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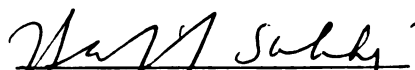
thesis entitled

On subordination, sampling theorem
and "Past and Future" of some
classes of second-order Processes
presented by

Mohsen Pourahmadi S.A.

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Major professor

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ON SUBORDINATION, PART I
"PAST AND PRESENT"
CLASSES OF SUBORDINATION

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Michigan State University
in partial fulfillment of the requirements
for the degree

DOCTOR OF PHILOSOPHY

Department of Classics and Archaeology

1990

Approved by the Faculty

ON SUBORDINATION, SAMPLING THEOREM AND
"PAST AND FUTURE" OF SOME
CLASSES OF SECOND-ORDER PROCESSES

By

Mohsen Pourahmadi S.A.

By

Mohsen Pourahmadi S.A.

In this thesis, three independent problems (subordination, sampling theorem and "Past and Future" concerning harmonizable and stationary processes are studied.

Chapter I contains some results regarding second order processes along with a necessary and sufficient condition for strong subordination of q -variate stationary processes which are cross-correlated.

The problem of finding statistical conditions for subordination of harmonizable and periodically correlated sequences is studied

in Chapter II. Sufficient conditions for subordination of harmonizable sequences and a simple condition for subordination of periodically correlated sequences are not necessary. Submitted to Michigan State University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Department of Statistics and Probability

1980

sequences are derived.

In Chapter III, the problem of admittance of sampling theorem of a q -variate stationary process and its relation with the admittance of sampling theorem of its components is considered. It is shown that if the components of a

ABSTRACT
ON SUBORDINATION, SAMPLING THEOREM AND
"PAST AND FUTURE" OF SOME
 $h > 0$, then the CLASSES OF SECOND-ORDER PROCESSES

By
A sampling theorem for q -variate stationary process, under a periodicity condition Mohsen Pourahmadi S.A. spectral measure of the process, is proved in the spirit of Lloyd's work. This sampling theorem In this thesis, three independent problems (subordination, sampling theorem and "Past and Future") concerning harmonizable and stationary processes are studied. Chapter I contains some well-known results about such processes along with a necessary and sufficient conditions for strong subordination of q -variate stationary processes which are stationarily cross-correlated.

The problem of finding analytic conditions for subordination of harmonizable and periodically correlated sequences is studied. In Chapter II, Sufficient conditions for subordination of harmonizable sequences and a simple counter-example showing that these conditions are not necessary are given. In the case of periodically correlated sequences, which is a subclass of harmonizable sequences, necessary and sufficient conditions for subordination, mutual subordination of such processes in terms of their associated multi-variate stationary sequences are derived.

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In Chapter III, the problem of admittance of sampling theorem of a q -variate stationary process and its relation with the admittance of sampling theorem of its components is considered. It is shown that if the components of a q -variate process (not necessarily stationary) admits a sampling theorem with the same sample spacing $h > 0$, then the process itself admits a sampling theorem with the same h . A sampling theorem for q -variate stationary process, under a periodicity condition on the range of the spectral measure of the process, is proved in the spirit of Lloyd's work. This sampling theorem is used to show that if a q -variate stationary process admits a sampling theorem, then each of its components will do so. In Section 5, by using Abreu's theorem, the well-known sampling theorems for harmonizable processes is proved in an easier way with more explicit coefficients for the sampling series.

In Chapter IV, Helson-Sarason Theorem on "Past and Future" is generalized from the disk algebra to a Dirichlet algebra setting by using function-algebraic method. Advantages of our method as compared to Ohno-Yabuta's method [32] on the same problem is discussed. This theorem is used to answer a question of M. Rosenblatt on the strong mixing of multi-parameter Gaussian stationary processes.

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and the existence of a shift operator for the latter sequences, we have been able to find necessary and sufficient conditions for subordination of the former sequences. Sufficient conditions for the subordination of harmonizable sequences and a simple counter-example showing that these conditions are not necessary is given in Section 3.

INTRODUCTION

The concept of subordination was introduced, studied and used in prediction of univariate stationary sequences by A.N. Kolmogorov [18]. Analytic necessary and sufficient conditions for subordination of such processes were derived in [18]. Analogous analytic conditions for the subordination of q -variate and infinite-dimensional stationary sequences were derived by M. Rosenberg, Yu. Rosanov and others in [39], [41], [38], [23] and [22]. In [12], the notion and analytic characterization of subordination of stationary sequences have been used for optimal filtering of stationary signals.

The problem of finding such an analytic characterization for the subordination of harmonizable sequences which are harmonizably cross-correlated is studied in Chapter II. The study is carried out in such a way that when specialized to stationary sequences, the results will reduce to the corresponding results of Kolmogorov [18]. In Section 2, necessary and sufficient conditions for subordination, mutual subordination and necessary condition for strong subordination of periodically correlated sequences in terms of their associated multi-variate stationary sequences are derived. It is well-known that the class of periodically correlated sequences is a natural extension of stationary sequences but a subclass of harmonizable sequences. Because of the 1-1 correspondence between periodically correlated sequences and q -variate stationary sequences

and the existence of a shift operator for the latter sequences, we have been able to find necessary and sufficient conditions for subordination of the former sequences. Sufficient conditions for the subordination of harmonizable sequences and a simple counter-example showing that these conditions are not necessary is given in Section 3. It seems that, the fact that these conditions are not necessary can be attributed to the failure of existence of a shift operator for the harmonizable sequences.

The Appendix In Chapter III, the problem of sampling theorem for q -variate stationary and univariate harmonizable processes is considered.

Sections 2 and 3 contain some well-known results as well as some new results which play a crucial role in the proof of our sampling theorem for q -variate stationary processes. In Section 4, a sampling theorem for a q -variate stationary process, similar to that of Lloyd's [21], is proved under the condition that the range of the spectral measure of the process considered as a linear operator-valued function from C^q to C^q is periodic. Then, this sampling theorem is used to prove that if a q -variate stationary process admits a sampling theorem, then each of its components will do so. In Section 5, by using Abreu's theorem [1], we prove in an easier way as compared to [36] and [20], sampling theorems known for harmonizable process, with the advantage that in our proof the coefficients in the sampling series for such processes are exactly the coefficients of the sampling series of its associated stationary process.

The problem of strong mixing of multi-parameter Gaussian

stationary sequences was first studied by M. Rosenblatt [43]. In [43], some sufficient conditions for strong mixing of such process along the work of Kolmogorov and Rosanov [19] were given. The problem of strong mixing of such processes is yet open. In Chapter IV, first we generalize Helson-Sarason theorem on "Past and Future"

from the disk algebra to a Dirichlet algebra setting and then specialize this theorem to the torus to obtain a necessary and sufficient condition for strong mixing of multi-parameter Gaussian stationary sequences.

The Appendix explains some of the terminologies related to a Dirichlet algebra used in Chapter IV.

In the following we introduce some basic terminology and concepts in the spirit of the work of P. Masani [20]. P. Masani [22]. These are q -stationary processes.

For $q \in \mathbb{N}$, $q \geq 1$, a sequence $\{X_k\}_{k=0}^{\infty}$ of $\mathbb{C}^q$ -valued random variables on Ω with itself q -times, $\{X_k\}_{k=0}^{\infty}$ is called q -stationary if X_k and X_{k+1} are independent for $k \in \mathbb{N}$, for $k \geq 1$. A Gramian structure $\{A_k\}_{k=0}^{\infty}$ on $\mathbb{C}^q$ is defined to be the sequence $\{A_k\}_{k=0}^{\infty}$ of $q \times q$ matrices

One can easily see that

(1.1)

$$\begin{cases} (X_k, X_k) = A_k, \\ (X_k, X_{k+1}) = A_{k+1}, \\ (X_k, X_{k+2}) = A_{k+2}, \end{cases}$$

for any $X_k, X_{k+1}, X_{k+2} \in \mathbb{C}^q$ and $k \in \mathbb{N}$. If X_k is orthogonal to X_{k+1} in $\mathbb{C}^q$ for all $k \in \mathbb{N}$, then $\{A_k\}_{k=0}^{\infty}$ is

It is well-known that $\{A_k\}_{k=0}^{\infty}$ is a positive semi-definite sequence

inner product,

$$(1.2) \quad ((X,Y)) = \text{trace } (X,Y) = \sum_{j=1}^q (x^j, y^j).$$

CHAPTER I

NOTATIONS AND PRELIMINARIES

Let $(\Omega, \mathcal{B}, P)$ be a probability space. $H = L^2(\Omega, \mathcal{B}, P)$ denotes the Hilbert space of all complex-valued random variables on Ω with zero expectation and finite variance. The inner product in H is defined by $(x, y) = \int_{\Omega} x(\omega) \overline{y(\omega)} P(d\omega)$, $x, y \in H$.

In the following we introduce some basic terminologies and concepts in the spirit of the work of N. Wiener and P. Masani [50], P. Masani [23]. These are used in the study of q -variate stationary processes.

1.1 Definition: For $q \geq 1$, H^q denotes the Cartesian product of H with itself q -times, i.e. the set of all column vectors $X = (x^1, x^2, \dots, x^q)^T$ with $x^i \in H$, for $i = 1, 2, \dots, q$. As usual we endow the space H^q with a Gramian structure: For X and Y in H^q their Gramian (X, Y) is defined to be the $q \times q$ matrix $(X, Y) = [(x^i, y^j)]_{i,j=1}^q$.

One can easily verify that,

$$(1.1) \quad \begin{cases} (X, X) \geq 0, (X, X) = 0 \Leftrightarrow X = 0; \\ (\sum_{k=1}^m A_k X_k, \sum_{\ell=1}^n B_{\ell} Y_{\ell}) = \sum_{k=1}^m \sum_{\ell=1}^n A_k (X_k, Y_{\ell}) B_{\ell}^*, \end{cases}$$

1.2 Convention: Since the class of $q \times q$ matrices A_k, B_{ℓ} is a subspace of M_q , we say that X is orthogonal to Y in H^q if $(X, Y) = 0$.

It is well-known that H^q is a Hilbert space under the following

inner product,

$$(1.2) \quad ((X, Y)) = \text{trace } (X, Y) = \sum_{j=1}^q (x^j, y^j).$$

A closed subset $\bar{H}$ of H^q is called a subspace of H^q if it is a manifold, i.e. $AX + BY \in \bar{H}$ whenever $X, Y \in \bar{H}$ and A, B are $q \times q$ matrices. It is known [50] that $\bar{H}$ is a subspace of H^q if and only if there exists a subspace H of H such that $\bar{H} = H^q$. Thus, we obtain a structure for H^q which differs from but also closely resembles that of a Hilbert space, and which we shall call Hilbertian [23]. For any $x \in H$, its orthogonal projection on a subspace H of H is denoted by $(x|H)$. Given a vector $X = (x^1, \dots, x^q)^T \in H^q$, its projection on a subspace $\bar{H} = H^q$ is the vector $(X|\bar{H})$ whose i -th component is $(x^i|H)$ for each $i = 1, \dots, q$.

