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PIECEWISE LINEAR HOMEOMORPHISMS OF
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PIECEWISE LINEAR HOMEOMORPHISMS OF PERIOD 2^n ON THE SOLID KLEIN BOTTLE

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ABSTRACT

PIECEWISE LINEAR HOMEOMORPHISMS OF PERIOD 2^n ON THE SOLID KLEIN BOTTLE

By

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In this thesis we classify piecewise linear homeomorphisms of period 2^n on the solid Klein bottle.

It is shown that up to equivalence there are five distinct involutions on the solid Klein bottle, $\mathcal{K}$.

Also, for $n > 1$, there are only two equivalence classes of homeomorphisms of period 2^n on $\mathcal{K}$.

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INTRODUCTION

All spaces and maps will be in the PL category.

In this theses we classify piecewise linear homeomorphisms of period 2^n on the solid Klein bottle $\mathcal{K}$.

Two homeomorphisms $f : M \to M$ and $g : N \to N$ are said to be equivalent (written $f \sim g$) if there is a homeomorphism $k : M \to N$ such that $k^{-1}gk = f$.

If $k^{-1}gk = f^i$ for $i \neq 1$, then we say that f and g are weakly equivalent.

Let $\dot{\cup}$ denote disjoint union and $I = [0,1]$. Our results are as follows:

I - The involutions on $\mathcal{K}$ are determined up to equivalence by their fixed point set. The possible fixed point sets are $I \,\dot{\cup}\,$ pt., $D^2 \,\dot{\cup}\, I$, S', an anulus, a Möebius band.

II - For $n \geq 2$, there are exactly two weak equivalence classes of maps of period 2^n on $\mathcal{K}$.

In proving the above results, we make extensive use of the following theorems of P.K. Kim and J.L. Tollefson:

<u>Theorem</u>: ([5])

Let F be a compact surface and let h be a PL involution of $F \times I$ such that $h(F \times \partial I) = F \times \partial I$ (I denotes the unit interval). Then there exists a map g of F (with g^2 = identity) such that h is equivalent to the involution h' of $F \times I$ defined by $h'(x,t) = (g(x),\lambda(t))$ for $(x,t) \in F \times I$ and $\lambda(t) = t$ or $1-t$.

<u>Theorem</u>: ([6])

Let h be an involution on a compact 3-manifold M. Suppose that there exists a properly embedded disk D in M such that ∂D lies in a given component B of ∂M and ∂D does not bound a disk in B. Then there exists a disk S, properly embedded in M, with the properties:

(i) $\partial S \subset B$

(ii) ∂S does not bound a disk in B

(iii) either $h(S) \cap S = \emptyset$ or $h(S) = S$ and S lies in general position with respect to $\mathrm{Fix}(h)$.

CHAPTER 1

NOTATION AND PRELIMINARIES

All spaces and maps will be in the PL category.

A homeomorphism $h : M \to M$ is said to be periodic if h^m = identity for m an integer greater than 1.

An involution is a homeomorphism of period 2.

A periodic homeomorphism h is said to be free if h and h^i have no fixed points for all i for which $h^i \neq$ identity.

Given a periodic homeomorphism $h : M \to M$, the orbit space $M/\langle h \rangle$ is the quotient space formed by identifying x with $h^i x$ for all x in M and all i.

The following elementary result will be used without further notice:

Lemma 1.

Let $h : M \to M$ be a free periodic homeomorphism on a connected manifold M. Let $q : M \to M/\langle h \rangle$ be the natural projection. Let $x \in M/\langle h \rangle$ and $\tilde{x} \in q^{-1}\{x\}$. Then $M/\langle h \rangle$ is a connected manifold, q is a regular covering map and

$$\langle h\rangle \approx \frac{\pi_1(M/\langle h\rangle, x)}{q_*(\pi_1(M,\ddot{x}))}$$

where $\langle h\rangle$ denotes the group of homeomorphisms of M generated by h.

Proof:

See for example section 57 of [10].

Let $h : M \to M$. The fixed point set of $h = \{x \in M \mid x = hx\}$ will be denoted by $\text{Fix}(h)$ or by $F(h)$.

It is well known that if $h : M \to M$ is simplicial and periodic, and if M'' denotes the second barycentric subdivision, then:

1. For every i, $\text{Fix}(h^i)$ is a subcomplex of M''.

2. The natural cell structure of the orbit space $M''/\langle h\rangle$ and the projection $q : M'' \to M''/\langle h\rangle$ are simplicial.

3. q maps each simplex homeomorphically.

4. An h-invariant subcomplex of M'', has an h-invariant regular neighborhood.

The star of a vertex x of a simplicial complex M, will be denoted $St(x)$.

A compact, not necessarily connected 2-manifold F is said to be 2-sided in M if there is an embedding

$h : F \times [-1,1] \to M$ with $h(x,0) = x$ for all $x \in F$ and $h(F \times [-1,1]) \cap \partial M = h(\partial F \times [-1,1])$.

A surface F is properly embedded in M if $F \cap \partial M = \partial F$.

Let F be a two-sided surface properly embedded in the 3-manifold M. The manifold M' obtained by cutting M along F is the manifold whose boundary contains two copies of F, F_1 and F_2, such that there is a natural projection $g : (M', F_1 \cup F_2) \to (M,F)$ with the property that $g \mid M' - (F_1 \cup F_2)$ is a homeomorphism onto $M - F$. If $h : F \times [-1,1] \to M$ is an embedding then M' is homeomorphic to $M - h(F \times (-1,1))$. So in particular if R is a regular neighborhood of F, $M' \approx \overline{M - R}$.

A disk D which is properly embedded in M and such that ∂D does not bound a disk in ∂M, will be called a meridional disk.

The boundary of a meridional disk will be called a meridional simple closed curve (s.c.c.).

We will be using the following spaces:

$\mathbb{R}$ = set of real numbers

$\mathbb{C}$ = set of complex numbers

$D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$

$I = [0,1]$

$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

P^2 = projective plane

K = solid Klein bottle (as defined below)

N = the non-orientable 2-sphere bundle over S'.

By the solid Klein bottle we mean the quotient space $K = \frac{D^2 \times \mathbb{R}}{\sim}$ where $(z,t) \sim (\overline{z}, t+1)$. An element of $\mathcal{K}$ with representative (z,t) is denoted by $[(z,t)]$.

