CONTRIBUTIONS TO THE THEORY OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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This is to certify that the

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ABSTRACT

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By

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Let $\rho : (-\infty, 0] \to (0, \infty)$ be nondecreasing and integrable on $(-\infty, 0]$. Assume also that $\rho(u + v) \leq \rho(u)\rho(v)$ for all $u, v \leq 0$. Let $r > 0$.

Functional differential equations are discussed in the context of the phase space $X = \{\varphi : (-\infty, 0] \to \mathbb{R}^n \mid \varphi|_{[-r,0]}$ is continuous, $\varphi|_{(-\infty,-r)}$ is measurable and $\int_{-\infty}^{-r} |\varphi(u)| \rho(u)\, du < \infty\}$.

The adjoint theory for linear autonomous equations is considered from the point of view of adjoint semi-group theory. The adjoint equation is derived and the space decomposition at characteristic values is given in terms of an extension of the classic bilinear form known for finite delay equations.

General linear systems and their adjoints are discussed in a manner similar to that for finite delay equations. Let

$$\beta = \inf\{c \in \mathbb{R} \mid \int_{-\infty}^{0} e^{cs}\rho(s)ds < \infty\}.$$

It is shown that the spectrum of the usual solution operator $T(t,s)$, $t > s$ consists entirely of normal eigenvalues outside the circle of radius $e^{\beta(s-t)}$.

In the case of linear periodic systems, an extension of the Floquet theory known for finite delay FDE's is made to the space X. Under the assumption that $\beta < 0$, the usual criterion (in the context of characteristic exponents) for the stability of the zero solution is shown to be true. Also, the Fredholm Alternative is proved for nonhomogeneous systems. The projections associated with the space decompositions at characteristic multipliers are calculated in terms of an adjoint equation and bilinear form.

The behavior of solutions near periodic solutions to C^1 nonlinear FDE's is considered. Conditions are given under which the Poincaré map can be defined about nondegenerate periodic orbits. The Poincaré map is then used to discuss the stability of the periodic orbit.

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By

Harlan West Stech

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To Ruth Ellen Magnuson

for teaching me that understanding

is the real subject

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TABLE OF CONTENTS

CHAPTER I

PRELIMINARIES

§1. Introduction

In many biological and physical modeling problems the rate of change of a system can not be assumed to depend only upon the current state of the system. Rather, it depends also upon the past states or "history" of the system. The incorporation of past history into differential models leads to a class of equations that are now referred to as functional differential equations (FDE).

For example, the scalar equation

$$\dot{x}(t) = -\alpha[x(t) + 1]x(t - T)$$

where $\alpha, T > 0$, has been used by Hutchinson [19] to describe the growth of a single species population with a self-regulating mechanism that affects the birth rate after T units of time. Note that the historical effect occurs in the term "$x(t - T)$" and requires the assumption that all historic effects on the rate of change "$\dot{x}(t)$" of the system be concentrated at T units of time previous to the current time "t". As a relaxation of this assumption, May [27] has suggested that

$$\dot{x}(t) = -\alpha[x(t) + 1]\int_{-\infty}^{0} x(t + s)\eta(s)ds$$

is a more realistic model of that biological system. Here, η is a positive weight function integrable on $(-\infty,0)$. The term

$$"\int_{-\infty}^{0} x(t + s)\eta(s)ds"$$

encorporates the past history into the system since as s varies over $(-\infty,0)$, the values of x on $(-\infty,t)$ are used in evaluating the integral.

Hans-Otto Walther [39] has studied the similar model

$$\dot{x}(t) = -\alpha[x(t) + 1]\int_{-r}^{0} x(t + s)d\eta(s)$$

where $r > 0$ and η is a real-valued, nondecreasing function defined on $[-r,0]$.

Volterra [38] proposed the system

$$\dot{N}_1(t) = [\epsilon_1 - \gamma_1 N_2(t) - \int_{-\infty}^{0} F_1(s)N_2(t + s)ds]N_1(t)$$

$$\dot{N}_2(t) = [-\epsilon + \gamma_2 N_1(t) + \int_{-\infty}^{0} F_2(s)N_1(t + s)ds]N_2(t)$$

to study the interactions between predator and prey populations N_1 and N_2, respectively. The weight functions F_1 and F_2 are assumed to be integrable on $(-\infty,0)$.

May [27] has considered a similar model

$$\dot{H}(t) = [r - cH(t - T) - \alpha P(t)]H(t)$$

$$\dot{P}(t) = [-b + \beta H(t)]P(t)$$

in describing (among other things) the wolf-moose relationship on Isle Royale, Michigan.

The scalar equation

$$\dot{x}(t) = -\alpha\int_{-\infty}^{-1} \sin(x(t + s))\rho(s)ds$$

was investigated by Israelsson and Johnsson [20] as a model of autonomous plant oscillations. The weight function, ρ, was assumed again to be integrable on $(-\infty,-1)$ and "exponential" in form. See Klein [22] for further discussions.

For nonbiological examples, see Coleman and Mizel [6] for models arising in the theory of mechanics and thermodynamics of materials.

The models of Hutchinson, Walther and the wolf-moose model of May are said to be of "finite delay" type since the right hand sides of these equations rely only on a finite past history of the respective systems. The remaining examples are functional differential equations with infinite delay. Although these models can incorporate the entire past history into determining the rate of change of the systems, the "distant" past behavior has slight influence on the equation since the weight functions involved are assumed to be integrable on $(-\infty,0)$. Such FDE's are said to have a "fading memory". The finite delay situations are, of course, special cases in which the history past a fixed amount of time is completely "forgotten" and thus cannot influence the behavior in the future.

Functional differential equations with finite delay have received extensive study over the past half century. It is an area that is currently undergoing a rapid development. The monograph of Hale [11] serves as the best exposition on the qualitative theory of finite delay FDE's.

In 1974, Hale [12] initiated the study of a class of FDE's with infinite delays. The class includes as a proper subset all FDE's with finite delay and (with various assumptions on the respective weight functions) each of the infinite delay models mentioned above. In particular, FDE's were considered in which weighted averages of "distant" past effects were used in calculating the rate of change of the system while more precise effects of the "not-so-distant" past were allowed. This will be clarified in the next section. However, as an example of such an equation we might consider

$$\dot{x}(t) = -\alpha[x(t) + 1][\int_{-r}^{0} x(t+s)d\eta(s) + \int_{-\infty}^{-r} x(t+s)\eta(s)\rho(s)ds]$$

which generalizes the first three examples of this section. Here, η is assumed on $[-r,0]$ to be as in Walther's model and essentially bounded on $(-\infty,-r)$, while ρ is integrable on $(-\infty,-r)$ and (in some sense) "exponential" in form. (e.g., $\rho(s) = e^{-c|s|^n}$; $c,n > 0$.)

A paper of Naito [29] has investigated linear autonomous equations of this type and generalized to this class some of the results known for FDE's with finite delay.

In this thesis we have continued the study of this class of functional differential equations with infinite delay. Using the theory of finite delay equations as a guide, topics have been chosen whose counterparts in the theory of FDE's with finite delay have proved fundamental to that theory. The bulk of our study concerns the qualitative behavior of linear systems. However, a nonlinear topic is discussed in the final chapter.

In Chapter II we continue the study of autonomous linear FDE's initiated by Hale [12] and Naito [29]. We are most concerned with the adjoint theory and a function analytic derivation of the adjoint to a given FDE. The study of the adjoint problem makes heavy use of semi-group theory. The chapter includes a complete description of the adjoint semi-group, its infinitesimal generator and a discussion of the associated adjoint space. The final section concerns the calculation of the projections onto the eigenspaces associated with roots of the characteristic equations. An explicit description of these projections is important from an applications point of view since, for example, such information is used in the finite delay case in showing the existence of nonconstant periodic solutions to Hutchinson's model. See Grafton [10] and Chow and Hale [5] for related topics. In addition, we are able to present a new explanation (in function analytic terms) of the classic bilinear form known to the theory of finite delay equations.

Chapter III considers general linear systems. After a brief look at the existence, uniqueness and continuous dependence properties of these equations, the solution operator associated to such equations is discussed. Once again the adjoint equation is derived by function analytic techniques. Fortunately, many of the technical aspects of this chapter may be proved in a manner similar to the analogous results from the theory of FDE's with finite delay. Where possible, proofs have been omitted and replaced with references. The chapter ends with a discussion of the spectral properties of the solution operator and the strengthening of a result of Hale [12].

Chapter IV investigates the special case in which the linear system of the previous chapter is assumed to be periodic. The chapter starts with the definition and study of periodic families of bounded linear operators. These families play a role analogous to that of semi-groups of bounded linear operators associated with autonomous systems. The section makes no reliance on the theory of FDE's and is so presented to stress the fact that many of the general properties of periodic FDE's are valid in a much larger context. The chapter includes some of the first dividends of our function analytic approach. In particular, we are able to derive the Fredholm Alternative for forced periodic systems. We also discuss the classic criterion for the stability of linear periodic systems (in the context of characteristic exponents) and calculate the projection operators onto the invariant subspaces associated to the characteristic multipliers.

The last chapter concerns a brief study of the behavior near periodic solutions to smooth (C^1) nonlinear autonomous equations. By imposing a technical assumption on the equations under study, we are able to define the Poincaré Map about periodic orbits. The standard criterion concerning the stability of the periodic orbits is shown to generalize to this class of FDE's.

There are many justifications for the work that is to follow. In terms of modeling with FDE's, it is a first step towards removing the assumption that only a finite past history can influence the behavior of the system. In some instances, the mathematical analysis of models is greatly simplified if one includes the entire past history into the model. As was pointed out by May [27], the stability of the zero solutions of

$$\dot{x}(t) = \alpha[x(t) + 1]\int_{-k}^{0} x(t + s)se^{cs}\, ds$$

and

$$\dot{x}(t) = \alpha[x(t) + 1]\int_{-\infty}^{0} x(t + s)se^{cs}\, ds$$

for $c > 0$ relies on the location of the solutions, z, of the complex equations

$$\begin{aligned} z &= \alpha\int_{-k}^{0} se^{(z+c)s}\, ds \\ &= -\alpha(z + c)^{-2} + \alpha[k(z + c)^{-1}\cdot e^{-(z+c)k} \\ &\qquad + (z + c)^{-2}\cdot e^{-(z+c)k}] \end{aligned}$$

and

$$z = \alpha\int_{-\infty}^{0} se^{(z+c)s}\, ds = -\alpha(z + c)^{-2},$$

respectively. The infinite delay case is decidedly the friendlier of the two.

From a mathematical standpoint there are other reasons for such a study. Certainly, it can be viewed as a step towards creating a qualitative theory for general integro-differential equations. (In many situations proofs have been supplied that lend themselves to immediate generalization.) However, this work also sheds new light on the theory of finite delay FDE's. We mention the results concerning the classical bilinear forms (Chapter II, §8 and Chapter IV, §3,4) as specific situations where finite delay results become more meaningful when seen as special cases of results true for infinite delay equations.

It should also be mentioned that the work of Levin and Shea [24, 25, 26] also indicates the importance of the topic under consideration. They have shown, for example, that for a class of measures, $A(t,s)$, satisfying a periodicity assumption in t, the asymptotic behavior of bounded solutions to

$$\dot{x}(t) = \int_{-t}^{0} x(t + s)dA(t,s)$$

is describable in terms of the solutions of the "limit equation"

$$\dot{x}(t) = \int_{-\infty}^{0} x(t + s)dA(t,s).$$

Linear periodic systems of this type are discussed in Chapter IV.

§2. Notation

An effort has been made to use the basic notations from the theory of FDE's with finite delays.

If n is a positive integer, $\mathbb{R}^n$ will denote Euclidean n-space. Elements of $\mathbb{R}^n$ will be viewed as column vectors and $|\cdot|$ will denote the Euclidean norm. If $\xi \in \mathbb{R}^n$, then ξ^T will denote the same vector viewed as a row vector in the usual manner. If η is a row vector, then $\eta\xi$ will denote the usual inner product between η and ξ.

Let $\rho:(-\infty,0] \to (0,\infty)$ be continuous, nondecreasing and satisfy

$$\rho(u + v) \leq \rho(u)\rho(v) \quad \text{for} \quad u,v \leq 0 \tag{1.1}$$

$$\int_{-\infty}^{0} \rho(u)du < \infty. \tag{1.2}$$

For $r > 0$ we define $X = \{\varphi:(-\infty,0] \to \mathbb{R}^n \,|\, \varphi$ is continuous on $[-r,0]$, measurable on $(-\infty,-r)$ and

$$\int_{-\infty}^{-r} |\varphi(u)|\rho(u)du < \infty\}.$$

We endow the set X with the norm

$$|\varphi| = \sup_{[-r,0]} |\varphi(u)| + \int_{-\infty}^{-r} |\varphi(u)|\rho(u)du.$$

The use of $|\cdot|$ to also denote the norm on X should cause no confusion. With this norm X becomes a Banach Space.

The dual space, X^*, is given by $\{\psi:(-\infty,0] \to \mathbb{R}^{n^T} \mid \psi$ is essentially bounded and measurable on $(-\infty,-r)$, of bounded variation on $[-r,0]$, left continuous on $[-r,0)$, and satisfies $\psi(0) = 0\}$. For $\psi \in X^*$ we define $\psi(u) = 0$ if $u > 0$. If $\psi:(-\infty,0] \to \mathbb{R}^{n^T}$ is essentially bounded and measurable on $(-\infty,-r)$, of bounded variation on $[-r,0]$ and left continuous on $[-r,0]$ we define the element $\psi^0 \in X^*$ by

$$\psi^0(u) = \begin{cases} 0, & u = 0 \\ \psi(u), & u < 0. \end{cases}$$

The duality pairing between $\psi \in X^*$ and $\varphi \in X$ will be denoted by $\langle\psi,\varphi\rangle$ and is given by

$$(1.3) \qquad \langle\psi,\varphi\rangle = \int_{-\infty}^{-r} \psi(u)\varphi(u)\rho(u)\,du + \int_{-r}^{0} [d\psi(u)]\varphi(u).$$

The integral on $[-r,0]$ is of Lebesgue-Stieltjes type (see [17] or [34]). We will write

$$\int_a^{b^-} = \int_{[a,b)} \quad \text{and} \quad \int_a^b = \int_{[a,b]}.$$

The dual norm on X^* associated with (1.3) is given by

$$|\psi| = \max\{\operatorname*{ess\,sup}_{u<-r}|\psi(u)|, \operatorname*{var}_{[-r,0]}|\psi(u)|\}.$$

The symbol I will denote the $n \times n$ identity matrix or the identity operator on a Banach space. We shall make specific comments whenever confusion might arise.

§3. The ~ Representation of $\psi \in X^*$

In the sections to follow, we shall see that it is convenient to write the pairing (1.3) as

$$(1.4) \qquad \langle\psi,\varphi\rangle = \int_{-\infty}^{0} [d\tilde{\psi}(u)]\varphi(u)$$

where $\tilde{\psi}$ is defined by

$$\tilde{\psi}(u) = \begin{cases} \psi(u), & -r \leq u \leq 0 \\ \psi(-r) - \int_u^{-r} \psi(s)\rho(s)ds, & u < -r. \end{cases}$$

For example, if I is the $n \times n$ identity matrix and μ_i denotes its ith row we define ν_i by

$$\nu_i(u) = \begin{cases} \mu_i, & -r \leq u \leq 0 \\ 0, & u < -r. \end{cases}$$

Then

$$\nu_i^0(u) = \begin{cases} \mu_i, & -r \leq u < 0 \\ 0, & u < -r \quad \text{or} \quad u = 0 \end{cases}$$

and

$$\tilde{\nu}_i^0(u) = \begin{cases} \mu_i, & u < 0 \\ 0, & u = 0. \end{cases}$$

We have $\langle\nu_i^0,\varphi\rangle = -\varphi_i(0)$, the ith co-ordinate of $-\varphi(0)$, for $1 \leq i \leq n$. If we define the $n \times n$ matrix valued function δ^* by $\delta^* = \text{row}(\nu_1^0, \nu_2^0, \ldots, \nu_n^0)$ then

$$(1.5) \qquad \langle\delta^*,\varphi\rangle = -\varphi(0).$$

The following may be shown by elementary methods.

<u>Lemma 1.1</u>. (i) If $\psi_1, \psi_2 \in X^*$ and c_1, c_2 are scalars then $\widetilde{[c_1\psi_1 + c_2\psi_2]} = c_1\tilde{\psi}_1 + c_2\tilde{\psi}_2$.

(ii) If $\{\psi_m\} \subset X^*$ converges to $\psi \in X^*$, then $\tilde{\psi}_m \to \tilde{\psi}$ uniformly on compact subsets of $(-\infty, 0]$.

Finally, if $\varphi \in X$ has a continuous extension to $(-\infty, a)$ for some $a > 0$ we may define the element $\varphi_t \in X$ for $0 \leq t \leq a$ by $\varphi_t(u) = \varphi(t + u)$, $u \leq 0$.

CHAPTER II

LINEAR AUTONOMOUS SYSTEMS

§1. The Solution Semi-group

Let $L : X \to \mathbb{R}^n$ be bounded and linear. We can represent L in terms of an $n \times n$ matrix valued function, η, whose rows are elements of X^*. That is,

$$L\bar{\varphi} = \langle \eta, \bar{\varphi} \rangle = \int_{-\infty}^{-r} \eta(s)\bar{\varphi}(s)ds + \int_{-r}^{0} [d\eta(s)]\bar{\varphi}(s)$$

$$= \int_{-\infty}^{0} [d\tilde{\eta}(s)]\bar{\varphi}(s)$$

for $\bar{\varphi} \in X$.

We consider the system

$$\dot{x}(t) = Lx_t, \quad t > 0 \tag{2.1}$$

$$x_0 = \varphi \in X. \tag{2.2}$$

As shown in Naito [29], we may associate with (2.1)-(2.2) a strongly continuous semi-group of bounded linear operators, $T(t)$, $t \geq 0$, defined on X by $T(t)\varphi = x_t(\varphi)$, where $x(\varphi)(\cdot)$ denotes the solution to (2.1)-(2.2). Define

$$\beta = \inf\{c \in \mathbb{R} \mid \int_{-\infty}^{0} e^{cs} \rho(s)ds < \infty\}.$$

<u>Theorem 2.1</u> [29]. The infinitesimal generator, A, of $T(t)$, $t \geq 0$ is given by $A\varphi = \dot{\varphi}$ with the domain

$$\mathcal{D}(A) = \{\varphi \in X \mid \dot{\varphi} \in X \text{ and } \dot{\varphi}(0) = L\varphi\}.$$

Furthermore, the point spectrum of A is contained in the half plane $\{\lambda \in C \mid \mathrm{Re}\,\lambda \geq \beta\}$. Any λ with real part larger than β is in the point spectrum of A if it satisfies

$$\det[\lambda I - L(e^{\lambda \cdot} I)] = 0. \tag{2.3}$$

Otherwise, λ is in the resolvent set of A.

It follows from Theorem 16.7.2 of Hille and Phillips [18] that $\mu \neq 0$ is in the point spectrum of $T(t)$ for $t > 0$ if $\mu = e^{\lambda t}$ where λ is in the point spectrum of A. Define

$$\gamma(t) = \sup_{s<-r} \frac{\rho(s-t)}{\rho(s)}, \quad t > 0. \tag{2.4}$$

The following theorem may be found in Hale [12].

<u>Theorem 2.2</u>. Let $t > 0$. For any $\varepsilon > 0$ there is only a finite number of points $\mu = \mu(t)$ in the spectrum of $T(t)$ with modulus $> \gamma(t) + \varepsilon$. Each such μ is in the point spectrum of $T(t)$ and must be of the form $\mu = e^{\lambda t}$ for some λ satisfying (2.3). Also, the generalized eigenspace of λ is finite dimensional and there is an integer, k, such that

$$(2.5) \qquad X = \mathcal{N}(A - \lambda I)^k \oplus \mathcal{R}(A - \lambda I)^k$$

where $\mathcal{R}$, $\mathcal{N}$ denote the range and null spaces, respectively.

Define M_λ on X by

$$(2.6) \qquad [M_\lambda \varphi](s) = \int_s^0 e^{\lambda(s-u)} \varphi(u)du = \int_{-\infty}^0 M(s,u;\lambda)\varphi(u)\rho(u)du$$

for $s \leq 0$, where

$$(2.7) \qquad M(s,u;\lambda) = \begin{cases} 0, & u < s \\ e^{\lambda(s-u)} \dfrac{1}{\rho(u)} I, & s \leq u \leq 0 \end{cases}$$

From Naito [29] we have that M_λ is a bounded linear operator from X into X and $R_\lambda(A) \equiv [\lambda I - A]^{-1}$ is defined for all λ in the resolvent set of A by

$$(2.8) \qquad [R_\lambda(A)\varphi](s) = e^{\lambda s}\Delta^{-1}(\lambda)\{\varphi(0) + L(M_\lambda\varphi)\} + [M_\lambda\varphi](s)$$

for $\varphi \in X$, $s \leq 0$. Here, $\Delta(\lambda)$ is the $n \times n$ matrix defined by

$$(2.9) \qquad \Delta(\lambda) = \lambda I - L(e^{\lambda \cdot} I),$$

with I the $n \times n$ identity matrix.

§2. Calculation of $\mathcal{D}(A^*)$ and A^*.

In this section we turn our attention towards the calculation of the adjoint, A^*, of the infinitesimal generator A associated with the semi-group $T(t)$, $t \geq 0$.

A representation of A^* is essential to our study of the adjoint equation and semi-group.

By Phillips [31], $R_\lambda(A^*) = R_\lambda(A)^*$ whenever λ is in the resolvent set of A. It follows easily that $R_\lambda(A)^*$ maps X^* onto $\mathcal{D}(A^*)$. Thus, the problem of characterizing $\mathcal{D}(A^*)$ is equivalent to that of determining the range of $R_\lambda(A)^*$. For this reason, we first calculate the adjoint of $[\lambda I - A]^{-1}$.

For any $\psi \in X^*$ and $\varphi \in X$, it follows from (2.8) that

$$\begin{aligned} (2.10) \qquad \langle\psi, R_\lambda(A)\varphi\rangle &= \langle\psi, e^{\lambda\cdot}b\rangle + \langle\psi, M_\lambda\varphi\rangle \\ &= \langle\psi, e^{\lambda\cdot}\rangle b + \langle M_\lambda^*\psi, \varphi\rangle \end{aligned}$$

where M_λ^* is the adjoint of M_λ,

$$\begin{aligned} (2.11) \qquad b &= \Delta^{-1}(\lambda)\{\varphi(0) + L(M_\lambda(\varphi))\} = \Delta^{-1}(\lambda)\{\varphi(0) + \langle\eta, M_\lambda(\varphi)\rangle\} \\ &= \Delta^{-1}(\lambda)\{-\langle\delta^*, \varphi\rangle + \langle M_\lambda^*\eta, \varphi\rangle\} \end{aligned}$$

and δ^* is defined by (1.5). Now (2.10) and (2.11) imply

$$(2.12) \qquad R_\lambda(A)^*\psi = \langle\psi, e^{\lambda\cdot}\rangle\, \Delta^{-1}(\lambda)\{M_\lambda^*\eta - \delta^*\} + M_\lambda^*\psi.$$

Thus, we consider the calculation of M_λ^*.

<u>Lemma 2.3</u>. If $\psi \in X^*$, then

(i) $M_\lambda^*\psi$ is absolutely continuous on $[-r,0)$ with bounded variation, left continuous derivative,

(ii) $\rho M_\lambda^* \psi$ is locally absolutely continuous on $(-\infty,-r)$ with

$$\frac{1}{\rho(s)}\frac{d}{ds}(\rho(s)[M_\lambda^*\psi](s))$$

essentially bounded on $(-\infty,-r)$, and

(iii) $\rho(-r^-)[M_\lambda^*\psi](-r^-) = [\dot{M}_\lambda^*\psi](-r)$.

