 1 \rangle$ .

Definition 3.12: A lower chain in G is a sequence of subgroups, $\langle 1 \rangle = H_0 < H_1 < H_2 < \dots < H_n \leq G$, where for each i, $H_i$ is a maximal subgroup of $H_{(i+1)}$. Then lv(G), the lower variance of G, is the maximum, taken over all lower chains of the ratio $\mu(C)/\partial(C)$, where C denotes a lower chain, $\mu(C)$ denotes its length, and $\partial(C)$ the number of subnormal entries in C.

In a similar fashion, we define the function lh(G) as the length of the longest lower chain having only its terminal, i.e., largest, entry subnormal in G.

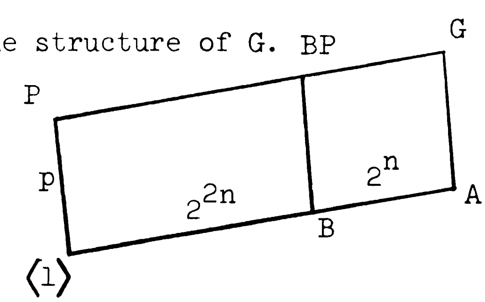
The difficulty that one encounters in working with these functions is that there does not seem to be a direct analogue to the theorem which states that if every maximal subgroup of G is normal, then G is nilpotent. A few results are available. Clearly if $lv(G) = 1$, then G is nilpotent, since every subgroup of G is subnormal. From the structure of the groups described in theorem 2.13, we have the following theorem.

Theorem 3.13: If $h(G) \leq 2$, then $lv(G) \leq 2$.

The converse is not true, in fact we can find a group G having $lv(G) \leq 2$ and $h(G)$ arbitrarily large. By theorem 3.7, $v(G)$ is then also arbitrarily large. Such a group can be exhibited as follows:

Theorem 3.14: Given $n \geq 2$, there exists a group G satisfying: $h(G) \geq n$, and $lv(G) \leq 2$.

Proof: Let p be a prime of the form $p = 2^n k + 1$. Let P be a group of order p. The automorphism group of P is cyclic of order $2^n k$, so contains a cyclic subgroup of order $2^n$. Let A be a cyclic group of order $2^{3n}$. $A = \langle a \rangle, |a| = 2^{3n}$. Let $B = \langle a^{2^n} \rangle$, then $|B| = 2^{2n}$. Let $G = \langle a, P \rangle$ such that B centralizes P, and A/B acts on P as a subgroup of the automorphism group of P. The following diagram illustrates the structure of G.




P is normal in G, B is normal in G, and every subgroup of A that is subnormal in G lies in B. This follows since every subgroup of A that is subnormal in G must lie in a subgroup in A that is normal in G, hence centralizes P. However $C_A(P) = B$. Consider an upper chain in G with A as its first entry. The first subnormal entry in the chain is B, so since $[A:B] = 2^n$, $h(G) \geq (n+1)$. However, if we consider lower chains, the situation is quite different. The only non-subnormal subgroups in G are the subgroups of A containing B. Therefore for a lower chain to contain a non-subnormal entry, it must pass through B. However, there will then be 2n entries in the chain, all subnormal, before a non-subnormal entry is encountered. Since the longest possible chain is $(3n + 1)$ entries long, we have $lv(G) \leq (3n + 1)/2n \leq 2$.∎

We define the notion of full variance in a fashion similar to the variance, the only additional requirement being that the upper chains under consideration must terminate with the identity subgroup. Theorem 3.13 and Theorem 3.14 can be restated with the term "lower variance" replaced by the term "full variance" and they are still valid. This shows that even if the full variance of a group is very small, the variance can be arbitrarily large.

Up to this point the concern has always been with solvable groups. It is natural to ask whether the hypotheses are sufficient to imply solvability. Again in theorem 2.13, it is clear that if $h(G) \leq 2$, then G is solvable. Deskins [3] showed that $v(G) < 4$ implies solvability for G. The proofs of the theorems in [3] can be effectively duplicated to prove the following theorem.