Products of maps are also defined in a standard way, so for example $-z \times (t+1) : D^2 \times \mathbb{R} \to D^2 \times \mathbb{R}$ is the function which sends (z,t) to $(-z, t+1)$.

When the domain of a product map is not given it will be assumed to be $D^2 \times \mathbb{R}$.

It is easy to check that $-z \times (-t)$, $z \times (-t)$, $-z \times (t+1)$, $-z \times t$ and $\overline{z} \times t$ all induce maps on $\mathcal{K}$. The induced maps will be denoted by $[-z \times (-t)]$, $[z \times (-t)]$, and so on.

The above induced maps are involutions. Their fixed point sets are given in the following list:

$\mathrm{Fix}([-z \times (-t)]) \approx I \dot{\cup}\, pt$

$\mathrm{Fix}([z \times (-t)]) \approx D^2 \dot{\cup}\, I$

$\mathrm{Fix}([-z \times (t+1)]) \approx$ a Möebius band

$\mathrm{Fix}([-z \times t]) \approx S'$

$\mathrm{Fix}([\overline{z} \times t]) \approx$ an anulus.

With the above notation, and for $n > 1$, let $\varphi_1 = [z \times (t + \frac{1}{2^{n-1}})]$ and $\varphi_2 = [-z \times (t + \frac{1}{2^{n-1}})]$. Then for $i = 1,2$ and $n > 1$, φ_i is a map of period 2^n on $\mathcal{K}$, with $\mathrm{Fix}(\varphi_i^{2^{n-1}}) \approx$ anulus.

If M is a manifold with boundary, the manifold obtained by taking two disjoint copies of M and identifying corresponding boundary points is called the double of M and is denoted $2M$.

A 3-manifold M is said to be irreducible if every embedded 2-sphere in M bounds a 3-cell.

If a 3-manifold M is irreducible and does not contain any two-sided P^2, then M is said to be P^2-irreducible.

CHAPTER II

STATEMENT AND PROOF OF THE MAIN RESULTS

Our main results are:

Theorem 1.

An involution on $\mathcal{K}$ is equivalent to exactly one of:

1 - $[-z \times (-t)]$

2 - $[z \times (-t)]$

3 - $[-z \times (t+1)]$

4 - $[-z \times t]$

5 - $[\bar{z} \times t]$.

Theorem 2.

A homeomorphism on $\mathcal{K}$, of period 2^n $(n \geq 2)$ is equivalent to exactly one of:

1 - $[z \times (t + \frac{1}{2^{n-1}})]$

2 - $[-z \times (t + \frac{1}{2^{n-1}})]$

Both theorems above will be shown simultaneously in what follows.

Proof of Main Results:

Let h be a homeomorphism of period 2^n on $\mathcal{K}$. Then $h^{2^{n-1}}$ is an involution on $\mathcal{K}$.

From Smith theory (see Theorem 12.1 of [11]) we know that since $\mathcal{K}$ is a homology 1-sphere then $F(h^{2^{n-1}})$ must be a homology r-sphere where $-1 \leq r \leq 1$. Hence $F(h^{2^{n-1}})$ must be within the list ϕ, S^0, pt $\dot{\cup}$ I, pt $\dot{\cup}$ D^2, I $\dot{\cup}$ D^2, I $\dot{\cup}$ I, $D^2 \dot{\cup} D^2$, S', an anulus, a Möebius band.

Note that $2\mathcal{K} \approx N$ and so $h^{2^{n-1}}$ induces an involution $\hat{h}$ on N. Going through the list of the possible involutions $\hat{h}$ on N ([3]), we see that there is none with $F(\hat{h}) = S^0 \dot{\cup} S^0$, $S^0 \dot{\cup} S^2$ or $S^2 \dot{\cup} S^2$. Hence $F(h^{2^{n-1}})$ cannot be one of S^0, pt $\dot{\cup}$ D^2 or $D^2 \dot{\cup} D^2$.

We are left then with the following seven cases: $F(h^{2^{n-1}}) = \phi$, I $\dot{\cup}$ pt, $D^2 \dot{\cup}$ I, I $\dot{\cup}$ I, S', an anulus, a Möebius band.

Case 1.

Suppose $Fix(h^{2^{n-1}}) = \phi$.

Lemma 2.

There is no free involution on $\mathcal{K}$.

Proof:

Suppose otherwise. Let f be an involution on $\mathcal{K}$ with $F(h) = \phi$. Let $\mathcal{O}$ be the orbit space $\mathcal{K}/\langle f\rangle$ and $p: \mathcal{K} \to \mathcal{O}$ be the natural projection. Since f is fixed point free, p is a double covering map.

Note that $\mathcal{O}$ is compact, non-orientable, irreducible and has as boundary a 2-dimensional Klein bottle. Further, $\mathcal{O}$ does not contain any two-sided projective plane P, since otherwise:

> $p^{-1}P$ double covers P and hence consists of either two 2-sided copies of P or a sphere. Since $\mathcal{K}$ is P^2-irreducible, $p^{-1}P$ is a sphere which bounds a 3-cell C. Then p(C) is a 3-manifold bounded by a projective plane which is impossible.

Since p is a double cover, $p_*(\pi_1\mathcal{K}) \approx \mathbb{Z}$ has index 2 in $\pi_1\mathcal{O}$ and so by Lemma 11.4 of [1], $\exists$ a finite normal subgroup H of $\pi_1\mathcal{O}$ such that $\frac{\pi_1(\mathcal{O})}{H} \approx \mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.

Now, the exact sequence $1 \to H \to \pi_1\mathcal{O} \to \frac{\pi_1\mathcal{O}}{H} \to 1$ and the manifold $\mathcal{O}$ satisfy the hypotheses of Theorem 11.1 (part 3) of [1]. Thus we must have $\mathcal{O} \approx \mathcal{K}$. But this is impossible since $\pi_1(\mathcal{K}) = \mathbb{Z}$ has a unique subgroup of index 2 and hence a unique double cover (the orientable double cover). □

Hence case 1 does not arise.

<u>Case 2</u>.