<u>Proof</u>: For $\psi \in X^*$ and $\varphi \in X$

$$\langle\psi, M_\lambda\varphi\rangle = \int_{-\infty}^{-r} \psi(s)[M_\lambda\varphi](s)\rho(s)ds + \int_{-r}^{0} [d\psi(s)][M_\lambda\varphi](s)$$

$$\equiv I_1 + I_2.$$

Applying (2.6), (2.7) and Fubini's Theorem [34] to I_1,

$$\begin{aligned} I_1 &= \int_{-\infty}^{-r} \psi(s)\rho(s)\int_{-\infty}^{0} M(s,u;\lambda)\varphi(u)\rho(u)duds \\ &= \int_{-\infty}^{0} [\int_{-\infty}^{-r} \psi(s)\rho(s)M(s,u;\lambda)ds]\varphi(u)\rho(u)du \\ &= \int_{-\infty}^{-r} [\int_{-\infty}^{u} \psi(s)e^{\lambda(s-u)} \rho(s)ds]\varphi(u)du \\ &\quad + \int_{-r}^{0} [\int_{-\infty}^{-r} \psi(s)e^{\lambda(s-u)} \rho(s)ds]\varphi(u)du. \end{aligned}$$

To the integral I_2 we apply integration by parts [17].

$$\begin{aligned} I_2 &= \psi(s)\int_s^0 e^{\lambda(s-u)} \varphi(u)du\Big|_{-r}^{0} \\ &\quad - \int_{-r}^{0} \psi(s)[\lambda\int_s^0 e^{\lambda(s-u)} \varphi(u)du - \varphi(s)]ds \\ &= -\psi(-r)\int_{-r}^{0} e^{\lambda(-r-u)} \varphi(u)du + \int_{-r}^{0} \psi(u)\varphi(u)du \\ &\quad - \lambda\int_{-r}^{0} \psi(s)\int_s^0 e^{\lambda(s-u)} \varphi(u)duds. \end{aligned}$$

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Integration of the last integral by parts yields

$$\begin{aligned}
I_2 &= \int_{-r}^{0} [\psi(u) - \psi(-r)e^{\lambda(-r-u)}]\varphi(u)\,du \\
&\quad - \lambda\Big[\int_{-r}^{s} \psi(v)e^{\lambda v}\,dv \cdot \int_{s}^{0} e^{-\lambda u}\varphi(u)\,du\Big|_{-r}^{0} \\
&\quad + \int_{-r}^{0}\int_{-r}^{u} \psi(v)e^{\lambda v}\,dv \cdot e^{-\lambda u}\varphi(u)\,du\Big] \\
&= \int_{-r}^{0} [\psi(u) - \psi(-r)e^{\lambda(-r-u)} - \lambda\int_{-r}^{u} \psi(v)e^{\lambda(v-u)}\,dv]\varphi(u)\,du.
\end{aligned}$$

Combining the expressions for I_1 and I_2, we obtain

$$\begin{aligned}
(2.13)\qquad \langle M_\lambda^*\psi,\varphi\rangle &= \int_{-\infty}^{-r} [\int_{-\infty}^{u} \psi(s)e^{\lambda(s-u)}\frac{\rho(s)}{\rho(u)}\,ds]\varphi(u)\rho(u)\,du \\
&\quad + \int_{-r}^{0} [\psi(u) - \psi(-r)e^{\lambda(-r-u)} \\
&\quad - \lambda\int_{-r}^{u} \psi(s)e^{\lambda(s-u)}\,ds \\
&\quad + \int_{-\infty}^{-r} \psi(s)\rho(s)e^{\lambda(s-u)}\,ds]\varphi(u)\,du.
\end{aligned}$$

Statement (i) follows from (2.13) since this equation must hold for every $\varphi \in X$. As for (ii), (2.13) shows

$$[M_\lambda^*\psi](u) = \frac{1}{\rho(u)}\int_{-\infty}^{u} \psi(s)e^{\lambda(s-u)}\rho(s)\,ds$$

for $u < -r$. It is an easy computation to show that

$$\frac{1}{\rho(u)}\frac{d}{du}[\rho(u)[M_\lambda^*\psi](u)] = \psi(u) - \lambda[M_\lambda^*](u)$$

for a.e. $u < -r$. Thus, (ii) follows immediately. Statement (iii) follows upon inspection of (2.13) also. $\sqcap$

Corollary 2.4. If $\psi \in X^*$; then

(i) $R_\lambda(A)^*\psi$ is absolutely continuous on $[-r,0)$ with bounded variation, left continuous derivative,

(ii) $\rho R_\lambda(A)^*\psi$ is locally absolutely continuous on $(-\infty,-r)$ with

$$\frac{1}{\rho(s)}\frac{d}{ds}[\rho(s)[R_\lambda(A)^*\psi](s)]$$

essentially bounded on $(-\infty,-r)$, and

(iii) $\rho(-r^-)[R_\lambda(A)^*\psi](-r^-) = [R_\lambda(A)^*\psi]^{\bullet}(-r)$.

Proof: All three statements follow from (2.12), the corresponding statements of Lemma 2.3 and the form of δ^* given by (1.5). □

Theorem 2.5. Let A be as in §1. The adjoint, A^*, of A is given by

$$(2.14)\qquad (A^*\psi)(s) = \begin{cases} 0, & s = 0 \\ -\dot{\psi}(s) - \psi(0^-)\eta(s), & -r \le s < 0 \\ -\dfrac{1}{\rho(s)}\dfrac{d}{ds}[\rho(s)\psi(s)] - \psi(0^-)\eta(s), & s < -r \end{cases}$$

with $\mathcal{D}(A^*)$ consisting of exactly those $\psi \in X^*$ satisfying

$$(2.15)\quad \begin{cases} \text{(i) } \psi \text{ is absolutely continuous on } [-r,0) \text{ with bounded variation, left continuous derivative,} \\ \text{(ii) } \rho\psi \text{ is locally absolutely continuous on } (-\infty,-r) \text{ with } \rho^{-1}(\rho\dot{\psi}) \text{ essentially bounded on } (-\infty,-r), \text{ and} \\ \text{(iii) } \rho(-r^-)\psi(-r^-) = \dot{\psi}(-r). \end{cases}$$

<u>Proof</u>: If $\psi \in \mathcal{D}(A^*)$, then $\psi = R_\lambda(A)^*\bar{\psi}$ for some $\bar{\psi} \in X^*$. From Corollary 2.4 we see that ψ must satisfy (2.15).

On the other hand, if ψ satisfies (2.15) and $\varphi \in \mathcal{D}(A)$, then for any $k > r$

$$\langle\psi, A\varphi\rangle = \int_{-\infty}^{-r} \psi(u)\rho(u)\dot{\varphi}(u)du + \int_{-r}^{0} [d\psi(u)]\dot{\varphi}(u)$$

$$= \int_{-\infty}^{-k} \psi(u)\rho(u)\dot{\varphi}(u)du + \int_{-k}^{-r} \psi(u)\rho(u)\dot{\varphi}(u)du$$

$$+ \int_{-r}^{0^-} \dot{\psi}(u)\dot{\varphi}(u)du - \psi(0^-)\dot{\varphi}(0)$$

since ψ has a jump at $u = 0$. Integrating by parts, one obtains

$$(2.16)\quad \langle\psi, A\varphi\rangle = \int_{-\infty}^{-k} \psi(u)\rho(u)\dot{\varphi}(u)du + \psi(-r^-)\rho(-r^-)\varphi(-r)$$

$$- \psi(-k)\rho(-k)\varphi(-k) - \int_{-k}^{-r} (\psi(u)\rho(u))^{\cdot}\varphi(u)du$$

$$+ \dot{\psi}(0^+)\varphi(0) - \dot{\psi}(-r)\varphi(-r) - \int_{-r}^{0} [d\dot{\psi}(u)]\varphi(u)$$

$$- \psi(0^-)\langle\eta, \varphi\rangle$$

$$= \int_{-\infty}^{-k} \psi(u)\rho(u)\dot{\varphi}(u)du - \psi(-k)\rho(-k)\varphi(-k)$$

$$- \int_{-k}^{-r} (\psi(u)\rho(u))^{\cdot}\varphi(u)du - \int_{-r}^{0} [d\dot{\psi}(u)]\varphi(u)$$

$$- \langle\psi(0^-)\eta, \varphi\rangle$$

using the properties (2.15) of ψ. By elementary arguments it follows that we may let $k \to -\infty$ in (2.16) to obtain

$$(2.17)\quad \langle\psi, A\varphi\rangle = -\int_{-\infty}^{-r} [(\psi(u)\dot{\rho}(u)) + \psi(0^-)\eta(u)\rho(u)]\varphi(u)\,du$$

$$-\int_{-r}^{0} [d(\dot{\psi}(u) + \psi(0^-)\eta(u))]\varphi(u)$$

Since (2.17) holds for every $\varphi \in \mathcal{D}(A)$, we conclude that $\psi \in \mathcal{D}(A^*)$ and $A^*\psi$ is given as in (2.14). □

§3. The Adjoint Semi-group - General Theory

Before we continue, we briefly compile some relevant facts from the theory of function analytic semi-groups. As general references, we mention Phillips [31] and Hille and Phillips [18].

If $T(t)$, $t \geq 0$ is a strongly continuous semi-group of bounded linear operators on a Banach space E, then its infinitesimal generator, $\mathcal{A}$, is closed and densely defined. However, the adjoint $\mathcal{A}^*$ of $\mathcal{A}$ need not be densely defined. In general, $\mathcal{D}(\mathcal{A}^*)$ may be characterized as $\{\psi \in X^* \mid \lim_{t\to 0^+} t^{-1}\langle T^*(t)\psi - \psi, \varphi\rangle$ exists for all $\varphi \in E\}$. The limit is given by $\langle \mathcal{A}^*\psi, \varphi\rangle$.

Although $T^*(t)$, $t \geq 0$ defines a semi-group of bounded linear operators on E^* it is, in general, not strongly continuous in t on all of E^*. In fact, it is known that $\overline{\mathcal{D}(\mathcal{A}^*)}$ is the largest subspace of E^* on which $T^*(t)$, $t \geq 0$ is strongly continuous.

Definition 2.6. The space $\overline{\mathcal{D}(\mathcal{A}^*)}$ is called the adjoint space associated to the semi-group $T(t)$, $t \geq 0$ and will be denoted by E^+.

If we define $T^+(t) = T^*(t)|_{E^+}$, then $T^+(t)$, $t \geq 0$ is a strongly continuous semi-group of bounded linear operators on E^+. The infinitesimal generator $\mathcal{A}^+$ associated with $T^+(t)$, $t \geq 0$ is closed and densely defined in E^+. In fact, $\mathcal{A}^+ = \mathcal{A}^*|_{\mathcal{D}(\mathcal{A}^+)}$ where

$$\mathcal{D}(\mathcal{A}^+) = \{\psi \in \mathcal{D}(\mathcal{A}^*) | \mathcal{A}^*\psi \in E^+\}.$$

§4. Calculation of X^+

We now give characterizations of $\mathcal{D}(A^*)$ and X^+ derived from the semi-group associated with (2.1)-(2.2). Note that $\mathcal{D}(A^*)$, as described in Theorem 2.5 is independent of the operator L in (2.1). Thus, it suffices to consider the trivial FDE $\dot{x}(t) = 0$. The associated semi-group will be denoted by $S(t)$ and is given by

$$(2.18) \qquad [S(t)(\varphi)](u) = \begin{cases} \varphi(t+u), & u < -t \\ \varphi(0), & -t \leq u \leq 0. \end{cases}$$

The adjoint $S^*(t)$ is easily calculated. In fact, if $t > 0$, $\psi \in X^*$ and $\varphi \in X$, then

$$\langle\psi, S(t)\varphi\rangle = \int_{-\infty}^{0} [d\tilde{\psi}(u)][S(t)\varphi](u)$$

$$= \int_{-\infty}^{-t} [d\tilde{\psi}(u)]\varphi(t+u) + \int_{-t^+}^{0} [d\tilde{\psi}(u)]\varphi(0)$$

$$= \int_{-\infty}^{0} [d\tilde{\psi}(s - t)]\varphi(s) + [\tilde{\psi}(0) - \tilde{\psi}(-t^{+})]\varphi(0)$$

$$= \int_{-\infty}^{0} [d\{\tilde{\psi}_{-t}\}^{0}(s)]\varphi(s) + [\tilde{\psi}(-t^{-}) - \tilde{\psi}(-t^{+})]\varphi(0).$$

Thus

$$\text{(2.19)} \qquad \widetilde{[S^{*}(t)\psi]}(u) = \begin{cases} 0, & u = 0 \\ \tilde{\psi}(u - t) + [\tilde{\psi}(-t^{+}) - \tilde{\psi}(-t)]I, & u < 0. \end{cases}$$

This is also true when $t = 0$. For $u < -r$, (2.19) implies

$$\text{(2.20)} \qquad [S^{*}(t)\psi](u) = \frac{\rho(u - t)}{\rho(u)}\,\psi(u - t).$$

If we denote the associated infinitesimal generator as A_0, then A_0^{*} follows from (2.14) upon setting $\eta = 0$.

<u>Theorem 2.7</u>. Let A_0 be the infinitesimal generator associated with $S(t)$; $t \geq 0$ and A_0^{*} its adjoint. Let A be as in Theorem 2.5. Then

(i) $\mathcal{D}(A^{*}) = \mathcal{D}(A_0^{*}) = \{\psi \in X^{*} \mid \lim_{t\to 0^{+}} t^{-1}\langle S(t)\psi - \psi, \varphi\rangle$ exists for all $\varphi \in X\}$. X^{+} is the largest subspace of X^{*} on which $S^{*}(t)$ is strongly continuous.

(ii) If $\psi \in X^{*}$, then $\psi \in X^{+}$ if and only if ψ is absolutely continuous on $[-r,0)$ and the map associating $t \in [0,\infty)$ to the restriction of $\rho^{-1}\rho_{-t}\psi_{-t}$ to $(-\infty,-r)$ is continuous as a function from $[0,\infty)$ into $L^{\infty}(-\infty,-r)$.

<u>Proof</u>: Only (ii) requires further argument. The characterization of ψ on $[-r,0)$ may be found in the finite delay case in Henry [16] or derived from Theorem 1.4.9 of Butzer and Berens [4]. The characterization of ψ on $(-\infty,-r)$ is simply a restatement of the later portion of (i) taking (2.20) into account. $\square$

We remark at this point that if $\psi \in \mathcal{D}(A^*)$ then, as indicated in §3,

$$\lim_{t\to 0^+} t^{-1}\langle S^*(t)\psi - \psi,\varphi\rangle = \langle A_0^*\psi,\varphi\rangle.$$

Note also that if $\psi \in X^+$, the norm of ψ is given by

$$|\psi| = \max\{\operatorname*{ess\,sup}_{u<-r}|\psi(u)|,\ \int_{-r}^{0} |\dot{\psi}(u)|du + |\psi(0^-)|\}.$$

While X^+ does not depend on L, it does vary with ρ. The following examples illustrate the dependence.

<u>Example 2.8</u>. Let $k > 0$ and $\rho(u) = e^{ku}$. If $\psi \in \mathcal{D}(A^*)$, the requirement that $\rho^{-1}(\rho\dot{\psi})$ be essentially bounded on $(-\infty,-r)$ becomes that $\dot{\psi} + k\psi$ be essentially bounded on $(-\infty,-r)$. Thus $\dot{\psi}$ is essentially bounded on $(-\infty,-r)$ and elementary arguments show $X^+ = \{\psi \in X^* | \psi$ is uniformly continuous on $(-\infty,-r)$ and absolutely continuous on $[-r,0)$. The same result is true if $\rho^{-1}\dot{\rho}$ is essentially bounded.

<u>Example 2.9</u>. Let $k > 0$ and $\rho(u) = e^{-ku^2}$. If $\psi \in \mathcal{D}(A^*)$, then $\dot{\psi}(u) = 2ku\psi(u)$ must be essentially bounded on $(-\infty,-r)$. Since $\mathcal{D}(A^*)$ clearly contains all continuously differentiable functions with compact support, X^+ contains $\{\psi \in X^* \mid \psi$ is absolutely continuous on $[-r,0)$, uniformly continuous on $(-\infty,-r)$ and $\psi(u) \to 0$ as $u \to -\infty\}$. In fact, X^+ is precisely equal to this set. To see this, we define

$$\mu(u) = \frac{1}{\rho(u)} \frac{d}{du} (\rho(u)\psi(u))$$

for $u < -r$ and $\psi \in \mathcal{D}(A^*)$. Thus μ is essentially bounded on $(-\infty,-r)$. Since

$$\psi(u) = \frac{1}{\rho(u)} \int_{-\infty}^{u} \rho(s)\mu(s)ds$$

for $u < -r$, it follows that

$$|\psi(u)| \le \left(\frac{1}{\rho(u)} \int_{-\infty}^{u} \rho(s)ds\right) \cdot \text{constant}$$

$$= \left(\int_{-\infty}^{0} e^{-k(2u+s)s}\, ds\right) \cdot \text{constant},$$

which tends to zero as $u \to -\infty$. The same is true if

$$\frac{1}{\rho(u)} \int_{-\infty}^{0} \rho(s+u)ds \to 0 \tag{2.21}$$

as $u \to -\infty$.

§5. <u>The Second Adjoint Space - General Theory</u>

Let $T(t)$, $t \ge 0$ be the strongly continuous semi-group on E discussed in §3. If $T^+(t)$, $t \ge 0$ is the strongly continuous semi-group defined on E^+, then its adjoint space will be a subspace of $(E^+)^*$ and will be denoted by E^{++}. Although our representation of X^+ is not exact enough to give a precise representation of $(X^+)^*$ we can show that in some cases "$X = X^{++}$". To make this statement more precise we must introduce a new topology on X.

<u>Definition 2.10</u> [18]. For $\varphi \in E$ define

$$|\varphi|' = \sup\{|\langle\psi,\varphi\rangle| \mid \psi \in E^+, \quad |\psi| \le 1\}.$$

By Theorem 14.2.1, Hille and Phillips [18], $|\cdot|'$ is a norm on E equivalent to $|\cdot|$ if E^+ is total.

<u>Definition 2.11</u> [18]. The (+)-weak topology on E is defined by a neighborhood basis of the form

$$N(\varphi_0;\psi_1,\ldots,\psi_m;\epsilon) = \{\varphi \in E \mid |\langle\psi_k,\varphi - \varphi_0\rangle| < \epsilon$$
$$\text{for } k = 1,\ldots,m\}$$

where $\{\psi_1,\ldots,\psi_m\}$ is any finite subset of E^+ and ϵ is an arbitrary positive number.

A sequence $\{\varphi_i\} \subset E$ converges to $\varphi \in E$ in the (+)-weak topology if and only if $\langle\psi,\varphi - \varphi_i\rangle \to 0$ as $i \to \infty$ for every $\psi \in E^+$.

It is well known that there is a natural imbedding of E into E^{**} and that $T^{**}(t)$ defines a continuous extension of $T(t)$ from E to E^{**}. Since $E^+ \subset E^*$ we have $(E^+)^* \supset E^{**}$ and therefore E (by the natural injection) may be viewed as a subset of $(E^+)^*$. By "$E = E^{++}$" it will be meant that the natural imbedding of E into $(E^+)^*$ is an isometric isomorphism of E onto E^{++}. It is known that $E = E^{++}$ whenever E is reflexive. More generally,

Theorem 2.12 [18]. Let the norm of E be given by $|\cdot|'$. Then $E = E^{++}$ if and only if $R_\lambda(A)$ is (+)-weakly compact. That is, $R_\lambda(A)$ takes bounded subsets of E into (+)-weakly compact subsets of E.

§6. $X = X^{++}$?

This section considers the problem of determining when $X = X^{++}$ for our function spaces. For reflexive spaces of initial functions this equality is always valid. The arguments of the following theorem show also that the equality holds in those initial function spaces similar to X in which the term

$$\int_{-\infty}^{-r} |\varphi(u)|\rho(u)du$$

in the expression for $|\varphi|$ is replaced by

$$[\int_{-\infty}^{-r} |\varphi(u)|^p \rho(u)du]^{1/p}$$

with $p > 1$.

Theorem 2.13. Let X be given the norm $|\cdot|'$. If condition (2.21) holds, then $X = X^{++}$.

Proof: We shall apply Theorem 2.12.

Let λ be an element of the resolvent set of A. From (2.8), $R_\lambda(A)$ is a one-dimensional perturbation of M_λ defined by (2.6). Thus, it suffices to show that M_λ is (+)-weakly compact. Following Theorem 2.9.6 of Hille and Phillips [18], we need only show that for every bounded sequence $\{\varphi_i\} \subset X$ the set $\{M_\lambda \varphi_i\}$ has a subsequence that converges (+)-weakly to some element of X. To this end, we assume $k > 0$ and that $|\varphi_i| \leq k$ for $i \geq 1$.

Let $N > 0$ and $C([-N,0],\mathbb{R}^n)$ denote the Banach space of continuous, $\mathbb{R}^n$-valued functions on $[-N,0]$ with the supremum norm. It is well known that the mapping associating $\varphi \in X$ to the restriction of $M_\lambda \varphi$ to $[-N,0]$ is a compact map from X into $C([-N,0],\mathbb{R}^n)$. Thus, by a standard diagonalization argument, $\{M_\lambda \varphi_i\}$ has a subsequence $\mu_j = M_\lambda \varphi_{i_j}$; $j = 1,2,\ldots$ that converges uniformly on compact subsets of $(-\infty,0]$ to a continuous function, μ.

Claim. $\mu \in X$.

Proof of claim: Assume $N > r$. Then

$$\int_{-N}^{-r} |\mu(s)|\rho(s)ds + \sup_{[-r,0]} |\mu(s)|$$

$$\leq \int_{-N}^{-r} |\mu_j(s) - \mu(s)|\rho(s)ds + \sup_{[-r,0]} |\mu_j(s) - \mu(s)|$$

$$+ \int_{-N}^{-r} |\mu_j(s)|\rho(s)ds + \sup_{[r,0]} |\mu_j(s)| \leq 1 + |M_\lambda \varphi_{i_j}|$$

for all j sufficiently large ($\mu_j \to \mu$ uniformly on $[-N,0]$ as $j \to \infty$). Since $|M_\lambda \varphi_{i_j}| \leq \|M_\lambda\| \cdot k$ and $N > r$ was arbitrary we see that $|\mu| < \infty$. The claim is verified.

Finally, to show $\mu_j \to \mu$ (+)-weakly in X, we choose $\psi \in X^+$, $N > r$ and $\varepsilon > 0$. Then

$$\begin{aligned}
|\langle\psi, \mu_j - \mu\rangle| &\leq \int_{-\infty}^{-N} |\psi(s)|\,|\mu_j(s) - \mu(s)|\rho(s)ds \\
&\quad + \int_{-N}^{-r} |\psi(s)|\,|\mu_j(s) - \mu(s)|\rho(s)ds \\
&\quad + \sup_{[-r,0]} |\mu_j(s) - \mu(s)| \cdot |\psi| \\
&\leq \sup_{u \leq -N} |\psi(u)| \cdot [\,|\mu_j| + |\mu|\,] \\
&\quad + \int_{-N}^{-r} \psi(s)|\mu_j(s) - \mu(s)|\rho(s)ds \\
&\quad + \sup_{[-r,0]} |\mu_j(s) - \mu(s)| \cdot |\psi| .
\end{aligned}$$

By Example 2.9, condition (2.21) implies $|\psi(u)| \to 0$ as $u \to -\infty$. Since $|\mu_j| \leq \|M_\lambda\| \cdot k$, the first term may be made arbitrarily small (for all j) by choosing N large enough. The last two terms tend to 0 as $j \to \infty$ since $\mu_j \to \mu$ uniformly on compact subsets of $(-\infty,0]$. $\square$

<u>Example 2.14</u>. There are situations in which the space of initial functions is not reflexive in the semi-group sense. Consider the special case when $n = 1$, $r = \frac{1}{2}$ and $\rho(u) = e^u$. By Theorem 2.12 it suffices to show that $R_\lambda(A)$ is not (+)-weakly compact. Without loss of generality

we may assume that $\lambda = 0$ is in the resolvent set of A. The argument of the previous theorem shows that we need only show that M_0 is not (+)-weakly compact. To this end, define

$$\varphi_m(u) = \begin{cases} e^m, & u < -m \\ 0, & -m \le u \le 0 \end{cases}$$

for $m = 1, 2, \cdots$. Then $|\varphi_m| = 1$ for all m and

$$[M_0\varphi_m](n) = \begin{cases} -(m + u)e^m, & u < -m \\ 0, & -m \le u \le 0. \end{cases}$$

<u>Claim</u>. 0 is the only possible (+)-weak limit point of $\{M_0\varphi_m\}$.

<u>Proof of claim</u>: Let μ be a (+)-weak limit point of $\{M_0\varphi_m\}$ and $N > 1$ be arbitrary. Assume $\{\mu_i\} \subset \{M_0\varphi_m\}$ and $\mu_i \to \mu$ (+)-weakly. Then for any $\psi \in X^+$ with support in $[-N, 0]$, $\langle\psi, \mu_i - \mu\rangle = \langle\psi, -\mu\rangle$ for all $i > N$. Because $\langle\psi, \mu_i - \mu\rangle \to 0$ (by hypothesis) as $i \to \infty$ we must conclude that $\langle\psi, \mu\rangle = 0$. It follows easily that $\mu = 0$ on $[-N, 0]$. Since $N > 1$ was arbitrary, the claim is verified.

Finally, we note that 0 is not a (+)-weak limit point of $\{M_0\varphi_m\}$ since for $\psi \equiv 1 \in X^+$ we have

$$\langle\psi, M_0\varphi_m\rangle = -\int_{-\infty}^{-m} (m + u)e^{m+u}\, du$$

$$= -\int_{-\infty}^{0} se^s\, ds > 0.$$

§7. Representation of $T^+(t)$

In this section we seek a representation of the adjoint semi-group. This result will be reproved (by more complicated means) in the context of linear non-autonomous systems. However, the theory of semi-groups affords us a more direct proof in the autonomous case.

Our calculations become less tedious if the $\sim$ representation in X^+ is used. Thus, as preparation, we phrase some of the facts known about the adjoint semi-group in terms of that representation.

