

AN EVALUATION OF RELATIVISTIC THERMODYNAMICS

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ABSTRACT

AN EVALUATION OF RELATIVISTIC THERMODYNAMICS

By

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The problem of which of the formulations of relativistic thermodynamics that have been proposed by Planck, Eckart, Ott, and Landsberg is correct is examined. Due to the lack of experimental data on relativistic thermodynamic systems, it is not possible to compare predictions made by the various formulations with experimental data. But, using as a model the process of evaporation, I found that it is possible to study the consistency between the transformation laws for temperature and for mechanical energy. The result obtained is that only the formulation by Ott is free of contradiction.

Ott's proposed transformation laws are further evaluated in terms of their compatibility with relativistic formulations of fluid dynamics and statistical mechanics. Compatibility is to be expected because thermodynamics, fluid dynamics, and statistical mechanics are compatible in their non-relativistic formulations. The lack of contradiction between Ott's formulation and the transformation law for mechanical energy, and its compatibility with formulations of relativistic fluid dynamics and statistical mechanics, give support to a conclusion that the Ott formulation of relativistic thermodynamics is correct.

AN EVALUATION OF RELATIVISTIC

THERMODYNAMICS

Ву

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DEDICATION

I want to dedicate this thesis to my wife Beth for giving me the support needed during this work.

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I. HISTORY

A. Planck-Einstein

In 1907 and 1908 Max Planck^{1,2} published papers on relativity theory in which a formulation of relativistic thermodynamics was developed. He approached the problem using a variational method that was a generalization of the work by Helmholtz. He generalized Helmholtz's Lagrangian,

$$L = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - F, \qquad (1)$$

where

$$F = U - TS$$
 (2)

is the Helmholtz free-energy, to

$$L = -(mc^{2} + F^{\circ}) \left[1 - (\frac{dx}{dt})^{2} / c^{2}\right]^{\frac{1}{2}}$$
(3)

Equations (1) and (3) are for one-dimensional motion and F° is the rest-frame free energy. This is the same generalization that he used in 1906³ to develop a relativistic Lagrangian formulation of mechanics.

The transformation law for the temperature was calculated as follows. Let T° be the rest-frame temperature and T the temperature in the moving frame of reference. Now write

$$T = DT^{\circ}, \qquad (4)$$

where D is an undetermined function of the relative velocities between the two frames of reference. This function must be even with respect to the velocity \underline{v} and become unity when \underline{v} approaches zero. The entropy is defined to be $S = \frac{\partial L}{\partial T}$, where T is the temperature. Thus, taking the partial derivative of Eqn (2) with respect to the temperature T and using Eqn (4), one gets

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$$\frac{\partial \mathbf{L}}{\partial \mathbf{T}} = -\gamma^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{T}}^{\circ} = -\frac{\gamma^{-1}}{D} \frac{\partial \mathbf{F}}{\partial \mathbf{T}}^{\circ} = \frac{\gamma^{-1}}{D} \mathbf{S}^{\circ}, \qquad (5)$$

where $\gamma^{-1} = [1-\beta^2]^{\frac{1}{2}}$ and $\beta = v/c$. Planck argued⁴, as follows below, that a reversible adiabatic acceleration would leave the entropy constant. Consider a thermodynamic system at rest in the frame Σ° . Reversibly and adiabatically accelerate the system until it has the same velocity \underline{v} as the system Σ relative to Σ° . Since the acceleration was adiabatic, the system still has the same entropy (S°) relative to Σ° . However, the system now has zero velocity with respect to Σ . Thus it is now in the same situation with respect to Σ as it previously was to Σ° . Hence, the entropy (S) of the system relative to Σ° . Therefore, the entropy of a system is an invariant. Then, since S=S°

 $\frac{\partial \mathbf{L}}{\partial \mathbf{T}} = \frac{\partial \mathbf{L}^{\circ}}{\partial \mathbf{T}^{\circ}} ,$



and therefore

or,

$$\frac{\gamma^{-1}}{D} = 1,$$

$$D = \gamma^{-1}.$$
(6)

Hence, from Eqn (4),

$$T = \gamma^{-1} T^{\circ}.$$
 (7)

From this transformation law and the invariance of the entropy, it follows that the transformation law for heat (Q) is the same as for the temperature.

Therefore, the transformation laws for the thermodynamic quantities are, in the Planck formulation.

$$S = S^{\circ},$$
 (8.1)

$$Q = \gamma^{-1} Q^{\circ}, \qquad (8.2)$$

$$\mathbf{T} = \gamma^{-1} \mathbf{T}^{\circ}, \qquad (8.3)$$

$$P = P^{\circ},$$
 (9.1)

$$V = \gamma^{-1} V^{\circ}. \qquad (9.2)$$

The transformation laws for the pressure (P) and the volume (V) were determined from mechanical considerations and thus are numbered differently.

An equivalent approach to the transformation law for heat (and hence for the temperature) follows. It is used to show the need, in the Planck formulation, for an arbitrary force called the <u>Führungskraft</u>. Planck defined heat to be pure energy. Hence it made no sense to talk about the velocity of heat. This meant that to all observers the heat reservoir and the transferred heat have no momentum, effectively keeping them at rest. (This might now be strange because of the equivalence between mass and energy; however, in 1907 this equivalence was probably not completely appreciated.)

We examine, in the rest frame, the process of the transfer of an amount of heat (ΔQ°) from the reservoir to a body. Since heat is pure energy, the body increases its rest-energy by the amount

$$\Delta E^{\circ} = \Delta Q^{\circ}. \tag{10}$$

This implies an increase in the rest mass of

$$\Delta m^{\circ} = \Delta Q^{\circ} / c^{2}. \qquad (11)$$

We now examine this process from a frame of reference in which the body is at rest, but has a velocity \underline{v} with respect to the reservoir. Since the heat is energy, the body increases its energy by an amount,

$$\Delta \mathbf{E} = \gamma \Delta \mathbf{E}^{\circ} . \tag{12}$$

This energy carries no momentum, in accordance with the postulate about heat, but it does increase the mass of the body by an amount,

$$\Delta m = \gamma \Delta m^{\circ}. \tag{13}$$

Thus, the rest frame observer would see the body slow down due to the increase in its mass. However, to an observer in the body's frame of reference there would be no change because the body had no momentum in the beginning and the heat transferred into the body carried no momentum. This paradox is resolved by creating a force, called the <u>Führungskraft</u>, that increases the body's momentum such that its velocity \underline{v} remains constant with respect to the rest observer. This force is

$$\underline{F}_{F} = \underline{v} \frac{dm}{dt}, \qquad (14)$$

or,

$$\frac{F_{\rm F}}{F} = (\underline{v} \ \gamma/c^2) \ \frac{dQ^{\circ}}{dt} , \qquad (15)$$

The work done on the body (- ΔW) by this force in the time Δt is

$$-\Delta W = \underline{v} \cdot \underline{F}_{F} \Delta t = \beta^{2} \gamma \Delta Q^{\circ}. \qquad (16)$$

Hence, from the first law of thermodynamics,

 $\Delta Q = \Delta E + \Delta W,$

we have,

$$\Delta Q = \gamma \Delta Q^{\circ} - \gamma \beta^{2} \Delta Q^{\circ} = \gamma (1 - \beta^{2}) \Delta Q^{\circ},$$

or,



 $\Delta Q = \gamma^{-1} \Delta Q^{\circ}. \qquad (17)$

This is the same as Eqn (8.2).

In a 1907 paper⁵, Einstein attacked the problem of relativistic thermodynamics. Instead of Planck's variational approach, he examined contributions to the thermodynamic quantities that were simultaneous in the rest frame of the system (Σ). He then compared these results with those obtained by similar calculations made from a moving frame of reference (Σ). His results for the transformation laws of the thermodynamic quantities agreed with Planck's, and the agreement has been used in recent literature⁶ to add credence to this formulation, which today is commonly known as the Planck-Einstein formulation. However, it is never mentioned that when Einstein wrote the first law of thermodynamics in his paper, he included the work contributed by the Führungskraft (16). This inclusion guaranteed that Einstein's solution agree with Planck's. But, because of the inability of Planck's heat to carry momentum, his and Einstein's formulation is noncovariant. This is not surprising, because the development of four-dimensional tensors was just beginning in 1908. This development, started by Minkowski⁷, was finished after his death by Sommerfeld with the publication of two papers⁸ in 1910. The power and significance of their work were only appreciated by physicists after Einstein's proposal of the General Theory of Relativity in

1916 and the development of Relativistic Quantum Mechanics at the end of the next decade.