Suppose $\mathrm{Fix}(h^{2^{n-1}}) \approx I \dot{\cup}$ pt.

We may assume that the isolated fixed point of $F(h^{2^{n-1}})$ is a vertex x_0.

Consider the action of h on the boundary sphere of $St(x_0)$. $h \mid \partial(St(x_0))$ acts freely with period 2^n on a sphere. This is only possible if $n = 1$.

Lemma 3.

An involution h on $\mathcal{K}$ with $Fix(h) \approx I \dot{\cup} pt$ is equivalent to $[-z \times (-t)]$.

Proof:

By [6] either $\exists$ a meridional disk $D \ni D \cap h(D) = \phi$ or $D = h(D)$ and D is in general position with respect to $Fix(h)$.

Suppose $D = h(D)$. Then by general position, $D \cap Fix(h)$ consist of a single point x. If $x \neq x_0$ then by [5] we get a contradiction. Hence $x = x_0$. Now, using a small enough regular neighborhood of D, we get a meridional disk $D' \ni D' \cap h(D') = \phi$.

So we may assume $D \cap h(D) = \phi$.

Cut along $D \cup h(D)$ to obtain components U_1, U_2, each homeomorphic to $D^2 \times I$. Since $F(h) \neq \phi$, we must have $h(U_1) = U_1$ and $h(U_2) = U_2$. Further by [5] we may assume $h \mid U_1 \sim \bar{z} \times (1-t)$ and $h \mid U_2 \sim -z \times (1-t)$, were both $\bar{z} \times (1-t)$ and $-z \times (1-t)$ have $D^2 \times I$ as domain. Thus we can find two disks D_1, D_2 with $D_1 \subset U_1$, $D_2 \subset U_2$, such that $h(D_i) = D_i$ and $F(h) \subset D_1 \cup D_2$.

Let V_1 and V_2 be the closures of the components of $\mathcal{K} - (D_1 \cup D_2)$. Then we must have $h(V_1) = V_2$.

Let $\bar{h} = [-z \times (-t)]$. It is easy to check that $\bar{h}$ is an involution with $F(\bar{h}) = I \dot{\cup}$ pt.

Let $\bar{D}_1 = \{[(z,1/2)] \mid z \in D^2\} \subset \mathcal{K}$ and $\bar{D}_2 = \{[(z,0)] \mid z \in D^2\} \subset \mathcal{K}$. Note that the $\bar{D}_i$ are disjoint disks and $Fix(\bar{h}) \subset \bar{D}_1 \cup \bar{D}_2$. Let $\bar{V}_1, \bar{V}_2$ be the closures of the components of $\mathcal{K} - (\bar{D}_1 \cup \bar{D}_2)$.

Since both $h \mid D_2$ and $\bar{h} \mid \bar{D}_2$ fix only one point and both $h \mid D_1$ and $\bar{h} \mid \bar{D}_1$ fix a properly embedded line segment, we have a map $t : D_1 \to \bar{D}_1$, $t : D_2 \to \bar{D}_2$ such that $th = \bar{h}t$. Extend t to a homeomorphism $t : V_1 \to \bar{V}_1$. On V_2, define t by $\bar{h}th$. Note that the definitions of t on V_1 and on V_2 agree on $V_1 \cap V_2 = D_1 \cup D_2$ (since here $th = \bar{h}t$).

Let $x \in \mathcal{K}$. If $x \in V_2$ then $tx = \bar{h}thx$ and so $t^{-1}\bar{h}tx = hx$. If $x \in V_1$ then since $h(V_1) = V_2$, $\bar{h}thx = \bar{h}(t \mid V_2)(hx) = \bar{h}(\bar{h}th)(hx) = tx$ and so $t^{-1}\bar{h}tx = hx$.

Therefore $h \sim [-z \times (-t)]$. □

Case 3.

Suppose $Fix(h^{2^{n-1}}) \approx D^2 \dot{\cup} I$.

Since $F(h^{2^{n-1}}) \approx D^2 \dot{\cup} I$ is h-invariant, we must have $h(D) = D$ (where $D^2 \approx D \subset F(h^{2^{n-1}})$). Let U be an h-invariant regular neighborhood of D such that $U \cap F(h^{2^{n-1}}) = D$. Then $U \cap \overline{\mathcal{K} - U}$ consists of disjoint meridional disks D', D'' with the property that $h(D') = D''$. Clearly $D \cap (D' \cup D'') = \phi$.

Note that $h^2(D') = D'$ and so h^2 leaves a point fixed in D'. Then $h^{2^{n-1}}$ leaves the same point fixed in D'. This is impossible unless $h^{2^{n-1}} = h^2 =$ identity. Hence h is an involution.

<u>Lemma 4</u>.

If h is an involution on $\mathcal{K}$ with $\mathrm{Fix}(h) \approx D^2 \times I$ then $h \sim [z \times (-t)]$.

<u>Proof</u>:

Let D' be as above. Then $D' \cap hD' = \phi$.

Cut $\mathcal{K}$ along $D' \cup hD'$ to get two components U_1 and U_2 each homeomorphic to $D^2 \times I$. Note that by [5] we may assume that $h \mid U_1 \sim z \times (1-t)$ and $h \mid U_2 \sim \bar{z} \times (1-t)$ where both $z \times (1-t)$ and $\bar{z} \times (1-t)$ are maps on $D^2 \times I$. Now proceed as in the proof of Lemma 3. □

<u>Case 4</u>.

Suppose $\mathrm{Fix}(h^{2^{n-1}}) \approx I \dot{\cup} I$.

<u>Lemma 5</u>.

There is no involution on $\mathcal{K}$ leaving fixed two line segments.

Proof:

Suppose otherwise. Let f be an involution on $\mathcal{K}$ with $F(f) = I \dot{\cup} I$.

