ROLE OF THE RATIONALS IN THE MARKOV AND LAGRANGE SPECTRA

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ABSTRACT

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For each infinite sequence of positive integers $\xi = \{x_i\}$, we let

$$[x_0;x_1,x_2,\ldots,x_n] = x_0 + \frac{1|}{|x_1} + \frac{1|}{|x_2} + \ldots + \frac{1|}{|x_n},$$

$$[x_0;x_1,x_2,x_3,\ldots] = \lim_{n\to\infty} [x_0;x_1,x_2,\ldots,x_n],$$

$$M(\xi,k) = [x_k;x_{k+1},x_{k+2},\ldots] + [0;x_{k-1},x_{k-2},\ldots],$$

$$M(\xi) = \sup_k M(\xi,k), \quad L(\xi) = \overline{\lim_{k\to\infty}} \ M(\xi,k) .$$

The range of $M(\xi)$ is known as the Markov spectrum and the range of $L(\xi)$ as the Lagrange spectrum.

In section 1 it is shown how to construct rationals in the difference set $E_2 \ominus E_2$ and the sum set $E_2 \oplus E_2$ by purely periodic pairs in $E_2 \otimes E_2$. Non-denumerably many pairs in $E_2 \otimes E_2$ are found for each rational so obtained in the difference set. It is also shown that such rationals are dense in the difference set. This shows that there are infinitely many rational values in the Lagrange spectrum.

In section II, it is shown how to choose an interval of the complement of the Lagrange spectrum containing uncountably many points in the Markov spectrum. In fact a non-denumerable set P of points in the Markov spectrum and not in the Lagrange spectrum is found. Moreover, every point in P is a limit point of P.

In section III, the dimension of some level sets corresponding to small values of the Lagrange spectrum are shown to be small.

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CHAPTER I

INTRODUCTION

The Markov and Lagrange spectra are defined as follows: For $x \in A$, the set of all sequences $x_0, x_1, x_2, \ldots, x_k, \ldots$ of natural numbers, we let

$$[x_0; x_1, x_2, \ldots, x_n] = x_0 + \frac{1|}{|x_1} + \frac{1|}{|x_2} + \ldots + \frac{1|}{|x_n}$$

$$[x_0; x_1, x_2, x_3, \ldots] = \lim_{n\to\infty} [x_0; x_1, x_2, \ldots, x_n] \ .$$

Let D be the set of doubly infinite sequences of positive integers $\xi = \{x_i\}_{-\infty<i<\infty}$. Define the shift transformation $L^j\xi = \{x_{i+j}\}_{-\infty<i<\infty}$, $j = \pm 1, \pm 2, \ldots$ We let

1) $M(\xi, k) = [x_k; x_{k+1}, x_{k+2}, \ldots] + [0; x_{k-1}, x_{k-2}, \ldots]$,

2) $M(\xi) = \sup\limits_k M(\xi, k)$, $L(\xi) = \overline{\lim\limits_{k\to\infty}} M(\xi, k)$.

The range of $M(\xi)$ is known as the Markov spectrum, the range of $L(\xi)$ is known as the Lagrange spectrum. The first function arises in the study of the maxima of binary quadratic forms [2], [8], [15], the second in the approximation of real numbers by rationals. O. Perron [2] noted the connection between the two and showed that the lowest parts of the spectra are a discrete sequence of values approaching 3 from below, and corresponding to the numbers most poorly approximable by rationals. He also noted that the range where the x_i are restricted to 1 and 2 is always less than the range where threes appear in the expansions. Hall [13] showed that the sum and difference sets of the Cantor sets formed by continued fractions allowing 1,2,3,4 as partial quotients filled an interval, from which he deduced [19] that the ranges take on all values above 5.

The work of Hall was difficult to extend to more restricted sets of continued fractions but [18], [14] and [4] showed that the lower ranges of the spectra are relatively sparse, in fact, the portion below $\sqrt{10}$ is of measure zero. It was recently established [6] that the Lagrange spectrum is the closure of the range over periodic sequences. The Markov spectrum was known to include that of Lagrange and also to be closed. It was only recently established [11], [5] that the Markov and Lagrange spectra do not coincide. Kogonija [20] gave a sufficient condition for the two to coincide above $\sqrt{10}$, namely that the lengths of the repeated 12 blocks in the continued fraction expansion be bounded. The sufficient conditions for the two to not coincide was given in [6] recently. Some of these conditions are shown [5] to be necessary in certain intervals. Nevertheless, this theorem [5] does not insure the existence of a non-coincidence in these intervals. Hightower [15] found countably many gaps above 3.

In section 1 it is shown that the rationals are dense in the difference set $E_2 \ominus E_2$ and sum set $E_2 \oplus E_2$, where E_2 is the Cantor set of continued fractions with entries 1,2. It is shown how to construct rationals in the sum and difference sets by purely periodic pairs.

A point $w \in E_2$ with continued fraction expansion $[a_0;a_1,a_2,\ldots]$ is said to be purely periodic if there exists a non-negative integer k such that $a_i = a_{i+k}$ for all integers i. Two distinct pairs in $E_2 \otimes E_2$ are found for each rational so obtained in the difference set. An interleaving technique is developed showing that the set of pairs in $E_2 \otimes E_2$ whose difference is such a rational is non-denumerable. A way of finding another rational in the difference set if one rational in the difference set is known is also given. Many rationals in the sum set can be obtained from the difference set. The hope here is to extend the methods of [18], [14], [4], to restrict the dimension of the spectra in the range $[3, 2\sqrt{3}]$.

In section II, the non-coincidence of the Markov and Lagrange spectra is considered. It is shown how to choose an interval with uncountably many Markov points which does not contain any Lagrange points. In fact a non-denumerable set of points in the Markov spectrum with no limit point in the Lagrange spectrum is found.

In section III level sets of the Lagrange spectrum are discussed. The dimension of one of those corresponding to small values of the Lagrange spectrum are shown to be small.

CHAPTER II

RATIONAL APPROXIMATION

If $w \in [0,1)$ we define $Tw = \begin{cases} \frac{1}{w} - [\frac{1}{w}] & \text{if } w \neq 0 \\ 0 & \text{if } w = 0 \end{cases}$

$a_n(w) = [\frac{1}{T^{n-1}w}]$ $n = 1,2,\ldots$ where $[x]$ is greatest integer $< x$.

$p_{-1}(w) = 1,\ p_0(w) = 0,\ q_{-1}(w) = 0,\ q_0(w) = 1$

$p_n(w) = a_n(w)p_{n-1}(w) + p_{n-2}(w),\ q_n(w) = a_n(w)q_{n-1}(w) + q_{n-2}(w)$

i.e.

$\frac{p_n(w)}{q_n(w)} = [0;a_1(w),a_2(w),\ldots,a_n(w)]$

$w \in E_2$ if and only if $a_n(w) = 1$ or 2 $n = 1,2,\ldots$

If a and b are the words $a_1a_2\ldots a_n$ and $b_1b_2\ldots b_m$ respectively, we

define a^{-1} to be the inverse word $a_na_{n-1}\ldots a_2a_1$,

$a \circ b$ the composition word $a_1a_2\ldots a_nb_1b_2\ldots b_m$,

La the left shift word $a_2a_3\ldots a_na_1$,

$a^n = \overbrace{a \circ a \circ \ldots \circ a}^{n \text{ times}}$.

$\bar{a} = [0;\overline{a_1,a_2,\ldots,a_n}] = [0;a_1,a_2,\ldots]$ where $a_{n+j} = a_j$,

and we define $b*w = [0;b_1,b_2,\ldots,b_m,a_1(w),a_2(w),\ldots]$,

If $P_1,P_2,\ldots$ is a sequence of words, we define

$$\prod_{i=1}^{\infty} P_i = P_1 * P_2 * \ldots = [0;P_{11},P_{12},\ldots,P_{1n_1},P_{21},P_{22},\ldots,P_{2n_2},\ldots]$$

where $P_i = P_{i1}P_{i2}\ldots P_{in_i}$.

$M_k = \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}$ where k is an integer

$M_a = M_{a_1} \cdot M_{a_2} \ldots M_{a_n}$.

Remark 0.0. $a_1(w),a_2(w),\ldots$ are the partial quotients in the continued-fraction expansion of w.