1.1 Definition: A sequence $x_n, n \in \mathbb{Z}$ ($X_n, n \in \mathbb{Z}$) of elements of $H(H^q)$ is called a univariate (q-variate) stochastic sequence.

For convenience we may abbreviate $x_n, n \in \mathbb{Z}$ ($X_n, n \in \mathbb{Z}$) by $x_n(x_n)$ or simply by $x(X)$. Also, throughout we use small $x, y, \dots$ to denote univariate and capital $X, Y, \dots$ for q -variate sequences.

For random variables $\{x_j\}_{j \in J}$ in H we denote by $\sigma\{x_j; j \in J\}$ the subspace spanned by x_j , for all j in the indexed set J . Similarly for random vectors $\{X_j\}_{j \in J}$ in H^q , $\sigma\{X_j; j \in J\}$ is the subspace of H^q spanned by all $X_j, j \in J$ with matrix coefficient.

1.2 Convention: Since the class of univariate stochastic sequences is a subclass of q -variate stochastic sequences ($q = 1$), here, we only introduce notations and known results about q -variate sequences. The

$$H(Y, n) \subset H(X, n), \quad n \in \mathbb{Z}$$

corresponding notations and results for univariate sequences can be obtained by obvious specialization.

1.3 Definition: To every q -variate sequence X , we associate the present and past subspaces $\bar{H}(X, n)$, $n \in \mathbb{Z}$, defined by,

(1.3) Definition: A q -variate stochastic sequence X is said to be stationary if the covariance function $R(m, n) = \sigma\{X_k, X_{k+m}\}$ depends on

and the terminal subspace $\bar{H}(X)$, defined by,

(1.4) Definition: $\bar{H}(X) = \bar{H}(X, \infty) = \sigma\{X_k; \text{all } k\}$.

Also, we define $H(X, n)$, $n \in \mathbb{Z}$, and $H(X)$ by,

(1.5) We note that $H(X, n) = \sigma\{x_k^i; k \leq n, 1 \leq i \leq q\} \subseteq H$, considered as a set of q univariate sequences.

(1.6) $H(X) = H(X, \infty) = \sigma\{x_k^i; \text{all } k, 1 \leq i \leq q\}$.

To introduce the concept of subordination of q -variate stochastic sequences, it is easy to check that,

(1.7) $\bar{H}(X, n) = \bar{H}^q(X, n)$, $n \in \mathbb{Z}$.

In simultaneous treatment of two q -variate stochastic sequences, the concept of subordination plays an important role. Here, we define subordination and some related notions for two q -variate sequences X and Y .

1.4 Definition: Suppose X and Y are q -variate stochastic sequences.

We say that,

(i) Y is subordinate to X if and only if $H(Y) \subset H(X)$.

(ii) Y is strongly subordinate to X if and only if

$$H(Y, n) \subset H(X, n), \quad n \in \mathbb{Z}.$$

(iii) Y and X are mutually subordinate or equivalent if and only if $H(Y) = H(X)$.

(1) In the following, we assume that our q -variate stochastic sequence is stationary in the sense defined below.

1.5 Definition: A q -variate stochastic sequence X is said to be stationary if the covariance function $R(m, n) = (X_m, X_n)$ depends on $m-n$ alone.

1.6 Definition: Two q -variate stationary sequences X and Y are said to be stationarily cross-correlated if the qxq Gram matrix (X_m, Y_n) depends on $m-n$ alone.

We note that a q -variate stationary sequence can be considered as a set of q univariate stationary and stationarily cross-correlated sequences.

To introduce the known results about spectral analysis of q -variate stationary sequences, and for later use, we need the following concepts.

(1.8) Let $\mathcal{B}$ be a σ -algebra of subsets of a space Ω . M is said to be a qxq matrix-valued signed measure on $(\Omega, \mathcal{B})$ if for each $A \in \mathcal{B}$, $M(A)$ is a qxq matrix, with finite complex entries and $M(A) = \sum_{k=1}^{\infty} M(A_k)$, whenever $A_1, A_2, \dots$ is a sequence of disjoint sets in $\mathcal{B}$ whose union is A .

1.7 Definition: A qxq matrix-valued signed measure M is called a qxq matrix-valued measure if $M(A)$ is a nonnegative hermitian matrix for each $A \in \mathcal{B}$.

1.8 Definition: Let $\phi = (\phi_{ij})$ be a matrix-valued function on Ω and μ a nonnegative real-valued measure on $\mathcal{B}$.

(i) We say that ϕ is $\mathcal{B}$ -measurable if each function ϕ_{ij} is $\mathcal{B}$ -measurable.

(ii) $L_{1,\mu}$ is the class of all ϕ such that each ϕ_{ij} is integrable with respect to μ .

(iii) For $\phi \in L_{1,\mu}$, we define $\int_{\Omega} \phi d\mu = (\int_{\Omega} \phi_{ij} d\mu)$.

1.9 Definition: We say that the $q \times q$ matrix-valued signed measure M is absolutely continuous (a.c.) with respect to (w.r.t.) a σ -finite nonnegative real-valued measure μ on $(\Omega, \mathcal{B})$ if the entries of M , is defined by,

i.e. M_{ij} 's are a.c. w.r.t. μ . We write $M'_{\mu} = \frac{dM}{d\mu} = (\frac{dM_{ij}}{d\mu})$ for

the Radon-Nikodym derivative of M w.r.t. μ .

Now, using Definition 1.8 (iii) we define integrals of the form

$\int_{\Omega} \phi(\lambda) M(d\lambda) \psi(\lambda)$, where M is any matrix-valued signed measure and ϕ and ψ are suitable functions, by

$$(1.8) \quad \int_{\Omega} \phi(\lambda) M(d\lambda) \psi(\lambda) = \int_{\Omega} \phi(\lambda) \frac{dM}{d\mu}(\lambda) \psi(\lambda) \mu(d\lambda)$$

where μ is some nonnegative real-valued σ -finite measure on $(\Omega, \mathcal{B})$ such that M is a.c. w.r.t. μ . It can be shown that the definition of the integral does not depend on the choice of μ . When M is a $q \times q$ matrix-valued measure it is customary to choose μ to be $\tau M = \text{trace } M$. In this case, we denote $\frac{dM}{d\tau M} = M'_{\tau}$ by M' .

In the following, we take $\Omega = [0, 2\pi) = T$, $\mathcal{B}$ the σ -algebra of Borel subsets of $T = [0, 2\pi)$ and as usual identify T with the unit circle $\{z \in \mathbb{C}; |z| = 1\}$ in the complex-plane.

It is known that (cf. [18, Theorem 1], [38, page 14]) if X_n and Y_n are q -variate stationary and stationarily cross-correlated sequences, then there exists a unitary operator U on the subspace $\sigma(X_n^i, Y_n^i; \text{ all } n, 1 \leq i \leq q) \subset H$ onto itself such that $UX_n^i = X_{n+1}^i$ and $UY_n^i = Y_{n+1}^i$, $1 \leq i \leq q$. This operator U has a spectral resolution;

$$(1.9) \quad U = \int_{\mathbb{T}} e^{-i\lambda} E(d\lambda)$$

where E is a projection-valued measure over $(\mathbb{T}, \mathcal{B})$. The operator U may be extended to a unitary operator on H onto H in many ways, we denote this extension again by U . The inflation of U denoted by $\bar{U}$ is defined by,

$$(1.10) \quad \bar{U}(X) = (UX^1, \dots, UX^q), \quad X = (x^i)_{i=1}^q \in H^q.$$

By taking the inflation $\bar{E}$ of E analogously, we can define the following qxq matrix-valued signed measures.

1.10 Definition: With each pair of stationary and stationarily cross-correlated sequences X_n and Y_n we associate the qxq matrix-valued cross-measure M_{XY} , not necessarily hermitian-valued, and qxq cross-spectral distribution F_{XY} defined by,

$$(1.11) \quad M_{XY}(A) = (\bar{E}(A)X_0, \bar{E}(A)Y_0), \quad A \in \mathcal{B},$$

$$(1.12) \quad F_{XY}(\lambda) = 2\pi M_{XY}(0, \lambda], \quad \lambda \in \mathbb{T}.$$

It is clear that $M_{YX}(A) = M_{XY}^*(A)$, $A \in \mathcal{B}$, and

$$(1.13) \quad R_{XY}(n) = (X_n, Y_0) = \int_T e^{-in\lambda} M_{XY}(d\lambda) = \frac{1}{2\pi} \int_T e^{-in\lambda} dF_{XY}(\lambda),$$

where these integrals are defined as in (1.8) with $\phi(\lambda) = e^{-in\lambda}I$ and $\psi(\lambda) = I$ and I is the $q \times q$ identity matrix.

In the special case, when $X = Y$, from (1.11) it is obvious that $M(\cdot) = M_{XX}(\cdot)$ is a $q \times q$ matrix-valued measure (cf. Definition 1.7).

1.11 Definition: The $q \times q$ nonnegative hermitian matrix-valued function F defined by,

$$(1.14) \quad F(\lambda) = 2\pi M(0, \lambda], \quad \lambda \in T,$$

is called the spectral distribution of the stationary sequence X .

1.12 Definition: By the spectral representation of the stationary sequence X_n and its covariance $R(n)$ we mean

$$(1.15) \quad X_n = \int_T e^{-in\lambda} \bar{E}(d\lambda) X_0 = \int_T e^{-in\lambda} \xi(d\lambda)$$

$$(1.16) \quad R(n) = (X_n, X_0) = \int_T e^{-in\lambda} M(d\lambda) = \frac{1}{2\pi} \int_T e^{-in\lambda} dF(\lambda),$$

where in (1.15), $\xi(A) = \bar{E}(A)X_0$, $A \in B$, is an H^q -valued countably additive, orthogonally scattered (c.a.o.s.) measure, so-called because of its decisive property, $A, B \in B$ and A, B disjoint implies $\xi(A) \perp \xi(B)$. The last integral in (1.15) is defined as

$$\left(\int_T e^{-in\lambda} \bar{J}_\xi(d\lambda) \right)^q_{j=1}.$$

With the definition of integral for matrix-valued functions as in (1.8), we define the L_2 class of such functions with respect to a matrix-valued measure M associated to a q -variate stationary sequence by,

$$(1.17) \quad L_{2,F} = L_{2,M} = \{ \phi ; \int \phi(\lambda) M(d\lambda) \phi^*(\lambda) \text{ exists} \}.$$

We put the following natural norm on $L_{2,F}$;

$$(1.18) \quad \|\phi\|_F = [\text{trace} \int \phi(\lambda) M(d\lambda) \phi^*(\lambda)]^{1/2}.$$

It is known that $L_{2,F}$ is complete under this norm (cf. [40], [38, page 30]).