An important aspect of the work in General Relativity and Relativistic Quantum Mechanics is that the Relativity Principle was reformulated. Instead of saying that all the laws of physics must be of the same form in all inertia frames, the more general statement is that the laws of physics belong to an irreducible representation of the Lorentz Group. Thus, all the operators and variables in the equation stating a physical law are such that each term in the equation belongs to the same irreducible representation of the Lorentz Group. This means that in every frame it satisfies the same irreducibility condition. Because the Planck definition of heat does not allow the heat to ever carry momentum, the heat only appears in the time-like component of a four-vector. This means that this formulation of relativistic thermodynamics does not satisfy this condition of covariance or irreducibility.

The fact that relativistic thermodynamics was ignored during the 1920's is not surprising. The major thrust of research was in the development of Quantum Mechanics. In areas related to relativistic thermodynamics, relativistic fluid dynamics and cosmology, the problem of relativistic thermodynamics was of secondary importance. Relativistic fluid dynamics was concerned with adiabatic flow and barotropic flow. Eddington's⁹ view of

the role of relativistic thermodynamics in cosmology was that " . . . the transformation to moving axes introduces great complications without any evident advantages, and is of little interest except as an analytical exercise."

B. Eckart

In 1940 C. Eckart¹⁰ examined the possibility of constructing a systematic theory of irreversible processes. In this development he used what he called an ε -substitution, where ε is the internal energy per unit mass of substance, to verify certain equations. In this method one replaces ε by ε + f($\Sigma_{K=1}^{N}M_{k}C_{k}$), where f is arbitrary,

 M_R is the molecular weight of the substance k and C_k is its concentration. Because $\Sigma_{K=1}^N M_k C_k = 1$ and the zero point of the energy is arbitrary, the equations must be invariant when the ε -substitution is made. In a footnote¹¹ he voiced concern that the equivalence of mass and energy made the status of the ε -substitution unclear.

Because of this concern he¹² did develop a relativistically invariant theory of the simple fluid. This formulation is completely different from the Planck-Einstein formulation and hence will be presented.

Eckart assumed a Galilean metric of the form,

$$g\alpha\beta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
(18)

with $(x^{\circ} = ct, x^{1}, x^{2}, x^{3})$. (Greek indices range from 0 to 3 and Latin indices range from 1 to 3.) He used the four-vector m^{α} to represent matter. It has units of gm/cc. He then defined two projection operators,

$$U^{\alpha} = m^{\alpha} / (m^{\gamma} m_{\gamma})^{\frac{1}{2}}$$
(19)

and

$$s_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} + U^{\alpha}U_{\beta}, \qquad (20)$$

such that for any vector F^{α} ,

$$f = -U_{\alpha} F^{\alpha}$$
(21)

is the projection of \textbf{F}^{α} on the proper-time axis, and

$$f^{\alpha} = S_{\beta}^{\alpha} F^{\beta}$$
 (22)

is the projection of F^{α} into proper-space. He then defined the proper-rate of change operator (D) as

$$D \equiv U^{\alpha} \frac{\partial}{\partial \mathbf{x}^{\alpha}} .$$
 (23)

He used these quantities in the law of conservation of energy-momentum,

$$\frac{\partial w^{\alpha\beta}}{\partial \chi^{\alpha}} = 0, \qquad w^{\alpha\beta} = w^{\beta\alpha}, \qquad (24)$$

where $W^{\alpha\beta}$ has units of erg/cm³, to derive a statement of the first law of thermodynamics. He used the proper components of $W^{\alpha\beta}$,

$$w = W^{\alpha\beta} U_{\alpha} U_{\beta}, \qquad (25)$$

$$w^{\alpha} = -S_{\beta}^{\alpha} W^{\beta \gamma} U_{\gamma}, \qquad (26)$$

and

$$w^{\alpha\beta} = s_{\gamma}^{\alpha} s_{\beta}^{\beta} W^{\gamma\beta}, \qquad (27)$$

to define the internal energy, ϵ , and the heat flow, q^{α} , as follows:

$$m(\varepsilon + a) = w, \qquad (28)$$

where ε has units of erg/gm and a is an arbitrary constant;

$$q^{\alpha} = cw^{\alpha}$$
 (29)

and has units of erg/cm² sec. Then by taking the scalar product,

$$-U_{\beta} \left(\frac{\partial W^{\alpha\beta}}{\partial x^{\alpha}}\right) = 0, \qquad (30)$$

and using the proper-components of $W^{\alpha\beta}$, the definitions of the internal energy, ε , and the heat flow, q^{α} , he derived a statement of the first law of the form,

$$mD\varepsilon + \frac{1}{c}[(\partial q^{\alpha}/\partial x^{\alpha}) + q^{\alpha}DU^{\alpha}] + w^{\alpha\beta}(\partial U_{\beta}/\partial x^{\alpha}) = 0. \quad (31)$$

This is quite similar to the following classical form of the first law

$$m(D\varepsilon/Dt) + \nabla \cdot q - (P \cdot \nabla) \cdot V = 0,$$

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where \underline{q} is the classical heat flow, \underline{V} the velocity, and P the total stress. He interpreted the term $\underline{q}^{\alpha}_{c} DU^{\alpha}$ in Eqn (31) as the work done by a flow of heat through accelerated matter.

To introduce the quantities temperature and entropy he defined the hydrostatic pressure (p) as

$$p = \frac{1}{3} w_{\alpha}^{\alpha}, \qquad (32)$$

and the viscous stress tensor ($P^{\alpha\beta}$) as

$$P^{\alpha\beta} = -w^{\alpha\beta} + pS^{\alpha\beta}.$$
 (33)

By solving Eqn (33) for $w^{\alpha\beta}$ and substituting this into Eqn (31), the first law can be rewritten as

$$m(D_{\varepsilon} + pDv) + \frac{1}{c} [(\partial q^{\alpha} / \partial x^{\alpha}) + q^{\alpha}DU^{\alpha}] + -P^{\alpha\beta}(\partial U_{\beta} / \partial x^{\alpha}) = 0.$$

The term mpDv appears because

$$S^{\alpha\beta}(\partial U_{\beta}/\partial x^{\alpha}) = (\partial U^{\alpha}/\partial x^{\alpha}) = mDv,$$

where $v \equiv 1/m$ is the invariant specific volume.

Since ε is only a function of p and v for a simple fluid, Eckart used a standard argument to show that there are two functions θ and η such that

$$D\varepsilon + pDv = \theta D\eta. \tag{34}$$

He then identified θ as being the temperature and η as being the specific entropy. The significant property of θ and η is that they are true scalars because ε , p and v are true scalars. In fact, his entire approach was such that he formulated a relativistically invariant scalar thermodynamics.

His scalar temperature (θ) is related to Planck's temperature (T) by the following equation,

$$T = \theta U^{\circ}, \qquad (35)$$

and his definition of heat flow $(q^{\alpha} = cw^{\alpha})$ is such that it does involve momentum terms, in contrast to Planck's heat which is not able to "carry" momentum. That Eckart's definition of heat flow "carries" momentum is easily shown. Let the fluid be at rest in a Galilean frame x^{α} whose coordinates are parallel to the proper time coordinates at the point 0. Thus, at this point,

$$U^{\alpha} = (1, 0, 0, 0), \qquad (36)$$

and

$$\mathbf{S}^{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
(37)

This means that the conservation of matter equation,

$$\partial m^{\alpha} / \partial x^{\alpha} = 0$$
,

becomes

$$\partial m/\partial t = 0,$$
 (38)

so that m is only a function of χ^1 , χ^2 , χ^3 . The components of the energy-momentum tensor ($W^{\alpha\beta}$) are,

$$W^{\circ \circ} = m(\varepsilon + a),$$
$$W^{\circ j} = \underline{qj}_{c} = W^{j^{\circ}}, \quad j = 1, 2, 3$$
$$W^{j\mu} = p_{\delta}^{j\mu}.$$

With these components, the energy-momentum principle, Eqn (24), becomes

$$m\partial_{\varepsilon}/\partial t + (\partial q^{j}/\partial \chi^{j}) = 0, \qquad (39)$$

and

$$\frac{\partial}{\partial t} (q^j/c^2) + \frac{\partial}{\partial \chi^j} = 0.$$
 (40)

Eqn (39) is the law of conservation of energy and Eqn (40) are the conservation of momentum equations. Since the time rate of increase of (q^j/c^2) is equal to the negative gradient of the pressure, it is seen that (q^j/c^2) is a momentum.