(0-1) $\quad p_{n-1}(w)q_n(w) - p_n(w)q_{n-1}(w) = (-1)^n \qquad n \geq 0$

(0-2) $\quad M_k^{-1} = \begin{pmatrix} -k & 1 \\ 1 & 0 \end{pmatrix}$

(0-3) $\quad M_a = \begin{pmatrix} p_{n-1}(\bar{a}) & p_n(\bar{a}) \\ q_{n-1}(\bar{a}) & q_n(\bar{a}) \end{pmatrix}$

(0-4) $\quad M_{a^{-1}} = M_a^t = \begin{pmatrix} p_{n-1}(\bar{a}) & q_{n-1}(\bar{a}) \\ p_n(\bar{a}) & q_n(\bar{a}) \end{pmatrix}$

since $(AB)^t = B^tA^t$ for any matrices A and B

and $M_k^t = M_k$.

Lemma 1. Let a be the word $a_1a_2\ldots a_n$, $x = \bar{a}$ and $y = \overline{La^{-1}}$. Then

$$q_{n-1}(y) = q_{n-1}(x),\ p_{n-1}(y) = q_n(x) - a_nq_{n-1}(x)$$

$$q_n(y) = p_{n-1}(x) + a_nq_{n-1}(x) \quad \text{and} \quad p_n(y) = p_n(x) - a_n(p_{n-1}(x) - q_{n-2}(x))$$

Proof. By preceding remarks,

$$M_{La^{-1}} = M_{a_n}^{-1}M_{a^{-1}}M_{a_n} = \begin{pmatrix} -a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_{n-1}(x) & q_{n-1}(x) \\ p_n(x) & q_n(x) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$$

$$= \begin{pmatrix} -a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_{n-1}(x) & p_{n-1}(x) + a_nq_{n-1}(x) \\ q_n(x) & p_n(x) + a_nq_n(x) \end{pmatrix}$$

$$= \begin{pmatrix} q_n(x) - a_nq_{n-1}(x) & p_n(x) - a_np_{n-1}(x) + a_n(q_n(x) - a_nq_{n-1}(x)) \\ q_{n-1}(x) & p_{n-1}(x) + a_nq_{n-1}(x) \end{pmatrix}.$$

Theorem 1. If $x = \bar{a}$ is purely periodic in E_2 with period $a = a_1a_2\ldots a_n$, $n \geq 2$ and $y = \overline{La^{-1}}$, then $x - y$ is a rational and

$$x - y = [p_{n-1}(x) - q_{n-2}(x)]/q_{n-1}(x) = [0;a_1,\ldots,a_{n-1}] - [0;a_{n-1},\ldots,a_1].$$

Proof: Let $y = \dfrac{-B + \sqrt{D}}{Q}$. By the preceding lemma,

$$Q = 2q_{n-1}(x),\ B = p_{n-1}(x) - q_n(x) + 2a_nq_{n-1}(x)$$

$$D = [(p_{n-1}(x) - q_n(x)) + 2a_nq_{n-1}(x)]^2 + 4q_{n-1}(x)[p_n(x) - a_n(p_{n-1}(x) - q_n(x) + a_nq_{n-1}(x))] = (q_n(x) - p_{n-1}(x))^2 + 4q_{n-1}(x)p_n(x).$$

But $x = (p_{n-1}(x) - q_n(x) + \sqrt{D})/2q_{n-1}(x)$.

Hence $x - y = [2(p_{n-1}(x) - q_n(x)) + 2a_nq_{n-1}(x)]/2q_{n-1}(x)$

$$= [p_{n-1}(x) - q_{n-2}(x)]/q_{n-1}(x) .$$

Corollary 1. If $X = \dfrac{-P + \sqrt{D}}{Q}$ is purely periodic in E_2 with period $a = a_1a_2...a_n$ where P, Q and D are positive integers, then

1. $\frac{P}{Q} = 1$ iff $a = La^{-1}$ and $a_n = 2$

2. $\frac{P}{Q} = \frac{1}{2}$ iff $a = La^{-1}$ and $a_n = 1$.

Proof: $x = [p_{n-1}(x) - q_n(x) + \sqrt{D}]/2q_{n-1}(x)$ for some positive integer D.

By preceding theorem,

$a = La^{-1}$

if and only if $p_{n-1}(x) - q_{n-2}(x) = 0$ i.e. $p_{n-1}(x) - q_n(x) = -a_nq_{n-1}(x)$

if and only if $x = -\dfrac{a_n}{2} + \dfrac{\sqrt{D}}{2q_{n-1}(x)}$.

Corollary 2. If $x = \bar{a}$ is purely periodic in E_2 with period $a = a_1a_2...a_{n-1}a_n$, $n \geq 2$, then $x' = \bar{a}'$ with $a' = a_1,...,a_{n-1}a_n'$ and $a_n' = 3 - a_n$ is purely periodic in E_2 such that

$$x' - \overline{La'^{-1}} = x - \overline{La^{-1}}.$$

Proof: This result follows from Theorem 1 since $x'\text{-}L\overline{a'^{-1}}$ and $x\text{-}L\overline{a^{-1}}$ are independent of a'_n and a_n respectively.

Lemma 2. $[0;a_1,a_2,\ldots,a_n+t] = \dfrac{p_n + tp_{n-1}}{q_n + tq_{n-1}}$ holds for all positive integers n and any number t, where $\dfrac{p_n}{q_n} = [0;a_1,a_2,\ldots,a_n]$, $q_0=1$, $p_0=0$.

Remark 1. $\dfrac{a + Db}{c + Dd} + \dfrac{a-Db}{c-Db} = \dfrac{2(ac - D^2bd)}{c^2 - D^2d^2}$ for any numbers a,b,c,d,D.

Lemma 3. If $x \in E_2$ and $(a_1(x),a_2(x)) \neq (1,2)$ then $1 - x \in E_2$.

Proof: If $x = \dfrac{1}{1 + \frac{1}{1+t}}$ where $t \in E_2$ then

$$1 - x = 1 - \frac{1}{1 + \frac{1}{1+t}} = 1 - \frac{1 + t}{2 + t} = \frac{1}{2+t} \in E_2 .$$

Theorem 2. If P is purely periodic in E_2 with period $p = 1p_2p_3\ldots p_{n-1}2$ and if $p = Lp^{-1}$, then there exist positive integers m and D such that $P = -1 + \dfrac{\sqrt{D}}{m}$. Let $x = [0;a_1,a_2,\ldots,a_{n-1}, 2 + \frac{m}{\sqrt{D}}]$ and let $y = [0;a_1,a_2,\ldots,a_{n-1}, 2 - \frac{m}{\sqrt{D}}]$ where $a_i = 1$ or 2 for all $i<n$ and any positive integer n, then $x + y$ is rational in $E_2 \oplus E_2$.

Proof: By Corollary 1, $P = -1 + \dfrac{\sqrt{D}}{m}$ for some positive integers m and D.

$$\frac{m}{\sqrt{D}} = \frac{1}{1+p} = \frac{1}{1 + \frac{1}{1+t}} \quad \text{for some } t \in E_2.$$

So $\frac{m}{\sqrt{D}} \in E_2$ and by Lemma 3, $1 - \frac{m}{D} \in E_2$ and hence $x + y \in E_2 \oplus E_2$.

By Lemma 2

$$x = \frac{p_n + \frac{m}{\sqrt{D}} p_{n-1}(x)}{q_n + \frac{m}{\sqrt{D}} q_{n-1}(x)} \quad \text{and} \quad y = \frac{p_n - \frac{m}{\sqrt{D}} p_{n-1}(x)}{q_n - \frac{m}{\sqrt{D}} q_{n-1}(x)}$$

where $p_n = 2p_{n-1}(x) + p_{n-2}(x)$ and $q_n = 2q_{n-1}(x) + q_{n-2}(x)$. Also by Remark 1

$$x + y = \frac{2(p_n \cdot q_n - \frac{m^2}{D} p_{n-1}(x) q_{n-1}(x))}{q_n^2 - \frac{m^2}{D} q_{n-1}^2(x)} .$$

Therefore $x + y$ is rational.