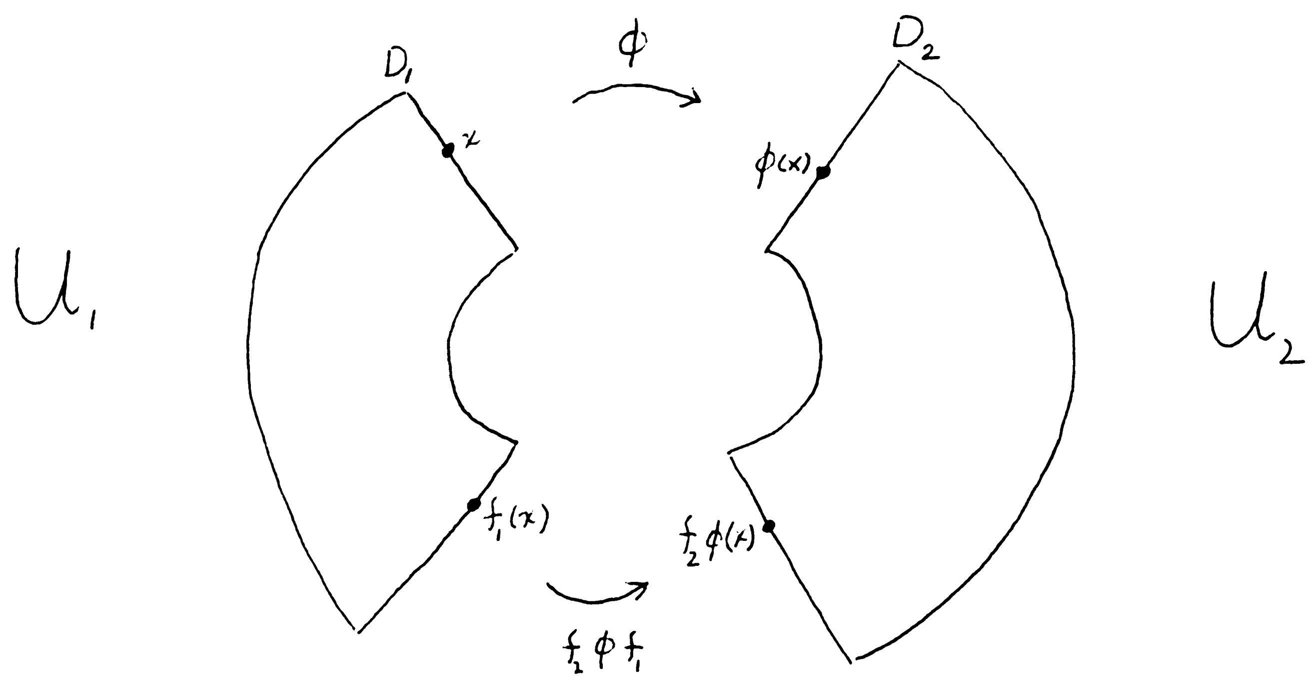
By [5] and [6], $\exists$ a meridional disk D in $\mathcal{K} \ni D \cap f(D) = \phi$. Let N be a regular neighborhood of $D \ni N \cap F(f) = \phi$. Then $\overline{\mathcal{K} - N}$ contains two copies of D, D_1 and D_2, such that $\mathcal{K}$ is obtained back from $\overline{\mathcal{K} - N}$ by identifying D_1 with D_2 via a homeomorphism ϕ.

Note that $\overline{\mathcal{K} - N - f(N)}$ consists of components U_i, $i = 1,2$, each homeomorphic with $D^2 \times I$. Further note that $f(U_i) = U_i$.

Let $f \mid U_i = f_i$. By [5], $f_i \sim \bar{z} \times (1-t)$ (on $D^2 \times I$).

Now, we may obtain $\mathcal{K}$ back from U_1 and U_2 in a two step procedure.

First, identify D_1 with D_2 via ϕ, to obtain a manifold homeomorphic to $D^2 \times I$ (see diagram).



In this new space, the map on $f_1D_1 (\approx D^2 \times 0)$ induced by $f_2 \phi f_1$ is the same as the map induced by f_2f_1. This map (f_2f_1) can be considered as being from $D^2 \times 0$ to $D^2 \times 1$ in $D^2 \times I$.

Finally, we should obtain $\mathcal{K}$ from $D^2 \times I$ by identifying x in $D^2 \times 0$ with f_2f_1x in $D^2 \times 1$. But using the fact that $f_i \sim \bar{z} \times (1-t)$ we see that h_2h_1 is orientation preserving, which is impossible.

Thus no involution on $\mathcal{K}$ leaves two fixed line segments. □

From the above lemma, we see that case 4 does not arise.

<u>Case 5</u>.

Suppose $Fix(h^{2^{n-1}}) \approx S'$.

<u>Lemma 6</u>.

Suppose h is an involution on $\mathcal{K}$ with $Fix(h) \approx S'$. Then $h \sim [-z \times t]$.

<u>Proof</u>:

By [5] and [6], $\exists$ a meridional disk $D \ni D = h(D)$ and $Fix(h)$ is in general position with respect to D.

Cutting $\mathcal{K}$ along D we obtain a component $U \approx D^2 \times I$ where $\mathcal{K}$ is obtained from $D^2 \times I$ by identifying $(x,0)$ with $(\phi(x),1)$ where ϕ is an orientation reversing map of D^2. Also, by [5], $h \mid U \sim (-z \times t) \mid D^2 \times I$.

Let p be the orbit map of h. Let ϕ' be the map of $D^2 (\approx p(D))$ induced by ϕ such that $\phi' p = p\phi$. Then the orbit space of h is viewed as $\frac{D^2 \times I}{\phi'}$.

Now, $\frac{D^2 \times I}{\phi'}$ is either a solid torus or a solid Klein bottle. It can't be a solid torus since otherwise $p \mid \partial\mathcal{K}$ would be a double cover of the 2-dimensional torus which is impossible. Hence $\mathcal{K}/\langle h\rangle \approx \frac{D^2 \times I}{\phi'} \approx \mathcal{K}$.

Let $f = [-z \times t]$. The above argument gives $\mathcal{K}/\langle f\rangle \approx \mathcal{K}$.

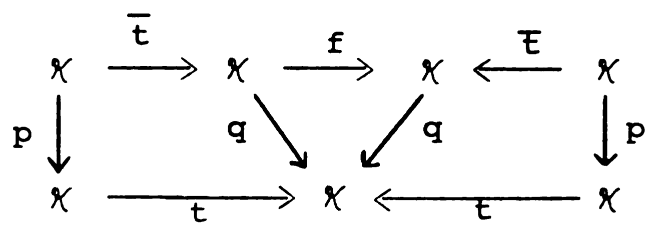
Let $q : \mathcal{K} \to \mathcal{K}/\langle f\rangle$ be the orbit projection. Note that we may assume that $p(F(h))$ is the "core" of $\mathcal{K}$. Similarly for $q(F(f))$. Then there is a homeomorphism $t : \mathcal{K}/\langle h\rangle \to \mathcal{K}/\langle f\rangle$ mapping $p(F(h))$ onto $q(F(f))$.