The significance of Eckart's work is that he showed that the presence of heat flux can be accounted for by the addition of a symmetric tensor, $Q^{\alpha\beta}$, to the usual stress-energy tensor, $T^{\alpha\beta}$. Thus, in his work

ī

$$W^{\alpha\beta} = T^{\alpha\beta} + Q^{\alpha\beta}, \qquad (41)$$

where

$$T^{\alpha\beta} = m(1 + h/c^2) U^{\alpha}U^{\beta} - pg^{\alpha\beta},$$

with h being the specific enthalpy and p the pressure. So, we have the heat flux, q^{α} , as

$$q^{\alpha} = U_{\beta} Q^{\beta \alpha}.$$
 (42)

In 1951 Boris Leaf¹³ published a paper in which he showed that Eckart's formulation of a relativistic thermodynamics was related to the time components of the energymomentum principle,

$$\frac{\partial W^{\alpha\beta}}{\partial \chi^{\alpha}} = 0, \qquad (24)$$

and that the laws of dynamics were related to the spatial components.

In the 1952 edition of his book on relativity theory, von Laue was probably aware of Eckart's work (although perhaps not Leaf's, because of the time delay between writing and printing), since in the sections on relativistic thermodynamics, in which the Planck-Einstein formulation is developed, he comments in a footnote¹⁴ on the possibilities of formulating a relativistic thermodynamics with an invariant temperature. However, after doing this he adds the following statement: " . . . this change, if one attempts to carry it out, so deeply affects our perception of and methods of expressing things that one would be wise to use the definition of temperature given in the text."

C. Ott

Other than von Laue's footnote, I have found nothing further in the literature until 1963 when H. Ott published a paper¹⁵ in which he proposed a set of transformation laws for the thermodynamic quantities that were different than those in either the Planck-Einstein or Eckart formulations.

Ott agreed with the Planck-Einstein formulation in that he also felt that the relativistically transformed thermodynamic variables should be related by the same laws of thermodynamics that had been established for the nonrelativistic variables. In his paper he used many different phenomena to argue for the following transformation laws for the thermodynamic quantities:

$$Q = \gamma Q^{\circ}, \qquad (41.1)$$

$$T = \gamma T^{\circ}, \qquad (41.2)$$

$$S = S^{\circ},$$
 (41.3)

where the ° superscript is for the rest observer.

Because of the invariance of the entropy, the temperature and heat must transform in the same manner. This argument is the same as in the Planck-Einstein formulation. However, in the Ott formulation the heat is allowed to "carry" momentum. Thus, if the heat arrives from a reservoir that is traveling at a velocity \underline{v} , the heat carries the momentum $(\gamma \Delta Q^{\circ}/c^2) \underline{v}$ with it. Hence, there is no need for the Fuhrungskraft, as in the Planck-Einstein formulation, if the body receiving the heat and the reservoir are traveling with the same velocity. Therefore, an observer moving uniformly at a velocity \underline{v} past the reservoir-body system would write the first law of thermodynamics as

$$\Delta Q = \Delta E, \qquad (42)$$

and, since from Eqn (10) $\Delta E = \gamma \Delta E^{\circ} = \gamma \Delta Q^{\circ}$,

$$\Delta Q = \gamma \Delta Q^{\circ}. \tag{43}$$

Figure 1 on page 17 shows the differences between the Planck-Einstein results and Ott's.

Ott was very critical of the Planck-Einstein formulation. He was very disturbed by two aspects of their formulation. First, he argued that heat cannot be uniquely defined in terms of only the first law of thermodynamics, since the law only says that the heat is a form of energy. Second, he felt that the Planck-Einstein form of the equation of motion for a body of variable rest mass, M, that is absorbing heat from a reservoir at a rate $-d'Q_r/d\tau$, measured in the rest frame of M, or at the rate $-D'Q_r \equiv -(d'Q_r/d\tau)/\gamma$ measured in the observers frame, is



Figure 1.--A pictoral comparison of the Planck-Einstein and Ott formulation. incorrect. (d' is an operator that gives a small change in a quantity and is not a perfect differential.) This differential equation of motion has the following form:

$$d(Mv^{\mu})/d\tau = F^{\mu}, \qquad (44a)$$

where

$$\mathbf{F}^{\mu} = \gamma \left[\mathbf{c}^{-1} \left(\underline{\mathbf{v}} \cdot \underline{\mathbf{F}} - \mathbf{D}' \mathbf{Q}_{\mathbf{r}} \right), \, \underline{\mathbf{F}} \right].$$
(44b)

Note that the timelike component of F^{μ} (F°) contains -(γ/c)D'Q_r. Møller¹⁶ shows that when the heat transfer is taken into account in this manner the Planck-Einstein transformation law for the heat, Eqn (8.2), follows.

In the Ott formulation the equation of motion has the following form:

$$d(Mv^{\mu})/d\tau = F^{\mu} - d'Q_{r}/d\tau$$
, (45a)

where

$$\mathbf{F}^{\mu} = \gamma \left[\left(\underline{\mathbf{v}} \cdot \underline{\mathbf{F}} / \mathbf{c} \right), \ \underline{\mathbf{F}} \right], \qquad (45b)$$

and

$$Q^{\mu} = Q[1, v/c]$$
 (45c)

Whereas Ott thinks that Eqn (44a) is wrong and that Eqn (45a) is correct, it is obvious that the "correct" form of the equation of motion is dependent on the definition of heat used. This is easily seen when one realizes that the Planck-Einstein definition of heat does <u>not</u> allow the heat to carry momentum and therefore heat can only appear in the timelike component, while the Ott definition allows the heat to carry momentum and therefore the heat forms a four-momentum.

In Ott's formulation the temperature, along with the heat, is also written as a four-vector:

$$T^{\mu} = T(1, v/c).$$
 (46)

Thus, the second law of thermodynamics can be written as a four-equation:

$$\mathbf{T}^{\mu}\mathbf{ds} = \mathbf{dQ}^{\mu}. \tag{47}$$

Hence, the Ott formulation forms a covariant representation of thermodynamics, as was discussed earlier.

In 1965, H. Arzeliès¹⁷ published a paper whose results agree with Ott's. In this paper he also argued that an observer in relative motion with respect to a <u>Hohlraum</u> would observe as the total energy (U),

$$U = \Upsilon U_{0}, \qquad (48)$$

where U_o is the total internal energy as observed by an observer at rest with respect to the <u>Hohlraum</u>, rather than

$$U = \frac{1 + \frac{1}{3} \beta^{2}}{\left(1 - \beta^{2}\right)^{\frac{1}{2}}} U_{\circ}, \qquad (49)$$

which is just the usual¹⁸

$$U = \frac{U_{\circ} + P_{\circ}V_{\circ}\beta^{2}}{(1 - \beta^{2})\frac{1}{2}}$$

with $P_{o}V_{o} = \frac{1}{3} U_{o}$ for a <u>Hohlraum</u>. Instead of using Arzeliès' argument, which is rather obscure, it will be easier to repeat the method used in an article by A. Gamba¹⁹ that appeared shortly after Arzeliès' and is subtitled "Beware of Jacobians!"

Gamba's method is to pair photons of equal and opposite momenta. The sum over all these pairs, in the rest system of the <u>Hohlraum</u>, yields a total energy E and a null momentum, $P_o = 0$. Then, the energy transformation would yield Eqn (48) rather than Eqn (49). Gamba showed that the extra term $(\frac{1}{3} \beta^2)$ came from an integration over a solid angle (d Ω) of the type

$$\int (1 - \beta \cos \alpha)^2 d\Omega = 2\pi \int (1 - \beta \cos \alpha)^2 d(\cos \alpha)$$
$$= 4\pi (1 + \frac{1}{3}\beta^2),$$

where the integrand is the result of Doppler effects (D),

$$D = \frac{1 - \beta \cos \alpha}{\sqrt{1 - \beta^2}}$$

However, there is still an integration over the volume to get the total internal energy. For the moving observer, one writes

$$dV' = dV \sqrt{1 - \beta^2};$$

however, Gamba claimed that the correct volume transformation is
$$dV' = \frac{dV \sqrt{1 - \beta^2}}{1 - \beta \cos \alpha} .$$

When this is incorporated into the solid angle integral one gets

$$\int (1 - \beta \cos \alpha)^2 d\Omega = 2\pi \int (1 - \beta \cos \alpha)^2 d(\cos \alpha) = 0$$

which removes the "unwanted term $\beta/3$."