Lemma 2.15. Assume $\psi \in \mathcal{D}(A^*)$. Then

(i) $\widetilde{[A^*\psi]}(u) = -\frac{d}{du}[\tilde{\psi}(u)] - \psi(0^-)\tilde{\eta}(u)$

for $u < 0$. If also $\psi \in \mathcal{D}(A^+)$, then

(ii) $\frac{d}{dt}(T^+(t)\psi)$ exists for $t \geq 0$ and

$$\widetilde{\left[\frac{d}{dt}(T^+(t)\psi)\right]} = \frac{d}{dt}\,\widetilde{[T^+(t)\psi]}.$$

Proof: For $-r \leq u < 0$, (i) follows from (2.14). If $u < -r$, (2.14) gives

$$[A^*\psi](u) = -\frac{1}{\rho(u)}\frac{d}{du}[\rho(u)\psi(u)] - \psi(0^-)\eta(u).$$

Thus, using (2.14) and (2.15),

$$[\widetilde{A^* \psi}](u) = [A^* \psi](-r) - \int_u^{-r} \{- \frac{1}{\rho(s)} \frac{d}{ds} [\rho(s)\psi(s)]$$

$$- \psi(0^-)\eta(s)\}\rho(s)ds$$

$$= -\dot{\psi}(-r) - \psi(0^-)\eta(-r) + \int_u^{-r} (\rho(s)\dot{\psi}(s))ds$$

$$+ \psi(0^-)\int_u^{-r} \eta(s)\rho(s)ds$$

$$= -\dot{\psi}(-r) + \rho(-r)\psi(-r) - \rho(u)\psi(u) - \psi(0^-)\tilde{\eta}(u)$$

$$= -\rho(u)\psi(u) - \psi(0^-)\tilde{\eta}(u)$$

$$= - \frac{d}{du} [\tilde{\psi}(u)] - \psi(0^-)\tilde{\eta}(u).$$

As for (ii), the existence of

$$\frac{d}{dt} T^+(t)\psi = \lim_{h \to 0^+} h^{-1}[T^+(t + h)\psi - T^+(t)\psi]$$

for $\psi \in \mathcal{D}(A^+)$ may be found in Hille and Phillips [18] and is a consequence of the general theory of strongly continuous semi-groups. In fact,

$$\frac{d}{dt} T^+(t)\psi = A^+T^+(t)\psi = T^+(t)A^+\psi \tag{2.22}$$

for $t \geq 0$. Finally, Lemma 1.1 implies that for $t \geq 0$ and $u \leq 0$,

$$\frac{d}{dt} [\widetilde{T^+(t)\psi}](u) = \lim_{h \to 0^+} \frac{[\widetilde{T^+(t + h)\psi}](u) - [\widetilde{T^+(t)\psi}](u)}{h}$$

$$= \lim_{h \to 0^+} [\widetilde{h^{-1}(T^+(t + h)\psi - T^+(t)\psi)}](u)$$

$$= \widetilde{[\lim_{h \to 0^+} h^{-1}(T^+(t + h)\psi - T^+(t)\psi)](u)}$$

$$= \widetilde{[\frac{d}{dt} T^+(t)\psi]}(u). \quad \square$$

For $\psi \in X^*$, we consider the problem

$$y(t) + \int_0^t y(u)\tilde{\eta}(u - t)du = \tilde{\psi}(-t), \quad t > 0 \tag{2.23}$$

$$y(0) = \psi(0^-). \tag{2.24}$$

As shown in the finite delay case, (2.23)-(2.24) has a unique solution, y, defined for $t \geq 0$, that is of bounded variation on compact subsets of $[0, \infty)$. These solutions vary continuously with changes in the initial data in the sense that if $\psi_m \to \psi$ in X^*, then the corresponding solutions, y_m, of (2.23)-(2.24) with "initial data" ψ_m converge to the solution of (2.23)-(2.24) uniformly on compact subsets of $[0, \infty)$. See Hale [11].

We now state the principal result of this section. The argument follows closely that of Burns and Herdman [3] in their study of a semi-group associated with a linear integro-differential equation in a different function space setting.

<u>Theorem 2.16</u>. For $\psi \in X^+$, $T^+(t)\psi$ is defined for $t \geq 0$ by

$$[\widetilde{T^+(t)\psi}](s) = \tilde{\psi}(s - t) - \int_0^t y(u)\tilde{\eta}(u + s - t)du \tag{2.25}$$

for $s < 0$, where

(2.26) $$y(t) = [T^{+}(t)\psi](0^{-}), \quad t \geq 0$$

satisfies the adjoint equation

(2.27) $$y(t) = \tilde{\psi}(-t) - \int_0^t y(u)\tilde{\eta}(u - t)du$$

for $t > 0$ and

(2.28) $$y(0) = \psi(0^{-}).$$

<u>Proof</u>: By Lemma 1.1 and the continuous dependence of solutions to (2.27)-(2.28) on initial data, it suffices to show (2.25)-(2.28) for $\psi \in \mathcal{D}((A^{+})^{2})$, which is dense in X^{+} by Butzer and Berens [4]. The map associating $t \geq 0$ to $T^{+}(t)\psi$ is differentiable with Lipschitz continuous derivative.

For $s < 0$, $t \geq 0$ and $u \geq 0$ define $G(u) = [\widetilde{T^{+}(u)\psi}](u + s - t)$. By Lemmas 2.15 and 1.1, G is differentiable and, in fact,

$$\begin{aligned}\frac{d}{du} G(u) &= [\frac{d}{du} (\widetilde{T^{+}(u)\psi})](u + s - t) + \frac{d}{dv} [\widetilde{T^{+}(u)\psi}](v)\big|_{v=u+s+t} \\ &= [\widetilde{A^{+}T^{+}(u)\psi}](u + s - t) + \frac{d}{dv} [\widetilde{T^{+}(u)\psi}](v)\big|_{v=u+s-t}\end{aligned}$$

by Lemma 2.15 and (2.22). Thus

$$\begin{aligned}\frac{d}{du} G(u) = &- \frac{d}{dv} [\widetilde{T^{+}(u)\psi}](v)\big|_{v=u+s-t} \\ &- [T^{+}(u)\psi](0^{-})\tilde{\eta}(u + s - t) \\ &+ \frac{d}{dv} [\widetilde{T^{+}(u)\psi}](v)\big|_{v=u+s-t} = -y(u)\tilde{\eta}(u + s - t)\end{aligned}$$

where y is given by (2.26). Integrating over $[0,t]$,

$$G(t) - G(0) = -\int_0^t y(u)\tilde{\eta}(u + s - t)du.$$

Therefore, for $s < 0$,

$$[\widetilde{T^+(t)\psi}](s) - \tilde{\psi}(s - t) = -\int_0^t y(u)\tilde{\eta}(u + s - t)du.$$

Equation (2.28) follows from (2.27) by letting $s \to 0^-$. □

§8. The Adjoint Equation and Bilinear Form

In this section we will study a "differential" form of the adjoint equation and show that for a special class of $\psi \in X^+$ the solution of the adjoint equation actually solves a delay differential equation whose form is quite similar to (2.1). For this class of ψ the duality pairing $\langle\psi,\varphi\rangle$ will be seen to reduce to the classic bilinear form that has played such a prominent role in the theory of FDE's with finite delay, see Hale [11].

Lemma 2.17. If $\psi \in X^+$, the solution to (2.27)-(2.28) is locally absolutely continuous and solves

$$\dot{y}(t) = \int_{-t}^{0} y(t + u)d\tilde{\eta}(u) - \dot{\tilde{\psi}}(-t) \tag{2.29}$$

for $t > 0$ with the initial value, $y(0)$, given by (2.28).

Proof: If $\psi \in X^+$, Theorem 2.7 implies that $\tilde{\psi}$ is locally absolutely continuous on $(-\infty,0)$. The problem (2.28)-(2.29), viewed as a finite delay system of Caratheodory type, has a unique continuous solution on $[0,\infty)$ which is locally absolutely continuous on $(0,\infty)$. Integration by parts in (2.29) shows y to solve

$$\begin{aligned}\dot{y}(t) &= -y(0)\eta(-t) - \int_{-t}^{0} \dot{y}(t+s)\tilde{\eta}(s)ds + \frac{d}{dt}(\tilde{\psi}(-t)) \\ &= \frac{d}{dt}\left[-\int_{-t}^{0} y(t+s)\tilde{\eta}(s)ds + \tilde{\psi}(-t)\right]\end{aligned}$$

for $t > 0$. Therefore, for $t > 0$

$$y(t) = -\int_{-t}^{0} y(t+s)\tilde{\eta}(s)ds + \tilde{\psi}(-t) + \text{constant.}$$

Letting $t \to 0^+$ and using (2.28) we see that the constant is zero. Equation (2.27) is seen to be satisfied upon setting $u = t + s$ in the above integral. Thus, the solution to (2.28)-(2.29) is the unique solution to the adjoint equations (2.27)-(2.28). □

Consider the problem

$$\dot{z}(t) = \int_{-\infty}^{0} z(t+u)d\tilde{\eta}(u) \tag{2.30}$$

for $t > 0$ with initial condition given by

$$z_0 = \alpha \tag{2.31}$$

where $\alpha^T \in X$. Note that if $\psi \in X^+$ then (loosely speaking) $\dot{\tilde{\psi}}(-t) \to 0$ as $t \to \infty$. Thus (2.30) is the "limit equation" associated with (2.29), see Levin and Shea [24,25,26].

<u>Theorem 2.18</u>. If z solves (2.30)-(2.31), then there is a $\psi \in X^+$ for which z solves (2.28)-(2.29).

<u>Proof</u>: Clearly, (2.30)-(2.31) has a solution since (2.1)-(2.2) enjoys this property. From (2.30),

$$\dot{z}(t) = \int_{-t}^{0} z(t+u)d\tilde{\eta}(u) + \int_{-\infty}^{-t^-} z(t+u)d\tilde{\eta}(u)$$

for $t > 0$. Define, for $s < 0$

$$\omega(s) = \int_{-\infty}^{s^-} z(u-s)d\tilde{\eta}(u) = \int_{-\infty}^{0^-} \alpha(u)d\tilde{\eta}(u+s).$$

<u>Claim</u>. ω in integrable on $[-r,0]$ and $\rho^{-1}\omega$ is essentially bounded on $(-\infty,-r)$.

<u>Proof of claim</u>: Clearly, ω is measurable. For $-r \leq s < 0$,

$$\begin{aligned}
|\omega(s)| &\leq \int_{-\infty}^{0} |\alpha(u)|\cdot|d\tilde{\eta}(u+s)| \\
&\leq \int_{-\infty}^{-r-s} |\alpha(u)|\rho(u+s)|\eta(u+s)|du \\
&\quad + \int_{-r-s}^{0} |\alpha(u)|\cdot|d\tilde{\eta}(u+s)| \\
&\leq \int_{-\infty}^{0} |\alpha(u)|\rho(u)du \cdot \rho(s) \cdot \|L\| + \max_{[-r,0]} |\alpha(u)|\cdot\|L\| \\
&\leq \text{constant} \cdot |\alpha^T|
\end{aligned}$$

using (1.1) and (1.2). For $s < -r$,

$$\begin{aligned}
\rho^{-1}(s)|\omega(s)| &= \rho^{-1}(s)\left|\int_{-\infty}^{0^-} \alpha(u)\rho(u+s)\eta(u+s)ds\right| \\
&\leq \rho^{-1}(s)\int_{-\infty}^{0} |\alpha(u)|\rho(u)\rho(s)|\eta(u+s)|ds \\
&\leq \int_{-\infty}^{0} |\alpha(u)|\rho(u)du \cdot \|L\|.
\end{aligned}$$

The claim has been verified.

Thus, we can define an element $\psi \in X^*$ by

$$\dot{\psi}(s) = -\int_{-\infty}^{0^-} \alpha(u)d\tilde{\eta}(u + s) \tag{2.32}$$

for $s < 0$ and

$$\psi(0^-) = \alpha(0). \tag{2.33}$$

The function z is easily seen to solve (2.28)-(2.29) with ψ so defined.

To see that $\psi \in X^+$, we use Theorem 2.7. The fact that ψ is absolutely continuous on $[-r,0)$ was shown by the claim. Arguing in a manner similar to the claim one can show that for $t < \tau$ and $s < -r$

$$\begin{aligned} &|\rho^{-1}(s)\rho(t - s)\psi(t - s) - \rho^{-1}(s)\rho(\tau - s)\psi(\tau - s)| \\ &\quad \le \int_{-\infty}^{-\tau} |\alpha(t + u) - \alpha(\tau + u)|\rho(u)du \cdot \|L\| \\ &\quad + \max_{[-r,0]} |\alpha(u)| \cdot |t - \tau| \cdot \|L\| \cdot \rho(1). \end{aligned}$$

The conditions of Theorem 2.7 follow immediately. □

We remark that the proof of the previous theorem shows that the map that associates α to the element it defines via equations (2.32)-(2.33) is continuous when viewed as a function from X into X^*. In the future, we shall say "α defines ψ" or "ψ is defined by α" if ψ is given in terms of α by way of equations (2.32)-(2.33).

<u>Lemma 2.19</u>. Let $\alpha^T \in X$ and ψ be defined by α. Then, for any $\varphi \in X$,

$$-\langle\psi,\varphi\rangle = \alpha(0)\varphi(0) + \int_{-\infty}^{0}\int_{v}^{0} \alpha(v - s)d\tilde{\eta}(v)\varphi(s)ds. \tag{2.34}$$

<u>Proof</u>: For ψ above, $\tilde{\psi}$ is locally absolutely continuous on $(-\infty,0)$ and has a jump discontinuity at 0. Thus,

$$\begin{aligned}-\langle\psi,\varphi\rangle &= -\int_{-\infty}^{0} \dot{\tilde{\psi}}(s)\varphi(s)ds + \psi(0^-)\varphi(0) \\ &= \alpha(0)\varphi(0) + \int_{-\infty}^{0}\int_{-\infty}^{0^-} \alpha(u)[d\tilde{\eta}(u + s)]\varphi(s)ds\end{aligned}$$

by (2.32)-(2.33). Define

$$\nu(v) = \begin{cases} 0, & v \geq 0 \\ \alpha(v), & v < 0. \end{cases}$$

Then, using Fubini's Theorem [34],

$$\begin{aligned}-\langle\psi,\varphi\rangle &= \alpha(0)\varphi(0) + \int_{-\infty}^{0}\int_{-\infty}^{s^-} \alpha(u - s)[d\tilde{\eta}(u)]\varphi(s)ds \\ &= \alpha(0)\varphi(0) + \int_{-\infty}^{0} \{\int_{-\infty}^{0} \nu(u - s)[d\tilde{\eta}(u)]\}\varphi(s)ds \\ &= \alpha(0)\varphi(0) + \int_{-\infty}^{0} \{\int_{-\infty}^{0} \nu(u - s)[d\tilde{\eta}(u)]\varphi(s)ds\} \\ &= \alpha(0)\varphi(0) + \int_{-\infty}^{0}\int_{u}^{0} \alpha(u - s)[d\tilde{\eta}(u)]\varphi(s)ds\end{aligned}$$

since $\nu(u - s) = 0$ if $s \leq u$. □

We define the bilinear pairing (α,φ) between $\varphi \in X$ and $\alpha^T \in X$ by

$$(\alpha,\varphi) = \alpha(0)\varphi(0) + \int_{-\infty}^{0}\int_{u}^{0} \alpha(u - s)[d\tilde{\eta}(u)]\varphi(s)ds. \tag{2.35}$$

In the finite delay case, $\tilde{\eta}(u) = \eta(-r)$ for $u \leq -r$ and (α, φ) reduces to the classic bilinear form

$$\alpha(0)\varphi(0) + \int_{-r}^{0}\int_{u}^{0} \alpha(u - s)[d\tilde{\eta}(u)]\varphi(s)\,ds.$$

For the problem (2.30)-(2.31) we can define the solution semi-group analogous to that for (2.1)-(2.2). That is, $T^0(t)\alpha = z_t(\alpha)$, $t \geq 0$, where $z(\alpha)(\cdot)$ denotes the solution to (2.30)-(2.31). The connection between $T^0(t)$ and $T^+(t)$ is given by

<u>Theorem 2.20</u>. Let $\alpha^T \in X$, $\varphi \in X$ and $\psi \in X^+$ be defined by α. Then, for any $t \geq 0$, $T^0(t)\alpha$ defines $T^+(t)\psi$ and

$$-\langle T^+(t)\psi, \varphi\rangle = (T^0(t)\alpha, \varphi). \tag{2.36}$$

<u>Proof</u>: Let z solve (2.30)-(2.31). By Theorem 2.18 and Lemma 2.17, z solves the adjoint equations (2.27)-(2.28) and by Theorem 2.16

$$\begin{aligned}[\widetilde{T^+(t)\psi}](s) &= \tilde{\psi}(s - t) - \int_0^t z(v)\tilde{\eta}(v + s - t)\,dv \\ &= \tilde{\psi}(s - t) - \int_s^{s+t} z(u - s)\tilde{\eta}(u - t)\,du\end{aligned}$$

for $s < 0$, $t > 0$. Since ψ, $T^+(t)\psi \in X^+$, we may differentiate with respect to s to find

$$\begin{aligned}\frac{d}{ds}[\widetilde{T^+(t)\psi}](s) = \dot{\tilde{\psi}}(s - t) &- [z(t)\tilde{\eta}(s) - z(0)\tilde{\eta}(s - t)] \\ &+ \int_s^{s+t} \dot{z}(u - s)\tilde{\eta}(u - t)\,du\end{aligned}$$

$$= \dot{\tilde{\psi}}(s - t) - z(t)\tilde{\eta}(s) + z(0)\tilde{\eta}(s - t)$$

$$+ z(t)\tilde{\eta}(s^+) - z(0)\tilde{\eta}(s - t)$$

$$- \int_s^{s+t} z(u - s)d\tilde{\eta}(u - t)$$

using the Lebesgue-Stieltjes integration by parts formula. Thus,

$$\frac{d}{ds}\,[\widetilde{T^+(t)\psi}](s) = \dot{\tilde{\psi}}(s - t) + z(t)[\tilde{\eta}(s^+) - \tilde{\eta}(s^-)]$$

$$- \int_0^t z(u)d\tilde{\eta}(u - t + s)$$

$$= \dot{\tilde{\psi}}(s - t) - \int_0^{t^-} z(u)d\tilde{\eta}(u - t + s)$$

$$= -\int_{-\infty}^{0^-} \alpha(u)d\tilde{\eta}(u + s - t)$$

$$- \int_0^{t^-} z(u)d\tilde{\eta}(u - t + s)$$

by (2.32). Therefore, using the definition of $z(\cdot)$,

$$\frac{d}{ds}\,[\widetilde{T^+(t)\psi}](s) = -\int_{-\infty}^{0^-} z_t(u)d\tilde{\eta}(u + s)$$

for $s < 0$. Now, by (2.26), $[T^+(t)\psi](0^-) = z(t)$. Thus, $z_t(u)$ is seen to define $T^+(t)\psi$ and, using Lemma 2.19,

$$-\langle T^+(t)\psi, \varphi\rangle = (z_t, \varphi)$$

for any $\varphi \in X$. $\square$

Define A^0 to be the infinitesimal generator associated with $T^0(t)$, $t \geq 0$. It follows from Theorem 2.1

that $\mathcal{D}(A^O) = \{\alpha^T | \alpha \in \mathcal{D}(A)\}$. The connection between A^+ and A^O is similar to that relating $T^+(t)$ to $T^O(t)$ in the previous theorem.

Theorem 2.21. If $\alpha \in \mathcal{D}(A^O)$ and ψ is defined by α, then $\psi \in \mathcal{D}(A^+)$. In addition, $A^O\alpha$ defines $A^+\psi$ and

$$-\langle A^+\psi, \varphi\rangle = (A^O\alpha, \varphi) \tag{2.37}$$

for all $\varphi \in X$.

Proof: It can be shown exactly as in Hale [11], page 105, that if $\alpha \in \mathcal{D}(A^O)$ and $\varphi \in \mathcal{D}(A)$ then $(A^O\alpha, \varphi) = (\alpha, A\varphi)$. Thus, if $\nu \in X^+$ is defined by $A^O\alpha$, then $\langle \nu, \varphi\rangle = \langle \psi, A\varphi\rangle$ for every $\varphi \in \mathcal{D}(A)$. By definition of A^* we must conclude that $\psi \in \mathcal{D}(A^*)$ and $A^*\psi = \nu$. Because $\psi, A^*\psi \in X^+$ we see that $\psi \in \mathcal{D}(A^+)$. Thus $\psi \in \mathcal{D}(A^+)$ and $A^+\psi$ is defined by $A^O\alpha$. The last assertion follows immediately from Lemma 2.19. □

§9. Decomposition of X and X^*

Let $\gamma(t)$ be defined by (2.4). By Theorem 2.2, if λ solves (2.3) and $|e^{\lambda t}| > \gamma(t)$, then the generalized eigenspace $\mathcal{N}(A - \lambda I)^k$ is finite dimensional and

$$X = \mathcal{N}(A - \lambda I)^k \oplus \mathcal{R}(A - \lambda I)^k.$$

In this section we consider the problem of computing a projection of X onto $\mathcal{N}(A - \lambda I)^k$. We make use of the bilinear form given by equation (2.35). Since the arguments closely parallel those of Hale [11], Section 21, we omit the proofs whenever possible. See Naito [28] for a different approach to calculating the projections.

As in Hale [12] and Naito [29] we define the $(nk) \times (nk)$ matrix A_k by

$$A_k = \begin{pmatrix} P_1 & P_2 & \cdots & P_k \\ 0 & P_1 & & P_{k-1} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & P_1 \end{pmatrix}$$

where, for $j = 0,1,2,\ldots,$

$$P_{j+1} = \frac{1}{j!} \frac{d^j}{d\lambda^j} \Delta(\lambda).$$

The first assertion of the following lemma was shown in Naito [29]. The characterization of $\mathcal{N}(A^0 - \lambda I)^k$ follows by similar arguments.

<u>Lemma 2.22</u>. (i) $\varphi \in \mathcal{N}(A - \lambda I)^k$ if and only if φ is of the form

$$\varphi(u) = \sum_{j=0}^{k-1} a_{j+1} \frac{u^j}{j!} e^{\lambda u}$$

where $a = \mathrm{col}(a_1,\ldots,a_k)$ satisfies $A_k a = 0$.

(ii) $\alpha \in \mathcal{N}(A^0 - \lambda I)^k$ if and only if α is of the form

$$\alpha(u) = \sum_{j=0}^{k-1} b_{j+1} \frac{u^j}{j!} e^{\lambda u}$$

where $b = \mathrm{row}(b_k,b_{k-1},\ldots,b_1)$ satisfies $bA_k = 0$.

Lemma 2.23. $\varphi \in \mathcal{R}(A - \lambda I)^k$ if and only if $(\alpha, \varphi) = 0$ for every $\alpha \in \mathcal{N}(A^O - \lambda I)^k$.

Proof: See Lemma 21.2 of Hale [11]. The necessary modifications are obvious. □

Theorem 2.24. For $\lambda \in \sigma(A)$ satisfying $|e^{\lambda t}| > \gamma(t)$, one has $\dim \mathcal{N}(A - \lambda I)^k = \dim \mathcal{N}(A^O - \lambda I)^k$. If $\Phi_\lambda = (\varphi_1, \ldots, \varphi_p)$ and $\Omega_\lambda = \mathrm{col}(\alpha_1, \ldots, \alpha_p)$ are basis "vectors" for $\mathcal{N}(A - \lambda I)^k$ and $\mathcal{N}(A^O - \lambda I)^k$, respectively, then $(\Omega_\lambda, \Phi_\lambda) = [(\alpha_i, \varphi_j)]$ is nonsingular and thus may be taken as the identity. The projection $\Pi_\lambda : X \to \mathcal{N}(A - \lambda I)^k$ is given by

$$\Pi_\lambda \varphi = \Phi_\lambda (\Omega_\lambda, \varphi).$$

Proof: See Lemma 21.4 of Hale [11]. □

Corollary 2.25. Let Φ_λ, Ω_λ be as in Theorem 2.24. Let ψ_i be defined by $-\alpha_i$; $i = 1, 2, \ldots, p$, and $\Psi_\lambda = \mathrm{col}(\psi_1, \ldots, \psi_p)$. The projection Π_λ is given by

$$\Pi_\lambda \varphi = \Phi_\lambda \langle \Psi_\lambda, \varphi \rangle.$$

Proof: This is an immediate consequence of Lemma 2.19 and the previous theorem. □

Corollary 2.26. Let Φ_λ, Ω_λ, Ψ_λ be as in Corollary 2.25. Then $\dim \mathcal{N}(A^* - \lambda I)^k$ is equal to $\dim \mathcal{N}(A - \lambda I)^k$

and Ψ_λ defines a basis for $\mathcal{N}(A^* - \lambda I)^k$. The projection $\Pi_\lambda^*: X^* \to \mathcal{N}(A^* - \lambda I)^k$ is onto and is given by

$$\Pi_\lambda^* \psi = \langle \psi, \Phi_\lambda \rangle \Psi_\lambda.$$

<u>Proof</u>: Let $1 \leq i \leq p$ and $-\alpha_i$ define ψ_i. It is an easy consequence of Theorem 2.21 that $(A^* - \lambda I)^m \psi_i \in \mathcal{D}(A^*)$ for $m = 0, 1, \ldots, k-1$ and $\langle (A^* - \lambda I)^k \psi_i, \varphi \rangle =$ $-(-(A^O - \lambda I)^k \alpha_i, \varphi) = 0$ for every $\varphi \in X$. Thus, $\psi_i \in \mathcal{N}(A^* - \lambda I)^k$. Because

$$\langle \psi_i, \varphi_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases}$$

it follows that $\{\psi_1, \psi_2, \ldots, \psi_p\}$ is a linearly independent set.