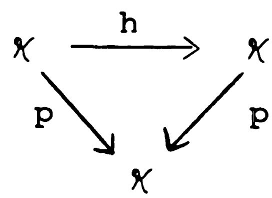
Let S be the 2-dimensional Klein bottle. Clearly $t \circ p : \mathcal{K} - F(h) \to \mathcal{K}/\langle f\rangle - q(F(f))$ and $q : \mathcal{K} - F(f) \to \mathcal{K}/\langle f\rangle - q(F(f))$ are double covering maps. Since $\pi_1(K/\langle f\rangle - q(F(f))) \approx \pi_1(S)$ has a unique subgroup of index 2 isomorphic to $\pi_1(S)$, $\exists$ a homeomorphism $\bar{t}$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{K} - F(h) & \xrightarrow{\ \bar{t}\ } & \mathcal{K} - F(f) \\ \downarrow p & & \downarrow q \\ \mathcal{K}/\langle h\rangle - p(F(h)) & \xrightarrow{\ t\ } & \mathcal{K}/\langle f\rangle - q(F(f)) \end{array}$$

Since t maps $p(F(h))$ homeomorphically onto $q(F(f))$, we may extend $\bar{t}$ to a homeomorphism on all of $\mathcal{K} \ni q\bar{t} = tp$.

The following diagrams are commutative:





Hence $P = P\bar{t}^{-1}f\bar{t}$ and since on $\mathcal{K} - F(h)$, p is a double covering map, it follows that $\bar{t}^{-1}f\bar{t}$ is the unique non-trivial covering translation. That is, $\bar{t}^{-1}f\bar{t} = h$ on $\mathcal{K} - F(h)$. Also, since $\bar{t}$ maps $F(h)$ onto $F(f)$, $\bar{t}^{-1}f\bar{t} = h$ on all of $\mathcal{K}$. □

Lemma 7.

There is no homeomorphism h of period 4 on $\mathcal{K}$ with $F(h^2) \approx S'$.

Proof:

Suppose otherwise.

Since h^2 is an involution, the same argument of Lemma 6 gives that $\mathcal{K}/\langle h^2 \rangle \approx \mathcal{K}$. Clearly h induces an involution $\bar{h}$ on $\mathcal{K}/\langle h^2\rangle$. It is easy to see that $F(\bar{h}) \subset P(F(h^2)) \approx S'$. Further, by [3], $F(\bar{h})$ can't be 0-dimensional. From case 1, $F(\bar{h}) \neq \phi$. Hence $F(\bar{h}) \approx S'$.

Applying Lemma 6 again to $\mathcal{K} \approx \mathcal{K}/\langle h^2\rangle$ and $\bar{h}$ we get that $\mathcal{K}/\langle h\rangle \approx \mathcal{K}/\langle \bar{h}\rangle \approx \mathcal{K}$. Let $P_1 : \mathcal{K} \to \mathcal{K}/\langle h^2\rangle$ and $P_2 : \mathcal{K}/\langle h^2\rangle \to \mathcal{K}/\langle \bar{h}\rangle$ be the orbit projections.

Note that $\bar{h} \sim [-z \times t]$ and so $\exists$ a meridional disk D in $\mathcal{K}/\langle h\rangle$ such that $P_2^{-1}(D)$ is a disk D'. The set $P_1^{-1}(D')$ consists of either a single disk or two disks meeting at a common interior point.

Consider $P_1 \mid \partial\mathcal{K}$. This is a double cover of $\partial\mathcal{K}$ by $\partial\mathcal{K}$. Let $\alpha : I \to \partial\mathcal{K}$ be a loop which traverses once around $\partial D'$. Let $b_0 = \alpha(0) = \alpha(1)$. Then $\pi_1(\partial\mathcal{K}, b_0) \approx \langle a,b \mid bab^{-1} = a^{-1}\rangle$ where a is represented by α and b is represented by an orientation reversing loop β which meets α transversely once at b_0. Let $\tilde{b}_0 \in P_1^{-1}(b_0)$. Since $\pi_1(\partial\mathcal{K})$ has a unique subgroup of index 2 isomorphic to $\pi_1(\partial\mathcal{K})$, we must have $(P_1 \mid \partial\mathcal{K})_*(\pi_1(\partial\mathcal{K}, \tilde{b}_0)) = \langle a^2, b \mid bab^{-1} = a^{-1}\rangle$.

Let $\gamma : I \to \partial\mathcal{K}$ be a loop which travels once around the component of $P_1^{-1}(\partial D')$ containing $\tilde{b}_0$, with $\gamma(0) = \gamma(1) = \tilde{b}_0$. Then $(P_1 \mid \partial\mathcal{K})_*([\gamma]) = a^2$ and so $P_1 \mid \gamma(I)$ double covers $\alpha(I) = \partial D'$. Therefore $P_1^{-1}(D')$ is an invariant meridional disk D''.

Now cut $\mathcal{K}$ along D'' to obtain a component $U \approx D^2 \times I$. It follows that $h \mid U \sim (iz,t)$. This is impossible since $h|u$ should induce a map on $\frac{D^2 \times I}{\phi}$, where ϕ is an orientation reversing map ($\phi : D^2 \times 0 \to D^2 \times 1$), but h does not commute with ϕ.

Therefore, there is no h as assumed.

Lemma 8.

There is no homeomorphism h of period $2^n (n \geq 2)$ on $\mathcal{K}$ with $\mathrm{Fix}(h^{2^{n-1}}) \approx S'$.

Proof:

The proof is by induction on n.

Lemma 7 provides the initial step of induction.

Suppose there is no periodic homeomorphism $\hat{h} : \mathcal{K} \to \mathcal{K}$ of period 2^n with $F(h^{2^{n-1}}) \approx S'$ $(n \geq 2)$.