T. Kibble²⁰ pointed out that Gamba's approach is such that the observer moving with respect to the <u>Hohlraum</u> is required to integrate over the rest observer's hypersurface of simultaneity rather than the hypersurface of simultaneity in his frame. Because Gamba requires that the contributions, in the frame of the moving observer, to the total energy be simultaneous in the rest frame of the Hohlraum, this total energy is not necessarily an additive constant of the motion for the moving observer. Whereas, if the moving observer adds all the contributions to the energy on his hypersurface of simultaneity the total energy is an additive constant of the motion in his frame. Thus, if one wishes to keep the total energy an additive constant of the motion in all frames of reference, Eqn (50) must be used rather than Eqn (48).

Along with agreeing with the Ott and Arzeliès transformations Kibble also pointed out that the Planck-Einstein formulation implies that work can be done on a system when neither the volume or the pressure changes. This is seen by writing the first and second laws,

$$dQ = dE - dW, \qquad (51.1)$$

and

$$dS >, dQ/T,$$
 (51.2)

and defining two new variables (T $_1$ and Q $_1$) as

$$T = T_{1}g(\beta), \qquad dQ = dQ_{1}g(\beta)$$

where T and dQ are the Ott temperature and heat and $g(\beta)$ is any function of the velocity, $\beta = v/c$. These new variables transform as

$$T_{1} = \gamma T_{o}/g(\beta); \qquad dQ_{1} = \gamma dQ_{o}/g(\beta). \qquad (52)$$

The two laws of thermodynamics, Eqn (51.1) and Eqn (51.2), will be satisfied in terms of the two new variables if we redefine the work done on the system as

$$dW_{1} = dW + dQ_{1}[g(\beta) - 1].$$
 (53)

Upon comparing Eqn (8.2) and Eqn (8.3) with Eqn (52), it is seen that the two new variables correspond to the Planck-Einstein variables when $g(\beta) = \gamma^2$. But, from Eqn (53) one sees that this means that dW_1 can be non-zero when neither the pressure or the volume are changed (dW = 0). At this same time A. Børs²¹, using a microconical distribution and statistical mechanics methods, argued for the Ott transformation laws for heat and temperature. In his development he assumed that the form of the distribution function was the same for the moving observer as for the rest observer.

In an obvious response to this article and those by Ott, Arzeliès, and Gamba, R. Pathria⁶ used statistical mechanics arguments to support the Planck-Einstein transformation laws. Also, in the same vein, R. Penney²² published a rebuttal in favor of this formulation and against the Ott formulation.

This flurry of papers was climaxed by a paper by F. Rohrlich²³ which reviewed and compared the two formula-tions. His conclusion was:

"There is no way to choose between the conventional (Planck-Einstein) and the manifestly covariant (Ott) formulation of relativistic thermodynamics in terms of logical arguments. Both descriptions are consistent with special relativity and classical thermodynamics."

D. Landsberg

However, Rohrlich's paper did not end the debate concerning the transformation laws of thermodynamic variables. In 1966 P. T. Landsberg²⁴ began publishing a series of articles in which yet another special relativistic formulation of thermodynamics was presented. To avoid problems about how a moving observer would make

thermodynamic measurements (temperature, entropy, or internal energy, for example), he proposed a formulation in which these measurements need only be done in the rest frame of the system. In other words, he proposed a formulation of relativistic thermodynamics in which all thermodynamic variables were Lorentz invariant. He, like Børs and Pathria, used statistical mechanical arguments.

His development is as follows: Let Π_{\circ} be the maximum entropy distribution function in the rest frame. This is

$$\ln \left[\Pi_{\circ} \left(V_{\circ}, N_{\circ}, E_{\circ}, P_{\circ} \right) \right] = -\ln \left(Q_{\circ} \right) - \alpha_{10} N_{\circ}$$
$$- \alpha_{20} E_{\circ} - \alpha_{30} P_{\circ}$$
(54)

where V_o is the volume, N_o the number of particles in the system, E_o the total energy of the system, P_o the total momentum of the system, Q is a partition function and the α 's are Lagrangian multipliers.

He assumed that in an inertial frame traveling at a velocity \underline{v} in the +x direction with respect to the rest frame, the same form of Eqn (54) would be true. Thus,

$$\ln[\Pi(V,N,E,P)] = -\ln(Q) - \alpha_1 N - \alpha_2 E - \alpha_3 P. \quad (54.1)$$

Now, if the variables of Eqn (54) and Eqn (54.1) refer to the same state of the system one can use the special relativity transformations of N (N = N_o), E, and P to write Eqn (54.1) in terms of the corresponding rest frame quantities. Since the entropy is defined such that it is a Lorentz invariant in this formulation, the two probabilities given by Eqn (54) and Eqn (54.1) must be the same. This means that the Lagrange multipliers are related as follows:

$$\alpha_1 = \alpha_{10}; \quad \alpha_2 = \gamma [\alpha_{20} + \frac{\nu \alpha_{30}}{c^2}];$$
(55)

$$\alpha_3 = \gamma \left[\alpha_{30} + v \alpha_{20} \right].$$

Also, the partition function must be an invariant:

$$Q = Q_{o}$$
(56)

From the way the Lagrange multipliers α_2 and α_3 transform, one sees that they form a four-vector

$$\alpha^{\mu} = [c\alpha_2, \underline{\alpha}_3].$$
 (57)

From the definition of the entropy, $S_{\circ} = -k\Sigma_{N_{\circ}, E_{\circ}, P_{\circ}} \Pi_{\circ} \ln \Pi_{\circ}$, one obtains

$$S_{\circ} = -k\{\ln(Q_{\circ}) + \alpha_{10}\overline{N}_{\circ} + \alpha_{20}\overline{E}_{\circ} + \alpha_{30}\overline{P}_{\circ}\}$$
(58)

where the bars denote averages. In the rest frame of the system $\overline{P}_{o} = 0$.

To identify the Lagrange multipliers Landsberg assumed that the entropy depended not on \overline{E} and \overline{P} separately, but instead on some combination of them that is called the internal energy (U). Hence, from Eqn (58)

$$\alpha_{10} = \frac{1}{k} \left(\frac{\partial S_{o}}{\partial \overline{N}_{o}} \right)_{V_{o}, U_{o}}$$
(59.1)

$$\alpha_{20} = \frac{1}{k} \left(\frac{\partial S_{o}}{\partial \overline{E}_{o}} \right) V_{o}, \overline{N}_{o}, \overline{P}_{o}$$
(59.2)

$$= \frac{1}{k} \left(\frac{\partial S_{\circ}}{\partial \overline{V}_{\circ}} \right) V_{\circ}, \overline{N}_{\circ} \left(\frac{\partial U_{\circ}}{\partial \overline{E}_{\circ}} \right) V_{\circ}, \overline{N}_{\circ}, \overline{P}_{\circ}'$$
(59.3)

$$\alpha_{30} = \frac{1}{k} \left(\frac{\partial S_{o}}{\partial \overline{P}_{o}} \right) \overline{V}_{o}, \overline{N}_{o}, \overline{E}_{o}$$
$$= \frac{1}{k} \left(\frac{\partial S_{o}}{\partial V_{o}} \right)_{V_{o}}, \overline{N}_{o} \left(\frac{\partial U_{o}}{\partial \overline{P}_{o}} \right)_{V_{o}}, \overline{N}_{o}, \overline{E}_{o}.$$
(59.3)

For the internal energy in the moving frame, he proposed the total energy (\overline{E}) in

$$U \equiv \overline{E} \left[1 - \left(\frac{c\overline{P}}{\overline{E}}\right)^2 \right]^{\frac{1}{2}} - B, \qquad (60)$$

where B is an invariant energy depending at most on \overline{N} . Also, since $P_o = 0$, from the transformation law for momentum one sees that $\frac{V}{C} = c\overline{P}/\overline{E}$. Thus, one can write the internal energy as

$$U = \overline{E}/\gamma - B.$$

Using the fact that α_{30} is the spatial component and α_{20} the timelike component of a four-vector, along with the definition for the internal energy (U) and Eqn (59.2) and (59.3), one finds that

$$[c\alpha_{20}, \underline{\alpha}_{30}] = \frac{1}{k} (\frac{\partial S}{\partial U})_{V, \overline{N}} \gamma[c, -\underline{v}].$$
(61)

This is a true four-vector equation if $(\frac{\partial S}{\partial U})_{V,\overline{N}}$ is Lorentz

invariant. Thus, since the absolute temperature is usually defined in terms of this partial derivative as

$$\frac{1}{T} \equiv \left(\frac{\partial S}{\partial U}\right)_{V,\overline{N}},$$
(62)

one sees that it is also Lorentz invariant. He also argued that since the entropy was a Lorentz invariant the heat (Q) must also be Lorentz invariant.