(1.22) We can introduce in $L_{2,F}$ a matricial and scalar-valued inner products by,

$$(1.19) \quad (\phi, \psi)_F = (\phi, \psi)_M = \int \phi(\lambda) M(d\lambda) \psi^*(\lambda), \quad \phi, \psi \in L_{2,F},$$

$$(1.20) \quad ((\phi, \psi))_F = ((\phi, \psi))_M = \text{trace} (\phi, \psi)_F.$$

Thus, the norm introduced in (1.18) can then be written as

$$(1.21) \quad \|\phi\|_F = [((\phi, \phi))_F]^{1/2}$$

The following theorem of [40] connects $L_{2,F}$ and $\tilde{H}(X) \subseteq H^q$.
 (1.24) The integral appearing in the theorem is defined in [40].

1.13 Theorem. For a q -variate stationary sequence X_n , the correspondence $\phi \rightarrow \int \phi(\lambda) \bar{E}(d\lambda) X_0$ is an isomorphism on the space $\bar{H}(X) \subseteq H^q$.

The following theorem is an extension of Kolmogorov's Theorems 8, 9 and 10 [18] in a form which is given in [23]. Actually, this theorem gives analytical necessary and sufficient conditions for subordination and mutual subordination in terms of the spectral measures of two q -variate stationary and stationarily cross-correlated sequences.

1.14 Theorem. Suppose X_n and Y_n are stationary and stationarily cross-correlated sequences, then

(i) Y_n is subordinate to X_n if and only if there exists a $\phi \in L_{2,F_{XX}}$ such that

$$(1.22) \quad dF_{YY}(\lambda) = \phi(\lambda) dF_{XX}(\lambda) \phi^*(\lambda),$$

$$(1.23) \quad dF_{YX}(\lambda) = \phi(\lambda) dF_{XX}(\lambda).$$

Proof. Suppose

and since In the sense that for any $A \in \mathcal{B}$,

$$\phi \in H_{XX}^{2,F_{XX}} \text{ such that } F_{YY}(A) = \int_A \phi(\lambda) dF_{XX}(\lambda) \phi^*(\lambda) \text{ and } F_{YX}(A) = \int_A \phi(\lambda) dF_{XX}(\lambda).$$

(1.13). Thus for all $A \in \mathcal{B}$

(ii) Let Y_n be subordinate to X_n and ϕ as in (i). Then X_n and Y_n are mutually subordinate if and only if,

$$(1.24) \quad \text{rank}\left\{\phi(\lambda) \frac{dF_{XX}}{d\tau_{F_{XX}}}(\lambda) \phi^*(\lambda)\right\} = \text{rank}\left\{\frac{dF_{XX}}{d\tau_{F_{XX}}}(\lambda)\right\} \text{ a.e.}(\tau_{F_{XX}}).$$

which implies (1.25) and (1.26). Suppose

Condition for strong subordination of stationary and stationarily cross-correlated sequences is not available in the literature. In the following, by using Theorem 1.14, we give necessary and sufficient conditions for strong subordination of such sequences.

For F the spectral distribution of a q -variate stationary sequence we define $H_{2,F} = \sigma\{e^{-in\lambda}I; n \leq 0\}$ in $L_{2,F}$. In the special case, when $dF(\lambda) = Id\lambda$, $H_{2,F}$ is the usual matricial Hardy class of functions denoted by H_2 .

1.15 Theorem. Suppose X_n and Y_n are stationary and stationarily cross-correlated sequences, then Y_n is strongly subordinated to X_n if and only if there exists a function $\phi \in H_{2,F_{XX}}$ such that

$$(1.25) \quad dF_{YY}(\lambda) = \phi(\lambda)dF_{XX}(\lambda)\phi(\lambda)^*$$

$$(1.26) \quad dF_{YX}(\lambda) = \phi(\lambda)dF_{XX}(\lambda).$$

Proof. Suppose Y_n is strongly subordinate to X_n , then $Y_0 \in \bar{H}(X,0)$ and since $\bar{H}(X,0)$ and $H_{2,F_{XX}}$ are isomorphic, there exists a function $\phi \in H_{2,F_{XX}}$ such that $Y_n = \int_{\mathbb{T}} e^{in\lambda} \phi(\lambda) E(d\lambda) X_0$, for every n (c.f. Theorem 1.13). Thus for all integers m and n we have;

$$\frac{1}{2\pi} \int_{\mathbb{T}} e^{i(m-n)\lambda} dF_{YY}(\lambda) = (Y_m, Y_n) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{i(m-n)\lambda} \phi(\lambda) dF_{XX}(\lambda) \phi(\lambda)^*,$$

$$\frac{1}{2\pi} \int_{\mathbb{T}} e^{i(m-n)\lambda} dF_{YX}(\lambda) = (Y_m, X_n) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{i(m-n)\lambda} \phi(\lambda) dF_{XX}(\lambda).$$

Which implies (1.25) and (1.26) respectively.

Conversely, suppose that there exists a function $\phi \in H_2, F_{XX}$ satisfying (1.25) and (1.26). Then by Theorem 1.14 (i) Y_n is subordinate to X_n . Define $Z_n = \int_T e^{-in\lambda} \phi(\lambda) \bar{E}(d\lambda) X_0$, then it is easy to check that,

$$(1.27) \quad dF_{ZZ}(\lambda) = dF_{YY}(\lambda) \quad \text{and} \quad dF_{XZ}(\lambda) = dF_{XY}(\lambda).$$

Thus by Lemm 8.1 [38, page 35], it follows that;

$Y_n = Z_n = \int_T e^{-in\lambda} \phi(\lambda) \bar{E}(d\lambda) X_0$, for all integers n , which shows that $Y_n \in \bar{R}(X, n)$ for all n thus Y_n is strongly subordinate to X_n by (1.7) and Definition 1.4 (ii). Q.E.D.

(1.28) Using Theorem 1.15, we note that if F_{XX} is a.c. w.r.t. the Lebesgue measure $d\lambda$ and $F'_{XX} = \frac{dF_{XX}}{d\lambda} = \psi\psi^*$ a.e. $(d\lambda)$, where $\psi \in H_2$, then (1.25) implies that $F'_{YY}(\lambda) = \phi\psi\psi^*\phi^* = (\phi\psi)(\phi\psi)^*$ a.e. $(d\lambda)$, i.e. if Y_n is strongly subordinate to X_n and X_n is purely nondeterministic (cf. [23], Theorem 9.7), then knowledge of ϕ and ψ facilitates the task of finding an optimal factor of F'_{YY} . It is known that this type of analytic factorization plays a major role in prediction theory of stationary sequences, (cf. [23], section 13).

(1.30) Next, we define a class of H -valued stochastic sequences, which are a natural generalization of univariate stationary sequences and closely related to q -variate stationary sequences.

1.16 Definition. A stochastic sequence x_n is said to be periodically correlated of period q if the function $\tilde{R}(m, n) = (x_{m+n}, x_m) = R(m+n, n)$ is periodic in m of period q (we note that when $q = 1$ the sequence

is stationary). Since $\tilde{R}(m, n)$ is periodic in m of period q , one can write $\tilde{R}(m, n) = \sum_{k=1}^q R_k(n) \exp(\frac{2\pi i k m}{q})$.

For convenience we extend the definition of these functions $R_k(n)$, $k = 1, 2, \dots, q$, to all integers by $R_k(n) = R_{k+q}(n)$.

It is shown in [10] that each $R_k(n)$ has the representation

$$(1.27) \quad R_k(n) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-in\lambda} dF_k(\lambda),$$

1.17 Lemma. x_n is periodically correlated with period q if and where each $F_k(\cdot)$ is a complex-valued measure on $\mathbb{T}$. Let $F(\cdot)$ be the $q \times q$ matrix-valued measure, given on intervals by

$$(1.28) \quad F(\lambda_1, \lambda_2) = [F_{k-j}(\frac{\lambda_1 + 2\pi j}{q}, \frac{\lambda_2 + 2\pi j}{q})]_{j,k=0}^{q-1}, \quad \lambda_1 \leq \lambda_2.$$

The following theorem gives the relation between this measure

It is proved in [10] that $F(\cdot)$ is a matrix-valued measure.

It is also shown that

$$(1.29) \quad \tilde{R}(m, n) = \frac{1}{4\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} e^{-i(m+n)\lambda_1 + im\lambda_2} dF(\lambda_1, \lambda_2),$$

where the spectral measure $F(\cdot, \cdot)$ is given by

$$(1.30) \quad F(A, B) = \sum_{k=-q+1}^{q-1} \int_{A \cap (B - \frac{2\pi k}{q})} dF_k(\lambda).$$

In other words the spectral measure $F(\cdot, \cdot)$ is concentrated on $2q-1$ straight line segments $\lambda_1 - \lambda_2 = \frac{2\pi k}{q}$, $k = -q+1, \dots, q-1$, contained inside the square $\mathbb{T}^2$, and the measures $F_k(\cdot)$ give the mass of $F(\cdot, \cdot)$ on these lines according to (1.30).

With any H -valued sequence x_n we associate the H^q -valued sequence X_n whose i -th coordinate x_n^i is given by x_{nq+i} , $i = 0, 1, 2, \dots, q-1$. This correspondence establishes a one-to-one linear transformation from the H -valued sequences onto the H^q -valued sequences and we have,

$$X_n = (x_{nq}, x_{nq+1}, \dots, x_{nq+q-1})^T.$$

1.17 Lemma. x_n is periodically correlated with period q if and only if X_n is a q -variate stationary sequence.

By Definition 1.12, this associated q -variate stationary sequence has a spectral measure F which is a qxq nonnegative definite matrix-valued measure such that $R(n) = (X_n, X_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\lambda} dF(\lambda)$.

The following theorem which gives the relation between this measure F and the measure F given in (1.28) can be found in [10] and [27].