To show that $\mathcal{N}(A^* - \lambda I)^k$ is spanned by $\{\psi_1, \ldots, \psi_p\}$ it suffices to show that for any $\psi \in \mathcal{N}(A^* - \lambda I)^k$ and $\varphi \in X$, $\langle \Psi_\lambda, \varphi \rangle = 0$ implies $\langle \psi, \varphi \rangle = 0$. However, since $\langle \Psi_\lambda, \varphi \rangle =$ $-(\Omega_\lambda, \varphi)$, we may apply Lemma 2.23 to conclude that any φ satisfying $\langle \Psi_\lambda, \varphi \rangle = 0$ must lie in $\mathcal{R}(A - \lambda I)^k$. Thus, $\varphi = (A - \lambda I)^k \nu$ for some $\nu \in X$ and $\langle \psi, \varphi \rangle =$ $\langle \psi, (A - \lambda I)^k \nu \rangle = \langle (A^* - \lambda I)^k \psi, \nu \rangle = 0$.

Finally, if $\varphi \in X$ and $\psi \in X^*$,

$$\begin{aligned} \langle \Pi_\lambda^* \psi, \varphi \rangle &= \langle \psi, \Pi_\lambda \varphi \rangle = \langle \psi, \Phi_\lambda \langle \Psi_\lambda, \varphi \rangle \rangle = \langle \psi, \Phi_\lambda \rangle \langle \Psi_\lambda, \varphi \rangle \\ &= \langle \langle \psi, \Phi_\lambda \rangle \Psi_\lambda, \varphi \rangle. \end{aligned}$$

Thus, $\Pi_\lambda^* \psi = \langle \psi, \Phi_\lambda \rangle \Psi_\lambda$. Clearly, $(\Pi_\lambda^*)^2 = \Pi_\lambda^*$ and $\mathcal{R}(\Pi_\lambda^*) = \mathcal{N}(A^* - \lambda I)^k$. $\square$

CHAPTER III

GENERAL LINEAR SYSTEMS

§1. Existence, Uniqueness and Continuous Dependence

Consider the linear nonhomogeneous system

$$\dot{x}(t) = L(t,x_t) + h(t) \tag{3.1}$$

for $t > \sigma$, with initial data given by

$$x_\sigma = \varphi \in X. \tag{3.2}$$

The associated homogeneous system is

$$\dot{x}(t) = L(t,x_t). \tag{3.3}$$

Throughout this chapter it will be assumed that $L:\mathbb{R} \times X \to \mathbb{R}^n$ is continuous in each variable and linear in the second, while h will be locally integrable. These hypotheses are by no means minimal but suffice for our subsequent applications. It can be easily shown that $L(t,\bar{\varphi})$ is continuous in $(t,\bar{\varphi})$.

As in the autonomous case, $L(t,\cdot)$ may be represented in integral form. In particular, there exists for each t an $n \times n$ matrix valued function $\eta(t,\cdot):\mathbb{R} \to \mathbb{R}^{n \times n}$ satisfying

(i) $L(t,\bar{\varphi}) = \int_{-r}^{0} [d_u\eta(t,u)]\bar{\varphi}(u) + \int_{-\infty}^{-r} \eta(t,u)\bar{\varphi}(u)\rho(u)du$ for every $\bar{\varphi} \in X$,

(ii) each row of $\eta(t,u)$ is of bounded variation in u on $[-r,0]$ and left continuous in u on $[-r,0)$,

(iii) $\eta(t,u) = 0$ for $u \geq 0$, and

(iv) $\eta(t,u)$ is measurable in u and essentially bounded on $(-\infty,-r)$.

As in the previous chapters, we define

$$\tilde{\eta}(t,u) = \begin{cases} \eta(t,u), & u \geq -r \\ \eta(t,-r) - \int_{u}^{-r} \eta(t,s)\rho(s)ds, & u < -r \end{cases}$$

and therefore represent L by

$$L(t,\bar{\varphi}) = \langle\eta(t,\cdot),\bar{\varphi}\rangle = \int_{-\infty}^{0} [d_u\tilde{\eta}(t,u)]\bar{\varphi}(u).$$

It follows from Kato [21] that $\tilde{\eta}(t,u)$ is measurable in (t,u).

<u>Theorem 3.1</u>. Under the above assumptions on L and h, the initial value problem (3.1)-(3.2) has a unique solution $x(\cdot) = x(\sigma,\varphi,h)(\cdot)$ defined on (σ,∞). The solution depends continuously on φ and h in the sense that if $\tau > \sigma$, $\varphi^{(n)} \to \varphi$ in X and $h^{(n)} \to h$ in $L^1[\sigma,\tau]$, then $x(\sigma,\varphi^{(n)},h^{(n)}) \to x(\sigma,\varphi,h)$ uniformly on $[\sigma,\tau]$.

<u>Proof</u>: First, consider the problem of solving (3.1)-(3.2) on the interval $[\sigma, \sigma + r]$. Any solution of (3.1)-(3.2) on this interval corresponds to a solution of the finite delay equation

$$\dot{x}(t) = \int_{-r}^{0} [d_u \eta(t,u)] x_t(u) + g(t) \tag{3.4}$$

where

$$g(t) = \int_{-\infty}^{-r} [d_u \tilde{\eta}(t,u)] \varphi_{t-\sigma}(u) + h(t) \tag{3.5}$$

and $x_\sigma \in C([-r,0], \mathbb{R}^n)$ is defined by

$$x_\sigma = \varphi|_{[-r,0]}. \tag{3.6}$$

Note that g is clearly an element of $L^1[\sigma, \sigma + r]$. By the assumptions on L and h, Theorems 16.1 and 5.1 of Hale [11] apply to (3.4)-(3.6) and the conclusions of this theorem are therefore valid for $\sigma \leq \tau \leq \sigma + r$.

The argument may now be repeated on successive intervals $[\sigma + r, \sigma + 2r]$, $[\sigma + 2r, \sigma + 3r], \ldots$ to obtain the full conclusion of the theorem. $\square$

By the theorem just proved, we may define for $t \geq \sigma$ a linear map $T(t,\sigma): X \to X$ by $T(t,\sigma)\varphi = x_t(\sigma,\varphi,0)$. In fact, the following lemmas show that this operator is continuous.

<u>Lemma 3.2</u>. Assume $\sigma \leq t$ and $x: (-\infty, t] \to \mathbb{R}^n$ is continuous on $[\sigma, t]$ and satisfies $x_\sigma \in X$. Then there

exists a constant $c > 0$ (dependent only on ρ) such that

$$|x_t| \leq c|x_\sigma| + c \max_{[\sigma,t]} |x(u)|.$$

<u>Proof</u>: $$|x_t| = \max_{[-r,0]} |x_t(u)| + \int_{-\infty}^{-r} |x(t+u)|\rho(u)du$$

$$= \max_{[-r,0]} |x_t(u)| + \int_{-\infty}^{-r} |x_\sigma(s)|\rho(s+\sigma-t)ds$$

$$+ \int_{\sigma-t-r}^{-r} |x_t(u)|\rho(u)du \leq \max_{[-r,0]} |x_t(u)|$$

$$+ \rho(\sigma-t)\int_{-\infty}^{r} |x_\sigma(s)|\rho(s)ds$$

$$+ \int_{\sigma-t-r}^{-r} |x(t+u)|\rho(u)du$$

by property (1.1) of ρ. If $\sigma < t \leq \sigma + r$, then

$$\max_{[-r,0]} |x(t+u)| + \int_{\sigma-t-r}^{-r} |x(t+u)|\rho(u)du$$

$$\leq \max_{[\sigma,t]} |x(u)| + [1 + \int_{\sigma-t-r}^{-r} \rho(u)du] \cdot \max_{[-r,0]} |x_\sigma(u)|.$$

If $\sigma + r < t$, then

$$\max_{[-r,0]} |x(t+u)| + \int_{\sigma-t-r}^{-r} |x(t+u)|\rho(u)du$$

$$\leq \max_{[-r,0]} |x_\sigma(u)| \cdot \int_{-r}^{0} \rho(\sigma-t+s)ds$$

$$+ [1 + \int_{\sigma-t}^{-r} \rho(u)du] \cdot \max_{[\sigma,t]} |x(u)|.$$

In either case the conclusion of the lemma follows from (1.2) with

$$c = \max\{\rho(1), 1 + \int_{-\infty}^{0} \rho(u)du\}. \quad \square$$

By the Uniform Boundedness Principle it follows that for any fixed $\sigma \in \mathbb{R}$ and $\tau \geq \sigma$, $\sup\{\|L(s,\cdot)\| \,|\, \sigma \leq s \leq \tau\}$ is finite. Define m to be a real, locally bounded, continuous function such that $\|L(s,\cdot)\| \leq m(s)$ for $s \in \mathbb{R}$.

Lemma 3.3. Let c be as in the previous lemma. If $x = x(\sigma,\varphi,0)$ is the solution to (3.2)-(3.3) on $[\sigma,\infty)$, then for any $t \geq \sigma$,

$$|x_t| \leq |x_\sigma| \cdot 2c\, \mathrm{Exp}[c\int_\sigma^t m(s)ds]. \tag{3.7}$$

Proof: For any $u \in [\sigma,t]$

$$x(u) = x(\sigma) + \int_\sigma^u L(s,x_s)ds.$$

Therefore

$$|x(u)| \leq |x(\sigma)| + \int_\sigma^u m(s)|x_s|ds$$
$$\leq |x_\sigma| + \int_\sigma^t m(s)|x_s|ds$$

and, consequently,

$$\max_{[\sigma,t]} |x(u)| \leq |x_\sigma| + \int_\sigma^t m(s)|x_s|ds.$$

The previous lemma implies

$$|x_t| \leq 2c|x_\sigma| + c\int_\sigma^t m(s)|x_s|ds$$

and (3.7) now follows from Gronwall's Inequality. See Coppel [7], page 19. □

<u>Corollary 3.4</u>. $T(t,\sigma)$ is a bounded linear operator on X with

$$\|T(t,\sigma)\| \leq 2c \cdot \mathrm{Exp}[c\int_{\sigma}^{t} m(s)ds].$$

§2. <u>Representation of Solutions</u>

In this section we will generalize the representation theorem of Banks [1] known for FDE's with finite delay. Fortunately, many of the results of this section can be derived directly from the theory of finite delay equations.

Consider the problem

$$z(s,t) + \int_{s}^{t} z(u,t)\tilde{\eta}(u,s-u)du = \tilde{\Psi}(t-s) \tag{3.8}$$

for $s < t$,

$$z(s,t) = 0 \tag{3.9}$$

for $s > t$, and

$$z(t,t) = \psi(0^-), \tag{3.10}$$

where $\psi \in X^*$.

<u>Lemma 3.5</u>. For any $\psi \in X^*$ there exists a unique solution, z, of (3.8)-(3.10) that is locally of bounded variation in s.

<u>Proof</u>: See Theorem 32.1 of Hale [11]. □

The assertions of the following lemma are verified in the proof of Theorem 32.2 of Hale [11].

Lemma 3.6. There exists a unique $n \times n$ matrix function $Y(s,t)$ satisfying

$$Y(s,t) + \int_s^t Y(u,t)\tilde{\eta}(u,s-u)du = I \tag{3.11}$$

for $s \leq t$, and

$$Y(s,t) = 0 \tag{3.12}$$

for $s > t$, where I is the $n \times n$ identity matrix. In addition, $Y(s,t)$ is locally absolutely continuous in t (except at $t = s$) and locally of bounded variation in s.

Theorem 3.7 (Representation Theorem). If x solves (3.1)-(3.2) on (σ,∞), then for any $t \geq \sigma$

$$\begin{aligned} x(t) = Y(\sigma,t) &+ \int_{-\infty}^{\sigma^-} [d_u\{\int_\sigma^t Y(s,t)\tilde{\eta}(s,u-s)ds\}]x(u) \\ &+ \int_\sigma^t Y(u,t)h(u)du \end{aligned} \tag{3.13}$$

where Y satisfies (3.11)-(3.12).

Proof: Assume first that $\varphi \in X$ is continuous on $(-\infty,0]$ and choose $R > t - \sigma + r$. Then x corresponds to the solution of

$$\begin{aligned} \dot{x}(t) = \int_{-R}^{0} [d_s\tilde{\eta}(t,s)]x_t(s) &+ \int_{-\infty}^{-R} \eta(t,s)\rho(s)x_t(s)ds \\ &+ h(t) \end{aligned} \tag{3.14}$$

with initial value

$$x_\sigma|_{[-R,0]} = \varphi|_{[-R,0]}. \tag{3.15}$$

The assumptions on η and h allow us to apply Theorem 32.2, Hale [11], to (3.14)-(3.15) and conclude

$$\begin{aligned} x(t) &= Y(\sigma,t)x(\sigma) + \int_{\sigma-R}^{\sigma^-} [d_u\{\int_\sigma^t Y(s,t)\tilde{\eta}(s,u-s)ds\}]x(u) \\ &\quad + \int_\sigma^t Y(u,t)h(u)du \\ &\quad + \int_\sigma^t Y(u,t)[\int_{-\infty}^{-R} \eta(u,s)\rho(s)x(u+s)ds]du \end{aligned} \tag{3.16}$$

where Y is given in the previous lemma. Using property (1.2) of ρ,

$$\begin{aligned} |\int_{-\infty}^{-R} \eta(u,s)\rho(s)x(u+s)ds| &\le m(u)\int_{-\infty}^{-R+u-\sigma} \rho(v+\sigma-u)\,|x_\sigma(v)|\,dv \\ &\le m(u)\rho(\sigma-u)\int_{-\infty}^{-R+t-\sigma} \rho(v)\,|x_\sigma(v)|\,dv \end{aligned}$$

since $\sigma \le u \le t$. Thus, the last integral in (3.16) is seen to tend to 0 as $R \to +\infty$. Equation (3.14) is verified.

Since the continuous elements of X are dense in X, the full assertion of the theorem will be proved once

$$\begin{aligned} x_\sigma \to &\int_{-\infty}^{\sigma-r} [d_u\{\int_\sigma^t Y(s,t)\tilde{\eta}(s,u-s)ds\}x(u) \\ &= \int_{-\infty}^{\sigma-r} \{\int_\sigma^t Y(s,t)\eta(s,u-s)\rho(u-s)ds\}x(u)du \end{aligned}$$

is shown to be continuous from X into $\mathbb{R}^n$. By linearity, it suffices to show continuity at $x_\sigma = 0$. However,

$$\left|\int_{-\infty}^{\sigma-r} \{\int_{\sigma}^{t} Y(s,t)\eta(s,u-s)\rho(u-s)ds\}x(u)du\right|$$

$$\leq \int_{-\infty}^{\sigma-r} [\int_{\sigma}^{t} |Y(s,t)|m(s)\rho(\sigma-s)ds]\rho(u-\sigma)|x(u)|du$$

$$\leq \int_{-\infty}^{-r} \rho(v)|x_\sigma(v)|dv \cdot \int_{\sigma}^{t} |Y(s,t)|m(s)\rho(\sigma-s)ds$$

$$\leq |x_\sigma| \cdot \text{constant}$$

since $Y(s,t)$, $m(s)$ and $\rho(\sigma - s)$ are bounded for $\sigma \leq s \leq t$. □

Define the operator $K(t,\sigma):L^1[\sigma,t] \to X$ by

$$(3.17) \quad [K(t,\sigma)h](u) = \begin{cases} 0, & t+u < \sigma \\ \int_{\sigma}^{t+u} Y(s,t+u)h(s)ds, & \sigma \leq t+u \leq t. \end{cases}$$

By the previous theorem, the solution $x_t(\sigma,\varphi,h)$ may be written as

$$x_t(\sigma,\varphi,h) = T(t,\sigma)\varphi + K(t,\sigma)h = x_t(\sigma,\varphi,0) + x_t(\sigma,0,h).$$

The continuous dependence assertions of Theorem 3.1 imply that $K(t,\sigma)$ is a bounded operator from $L^1(\sigma,t)$ into X.

§3. The Adjoint Problem

Since $T(t,\sigma):X \to X$ and $K(t,\sigma):L^1[\sigma,t] \to X$ have been shown to be bounded and linear, their adjoints $T^*(t,\sigma):X^* \to X^*$ and $K^*(t,\sigma):X^* \to L^\infty[\sigma,t]$ are bounded and linear. In this section we will obtain representations for these operators.

Theorem 3.8. For $\psi \in X^*$, $T^*(t,\sigma)\psi$ is defined by

$$(3.18) \qquad [\widetilde{T^*(t,\sigma)\psi}](u) = \tilde{\psi}(u + \sigma - t) - \int_{\sigma}^{t} y(s,t)\tilde{\eta}(s,\sigma + u - s)ds$$

for $u < 0$, where

(3.19) $y(s,t) = [T^*(t,s)\psi](0^-)$ solves the "adjoint equation"

$$(3.20) \qquad y(s,t) = -\int_{s}^{t} y(u,t)\tilde{\eta}(u,s - u)du + \tilde{\psi}(s - t)$$

for $s < t$ and

$$(3.21) \qquad y(t,t) = \psi(0^-).$$

Proof: The theorem may be proved in a manner similar to the proof of Theorem 33.1 of Hale [11]. □

Also similar in proof to its counterpart from the theory of FDE's with finite delay is

Corollary 3.9. For any $\psi \in X^*$, $K^*(t,\sigma)\psi \in L^\infty[\sigma,t]$ is defined by

$$(3.22) \qquad [K^*(t,\sigma)\psi](s) = -[T^*(t,s)\psi](0^-)$$

$$= -\int_{-\infty}^{0} [d\tilde{\psi}(u)]Y(s,t + u)$$

for almost every $s \in [\sigma,t]$.

It should be remarked that since $[T^*(t,s)\psi](0^-)$ is the solution to the adjoint equation (3.20) for $s < t$,

$[K^*(\sigma,t)\psi](s)$ is actually of bounded variation in s on $[\sigma,t]$.

Corollary 3.10. Let I denote the $n \times n$ identity matrix and define

$$\delta(u) = \begin{cases} 0, & u < 0 \\ I, & u = 0 \end{cases}$$

and

$$\delta^*(u) = \begin{cases} I, & -r \leq u < 0 \\ 0, & u = 0 \text{ or } u < -r. \end{cases}$$

Then

$$(3.23) \qquad [T(t,\sigma)\delta](u) = Y(\sigma,t+u) = [T^*(t+u,\sigma)\delta^*](0^-)$$

for $\sigma \leq t$ and $u \leq 0$.

Proof: Clearly, the solution operator $T(t,\sigma)\varphi$ is defined for such φ as are only piecewise continuous on $[-r,0]$ (but otherwise satisfy the requirements needed in order to belong to X). Thus, $T(t,\sigma)\delta$ makes sense. For any $\psi \in X^*$

$$\begin{aligned} \langle \psi, Y(\sigma,t+\cdot)\rangle &= \int_{-\infty}^{0} [d\tilde{\psi}(u)]Y(\sigma,t+u) \\ &= \langle T^*(t,\sigma)\psi, \delta\rangle \end{aligned}$$

by (3.22). Therefore, $\langle \psi, Y(\sigma,t+\cdot)\rangle = \langle \psi, T(t,\sigma)\delta\rangle$ for every $\psi \in X^*$. It follows that for almost every $u \leq 0$,

$Y(\sigma, t+u) = [T(t,\sigma)\delta](u)$. Since both sides of this equality are continuous for $t + u < \sigma$ (where both are 0) and $t \geq t + u \geq \sigma$, the equality holds pointwise.

The final assertion follows from Lemma 3.6 and Theorem 3.8 applied to δ^*. In fact, $[T^*(t+u,\sigma)\delta^*](0^-)$ solves (by equations (3.20)-(3.21)) equations (3.11)-(3.12) and, therefore, $[T^*(t+u,\sigma)\delta^*](0^-) = Y(t+u,\sigma)$. □

In light of (3.13), (3.17) and the previous corollary, the solution $x_t(\sigma,\varphi,h)$ of (3.1)-(3.2) may be expressed in "Variation of Constants" form

$$x_t = T(t,\sigma) + \int_\sigma^t T(t,u)\delta h(u)du \tag{3.24}$$

where for $s \leq 0$,

$$\left[\int_\sigma^t T(t,u)\delta h(u)du\right](s) = \int_\sigma^t [T(t,u)\delta](s)h(u)du. \tag{3.25}$$

Finally, we remark in the autonomous case that $T(t,\sigma) = T(t-\sigma)$, while the solution $y(s,t)$ of the adjoint equations (3.20)-(3.21) satisfies $y(s,t) = y(t-s)$. Without loss of generality we may set $s = 0$ and $T(t)$, $t \geq 0$ now corresponds to the strongly continuous semigroup studied in the previous chapter. For the nonhomogeneous autonomous case (3.24) reduces to

$$x_t = T(t)\varphi + \int_0^t T(t-u)\delta h(u)du. \tag{3.26}$$

Comparing Theorems 2.16 and 3.8 we see that the representation of the adjoint semi-group operators $T^{+}(t)$, $t \geq 0$ extends in an unaltered form to the representation of the adjoint operators $T^{*}(t)$, $t \geq 0$ defined on X^{*}.

§4. Normal Eigenvalues of the Solution Operator

In this section we will discuss some of the spectral properties of $T(t,s)$. For example, we will show that there is a real number $r_{\rho}(t - s)$, $0 \leq r_{\rho} \leq 1$, such that for any $\varepsilon > 0$, $\sigma(T(t,s)) \cap \{\lambda \in C \mid |\lambda| \geq r_{\rho} + \varepsilon\}$ is finite and consists (if it is not empty) entirely of normal eigenvalues of $T(t,s)$.

Definition 3.11 [9]. A complex number, λ, is said to be a normal eigenvalue of a bounded, linear operator, T, on a Banach space, E, provided it is an eigenvalue of T with finite dimensional generalized eigenspace, $\mathcal{N}(T - \lambda I)^{k}$, and

$$E = \mathcal{N}(T - \lambda I)^{k} \oplus \mathcal{R}(T - \lambda I)^{k}$$

where $\mathcal{R}(T - \lambda I)^{k}$ is invariant under T. A point, λ, is called a normal point of T if it is either a normal eigenvalue of T or in the resolvent set of T.

Definition 3.12 [2]. The essential spectrum, $\mathrm{ess}(T)$, is defined to be the set of all λ in the spectrum of T, $\sigma(T)$, for which at least one of the following holds:

(i) $\mathcal{R}(T - \lambda I)$ is not closed;

(ii) $\lambda \in \overline{\sigma(T) \setminus \{\lambda\}}$;

(iii) $\bigcup_{k\geq 1} \mathcal{N}(T - \lambda I)^k$ is infinite dimensional.

We define the <u>essential radius,</u> $r_e(T)$, of T as

$$r_e(T) = \sup\{|\lambda| \,|\, \lambda \in \mathrm{ess}(T)\}.$$

The following result of Gohberg and Kreĭn [9] shows that the normal points of T are precisely those points not in $\mathrm{ess}(T)$.

<u>Lemma 3.13</u>. A necessary and sufficient condition that λ be a normal eigenvalue of T is that

(i) $\mathcal{R}(T - \lambda I)$ is closed,

(ii) λ is an isolated point of $\sigma(T)$, and

(iii) the generalized eigenspace associated with λ is finite dimensional.

Also in Gohberg and Kreĭn [9] may be found this result concerning the behavior of the set of normal points when the operator is perturbed by a completely continuous operator.

<u>Lemma 3.14</u>. Let $\mathcal{J}:E \to E$ be bounded and linear and $U:E \to E$ be completely continuous and linear. Any unbounded connected component of normal points of $\mathcal{J}$ is a connected component of normal points for $\mathcal{J} + U$.

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Definition 3.15 [8,33]. Kuratowski's measure of noncompactness, $\alpha(B)$, of a bounded subset, B, of a Banach space, E, is defined as

$$\alpha(B) = \inf\{d > 0 \mid B \text{ has a finite cover of diameter } < d\}.$$

For any continuous $T:E \to E$ we define

$$\alpha(T) = \inf\{k \mid \alpha(TB) \leq k\alpha(B) \text{ for all bounded subsets } B \subset E\}.$$

The map, T, is said to be an α-contraction of order k if $\alpha(T) \leq k$.

The operator, T, is compact if and only if $\alpha(T) = 0$. The connection between the Kuratowski measure and the essential radius of a bounded, linear $\mathcal{J}:E \to E$ is given by the following result of Nussbaum [30].

Lemma 3.16. If $\mathcal{J}:E \to E$ is bounded and linear, then $r_e(\mathcal{J}) = \lim_{n\to\infty} \sqrt[n]{\alpha(\mathcal{J}^n)}$.

We now consider the solution operator, $T(t,s)$ defined in §1 of this chapter. The study of the normal points of $T(t,s)$ was initiated by Hale [12]. As in that paper we decompose $T(t,s)$ as $T(t,s)\varphi = \mathcal{J}(t-s)\varphi + U(t,s)\varphi$ where $\mathcal{J}(\tau):X \to X$, $\tau \geq 0$ and $U(t,s):X \to X$, $t \geq s$ are defined by

$$(3.27)\qquad [\mathscr{J}(\tau)\varphi](u) = \begin{cases} 0, & \tau + u \geq 0 \\ \varphi(\tau + u) - \varphi(0), & \tau + u < 0 \end{cases}$$

and

$$(3.28)\qquad [U(t,s)\varphi](u) = \begin{cases} \varphi(0) + \int_s^{t+u} L(v,T(v,s)\varphi)dv, & \tau + u \geq 0 \\ \varphi(0), & \tau + u < 0 \end{cases}$$

for $u \leq 0$. The following result is a special case of what may be found in Hale [12].