Thus, in Landsberg's formulation the thermodynamic quantities are invariant, but the form of the first law of thermodynamics is changed. It is written as,

$$TdS = dU + \gamma p dV - \mu dN$$
 (63)

instead of in the usual form,

$$TdS = dU + pdV - \mu dN$$
 (64)

The γ factor appears in the pdV term because to the moving observer $p = p_o$ and $dV = \gamma^{-1} dV_o$. So, for the pdV term to be Lorentz invariant the γ factor is needed to cancel the γ^{-1} that the moving observer uses in the volume element.

The second law still has the usual form

$$ds > dQ/T.$$
(65)

However, in a later paper²⁵ Landsberg decided to redefine his heat, so that it is defined as in the PlanckEinstein formulation. This meant that he had to rewrite the second law, Eqn (65) as,

$$ds > \gamma dQ/T,$$
(66)

since $dQ = \gamma^{-1} dQ_0$ in the Planck-Einstein formulation and the temperature and entropy are still to be Lorentz invariant. Thus, the γ factor in Eqn (66) enters into the equation for the same reason that it entered Eqn (63).

E. van Kampen

In 1968 another formulation was proposed by N. van Kampen²⁶. In his formulation, the first law is

$$dQ_{\mu} = dU_{\mu} - K_{\mu}d\tau + dA_{\nu_{\mu}},$$
 (67)

where dQ_{μ} is called the thermal energy-momentum transfer, $U_{\mu} = v_{\mu}U^{\circ}$, where v_{μ} is the four-velocity of the system and U° the internal energy of the rest system, K_µ is the Minkowski force acting on the system, τ is the proper time of the system, and $dA_{v_{\mu}}$ is the mechanical energy and momentum transferred from the system. He then defined a "heat supply," dQ°, as

$$dQ^{\circ} = v^{\mu} dQ_{\mu}.$$
 (68)

In other words, the heat supply is the component of the thermal-energy momentum transfer parallel to the four-velocity. This is a true scalar.

By using Eqn (68) in the second law, one can define a scalar invariant temperature T such that,

$$ds = dQ'/T.$$
 (69)

He also suggested that one could define a fourvector temperature $\beta_{\mu} = v_{\mu}/T$. Using this it is possible to rewrite Eqn (69) as

$$ds = \beta_{\mu} dQ \quad . \tag{70}$$

However, if one compares this equation with the covariant statement of the second law in the Ott formulation, $T^{\mu}ds = dQ^{\mu}$, one sees that the only difference between van Kampen's formulation and Ott's is the definition of the temperature four-vector. Ott's definition is

$$\mathbf{T}^{\mu} = \mathbf{T}^{\circ} \mathbf{v}^{\mu} \,. \tag{46}$$

and van Kampen's is

$$\beta^{\mu} = v^{\mu}/T^{\circ}. \tag{71}$$

Hence,

$$\mathbf{T}^{\mu} = \mathbf{T}^{\circ 2} \boldsymbol{\beta}^{\mu}. \tag{72}$$

F. Historical Resumè

In view of all the arguments that have been raised by proponents of different formulations of relativistic thermodynamics since Ott's work one can only wonder why the work of Eckart and Leaf was ignored. That their work was ignored both at the time it was done and then, more than ten years later, during the debates precipitated by Ott's work is puzzling. The explanation that the neglect is due to incomplete surveys of the literature may seem improbable because of the number of different people that have participated in the debates. But in fact, since each proponent's concern has been only to support one formulation rather than another, extensive surveys of the literature were probably not made. Another reason is that Eckart's and Leaf's work is not readily interpreted in terms of the familiar thermodynamic variables (Q,T,P,V) because it involves nonintuitive entropy flow tensors and the nonintuitive temperature θ .

Of the four formulations of relativistic thermodynamics (Planck-Einstein, Eckart-Leaf, Ott - vanKampen, and Landsberg), only Ott's and Eckart's are manifestly covariant. That the Planck-Einstein formulation is not manifestly covariant is not surprising, because it was developed before the meaning of covariance was completely understood.

The noncovariance of the Landsberg formulation appears to be the result of his concern about how an observer would measure the temperature of a system that is moving at a uniform velocity with respect to himself. He says²⁷ that the only way this problem can be solved is by requiring that the temperature be invariant. This, along with his acceptance of the Planck definition of heat,

guarantees the lack of covariance. His justification²⁸ for the lack of covariance in his formulation is that he thinks that the condition of manifest covariance is too restrictive. His definition of covariance is the same as that used in the Planck-Einstein formulation: The laws of physics must be of the same form in all inertial frames of reference.

Figure 2, page 32, gives a short resume of the transformation laws and form of the first two laws of thermodynamics for the four proposed formulations of relativistic thermodynamics.

PROPONENTS	dQ	Т	U	LAWS OF THERMODYNAMICS
PLANCK- EINSTEIN	γ ^{−1} dQ°	γ ^{−1} T°	γ(U°+β ² PV°)	dQ = dU + dW T $dS = dQ$
ОТТ	γdQ°	γT°	γU°	dQ = dU + dW T $dS = dQ$
LANDSBERG	γ ^{−1} dQ°	т°	U°[1-(<u>CP</u>) ²] ¹ / ₂	$dQ = dU + \gamma P dV$ T $dS = \gamma dQ$
ECKART	γdQ°	۳°	۵°	*See Note

*Because the form of the first and second laws of thermodynamics is not the usual form in Eckart's formulation, they are not included in this figure.

Figure 2.--A review of the proposed transformations.

II. THE PROBLEM

My problem is to determine which, if any, of the formulations of relativistic thermodynamics proposed by Planck-Einstein, Eckart, Ott, and Landsberg is the correct formulation.

The usual approach of comparing predictions made by a theory to the results of experiments cannot be made here because there is no experimental data on relativistic thermodynamic systems. It must be remembered that I am considering systems which are non-relativistic in their rest frames and any relativistic affects are due only to the center of mass of the system of interest having a relativistic velocity with respect to the observer. This means that the results of experiments on relativistic plasmas are of no use because the plasma has a zero center of mass velocity with respect to the observer. Thus, any relativistic effects in a plasma are due to a sizable fraction of the number of particles in the system having relativistic velocities and are not due to the center of mass of the system having a relativistic velocity with respect to the observer. The lack of experimental information means that we must examine the different relativistic thermodynamic formulations in terms of certain abstract

properties, to be discussed below, that theories are usually assumed to satisfy.

A. Consistency

The first, and probably most important consideration is whether or not a theory is consistent. Consistency is not easily shown rigorously; however, a necessary condition is that all predictions for a given situation are consistent with each other. It is possible to think of consistency in two ways. First, a theory must be selfconsistent; that is, no predictions made by the theory may contradict other predictions made by the theory or any hypothesis upon which the theory is based. Second, because the domains of physical theories overlap, there must be no contradictions between predictions made by the theory of interest and other, established theories when application is made to physical situations in the region where the theories overlap. Since this second concern is with the consistency of physics, rather than with self-consistency of a single theory, I call this latter consistency a physical-consistency.

In accordance with the preceding considerations, an attempt was made to test the physical-consistency of the formulations of relativistic thermodynamics proposed by Planck-Einstein, Eckart, Ott, and Landsberg. This attempt involved relating mechanical and thermodynamical

quantities. This procedure allows the comparison of the Lorentz transformation laws of thermodynamic quantities, in the aforementioned formulations, with those of mechanical quantities, for which the Lorentz transformation laws are well established and confirmed. However, mechanics deals with individual particles and thermodynamics deals with the bulk properties of a system composed of a large number of individual particles. This gap is bridged by statistical mechanics. But, because of the apparent failure, as seen in papers by Børs²¹, Pathria⁶, and Landsberg²⁴, of statistical mechanics to discern the differences between the different formulations of relativistic thermodynamics, I looked for a model of a physical process that would allow the direct comparison of mechanical and thermodynamic quantities. In this comparison the model is used to develop an equation that relates specific mechanical and thermodynamical quantities.