1.18 Theorem. With the notations as above, we have;

$$F(A) = \int_A q U^*(\lambda) dF(\lambda) U(\lambda), \quad A \in \mathcal{B}.$$

Where U is a unitary matrix-valued function whose (j,k) -th entry is given by $q^{-1/2} \exp[\frac{2\pi i j k + i k \lambda}{q}]$.

In [12], the notion of periodic correlation of subordination of stationary sequences was used to show that the output of a stationary signal, in the sense of [12], is a stationary process. In [12] it is shown that under subordination conditions the output of a linear system is a stationary process.

In this chapter we give analytic conditions for subordination of periodically correlated and harmonizable sequences in the spirit of

CHAPTER II

Kolmogorov [18, section 4], see also Theorems 1.14 and 1.15.

SUBORDINATION OF HARMONIZABLE SEQUENCES

In section 2, necessary and sufficient conditions for

subordination, mutual subordination and necessary conditions for strong

2.1 Introduction: The concept of subordination was introduced, studied and used in prediction of univariate stationary sequences by associated multi-variate stationary sequences of Lemma 1.17 of A.N. Kolmogorov [18]. Conditions for subordination in terms of the spectral measures of the sequences were derived in [18]. Analogous conditions for the subordination of q -variate stationary sequences were derived by M. Rosenberg [39,41], Yu. A. Rosanov [38] and P. Masani [23] and for infinite-dimensional stationary sequences by V. Mandrekar and H. Salehi [22]. In [41] and [22] the notion of subordination have been used to gain some new insight into some problems in analysis.

T.N. Siraya [48] gives conditions for subordination and strong subordination (cf. Definition 1.4) of second-order (not necessarily stationary) processes in terms of their covariances and corresponding reproducing kernel Hilbert spaces. In [49] conditions for subordination and strong subordination of one second-order process to another such process with orthogonal increments, in terms of the structural measure of the latter has been derived.

In [12], the notion and analytic characterization of subordination of stationary sequences have been used for optimal filtering of stationary signals. In [3] it is shown that under some general conditions the output of a linear system is a harmonizable stochastic process.

Proof. In this chapter we give analytic conditions for subordination of periodically correlated and harmonizable sequences in the spirit of Kolmogorov [18, section 4], see also Theorems 1.14 and 1.15.

In section 2, necessary and sufficient conditions for subordination, mutual subordination and necessary conditions for strong subordination of periodically correlated sequences in terms of their associated multi-variate stationary sequences (cf. Lemma 1.17) is studied. Sufficient conditions for subordination of harmonizable sequences and a counter-example showing that these conditions are not necessary along with the problem of linear transformation of harmonizable sequences is discussed in section 3.

2.2 Subordination of Periodically Correlated Sequences: Throughout this section we assume that x_n and y_n are periodically correlated sequences with period q and that they are periodically cross-correlated i.e. the function $R_{xy}(n, k) = (x_{n+k}, y_n)$ is periodic in n of period q .

2.2.1 Remark. If x_n and y_n are periodically cross-correlated with period q , then $\begin{bmatrix} x_n \\ y_n \end{bmatrix}, n \in \mathbb{Z}$, is a two-dimensional periodically correlated sequence. Thus by [10], $R_{xy}(\cdot, \cdot)$ has an spectral representation similar to the spectral representation of the covariance of x_n (cf. 1.29).

2.2.2 Lemma. If x_n and y_n are periodically cross-correlated with period q and X_n, Y_n are their associated q -variate stationary sequences. Then X_n and Y_n are stationarily cross-correlated.

Proof. It is easy to check that, for all integers m, n ;

$$(X_m, Y_n) = \left[(X_{(m-n)q+i}, Y_j) \right]_{i,j=0}^{q-1}$$

which depends on $m-n$ alone.

Q.E.D.

Proof: For x_n a periodically correlated sequence and x_n its associated q -variate stationary sequence we have for all integers n ,

$$(1) \quad H(X, n) = \sigma\{x_{mq+i}; m \leq n, 0 \leq i \leq q-1\} \\ = \sigma\{x_k; k \leq nq + q - 1\} = H(x; nq + q - 1).$$

Thus, letting $n \rightarrow \infty$, we get the following important equality,

$$(2) \quad H(X) = H(x).$$

In the following theorem we give necessary and sufficient conditions for subordination and mutual subordination of periodically correlated sequences in terms of their associated q -variate stationary sequences.

Necessary and sufficient conditions in terms of matricial spectral measures for subordination and mutual subordination of periodically correlated sequences can be obtained by using Theorems 1.14 and 1.18.

22.3 Theorem. Suppose x_n and y_n are periodically cross-correlated sequences of period q and x_n, y_n their associated q -variate stationary sequences, then

where μ is a countably additive K -valued measure (not necessarily orthogonally scattered) on T and for $A, B \in \mathcal{E}$, $\mu(A, B) = (\mu(A), \mu(B))$ extends to a complex-valued measure of bounded variation on $\mathcal{E}$.

(ii) y_n and x_n are mutually subordinate if and only if y_n and x_n are mutually subordinate.

2.3.2 Remark. Comparison of (1) and (1.29) reveals that the class of

Proof: (i) and (ii) are obvious because of Lemma 2.2.2 and relation

(2). Q.E.D.

2.2.4 Remark: If y_n is strongly subordinate to x_n , then by relation

(1), y_n is also strongly subordinate to x_n . But, the converse is

not necessarily true. For an example, let ε_n be a periodically correlated sequence of period $q = 2$ with $\varepsilon_n \notin \sigma\{\varepsilon_k; k \leq n-1\}$.

Define x_n and y_n by $x_{2n} = \varepsilon_{2n-1}$, $x_{2n+1} = \varepsilon_{2n}$, $y_{2n} = \varepsilon_{2n}$ and $y_{2n+1} = \varepsilon_{2n-1}$, then it is clear that y_n is strongly subordinate to x_n , but $H(y, 2n) \supsetneq H(x, 2n)$ i.e. y_n is not strongly subordinate to x_n .

2.3 Subordination of Harmonizable Sequences: In this section we study the problem of subordination of harmonizable sequences and its relation with linear transformation of such sequences. First we develop a few concepts which are essential in this study.

2.3.1 Definition: A stochastic sequence x_n is said to be harmonizable if $x_n = \int_T e^{-in\lambda} n(d\lambda)$ and

$$(1) \quad R(m, n) = \frac{1}{4\pi^2} \int_T \int_T e^{-im\lambda_1 + in\lambda_2} 2_{\mu}(d\lambda_1, d\lambda_2),$$

Let $A^2(du)$ be the completion of $A^2(du)$ in the Hilbert space with an inner product

where η is a countably additive H -valued measure (not necessarily orthogonally scattered) on T and for $A, B \in \mathcal{B}$, $\mu(A, B) = (\eta(A), \eta(B))$ extends to a complex-valued measure of bounded variation on T^2 . μ is called the spectral measure of the sequence.

2.3.2 Remark. Comparison of (1) and (1.29) reveals that the class of periodically correlated sequences is a special subclass of harmonizable sequences.

2.3.3 The Hilbert Space $L^2(d\mu)$. For φ, ψ measurable functions on T , $\varphi \otimes \psi$ will denote the tensor product of φ and ψ i.e.

$$(\varphi \otimes \psi)(\lambda_1, \lambda_2) = \varphi(\lambda_1)\psi(\lambda_2), \text{ for } \lambda_1, \lambda_2 \in T.$$

Let S be the class of all step functions on T , it is clear that S is a linear space and for all $\varphi, \psi \in S$, the double integrals

$$\iint_{T^2} \varphi \otimes \bar{\psi} d\mu = \iint_{T^2} \varphi(\lambda_1)\bar{\psi}(\lambda_2)\mu(d\lambda_1, d\lambda_2) \text{ is defined in the obvious way. } (\mu \text{ is a measure satisfying (1)}).$$

Two step functions φ and ψ will be considered identical if,

$$\iint_{T^2} (\varphi - \psi) \otimes (\overline{\varphi - \psi}) d\mu = 0.$$

If we define for $\varphi, \psi \in S$, $\langle \varphi, \psi \rangle = \iint_{T^2} \varphi \otimes \bar{\psi} d\mu$, then $(S, \langle \cdot, \cdot \rangle)$ is an inner product space. In fact, it is obvious that $\langle \varphi, \psi \rangle$ has the ordinary bilinear and conjugate symmetric properties and further $\langle \varphi, \varphi \rangle \geq 0$ (this follows from property of μ), and $\langle \varphi, \varphi \rangle = 0$ only when $\iint_{T^2} \varphi \otimes \bar{\varphi} d\mu = 0$ i.e. when φ is identical with 0. Also, it follows from $\langle \varphi, \varphi \rangle \geq 0$, that we have the Cauchy-Schwartz inequality i.e.

$$|\langle \varphi, \psi \rangle|^2 \leq \langle \varphi, \varphi \rangle \langle \psi, \psi \rangle.$$

Let $L^2(d\mu)$ be the completion of $(S, \langle \cdot, \cdot \rangle)$ so that it is a Hilbert space with an inner product denoted again by $\langle \cdot, \cdot \rangle$.

Elements in $\Lambda^2(d\mu)$ may no longer be functions on T . A typical element in $\Lambda^2(d\mu)$ can be realized as a Cauchy sequence of step functions. However, we treat elements in $\Lambda^2(d\mu)$ as "formal" functions on T and use the improper but suggestive notation $\iint_T \varphi \otimes \bar{\psi} d\mu$ for the inner product $\langle \varphi, \psi \rangle$ with $\varphi, \psi \in \Lambda^2(d\mu)$.

Of course, $\iint_T \varphi \otimes \bar{\psi} d\mu = \lim_n \iint_T \varphi_n \otimes \bar{\psi}_n d\mu$, where φ_n and ψ_n are Cauchy sequences of step functions from S converging to φ and ψ , respectively, in the norm of $\Lambda^2(d\mu)$.

Let $\Lambda(d\mu) = \{\text{all measurable functions } \varphi \text{ on } T; \iint_T |\varphi \otimes \bar{\varphi}| d\mu < \infty \text{ and } \iint_T |\varphi| d|\mu| < \infty\}$, where $|\mu|$ denotes the total variation measure of μ and the double integrals are in the sense of Lebesgue.