We'll show that no homeomorphism of period 2^{n+1} on $\mathcal{K}$ exists, with $F(h^{2^n}) \approx S'$.

Suppose otherwise. Then h^{2^n} is an involution on $\mathcal{K}$ with $F(h^{2^n}) \approx S'$ and as in Lemma 6, $\mathcal{K}/\langle h^{2^n} \rangle \approx \mathcal{K}$. Now, h induces a map $\bar{h}$ on $\mathcal{K}/\langle h^{2^n} \rangle$ of period 2^n. Let $p : \mathcal{K} \to \mathcal{K}/\langle h^{2^n} \rangle$ be the orbit projection. It is straightforward to show that $F(\bar{h}^{2^{n-1}}) \subset P(F(h^{2^n})) \approx S'$. Further, since $\bar{h}^{2^{n-1}}$ is an involution on $\mathcal{K}/\langle h^{2^n} \rangle \approx \mathcal{K}$, previous arguments apply to show that $F(\bar{h}^{2^{n-1}})$ is neither 0-dimensional nor empty. Hence $F(\bar{h}^{2^{n-1}}) \approx S'$.

But then $\bar{h}$ is a map of period 2^n on $\mathcal{K}$ with $F(\bar{h}^{2^{n-1}}) \approx S'$, a contradiction to the induction hypotheses.

This completes the proof of Lemma 8. □

The above three lemmas cover all possibilities in case 5.

Case 6.

Suppose $\mathrm{Fix}(h^{2^{n-1}}) \approx$ a Möebius band.

Lemma 9.

If h is an involution on $\mathcal{K}$ with $\mathrm{Fix}(h) \approx$ a Möebius band then $h \sim [-z \times (t+1)]$.

Proof:

Let $h_1 = h$, $h_2 = [-z \times (t+1)]$, $M_i = F(h_i)$ and $p_i : \mathcal{K} \to \mathcal{K}/\langle h_i \rangle$ be the orbit projections. By [5] and [6], $\exists$ meridional disks $D_i \ni D_i = h_i(D_i)$. For $i = 1,2$, cut $\mathcal{K}$ along D_i to obtain a component $U_i \approx D^2 \times I$. By [5], $h_i \mid U_i \sim \bar{z} \times t$.

Note that $p_i(M_i)$ is a Möebius band in $\partial(\mathcal{K}/\langle h_i \rangle)$. Since we can obtain $\mathcal{K}/\langle h_i \rangle$ as an identification space from $u_i/\langle h_i \rangle$ ($\approx D^2 \times I$), we must have that $\mathcal{K}/\langle h_i \rangle \approx \mathcal{K}$.

Let D_i' be h_i-invariant meridional disks in U_i. Note that $P_i(D_i')$ is a meridional disk in $\mathcal{K}/\langle h_i \rangle$. Let $\alpha_i = \partial(P_i(D_i'))$ and $\beta_i = \partial(P_i(M_i))$. Then β_i separates $\partial(\mathcal{K}/\langle h_i \rangle)$ into two Möebius bands. Also, β_i meets α_i in exactly 2 points in such a way that $\partial(\mathcal{K}/\langle h_i \rangle) - \alpha_i - \beta_i$ consists of two open rectangles.

Now take any homeomorphism from $\alpha_1 \cup \beta_1$ onto $\alpha_2 \cup \beta_2$ and extend it to a homeomorphism $t : \partial(\mathcal{K}/\langle h_1 \rangle) \to \partial(\mathcal{K}/\langle h_2 \rangle)$ in such a way that $P_1(M_1)$ goes onto $P_2(M_2)$. We can further extend t on $p_1 D_1'$. Finally, noting that $\mathcal{K}/\langle h_1 \rangle - \partial(\mathcal{K}/\langle h_1 \rangle) - p_1 D_1'$ is an open 3-cell, we can extend t to a homeomorphism $t : \mathcal{K}/\langle h_1 \rangle \to \mathcal{K}/\langle h_2 \rangle$.

Since β_i separates $\partial(\mathcal{K}/\langle h_i \rangle)$ then ∂M_i separates $\partial\mathcal{K}$ and so M_i separates $\mathcal{K}$ into two components. Each of the components of $\mathcal{K} - M_i$ is mapped homeomorphically onto $\mathcal{K}/\langle h_i \rangle$ by P_i. Hence, since $t(P_1 M_1) = P_2 M_2$, we have a homeomorphism $\bar{t}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\ \bar{t}\ } & \mathcal{K} \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\ \mathcal{K}/\langle h_1 \rangle & \xrightarrow[\ t\ ]{} & \mathcal{K}/\langle h_2 \rangle \end{array} \quad .$$

Now as in Lemma 6 we can conclude that $\bar{t}^{-1} h_2 \bar{t} = h_1$. □

Lemma 10.

There is no homeomorphism h of $\mathcal{K}$ of period 2^n $(n > 1)$ with $\mathrm{Fix}(h^{2^{n-1}}) \approx$ Möebius band.

Proof:

Suppose otherwise.

Let $c = \partial(F(h^{2^{n-1}}))$. Let N be an h-invariant regular neighborhood of c in $\partial(\mathcal{K})$. From Lemma 9, we see that N is an anulus. So we have that $h \mid N$ is a map of period 2^n on an anulus with $(h \mid N)^{2^{n-1}}$ orientation reversing. Since $n > 1$, this is impossible. □

Lemmas 9 and 10 complete case 6.

Case 7.

Suppose $\mathrm{Fix}(h^{2^{n-1}}) \approx$ anulus.

As before, we have $D \ni D = h^{2^{n-1}}(D)$ and such that $\mathcal{K}$ cut along D is a component $U \approx D^2 \times I$ with $h^{2^{n-1}} \mid U \sim (\bar{z} \times t) \mid D^2 \times I$. Then $\mathcal{K}/\langle h^{2^{n-1}} \rangle$ is either a solid torus or a solid Klein bottle.

Let $p : \mathcal{K} \to \mathcal{K}/\langle h^{2^{n-1}} \rangle$ be the projection onto the orbit space.

Lemma 11.

If h is an involution on $\mathcal{K}$ leaving an anulus fixed, then $h \sim [\bar{z} \times t]$.