The role of the model, that is, to give an equation relating specific mechanical and thermodynamic quantities, requires that the model satisfy the following conditions. First, since statistical mechanics is the bridge between mechanics and thermodynamics, the model must be based on statistical mechanical considerations. Second, the equation developed from the model can not include the ratios of thermodynamic quantities that have the same Lorentz transformation laws. This is because the transformation coefficients of the thermodynamic quantities, if any, would cancel, thereby yielding no information about the physical-consistency of the relativistic thermodynamics formulation being considered. It was found that the usual model for the process of evaporation satisfies the two requirements.

This model is satisfactory because, even though the process of evaporation is complex, the equilibrium point is determined only by quantities associated with the system itself. Therefore, it can be argued that the equilibrium point must be independent of the frame of reference from which it is observed. In any event, since the position of equilibrium determines the entropy, the invariance of the equilibrium point follows from the invariance of the entropy, which is asserted in all of the relativistic formulations of thermodynamics.

Mechanical quantities are brought into the calculation by interpreting the canonical distribution function as giving the probability of finding a particle with a given kinetic energy (ε), rather than making the usual interpretation of it giving the number of particles in the system with a given kinetic energy. This is done by dividing the distribution function by N, the number of particles in the system. Because all observers agree on the number of particles in the system, N, this renormalization of the distribution function does not affect any

predictions made in terms of the function. The change in entropy of the system resulting from the evaporation of a particle from the liquid to the vapor state of the system is given in terms of the change in kinetic energy of the particle and the temperature of the system. Hence, since the Lorentz transformation law for energy is agreed upon, the physical-consistency of the Lorentz transformation law for the temperature in the formulations of relativistic thermodynamics proposed by Planck-Einstein, Eckart, Ott, and Landsberg can be examined.

In the calculations to follow, the evaporation equation as given by Rodebush²⁹ is followed. Because of the invariance of the distribution function³⁰, it will be necessary only to carry out the necessary calculations in the system rest frame. The distribution function in that frame is

$$P(\varepsilon_1) = K \exp\{-\varepsilon_1/kT\}, \qquad (73)$$

where $P(\varepsilon_1)$ is the probability of finding a particle with the kinetic energy ε_1 arising from velocity components normal to the liquid surface. As the particle leaves the liquid surface it loses the energy $\Delta \varepsilon$. The remaining kinetic energy of the particle is ε_2 , where

$$\boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_1 - \Delta \boldsymbol{\varepsilon}. \tag{74}$$

Because of equilibrium between the liquid and vapor phases,

the same distribution function holds for both of the phases. Hence, the probability of finding the particle with a kinetic energy ε_2 in the vapor phase is

$$P(\varepsilon_2) = K \exp\{-\varepsilon_2/kT\}.$$
 (75)

This means that the change in entropy (ΔS) associated with the evaporation of this particle is

$$\Delta S = k \ln P(\varepsilon_2) - k \ln P(\varepsilon_1).$$
 (76)

Upon substituting Eqn (73), (74) and (75) into Eqn (76), we get that

$$\Delta S = \Delta \varepsilon / T.$$
 (77)

Due to the invariance of the distribution function an observer moving uniformly with respect to the liquid-vapor system would conclude that in his frame of reference (primed) the change in entropy is

$$\Delta S' = \Delta \varepsilon' / T'. \tag{78}$$

Since the equilibrium point is Lorentz invariant, the entropy, and hence entropy changes, are invariant, and we can equate Eqn (77) and (78). This yields,

$$\Delta \varepsilon / T = \Delta \varepsilon' / T'$$

or

$$\Delta \varepsilon' / \Delta \varepsilon = T' / T.$$
 (79)

Because of the way we use the distribution function for individual particles, $\Delta \epsilon$ and $\Delta \epsilon$ ' denote mechanical energies. This means that

$$\Delta \varepsilon' / \Delta \varepsilon = 1 / \sqrt{1 - v^2 / c^2} = \gamma, \qquad (80)$$

where \underline{v} is the velocity of the moving observer with respect to the system rest frame observer. Thus, examination of Eqn (79) and (80) shows that

$$T'/T = \gamma$$
(81)

must be true for each of the four formulations of relativistic thermodynamics being considered. Figure 3 shows the Lorentz transformation laws of the entropy and the temperature for the formulations of Planck-Einstein, Eckart, Ott, and Landsberg. We see that even though all four formulations have the Lorentz invariance of entropy which was used to equate Eqn (77) and (78), only one of them, the Ott formulation, has the value of γ for the ratio of the temperatures as seen by the moving and rest frame system observers respectively.

Consequently, because there can be <u>no</u> contradiction between different theories, I must conclude that the contradictions between the Planck-Einstein, Eckart, and Landsberg temperature transformation laws and that for mechanical energy, when the model for the process of evaporation is used to relate mechanical and thermodynamic

Formulation	s'/s	т'/т
Planck-Einstein	1	γ ⁻¹
Eckart	1	1
Ott	1	γ
Landsberg	1	1

Figure 3.--Ratio of entropy and temperature transformations for the formulations of relativistic thermodynamics being considered. quantities, eliminate them from consideration as being the correct formulation of relativistic thermodynamics. This leaves only Ott's formulation as a contender for being the correct formulation of relativistic thermodynamics.

B. Covariance

Even though the Planck-Einstein and Landsberg formulations are not manifestly covariant, the fact that the Eckart formulation is not physically consistent with the invariance requirement of a specific model including the transformation of mechanical energy shows that developing a manifestly covariant theory does not protect it from contradictions in application. Hence, the significance of manifest covariance must not be related to physical-consistency as a physical theory.

If a theory is cast in a manifestly covariant form we know that the theory is automatically in proper relativistic form, as discussed earlier. However, it is possible to formulate a theory in a non-manifestly covariant form and still have it be relativistically correct. An example of this is Maxwell's Equations as usually presented at the undergraduate level. But, because of the intimate relation between manifest covariance and the space-time geometry required by the relativity postulates all theories, if they are relativistically correct, must be able to cast in a manifestly covariant form. This means that the significance of manifest covariance, in the evaluation of a theory, is that with respect to its being relativistically correct, but is not a guarantee of its yielding physically consistent results.

Since it was discussed in the first chapter, I will not discuss the covariance of Ott's formulation. However, it will be remembered that it was stated that his formulation is manifestly covariant. Thus, we know that it is relativistically correct.

C. Compatability with Related Theories

Even though it has been shown that three of the four formulations of relativistic thermodynamics are inconsistent, the fourth formulation, Ott's, must still be examined to see if it is compatible with the related fields of relativistic fluid dynamics and statistical mechanics. This compatibility is very important because the non-relativistic formulations of thermodynamics, fluid dynamics, and statistical mechanics are closely related. In fact, because of the different hypotheses upon which fluid dynamics and statistical mechanics are developed, it is possible to think of thermodynamics as being the bridge that relates the two. It would be expected that the correct formulation of relativistic thermodynamics also played this role with respect to relativistic fluid dynamics and statistical mechanics.

1. Statistical Mechanics

In order to show the relationship between Ott's formulation and relativistic statistic mechanics I must give a rough outline of the way relativistical statistical mechanics is developed.

An early formal attempt to develop a relativistically correct form of statistical mechanics was carried out in 1951 by Peter Bergmann³¹. It is fascinating to note that he states that his reason for developing a relativistical statistical mechanics was to remove relativistic thermodynamics from the phenomenological level and place it on the underlying statistical concepts. His approach is to include the time as one of the canonical variables, with the negative of the Hamiltonian being its canonical conjugate. Since the time is involved a system is now represented by its trajectory in the expanded phase space and not by a point, as in non-relativistic statistical mechanics. However, because the negative of the Hamiltonian is one of the coordinates of what he calls "expanded phase space" the system is required to travel on a hypersurface of the space. This means that an ensemble has to be described in terms of the density of a field of system trajectories on a hypersurface. In spite of the complexities of the problem, Bergmann managed to develop a distribution function of the form

$$\mu = z^{-1} \exp\{-\Sigma_{i}\beta_{i}A_{i}\}, \qquad (82)$$

with

$$Z = \int (\theta, H) J \exp\{-\Sigma_{i} \beta_{i} A_{i} d\xi' \dots d\xi^{2f}, \qquad (83)$$

where (θ, H) is the Poisson bracket of the expanded phase space, θ is an arbitrary parameter $\theta(q_i, p_i, t, -H)$, J is the Jacobian of the equations leading from the canonical coordinates to the generalized parameters $(\xi^i, \theta, \text{ and } H)$, A_i are the additive constants of the motion for the specified system and β_i are a set of parameters determined by the specified values of the ensemble averages of the additive constants of the motion.