We say that the function φ in $\Lambda(d\mu)$ represents an element in $\Lambda^2(d\mu)$ if there exists a $\varphi' \in \Lambda^2(d\mu)$ such that for all $\psi \in S$,

$$\langle \varphi', \psi \rangle = \iint_T \varphi(\lambda_1) \overline{\psi(\lambda_2)} \mu(d\lambda_1, d\lambda_2).$$

We note that if such φ' exists, it is unique, since S is dense in $\Lambda^2(d\mu)$. Then, we denote φ' by φ and write $\varphi \in \Lambda^2(d\mu)$. With this convention and Theorem 1.1 of [4], $\Lambda(d\mu)$ is a dense subset of $\Lambda^2(d\mu)$ and if $\varphi_1, \varphi_2 \in \Lambda(d\mu)$ with $\iint_T |\varphi_1(\lambda_1) \varphi_2(\lambda_2)| |\mu|(d\lambda_1, d\lambda_2) < \infty$, then

$$\langle \varphi_1, \varphi_2 \rangle = \iint_T \varphi_1(\lambda_1) \overline{\varphi_2(\lambda_2)} \mu(d\lambda_1, d\lambda_2),$$

where the double integral is in the sense of Lebesgue. For n , as in the Definition 2.4.1, we define $H(n) = \sigma\{n(A); A \in \mathcal{B}\}$ in H .

$$\iint_T \varphi \otimes \bar{\varphi}_n d\mu = f(n), \text{ proves the claim.}$$

It is shown in [5] that $\{e_n(\lambda) = e^{-in\lambda}; n \in \mathbb{Z}\}$ forms a basis in $\Lambda^2(d\mu)$, $H(n) = H(x)$ and further that there exists an isomorphism between $\Lambda^2(d\mu)$ and $H(n)$ defined by $\varphi \rightarrow \int \varphi(\lambda) n(d\lambda)$, for $\varphi \in \Lambda^2(d\mu)$.

2.3.4 The RKHS For R. In the study of problems related to harmonizable sequences, it is useful to have an explicit representation of the elements of the corresponding reproducing kernel Hilbert space (RKHS). Siraya [49] gives such a representation with no proof, in the case when μ is absolutely continuous with respect to Lebesgue measure on T^2 . In the following we give an explicit representation of the elements of the RKHS corresponding to a harmonizable sequence with covariance R and spectral measure μ (cf. Definition 2.3.1).

2.3.5 Lemma. The RKHS corresponding to the covariance R is given by $H(R) = \{f \in L^2; f(n) = \iint \varphi \otimes \bar{e}_n d\mu, \varphi \in \Lambda^2(d\mu)\}$, with inner product $(f, g)_R = \iint_2 \varphi \otimes \bar{\psi} d\mu$, for $g \in H(R)$ with $g(n) = \iint_2 \psi \otimes \bar{e}_n d\mu$.

Proof. First we show that $H(R)$ is complete. Let $f_k \in H(R)$ be a Cauchy sequence, then there exists a sequence φ_k in $\Lambda^2(d\mu)$ such that $f_k(n) = \iint_2 \varphi_k \otimes \bar{e}_n d\mu$. Since $\{f_k\}$ is Cauchy in $H(R)$ and $\|\varphi_k - \varphi_l\|_{\Lambda^2(d\mu)} = \|f_k - f_l\|_R$, we conclude that $\{\varphi_k\}$ is Cauchy in $\Lambda^2(d\mu)$. But $\Lambda^2(d\mu)$ is complete, so there exists $\varphi \in \Lambda^2(d\mu)$ such that $\|\varphi_k - \varphi\|_{\Lambda^2(d\mu)} \rightarrow 0$ as $k \rightarrow \infty$. Now, let $f(n) = \iint_2 \varphi \otimes \bar{e}_n d\mu$, then $f \in H(R)$ and $\|f_k - f\|_R = \|\varphi_k - \varphi\|_{\Lambda^2(d\mu)} \rightarrow 0$ as $k \rightarrow \infty$ i.e. $f_k \rightarrow f$ in $H(R)$, so $H(R)$ is complete.

The fact that $R(m, \cdot) = R(m, n) = \iint e_m \otimes \bar{e}_n d\mu \in H(R)$ and for $f \in H(R)$ with $f(m) = \iint \varphi \otimes \bar{e}_m d\mu, \varphi \in \Lambda^2(d\mu)$ we have $(f, R(m, \cdot))_R = \iint_2 \varphi \otimes \bar{e}_m d\mu = f(m)$, proves the Lemma.

2.3.9 The Careful scrutiny of Kolmogorov and Siraya's work and results in cross section 2.2 reveal that, in problems of subordination, a major role is played by cross-correlation of the sequences under study. In the following we assume that x_n and y_n are harmonizable sequences with covariances,

(2)

$$R(m, n) = (x_m, x_n) = \frac{1}{4\pi^2} \iint_{T^2} e_m \otimes \bar{e}_n dF_{xx},$$

2.3.10 Remark. By (2) we mean, for any $x, y \in \mathcal{H}$,

$$B(m, n) = (y_m, y_n) = \frac{1}{4\pi^2} \iint_{T^2} e_m \otimes \bar{e}_n dF_{yy}.$$

2.3.6 Definition. We say that x_n and y_n are harmonizably cross-correlated if there exists a complex measure $F_{xy}(\cdot, \cdot)$ on T^2 such that,

Proof. Suppose there $C(m, n) = (x_m, y_n) = \frac{1}{4\pi^2} \iint_{T^2} e_m \otimes \bar{e}_n dF_{xy}$.

2.3.7 Definition. We say that the harmonizable sequence y_n is obtained from x_n by means of a linear transformation, if there exists a function $\varphi \in \Lambda(dF_{xx})$ such that

$$y_n = \int_T \bar{e}^{in\lambda} \varphi(\lambda) n(d\lambda), \text{ for all integers } n.$$

2.3.8 Remark. From definitions 1.4(i) and 2.3.7, it is easy to see that when y_n is obtained from x_n by means of a linear transformation then y_n is subordinate to x_n .

By using this remark and the following Theorem which is the analog of Theorem 8.1 for stationary sequences [38, page 36], we obtain sufficient conditions for subordination of harmonizable sequences which are harmonizably cross-correlated.

2.3.9 Theorem. Suppose x_n and y_n are harmonizable and harmonizably cross-correlated, then y_n is obtainable from x_n by means of a linear transformation, if and only if there exists a function $\varphi \in \Lambda(dF_{xx})$ such that,

$$(2) \quad \begin{cases} dF_{yy} = \varphi \otimes \bar{\varphi} dF_{xx} , \\ dF_{xy} = \bar{\varphi} dF_{xx} . \end{cases}$$

2.3.10 Remark. By (2) we mean, for any $A, B \in \mathcal{B}$,

$$F_{yy}(A, B) = \iint_A \varphi(\lambda_1) \overline{\varphi(\lambda_2)} dF_{xx}(\lambda_1, \lambda_2),$$

$$F_{xy}(A, B) = \iint_A \overline{\varphi(\lambda_2)} dF_{xx}(\lambda_1, \lambda_2).$$

Proof. Suppose there exists $\varphi \in \Lambda(dF_{xx})$ such that,

$$y_n = \int_T \bar{e}^{in\lambda} \varphi(\lambda) \eta(d\lambda) , \quad n \in \mathbb{Z}.$$

Since y_n is harmonizable, it has its own spectral representation, i.e. there exists an H -valued measure (cf. Definition 2.3.1) $\xi(\cdot)$ such that

$$y_n = \int_T \bar{e}^{in\lambda} \xi(d\lambda).$$

Thus, for all integers m and n we have;

$$\int_T e_m \otimes \bar{e}_n dF_{xy} = (x_m, y_n) = \int_T e_m \otimes \bar{e}_n \bar{\varphi} dF_{xx}$$

$$\int_T e_m \otimes \bar{e}_n dF_{yy} = (y_m, y_n) = \int_T e_m \otimes \bar{e}_n \varphi \otimes \bar{\varphi} dF_{xx} ,$$

which implies (2).

Conversely, suppose that there exists a $\varphi \in \Lambda(dF_{xx})$ such that (2) holds. We define $Z_n = \int_T e^{-in\lambda} \varphi(\lambda) \eta(d\lambda)$, then it is easy to check that,

$$(x_m, y_n) = (x_m, Z_n) \quad \frac{1}{2\pi} \int_0^{2\pi} e^{-in\lambda} f(\lambda) d\lambda.$$

It is easy to check that x_n and y_n are harmonizable and harmonizably cross-correlated.

Thus, by a slight extension of Lemma 8.1 [38, page 35], we get

$y_n = \int_T e^{-in\lambda} \varphi(\lambda) \eta(d\lambda)$, i.e. y_n is obtainable from x_n by means of a linear transformation. Q.E.D.

2.3.11 Theorem. Suppose x_n and y_n are harmonizable and harmonizably cross-correlated. If there exists a function $\varphi \in \Lambda(dF_{xx})$ such that

$$dF_{yy} = \varphi \otimes \bar{\varphi} dF_{xx}$$

$$(3) \quad dF_{xy} = \bar{\varphi} dF_{xx},$$

then y_n is subordinate to x_n .

Proof of this theorem is an easy consequence of Theorem 2.3.9 and Remark

2.3.8.3.13 Remark. Theorem 2.3.11 is a special case of Theorem 2.3.12.

2.3.12 A counter example. Here we give an example which shows that, unlike the stationary and periodically correlated sequences, the conditions of Theorem 2.3.11 are not necessary for subordination of harmonizable sequences.

Let ξ be a random variable on some probability space with $E \xi = 0$ and $E |\xi|^2 = 1$. Let $f, g \in L^1(T, d\lambda)$ where f is not identically zero. Define the following stochastic sequences $x_n = \hat{f}(n)\xi$ and $y_n = \hat{g}(n)\xi$, where $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\lambda} f(\lambda) d\lambda$.

It is easy to check that x_n and y_n are harmonizable and harmonizably cross-correlated with $dF_{xx} = f \otimes \bar{f} dm$, $dF_{yy} = g \otimes \bar{g} dm$ and $dF_{xy} = f \otimes \bar{g} dm$, where m is Lebesgue measure on T^2 .

For any choice of such functions f and g we have $H(y) \subseteq H(x)$ i.e. y_n is subordinate to x_n . But, in the following, we show that it is possible to choose f and g in such a way that none of the relations in (2) (or (2')) can hold.

Suppose, there exists a $\varphi \in \Lambda(dF_{xx})$ such that conditions in Theorem 2.3.11 are satisfied, then, for $A = B$ we have;

$$(3) \quad \left| \int_A g(\lambda) d\lambda \right|^2 = \left| \int_A \varphi(\lambda) f(\lambda) d\lambda \right|^2, \quad A \in B.$$

For $A = [0, \pi]$ choose $g \in L^1(T, d\lambda)$ such that $\int_A g(\lambda) d\lambda \neq 0$. Then with $f = \chi_{[\pi, 2\pi]}$ we have $\int_A \varphi(\lambda) f(\lambda) d\lambda = 0$, which contradicts (3).