Theorem 3. If P is purely periodic in E_2 with period $p = 2p_2p_3 \cdots p_{n-1}1$ and $p = Lp^{-1}$, then there exist positive integers D and m such that $P = -\frac{1}{2} + \frac{\sqrt{D}}{m}$. Let $x = [0; a_1, a_2, \ldots, a_{n-2}, 1, 2 + p]$ and let $y = [0; a_1, a_2, \ldots, a_{n-2}, 1, 1-p]$ for any positive integer n where $a_i = 1$ or 2 $\forall$ $i < n$, then $x + y$ is rational in $E_2 \oplus E_2$.

Proof: By Corollary 1, $P = -\frac{1}{2} + \frac{\sqrt{D}}{m}$ for some positive integers D and m. Clearly $x \in E_2$ and $\frac{1}{1-p} = \frac{2+s}{1+s} = 1 + \frac{1}{1+s}$ where $P = \frac{1}{2+s}$ for some

$s \in E_2$ so $y = [0;a_1,\ldots,a_{n-2},2,1+s] \in E_2$.

$x = [0;a_1,a_2,\ldots,a_{n-2},1,\ 1+(\frac{1}{2}+\frac{\sqrt{D}}{m})]$, $y = [0;a_1,a_2,\ldots,a_{n-2},1,1+(\frac{1}{2}-\frac{\sqrt{D}}{m})]$.

By Lemma 2

$$x = \frac{p_n + (\frac{1}{2}+\frac{\sqrt{D}}{m})p_{n-1}(x)}{q_n + (\frac{1}{2}+\frac{\sqrt{D}}{m})q_{n-1}(x)}\ ,\ y = \frac{p_n + (\frac{1}{2}-\frac{\sqrt{D}}{m})p_{n-1}(x)}{q_n + (\frac{1}{2}-\frac{\sqrt{D}}{m})q_{n-1}(x)}$$

where $p_n = p_{n-1}(x) + p_{n-2}(x) = p_{n-1}(y) + p_{n-2}(y)$,

$q_n = q_{n-1}(x) = q_{n-2}(x) = q_{n-1}(y) + q_{n-2}(y)$.

By Remark 1,

$$x + y = \frac{2[(p_n + \frac{1}{2}p_{n-1}(x))(q_n + \frac{1}{2}q_{n-1}(x)) - \frac{D}{m^2}p_{n-1}(x)q_{n-1}(x)]}{(q_n + \frac{1}{2}q_{n-1}(x))^2 - \frac{D}{m^2}q_{n-1}^2(x)}\ .$$

Therefore $x + y$ is rational in $E_2 \oplus E_2$.

Theorem 4. If $y, y' \in E_2$ and x is purely periodic in E_2 with period $a = a_1a_2\ldots a_n$ and if $y - y' = x - \overline{La^{-1}}$ then $a * y - La^{-1} * y' = x - \overline{La^{-1}}$.

Proof: By Theorem 1, $x - \overline{La^{-1}} = \dfrac{p_{n-1}(x) - q_{n-2}(x)}{q_{n-1}(x)} = y - y'$ ... (A)

$$a * y = \frac{p_{n-1}(x)y + p_n(x)}{q_{n-1}(x)y + q_n(x)} ,$$

$$La^{-1} * y' = \frac{p_{n-1}(\overline{La^{-1}})\cdot y' + p_n(\overline{La^{-1}})}{q_{n-1}(\overline{La^{-1}})\cdot y' + q_n(\overline{La^{-1}})}$$

$$= \frac{q_{n-2}(x)(y - \frac{p_{n-1}(x) - q_{n-2}(x)}{q_{n-1}(x)}) + p_n(x) - a_n(p_{n-1}(x) - q_{n-2}(x))}{q_{n-1}(x)(y - \frac{p_{n-1}(x) - q_{n-2}(x)}{q_{n-1}(x)}) + p_{n-1}(x) + a_n q_{n-1}(x)}$$

$$= \frac{q_{n-2}(x)y + p_n(x) - (\frac{q_{n-2}(x)}{q_{n-1}(x)} + a_n)(p_{n-1}(x) - q_{n-2}(x))}{q_{n-1}(x)y + q_{n-2}(x) + a_n q_{n-1}(x)}$$

$$= \frac{q_{n-2}(x)y + p_n(x) - \frac{q_n(x)}{q_{n-1}(x)}(p_{n-1}(x) - q_{n-2}(x))}{q_{n-1}(x)y + q_n(x)}, \text{ by Lemmas 1, 2.}$$

$$a * \bar{b} - La^{-1} * \overline{Lb^{-1}} = (p_{n-1}(x) - q_{n-2}(x))(y + \frac{q_n(x)}{q_{n-1}(x)})/[q_{n-1}(x)y + q_n(x)]$$

$$= \frac{p_{n-1}(x) - q_{n-2}(x)}{q_{n-1}(x)} = x - \overline{La^{-1}} .$$

Corollary 3. If x and y are purely periodic in E_2 with periods $a = a_1a_2\ldots a_n$ and $b = b_1b_2\ldots b_m$ respectively and

$x - \overline{La^{-1}} = y - \overline{Lb^{-1}}$ and if $p_1, p_2, \ldots, p_n, \ldots$ is a sequence of words

with $p_i = a$ or b, $i = 1,2,\ldots$ then

(A) $\overline{p_1 \circ p_2 \circ \ldots \circ p_n} - \overline{Lp_1^{-1} \circ Lp_2^{-1} \circ \ldots \circ Lp_n^{-1}} = x - \overline{La^{-1}}$

(B) $\prod_{i=1}^{\infty} p_i - \prod_{i=1}^{\infty} Lp_i^{-1} = x - \overline{La^{-1}}$

Proof: By the preceding theorem, $p_n * \bar{a} - Lp_n^{-1} * \overline{La^{-1}} = x - \overline{La^{-1}}$. Again, by the preceding theorem, $p_{n-1} * p_n * \bar{a} - Lp_{n-1}^{-1} * Lp_n^{-1} * \overline{La^{-1}} = x - \overline{La^{-1}}$. Continuing this procedure, we have $p_1 * p_2 * \ldots * p_n * \bar{a} - Lp_1^{-1} * Lp_2^{-1} * \ldots * Lp_n^{-1} * \overline{La^{-1}} = x - \overline{La^{-1}}$ i.e. $(p_1 \circ p_2 \circ \ldots \circ p_n) * \bar{a} - (Lp_1^{-1} \circ Lp_2^{-1} \circ \ldots \circ Lp_n^{-1}) * \overline{La^{-1}} = x - \overline{La^{-1}}$. If we go through the whole procedure one more time, we get

$$(p_1 \circ p_2 \circ \ldots \circ p_n)^2 * \bar{a} - (Lp_1^{-1} \circ Lp_2^{-1} \circ \ldots \circ Lp_n^{-1})^2 * \overline{La^{-1}} = x - \overline{La^{-1}}.$$

Using the same procedure, we get

$$(p_1 \circ p_2 \circ \ldots \circ p_n)^n * \bar{a} - (Lp_1^{-1} \circ Lp_2^{-1} \circ \ldots \circ Lp_n^{-1})^n * \overline{La^{-1}} = x - \overline{La^{-1}}.$$

Since $\overline{p_1 \circ p_2 \circ \ldots \circ p_n} = \lim_{n\to\infty} (p_1 \circ p_2 \circ \ldots \circ p_n)^n * \bar{a}$,

$$\overline{Lp_1^{-1} \circ Lp_2^{-1} \circ \ldots \circ Lp_n^{-1}} = \lim_{n\to\infty} (Lp_1^{-1} \circ Lp_2^{-1} \circ \ldots \circ Lp_n^{-1})^n * \overline{La^{-1}},$$

$$\prod_{i=1}^{\infty} p_i = \lim_{n\to\infty} p_1 * p_2 * \ldots * p_n * \bar{a},$$

and $\prod_{i=1}^{\infty} Lp_i^{-1} = \lim_{n\to\infty} Lp_1^{-1} * Lp_2^{-1} *\ldots* Lp_n^{-1} * \overline{La^{-1}}$.

Hence $\overline{p_1 \circ p_2 \circ\ldots\circ p_n} - \overline{Lp_1^{-1} \circ Lp_2^{-1} \circ\ldots\circ Lp_n^{-1}} = x - \overline{La^{-1}}$

and $\prod_{i=1}^{\infty} p_i - \prod_{i=1}^{\infty} Lp_i^{-1} = x - \overline{La^{-1}}$.