If we specify that there be no angular momentum in the system, but only linear momentum, the only additive constant of the motion for the system is the magnitude of its four-momentum (P^{μ}). This means that the distribution function has the form

$$\mu = Z^{-1} \exp\{-\beta_{\mu} P^{\mu}\}.$$
 (84)

We can give physical meaning to β_{μ} by comparing Eqn (84) to the non-relativistical distribution function for a system with a non-zero linear momentum (P). This function is

$$\mu = Z^{-1} \exp\{\frac{1}{kT}(E - \underline{v} \cdot \underline{P})\}.$$
(85)

Upon comparing Eqn (84) and (85) we see that we can identify the four-vector β^{μ} as

$$\beta^{\mu} = \frac{1}{kT} v^{\mu}.$$
 (86)

This is, except for Boltzman's constant, van Kampen's fourvector temperature, Eqn (71), which was earlier shown to be simply related to Ott's four-vector temperature.

It is interesting to note that Bergmann completely ignored temperature in his paper, but did show that the entropy was an invariant quantity and that heat carried linear momentum. This is contrary to Planck's definition of heat; however, he made no mention of this fact.

In 1963, W. Israel³² published a paper in which he develops a relativistic kinetic theory of an ideal gas. He uses a four-vector approach; however rather than relating the four-vector directly to the temperature, he defines a scalar temperature

$$T = 1/k (\beta^{\mu}\beta_{\mu})^{\frac{1}{2}}.$$
 (87)

His reasoning behind this is not clear to me.

Starting in 1967 R. Hakim³³ began publishing what has turned out to be virtually a book in which he has developed a relativistic statistical mechanics that is much more complex than Bergmann's. In this development he is concerned with plasmas. He starts with the Lorentz-Dirac equations of motion for a charged particle in a system of N identical particles. He then defines a phase space of 12N-dimensions in which he defines his "Gibbs" ensembles. His development from then on is strictly a Green's function approach with its associated perturbation expansions and "bubble" diagrams. However, it is interesting to note that even though he suggests that, because certain quantities are arbitrarily defined, more than one temperature be defined, one of these is a four-vector as in Bergmann's This temperature is what he calls the kinetic temwork. perature, and is based on Lagrange multipliers of the fourmomentum when the system is in equilibrium. However, the other temperature, which he calls the equilibrium temperature, is defined through the Jüttner-Synge distribution function and is neither a scalar or a vector, and is the same as Planck's temperature. This defining of two temperatures is probably due to the fact that he has problems with defining what is meant by equilibrium, since the concept of temperature only has a unique meaning when the system is in equilibrium in non-relativistic statistical mechanics. The equilibrium problem arises from the inclusion of the electromagnetic fields in the equation of motion. Hakim also states, without proof or explanation, that information theory requires that the entropy be a Then he states that the Lorentz transformatrue scalar. tion laws are uniquely determined for all thermodynamic variables when the law for the temperature transformation is combined with the scalar property of the entropy.

Even though there are obvious problems in relating Ott's formulation, with its four-vector temperature and heat, to Hakim's formulation of relativistic statistical mechanics, Ott's formulation is compatible with Bergmann's and Israel's work in relativistic statistical mechanics where there are no problems with equilibrium.

2. Fluid Dynamics

The relation between Ott's formulation of relativistic thermodynamics and relativistic fluid dynamics at first seems to be very poor. This is because the usual formulations³⁵ of relativistic fluid dynamics use a scalar formulation of thermodynamics similar to Eckart's work and not a four-vector formulation as in Ott's. However, in 1966 L. Schmid³⁶ started publishing a series of papers in which he uses variational techniques to develop a formulation of relativistic fluid dynamics with which Ott's formulation of relativistic thermodynamics is compatible.

His approach is to start with the four-vector Euler equation for adiabatic flow of an ideal gas (this is a gas with no viscosity or thermal conduction),

$$\frac{\mathrm{d}}{\mathrm{d}\,\tau} \,(\mathrm{m}^{\star}\mathrm{v}^{\mu}) = \frac{1}{\mathrm{n}} \,\partial^{\mu} \,\mathrm{P}, \qquad (88)$$

where $m^* = m(1 + h/c^2)$, with m being the particle rest mass and h the enthalpy per unit mass, n the number of particles per unit volume in the rest frame, and P the

pressure. He then uses the following thermodynamic identity,

$$\frac{1}{n} \partial_{\mu} P = m \partial_{\mu} h - m T \partial_{\mu} s \qquad (89)$$

where T is the scalar temperature and s the entropy per unit mass, to eliminate the particle density n from Eqn (88). This procedure yields

$$\frac{d}{d\tau} (m^* v^{\mu}) = m \partial^{\mu} h - m T \partial^{\mu} s$$

$$= \partial^{\mu} (m^* c^2) - m T \partial^{\mu} s.$$
(90)

The temperature T is a scalar here because it refers to the local rest frame of the fluid. Schmid then eliminates the temperature by defining a scalar function, θ ,

$$\frac{d\theta}{d\tau} = v^{\mu} \partial_{\mu} \theta \equiv T$$
 (91)

The function θ is commonly called the temperature integral. Applying θ , Eqn (91), and the **a**diabatic constraint,

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = 0, \qquad (92)$$

to Eqn (90) yields

$$\mathbf{v}_{\mu} \{ \left[\partial^{\mu} \left(\mathbf{m}^{*} \mathbf{v}^{\gamma} \right) - \partial^{\gamma} \left(\mathbf{m}^{*} \mathbf{v}^{\mu} \right) \right] - \mathbf{m} \left[\left(\partial^{\mu} \mathbf{s} \right) \left(\partial^{\gamma} \theta \right) + \left(\partial^{\gamma} \mathbf{s} \right) \left(\partial^{\mu} \theta \right) \right] \} = 0.$$
(93)

Because the terms in the braces on the left side of Eqn (93) are orthogonal to the four-velocity, it is possible

to relate these terms to a tensor $\omega^{\mu\gamma}$ in the following way:

$$[\partial^{\mu}(\mathbf{m} \star \mathbf{v}^{\gamma}) - \partial^{\gamma}(\mathbf{m} \star \mathbf{v}^{\mu})] - \mathbf{m}[\partial^{\mu}\mathbf{s})(\partial^{\gamma}\theta) - (\partial^{\gamma}\mathbf{s})(\partial^{\mu}\theta)] + \\ + 2\mathbf{m} \star \omega^{\mu\gamma} = 0, \qquad (94)$$

where $\omega^{\mu\gamma}$ is antisymmetric and completely unspecified except that,

$$v_{\mu}\omega^{\mu\gamma} = -\omega^{\mu\gamma}v_{\gamma} = 0.$$
 (95)

Schmid calls $\omega^{\mu\gamma}$ the intrinsic vorticity because when h, s and θ are constants,

$$\omega^{\mu\gamma} = -\frac{1}{2} \left(\partial^{\mu} v^{\gamma} - \partial^{\gamma} v^{\mu} \right), \qquad (96)$$

which is the relativistic vorticity. Thus he regards $\omega^{\mu\gamma}$ as the contribution to the total fluid vorticity that is not produced by thermal effects.