2.3.13 Remark. Theorem 2.3.11 can also be proved by using Theorem 1 of [48] and Lemma 2.3.5.

2.3.14 Remark. In Definition 2.3.1, if μ is a measure which is concentrated on the main diagonal of the square T^2 , then the corresponding process x_n is stationary. In this case, we can think of μ as a nonnegative measure on T , then it is easy to see that $\Lambda(d\mu)$ (as defined in 2.3.3) is the same as the space of measurable functions on T which are square integrable with respect to μ i.e.

$\Lambda(du) = L^2(du)$. Thus Theorems 2.3.9 and 2.3.11 specialized to the case when x_n and y_n are stationary and stationarily cross-correlated will reduce to Theorem 8.1 of [38, page 36] and sufficient part of Theorem 9 of [18], respectively.

2.3.15 Remark. We note that for stationary sequences, the property that y_n is obtainable from x_n is equivalent to the subordination of y_n to x_n [38, Theorem 8.1] and Theorem 1.14. But, this is not the case for harmonizable sequences, as counter example 2.3.12 shows.

2.3.16 An Open Problem. It is conjectured that the assertions of Theorems 2.3.9 and 2.3.11 are true even when $\varphi \in \Lambda^2(dF_{xx})$ instead of belonging to $\Lambda(dF_{xx})$. Although this can be established formally, we have been unable to prove it rigorously. It seems that a rigorous proof of these assertions in this new setting hinges on giving a proper meaning to the relation (2) in Theorem 2.3.9.

This sampling theorem, which is also almost surely, enables one to reconstruct from its sample values.

Of course, a sampling theorem for free reconstruction from its samples, and sufficient conditions for admitting such a reconstruction.

More precisely, a sequence $\{x_n\}$ can in principle be exactly reconstructed if $H(x) = H_g(x)$, where $H_g(x) =$

3.1.1 Definition. For a fixed λ , the sequence $\{x_n\}$ admits a sampling theorem if

Lloyd uses the terminology "samples" when $H(x) = H_g(x)$. We also use the term "samples" when $H(x) = H_g(x)$.

CHAPTER III

SAMPLING THEOREM FOR q-VARIATE

STATIONARY AND UNIVARIATE HARMONIZABLE

PROCESSES

3.1 Introduction. It is well-known that a stationary stochastic process $x(t) \in H$, $t \in R$, has the sampling series

$$x(t) = \sum_{n=-\infty}^{\infty} x(nh) \frac{\sin \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)}$$

if the spectral measure μ of $x(t)$ is supported by the interval

$(-\frac{h^{-1}}{2}, \frac{h^{-1}}{2})$. This so called "sampling theorem" dates back to Cauchy and is of considerable importance in communication and information theory [11] and [29]. Such processes with bounded spectra are called "band limited".

This sampling series, which converges in mean-square and also almost surely, enables a band-limited process to be exactly reconstructed from its sample $\{x(nh); n \in Z\}$.

Of course, a process need not be band-limited to admit an error-free reconstruction from its samples. S.P. Lloyd [21] gave a necessary and sufficient condition on the spectral measure for a stationary process to admit such a reconstruction.

More precisely, a process $x(t) \in H$ (not necessarily stationary) can in principle be exactly reconstructed from its samples $\{x(nh); n \in Z\}$ if $H(x) = H_S(x)$, where $H_S(x) = \sigma\{x(nh); n \in Z\}$ in H .

3.1.1 Definition. For a fixed $h > 0$, we say that the process $x(t)$, $t \in R$, admits a sampling theorem if $H(x) = H_S(x)$.

Lloyd uses the terminology that " x is linearly determined by its samples" when $H(x) = H_S(x)$. We will refer to the fixed positive h as

"sample spacing" and to the set $\{t_n\} = \{nh; n \in \mathbb{Z}\}$ as "sample times".

In [21], Lloyd proved the following remarkable results for any stationary stochastic process.

3.1.2 Theorem. Let $x(t)$, $t \in \mathbb{R}$, be a stationary process with spectral measure μ , then

(i) $x(t)$ admits a sampling theorem if and only if μ has a support Λ such that the translates of Λ by nh^{-1} i.e. $\{\Lambda + nh^{-1}; n \in \mathbb{Z}\}$ are mutually disjoint.

(ii) If the measure μ has an open support Λ whose translates $\{\Lambda + nh^{-1}; n \in \mathbb{Z}\}$ are mutually disjoint, then we have

that if the components of a q -variate process $x(t) = \text{l.i.m.} \sum_{n=-N}^N (1 - \frac{|n|}{N}) x(nh)K(t-nh)$, $t \in \mathbb{R}$, sampling theorem with $N \rightarrow \infty$ holds, then the process is

where $K(t) = h \int_{\Lambda} \frac{2\pi i \lambda t}{e} d\lambda$, $t \in \mathbb{R}$, and l.i.m. stands for limit in mean square.

(iii) If the Λ from (ii) is a finite union of intervals, or, more generally if $\sup_t |tK(t)| < \infty$, then

section 5, by using

$$x(t) = \text{l.i.m.} \sum_{n=-N}^N x(nh)K(t-nh), \quad t \in \mathbb{R}.$$

sampling theorem

harmonizable stochastic processes. For more information on sampling theorems and its applications in different fields, as well as a complete bibliography of this subject,

[17] may be consulted. The extension of sampling theorem for multi-parameter stationary processes have been studied by Parzen [23], Miyakawa [28] and others [17].

For q -variate stationary processes no sampling theorem is available in the literature. Due to the importance of such processes in application, it is important to have theorems similar to 3.1.2 for q -variate stationary

processes. Also, it is important to know whether anything is gained by studying sampling theorem and sampling series for q -variate stationary processes. It is useful to know whether there is any connection between admittance of a sampling theorem for a q -variate process and its components. If it is so, then it is desirable to know something about the rate of convergence of the q -variate sampling series and its relation with the rate of convergence of its individual component's sampling series.

In sections 2 and 3, using the ideas of Lloyd, we develop the necessary machinery which is needed to prove a sampling theorem for a q -variate stationary stochastic process. Also, we show that if the components of a q -variate process (not necessarily stationary) admits a sampling theorem with the same sample spacing $h > 0$, then the process itself admits a sampling theorem with the same h . In section 4 we prove a sampling theorem for q -variate stationary process and use this sampling theorem to show that if a q -variate stationary process admits a sampling theorem then each of its components will admit a sampling theorem. In section 5, by using Theorem 3.1.2 and Abreu's Theorem [1] we obtain a sampling theorem and a sampling series with explicit coefficients for harmonizable stochastic processes.

3.2 Preliminaries. In the study of sampling theorem for q -variate stationary stochastic processes the notion of absolute continuity of a matrix-valued signed measure (defined in Chapter I) with respect to another such measure plays an important role. The problem of defining a "proper" notion of absolute continuity for such measures was first posed by P. Masani [23]. Later J. B. Robertson and M. Rosenberg [37] dealt with this question and obtained a satisfactory solution to it.

Here, we will briefly review some of their results and some other concepts which are needed for the proof of our sampling theorem.

Throughout this Chapter $\Omega = \mathbb{R}$ and $\mathcal{B}$ is the σ -algebra of Borel subsets of $\mathbb{R}$. The customary definition of absolute continuity for matrix-valued signed measures does not guarantee the existence of a Radon-Nikodym derivative.

3.2.1 Definition. Let M_1 and M_2 be $q \times q$ matrix-valued signed measures on $(\Omega, \mathcal{B})$ respectively, let μ be any σ -finite nonnegative real-valued measure on $(\Omega, \mathcal{B})$ such that M_1 and M_2 are a.c. w.r.t. μ . We say that M_2 is strongly absolutely continuous (s.a.c) w.r.t. M_1 if,

$$N(M'_{1,\mu}(\lambda)) \subset N(M'_{2,\mu}(\lambda)) \quad \text{a.e. } (\mu),$$

where for each matrix M , $N(M) = \{X; MX = 0\}$.

It can be shown that this definition is independent of μ . Hence, we suppress the dependence of $M'_{1,\mu}$ and $M'_{2,\mu}$ on μ i.e. we only write M'_i for $M'_{i,\mu}$, $i = 1, 2$.

The following theorem is proved in [37].

3.2.2 Theorem (Robertson-Rosenberg). Let M_1 and M_2 be $q \times q$ matrix-valued signed measures on $(\Omega, \mathcal{B})$ then,

(i) M_2 is s.a.c. w.r.t. M_1 if and only if there exists a measurable $q \times q$ matrix-valued function ϕ on Ω such that for all $A \in \mathcal{B}$

$$M_2(A) = \int_A \phi dM_1.$$

(ii) Let ϕ and ψ be measurable $q \times q$ matrix-valued functions on Ω . Then for each $A \in \mathcal{B}$, $\int_A \phi dM_1 = \int_A \psi dM_1$ if and only if $\phi J = \psi J$ a.e. (μ) .

where J is the orthogonal projection matrix-valued function onto the range of M_1' and μ is any σ -finite nonnegative real-valued measure on $(\Omega, \mathcal{B})$ w.r.t. which M_1 is a.c..

Thus, if M_2 is s.a.c. w.r.t. M_1 , then by Theorem 3.2.2 (i) there exists a measurable matrix-valued function ϕ such that for each $A \in \mathcal{B}$, $M_2(A) = \int_A \phi \, dM_1$. ϕ is called the Radon-Nikodym derivative of M_2 w.r.t. M_1 and will be denoted by $\frac{dM_2}{dM_1}$. To make this notation more clear and for later use we need to introduce the concept of generalized inverse of matrices due to R. Penrose [34].

3.2.3 Theorem (Penrose). Let A be any $q \times q$ matrix, then there exists a unique $q \times q$ matrix X such that,

$$A = AXA, X = XAX, (AX)^* = AX \text{ and } (XA)^* = XA.$$

3.2.4 Definicion. The matrix X in Theorem 3.2.3 is called the generalized inverse of A , and will be denoted by A^- .

It can be shown that the generalized inverse of a matrix A has the following important properties:

$$AA^- = P_{R(A)} = P_{N(A^*)^\perp},$$

$$A^-A = P_{R(A^*)} = P_{N(A)^\perp}.$$

Where $R(A)$ stands for the range of the matrix A considered as an operator from $\mathcal{C}^q$ to $\mathcal{C}^q$ and P denotes orthogonal projection.