Theorem 5. If x and y satisfy the hypothesis of the preceding corollary and $a \neq La^{-1}$ then there exists a non-denumerable set of pairs (α, β) in $E_2 \otimes E_2$ such that $\alpha - \beta = x - \overline{La^{-1}}$.

Proof: Without loss of generality, we may assume $a_k \neq b_k$ for some k.

If $w \in (0,1)$ has the binary expansion $w = \sum_{i=1}^{\infty} w_i \cdot 2^{-i}$, we let $\alpha_w = p(w_1) * p(w_2) * p(w_3) *\ldots= \prod_{i=1}^{\infty} p(w_i)$ where $p(0) = a$, $p(1) = b$ and let $\beta_w = \prod_{i=1}^{\infty} L(p(w_i))^{-1}$. Then $\alpha_w - \beta_w = x - \overline{La^{-1}}$.

If $w, w' \in (0,1)$ with $w \neq w'$ i.e. there exists a first integer ℓ such that $w_\ell \neq w'_\ell$. Then

$$(p(w_\ell))_k \neq (p(w'_\ell))_k \text{ and thus } \alpha_w \neq \alpha_{w'}.$$

Hence there exists a non-denumerable set of pairs (α, β) in $E_2 \otimes E_2$ with

$$\alpha - \beta = x - \overline{La^{-1}}.$$

Notation: Let Q be the set of all rationals obtained from Theorem 1.

Corollary 4. If $q \in Q$, then there exists a non-denumerable set of pairs (α, β) in $E_2 \otimes E_2$ such that $\alpha - \beta = q$.

Proof: Immediate result from Corollary 2 and Theorem 5.

Theorem 6. If x and y are purely periodic in E_2 with periods $a = a_1 \dots a_{n-1} 1$ and $b = a_1 \dots a_{n-1} 2$ respectively and $k = 1$ or 2 then

$$\overline{k \circ a} - \overline{L(k \circ a)^{-1}} = \overline{k \circ b} - \overline{L(k \circ b)^{-1}} = \frac{q_{n-1}(x) - kq_{n-2}(x) - p_{n-2}(x)}{p_{n-1}(x) + kq_{n-1}(x)}.$$

Proof: By Theorem 1,

$$\overline{k \circ a} - \overline{L(k \circ a)^{-1}} = \overline{k \circ b} - \overline{L(k \circ b)^{-1}}$$

$$= \frac{p_n(\overline{k \circ a})}{q_n(\overline{k \circ a})} - \frac{q_{n-1}(\overline{k \circ a})}{q_n(\overline{k \circ a})}$$

$$= \frac{1}{k + \dfrac{p_{n-1}(x)}{q_{n-1}(x)}} - \frac{\frac{1}{k} p_{n-2}(x) + q_{n-2}(x)}{\frac{1}{k} p_{n-1}(x) + q_{n-1}(x)}$$

$$= \frac{q_{n-1}(x)}{p_{n-1}(x) + kq_{n-1}(x)} - \frac{p_{n-2}(x) + kq_{n-2}(x)}{p_{n-1}(x) + kq_{n-1}(x)}$$

$$= \frac{q_{n-1}(x) - kq_{n-2}(x) - p_{n-2}(x)}{p_{n-1}(x) + kq_{n-1}(x)} .$$

Theorem 7. $E_2 \ominus E_2 = \overline{Q}$.

Proof: If $x \in E_2 \ominus E_2$, then there exist $\alpha, \beta \in E_2$ such that

$x = \alpha - \beta$.

If $\varepsilon > 0$, there exist positive integers n and m such that

$|\alpha-[0,a_1(\alpha),a_2(\alpha),\ldots,a_n(\alpha)]| < \frac{\varepsilon}{2}$ and $|\beta-[0;a_1(\beta),a_2(\beta),\ldots,a_m(\beta)]| < \frac{\varepsilon}{2}$

Let w be the word $a_1(\alpha)a_2(\alpha)\ldots a_n(\alpha)a_m(\beta)a_{m-1}(\beta)\ldots a_1(\beta)1$.

Let $\alpha^* = \bar{w}$ and $\beta^* = \overline{Lw^{-1}}$. Then $\alpha^* - \beta^* \in Q$ and

$$|\alpha - \beta - (\alpha^* - \beta^*)| \leq |\alpha - \alpha^*| + |\beta - \beta^*| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies the statement of the theorem.

Lemma 4. If $x,y \in E_2$ with rational $x - y$ and either

$(a_1(x),a_2(x)) \neq (1,2)$ or $(a_1(y),a_2(y)) \neq (1,2)$ then either $y - x + 1$

or $x - y + 1$ is rational in $E_2 \oplus E_2$.

Proof:

If $(a_1(x),a_2(x)) \neq (1,2)$ then by Lemma 3, $1 - x \in E_2$. Thus

$y - x + 1$ is a rational in $E_2 \oplus E_2$. Similarly, $(a_1(y),a_2(y)) \neq (1,2)$

implies $x - y + 1$ is a rational in $E_2 \oplus E_2$.

Examples

Example 1. Let a and b be the words 121 and 122 respectively and let

$x = \bar{a} = [0;\overline{1,2,1}]$, $y = \bar{b} = [0;\overline{1,2,2}]$. Then $x = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1+x}}} = \dfrac{2x+3}{3x+4}$, so

$3x^2 + 2x - 3 = 0$, so $x = \dfrac{-1+\sqrt{10}}{3}$.

$$\overline{La^{-1}} = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \overline{La^{-1}}}}} = \frac{\overline{La^{-1}} + 2}{3\overline{La^{-1}} + 5}, \quad 3\overline{La^{-1}}^2 + 4\overline{La^{-1}} - 2 = 0 \text{ so}$$

$\overline{La^{-1}} = \dfrac{-2+\sqrt{10}}{3}$. Thus $x - \overline{La^{-1}}$ is the rational $\dfrac{1}{3} = \dfrac{p_2(x) - q_1(x)}{q_2(x)} =$

$\dfrac{2-1}{3}$. $y = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2+y}}} = \dfrac{2y+5}{3y+7}$, $3y^2 + 5y - 5 = 0$, so $y = \dfrac{-5+\sqrt{85}}{6}$.

$$\overline{Lb^{-1}} = \cfrac{1}{2 + \cfrac{1}{2 + \overline{Lb^{-1}}}} = \frac{\overline{Lb^{-1}} + 3}{3\overline{Lb^{-1}} + 8}, \quad 3\overline{Lb^{-1}}^2 + 7\overline{Lb^{-1}} - 3 = 0, \text{ so } \overline{Lb^{-1}} = \frac{-7+\sqrt{85}}{6}.$$

Thus $y - \overline{Lb^{-1}}$ is also equal to $\dfrac{1}{3} = \dfrac{p_2(y) - q_1(y)}{q_2(y)} = \dfrac{2-1}{3}$. Moreover,

$p_2(x) = p_2(y) = 2$, $q_2(x) = q_2(y) = 3$. By Theorem 6,

$$[0;\overline{1,1,2,1}] - [0;\overline{2,1,1,1}] = [0;\overline{1,1,2,2}] - [0;\overline{2,1,1,2}]$$

$$= \frac{3 - 1 - 1}{2 + 3}$$

$$= \frac{1}{5}.$$

We observe that there are two distinct pairs (α,β) with $\alpha \neq \beta$ for both $\alpha - \beta = \frac{1}{3}$ and $\alpha - \beta = \frac{1}{5}$. By Theorem 5, there is a non-denumerable set of pairs (α,β) for both $\alpha - \beta = \frac{1}{3}$ and $\alpha - \beta = \frac{1}{5}$. In fact, by Lemma 4, there is a non-denumerable set of pairs (α,β) for both $\alpha + \beta = \frac{4}{3}$ and $\alpha + \beta = \frac{6}{5}$ in $E_2 \oplus E_2$.

Example 2. Let p be the word $1\ 2$ and $p = \bar{p}$. Then $p = Lp^{-1}$ and $p = [0,\overline{1,2}] = -1 + \sqrt{3}$. Let $x = \dfrac{1}{2 + \frac{1}{\sqrt{3}}}$ and $y = \dfrac{1}{2 - \frac{1}{\sqrt{3}}}$. Since

$\dfrac{1}{\sqrt{3}} = \dfrac{1}{1+p} = [0;\ 1,\overline{1,2}]$, $x \in E_2$. and

$$y = \cfrac{1}{1 + 1 - \frac{1}{\sqrt{3}}} = \cfrac{1}{1 + \cfrac{1}{\frac{\sqrt{3}}{\sqrt{3}-1}}} = \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{\sqrt{3}-1}}} = \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{p}}} = [0;1,2,\overline{2,1}]\ .$$

Thus $x + y = \dfrac{4}{4-\frac{1}{3}} = \dfrac{12}{11}$ is a rational in $E_2 \oplus E_2$.