Theorem 3.15: If G is a finite group, and $h(G) \leq 3$, then G is solvable. Moreover if $h(G) \leq 4$ and $(|G|, 3) = 1$, then G is solvable.

Note that if $v(G) < 4$, then $h(G) \leq 3$. The hypothesis $h(G) \leq 4$ is not sufficient to imply solvability. The simple group $A_5$ of order sixty has $h(A_5) = 4$.

## INDEX OF NOTATION

I. Relations:

| | |
|---|---|
| $\leq$ | Is a subgroup of |
| $<$ | Is a proper subgroup of ( $\lneq$ for emphasis) |
| $\triangleleft$ | Is a normal subgroup of |
| $\triangleleft \triangleleft$ | Is a subnormal subgroup of |
| $\in$ | Is an element of |

II. Operations:

| | |
|---|---|
| $G^x$ | $x^{-1}Gx$ |
| $a^x$ | $x^{-1}ax$ |
| $G/H$ | Factor group |
| $[G:H]$ | Index of H in G |
| $\lvert G \rvert$ | The number of elements in G |
| $\lvert x \rvert$ | The order of the element x |
| $\langle \ \rangle$ | Subgroup generated by |
| $\times$ | Direct product of groups |
| $\{ \ \}$ | Set whose elements are |

III. Groups:

| | |
|---|---|
| $Z(G)$ | The center of G |
| $N_G(H)$ | The normalizer of H in G |
| $C_G(H)$ | The centralizer of H in G |
| $\Phi(G)$ | The Frattini subgroup of G |
| $F(G)$ | The Fitting subgroup of G |
| $D(G)$ | The hypercommutator subgroup of G |
| $G'$ | The derived group of G |
| $S_n$ | The symmetric group of degree n |
| $A_n$ | The alternating group of degree n |

IV. Functions:

| | |
|---|---|
| $h(G)$ | Defined on page 16 |
| $v(G)$ | The variance of G, defined on page 37 |
| $\ell(G)$ | The nilpotent length of G |
| $\pi(G:H)$ | The number of distinct prime divisors of $[G:H]$ |
| $r(G)$ | The minimal number of generators of G |
| $lv(G)$ | The lower variance of G, defined on page 39 |
| $h(G:H)$ | Defined on page 37 |
| $v(G:H)$ | Defined on page 37 |

## BIBLIOGRAPHY


1. R. Baer, Sylowturmgruppen II, Mathematische Zeitshcrift, 92 (1966), pp. 256-268.

2. R. W. Carter, On a class of finite solvable groups, Proceedings of the London Mathematical Society, 36 (1959), pp. 623-640.

3. W. E. Deskins, A condition for the solvability of a finite group, Illinois Journal of Mathematics, 2 (1961), pp. 306-313.

4. W. Gaschütz, Zur Theorie der endlichen auflösbaren Gruppen, Mathematische Zeitschrift, 80 (1963), pp. 300-305.

5. P. Hall, A characteristic property of soluble groups, Journal of the London Mathematical Society, 12 ( 1937), pp. 198-200.

6. ______, On the Sylow systems for a soluble group, Proceedings of the London Mathematical Society, 43 ( 1937 ), pp. 316-323.

7. ______, On the system normalizers of a soluble group, Proceedings of the London Mathematical Society, 43 ( 1937), pp. 507-528.

8. B. Huppert, Normalteiler und maximale Untergruppen endlichen Gruppen, Mathematische Zeitschrift, 60 (1954), pp. 409-434.

9. Z. Janko, Finite groups with invariant fourth maximal subgroups, Mathematische Zeitschrift, 82 (1963), pp. 82-89.

10. J. Rose, The influence on a finite group of its proper abnormal structure, Journal of the London Mathematical Society, 40 (1965), pp. 348-361.

11. W.R. Scott, Group Theory, (Englewood Cliffs, New Jersey: Prentice Hall, 1964)