From Theorem 3.2.3 and Definicion 3.2.4, if M_2 is s.a.c. w.r.t. M_1 we define the Radon-Nikodym derivative $\frac{dM_2}{dM_1} = dM_2 \cdot dM_1^-$ by

$\frac{dM_2}{dM_1}(\lambda) = M_2'(\lambda) \cdot M_1'^{-1}(\lambda)$ a.e. (μ), where μ is any nonnegative measure such that M_1 is a.c. w.r.t. μ .

Proof: Next, we introduce some basic notions about continuous time q -variate stationary processes. We note that all definitions and results of Chapter I are still valid for continuous time stationary processes, if n is replaced by t and the region of integration by $R = (-\infty, \infty)$, [38, Chapter I].

To be consistent with the literature on sampling theorem, throughout this chapter, we replace $e^{i\lambda t}$ by $e^{2\pi i\lambda t}$ contrary to our standard notation of earlier chapters.

theorem. Let $X(t)$, $t \in R$, be a q -variate mean continuous stationary stochastic process with the spectral distribution, $q \times q$ matrix-valued function, F defined on Ω . Then, $X(t)$ has the spectral representation $X(t) = \int_{-\infty}^{\infty} e^{-2\pi i\lambda t} \bar{E}(d\lambda) X(0)$, (c.f. Definition 1.12). By Theorem 1.13, under the map $X(t) \rightarrow e^{-2\pi i\lambda t} I$, $t \in R$, where I is the $q \times q$ identity matrix, $\bar{H}(X)$ is isometric to $L_{2,F}$.

period For fixed $h > 0$, by the samples of the process $X(t)$, we mean the collection $\{X(nh); n \in Z\}$ of random vectors. The samples $\{X(nh); n \in Z\}$ span a closed subspace of $\bar{H}(X)$. We denote this subspace by $\bar{H}_S(X)$. The random vectors in $\bar{H}_S(X)$ are those determined linearly by the samples with matrix coefficients.

3.2.5 Definition. We say that the q -variate stochastic process $X(t)$ admits a sampling theorem if $\bar{H}(X) = \bar{H}_S(X)$.

support Now, we prove the following important but simple theorem.

3.2.6 Theorem. If the components of $X(t)$ i.e. $x^i(t)$, $1 \leq i \leq q$ admits

a sampling theorem with the same h , then $X(t)$ admits a sampling theorem with the same h .

Proof: From Definitions 3.2.5 and 1.3 it follows that $X(t)$ admits a sampling theorem if and only if $H(X) = H_S(X)$, where $H_S(X) = \sigma\{x^i(nh); 1 \leq i \leq q, n \in \mathbb{Z}\}$ in H . From this observation and the fact $H(x^i) = H_S(x^i)$, $1 \leq i \leq q$, it follows that $H(X) = \sigma\{H(x^i); 1 \leq i \leq q\} = \sigma\{H_S(x^i), 1 \leq i \leq q\} = H_S(X)$ i.e. $X(t)$ admits a sampling theorem. Q.E.D.

We note that this theorem holds for any second-order q -variate process. The converse of this theorem is not that easy. In the case of q -variate stationary processes we get that as a corollary of our main theorem.

We denote by $L_{2,F,S}$ the image in $L_{2,F}$ of $\bar{H}_S(X)$ under the isomorphism. According to this isomorphism to $X(nh) \in \bar{H}_S(X)$ corresponds $\bar{e}^{2\pi i n h \lambda} I \in L_{2,F,S}$, $n \in \mathbb{Z}$. Since for each $n \in \mathbb{Z}$, $\bar{e}^{2\pi i n h \lambda} I$ is periodic with period h^{-1} in λ , it is tempting to characterize $L_{2,F,S}$ as equivalent classes of all matrix-valued functions in $L_{2,F}$ which are periodic with period h^{-1} . But, this is not true in general.

Next, we put enough conditions on F which guarantees that $L_{2,F,S}$ is the equivalent classes of matrix-valued functions which are periodic with period h^{-1} .

3.2.7 Assumption. Throughout this chapter we assume that the spectral distribution F is such that $R(F'(\lambda))$ is periodic in λ a.e. (τ) with period h^{-1} (i.e. $R(F'(\lambda)) = R(F'(\lambda + nh^{-1}))$ if $\lambda, \lambda + nh^{-1} \in$ support of τ) where $\tau = \text{trace } F$ and $F' = \frac{dF}{d\tau}$ a.e. (τ) .

It is obvious that when F' is of full-rank or F' has constant range or F' has periodic entries, on the support of τ , then Assumption 3.2.7 is satisfied.

Now, we show that under Assumption 3.2.7, $L_{2,F,s}$ can be identified as equivalent classes of functions in $L_{2,F}$ which are periodic with period h^{-1} .

3.2.8 Lemma. Under Assumption 3.2.7, $L_{2,F,s}$ consists of equivalent classes of matrix-valued functions in $L_{2,F}$ which are periodic with period h^{-1} .

Proof. First we note that $L_{2,F,s} = \sigma\{e^{-2\pi i n h \lambda} I; n \in \mathbb{Z}\}$ in $L_{2,F}$. Thus, for $\phi \in L_{2,F,s}$ there exists a sequence ϕ_n of matrix-valued functions which are periodic with period h^{-1} such that $\phi_n \rightarrow \phi$ in $L_{2,F}$ or what is the same $\phi_n \sqrt{F'} \rightarrow \phi \sqrt{F'}$ in $L_{2,\tau I}$. This implies that there exists a subsequence ϕ_{n_i} such that

$$\phi_{n_i} \sqrt{F'} \rightarrow \phi \sqrt{F'} \quad \text{a.e. } (\tau).$$

Thus, $\phi_{n_i} F' \rightarrow \phi F' \quad \text{a.e. } (\tau)$,

which implies that

$$\phi_{n_i} F' x \rightarrow \phi F' x \quad \text{a.e. } (\tau), \quad x \in \mathbb{C}^q.$$

therefore,

$$(1) \quad \phi_{n_i} \rightarrow \phi \quad \text{a.e. } (\tau) \text{ on } R(F').$$

Now, we show that ϕ , as a function in $L_{2,F}$, is periodic with period h^{-1} .

From (1), we have for almost all λ ,

$$(2) \quad \phi_{n_i}(\lambda) \rightarrow \phi(\lambda) \quad \text{on } R(F'(\lambda))$$

$$(3) \quad \phi_{n_i}(\lambda) = \phi_{n_i}(\lambda + nh^{-1}) \rightarrow \phi(\lambda + nh^{-1}) \text{ on } R(F'(\lambda + nh^{-1})) = R(F'(\lambda))$$

(by Assumption 3.2.7).

Thus, (2) and (3) implies that for almost all λ

$$\phi(\lambda + nh^{-1}) = \phi(\lambda) \quad \text{on } R(F'(\lambda)).$$

Thus, $L_{2,F,S}$ is contained in the collection of all equivalent classes of matrix-valued functions in $L_{2,F}$ which are periodic with period h^{-1} .

Next, suppose that $0 \neq \phi \in L_{2,F}$ is periodic with period h^{-1} such that,

$$(4) \quad \int_{-\infty}^{\infty} e^{2\pi i n h \lambda} dF(\lambda) \phi(\lambda) = 0, \text{ for all integers } n.$$

By periodicity of ϕ , (4) is the same as

$$\int_0^{h^{-1}} e^{2\pi i n h \lambda} \left(\sum_{m=-\infty}^{\infty} dF(\lambda + mh^{-1}) \right) \phi(\lambda) = 0, \text{ for all } n, \text{ which implies}$$

that $\phi \equiv 0$ in $L_{2,F}$. This contradiction proves that $L_{2,F,S}$ contains all equivalent classes of matrix-valued functions in $L_{2,F}$ which are periodic with period h^{-1} . Q.E.D.

3.3 Projection on $L_{2,F,S}$. For the proof of our main result, Theorem

3.4.1, we need to have an explicit form for the operator P projecting $L_{2,F}$ onto $L_{2,F,S}$ (Lemma 3.3.5). In this section we find such a form for P along the line of Lloyd's Lemma [21].

Let $\mathcal{B}_b$ denote the family of all bounded sets in $\mathcal{B}$. For $A \in \mathcal{B}_b$ and given $\phi \in L_{2,F}$ we define the following countably additive and σ -finite set functions on $\mathcal{B}_b$:

$$M(A) = \sum_{n=-\infty}^{\infty} \int_A dF_n(\lambda) \quad \text{and} \quad M_\phi(A) = \sum_{n=-\infty}^{\infty} \int_A \phi_n(\lambda) dF_n(\lambda),$$

where $F_n(\lambda) = F(\lambda + nh^{-1})$ and $\phi_n(\lambda) = \phi(\lambda + nh^{-1})$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{R}$ are translates of F and ϕ . We note that M_ϕ is equal to M , when $\phi = I$, the $q \times q$ constant matrix. These functions are determined by their values for sets in $\mathcal{B}_b$. Let $A \in \mathcal{B}_b$ have diameter less than h^{-1} , so that its translates $\{A_n = A - nh^{-1}; n \in \mathbb{Z}\}$ are mutually disjoint.

Then, $M_\phi(A) = \int_{\bigcup_n A_n} \phi dF$, and the countable-additivity and σ -finiteness of

M_ϕ follows from this and the fact that each set in $\mathcal{B}_b$ can be written as finite union of Borel sets with diameter less than h^{-1} . Due to this latter fact, without loss of generality, we assume throughout this chapter that $A \in \mathcal{B}$ has diameter less than h^{-1} .

3.3.1 Remark. Here, we note that although M and M_ϕ are not (necessarily) defined on the σ -algebra $\mathcal{B}$, nevertheless, the assertions of Theorem 3.2.2, concerning s.a.c. and Radon-Nikodym derivative and its uniqueness, are still valid when M_1, M_2 and $\mathcal{B}$ are replaced by M, M_ϕ and $\mathcal{B}_b$, respectively. This can be proved by applying Theorem 3.2.2 to each bounded Borel set and the σ -algebra of its Borel subsets.

3.3.2 Lemma. M_ϕ is s.a.c. w.r.t. M .

Proof. We must find a σ -finite nonnegative measure μ such that

$M_\phi \ll \mu$ and $M \ll \mu$ and then show that:

$$N(M'(\lambda)) \subseteq N(M'_\phi(\lambda)) \quad \text{a.e. } (\mu).$$

Let $\mu = \sum_n \frac{1}{2^{|n|}} \tau_n$, where $\tau_n = \tau F_n$ ($\tau F_n = \text{trace } F_n$), then it

is clear that $F_n \ll \tau_n \ll \mu$. Thus, we can define

$F'_n = \frac{dF_n}{d\mu}$ a.e. (μ) , and we have:

$$M(A) = \int_A (\sum_n F'_n) d\mu, \quad M_\phi(A) = \int_A (\sum_n \phi_n F'_n) d\mu,$$

which implies $M' = \sum_n F'_n$ a.e. (μ) and $M'_\phi = \sum_n \phi_n F'_n$ a.e. (μ) .