Example 3. Let p' be the word $2\ 1$ and $p' = \overline{p'}$. Then $p' = Lp'^{-1}$ and $p' = [0;\overline{2,1}] = \dfrac{-1 + \sqrt{3}}{2}$. Let $x = [0;1,2 + p']$ and $y = [0;1,1-p']$

Then $x = \dfrac{1}{1 + \dfrac{1}{2 + \dfrac{\sqrt{3}-1}{2}}} = \dfrac{3 + \sqrt{3}}{5 + \sqrt{5}}$ and $y = \dfrac{1}{1 + \dfrac{1}{1 - \dfrac{\sqrt{3}-1}{2}}} = \dfrac{3 - \sqrt{3}}{5 - \sqrt{3}}$.

Clearly $x \in E_2$.

$1 - p' = 1 - \dfrac{1}{2+p} = \dfrac{1+p}{2+p} = \dfrac{1}{1 + \dfrac{1}{1+p}}$ where $p = [0;\overline{1,2}]$ so

$$y = \frac{1}{1 + \frac{1}{1-p'}} = \frac{1}{1 + 1 + \frac{1}{1+p}} = [0;2,1,\overline{1,2}] \in E_2.$$

$$x + y = \frac{2(15-3)}{25-3} = \frac{24}{22} = \frac{12}{11} .$$

Thus there is another pair (x,y) with $x + y = \frac{12}{11}$ in $E_2 \oplus E_2$.

CHAPTER III

NON-COINCIDENCE OF THE MARKOV AND LAGRANGE SPECTRA

For each $x = \{x_k\} \in A$, we define $M(x,k) = [x_k;x_{k+1},\ldots] + [0;x_{k-1},x_{k-2},\ldots,x_1,x_0]$. $M(x) = \sup_k M(x,k)$ and $L(x) = \overline{\lim_k} M(x,k)$.

The ranges of M and L are known as the Markov spectrum and the Lagrange spectrum respectively. It is known that the Markov spectrum is closed and contains the Lagrange spectrum.

For each $x = \{x_k\} \in A$, we define

$$I(x,n) = \{y = \{y_k\} \in A : y_k = x_k,\ k = 1,2,\ldots,n\}.$$

Let $\varepsilon_\ell(n) = L\cdot(1+\sqrt{2})^{-2n}$ and $\varepsilon_u(n) = K\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{-2n}$.

We then state the following two lemmas without proof.

Lemma 1. If $y \in I(x,n) \cap [I(x,n+1)]^c$, then $\varepsilon_\ell(n) < |M(x,0) - M(y,0)| < \varepsilon_u(n)$

Remark 1. $|M(x,0) - M(y,0)| < 10^{-7}$ if $n = 19$

and

$|M(x,0) - M(y,0)| < 10^{-11}$ if $n = 29$,

Lemma 2. For $\xi,\eta \in E_2$

$$[0;x_1,x_2,\ldots,x_{2n+1},2+\xi] \geq [0;x_1,\ldots,x_{2n+1},1+\eta]$$

$$[0;x_1,x_2,\ldots,x_{2n},1+\eta] \geq [0;x_1,\ldots,x_{2n},2+\xi] .$$

Let $\xi = \{\xi_i\} \in D$ be a sequence such that $\xi_i = \xi_{i-7}$, $i \geq -4$;

$\xi_{-5} = \xi_{-6} = 1$, $\xi_{-7} = 2$, $\xi_{-8} = 1$, $\xi_{-9} = \xi_{-10} = \xi_{-11} = \xi_{-12} = \xi_{-13} = \xi_{-14} =$

$\xi_{-15} = \xi_{-16} = 2$, $\xi_{-17} = 1$, $\xi_{-18} = 2$, $\xi_{-19} = \xi_{-20} = 1$; $\xi_i = \xi_{i+7}$, $i \leq -21$.

Let a be the word $1\ 1\ 2\ 2\ 2\ 1\ 2$.

$$M(\xi,0) = \bar{a} + 2 + [0;1,2,2,2,1,1,2,1,2,2,2,2,2,\overline{2,2,2,1,2,1,1}] \simeq 3.2930442654$$

Let $\alpha = M(\xi,0)$. Clearly, $M(\xi,k) < M(\xi,0)$ if $\xi_k = 1$, $(\xi_{k-1},\xi_k) =$ $(2,2)$ or $(\xi_k,\xi_{k+1}) = (2,2)$. Without loss of generality, due to the symmetry $\xi_{-12+i} = \xi_{-13-i}$, $i = 0,1,2,\ldots$ we may assume that $M(\xi)$ can only occur at ξ_k for $k = 7n$, $n = -1,0,1,2,\ldots$. Since $(L^{-7}\xi)_i = \xi_i$, $i \geq -4$, $(L^{-7}\xi)_{-5} = 2$ and $\xi_{-5} = 1$ we have, by Lemma 2,

$$M(L^{-7}\xi,0) < \alpha \quad \text{i.e.}$$

$$M(\xi,-7) < \alpha.$$

Since $(L^{7n}\xi)_i = \xi_i$, $i \geq -11$, $\xi_{-12} = 2$ and $(L^{7n}\xi)_{-12} = 1$, $n = 1,2,\ldots$ we have, by Lemma 2

$$M(\xi,7n) < \alpha, \; n = 1,2,\ldots$$

Hence $M(\xi) = \alpha$.

We define words $W^i = W_1^i W_2^i \ldots W_{n_i}^i$, $i = 1,2,\ldots$, as follows:

$W^1 = 2\ 1\ 2\ 1\ 2$

$W^2 = 2\ 1\ 2\ 1\ 1\ 2$

$W^3 = 1\ 2\ 1\ 2\ 1\ 1\ 2$

$W^4 = 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 1$

$W^5 = 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 2$

$W^6 = 2\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1$

$W^7 = 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 1$

$W^8 = 1\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2$

$W^9 = 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 2$

$W^{10} = 2\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2$

$W^{11} = 1\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2$

$W^{12} = 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 1$

$W^{13} = 1\ 2\ 2$

$W^{14} = 1\ 1\ 2\ 1\ 1$

$W^{15} = 2\ 2\ 1\ 2\ 1\ 1\ 2\ 1$

$W^{16} = 1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2$

$W^{17} = 2\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 2$

Lemma 3. If a sequence $x = \{x_k\}$ contains the word W^i with $W^i_k = x_j$ for some $i \leq 12$, where $j > 14$ for $x \in A$; and $k = 3$ if $i = 1$; $k = i + 1$ if $i = 2,3,4,5$ or 6; $k = i$ if $i = 7,8$ or 12 and $k = i - 1$ if $i = 9,10$ or 11, then, $M(x,j) > \alpha + 10^{-7}$.

Proof: For $i = 1$, for example, we have

$$M(x,y) > 2 + 2[0;1,2,2,1] = 3.4 > \alpha + 10^{-7} .$$

For the remaining i's, the inequality is obtained analogously.

Lemma 4. If a sequence $x = \{x_k\}$ contains the word w^i with $w^i_k = x_j$ for some i, $13 \leq i \leq 17$, where $j > 8$ for $x \in A$ and $k = i - 11$, then $M(x,j) < \alpha - 10^{-4}$.

Proof: For $i = 13$, for example, we have

$$M(x,j) < 2 + [0;1,2,1] + [0;2,2,1] = 2 + \frac{3}{4} + \frac{3}{7} < 3.18 < \alpha - 10^{-4} .$$

For remaining i's, the inequality is obtained analogously.

Remark 2. Lemma 3 and Lemma 4 are also true if x contains $W^{i^{-1}}$ in a similar way.

Lemma 5. If $x = \{x_k\}$ is a sequence such that $M(x,j) \in (\alpha - 10^{-7}, \alpha + 10^{-7})$ for some $j > 14$, then for any integer $n > 14$;

1) $x_{j+2} = 1$ and $M(x,\ell) \leq \alpha + 10^{-7}$ for $j - 10 \leq \ell \leq j + n$ implies

   $x_{j+\ell} = \xi_\ell$, $\ell = 0,1,2,\ldots,n$.