Proof:

Let $h_1 = h$, $P_1 = P$ and $h_2 = [\bar{z} \times t]$. Let $P_2 : \mathcal{K} \to \mathcal{K}/\langle h_2 \rangle$ be the natural projection. Let $D_1 = D$ and similarly define D_2 for h_2. Both h_i are involutions with $F(h_i)$ an anulus.

It is known (see [7] or [8]) that up to equivalence there is a unique involution on the 2-dimensional Klein bottle leaving two circles fixed. So in particular, $h_1 \mid \partial\mathcal{K} \sim h_2 \mid \partial\mathcal{K}$. Using this, it is easy to see that $\partial\mathcal{K} - \partial(F(h_i))$ is an open anulus A_i. Also, $P_i(F(h_i))$ is an anulus F_i in $\partial(\mathcal{K}/\langle h_i\rangle)$. From the remarks preceding the statement of this Lemma, we know that $\partial(\mathcal{K}/\langle h_i\rangle)$ is either a torus or a Klein bottle. Further, $\partial(\mathcal{K}/\langle h_i\rangle) = F_i \dot{\cup} P_i(A_i)$ where P_iA_i is either an open anulus or an open Möebius band. Hence the only possibility is for P_iA_i to be an open anulus and so $\partial(\mathcal{K}/\langle h_i\rangle)$ is a torus. Thus $\mathcal{K}/\langle h_i\rangle \approx D^2 \times S'$.

Note that $P_1(D_1)$ and $P_2(D_2)$ are meridional disks in $\mathcal{K}/\langle h_1\rangle$ and $\mathcal{K}/\langle h_2\rangle$ respectively. Let S_i be the segment $P_i(D_i) \cap F_i$ and $S_i' = \partial(P_i(D_i)) - S_i$ so that $S_i \cup S_i' = \partial(P_i(D_i))$. Let t be a homeomorphism of S_1 onto S_2. Extend t homeomorphically in steps as follows: first on F_1 onto F_2, then on S_1' onto S_2', next on P_1D_1 onto P_2D_2, followed by an extension on the open rectangle $\partial(\mathcal{K}/\langle h_1\rangle) - F_1 (= p_1A_1)$ onto the open rectangle $\partial(\mathcal{K}/\langle h_2\rangle) - F_2 (= P_2A_2)$, finally across the remaining open cell in $\mathcal{K}/\langle h_1\rangle$.

Since $P_i \mid (\mathcal{K} - F(h_i))$ double covers $\mathcal{K}/\langle h_i\rangle - F_i$ and since $\pi_1(\mathcal{K}/\langle h_i\rangle - F_i) \approx \mathbb{Z}$, we can lift t to a homeomorphism $\bar{t}$ such that:

$$\begin{array}{ccc} \mathcal{K} - F(h_1) & \xrightarrow{\;\;\bar{t}\;\;} & \mathcal{K} - F(h_2) \\ {\scriptstyle P_1}\Big\downarrow & & \Big\downarrow{\scriptstyle P_2} \\ \mathcal{K}/\langle h_1 \rangle - F_1 & \xrightarrow{\;\;t\;\;} & \mathcal{K}/\langle h_2 \rangle - F_2 \end{array}$$

commutes.

Since $P_2^{-1} t P_1$ maps $F(h_1)$ homeomorphically onto $F(h_2)$, we can extend $\bar{t}$ on $F(h_1)$ so that $P_2 \bar{t} = t P_1$. The simplicial nature of the maps insures the continuity of $\bar{t}$.

We can now conclude that $\bar{t}^{-1} h_2 \bar{t} = h_1$. □

The following result is well known (see [9]). Since a proof of it does not seem to appear in print, we will include it for completeness. For a possible alternate approach to the proof see [10] together with [11]. We will need to make use of it later on.

Lemma 12.

Let k be a homeomorphism of period m on $D^2 \times S'$. If $\langle k \rangle$ acts freely on $D^2 \times S'$ then k is weakly equivalent to one of:

$$k_1(z_1, z_2) = (z_1, wz_2)$$

or

$$k_2(z_1, z_2) = (\bar{z}_1, wz_2)$$

where $w = e^{\frac{2\pi i}{m}}$. Further, $\frac{D^2 \times S'}{\langle k_1 \rangle} \approx D^2 \times S'$ and $\frac{D^2 \times S'}{\langle k_2 \rangle} \approx \mathcal{K}$.

Proof:

Let $\mathcal{O}$ be the orbit space $\frac{D^2 \times S'}{\langle k \rangle}$ and let P be the projection onto the orbit space. Clearly, P is an m-fold covering map. Also, $P \mid \partial(D^2 \times S')$ is an m-fold cover and so depending on whether k preserves or reverses orientation, we get $\partial\mathcal{O} \approx S' \times S'$ or $\partial\mathcal{O} \approx \partial\mathcal{K}$ respectively.

Also note $\mathcal{O}$ is compact, irreducible and does not admit a 2-sided P^2.

By Lemma 11.4 of [4], we have a short exact sequence $1 \to N \to \pi_1(\mathcal{O}) \to Q \to 1$, where $Q \approx \mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$ and N is finite normal. By Theorem 11.1 of [4], we see that $N = 1$ and also if k is orientation reversing (preserving), $\mathcal{O} \approx \mathcal{K}$ (respectively $\mathcal{O} \approx D^2 \times S'$). This implies k is unique up to weak equivalence. □

We are now ready for the last result of this paper.

Lemma 13.

If h is a self homeomorphism of $\mathcal{K}$ of period 2^n $(n \geq 2)$ and if $\mathrm{Fix}(h^{2^{n-1}})$ is an anulus, then either $h \sim [z \times (t + \frac{1}{2^{n-1}})]$ or $h \sim [-z \times (t + \frac{1}{2^{n-1}})]$.

Proof:

Let $h : \mathcal{K} \to \mathcal{K}$ be a homeomorphism of period 2^n. Suppose $\mathrm{Fix}(h^{2^{n-1}})$ is an anulus.

If $x \in F(h)$ then we may assume that x is a vertex in the interior of $\mathcal{K}$ and so $\partial(St(x))$ is an h-invariant

2-sphere. Since $F(h^{2^{n-1}}) \cap \partial(St(x)) \approx S'$ and $n \geq 2$, we get the contradiction that $h^{2^{n-1}}$ is an orientation reversing map on $\partial(St(x))$. Hence $F(h) = \phi$.