Lemma 3.17. For $\mathscr{J}(\tau)$ and $U(t,s)$ as above, we have $\alpha(\mathscr{J}(\tau)) \leq \gamma(\tau)$, where γ is defined by (2.4), and $U(t,s)$ is completely continuous.

Armed with these lemmas, we are now able to state the principal result of this section.

Theorem 3.18. Define

$$(3.29)\qquad r_\rho(t - s) = \overline{\lim_{n\to\infty}} \sqrt[n]{\gamma(n(t - s))}.$$

Then $1 \geq r_\rho(t - s) \geq r_e(T(t,s))$. That is, for any $\epsilon > 0$, $\sigma(T(t,s)) \cap \{\lambda \in C \mid |\lambda| \geq r_\rho(t - s) + \epsilon\}$ is finite and consists entirely of normal eigenvalues.

Proof: By Lemma 3.16 and (3.27)

$$\lim_{n\to\infty} \sqrt[n]{\alpha(\mathscr{J}^n(t - s))} = \lim_{n\to\infty} \sqrt[n]{\mathscr{J}(n(t - s))}$$

$$\leq \overline{\lim_{n\to\infty}} \sqrt[n]{\gamma(n(t - s))} = r_\rho(t - s).$$

Thus, for any $\varepsilon > 0$, $\sigma(\mathcal{J}(t-s)) \cap \{\lambda \,|\, |\lambda| \geq r_\rho(t-s)+\varepsilon\}$ consists entirely of normal eigenvalues for $\mathcal{J}(t-s)$. Therefore, $\{\lambda \,|\, |\lambda| \geq r_\rho(t-s)+\varepsilon\}$ is contained in an unbounded, connected component of normal points of $\mathcal{J}(t-s)$. By Lemma 3.14 and the compactness of $U(t,s)$, any λ satisfying $|\lambda| > r_\rho(t-s)$ must be a normal point of $\mathcal{J}(t-s) + U(t,s) = T(t,s)$. Lemma 3.13 assures us that $\{\lambda \,|\, |\lambda| \geq r_\rho(t-s)+\varepsilon\} \cap \sigma(T(t,s))$ is finite. Finally, (2.4) and the monotonicity of ρ imply $\gamma(t-s) \leq 1$ for all $s < t$. Thus $r_\rho(t-s) \leq 1$ follows immediately. $\square$

There is a very close relationship between the quantities $r_\rho(t)$ and

$$\beta = \inf\{c \in \mathbb{R} \,|\, \int_{-\infty}^{0} e^{cs}\rho(s)ds < \infty\}.$$

Recall that the latter was used by Naito in his study of the linear autonomous systems of Chapter II. It follows by elementary means that under the assumptions (1.1)-(1.2) on ρ,

$$0 \geq \beta = \inf\{c \in \mathbb{R} \,|\, e^{cs}\rho(s) \to 0 \quad \text{as} \quad s \to -\infty\}.$$

<u>Lemma 3.19</u>. For $t > 0$,

$$r_\rho(t) = e^{\beta t} \tag{3.30}$$

where the right hand side of (3.30) is interpreted to mean 0 should $\beta = -\infty$.

<u>Proof</u>: It suffices to show for real numbers, k, that $e^{kt} > r_\rho(t)$ if and only if $k > \beta$. The only non-trivial case is when $k \leq 0$.

Assume there exists some $\varepsilon > 0$ such that $e^{kt} > r_\rho(t) + \varepsilon$. We must show that

$$\int_{-\infty}^{0} e^{ku}\rho(u)du < \infty.$$

Note that for $j = 0,1,2,\ldots$

$$\int_{-(j+1)t}^{-jt} e^{ku}\rho(u)du = \int_{-t}^{0} e^{k(s-jt)}\rho(s - jt)ds$$
$$\leq \gamma(jt)e^{-kjt}\int_{-t}^{0} e^{ks}\rho(s)ds.$$

Therefore

$$\int_{-\infty}^{0} e^{ku}\rho(u)du = \sum_{j=0}^{\infty} \int_{-(j+1)t}^{-jt} e^{ku}\rho(u)du$$
$$\leq \sum_{j=0}^{\infty} \gamma(jt)e^{-kjt}\int_{-t}^{0} e^{ks}\rho(s)ds$$

which converges since $e^{kt} - \varepsilon > r_\rho(t)$ implies $(1 - \frac{\varepsilon}{2} e^{-kt})^j > e^{-kjt} \gamma(jt)$ for all sufficiently large j (since $\varepsilon > 0$ may be taken $< 2e^{kt}$).

Conversely, if $k > \beta$ we may choose an $\varepsilon > 0$ such that

$$\int_{-\infty}^{0} e^{(k-\varepsilon)u} \rho(u)du < \infty.$$

Since ρ is monotone increasing, we conclude that $\lim_{j\to\infty} e^{-jt(k-\varepsilon)} \rho(-jt) = 0$ for any $t > 0$. By property (1.1) of ρ, we have $\gamma(jt) \leq \rho(-jt)$. Thus $e^{-(k-\varepsilon)jt} \gamma(jt) < 1$ for all j sufficiently large. It follows that $r_\rho(t) \leq e^{-\varepsilon t} e^{kt} < e^{kt}$. $\square$

<u>Corollary 3.20</u>. If ρ tends to zero "faster than every exponential" in the sense that $\beta = -\infty$, then $\sigma(T(t,s))$ is at most countable and $\sigma(T(t,s)) \setminus \{0\}$ consists entirely of normal eigenvalues of $T(t,s)$.

If we now apply the results of this section to the semi-group operator, $T(t)$, arising in the autonomous case, we can obtain a significant improvement in some of the results of Chapter II. We refer the reader to that chapter for the needed definitions.

<u>Theorem 3.21</u>. The conclusions of Theorems 2.2 and 2.24 remain valid if $\gamma(t)$ is replaced by $r_\rho(t)$.

<u>Proof</u>: See Theorem 1 of Hale [12] and Lemma 21.4 of Hale [11]. The necessary modifications to this situation are obvious once one observes (from Lemma 3.19) that $|e^{\lambda t}| > r_\rho(t)$ implies $\text{Re}\ \lambda > \beta$. $\square$

We conclude this section by returning to the general situation discussed earlier of a bounded linear operator, T, defined on E and compute the projection of E onto the generalized eigenspace, $\mathcal{N}(T - \lambda I)^k$, associated with any $\lambda \in \sigma(T) \setminus \mathrm{ess}(T)$. First, one final lemma is required. As usual, let E^* denote the dual of E and $\langle \psi, \varphi \rangle$ describe the duality pairing between $\varphi \in E$ and $\psi \in E^*$.

<u>Lemma 3.22</u>. If $T:E \to E$ is bounded and linear and $\lambda \in \sigma(T) \setminus \mathrm{ess}(T)$, then the dimensions of the spaces $\mathcal{N}(T - \lambda I)$ and $\mathcal{N}(T^* - \lambda I)$ are the same, as are the dimensions of the associated generalized eigenspaces $\mathcal{N}(T - \lambda I)^k$ and $\mathcal{N}(T^* - \lambda I)^k$. In addition, $\mathcal{R}(T - \lambda I)^k$ and $\mathcal{R}(T^* - \lambda I)^k$ are closed.

<u>Proof</u>: See Theorems 2.3 and 5.4 of Schechter [35]. □

Assume now that $T:E \to E$ is as above with $\lambda \in \sigma(T) \setminus \mathrm{ess}(T)$. By the previous lemma and Lemma 3.13, we may choose bases $\{\varphi_1, \ldots, \varphi_d\}$ and $\{\psi_1, \ldots, \psi_d\}$ for $\mathcal{N}(T - \lambda I)^k$ and $\mathcal{N}(T^* - \lambda I)^k$, respectively. We define the basis vectors $\Phi = (\varphi_1, \ldots, \varphi_d)$ and $\Psi = \mathrm{col}(\psi_1, \ldots, \psi_d)$.

<u>Claim</u>. The $d \times d$ matrix $\langle \Psi, \Phi \rangle$ is nonsingular.

<u>Proof of claim</u>: If c is a d-vector such that $\langle \Psi, \Phi \rangle c = 0$, then $\langle \Psi, \Phi c \rangle = 0$. By the closedness of $\mathcal{R}(T - \lambda I)^k$ (shown in the previous lemma), we see that Φc

is an element of $\mathcal{R}(T - \lambda I)^k$. Since clearly $\Phi c \in \mathcal{N}(T - \lambda I)^k$ also, we must conclude that $\Phi c = 0$. The linear independence of the φ_i imply that $c = 0$. The claim has been verified.

Thus, we may assume $\langle \Psi, \Phi \rangle = I$, the $d \times d$ identity matrix. The vector Ψ is easily seen to now be uniquely defined for each chosen basis for $\mathcal{N}(T - \lambda I)^k$. The desired projections

$$\Pi : E \to \mathcal{N}(T - \lambda I)^k$$

and

$$\Pi^* : E^* \to \mathcal{N}(T^* - \lambda I)^k$$

are given by $\Pi\varphi = \Phi\langle \Psi, \varphi \rangle$ and $\Pi^* \psi = \langle \psi, \Phi \rangle \Psi$.

By Theorem 3.18, this space decomposition is directly applicable to the solution operator $T(t,s)$ at any point $\lambda \in \sigma(T(t,s))$ satisfying $|\lambda| > r_\rho(t - s)$.

CHAPTER IV

LINEAR PERIODIC SYSTEMS

§1. Periodic Families of Bounded Linear Operators

Throughout this section E will denote a (complex or real) Banach space. The duality pairing between φ in E and ψ in the dual, E^* of E will be again denoted by $\langle\psi,\varphi\rangle$.

Definition 4.1. Let $T(t,s):E \to E$ be a family of bounded linear operators for $t \geq s$, satisfying:

(i) $T(s,s) = I$, the identity, for all $s \in \mathbb{R}$,

(ii) $T(t,u)T(u,s) = T(t,s)$ for all $t \geq u \geq s$,

(iii) there exists an $\omega > 0$ such that for any $t \geq s$, $T(t + \omega, s + \omega) = T(t,s)$, and

(iv) there exists a $\theta > 0$ such that $|T(t,s)| \leq \theta$ for all $0 \leq s \leq \omega$ and $s \leq t \leq s + \omega$.

The family $T(t,s)$, $t \geq s$ will be called an ω-periodic family of bounded linear operators, or, an ω-periodic family.

Note that property (iv) is easily seen to be valid (by the Uniform Boundedness Principle) if $T(t,s)$ is assumed to be strongly continuous in (t,s). For any $s \in \mathbb{R}$, we define the <u>period map</u>, $P(s):E \to E$, by $P(s) = T(s + \omega, s)$. From properties (ii) and (iii) it follows that for any $t \geq s$ and $k = 1,2,\ldots$

(4.1) $$T(t,s)P^k(s) = P^k(t)T(t,s).$$

Note that $P^k(s) = T(s + k\omega, s)$.

Before we continue, let us first make some basic observations concerning the point spectrum of $P(s)$. In particular, if $\mu \neq 0$ is an eigenvalue of $P(s)$ with associated eivenvector, φ, then (4.1) implies

(4.2) $$[P(t) - \mu I]T(t,s)\varphi = T(t,s)[P(s) - \mu I]\varphi$$

for any $t \geq s$. Note that $T(t,s)\varphi \neq 0$ for all $t \geq s$ since if $T(\tau,s)\varphi = 0$ we may choose m so that $s + m\omega \geq \tau$ and show $0 = T(s + m\omega, \tau)T(\tau,s)\varphi = T(s + m\omega, s)\varphi = P^m(s)\varphi = \mu^m\varphi$. This contradicts the facts that both μ and φ are nonzero. Thus, if $t \geq s$, $T(t,s)$ maps eigenvectors of $P(s)$ into eigenvectors of $P(t)$. In addition, any eigenvalue of $P(t)$ is also an eigenvalue of $P(s)$ since (as shown above), μ is an eigenvalue of $P(s + m\omega) = P(s)$ for all m such that $s + m\omega > t$. Thus, the nonzero point spectrum of $P(s)$ is seen to be independent of s, with the null spaces, $\mathcal{N}(P(s) - \mu I)$, $s \in \mathbb{R}$, all of the same (perhaps infinite) dimension.

<u>Definition 4.2</u>. The point $\mu \neq 0$ is said to be a <u>characteristic multiplier</u> of the ω-periodic family $T(t,s)$, $t \geq s$ provided it is a normal eigenvalue of $P(s)$ for all s. Any $\lambda \in C$ for which $\mu = e^{\lambda\omega}$ will be called a <u>characteristic exponent</u> of $T(t,s)$; $t \geq s$.

From section 4 of the last chapter, it follows that μ is a characteristic multiplier if $|\mu| > r_e(P(s))$ for $0 \leq s \leq \omega$. Should some iterate of $P(s)$ be compact (for some s) then it would follow that $r_e(P(t)) = 0$ for all t and, therefore, any nonzero element of $\sigma(P(t))$ is a characteristic multiplier.

Let μ be a characteristic multiplier and $\{\varphi_1, \ldots, \varphi_d\}$ be a basis for the generalized eigenspace $\mathcal{N}(P(0) - \mu I)^k$. Define the basis "vector" $\Phi = (\varphi_1, \ldots, \varphi_d)$.

<u>Theorem 4.3</u>. Let μ and Φ be as above. Then there exist $d \times d$ matrices B and $\Phi(t)$ such that $\sigma(e^{B\omega}) = \{\mu\}$, $\Phi(0) = \Phi$ and $\Phi(t + \omega) = \Phi(t)$ for $t \in \mathbb{R}$. If b is any d-vector, then $T(t,0)\Phi b$ is defined for all $t \in \mathbb{R}$ by

$$T(t,0)\Phi b = \Phi(t)e^{Bt}b. \tag{4.3}$$

Proof: The argument is essentially that of Stokes [36]. For convenience, we write $P = P(0)$. Since μ is a normal eigenvalue of P, we may write $E = \mathcal{N}(P - \mu I)^k \oplus \mathcal{R}(P - \mu I)^k$ with $\sigma(P|_{\mathcal{N}}) = \{\mu\}$ and $\sigma(P|_{\mathcal{R}}) = \sigma(P) \setminus \{\mu\}$. Because P is invariant on $\mathcal{N}(P - \mu I)^k$, there is a $d \times d$ matrix, M, such that $P\Phi = \Phi M$. The spectrum of M is exactly $\{\mu\}$ since $\sigma(P|_{\mathcal{N}}) = \{\mu\}$. Thus, there exists a $d \times d$ matrix, B, such that $M = e^{B\omega}$.

Define $\Phi(t) = T(t,0)\Phi e^{-Bt}$. Then

$$\Phi(t + \omega) = T(t + \omega, 0)\Phi e^{-B(t+\omega)} = T(t,0)T(\omega,0)\Phi e^{-B\omega}e^{-Bt}$$
$$= T(t,0)\Phi e^{B\omega}e^{-B\omega}e^{-Bt} = \Phi(t).$$

Therefore, $\Phi(t)$ is seen to be ω-periodic for $t \geq 0$. We extend the meaning of $\Phi(t)$ for $t < 0$ by defining $\Phi(t) = \Phi(t + m\omega)$ for any m such that $\omega m + t > 0$. The remainder of the theorem is clear. □

Lemma 4.4. Let μ be a characteristic multiplier. The dimension of the generalized eigenspace associated with $P(s)$ is independent of s. If $\Phi(t)$ is defined as in Theorem 4.3, then, for any t, $\Phi(t)$ defines a basis of the generalized eigenspace associated with $P(t)$ and μ.

Proof: The proof follows the lines of discussion directly before Definition 4.2. One uses the facts that

$(P(t) - \mu I)^k T(t,s) = T(t,s)(P(s) - \mu I)^k$ and that if $T(t,0)\Phi b = 0$ for some $t \geq 0$ then $b = 0$. □

For any $t \geq s$, we have that

$$T(t,s)\Phi(s) = \Phi(t)e^{B(t-s)} \tag{4.4}$$

since $T(t,s)\Phi(s) = T(t,s)T(s,0)\Phi e^{-Bs} = T(t,0)\Phi e^{-Bs} = \Phi(t)e^{B(t-s)}$. As in the proof of Theorem 4.3, we may define $T(t,s)\Phi(s)$ for $t < s$ once it is observed that $T(t,s)\Phi(s)e^{-B(t-s)} = \Phi(t)$ is ω-periodic in t. Thus, (4.4) holds for all t,s.

If μ is a characteristic multiplier, we may choose a basis, $\{\psi_1(t),\ldots,\psi_d(t)\}$, for $\mathcal{N}(P^*(t) - \mu I)^k$. Define $\Psi(t) = \mathrm{col}(\psi_1(t),\ldots,\psi_d(t))$. This basis vector is uniquely defined if we require, in addition, that $\langle\Psi(t),\Phi(t)\rangle = I$, the $d \times d$ identity matrix. See section 4 of the previous chapter.

<u>Theorem 4.5</u>. Let $\Psi(t)$, $\Phi(t)$ and B be defined as above. Then

$$T^*(t,s)\Psi(t) = e^{B(t-s)}\Psi(s) \tag{4.5}$$

for all $t \geq s$.

<u>Proof</u>: Let $t \geq s$. From property (ii) it follows that $[P^*(s) - \mu I]^k T^*(t,s) = T^*(t,s)[P^*(t) - \mu I]^k$. Thus

$T^*(t,s)$ maps $\mathcal{N}(P^*(t) - \mu I)^k$ into $\mathcal{N}(P^*(s) - \mu I)^k$. Note that

$$I = \langle \Psi(t), \Phi(t) \rangle = \langle \Psi(t), T(t,s)\Phi(s)e^{-B(t-s)} \rangle$$
$$= \langle T^*(t,s)\Psi(t), \Phi(s) \rangle e^{-B(t-s)}.$$

Therefore,

$$I = e^{-B(t-s)} \langle T^*(t,s)\Psi(t), \Phi(s) \rangle$$
$$= \langle e^{-B(t-s)} T^*(t,s)\Psi(t), \Phi(s) \rangle.$$

This implies that $\Psi(s) = e^{-B(t-s)}T^*(t,s)\Psi(t)$. Equation (4.5) follows immediately. $\square$

By considerations similar to those applied to (4.4), we can define $T^*(t,s)\Psi(t)$ for all t,s so that (4.5) is satisfied.

Having computed a precise description of $T^*(t,s)\varphi$ and $T^*(t,s)\psi$ for $\varphi \in \mathcal{N}(P(s) - \mu I)^k$ and $\psi \in \mathcal{N}(P^*(t) - \mu I)^k$, we now turn our attention towards estimating the growth of $T(t,s)\varphi$; $t \geq s$ for $\varphi \in \mathcal{R}(P(s) - \mu I)^k$. From Lemma 3.13 it follows that the characteristic multipliers of $T(t,s)$; $t \geq s$ are at most countable in number. We assume them ordered by decreasing modulus: $|\mu_1| \geq |\mu_2| \geq \cdots$. If we consider the first m characteristic multipliers, we may decompose E as

$$E = \mathcal{N}(P(s) - \mu_1 I)^{k_1} \oplus \mathcal{N}(P(s) - \mu_2 I)^{k_2} \oplus \cdots$$
$$\oplus \mathcal{N}(P(s) - \mu_m I)^{k_m} \oplus F_m(s).$$

Let $\Phi_i(s)$ and $\Psi_i(s)$ denote the basis vectors associated with $\mathcal{N}(P(s) - \mu_i I)^{k_i}$ and $\mathcal{N}(P^*(s) - \mu_i I)^{k_i}$; respectively. Then, the projection of E onto $F_m(s)$ is given by

$$\Pi_m'(s)\varphi = \varphi - \sum_{i=1}^{m} \Phi_i(s)\langle\Psi_i(s),\varphi\rangle.$$

Theorem 4.6. Let $\alpha > 0$ be chosen such that a finite number of characteristic multipliers, $\{\mu_1,\ldots,\mu_m\}$, satisfy $|\mu_i| \geq \alpha$, while any other multiplier, μ, satisfies $|\mu| < \alpha$. Then there exist constants $M = M(\alpha) > 0$ and $\nu = \nu(\alpha) > 0$ such that

$$|T(t,s)\varphi| \leq M(\alpha)e^{(\alpha-\nu(\alpha))(t-s)}|\varphi| \tag{4.6}$$

for any $\varphi \in F_m(s)$ and all $t \geq s$.

Proof: The argument follows closely the presentation in Hale [11]. Let s be fixed. For convenience, we write $P = P(s)|_{F_m(s)}$.

Because $\sigma(P) = \sigma(P(s)) \quad \{\mu_1,\ldots,\mu_m\}$, P has spectral radius, r', strictly smaller than $|\mu_m|$ (and equal to $|\mu_{m+1}|$ should there me $m + 1$ multipliers). Choose $\nu = \nu(\alpha) > 0$ such that $\nu(\alpha) < \alpha - \delta$ where $\delta \in \mathbb{R}$ satisfies $e^{\delta\omega} = r'$. As $j \to +\infty$ we have $|P^j|^{1/j} \to e^{\delta\omega}$. Thus, for some $\varepsilon > 0$ sufficiently small, $|P^j|^{1/j}e^{(\nu-\alpha)\omega} < 1 - \varepsilon$ for all sufficiently large j. Therefore, $|P^j|e^{(\nu-\alpha)j\omega} \to 0$ as $j \to \infty$.

Let $s \leq t$ and choose $j \geq 0$ such that $j\omega + s \leq t \leq (j + 1)\omega + s$. Then

$$T(t,s) = T(t,j\omega + s)T(j\omega + s,s) = T(t,j\omega + s)P^j(s)$$
$$= T(t - j\omega,s)P^j(s).$$

By property (iv) and the above, we have

$$(4.7) \qquad |T(t,s)\varphi| \leq \theta|P^j||\varphi|$$
$$\leq \theta e^{(\alpha-\nu)\omega j} \max_{n\geq 0} \{|P^n|e^{(\nu-\alpha)n\omega}\}|\varphi|$$

for $\varphi \in F_m(s)$. If $\alpha - \nu > 0$, then $\omega j < t - s$ implies $(\alpha - \nu)j\omega \leq (\alpha - \nu)(t - s)$. If $\alpha - \nu \leq 0$, then $t - s \leq \omega(j + 1)$ implies $(\alpha - \nu)(j + 1)\omega \leq (\alpha - \nu)(t - s)$. In either case, it follows that

$$(4.8) \qquad e^{(\alpha-\nu)\omega n} \leq e^{|\alpha-\nu|\omega} e^{(\alpha-\nu)(t-s)}.$$

From (4.7) and (4.8) we have

$$(4.9) \qquad |T(t,s)\varphi| \leq M(\alpha)e^{(\alpha-\nu)(t-s)}(\varphi)$$

for $t \geq s$ and $\varphi \in F_m(s)$. □

§2. Floquet Theory

We now apply the results of the previous section to the study of

$$(4.10) \qquad \dot{x}(t) = L(t,x_t) + h(t); \quad t > s$$

$$(4.11) \qquad x_s = \varphi \in X$$

under the assumptions on L and h found in Chapter III.

In addition, we make the periodicity assumption that there exists an $\omega > 0$ such that $L(t,\cdot) = L(t+\omega,\cdot)$ and $h(t) = h(t+\omega)$ for all t. As in Chapter III, we define $T(t,s)\varphi = x_t(s,\varphi,0)$ for $t \geq s$. By the uniqueness assertion of Theorem 3.1 and Corollary 3.4 we see that $T(t,s)$; $t \geq s$ satisfies the axioms of an ω-periodic family of bounded linear operators on X. In Chapter III, §4 it was shown that any $\mu \in \sigma(T(\omega,0)) \cap \{\lambda \mid |\lambda| > r_\rho(\omega)\}$ is a normal eigenvalue of $T(s+\omega,s) = P(s)$ for all s. Thus, any $\mu \in \sigma(P(0))$ satisfying $|\mu| > r_\rho(\omega)$ is a characteristic multiplier for the homogeneous analogue of (4.10)-(4.11).

The three theorems of the previous section each have an interpretation in the context of (4.10)-(4.11). The implications of Theorem 4.3 are considered in this section and given by

<u>Theorem 4.7</u>. Assume $|\mu| > r_\rho(\omega)$. Then $\mu = e^{\lambda\omega}$ is a characteristic multiplier if and only if there is a nonzero solution of (4.10) (with $h = 0$) of the form $x(t) = p(t)e^{\lambda t}$ where $p(t) = p(t+\omega)$.