2) $x_{j+2} = 2$ and $M(x,\ell) \leq \alpha + 10^{-7}$ for $\max(0,j-n) \leq \ell \leq j + 10$ implies

   $x_{j-\ell} = \xi_\ell$, $\ell = 0,1,2,\ldots,\min(n,j)$.

Proof: The proof is by contradiction.

Clearly, $x_j = 2$ and $(x_{j-1}, x_j, x_{j+1}) \neq (2,2,2)$.

$$x_{j-1} = x_{j+1} = 1 \quad \text{otherwise} \quad x_{j-1}x_jx_{j+1} = W^{13} \quad \text{or} \quad W^{13^{-1}}$$

By Lemma 4, $M(x,j) < x\text{-}10^{-4}$. This is a contradiction. For the remaining ℓ's, contradictions are obtained analogously according to the following table.

Case 1. $x_{j+2} = 1$

$x_{j+\ell} = \xi_\ell$ for $-10 \leq \ell < n$, otherwise x contains W^i as described in Lemma 3 or Lemma 4 which leads to a contradiction.

ℓ	2	-2	3	-3	4	-4	5	-5	6	-6	7	-7	8	-8	9	10	-9	-10	11	12	13	14
$x_{j+\ell}$	1	2	2	2	2	2	2	1	1	1	2	2	1	1	1	2	2	2	2	2	1	2
i		14	2	3	15	16	4	17	5	6	7	8	9	10	1	2	11	3	12	4	5	7

If $n \geq 18$, by repeating the last seven columns, we have

$$x_{j+\ell} = \xi_\ell, \; \ell = 18, 19, \ldots, n.$$

Case 2. $x_{j+2} = 2$.

We repeat the same argument as above, after replacing ℓ by $-\ell$, and interchanging column 1 and column 2 in the table. The rest of the columns remain unchanged, i.e.

$$\begin{array}{cccccc} -\ell & -2 & 2 & 3 & -3 & \\ x_{j+\ell} & 2 & 1 & 2 & 2 & \dots \\ i & & 1 & 2 & 3 & \end{array}$$

Then $x_{j-\ell} = \xi_\ell$, $\ell=0,1,2,\dots,\min(n,j)$.

Let $\rho = \{\rho_i\}$ be a doubly infinite sequence such that $\rho_i = \xi_i$, $i \geq -11$, $\rho_{-12} = \rho_{-13} = 1$, $\rho_{-14} = 2$, $\rho_i = \rho_{i+2}$, $i \leq -15$. Then

$$M(\rho,0) = [\overline{2;1,1,2,2,2,1}] + [0;1,2,2,2,1,1,2,1,2,2,2,1,\overline{1,2}] .$$

Let $\varepsilon = \alpha - M(\rho,0)$. Then $10^{-11} < \varepsilon < 10^{-7}$.

Lemma 6. If $x = \{x_k\}$ is a sequence with $x_{j+\ell} = \xi_\ell$ for some j, $\ell = 0,1,2,\dots$ then $M(x,n) \leq M(\rho,0)$ for all $n > j + 14$.

Proof: Clearly $M(x,n) \leq M(\rho,0)$ if $x_n = 1$ or $(x_{n-1},x_n) = (2,2)$ or $(x_n,x_{n+1}) = (2,2)$. It remains to prove $M(x,j + 7i) \leq M(\rho,0)$, $i = 2,3,\dots$

$$M(x,j + 7i) = \bar{a} + 2 + [0;1,2,2,2,1,1,2,1,2,2,2,1,1,2,\dots] , \quad i = 2,3,\dots$$

Since $(L^{j+7i}x)_k = \rho$, $k \geq -16$, $(L^{j+7i}x)_{-17} = 2$ and $\rho_{-17} = 1$ by Lemma 2, we have $M(x,j + 7i) \leq M(\rho,0)$ for all $i \geq 2$. Hence $M(x,n) \leq M(\rho,0)$ for all $n > j + 14$.

Theorem 1. The interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ does not contain a point of the Lagrange spectrum.

Proof: Suppose there is a sequence $\eta = \{\eta_i\}$ such that $L(\eta) \in (\alpha-\varepsilon,\alpha+\varepsilon)$. Then there exists an integer $k > 14$ such that $M(\eta,i) < \alpha + \varepsilon$ for all $i \geq k$, and there exist infinitely many integers $j > k$ such that $M(\eta,j) \in (\alpha - \varepsilon, \alpha + \varepsilon)$. By Lemma 1, there exists a smallest n_0 such that $\varepsilon_u(n_0) \leq \varepsilon_\ell(16)$. Let j_1 be the smallest one of such j's greater than $k + n_0$. Let j_2 be the smallest one of such j's greater than $j_1 + 21$. By Lemma 5, either $\eta_{j_2+\ell} = \xi_\ell$, $\ell = 0,1,2,\ldots$ which implies, by Lemma 6, $M(\eta,n) \leq M(\rho,0) = \alpha - \varepsilon$ for all $n > j_2 + 14$ or

$\eta_{j_2-\ell} = \xi_\ell$, $\ell = 0,1,2,\ldots,j_2-k$, which implies

$\eta_{j_1-\ell} = \xi_\ell, \ell = 0,1,2,\ldots,n_0$ and $M(\eta,j_1) = 2 + s + t$ where

$s = [0;1,1,2,2,2,1,2,1,1,2,2,2,1,2,\ldots,\eta_{j_1-n_0},\ldots,\eta_0] = [s_0;s_1,s_2,\ldots]$

$t = [0;1,2,2,2,1,1,2,1,2,2,2,1,1,2,1,2,2,2,1,1,2,\ldots,\eta_{j_2},\ldots] = [t_0;t_1,t_2,\ldots]$

Let $v = [0;1,2,2,2,1,1,2,1,2,2,2,1,1,2,\overline{1,2}] = [v_0;v_1,v_2,\ldots]$

$$T \in I(V,16) \cap [I(V,17)]^c \quad \text{and} \quad S \in I(\xi, n_0)$$

where $T = \{t_k\}$, $S = \{s_k\}$ and $V = \{v_k\}$.

By Lemma 1, $v - t > \varepsilon_\ell(16)$

$$s - \bar{a} < \varepsilon_u(n_0) \leq \varepsilon_\ell(16) < v - t \quad \text{and}$$

$$s + t < v + \bar{a}.$$

Since $M(\rho,0) = 2 + \bar{a} + v$,

$$M(\eta,j_1) < M(\rho,0) = \alpha - \varepsilon$$

Since both cases lead to a contradiction, $(\alpha - \varepsilon, \alpha + \varepsilon)$ does not contain a point of the Lagrange spectrum.

For each $\lambda \in [0,1]$ with its binary expansion $\lambda = \sum_{k=1}^{\infty} \frac{\lambda_k}{2^k}$ we define $\xi^\lambda = \{\xi_i^\lambda\}$ to be a doubly infinite sequence as follows:

$$\xi_i^\lambda = \xi_i \ \ -16 \leq i, \ \xi_{-17}^\lambda = \xi_{-18}^\lambda = 2 \ \text{ and}$$

$$\xi_{-(17+2k)}^\lambda = \xi_{-(18+2k)}^\lambda = \lambda_k + 1, \ k = 1,2,\ldots$$

Let $P = \{M(\xi^\lambda) \,|\, \lambda$ is an irrational number in $[0,1]\}$.

Theorem 2. P is an uncountable set consisting of points of the Markov spectrum which are not in the Lagrange spectrum. Every point of P is a limit point of P.