Arguments similar to the above permit us to conclude that $F(h^i) = \phi$ for $1 \leq i < 2^{n-1}$.

Since $h^{2^{n-1}}$ is an involution, Lemma 11 applies to give that $\mathcal{K}/\langle h^{2^{n-1}} \rangle \approx D^2 \times S'$. Let $\mathcal{K}/\langle h^{2^{n-1}} \rangle = \mathcal{T}$. Since $F(h^i) = \phi$ for $1 \leq i < 2^{n-1}$, h induces a free action $\bar{h}$ of period 2^{n-1} on $\mathcal{T}$.

Now we break the proof into cases depending on whether $\bar{h}$ is orientation preserving or reversing.

<u>Case a</u>:

Suppose $\bar{h}$ is orientation preserving.

Let $h_2 = [z \times (t + \frac{1}{2^{n-1}})]$. Then h_2 has period 2^n and $F(h_2^{2^{n-1}}) \approx$ an anulus. Hence $\mathcal{K}/\langle h_2^{2^{n-1}} \rangle \approx$ a solid torus $\mathcal{T}_2$. Also, $\bar{h}_2$, the map induced by h_2 on $\mathcal{K}/\langle h_2^{2^{n-1}} \rangle$ is free, orientation preserving and of period 2^{n-1}. By Lemma 12, $\mathcal{T}_2/\langle \bar{h}_2 \rangle \approx D^2 \times S' \approx \mathcal{T}/\langle \bar{h} \rangle$.

Let $h_1 = h$, $\mathcal{T}_1 = \mathcal{T}$ and $P_i : \mathcal{K} \to \mathcal{T}_i$, $q_i : \mathcal{T}_i \to \mathcal{T}_i/\langle \bar{h}_i \rangle$ be the natural projections. Note that $\mathcal{T}_i/\langle \bar{h}_i \rangle \approx \mathcal{K}/\langle h_i \rangle$.

It follows from Lemma 12 that for $i = 1,2$, $\exists$ a meridional disk D_i in $\mathcal{T}_i$ such that, for $\ell = 2^{n-1} - 1$, $D_i, \bar{h}_i D_i, \bar{h}_i^2 D_i, \ldots, \bar{h}_i^{\ell}(D_i)$ are all mutually disjoint.

Then D_i projects down via q_i onto a disk q_iD_i in $\mathcal{K}/\langle h_i\rangle$.

Also, $P_i(F(h_i^{2^{n-1}}))$ is an anulus A_i in $\partial\mathcal{T}_i$. Since $F(h_i^{2^{n-1}})$ is h_i-invariant, A_i is $\overline{h}_i$ - invariant. Since $\partial(\mathcal{K}/\langle h_i\rangle) \approx S' \times S'$, q_iA_i is an anulus in $\partial(\mathcal{K}/\langle h_i\rangle)$.

Now, using the D_i's, it is easy to construct a homeomorphism $t : \mathcal{K}/\langle h_1\rangle \to \mathcal{K}/\langle h_2\rangle$ which maps q_1A_1 onto q_2A_2.

Since the q_i are covering projections we may lift t to a homeomorphism $\overline{t} : \mathcal{T}_1 \to \mathcal{T}_2$ which maps A_1 onto A_2 and such that $q_2\overline{t} = tq_1$. Finally, exactly as with the homeomorphism t of Lemma 11, we may lift $\overline{t}$ to a homeomorphism $\overline{\overline{t}}$ which makes the following diagram commute:

$$\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\;\overline{\overline{t}}\;} & \mathcal{K} \\
{\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\
\mathcal{T}_1 & \xrightarrow{\;\overline{t}\;} & \mathcal{T}_2 \\
{\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\
\mathcal{K}/\langle h_1\rangle & \xrightarrow{\;t\;} & \mathcal{K}/\langle h_2\rangle
\end{array}$$

Note that since q_1 is a covering projection and since

$$\begin{array}{ccccccc} \mathcal{T}_1 & \xrightarrow{\bar{t}} & \mathcal{T}_2 & \xrightarrow{\bar{h}_2} & \mathcal{T}_2 & \xleftarrow{\bar{t}} & \mathcal{T}_1 \\ \downarrow q_1 & & & q_2 \searrow \quad \swarrow q_2 & & & \downarrow q_1 \\ \mathcal{K}/\langle h_1 \rangle & & \xrightarrow{t} & \mathcal{K}/\langle h_2 \rangle & \xleftarrow{t} & & \mathcal{K}/\langle h_1 \rangle \end{array}$$

commutes, we get that for some δ, $\bar{t}^{-1}\bar{h}_2\bar{t} = \bar{h}_1^{\delta}$. Further, since $p_i h_i = \bar{h}_i p_i$, it follows that: $p_1 h_1^{\delta} = \bar{h}_1^{\delta} p_1 = \bar{t}^{-1}\bar{h}_2\bar{t}p_1 = \bar{t}^{-1}\bar{h}_2 p_2 \bar{\bar{t}} = \bar{t}^{-1} p_2 h_2 \bar{\bar{t}} = p_1 \bar{\bar{t}}^{-1} h_2 \bar{\bar{t}}$. Thus if $x \notin \mathrm{Fix}(h^{2^{n-1}})$ then $h_1^{\delta+2^{n-1}}(x) = \bar{\bar{t}}^{-1} h_2 \bar{\bar{t}}(x)$. Also, since $p_1 | F(h_1^{2^{n-1}})$ is a homeomorphism, for $x \in F(h_1^{2^{n-1}})$ we get: $h_1^{\delta+2^{n-1}}(x) = h_1^{\delta}(x) = \bar{\bar{t}}^{-1} h_2 \bar{\bar{t}}(x)$. Therefore $h_1 \sim h_2$ (weakly) as desired.

<u>Case b</u>:

Suppose $\bar{h}$ is orientation reversing.

Then a proof similar to that of case a yields $h \sim [-z \times (t + \frac{1}{2^{n-1}})]$ (weakly). □

Lemmas 11 and 13 complete case 7 and with this we finish the proof of our classification theorems.

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