<u>Proof</u>: If μ is a characteristic multiplier, define $\Phi = (\varphi_1,\ldots,\varphi_d)$, where $\{\varphi_1,\ldots,\varphi_d\}$ is a basis for the generalized null space associated with $T(\omega,0) - \mu I$. For

$s = 0$, a solution of (4.10) with $x_0 \in \mathcal{N}(T(\omega,0) - \mu I)^k$ has the form $x_t = T(t,0)\Phi b = \Phi(t)e^{Bt}b$, where $\Phi(t) = T(t,0)\Phi e^{-Bt}$, B is the $d \times d$ matrix chosen such that $T(\omega,0)\Phi = \Phi e^{B\omega}$ and b is a d-vector. Therefore, for $u \leq 0$ and $t \geq 0$,

$$x_t(u) = [\Phi(t)](u)e^{Bt}b. \tag{4.12}$$

In fact, for $u \geq -t$ we have $x_t(u) = x_{t+u}(0) = [\Phi(t+u)](0)e^{B(t+u)}b$. Therefore, $[\Phi(t)](u) = [\Phi(t+u)](0)e^{Bu}$ for $-t \leq u \leq 0$. Replacing t by $t + j\omega$; j a positive integer, yields $[\Phi(t+j\omega)](u) = [\Phi(t+j\omega+u)](0)e^{Bu}$ for $-j\omega - t \leq u \leq 0$. Since $\Phi(t)$ is periodic in t, we obtain

$$[\Phi(t)](u) = \Gamma(t+u)e^{Bu} \tag{4.13}$$

for $t \geq 0$ and $u \leq 0$, where $\Gamma(\tau) = [\Phi(\tau)](0)$ for $\tau \in \mathbb{R}$.

It is immediate from (4.12) that for $t \geq 0$

$$x(t) = \Gamma(t)e^{Bt}b. \tag{4.14}$$

Equation (4.13) shows that (4.14) is also valid for $t < 0$. Note that $\Gamma(t)$ is periodic in t and, therefore, $\Gamma(t)e^{Bt}b$ takes the form of $e^{\lambda t}$ times a polynomial, $p(t)$, in t with ω-periodic coefficients.

The converse is trivial. $\square$

Corollary 4.8. If μ is a characteristic multiplier of (4.10)-(4.11), the basis vector $\Phi(t)$ associated with $\mathcal{N}(P(t) - \mu I)^k$ is given by

$$[\Phi(t)](u) = \Gamma(t + u)e^{Bu}$$

for $u \leq 0$, where B and Γ are as in Theorem 4.7.

§3. The Adjoint Equation for Periodic Systems

If μ is a characteristic multiplier of an ω-periodic family, Theorem 4.5 provides us with a description of $T^*(t,s)\psi$ when ψ is a generalized eigenvector for $T^*(t + \omega, t) - \mu I = P^*(t) - \mu I$. This information, along with the representation of $T^*(t,s)$ given by Theorem 3.8, can be used to obtain a precise description of the basis elements of $\mathcal{N}(P^*(t) - \mu I)^k$. Our efforts here will be preparatory to that description to be given in §4. We will study here a "limit adjoint equation" and bilinear form analogous to (2.30) and $(\cdot,\cdot)$ in the autonomous situation.

To motivate the discussions of this section we first apply Theorem 4.5 to equation (4.10). Let B be as in Theorem 4.7 and $\Psi(t)$ be the usual basis vector associated to $\mathcal{N}(P^*(t) - \mu I)^k$.

Lemma 4.9. Let μ be a characteristic multiplier and $\Psi(t)$, B be as above. Define $\Lambda(t) = [\Psi(t)](0^-)$. Then the solution $\Omega(s,t) = [T^*(t,s)\Psi(t)](0^-)$ of the adjoint equations

(4.15) $$\Omega(t,t) = [\Psi(t)](0^-)$$

(4.16) $$\Omega(s,t) + \int_s^t \Omega(u,t)\tilde{\eta}(u,s-u)du = [\Psi(t)](s-t);\ s < t$$

is locally of bounded variation in s and given by

(4.17) $$\Omega(s,t) = e^{B(t-s)}\Lambda(s).$$

<u>Proof</u>: This follows immediately from (4.5) and Theorem 3.8. □

Although $\Omega(s,t)$ solves (4.15)-(4.16) only for $s \leq t$, we will consider it defined for all s and t by equation (4.17). Note that if $\Omega(s,t)$ is known, then (4.16) can be viewed as a representation for $\Psi(t)$ which then could be used to perform the space decompositions described in §1. Therefore, it is important for $\Omega(s,t)$ to be characterized in a manner independent of $\Psi(t)$. In deriving this characterization, we first need

<u>Lemma 4.10</u>. Let μ be a characteristic multiplier with $|\mu| > r_\rho(\omega)$ and $\Omega(s,t)$ be as above. For $s < t$, $z(s) = \Omega(s,t)$ solves

(4.18) $$\frac{d}{ds}\left[z(s) + \int_0^r z(s+u)\eta(s+u,-u)du\right] - z(s+r)\eta(s+r,-r) + \int_r^\infty z(s+u)\eta(s+u,-u)\rho(-u)du = 0$$

where $z(t+u) = \Omega(t+u,t)$ for $u \geq 0$.

<u>Proof</u>: Let $s < t$ and j be a positive integer. The previous lemma shows $\Omega(s, j\omega + t)$ to solve

$$\Omega(s, j\omega + t) + \int_0^{t-s+j\omega} \Omega(s + u, t + j\omega)\tilde{\eta}(s + u, -u)du$$
$$= [\widetilde{\Psi(t + j\omega)}](s - t - j\omega)$$

for $s < t$. Equation (4.17) and the periodicity of $\Psi(t)$ in t immediately show

$$(4.19) \quad \Omega(s,t) + \int_0^{t-s+j\omega} \Omega(s + u, t)\tilde{\eta}(s + u, -u)du$$
$$= e^{-jB\omega}[\tilde{\Psi}(t)](s - t - j\omega)$$

for $s < t$. For j such that $t - s + j\omega > r$, the integral may be written as

$$\int_0^r \Omega(s + u, t)\eta(s + u, -u)du + \int_{s+r}^{t+j\omega} \Omega(u, t)\tilde{\eta}(u, s - u)du$$

and (4.19) may be differentiated with respect to s to yield

$$(4.20) \quad \frac{d}{ds}[\Omega(s,t) + \int_0^r \Omega(s + u, t)\eta(s + u, -u)du]$$
$$- \Omega(s + r, t)\eta(s + r, -r)$$
$$+ \int_r^{t-s+j\omega} \Omega(s + u, t)\eta(s + u, -u)\rho(-u)du$$
$$= e^{-B\omega j}\rho(s - t - j\omega)[\Psi(t)](s - t - j\omega)$$

for $s < t$. Note that

$$\operatorname*{ess\,sup}_{s \leq t} |\rho(s - t - j\omega)e^{-Bj\omega}[\Psi(t)](s - t - j\omega)|$$
$$\leq \gamma(j\omega)|e^{-Bj\omega}| \cdot \operatorname*{ess\,sup}_{s \leq t} |[\Psi(t)](s - t - j\omega)\rho(s - t)|$$
$$\leq \gamma(j\omega)|e^{-Bj\omega}| \cdot \text{constant}$$

which tends to 0 as $j \to \infty$ since $|\mu| > r_\rho(\omega)$ and $\sigma(e^{-B\omega}) = \{\frac{1}{\mu}\}$. Also, for all j sufficiently large,

$$\left|\int_r^{t-s+j\omega} \Omega(s+u,t)\eta(s+u,-u)\rho(-u)du\right|$$
$$\leq \int_r^{t-s+j\omega} |e^{B(t-s-u)}|\cdot|\Lambda(s+u)|\cdot|\eta(s+u,-u)|\rho(-u)du$$
$$\leq \int_{-r}^{t-s+j\omega} |e^{-Bu}|\rho(-u)du \cdot \text{constant}$$
$$\leq \int_{-r}^{\infty} |e^{-Bu}|\rho(-u)du \,.\, \text{constant} < \infty$$

by the properties of Ω and η. Therefore, we are justified in taking the limit as $j \to \infty$ in (4.20) to find

$$(4.21) \quad \frac{d}{ds}\left[\Omega(s,t) + \int_0^r \Omega(s+u,t)\eta(s+u,-u)du\right] - \Omega(s+r,t)\eta(s+r,-r) + \int_r^\infty \Omega(s+u,t)\eta(s+u,-u)\rho(-u)du = 0$$

for $s < t$. □

This result motivates the consideration of the "limit adjoint equation"

$$(4.22) \quad \frac{d}{ds}\left[z(s) + \int_0^r z(s+u)\eta(s+u,-u)du\right] - z(s+r)\eta(s+r,-r) + \int_r^\infty z(s+u)\eta(s+u,-u)\rho(-u)du = 0; \quad s < t,$$

with initial condition $z(t+u)$; $u \geq 0$ being defined by an element of $X^0 = \{\text{measurable } \alpha:[0,\infty) \to \mathbb{R}^{n^T} \mid \operatorname{var}_{[0,r]}(\alpha) + \int_r^\infty |\alpha(u)|\rho(-u)du < \infty\}$. To the set X^0 we give the norm

$$|\alpha| = |\alpha(0)| + \operatorname*{var}_{[0,r]}(\alpha) + \int_r^\infty |\alpha(u)|\rho(-u)du$$

and thereby make X^0 a Banach space.

Lemma 4.11. Equation (4.22), with initial date defined by

(4.23) $$z(t + u) = \alpha(u), \quad u \geq 0$$

where $\alpha \in X^0$, has a unique solution defined for all $s < t$ which is locally of bounded variation in s.

Proof: This result can be derived from Theorem 32.1 of Hale [11] since for $t - r \leq s < t$ equation (4.22) reduces upon integration to the form

$$z(s) + \int_s^t z(u)\eta(u,s - u)du = \chi(u)$$

where χ is of bounded variation on $[t - r,t]$. The argument can then be repeated on $[t - 2r,t - r]$, $[t - 3r, t - 2r]$, etc. $\square$

If z solves (4.22)-(4.23) we define for $s \leq t$ the function $z^s \in X^0$ by $z^s(u) = z(s + u)$ for $u \geq 0$. The map $T^0(t,s)z^t = z^s$ can easily be seen to be a linear operator defined on X^0. In fact, following arguments in Hale [11] and Hale [12] one can show

Lemma 4.12. The map $T^0(t,s):X^0 \to X^0$ is a bounded linear operator with $r_e(T^0(t,s)) \leq r_\rho(t - s)$. Thus, any $\mu \in \sigma(T^0(t,s)) \cap \{\lambda | |\lambda| > r_\rho(t - s)\}$ is a normal eigenvalue for $T^0(t,s)$.

The connection between solutions of (4.22)-(4.23) and the solutions of the adjoint equations (3.20)-(3.21) is given by

Lemma 4.13. If y solves (4.22)-(4.23) then there is a unique $\psi \in X^*$ defined by α for which y solves (3.20)-(3.21).

Proof: Assume $t - r \leq s < t$. We may integrate the equation

$$0 = \frac{d}{du}\Big[y(u) + \int_0^{t-u} y(v+u)\eta(v+u,-v)dv + \int_{t-u}^{r} \alpha(v+u-t)\eta(v+u,-v)dv\Big] - \alpha(u-t+r)\eta(u+r,-r) + \int_r^{\infty} \alpha(v+u-t)\eta(v+u,-v)dv$$

from s to t to find that

$$y(s) + \int_0^{t-s} y(s+v)\eta(s+v,-v)dv = \psi(s-t)$$

where $\psi(u)$; $-r \leq u \leq 0$ is defined by $\psi(0) = 0$ and

$$(4.24) \quad \psi(u) = \alpha(0) + \int_0^r \alpha(v)\eta(t+v,-v)dv - \int_0^{r+u} \alpha(v)\eta(t+v,u-v)dv + \int_{t+u}^{t} \Big[\int_r^{\infty} \alpha(w+v-t)\eta(v+w,-w)\rho(-w)dw - \alpha(v-t+r)\eta(v+r,-r)\Big]dv.$$

For $u < -r$, we define

$$\psi(u) = -\rho^{-1}(u)\int_0^{\infty} \alpha(v)\eta(t + v, u - v)\rho(u - v)dv. \tag{4.25}$$

As in the autonomous case (see Theorem 2.18), ψ is left continuous on $[-r,0)$, of bounded variation on $[-r,0]$ and essentially bounded on $(-\infty,-r)$. Differentiation of (3.20) for $s - t < -r$ with ψ defined by (4.25) reveals that y solves this "differentiated" form of (3.20) for $s < t - r$, with

$$y(t - r) = -\int_0^r y(t - r + u)\eta(t - r + u,-u)du$$
$$+ \psi(-r).$$

By the uniqueness of solutions to (3.21)-(3.22), the lemma is proved. $\square$

Note that if ψ is given in terms of $\alpha \in X^O$ by equations (4.24)-(4.25), then $\psi(0^-) = \alpha(0)$. We shall say that ψ is "defined by α at t" or "α defines ψ at t" provided α defines ψ via equations (4.24) and (4.25).

At this point, we introduce a bilinear form $(\alpha,\varphi)_t$ defined for $\alpha \in X^O$ and $\varphi \in X$ which plays a role analogous to that of (α,φ) in the autonomous case. In that situation, $\alpha^T \in X$. For $\alpha \in X^O$ and $\varphi \in X$ define

$$(\alpha,\varphi)_t = \alpha(0)\varphi(0) + \int_{-r}^{0} [d_s\int_0^{r+s} \alpha(u)\eta(t+u,s-u)du]\varphi(s)$$

$$+ \int_{-r}^{0} [\int_{r+s}^{\infty} \alpha(u)\eta(t+u,s-u)\rho(s-u)du$$

$$- \alpha(s+r)\eta(t+s+r,-r)]\varphi(s)ds$$

$$+ \int_{-\infty}^{-r} [\int_0^{\infty} \alpha(u)\eta(t+u,s-u)\rho(s-u)du]\varphi(s)ds.$$

In the special case of finite delay, $\eta(t+u,s) = 0$ for $s < -r$ and $s \geq 0$. The last two integrals reduce to

$$-\int_{-r}^{0} \alpha(s+r)\eta(t+s+r,-r)\varphi(s)ds$$

and $(\alpha,\varphi)_t$ becomes the classical bilinear form of Hale [11]. Clearly, $(\cdot,\cdot)_t = (\cdot,\cdot)_{t+\omega}$ for all t.

<u>Lemma 4.14</u>. If α defines ψ at t, then for all $\varphi \in X$, $\langle\psi,\varphi\rangle = -(\alpha,\varphi)_t$.

<u>Proof</u>: This is simply a matter of substituting the expressions (4.24)-(4.25) into the bilinear form $(\cdot,\cdot)_t$ and noting $\psi(0^-) = \alpha(0)$.

See the proof of Lemma 2.19. □

<u>Theorem 4.15</u>. If α defines ψ at t then $T^0(t,s)\alpha$ defines $T^*(t,s)\psi$ at s.

<u>Proof</u>: The proof is similar to that of Theorem 2.20.

Let $y(s,t) = [T^*(t,s)\psi](0^-)$. By the uniqueness of solutions to the adjoint equation and Lemma 4.13, $y(s,t) = y(s)$, where $y(s)$ is the solution of (4.22) with $y^t = \alpha$. By equation (3.18),

$$\widetilde{[T^*(t,s)\psi]}(u) = \tilde{\psi}(u + s - t) + \int_0^{t-s} y(s + v)\tilde{\eta}(s + v, u - v)dv$$

for $u < 0$. For $u < -r$, this may be differentiated with respect to u to find

$$\begin{aligned}
&\rho(u)[T^*(t,s)\psi](u) \\
&= \rho(u + s - t)\psi(u + s - t) \\
&\quad - \int_0^{t-s} y(s + v)\eta(s + v, u - v)\rho(u - v)dv \\
&= -\int_0^{\infty} \alpha(v)\eta(t + v, u + s - t - v)\rho(u + s - t - v)dv \\
&\quad - \int_0^{t-s} y(s + v)\eta(s + v, u - v)\rho(u - v)dv \\
&= -\int_0^{\infty} y(s + v)\eta(s + v, u - v)\rho(u - v)dv
\end{aligned}$$

by (4.25). Comparing with (4.25), we see that $[T^*(t,s)\psi](u)$ is defined by y^s at s for $u < -r$. Similar but more complicated calculations for $-r \leq u \leq 0$ show that $[T^*(t,s)\psi](u)$; $u \leq 0$ is indeed defined at s by y^s. □

<u>Corollary 4.16</u>. If ψ is defined at t by α, then for all $s \leq t$ and $\varphi \in X$,

$$(\alpha, T(t,s)\varphi)_t = -\langle\psi, T(t,s)\varphi\rangle = -\langle T^*(t,s)\psi, \varphi\rangle$$
$$= (T^0(t,s)\alpha, \varphi)_s.$$

Proof: This is immediate from Lemma 4.14 and the previous theorem. □

§4. The Dual Basis and Space Decomposition

We are now able to apply the results of the previous section to the calculation of the "dual" basis associated to the generalized eigenspace of $P^*(t) - \mu I$.

Theorem 4.17. Let μ be a characteristic multiplier with $|\mu| > r_\rho(\omega)$. Define B, $\Psi(t)$ and $\Lambda(t)$ as in Lemma 4.9. Then $\Psi(t)$ is defined at t by $\Omega^t(\cdot,t)$, which is a basis vector for the generalized eigenspace of $T^0(t+\omega,t) - \mu I = P^0(t) - \mu I$.

Proof: The fact that $\Psi(t)$ is defined at t by $\Omega^t(\cdot,t)$ follows immediately from Lemmas 4.10 and 4.13. The calculations of Lemma 4.10 show that the rows of $\Omega^t(\cdot,t)$ are elements of X^0.

To show the remainder of the theorem, first assume $\alpha \in X^0$ and $(P^0(t) - \mu I)^m\alpha = 0$ for some $m \geq 1$. If ψ is defined at t by α, then Theorem 4.15 and Corollary 4.16 show $0 = ((P^0(t) - \mu I)^m\psi, \varphi)_t = \langle (P^*(t) - \mu I)^m\psi, \varphi\rangle$ for all $\varphi \in X$. Thus, $\psi \in \mathcal{N}(P^*(t) - \mu I)^k$ and there exists a d-vector, b, such that $\psi = b\Psi(t)$. (We assume $d = \dim \mathcal{N}(P^*(t) - \mu I)^k$.) Therefore, ψ is seen to be defined at t by $b\Omega^t(\cdot,t)$.

<u>Claim</u>. $b\Omega^t(\cdot,t) \in \mathcal{N}(P^O(t) - \mu I)^k$.

<u>Proof of claim</u>: From (4.17) it follows that $\Omega^{t+\omega}(\cdot,t+\omega) = \Omega^t(\cdot,t)$. Using the definition of $T^O(t+\omega,t)$ and (4.17) we have

$$\begin{aligned} T^O(t+\omega,t)\Omega^t(\cdot,t) &= T^O(t+\omega,t)\Omega^{t+\omega}(\cdot,t+\omega) \\ &= \Omega^t(\cdot,t+\omega) = e^{B\omega}\Omega^{t+\omega}(\cdot,t+\omega) \\ &= e^{B\omega}\Omega^t(\cdot,t). \end{aligned}$$

Thus, $(P^O(t) - \mu I)^k\Omega^t(\cdot,t) = (e^{B\omega} - \mu I)^k\Omega^t(\cdot,t) = 0$ since $(e^{B\omega} - \mu I)^k = 0$. The claim is verified.

<u>Claim</u>. $\alpha = b\Omega^t(\cdot,t)$.

<u>Proof of claim</u>: Define $\nu = \alpha - b\Omega^t(\cdot,t)$. Then ν is an element of the generalized null space of $P^O(t) - \mu I$ and defines at t the element $0 \in X^*$. The solution of the adjoint equation (3.20) associated with 0 is 0 for $s < t$. Thus, the solution of (4.32) with initial value ν is 0 for $s < t$.

By Lemma 4.12, μ is a normal eigenvalue for $P^O(t)$. Arguing as in §1, the dimension of the generalized null space of $P^{O}(t) - \mu I$ is finite (say, equal to q) and invariant under $P^{O}(t)$. Let $\{\alpha_1, \ldots, \alpha_q\}$ be a basis for the generalized eigenspace $\eta(P^{O}(t) - \mu I)^{k}$ and $\chi = (\alpha_1, \ldots, \alpha_q)$ be the associated basis vector. There exists a $q \times q$ matrix, D, such that $P^{O}(t)\chi = D\chi$ and $\sigma(D) = \{\mu\}$. If b is the q-vector such that $\alpha = b\chi$, then for $m \geq 1$,

$$T^{O}(t + m\omega, t)b\chi = [P^{O}(t)]^{m}b\chi = bD^{m}\chi.$$

Considering the definition of $T^{O}(t,s)$, we must conclude that α is zero on $[0,m]$. However, since $m \geq 1$ was arbitrary, the claim holds true.

Thus, we have shown that $\eta(P^{*}(t) - \mu I)^{k}$ is spanned by linear combinations of the elements defined at t by the basis elements of $\eta(P^{O}(t) - \mu I)^{k}$. Finally, we note that the rows of $\Omega^{t}(\cdot,t)$ are linearly independent since if $b\Omega^{t}(\cdot,t) = 0$ for some d-vector, b, then $0 = (b\Omega^{t}(\cdot,t), \Phi(t))_{t} = \langle b\Psi(t), \Phi(t)\rangle = b.$ $\square$

The following may be "added in proof".

<u>Corollary 4.18</u>. If μ is as above, $\dim \eta(P^{*}(t) - \mu I)^{k} = \dim \eta(P^{O}(t) - \mu I)^{k}$. In fact, the linear mapping that associates to each $\alpha \in \eta(P^{O}(t) - \mu I)^{k}$ the element of $\eta(P^{*}(t) - \mu I)^{k}$ it defines at t is a 1-1, onto map.

We are now able to give a complete description of the projection operator $\Pi(t): X \to \mathcal{N}(P(t) - \mu I)^k$ for μ a characteristic multiplier with $|\mu| > r_\rho(\omega)$. We will assume, as before, that $d = \dim \mathcal{N}(P(t) - \mu I)^k$ and $\Phi(t)$, $\Psi(t)$ are the usual basis vectors.

As in the general setting of §1, $\Pi(t)\varphi = \Phi(t)\langle\Psi(t),\varphi\rangle$. For the FDE (4.10), Corollary 4.8 gives the general form for $\Phi(t)$. By the previous theorem, we may find a basis vector $\Omega^t(\cdot,t)$ for $\mathcal{N}(P^0(t) - \mu I)^k$ that defines $\Psi(t)$ at t. The general form of $\Omega^t(\cdot,t)$ follows from (4.17). Applying Lemma 4.14, we see

$$(4.26)\quad \Pi(t)\varphi = -\Phi(t)(\Omega^t(\cdot,t),\varphi)_t$$
$$= -\Gamma_t(\cdot)e^{B(\cdot)}(e^{-B(\cdot)}\Lambda^t(\cdot),\varphi)_t.$$

Similarly, $\Pi^*(t)\psi = \langle\psi,\Gamma_t(\cdot)e^{B(\cdot)}\rangle e^{-B(\cdot)}\Lambda^t(\cdot)$ for $\psi \in X^*$ and $\Pi^*(t)\psi = -(\alpha,\Gamma_t(\cdot)e^{B(\cdot)})_t e^{-B(\cdot)}\Lambda^t(\cdot)$ if ψ is defined by $\alpha \in X^0$ at t.

We close this section with the associated decomposition of the variation of constants formula (3.24) at a characteristic multiplier μ. As in Chapter III, §3, we define

$$\delta(u) = \begin{cases} 0, & u < 0 \\ I, & u = 0 \end{cases}$$

where I is the $n \times n$ identity matrix.

Theorem 4.19. If x is a solution of (4.10)-(4.11) for $t \geq s$ and μ is a characteristic multiplier of (4.10) with $|\mu| > r_\rho(\omega)$, then $x_t = \Pi(t)x_t + \Pi'(t)x_t$ with $\Pi(t)x_t$ and $\Pi'(t)x_t = [I - \Pi(t)]x_t$ satisfying the integral equations

$$(4.27) \quad \Pi(t)x_t = T(t,s)\Pi(s)x_s + \int_s^t T(t,u)\Pi(u)\delta h(u)du$$

$$(4.28) \quad \Pi'(t)x_t = T(t,s)\Pi'(s)x_s + \int_s^t T(t,u)\Pi'(u)\delta h(u)du,$$

respectively.

Proof: The fact that $T(t,s)\Pi(s)\delta$ is actually well defined follows by an argument similar to that of Section 36 of Hale [11].