The second term in Eqn (94) can be rewritten as

$$(\partial^{\mu}s) (\partial^{\gamma}\theta) - (\partial^{\gamma}s) (\partial^{\mu}\theta) = \partial^{\mu}(s\partial^{\gamma}\theta) - \partial^{\gamma}(s\partial^{\mu}\theta),$$

allowing us to rewrite Eqn (94) in the form,

$$2m^{*}\omega^{\mu\gamma} = -\left[\partial^{\mu}\left(m^{*}v^{\gamma}-ms\partial^{\gamma}\theta\right)-\partial^{\gamma}\left(m^{*}v^{\mu}-ms\partial^{\mu}\theta\right)\right].$$
(97)

This equation says that the tensor $2m^*\omega^{\mu\gamma}$ is expressible as the curl of a four-vector $[m^*v^{\gamma}-ms\partial^{\gamma}\theta]$ plus the fourgradient of a scalar function C. Calling this four-vector b^{μ} we have,

$$2\mathfrak{m}^{*}\omega^{\mu\gamma} = \partial^{\mu}\mathfrak{b}^{\gamma} - \partial^{\gamma}\mathfrak{b}^{\mu}, \qquad (98)$$

where

$$-\mathbf{b}^{\mu} = \mathbf{m}^{*}\mathbf{v}^{\mu} - \mathbf{m}\mathbf{s}\partial^{\mu}\theta + \partial^{\mu}\mathbf{c}.$$
 (99)

Schmid then develops an argument for allowing the fourvector b^{μ} to be written in terms of two unspecified constants of the fluid motion which he represents by M and ϕ . The resulting equation between M, ϕ and b^{μ} is,

$$b^{\mu} = M \partial^{\mu} \Phi, \qquad (100)$$

where because M and Φ are constants of the motion,

$$\frac{\mathrm{d}M}{\mathrm{d}\tau} = \frac{\mathrm{d}\Phi}{\mathrm{d}\tau} = 0.$$
 (101)

Upon substituting Eqn (100) into Eqn (99) one gets,

$$\mathbf{m}^{*}\mathbf{v}^{\mu} = -\partial^{\mu}\mathbf{C} + \mathbf{m}\mathbf{s}\partial^{\mu}\theta - \mathbf{M}\partial^{\mu}\Phi, \qquad (102)$$

which is the formal integration of Euler's equation of motion, Eqn (90).

Schmid then develops a variational formulation for the flow of a single ideal gas. This results in the following statements,

$$\partial \int L dV_{\Lambda} = 0, \qquad (103)$$

where dV_4 is a four-volume element, and,

$$\mathbf{L} = -\mathbf{n}\mathbf{m} \left(\mathbf{1} + \mathbf{U}/\mathbf{c}^{2}\right) \mathbf{c} \left(\mathbf{v}^{\mu}\mathbf{v}_{\mu}\right)^{\frac{1}{2}} + \\ +\mathbf{P}\left[\mathbf{1} - \left(\mathbf{v}^{\mu}\mathbf{v}_{\mu}\right)^{\frac{1}{2}}/\mathbf{c}\right] - \mathbf{n}\mathbf{v}^{\mu}\partial_{\mu}\mathbf{C} + \\ +\mathbf{n}\mathbf{m}\mathbf{s}\mathbf{v}^{\mu}\partial_{\mu}\theta - \mathbf{n}\mathbf{M}\mathbf{v}^{\mu}\partial_{\mu}\Phi, \qquad (104)$$

where U is the internal energy per unit mass. In Eqn (104), P, C, θ , M, ϕ and s are undetermined multipliers. The Euler-Lagrange equation that results from the variation of, as an example, n is

$$\partial_{\mu} \left[\frac{\partial \mathbf{L}}{\partial (\partial_{\mu} \mathbf{n})}\right] - \frac{\partial \mathbf{L}}{\partial \mathbf{n}} = 0.$$
 (105)

Upon varying θ and s one gets,

$$\partial \Phi: \frac{\mathrm{ds}}{\mathrm{d\tau}} = 0.$$
 (106)

S:
$$\frac{d\Phi}{d\tau} = \frac{\partial}{\partial s} [m(1 + U/c^2)]_n C^2 = (\frac{\partial U}{\partial S})_n \equiv T.$$
 (107)

Thus, the Lagrange multiplier θ is just the thermal integral defined in the development of the formal integration of the Euler equation of motion. However, Eqn (106) says that ϕ is the Lagrange multiplier associated with the adiabatic condition ($\frac{ds}{d\tau} = 0$).

Schmid then examined reversible heat transfer between two ideal gases. The Lagrangian densities for the two fluids were of the form of Eqn (104). He then shows that the Lagrangian density for the total system (L_{total}) is of the form,

$$L_{total} = L + L' + \sigma(\Phi - \Phi'), \qquad (108)$$

where primed quantities represent one fluid and unprimed quantities represent the other fluid. Due to the form of the Lagrangian density, Eqn (108), only the Euler-Lagrange equations for θ , θ' and σ will differ from the equations for the isolated fluids. The results of these three variations are,

$$\partial \theta: \partial_{\mu} (nmsv^{\mu}) = \sigma,$$
 (109)

$$\partial \theta': \partial_{\mu} (n'm's'v^{\mu'}) = -\sigma,$$
 (110)

$$\partial \sigma: \quad \theta = \theta'.$$
 (111)

The addition of Eqn (108) and (109) yields,

$$\partial_{\mu} [nmsv^{\mu} + n'm's'v^{\mu'}] = 0,$$
 (112)

which reiterates the requirement that the heat flow be reversible because it, Eqn (112), states that the total entropy flux is conserved. Thus, the meaning of σ is that if it is positive, it is the time rate of entropy increase per unit volume in the unprimed gas and the time rate of decrease of entropy per unit volume in the primed gas. Thus, it is seen that the role temperature usually plays in reversible heat transfer in non-relativistic thermodynamics, fluid dynamics, or statistical mechanics is in Schmid's formulation played by the <u>thermal integral</u>, θ . This means that for reversible heat transfer to occur between two systems in thermal contact, it is not the temperatures of the two systems that must be equal but their <u>thermal integrals</u> (θ and θ ') must be equal. Thus, since $v^{\mu} \neq v^{\mu}$ ', the scalar temperatures,

$$T = \frac{d\theta}{d\tau} = v^{\mu} \partial_{\mu} \theta, \qquad (113)$$

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$$\mathbf{T}' = \frac{d\theta'}{d\tau'} = \mathbf{v}^{\mu'} \partial_{\mu} \theta', \qquad (114)$$

are not necessarily equal.

From here Schmid goes on to show that the Euler equations for each of the two gases are of the form,

$$n \frac{d(m \star v^{\mu})}{d\hat{\tau}} = \partial^{\mu} P + \sigma \partial^{\mu} \theta, \qquad (115)$$

and,

$$n' \frac{d(m^{*'}v^{\mu'})}{d\tau'} = \partial^{\mu}P' - \sigma \partial^{\mu}\theta', \qquad (116)$$

where P and P' are the partial pressures of each of the gases. Upon comparison of Eqn (115) and (116) with those for the nonrelativistic case, it is seen that the four-vector $\partial^{\mu}\theta$ plays the role of a four-vector temperature T^{μ} . In fact, Schmid defines a four-vector temperature,

$$\mathbf{T}^{\mu} = \mathbf{c}\partial^{\mu}\theta, \qquad (117)$$

which is just Ott's four-vector temperature.

Thus, in spite of the fact that the usual formulations of relativistic fluid dynamics use a scalar thermodynamics, Schmid has developed an equivalent formulation which uses the four-vector relativistic thermodynamics formulated by Ott.



III. CONCLUSION

Since there is no experimental data on relativistic thermodynamic systems, the usual approach in the evaluation of a proposed theory, of comparing theoretical prediction with experimental data, can not now be undertaken. Hence, in order to evaluate the formulations of relativistic thermodynamics proposed by Planck-Einstein. Eckart, Ott, and Landsberg other criteria were selected. These were that the correct formulation of relativistic thermodynamics must be physically-consistent with relativistic mechanics and that it had to be compatible with the relativistic formulations of statistical mechanics and fluid dynamics. This latter compatibility requirement was used because of the compatibility between thermodynamics. fluid dynamics, and statistical mechanics in the non-relativistic region. With respect to the consistency requirement, it was found that the Planck-Einstein, Eckart, and Landsberg formulations were not physically-consistent with respect to the Lorentz transformation law for mechanical energy when application is made to the model of evaporation, which is assumed to be physically correct.

Hence, only the Ott formulation remained as a possible candidate for being the correct formulation of
relativistic thermodynamics. It was shown that there are formulations of relativistic statistical mechanics and fluid dynamics with which Ott's formulation is compatible. I must therefore conclude that Ott's formulation of relativistic thermodynamics is correct with respect to the criteria used to evaluate the proposed formulations of relativistic thermodynamics.

At the annual joint meetings of the American Physical Society and the American Association of Physics Teachers in February, 1969 there was a symposium on relativistic fluid dynamics. In an invited paper on relativistic shock waves, P.A. Koch predicted that experimentalists would have relativistic shock waves in the next five years. When these are observed it will be possible to evaluate the proposed formulations of fluid dynamics in the way all theories must in the final analysis be tested, by comparing theoretical predictions with experimental data. At that time it will be possible to compare predictions made by Ott's formulation with experimental data.

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