Proof: For ξ_k^λ with $k < -8$, 2's occur in paris, so $M(\xi^\lambda)$ can only occur at $\xi_k^\lambda = 2$ with $k \geq -8$. By an argument exactly the same as that leading to $M(\xi) = M(\xi,0)$, we have $M(\xi^\lambda) = M(\xi^\lambda,0)$. Since $(\xi^\lambda)_i = \rho_i$, $i \geq -11$, $\xi_{-12}^\lambda = 2$ and $\rho_{-12} = 1$ we have by Lemma 2, $M(\xi^\lambda,0) > M(\rho,0)$. Thus $M(\xi^\lambda) > \alpha - \varepsilon$. Since $(\xi^\lambda)_i = \xi_i$, $i \geq -16$, $\xi_{-17}^\lambda = 2$ and $\xi_{-17} = 1$ by Lemma 2, we have $M(\xi^\lambda,0) < \alpha$ and hence $\alpha - \varepsilon < M(\xi^\lambda) < \alpha$. By Theorem 1, $M(\xi^\lambda)$ is not in the Lagrange spectrum. Hence P does not contain a point of the Lagrange spectrum. Let $\lambda(k) =$

$\sum_{i=1}^{k} \frac{\lambda_i}{2^i} + \sum_{j=1}^{\infty} \frac{1-\lambda_{k+j}}{2^{k+j}}$, $j = 1,2,\ldots$ then $\lambda(k)$ is irrational in $[0,1]$ since λ is irrational in $[0,1]$, so

$$M(\xi^{\lambda(k)}) \in P \qquad \text{and}$$

$$M(\xi^{\lambda(k)}) = M(\xi^{\lambda(k)},0) \qquad \text{and} \qquad \lambda(k) \to \lambda \quad \text{as} \quad k \to \infty$$

$$M(\xi^{\lambda}) = \lim_{k\to\infty} M(\xi^{\lambda(k)}) \quad \text{since} \quad M(\xi^{\lambda},0) = \lim_{k\to\infty} M(\xi^{\lambda(k)},0).$$

Also $\xi^{\lambda(k)} \neq \xi^{\lambda}$ for all k. Thus $M(\xi^{\lambda},0) \neq M(\xi^{\lambda(k)},0)$ since $\xi_i^{\lambda} = \xi_i^{\lambda(k)}$ for all $i \geq 0$ i.e. $M(\xi^{\lambda}) \neq M(\xi^{\lambda(k)})$ for all $k = 1,2,\ldots$

If $\lambda \neq \lambda'$ then $M(\xi^{\lambda},0) \neq M(\xi^{\lambda'},0)$ since $\xi_i^{\lambda} = \xi_i^{\lambda'}$ for all $i \geq 0$ i.e. $M(\xi^{\lambda}) \neq M(\xi^{\lambda'})$. Hence every point of P is a limit point of P and P is nondenumerable.

CHAPTER IV

FRACTIONAL DIMENSION OF A LEVEL SET

This section provides an estimate of the Hausdorff-Besicovitch dimension of the level set $H = \{w \in E_2 \cap [0,1) : L(w) \leq \frac{\sqrt{221} + 5\sqrt{5} + 4}{10}\}$.

The Hausdorff-Besicovitch dimension of a set S, which we will write $\dim(S)$, is defined as follows: let (I_i) be a covering of S by intervals, and let $|I_i|$ be the length of I_i; then $\delta = \text{l.u.b.}|I_i|$ is called the norm of the covering;

$$\Gamma(\alpha, S) = \lim_{\delta \to 0} \text{g.l.b.} \Sigma |I_i|^\alpha,$$

where the greatest lower bound is taken over all coverings of norm δ, is the α-dimensional Hausdorff measure of S. $\dim(S)$ is the number such that, for every positive ε,

$$\Gamma(\dim(S) - \varepsilon, S) = \infty$$

and

$$\Gamma(\dim(S) + \varepsilon, S) = 0$$

Notation

$$\xi = [0;2,2,1,1,\ldots,1,1,\ldots],$$

$$p_n = p_n(\xi),\ q_n = q_n(\xi) .$$

If a is the word $a_1 a_2 \ldots a_n$, we define $\frac{p_i(a)}{q_i(a)} = [0;a_1,a_2,\ldots,a_i]$,

$i = 1,2,\ldots,n$ and $I_a = \{w \in [0,1) : a_1(w) = a_1,\ldots,a_n(w) = a_n\}$.

Let ξ_n be the word $2\ 2\ 1\ 1\ \ldots\ 1\ 1$ of length $2n$.

$\ell_n = |I_{\xi_n} \circ 22|$,

$\ell_{n,k} = |I_{\xi_n \circ \xi_k} \circ 22|$.

Remark: $p_{2n+1}(\xi_n \circ 2) = 2p_{2n} + p_{2n-1}$,

$$q_{2n+1}(\xi_n \circ 2) = 2q_{2n} + q_{2n-1},$$

$$q_{2n+2}(\xi_n \circ 22) = 2(2q_{2n} + q_{2n-1}) + q_{2n} = 5q_{2n} + 2q_{2n-1},$$

$$p_{2n+2}(\xi_n \circ 22) = 5p_{2n} + 2p_{2n-1},$$

$$p_{2(n+k)+1}(\xi_n \circ \xi_k \circ 2) = (2p_{2k} + p_{2k-1})p_{2n-1} + (2q_{2k} + q_{2k-1})p_{2n},$$

$$q_{2(n+k)+1}(\xi_n \circ \xi_k \circ 2) = (2p_{2k} + p_{2k-1})q_{2n-1} + (2q_{2k} + q_{2k-1})q_{2n},$$

$$p_{2(n+k)+2}(\xi_n \circ \xi_k \circ 22) = (5p_{2k} + 2p_{2k-1})p_{2n-1} + (5q_{2k} + 2q_{2k-1})p_{2n},$$

$$q_{2(n+k+1)}(\xi_n \circ \xi_k \circ 22) = (5p_{2k} + 2p_{2k-1})q_{2n-1} + (5q_{2k} + 2q_{2k-1})q_{2n},$$

$$\ell_n = \frac{1}{(5q_{2n} + 2q_{2n-1})(7q_{2n} + 3q_{2n-1})},$$

$\ell_{n,k} =$

$$\frac{1}{[(5p_{2k}+2p_{2k-1})q_{2n-1}+(5q_{2k}+2q_{2k-1})q_{2n}][(7p_{2k}+3p_{2k-1})q_{2n-1}+(7q_{2k}+3q_{2k-1})q_{2n}]}.$$

Lemma 1. $p_n = \frac{1}{2}[(1 + \frac{3}{\sqrt{5}})\lambda^{n-1} + (1 - \frac{3}{\sqrt{5}})\theta^{n-1}]$ and $q_n = (1 + \frac{4}{\sqrt{5}})\lambda^{n-1} + (1 - \frac{4}{\sqrt{5}})\theta^{n-1}$ where $\lambda = \frac{1 + \sqrt{5}}{2}$ and $\theta = \frac{1 - \sqrt{5}}{2}$ are roots of $x^2 = x + 1$.

Proof: Let $p_n = A\lambda^{n-1} + B\theta^{n-1}$

$$p_1 = 1 = A + B, \quad p_2 = 2 = A\lambda + B\theta = \frac{A+B}{2} + (A-B)\frac{\sqrt{5}}{2},$$

$$2 = \tfrac{1}{2} + (A + A - 1)\frac{\sqrt{5}}{2}, \quad 3 + \sqrt{5} = 2\sqrt{5}\,A,$$

so $A = \frac{1}{2}(1 + \frac{3}{\sqrt{5}})$ and $B = \frac{1}{2}(1 - \frac{3}{\sqrt{5}})$.

Hence $p_n = \frac{1}{2}[(1 + \frac{3}{\sqrt{5}})\lambda^{n-1} + (1 - \frac{3}{\sqrt{5}})\theta^{n-1}]$.

Similarly, $q_n = (1 + \frac{4}{\sqrt{5}})\lambda^{n-1} + (1 - \frac{4}{\sqrt{5}})\theta^{n-1}$

Lemma 2. If $1 = \sum_{k=2}^{\infty} (\frac{\ell_{n,k}}{\ell_n})^{\alpha}$ as $n \to \infty$ then $.2064 < \alpha < .20641$.

Proof: $\frac{p_n}{p_{n-1}} = \frac{A\lambda^{n-1} + B\theta^{n-1}}{A\lambda^{n-2} + B\theta^{n-2}} \to \lambda$ as $n \to \infty$ since $\theta \to 0$ as $n \to \infty$

where $A = \frac{1}{2}(1 + \frac{3}{\sqrt{5}})$ and $B = \frac{1}{2}(1 - \frac{3}{\sqrt{5}})$. Similarly, $\frac{q_n}{q_{n-1}} \to \lambda$ as $n \to \infty$.