Now, if x is as described, then by Chapter III, §2, $x_t = T(t,s)\varphi + K(t,s)h$. Therefore,

$$\begin{aligned}
\Pi(t)x_t &= \Pi(t)T(t,s)x_s + \Pi(t)K(t,s)h \\
&= \Phi(t)\langle\Psi(t),T(t,s)x_s\rangle + \Phi(t)\langle\Psi(t),K(t,s)h\rangle \\
&= \Phi(t)\langle T^*(t,s)\Psi(t),x_s\rangle + \Phi(t)\langle K^*(t,s)\Psi(t),h\rangle \\
&= T(t,s)\Phi(s)e^{-B(t-s)}\langle e^{B(t-s)}\Psi(s),x_s\rangle \\
&\quad + \Phi(t)\int_s^t [T^*(t,u)\Psi(t)](0^-)h(u)du
\end{aligned}$$

by equations (4.4), (4.5), (3.24) and Corollary 3.10. Consequently,

$$\Pi(t)x_t = T(t,s)\Phi(s)\langle\Psi(s),x_s\rangle + \int_0^t \Phi(t)\langle e^{B(t-u)}\Psi(u),\delta\rangle h(u)\,du$$

$$= T(t,s)\Pi(s)x_s + \int_s^t T(t,u)\Phi(u)e^{-B(t-u)}\cdot$$

$$\langle e^{B(t-u)}\Psi(u),\delta\rangle h(u)\,du$$

$$= T(t,s)\Pi(s)x_s + \int_s^t T(t,u)\Pi(u)\delta h(u)\,du.$$

Equation (4.28) follows since $\Pi'(t) = I - \Pi(t)$. □

As in the finite delay case, equation (4.27) is equivalent to an ordinary differential equation. That is, if $y(t)$ is the d-vector such that $\Phi(t)y(t) = \Pi(t)x_t$, then

$$\Phi(t)y(t) = T(t,s)\Phi(s)\langle\Psi(s),x_s\rangle$$

$$+ \Phi(t)\int_s^t [T^*(t,u)\Psi(t)](0^-)h(u)\,du$$

$$= \Phi(t)e^{B(t-s)}y(s) + \Phi(t)\int_s^t \Omega(u,t)h(u)\,du.$$

Therefore,

$$y(t) = e^{B(t-s)}y(s) + \int_s^t e^{B(t-u)}\Lambda(u)h(u)\,du$$

which may be differentiated with respect to t for $t > s$ to yield

$$\dot{y}(t) = By(t) + \Lambda(t)h(t).$$

§5. Stability of Linear Periodic Systems

Theorem 4.6, when applied to (4.10)-(4.11) can be used to derive a generalization of the usual criteria for the stability of linear periodic ordinary differential equations. (See Yoshizawa [40].) Throughout this section it will be assumed that $h \equiv 0$ in (4.10).

Definition 4.20. (i) The zero solution of (4.10) is called stable if for every $\epsilon > 0$ and $s \in \mathbb{R}$ there is a $\delta = \delta(\epsilon, s) > 0$ such that $|\varphi| < \delta$ implies $|x_t(s,\varphi)| < \epsilon$ for all $t \geq s$.

(ii) The zero solution of (4.10) is called asymptotically stable if it is stable and there exists an $H = H(s) > 0$ such that $|\varphi| < H$ implies $\lim_{t\to\infty} |x_t(s,\varphi)| = 0$.

(iii) The zero solution of (4.10) is called uniformly stable provided it is stable and the δ in (i) is independent of s.

(iv) The zero solution of (4.10) is called uniformly asymptotically stable if it is uniformly stable and for all $\nu > 0$ and $s \in \mathbb{R}$ there exists a $\tau = \tau(\nu) > 0$ (independent of s) and $K > 0$ (independent of s and ν) such that $|\varphi| < K$ implies $|x_t(s,\varphi)| < \nu$ for all $t > s + \tau(\nu)$.

From the linearity and periodicity of (4.10) we have

Lemma 4.21. The zero solution of (4.10) is uniformly stable if it is stable. The zero solution of (4.10) is uniformly asymptotically stable if it is asymptotically stable.

The proof is the same as for the analogous result for ordinary differential equations. See Yoshizawa [40].

As in Chapter II, §1 and Chapter III, §4 we define

$$\beta = \inf\{c \in \mathbb{R} \mid \int_{-\infty}^{0} e^{cs}\rho(s)ds < \infty\}.$$

Theorem 4.22. Assume $\beta < 0$. (i) The zero solution of (4.10) is uniformly asymptotically stable if and only if all characteristic multipliers of (4.10) have moduli less than 1.

(ii) The zero solution of (4.10) is uniformly stable if and only if all characteristic multipliers of (4.10) have moduli less than or equal to 1 and if μ is a characteristic multiplier with $|\mu| = 1$ then all solutions of (4.10)-(4.11) with initial value in $\mathcal{N}(T(\omega,0) - \mu I)^k$ are bounded.

Proof: By Lemma 3.19, $r_\rho(\omega) < 1$. Thus, there exists a $\nu > 0$ such that any $\mu \in \sigma(T(\omega,0))$ with

$|\mu'| > 1 - \nu$ is a characteristic multiplier of (4.10). The remainder of the argument follows along the lines of the proofs of Corallaries 35.1 and 35.2 of Hale [11]. □

Similar also to its counterpart in Hale [11] is

<u>Theorem 4.23</u>. Assume $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_i| \geq \cdots$ are the characteristic multipliers of (4.10) with $|\mu_i| > r_\rho(\omega)$ for $i = 1, 2, \cdots$. Let Π_i be the projection of X onto $\mathcal{N}(T(\omega,0) - \mu_i)^{k_i}$. If $\sum_i \Pi_i\varphi$ converges and $\bar{\varphi} = \varphi - \sum_i \Pi_i\varphi$, then $|x_t(0,\bar{\varphi})|e^{\alpha t} \to 0$ as $t \to \infty$ for all $\alpha > \beta$.

<u>Proof</u>: This is immediate from Lemma 3.19 and Theorem 4.6 since the projection of $\bar{\varphi}$ onto $\mathcal{N}(T(\omega,0) - \mu_i)^{k_i}$ is zero for any characteristic multiplier μ_i with $|\mu_i| > r_\rho(\omega)$. □

The analogue of the previous theorem for the autonomous system (2.1)-(2.2) is given by

<u>Theorem 4.24</u>. Define $\Delta(\lambda)$ by (2.9) and Π_λ as in Theorem 2.24. Let $S = \{\lambda \in C | \text{Re } \lambda > \beta \text{ and } \Delta(\lambda) = 0\}$. If $\sum_{\lambda \in S} \Pi_\lambda\varphi$ converges and $\bar{\varphi} = \varphi - \sum_{\lambda \in S} \Pi_\lambda\varphi$, then the solution $x_t(\bar{\varphi})$ of (2.1)-(2.2) with initial value $x_0 = \bar{\varphi}$ satisfies $|x_t(\bar{\varphi})|e^{\alpha t} \to 0$ as $t \to +\infty$ for every $\alpha > \beta$.

<u>Proof</u>: Let $T(t)$; $t \geq 0$ be the strongly continuous semi-group and A be the infinitesimal generator associated with (2.1)-(2.2). By Lemma 22.1 of Hale [11] and Theorem 3.21, if $\lambda \in S$, then $\mathcal{N}(T(1) - e^{\lambda}I)^k$ is the closed linear extension of the linearly independent eigenspaces $\mathcal{N}(A - \lambda_i I)^{k_i}$, where the λ_i are those elements in S satisfying $e^{\lambda_i} = e^{\lambda}$.

We define a 1-periodic family on X by $T(t,s) = T(t - s)$, $t \geq s$. It follows that the collection of characteristic multipliers, μ, of $T(t,s)$, $t \geq s$ with $|\mu| > r_\rho(1)$ is given by $\{\mu = e^{\lambda} | \lambda \in S\}$.

By Theorem 4.6, it suffices to show that $\bar{\varphi}$ has no nonzero projection onto any of the eigenspaces $\mathcal{N}(T(1,0) - e^{\lambda}I)^k = \mathcal{N}(T(1) - e^{\lambda}I)^k$ for $\lambda \in S$. However, this follows easily from the aforementioned lemma of Hale. □

§6. <u>The Fredholm Alternative for Forced Periodic Systems</u>

As an application of the previous sections we derive a generalization of the classical Fredholm Alternative known for linear periodic systems of ordinary differential equations. (See Hale [13].) Throughout this section, the β previously defined will be assumed negative. One lemma is needed in preparation.

Lemma 4.25. $T^*(\omega,0)\psi = \psi$ if, and only if ψ, is defined at ω by an ω-periodic solution to (4.22)-(4.23). That is, there exists a $z^\omega \in X^0$ which defines ψ at ω and $T^0(\omega,0)z^\omega = z^\omega$.

Proof: Assume ψ is defined at ω by $z^\omega \in X^0$ and $T^0(\omega,0)z^\omega = z^\omega$. Theorem 4.15 shows $T^*(\omega,0)\psi = \psi$.

Conversely, if $0 \neq \psi \in \mathcal{N}(T^*(\omega,0) - I)$, Lemma 3.22 and Theorem 3.18 show 1 to be a normal eigenvalue for $T(\omega,0)$. Because $\mathcal{N}(T(\omega,0) - I) \subset \mathcal{N}(T(\omega,0) - I)^k$, ψ is defined at 0 by some $\alpha \in \mathcal{N}(T^0(\omega,0) - I)^k$. Theorem 4.15 shows $T^0(\omega,0)\alpha$ to also define ψ at 0. By Corollary 4.18 we must conclude that $T^0(\omega,0)\alpha = \alpha$. The solution, z, of (4.22)-(4.23) with $z^\omega = \alpha$ is the desired periodic solution which defines ψ at ω. □

Without loss of generality, we may assume $s = 0$ in (4.10)-(4.11).

Theorem 4.26. Assume $\beta < 0$. The problem (4.10)-(4.11) has an ω-periodic solution if, and only if,

$$\int_0^\omega z(u)h(u)\,du = 0 \tag{4.29}$$

for every ω-periodic solution to (4.22)-(4.23).

<u>Proof</u>: Since the solution of (4.10)-(4.11) is given by $x_t = T(t,0)\varphi + K(t,0)h$, (4.10)-(4.11) has an ω-periodic solution if, and only if, $K(\omega,0)h \in \mathcal{R}(T(\omega,0) - I)$. Under the assumption $\beta < 0$, 1 will either be a normal eigenvalue for $T(\omega,0)$ or in the resolvent set of $T(\omega,0)$. In either case, $\mathcal{R}(T(\omega,0) - I)$ is closed. Thus (4.10) has an ω-periodic solution if and only if $\langle\psi,K(\omega,0)h\rangle = 0$ for all $\psi \in \mathcal{N}(T^*(\omega,0) - I)$.

Assume $\psi \in \mathcal{N}(T^*(\omega,0) - I)$. Recall from Corollary 3.9 that

$$\text{(4.30)} \quad \langle\psi,K(\omega,0)h\rangle = -\int_0^\omega [T^*(\omega,u)\psi](0^-)h(u)du$$

$$= -\int_0^\omega z(u)h(u)du$$

where z solves the adjoint equation (3.21) associated to ψ for $s < \omega$. Since $\psi \in \mathcal{N}(T^*(\omega,0) - I)^k$, Lemma 4.10 shows z to solve (4.22), and the previous lemma shows z to be ω-periodic. Conversely, if z is an ω-periodic solution to (4.22) and z^ω defines ψ at ω, then the previous lemma shows $\psi \in \mathcal{N}(T^*(\omega,0) - I)$.

In light of (4.30), we conclude that $\langle\psi,K(\omega,0)h\rangle = 0$ for all $\psi \in \mathcal{N}(T^*(\omega,0) - I)$ if, and only if, (42.9) holds for all ω-periodic solutions to (4.22)-(4.23). □

In some applications it is the case that the measure defining $L(t,\cdot)$ takes a special form from which it can be shown that the limit adjoint equation actually has absolutely continuous solutions. For example, consider the situation when

$$(4.31) \qquad L(t,\varphi) = \sum_{k=1}^{m} A_k(t)\varphi(-r_k) + \int_{-\infty}^{0} B(t,s)\rho(s)\varphi(s)ds$$

where $0 \leq r_1 \leq \cdots \leq r_m = r$ and the $n \times n$ matrices $A_k(t)$ are continuous and ω-periodic in t. We assume also that $B(t,\cdot)$ is essentially bounded on $(-\infty,0]$ for each t and $B(t,s)$ is continuous and ω-periodic for almost every fixed $s \leq 0$. Then $\tilde{\eta}(t,s)$ takes the form

$$(4.32) \qquad \tilde{\eta}(t,s) = -\sum_{k=1}^{m} A_k(t)\chi_{(-\infty,-r_k]}(s) - \int_{s}^{0} B(t,u)\rho(u)du$$

for $s < 0$, where $\chi_{(-\infty,-r_k]}(\cdot)$ is the characteristic function for the interval $(-\infty,-r_k]$. The system (4.10) becomes

$$(4.33) \qquad \dot{x}(t) = \sum_{k=1}^{m} A_k(t)x(t - r_k) + \int_{-\infty}^{0} B(t,s)\rho(s)x(t + s)ds + h(t)$$

and (by computations similar to those in Hale [11]), the adjoint system (4.22) becomes

$$(4.34) \qquad \dot{z}(s) = -\sum_{k=1}^{m} z(s + r_k)A_k(s + r_k) - \int_{0}^{\infty} z(s + u)B(s + u,-u)\rho(-u)du.$$

The initial value for the adjoint system may be assumed continuous on $[0,r]$ and integrable with respect to $\rho(-u)$ on $[r,\infty)$.

CHAPTER V

BEHAVIOR NEAR PERIODIC ORBITS

§1. Differentiability of the Solution Map

In this chapter we shall apply the results of the previous chapters to the study of the behavior near periodic solutions to autonomous non-linear FDE's. In particular, we consider

$$\dot{x}(t) = f(x_t), \quad t > 0 \tag{5.1}$$

where

$$x_0 = \varphi \in X \tag{5.2}$$

where $f:X \to \mathbb{R}^n$ is continuously Frechét differentiable (C^1). The derivative of f at $\varphi_0 \in X$ will be denoted by $Df(\varphi_0)$ or $D_{\varphi=\varphi_0} f(\varphi)$. It can be shown by standard techniques that (5.1)-(5.2) always has a unique solution, $x(\varphi)$, which depends continuously on its initial data. That is, if $\varphi^{(m)} \to \varphi$ in X, then $x(\varphi^{(m)}) \to x(\varphi)$ uniformly on compact subsets of its domain of definition.

If $x = x(\varphi_0)$ solves (5.1)-(5.2) with $x_0 = \varphi_0 \in X$, we define $L(t,\cdot):X \to \mathbb{R}^n$ by $L(t,\varphi) = Df(x_t(\varphi_0))\varphi$ for

$t \geq 0$. Under the assumption that f is C^1, $t \to L(t,\cdot)$ defines a continuous map of $[0,\infty)$ into $\mathcal{L}(X,\mathbb{R}^n)$, the Banach space of all bounded linear operators from X into $\mathbb{R}^n$. This clearly implies $L(t,\varphi)$ is continuous in t and φ. Thus, Theorem 3.1 assures us that the "linearized" system

$$\dot{z}(t) = L(t,z_t), \quad t > s \geq 0 \tag{5.3}$$

$$z_s = \varphi \in X \tag{5.4}$$

has a uniquely defined solution on (s,∞). For (5.3)-(5.4) we define the associated (linear) solution map by

$$T(\varphi_0;t,s)\varphi = z_t(s,\varphi)$$

for $t \geq s$, where $z(s,\varphi)$ is the solution of (5.3)-(5.4). As usual, we represent $L(t,\cdot)$ by the $n \times n$ matrix valued function $\eta(t,\cdot)$ described in Chapter III. We remark that under the assumptions on $L(t,\cdot)$, $\tilde{\eta}(t,u)$ can be shown to be continuous in t for fixed u. See Kato [21] and Riesz and Sz.-Nagy [32] for related matters.

<u>Theorem 5.1</u>. Let f be C^1 and $x(\varphi)$ solve (5.1)-(5.2). For all $t \geq 0$, the map $\varphi \to x_t(\varphi)$ is C^1 and, in fact,

$$D_{\varphi=\varphi_0}\, x_t(\varphi) = T(\varphi_0;t,0). \tag{5.5}$$

<u>Proof</u>: Since the assertion is trivial for $t = 0$, we will assume $t > 0$. For $\varphi_0 \in X$ and $\varphi \in X$ define $y(\varphi) = x(\varphi + \varphi_0) - x(\varphi_0)$. Then $y_0 = \varphi$ and for $t > 0$

$$\dot{y}(t) = \dot{x}(\varphi_0 + \varphi)(t) - \dot{x}(\varphi_0)(t) = f(x_t(\varphi_0 + \varphi))$$
$$- f(x_t(\varphi_0)) = D_{\nu=x_t(\varphi_0)} f(\nu)[y_t(\varphi)] + N(t, y_t(\varphi))$$

where

$$(5.6) \qquad N(t,\nu) = f(\nu + x_t(\varphi_0)) - f(x_t(\varphi_0)) - Df(x_t(\varphi_0))\nu.$$

Note that $D_{\nu=0} N(t,\nu) = N(t,0) = 0$ for all $t > 0$ and that $N(s, y_s(\varphi))$ is continuous in s for $0 \leq s \leq t$. Therefore, (writing $T(t,0)$ for $T(\varphi_0; t,0)$),

$$(5.7) \qquad y_t(\varphi) = T(t,0)\varphi + K(t,0)N(\cdot, y_\cdot(\varphi))$$
$$= T(t,0)\varphi + \int_0^t T(t,u)\,\delta N(u, y_u(\varphi))\,du$$

where for $s \leq 0$,

$$[\int_0^t T(t,u)\,\delta N(u,y_u(\varphi))du](s) = \int_0^t [T(t,u)\,\delta](s) N(u,y_u(\varphi))du.$$

It suffices to show

$$|\varphi|^{-1} \cdot \sup_{s \leq 0} \left|\int_0^t [T(t,u)\,\delta](s) N(u,y_u(\varphi))du\right| \to 0$$

as $\varphi \to 0$. However, for any $s \leq 0$,

$$\left|\int_0^t [T(t,u)\,\delta](s) N(u,y_u(\varphi))du\right| \leq \int_0^t |T(t,u)| \cdot |N(u,y_u(\varphi))|du$$
$$\leq \int_0^t |T(t,u)|du \cdot c(\varphi) \cdot |\varphi|$$

with the constant $c(\varphi) \to 0$ as $\varphi \to 0$ because of the continuous dependence of $y_u(\varphi)$ on φ at $\varphi = 0$ and the properties of N. □

Corollary 5.2. Let $\gamma_0 > 0$ and consider the system

$$\dot{x}(t) = f(x_t, \gamma), \quad t > 0 \tag{5.8}$$

$$x_0 = \varphi \in X \tag{5.9}$$

for $|\gamma| < \gamma_0$, where $f(\varphi, \gamma)$ is Frechét differentiable in (φ, γ). The map $(\varphi, \gamma) \to x_t(\varphi, \gamma)$ is C^1 for $|\gamma| < \gamma_0$ and $\varphi \in X$.

Proof: The above follows immediately from Theorem 5.1 if we consider the $n + 1$ dimensional system

$$\frac{d}{dt} \begin{bmatrix} \gamma \\ x(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(x_t, \gamma) \end{bmatrix}, \quad t > 0$$

with the initial data defined at $t = 0$ by

$$\begin{bmatrix} \gamma \\ \varphi \end{bmatrix}. \quad \square$$

§2. The Poincaré Map

Assume now that (5.1)-(5.2) has an ω-periodic solution, $p(t)$. That is, for all $t \in \mathbb{R}$,

$$\dot{p}(t) = f(p_t) \tag{5.10}$$

and

$$p(t + \omega) = p(t). \tag{5.11}$$

In order to study the behavior of (5.1)-(5.2) for initial values, φ, near p_0, we consider $y(t) = x(t) - p(t)$, where $x = x(\varphi)$ solves (5.1)-(5.2). Then for $t > 0$,

$$(5.12) \quad \dot{y}(t) = \dot{x}(t) - \dot{p}(t) = f(x_t) - f(p_t) = f(p_t + y_t) - f(p_t)$$

$$= Df(p_t)y_t + N(t,y_t)$$

and

$$(5.13) \qquad y_0 = \varphi \in X.$$

Here, N is defined as in (5.6) and is ω-periodic in t.

Let z be the solution of the associated "linearized" (periodic) system

$$(5.14) \qquad \dot{z}(t) = Df(p_t)z_t, \quad t > s$$

$$(5.15) \qquad z_s = \varphi \in X.$$

The solution map associated with (5.14)-(5.15) will be denoted by $T(t,s)$ (the dependence upon p_0 will be suppressed). Since (5.14)-(5.15) is a linear periodic system we will have the associated space decomposition at characteristic multipliers of the periodic family $T(t,s)$, $t > s$. In fact, from the differentiability of f, (5.10) and (5.11) it follows that $\dot{p}_t$ is an ω-periodic solution to (5.14) with $s = 0$ and $z_0 = \dot{p}_0$. In what follows, we will assume that the periodic solution, $p(t)$, of (5.1) is nonconstant. In addition, it will be assumed throughout the remainder of the chapter that

$$\beta = \inf\{c \in \mathbb{R} \mid \int_{-\infty}^{0} e^{cs}\rho(s)ds < \infty\}$$

is negative. Therefore, $\dot{p}_0 \in \mathcal{N}(T(\omega,0) - I)$ is non-zero and (by Theorem 3.18 and Lemma 3.19) 1 is seen to be a characteristic multiplier for (5.14)-(5.15). Following Chapter IV, we have

$$X = \mathcal{N}(T(\omega,0) - I)^k \oplus \mathcal{R}(T(\omega,0) - I)^k$$

with $\mathcal{N}(T(\omega,0) - I)^k$ the generalized eigenspace associated with 1. We will write

$$F = \mathcal{R}(T(\omega,0) - I)^k.$$

Recall that $T(\omega,0)$ is invariant on F and $\sigma(T(\omega,0)|_F) = \sigma(T(\omega,0)) \setminus \{1\}$.

<u>Definition 5.3</u>. The periodic orbit $\mathcal{O} = \bigcup_t \{p_t\}$ is said to be <u>nondegenerate</u> provided $\dim \mathcal{N}(T(\omega,0) - I)^k = 1$.

Because $\dim \mathcal{N}(T(t + \omega, t) - I)^k$ is constant in t, we see that nondegeneracy does not depend on the particular element from $\mathcal{O}$ we refer to as p_0. We will choose $\dot{p}_0$ as the basis vector for $\mathcal{N}(T(\omega,0) - I)$ and denote by ψ the unique element from X^* that spans $\mathcal{N}(T^*(\omega,0) - I)$ and satisfies $\langle\psi, \dot{p}_0\rangle = 1$.

<u>Lemma 5.4</u>. There exists an open neighborhood, U, of p_0 in X satisfying: for any $\varphi \in U$ there exists a $t \in (0, 2\omega)$ such that $x_t(\varphi) \in F + p_0 = \{\bar{\varphi} + p_0 \mid \bar{\varphi} \in F\}$.

<u>Proof</u>: For $s > -\omega$ and $\varphi \in X$ define the function $H(s,\varphi) = \langle\psi, x_{s+\omega}(\varphi) - p_0\rangle$, where ψ is as above. Clearly, $H(s,p_0)$ is differentiable in s and $D_{s=0}H(s,p_0) = \langle\psi,\dot{x}_{\omega}(p_0)\rangle = \langle\psi,\dot{p}_0\rangle = 1$. Thus, for s sufficiently small,

$$(5.16) \qquad \begin{aligned} H(s,p_0) &= H(0,p_0) + D_{s=0}H(s,p_0) + R(s) \\ &= 0 + s + R(s), \end{aligned}$$

where $|s|^{-1} \cdot |R(s)| \to 0$ as $s \to 0$.

Recall from Chapter III, §4 that the projection $\Pi: X \to \mathcal{N}(T(\omega,0) - I)$ is given by $\Pi\varphi = \dot{p}_0\langle\psi,\varphi\rangle$. By the closedness of F, the assertion will follow if we can find, for $|\varphi - p_0|$ sufficiently small, an $s \in (-\omega,\omega)$ for which $H(s,\varphi) = 0$. In light of (5.16), this is equivalent to finding an $s \in (-\omega,\omega)$ satisfying

$$s = H(s,p_0) - R(s) - H(s,\varphi).$$

That is, for $|\varphi - p_0|$ sufficiently small we must find a fixed point of the real valued map

$$G(s) = \langle\psi, x_{s+\omega}(p_0) - x_{s+\omega}(\varphi)\rangle - R(s).$$

Choose $s_0 < 1$ from $(0,\omega)$ sufficiently small so that $|R(s)| \le \frac{1}{2} s_0|s|$ for $|s| \le s_0$. By continuous dependence, we may choose $|\varphi - p_0|$ small enough so that

$$|\langle\psi, x_{s+\omega}(p_0) - x_{s+\omega}(\varphi)\rangle| \le \frac{1}{2} s_0$$

for $-\omega \le s \le -\omega + s_0$. Thus,

$$|G(s)| \leq \frac{1}{2} s_0 + \frac{1}{2} s_0 \leq s_0$$

for any $|s| \leq s_0$. By the continuity of G, there is a fixed point of G in $[-s_0, s_0]$. □

We now proceed to the definition of the Poincaré Map under the assumptions that $\beta < 0$ and $\omega > r$. As in §4 of Chapter II, let $S(t)$, $t \geq 0$ denote the solution semi-group associated with the trivial system $\dot{x} = 0$. Again, A_0^* will denote the adjoint of the infinitesimal generator of $S(t)$, $t \geq 0$. See Chapter II, §4.

<u>Lemma 5.5</u>. Let ψ be as above, and assume that $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all t in a neighborhood of ω. Then the map $H: X \times (0,\infty) \to \mathbb{R}$ defined by

$$H(\varphi, t) = \langle \psi, x_t(\varphi) - p_0 \rangle$$

is C^1 in a neighborhood of (p_0, ω).