From preceding remark,

$$\frac{\ell_{n,k}}{\ell_n} \to \frac{q_{2n-1}^2(5\lambda + 2)(7\lambda + 3)}{q_{2n-1}^2[\lambda(5q_{2k}+2q_{2k-1})+5p_{2k}+2p_{2k-1}][\lambda(7q_{2k}+3q_{2k-1})+7p_{2k}+3p_{2k-1}]}$$

as $n \to \infty$.

Let $\delta_k = \frac{\ell_{n,k}}{\ell_n}$. Then

$$\frac{\delta_{k+1}}{\delta_k} \to \frac{(5\lambda + 2)(q_{2k-1}\lambda + p_{2k-1})(7\lambda + 3)(q_{2k-1}\lambda + p_{2k-1})}{(5\lambda + 2)\lambda^2(q_{2k-1}\lambda + p_{2k-1})(7\lambda + 3)\lambda^2(q_{2k-1}\lambda + p_{2k-1})} = \frac{1}{\lambda^4} \quad \text{as} \quad k \to \infty$$

$$(5\lambda + 2)(7\lambda + 3) = 35\lambda^2 + 29\lambda + 6 = 35(\lambda + 1) + 29\lambda + 6 = 64\lambda + 41$$

$p_3 = 3$, $p_4 = 5$, $p_5 = 8$, $p_6 = 13$, $p_7 = 21$, $p_8 = 34$,

$q_3 = 7$, $q_4 = 12$, $q_5 = 19$, $q_6 = 31$, $q_7 = 50$, $q_8 = 81$.

Since the terms have nearly constant ratio, we approximate the tails of the series by the geometric series

$$\sum_{k=2}^{\infty} \delta_k^{\alpha} \doteq (64\lambda + 41)^{\alpha}\{[(60 + 14)\lambda + 25 + 6]^{-\alpha}[(84 + 21)\lambda + 35 + 9]^{-\alpha} +$$

$$[(155+38)\lambda+65+16]^{-\alpha}[(217+57)\lambda+91+24]^{-\alpha} +$$

$$\frac{[(405+100)\lambda+170+42]^{-\alpha}[(567+150)\lambda+238+63]^{-\alpha}}{1 - \lambda^{-4\alpha}}\}$$

$$= (64\lambda+41)^{\alpha}\{[(74\lambda+31)(105\lambda+44)]^{-\alpha} + [(193\lambda+81)(274\lambda+115)]^{-\alpha} +$$

$$\frac{[(505\lambda+212)(717\lambda+301)]^{-\alpha}}{1 - \lambda^{-4\alpha}}$$

$$= (64\lambda+41)^{\alpha}[(14281\lambda+9134)^{-\alpha} + (97271\lambda+62197)^{-\alpha} +$$

$$\frac{(666094\lambda+425897)^{-\alpha}}{1 - \lambda^{-4\alpha}}].$$

$$\sum_{k=2}^{\infty} \delta_k^{\alpha} \geq 1.000061078 \quad \text{when} \quad \alpha = .2064 \, ,$$

$$\sum_{k=2}^{\infty} \delta_k^{\alpha} \leq 0.9999675709 \quad \text{when} \quad \alpha = .20641 .$$

Hence $.2064 < \alpha < .20641$.

Let e_0 be the empty word and let e_n be the word $1\ 1\ \ldots\ 1\ 1$ of length $2n$. We define

$$T = \{e_{n_0} * \prod_{k=1}^{\infty} \xi_{n_k} : n_0 \geq 0 \text{ and } n_k \geq 2,\ k = 1,2,\ldots\} \cup \{\bar{1}\}.$$

The following theorem follows immediately from a theorem in Schweiger [24].

Theorem 1. If $1 = \sum_{k=2}^{\infty} \left(\frac{\ell_{n,k}}{\ell_n}\right)^{\alpha}$ as $n \to \infty$ then $\dim T = \alpha$.

Corollary 1. $.2064 < \dim T < .20641$.

Proof: This is an immediate result of the preceding theorem and Lemma 2.

Lemma 3. $T \subseteq H$.

Proof: Let $t = \dfrac{\sqrt{221} + 5\sqrt{5} + 4}{10} = [\overline{2;2,1,1}] + [0;\bar{1}]$. Let w be an element of T.

Case 1. There are infinitely many pairs $(a_i(w), a_{i+1}(w)) = (2,2)$.

Clearly, $M(w,k) < t$ when $a_k(w) = 1$. If $a_k(w) = 2$ then either $a_{k-1}(w) = 2$ or $a_{k+1}(w) = 2$. Without loss of generality, we may assume $a_{k+1}(w) = 2$. Since there exists $j < k$ such that $a_j(w) = 2$ and $k - j$ is even, $[0;a_{k-1}(w), a_{k-2}(w), \ldots, a_1(w)] < [0,\bar{1}]$

$a_{k+2}(w) = a_{k+3}(w) = 1$, since $w \in T$. If $a_{k+4}(w) = 1$ then $[a_k(w); a_{k+1}(w), \ldots] < [\overline{2;2,1,1}]$. Otherwise $(a_{k+4}(w), a_{k+5}(w), a_{k+6}(w), a_{k+7}(w)) = (2,2,1,1)$. Continuing this argument, unless $[a_k(w);a_{k+1}(w),\ldots] = [\overline{2;2,1,1}]$, there must exist a first i such that $a_{k+4i}(w) = 1$. This implies $[a_k(w);a_{k+1}(w),\ldots] < [\overline{2;2,1,1}]$. Thus $M(w,k) < t$ and $L(w) < t$ i.e. $w \in H$.

Case 2. There exists a positive interger k such that $a_i(w) = 1$ for all $i \geq k$. Then $M(w,i) < t$ for all $i \geq k$. Thus $L(w) < t$ and $w \in H$. Hence $T \subseteq H$.

Theorem 2. $\dim H > .2064$.

Proof: By preceding lemma, $\dim H \geq \dim T$. By Lemma 3, $\dim H > .2064$.

Alternative method of computing $\dim H$.

We approximate the covering of T which is found to be a generalized Cantor set, i.e. the set constructed by removing the middle interval. It is clear that $\max T = [0;\bar{1}]$. Let $b = \max T$. Let $s = \min T = [0;\overline{2,2,1,1}]$. That is $T \subset [s,b]$. Since

$$\max \{w \in T: a_1(w) = a_2(w) = 2\} = 2211 * b$$

and

$$\min \{w \in T: a_1(w) = a_2(w) = 1\} = 11 * s$$

we obtain our first stage covering Γ_1 of T by removing

$(2211 * b,\ 11 * s)$ from $[s,b]$ i.e. $\Gamma_1 = [s,\ 2211 * b] \cup [11s,b] \supset T$.

Similarly, we obtain our second stage covering of T.

$$\Gamma_2 = [s,(2211)^2 * b] \cup [(2211)*(11)*s, 2211*b] \cup [11*s,(11)*(2211)*b] \cup$$
$$[(11)^2 * s,b] \supset T.$$

Continuing this process, we obtain the n^{th} stage covering Γ_n of T for each positive integer n.

Let m be the length of the middle interval we removed, let ℓ be the length of the left hand interval and let r be the length of the right hand interval. Since the ratios $\frac{\ell}{\ell + m + r}$ and $\frac{r}{\ell + m + r}$ remain approximately the same each time we remove a middle interval from the remaining intervals, we denote the former by p and the latter by q. We can easily see, by induction, that at n^{th} stage we have $\binom{n}{j}$ intervals of size $p^{n-j}q^j(b-s)$ in $[s,b]$. Hence $\Gamma(\alpha,T)$ will be approximated by

$$\Gamma_n(\alpha,T) = \sum_{j=0}^{n} \binom{n}{j}[p^{n-j}q^j(b-s)]^\alpha = (b-s)^\alpha(p^\alpha + q^\alpha)^n.$$

So the α which makes $\Gamma_n(\alpha,T) = 1$, will be the dimension of T. Hence

$$\log(p^{\alpha} + q^{\alpha}) = \frac{-\alpha \log(b-s)}{n},$$

$$p^{\alpha} + q^{\alpha} \doteq 1 \qquad \text{as} \quad n \to \infty$$

A short computation shows:

$$.2064 < \alpha < .20644 \qquad \text{since} \quad p \doteq .004481486542 \quad \text{and}$$

$$q \doteq .1463587922$$

Again we proved $\dim H > .2064$.

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