<u>Proof</u>:

$$\begin{aligned} H(\varphi,t) &= \langle \psi, x_t(\varphi) - p_0\rangle = \langle \psi, S(t)\varphi\rangle + \langle \psi, x_t(\varphi) - S(t)\varphi\rangle \\ &= \langle S^*(t)\psi, \varphi\rangle + \int_{-t}^{0} [d\tilde{\psi}(u)][x_t(\varphi)(u) - \varphi(0)] \\ &= \langle S^*(t)\psi, \varphi\rangle + \int_{-t}^{-r} \psi(u)\rho(u)[x_t(\varphi)(u) - \varphi(0)]du \\ &\quad + \int_{-r}^{0} [d\psi(u)][x_t(\varphi)(u) - \varphi(0)]. \end{aligned}$$

Since $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all t in a neighborhood of ω, the first term is differentiable. In fact, if $t_0 < \omega$ and $S^*(t_0)\psi \in \mathcal{D}(A_0^*)$ then

$$\frac{d}{dt}\langle S^*(t)\psi,\varphi\rangle = \langle A_0^* S^*(t)\psi,\varphi\rangle = \langle A_0^* S^*(t-t_0)S^*(t_0)\psi,\varphi\rangle$$
$$= \langle S^*(t-t_0)A_0^* S^*(t_0)\psi,\varphi\rangle$$
$$= \langle A_0^* S^*(t_0)\psi, S(t-t_0)\varphi\rangle$$

for all $t > t_0$. By Theorem 2.5 and (2.19), $\rho\psi$ is left continuous and of bounded variation in a neighborhood of $-\omega$. It follows that the remaining terms are C^1 in t near ω. $\square$

Before we continue, we should comment on the condition "$S^*(t)\psi \in \mathcal{D}(A_0^*)$" that appears in the previous lemma. Our work in §3 and §4 of the previous chapter is in some cases helpful in verifying this hypothesis. Recall from (4.25) that for $s < -r$,

$$\psi(s)\rho(s) = -\int_0^\infty \alpha(u)\eta(u,s-u)\rho(s-u)\,du,$$

where α is the element of $\mathcal{N}(T^0(\omega,0) - I)$ that defines ψ at 0. Thus, from (2.19), if $t > r$ it follows that

$$[\dot{\widetilde{S^*(t)\psi}}](s) = [\dot{\tilde{\psi}}](s-t) = \rho(s-t)\psi(s-t)$$
$$= -\int_0^\infty \alpha(u)\eta(u,s-t-u)\rho(s-t-u)\,du$$

for $s < 0$. Since $S^*(t)\psi$ is absolutely continuous on $[-r,0)$, the three conditions of Theorem 2.5 reduce to that

$$w(s) = \int_0^\infty \alpha(u)\eta(u,s-t-u)\rho(s-t-u)\,du$$

is (i′) of bounded variation and left continuous on $[-r,0)$, and (ii′) locally absolutely continuous on $(-\infty,-r)$ with ρ^{-1} essentially bounded on $(-\infty,-r)$. These hypotheses are trivially satisfied for the finite delay case since $\eta(t,v) = 0$ for all $v < -r$.

For an infinite delay example we consider a model from mathematical biology. The scalar equation

$$\dot{x}(t) = -a[x(t) + 1]\int_{-\infty}^{-1} x(t + s)be^{b(s+1)}ds \tag{5.17}$$

where $a,b > 0$ arises in the description of population oscillations observed in single species communities. (See May [27].) For this equation, the obvious choice for ρ is $\rho(s) = e^{bs}$ and $r = 1$. Clearly $\beta = -b < 0$ and (5.17) takes the form (5.1) for

$$f(\varphi) = -a[\varphi(0) + 1]\int_{-\infty}^{-1} \varphi(s)be^{b}\ \rho(s)ds.$$

If $p(t)$ is a periodic solution to (5.1), then it is easily verified that

$$\begin{aligned} Df(p_t)\varphi = -a[p(t) + 1]&\int_{-\infty}^{-1} be^{b(s+1)}\varphi(s)ds \\ &- a[\int_{-\infty}^{-1} p(t + s)be^{b(s+1)}ds]\varphi(0). \end{aligned}$$

Thus, $\eta(t,u) = -a[p(t) + 1]e^{b}$ for $u < -1$. Conditions (i′) and (ii′) hold trivially since

$$\begin{aligned} w(s) &= \int_{0}^{\infty} \alpha(u)[a(p(u) + 1)e^{b}]e^{b(s-t-u)}du \\ &= \text{constant} \cdot e^{bs}. \end{aligned}$$

<u>Corollary 5.6</u>. For $\gamma_0 > 0$, consider the system (5.8)-(5.9) under the additional assumption that $f(\cdot,0) = f(\cdot)$. For ψ as above, if $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all t in a neighborhood of ω, the map $G: X \times (-\gamma_0, \gamma_0) \times (0,\infty) \to \mathbb{R}$ defined by $G(\varphi,\gamma,t) = \langle\psi, x_t(\varphi,\gamma) - p_0\rangle$ is C^1 in a neighborhood of $(p_0, 0, \omega)$.

The proof, which will be omitted, follows very closely that of Lemma 5.5 and makes use of Corollary 5.2.

<u>Lemma 5.7</u>. Assume β, ω and ψ are as above. Assume also that $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all t in a neighborhood of ω and the orbit, $\mathcal{O}$, is nondegenerate. There exists a neighborhood, U', of 0 in X and a C^1 real valued function, τ, defined on U' such that $\tau(0) = \omega$ and

$$x_{\tau(\varphi)}(\varphi + p_0) \in F + p_0$$

for all $\varphi \in U'$. If U' is chosen sufficiently small, τ is positive, bounded and unique.

<u>Proof</u>: Consider $G: X \times (0,\infty) \to \mathbb{R}$ defined by $G(\varphi,t) = \langle\psi, x_t(\varphi + p_0) - p_0\rangle$. Lemma 5.5 shows G to be C^1 in a neighborhood of $(0,\omega)$. Note that $G(0,\omega) = \langle\psi, x_\omega(0 + p_0) - p_0\rangle = \langle\psi, p_\omega - p_0\rangle = 0$ and

$$D_{t=\omega}\, G(p_O, t) = \langle \psi, \dot{x}_\omega(0 + p_O) \rangle = \langle \psi, \dot{p}_\omega \rangle = \langle \psi, \dot{p}_O \rangle = 1.$$

By the Implicit Function Theorem [23], there exists a neighborhood U' of $0 \in X$ and a C^1 function τ for which $\tau(0) = \omega$ and $G(\varphi, \tau(\varphi)) = 0$. That is,

$$\langle \psi, x_{\tau(\varphi)}(\varphi + p_O) - p_O \rangle = 0.$$

As remarked in Lemma 5.4, this is equivalent to saying

$$x_{\tau(\varphi)}(\varphi + p_O) - p_O \in F.$$

The uniqueness of τ follows from the statement of the Implicit Function Theorem. The other properties of τ follow from its smoothness near 0. □

We are now able to define the <u>Poincaré Map</u> in the neighborhood $U_F = F \cap U'$, where U' is given in the previous lemma. We define $\mathcal{P}: U_F \to F$ by

$$\mathcal{P}(\varphi) = x_{\tau(\varphi)}(\varphi + p_O) - p_O.$$

Then $\mathcal{P}$ is a C^1 function on U_F and $\mathcal{P}(0) = 0$. Note also that

$$\begin{aligned}
D\mathcal{P}(0) &= D_{\substack{\varphi=0\\ \varphi\in F}}\, x_{\tau(0)}(\varphi + p_O) + \dot{x}_{\tau(0)}(0 + p_O) \circ D_{\substack{\varphi=0\\ \varphi\in F}}\, \tau(\varphi) \\
&= D_{\substack{\varphi=0\\ \varphi\in F}}\, x_\omega(\varphi + p_O) + \dot{p}_\omega \cdot D_{\varphi\in F}\, \tau(0) \\
&= T(\omega, 0)\big|_F + \dot{p}_O \cdot D_{\varphi\in F}\, \tau(0)
\end{aligned}$$

by Theorem 5.1. Because $D\mathcal{P}(0): F \to F$ and $\dot{p}_O \notin F$ we must conclude that $D_{\varphi\in F}\, \tau(0) = 0$. Therefore, $D\mathcal{P}(0) = T(\omega, 0)\big|_F$.

Under the assumption of nondegeneracy,

$$\sigma(D\mathcal{P}(0)) = \sigma(T(\omega,0)|_F) = \sigma(T(\omega,0)) \setminus \{1\}.$$

§3. Nondegenerate Periodic Orbits

In this section we apply the Poincaré Map to the study of the behavior of solutions near a nondegenerate periodic orbit, $\mathcal{O} = \bigcup_t \{p_t\}$. As in the previous section, we shall assume throughout that $\beta < 0$ and $\psi \in X^*$ is the unique element spanning $\mathcal{N}(T^*(\omega,0) - I)$ that satisfies $\langle\psi,\dot{p}_0\rangle = 1$.

We first extend to our class of FDE's the Poincaré criteria for the "stability" of $\mathcal{O}$. See Coppel [7] for the analogous ODE result and Stokes [37] for the extension to finite delay equations.

Definition 5.8. (i) A periodic orbit, $\mathcal{O}$, is said to be orbitally stable provided: for any $\epsilon > 0$ there exists an open neighborhood V of $\mathcal{O}$ in X such that if $\varphi \in V$ then

$$\mathrm{dist}(x_t(\varphi),\mathcal{O}) = \max\{|x_t(\varphi) - p_\theta| \mid \theta \in [0,\omega]\} < \epsilon$$

for $t \geq 0$.

(ii) A periodic orbit, $\mathcal{O}$, is said to be orbitally asymptotically stable provided it is stable and

$$\mathrm{dist}(x_t(\varphi),\mathcal{O}) \to 0$$

as $t \to \infty$ for any $\varphi \in V$ (given in (i)).

<u>Lemma 5.9</u>. Assume $p(t)$ is a nonconstant ω-periodic solution to (5.1) with $\omega > r$. Assume also that the periodic orbit, $\mathcal{O}$, is nondegenerate and $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all t in a neighborhood of ω. If all characteristic multipliers (other than 1) of (5.14) have moduli strictly less than 1, then $\mathcal{O}$ is orbitally asymptotically stable.

<u>Proof</u>: Note that $\mathcal{O}$ is compact. Thus, it suffices to show there is an open neighborhood of p_0 such that if φ is taken from that neighborhood, $x_t(\varphi)$ satisfies the criteria for orbital asymptotic stability. Analogous neighborhoods may then be constructed at a finite number of $\rho_\theta \in \mathcal{O}$ to construct the neighborhood V.

By Lemma 5.4, there exists an open neighborhood, U, of p_0 in X such that if $\varphi \in U$, then $x_t(\varphi) \in F + p_0$ for some $t \in (0,2\omega)$. Therefore, it suffices to find an open neighborhood, B, of p_0 in $F + p_0$ such that the elements of B all satisfy the criteria for orbital asymptotic stability.

We choose U' to be that neighborhood given in Lemma 5.7 and $U_F = F \cap U'$. Recall that for $\varphi \in U_F$, $\mathcal{P}(\varphi) = x_{\tau(\varphi)}(\varphi + p_0) - p_0$ where τ is defined on $U' \supset U_F$ and $D\mathcal{P}(0) = T(\omega,0)|_F$. Note that $\mathcal{P}^2(\varphi) = \mathcal{P}(\mathcal{P}(\varphi)) =$

$x_{\tau(2;\varphi)}(\varphi + p_0) - p_0$ where $\tau(2;\varphi) = \tau(\varphi) + \tau(\mathscr{P}(\varphi))$.
More generally,

$$\mathscr{P}^k(\varphi) = x_{\tau(k;\varphi)}(\varphi + p_0) - p_0$$

where

$$\tau(k;\varphi) = \sum_{i=0}^{k-1} \tau(\mathscr{P}^i(\varphi)).$$

It follows that $D\mathscr{P}^k(0) = T(k\omega, 0)|_F$. Applying Theorem 4.6 to this ω-periodic family we see that there exists $\nu > 0$ and $M > 0$ such that

$$|T(k\omega, 0)|_F| \leq Me^{-\nu k\omega}.$$

Let j denote the smallest positive integer such that $Me^{-\nu j\omega} < 1$. By the differentiability of $\mathscr{P}^k$ at 0, we may find a, perhaps smaller, neighborhood (also to be called U_F) and a $c \in (0, 1 - Me^{-\nu j\omega})$ such that

$$|\mathscr{P}^k(\varphi)| \leq (Me^{-\nu j\omega} + c)|\varphi|$$

for $\varphi \in U_F$. We define $\xi = Me^{-\nu j\omega} + c$.

Now, let $\tau_0 = \sup\{\tau(j;\varphi) | \varphi \in U'\}$. By Lemma 5.7, τ_0 may be assumed finite. The differentiability of $x_t(\varphi)$ near p_0 implies the existence of a constant K (independent of t for $0 \leq t \leq \tau_0$) such that $|x_t(\varphi + p_0) - x_t(p_0)| \leq K|\varphi|$ for $\varphi \in U_F$. Thus, if $\varepsilon > 0$ is given and φ is an element of

$$B(\tfrac{\varepsilon}{K}) = \{\varphi \in F \,|\, |\varphi| \leq \tfrac{\varepsilon}{K}\},$$

then $|x_t(\varphi + p_0) - p_t| < \epsilon$ for $0 \leq t \leq \tau_0$. Since (for all ϵ sufficiently small) $B(\frac{\epsilon}{K})$ is contained in U_F, we have that

$$\mathcal{P}^j(B(\tfrac{\epsilon}{K})) \subset B(\xi \cdot \tfrac{\epsilon}{K})$$

and, therefore,

$$|x_{t+\tau(j;\varphi)}(\varphi + p_0) - x_t(p_0)| < \xi \cdot \epsilon$$

for $0 \leq t \leq \tau_0$. More generally,

$$|x_{t+\tau(mj;\varphi)}(\varphi + p_0) - p_t| < \xi^m \cdot \epsilon$$

for $0 \leq t \leq \tau$, $m \geq 0$ since $\xi \in (0,1)$.

The orbital and orbital asymptotic stability of $\mathcal{O}$ follow immediately since $\xi \in (0,1)$ and $\epsilon > 0$ was arbitrary. □

The hypotheses of the previous lemma are by no means minimal. First, the assumption that $\omega > r$ may be removed since we may view $p(t)$ as an $\ell\omega$-periodic solution to (5.1), where ℓ is a positive integer chosen large enough so that $\ell\omega > r$. It must then be assumed that $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all t in a neighborhood of $\ell\omega$. However, it is known [18] that $S^*(t_0)\psi \in \mathcal{D}(A_0^*)$ implies $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all $t > t_0$.

Since the characteristic multipliers of (5.14) correspond to normal eigenvalues of $T(\omega,0)$, the

Spectral Mapping Theorem shows the assumption that $\sigma(T(\omega,0))\setminus\{1\}$ lies strictly within the unit circle implies that $\sigma(T(\ell\omega,0))\setminus\{1\} = \sigma(T(\omega,0))^{\ell}\setminus\{1\}$ also lies strictly within the unit circle. Since $\beta < 0$, 1 is a normal eigenvalue for $T(\ell\omega,0)$.

Finally, if $\mathcal{O}$ is nondegenerate as an ω-periodic orbit, it is also nondegenerate as an $\ell\omega$-periodic orbit. In fact, if $\varphi \in \mathcal{N}(T(\ell\omega,0) - I)^k$ but is not an element of $\mathcal{N}(T(\omega,0) - I)^k = \mathcal{N}(T(\omega,0) - I)$, then $\bar{\varphi} = (T(\omega,0) - I)^k\varphi$ satisfies $[T^{\ell-1}(\omega,0) + \ldots + T^2(\omega,0) + T(\omega,0) + I]^k\bar{\varphi} = 0$. Thus, $0 \in \sigma([T^{\ell-1}(\omega,0) + \ldots + T(\omega,0) + I]^k)$ and, by the Spectral Mapping Theorem, there is some $\lambda \in \sigma(T(\omega,0))$ for which $0 = [\lambda^{\ell-1} + \ldots + \lambda^2 + \lambda + 1]^k = 0$. Therefore, $0 = [\lambda^{\ell-1} + \ldots + \lambda^2 + \lambda + 1]$ and (since all roots of this equation have moduli equal to 1) we conclude that $\lambda = 1$. This contradiction shows that $\dim \mathcal{N}(T(\ell\omega,0) - I)^k = 1$.

Combining these remarks, we have proved under the ever-present assumption: $\beta < 0$

Theorem 5.10. Assume $p(t)$ is a nonconstant ω-periodic solution to (5.1) and $\mathcal{O}$ is nondegenerate. Assume also that $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for some $t \geq 0$. If all characteristic multipliers (other than 1) of (5.14)

have moduli strictly less than 1, then $\mathcal{O}$ is orbitally asymptotically stable.

We next generalize a result of Hale [14] to the present class of equations.

Lemma 5.11. If $\mathcal{O}$ is nondegenerate there exists an open neighborhood V of $\mathcal{O}$ such that $V \setminus \mathcal{O}$ contains no ω-periodic solutions to (5.1).

Proof: As argued in Lemma 5.9, the compactness of $\mathcal{O}$ implies it suffices to find a neighborhood of p_0 that contains no ω-periodic solutions other than the elements of $\mathcal{O}$.

Assume the opposite. Then there exists a sequence of ω-periodic solutions to (5.1) that approach p_0, yet are not elements of $\mathcal{O}$. By Lemma 5.4 and the continuous dependence of solutions on initial data, there exists a sequence $\{\varphi^{(m)}\}_{m=1}^{\infty}$ of elements of F that approach 0 and $x_\omega(\varphi^{(m)} + p_0) = \varphi^{(m)} + p_0$ for each $m \geq 0$.

Consider the function $G:F \to F$ defined by $G(\varphi) = \Pi'[x_\omega(\varphi + p_0) - (\varphi + p_0)]$ where Π' is the projection of X onto F. Then, for $m \geq 1$, $G(\varphi^{(m)}) = 0$. Note, however, that $G(0) = 0$ that $DG(0) = \Pi'[T(\omega,0)|_F - I] = T(\omega,0)|_F - I$ which is an isomorphism under the assumptions that $\mathcal{O}$ is nondegenerate and $\beta < 0$. Thus, in a neighborhood of 0 in F, G is an isomorphism and, therefore, one-to-one. We must conclude that $\varphi^{(m)} = 0$ for all m sufficiently large. This contradicts the fact that the associated ω-periodic solutions to (5.1) were assumed not to be elements of $\mathcal{O}$. $\square$

Along the same lines as the previous lemma, we have

Lemma 5.12. Let $\psi \in X^*$ be as above, and assume as in Theorem 5.10 that $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for some $t \geq 0$. If $\mathcal{O}$ is nondegenerate then for $\epsilon > 0$ sufficiently small there exists an open neighborhood W of $\mathcal{O}$ such that $W \setminus \mathcal{O}$ contains no $\bar{\omega}$-periodic solutions with $|\omega - \bar{\omega}| < \epsilon$.

Proof: As argued after Lemma 5.9, we may assume without losing generality that $\omega > r$ and $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all t in a neighborhood of ω. Thus, we may define the Poincaré Map on the neighborhood U_F given in §2. That is, $\mathcal{P}(\varphi) = x_{\tau(\varphi)}(\varphi + p_0) - p_0$ where $\tau: U_F \to (0, \omega)$ is C^1 and satisfies $\tau(0) = \omega$. By further restricting U_F, we may assume $|\tau(\varphi) - \omega| < \epsilon$ for all $\varphi \in U_F$.

By arguments similar to those in the previous lemma, it suffices to show that for no $\bar{\varphi} \in U_F$ is $x_{\bar{\omega}}(\varphi + p_0) = \varphi + p_0$ for some $\bar{\omega} \in (\omega - \epsilon, \omega + \epsilon)$. It follows from the construction of τ (see Lemma 5.7) and the uniqueness assertion of the Implicit Function Theorem that if $\varphi \in U_F$ and $x_{\bar{\omega}}(\varphi + p_0) = \varphi + p_0$ for some $\bar{\omega}$ satisfying $|\omega - \bar{\omega}| < \epsilon$, then $\bar{\omega} = \tau(\varphi)$. Thus, periodic solutions to (5.1) with periods near ω correspond to fixed points of $\mathcal{P}$.

To show that $\varphi = 0$ is the only fixed point of $\mathcal{P}$, we define $G_1 : U_F \to F$ by $G_1(\varphi) = \mathcal{P}(\varphi) - \varphi$. Clearly, $G_1(0) = 0$ and $DG_1(0) = T(\omega,0)|_F - I$, which is an isomorphism since $\beta < 0$ and $\mathcal{O}$ is nondegenerate. The Inverse Function Theorem shows G_1 to be an isomorphism in a neighborhood of 0. The result follows immediately. □

Our final theorem is related to a result of Hale [14], where an analogue was shown for the finite delay situation by a different technique.

Theorem 5.13. For $\gamma_0 > 0$ consider the system (5.8)-(5.9) under the assumption that $f(\cdot,0) = f(\cdot)$. Let (5.1) have a nondegenerate periodic orbit, $\mathcal{O}$, whose period, ω, is larger than r. Let $\psi \in X^*$ be as usual, and assume $S^*(t)\psi \in \mathcal{D}(A_0^*)$ for all t in a neighborhood of ω. Then, there exists a neighborhood V of $\mathcal{O}$ such that for all γ sufficiently close to 0, (5.8)-(5.9) has a unique periodic orbit, $\mathcal{O}_\gamma$, in V of period $\omega(\gamma)$. The period, $\omega(\gamma)$, is C^1 in γ and

$\omega(0) = \omega$. Furthermore, $\mathcal{O}_\gamma$ varies continuously in γ and $\mathcal{O}_0 = \mathcal{O}$.

Proof: Define $G:(0,\infty) \times (-\gamma_0, \gamma_0) \times F \to \mathbb{R}$ by $G(t,\gamma,\varphi) = \langle\psi, x_t(\varphi + p_0, \gamma) - p_0\rangle$. Corollary 5.6 shows G to be C^1 in a neighborhood of $(\omega,0,0)$. Note that $G(\omega,0,0) = \langle\psi, x_\omega(0 + p_0, 0) - p_0\rangle = 0$ and $D_{t=\omega}(t,0,0) = \langle\psi, \dot{x}_\omega(0 + p_0), 0\rangle = \langle\psi, \dot{p}_\omega\rangle = 1$. The Implicit Function Theorem implies the existence of a neighborhood, U, of $(0,0)$ in $(-\gamma_0, \gamma_0) \times F$ and a unique C^1 function $\bar{\tau}(\gamma,\varphi)$ defined on U such that $\bar{\tau}(0,0) = \omega$ and $G(\bar{\tau}(\gamma,\varphi),\gamma,\varphi) = 0$ for all $(\gamma,\varphi) \in U$. The uniqueness assertion of the Implicit Function Theorem shows that $\bar{\tau}(0,\varphi) = \tau(\varphi)$ for $|\varphi|$ sufficiently small, where $\tau(\varphi)$ is defined in Lemma 5.7.

Since $\bar{\tau}$ is C^1 in U, we may define the C^1 function $H:U \to F$ by

$$H(\gamma,\varphi) = x_{\bar{\tau}(\gamma,\varphi)}(\varphi + p_0, \gamma) - p_0 - \varphi$$

Then $H(0,0) = x_{\bar{\tau}(0,0)}(p_0, 0) - p_0 = x_\omega(p_0) - p_0 = 0$ and

$$\begin{aligned} D_{\varphi=0}\, H(0,\varphi) &= D_{\varphi=0}\, [x_{\bar{\tau}(0,\varphi)}(\varphi + p_0, 0) - \varphi - p_0] \\ &= D_{\varphi=0}\, x_{\tau(\varphi)}(\varphi + p_0) - I = T(\omega,0)|_F - I. \end{aligned}$$

Since $\beta < 0$ and $\mathcal{O}$ is nondegenerate, this is an isomorphism. By the Implicit Function Theorem there exists a $\gamma_1 > 0$ and a C^1 function $\varphi(\gamma) \in F$ defined for $|\gamma| < \gamma_1$ satisfying $\varphi(0) = 0$ and $H(\gamma, \varphi(\gamma)) = 0$. We define $\omega(\gamma) = \bar{\tau}(\gamma, \varphi(\gamma))$. Then, considering the definition of H,

$$x_{\omega(\gamma)}(\varphi(\gamma) + p_0, \gamma) - \varphi(\gamma) - p_0 = 0.$$

Thus, $\varphi(\gamma) + p_0$ defines a $\omega(\gamma)$-periodic solution to (5.8)-(5.9). Clearly $\omega(\gamma)$ and $\varphi(\gamma)$ are C^1 in γ, $\omega(0) = \omega$ and $\varphi(0) = 0$. By the continuous dependence of $x_t(\varphi, \gamma)$ in (φ, γ), we see that $\mathcal{O}_\gamma = \bigcup_t \{x_t(\varphi(\gamma) + p_0, \gamma)\}$ varies continuously in γ and $\mathcal{O}_0 = \mathcal{O}$.

The fact that $\mathcal{O}_\gamma$ is the only periodic orbit of (5.8) in V follows from the uniqueness assertion of the Implicit Function Theorem and the form of H. $\square$

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