Let $X \in N(M')$, then $(\sum_n F'_n)X = 0$ which implies:

$$X^* (\sum_n F'_n) X = \sum_n X^* F'_n X = 0.$$

Since F'_n is nonnegative definite [40, Lemma 2.3], we get $X^* F'_n X = 0$ for every n . But, $X^* F'_n X = (\sqrt{F'_n} X)^*$. $(\sqrt{F'_n} X) = 0$, which implies that $\sqrt{F'_n} X = 0$, for every n . Thus,

$$M'_\phi X = (\sum_n \phi_n F'_n) X = \sum_n (\phi_n \sqrt{F'_n}) \sqrt{F'_n} X = 0 \text{ i.e. } X \in N(M'_\phi). \quad \text{Q.E.D.}$$

By Lemma 3.3.2, Theorem 3.2.2 and Remark 3.3.1 the Radon-Nikodym derivative $\frac{dM}{dM}^\phi$ exists. So we can define the operator P on $L_{2,F}$ into the space of matrix-valued functions by,

$$(P\phi)(\lambda) = \frac{dM}{dM}^\phi(\lambda) \text{ a.e. } (\mu), \quad \phi \in L_{2,F}.$$

It is clear that P is matricial linear, also since for each fixed $\phi \in L_{2,F}$, A and integer k , $M_\phi(A) = \sum_n \int_A \phi_n dF_n = \int_{\bigcup_n A_n} \phi dF =$

$\int_{\bigcup_n A_{n+k}} \phi dF = M_\phi(A + kh^{-1})$, it follows that $\frac{dM}{dM}^\phi$ can be chosen to be

periodic with period h^{-1} , this fact plays a key role in the proof of boundedness of P .

To show that $\frac{dM_\phi}{dM}$ is in $L_{2,F}$, it is enough to prove that P is norm bounded in $L_{2,F}$. For this, we need to prove the following matricial Cauchy-Schawrtz inequality for matricial inner product in $L_{2,F}$.

3.3.3 Lemma. For $\phi, \psi \in L_{2,F}$ with matricial inner product

$$(\phi, \psi)_F = \int_{-\infty}^{\infty} \phi \, dF \, \psi^* ,$$

we have,

$$(\phi, \psi)_F (\psi, \psi)_F^{-1} (\psi, \phi)_F \leq (\phi, \phi)_F .$$

Proof. For every $q \times q$ constant matrix A we have, c.f. (1.1),

$$(\phi + A\psi, \phi + A\psi)_F \geq 0$$

$$\text{or, } (\phi, \phi)_F + A (\psi, \psi)_F A^* + A (\psi, \phi)_F + (\phi, \psi)_F A^* \geq 0 .$$

For choice of $A = - (\phi, \psi)_F^{-1} (\psi, \psi)_F^{-1}$ and using the defining properties of the generalized inverse of matrices, (c.f. Theorem 3.2.3), we get the result. Q.E.D.

3.3.3 Lemma. P is a contraction on $L_{2,F}$ into $L_{2,F,S}$.

Proof. For $\phi \in L_{2,F}$ we have;

$$\begin{aligned}
 \|P\phi\|_F^2 &= \tau \int_{-\infty}^{\infty} \frac{dM}{dM} \phi \quad dF \left(\frac{dM}{dM} \phi \right)^* = \tau \int_{-\infty}^{\infty} \left(\frac{dM}{dM} \phi \right)^* \frac{dM}{dM} \phi \quad dF = \\
 &= \tau \sum_n \int_0^{h^{-1}} \left(\frac{dM}{dM} \phi (\lambda + nh^{-1}) \right)^* \frac{dM}{dM} \phi (\lambda + nh^{-1}) \quad dF (\lambda + nh^{-1}) \\
 &= \tau \int_0^{h^{-1}} \left(\frac{dM}{dM} \phi (\lambda) \right)^* \frac{dM}{dM} \phi (\lambda) \sum_n dF_n(\lambda) = \tau \int_0^{h^{-1}} \left(\frac{dM}{dM} \phi \right)^* \frac{dM}{dM} \phi \quad dM = \\
 &= \tau \int_0^{h^{-1}} \frac{dM}{dM} \phi \quad dM \left(\frac{dM}{dM} \phi \right)^* = \tau \int_0^{h^{-1}} dM_{\phi} \quad dM^{-} \quad dM \quad dM^{-} \quad dM_{\phi}^* = \\
 &= \tau \int_0^{h^{-1}} dM_{\phi} \quad dM^{-} \cdot dM_{\phi}^* .
 \end{aligned}$$

In this chain of equalities we have used the fact that $\frac{dM}{dM} \phi$ can be chosen to be periodic with period h^{-1} .

Since $M_{\phi}(A) = \int_{\bigcup_n A_n} \phi(\lambda) \quad dF(\lambda) \quad I$ with diameter of A less than h^{-1} , by letting $\phi_1 = \phi \chi_{\bigcup_n A_n}$ and $\psi = I \chi_{\bigcup_n A_n}$ we get $(\phi_1, \psi)_F = M_{\phi}(A)$,
 $(\psi, \psi)_F = \int_{\bigcup_n A_n} dF = M(A)$ and $(\phi_1, \phi_1)_F = \int_{\bigcup_n A_n} \phi(\lambda) \quad dF(\lambda) \quad \phi^*(\lambda) = N(A)$.

Thus, from Lemma 3.3.3 we get $M_{\phi}(A) \quad M^{-}(A) \quad M_{\phi}^*(A) \leq N(A)$, therefore;

$$\|P\phi\|_F^2 \leq \tau \int_0^{h^{-1}} dN = \tau \sum_n \int_0^{h^{-1}} \phi_n dF_n \quad \phi_n^* = \|\phi\|_F^2 .$$

Which shows that P is a contraction on $L_{2,F}$ into $L_{2,F}$. But, since $\frac{dM}{dM} \phi$ can be chosen to be periodic with period h^{-1} , it follows that the

range of P is inside $L_{2,F,S}$.

Q.E.D.

In the following, a bounded matricial linear operator P on $L_{2,F}$ is said to be a projection if $P^2 = P$. In this case P is the

identity operator on its range.

3.3.5 Lemma. The operator P is a projection onto $L_{2,F,S}$.

Proof. By Lemma 3.3.4 it is enough to show that P is (equivalent to) the identity operator on $L_{2,F,S}$. Since any $\phi \in L_{2,F,S}$ is equivalent to some $\hat{\phi} \in L_{2,F,S}$ which is periodic with period h^{-1} , (c.f. Lemma 3.2.8), thus by definition of M_ϕ , Lemma 3.3.2, Theorem 3.2.2 (i) and Remark 3.3.1:

$$M_\phi(A) = \sum_n \int_A \hat{\phi}_n(\lambda) dF_n(\lambda) = \int_A \hat{\phi} dM = \int_A \frac{dM}{dM} \phi dM.$$

Hence, by Theorem 3.2.2 (ii) we get;

$$\frac{dM}{dM} \phi J = \hat{\phi} J \quad \text{a.e. } (\mu),$$

where $J(\lambda)$ is the orthogonal projection matrix onto the range of $M'(\lambda)$ a.e. (μ) . Since $\frac{dM}{dM} \phi \in L_{2,F}$, (c.f. Lemma 3.3.4) and $R(F') \subseteq R(M')$, it follows that, $\frac{dM}{dM} \phi = \hat{\phi}$ a.e. (F) .

Thus, for $\phi \in L_{2,F,S}$ we have $P\phi = \phi$ a.e. (F) . Since range of P is contained in $L_{2,F,S}$ it follows that P is the projection onto $L_{2,F,S}$.

Next, we find a version of $\frac{dM}{dM} \phi$ which will play a major role in the proof of our main theorem. For each n , let F_n denotes the Lebesgue-Stieltjes matrix-valued measure induced by the functions $F_n(\lambda) = F(\lambda + nh^{-1})$, $\lambda \in \mathbb{R}$, $n \in \mathbb{Z}$. Each of these measures may be decomposed, by Crámer-Lebesgue theorem [37], into a τF -continuous part and a τF -singular part;

$$F_n(A) = \int_A f_n(\lambda) d\tau(\lambda) + F_n(A \cap S_n) \quad , \quad A \in \mathcal{B}, n \in \mathbb{Z},$$

where f_n , a $q \times q$ nonnegative definite matrix-valued function, is the Radon-Nikodym derivative of the τF -continuous part of F_n with respect to τF , and the τF -singular part of F_n is supported on the S_n i.e. $\tau(S_n) = 0$ ($F(S_n) = 0$).

Let $S = \bigcup_n S_n$, then $\tau(S) = 0$ and

$$F_n(A) = \int_A f_n(\lambda) d\tau(\lambda) + F_n(A \cap S), \quad A \in \mathcal{B}, n \in \mathbb{Z}.$$

thus, the measures M and M_ϕ will have the form,

$$M(A) = \int_A \left(\sum_n f_n(\lambda) \right) d\tau(\lambda) + M(A \cap S)$$

$$M_\phi(A) = \int_A \left(\sum_n \phi_n(\lambda) f_n(\lambda) \right) d\tau(\lambda) + M_\phi(A \cap S).$$

Hence, we arrive at the following important result.

3.3.6 Lemma. $(P\phi)(\lambda) = \frac{dM_\phi}{dM}(\lambda) = \left(\sum_n \phi_n(\lambda) f_n(\lambda) \right) \left(\sum_n f_n(\lambda) \right)^{-1}$ on $R \setminus S$, which is a.e. (τ) .

We note that this version of the projection is no longer formally periodic, but it plays a major role in the proof of Theorem 3.4.1.

3.4 A Sampling Theorem. From Definition 3.2.5, it easily follows that the statement that, for all values of $t \in \mathbb{R}$ not of the form nh , the random vector $X(t)$ can be obtained by linear combination of the sample random vectors $\{X(nh); n \in \mathbb{Z}\}$ with matrix coefficients. In this section we find necessary and sufficient conditions on the support of the spectral measure F or equivalently the trace measure of F so that the process admits a sampling theorem. By a support of a measure τ we mean any set $A \in \mathcal{B}$ whose complement has τ measure zero i.e. $\tau(R \setminus A) = 0$.

Here is our main theorem which is stated and proved in the spirit of Theorem 1 of [21].

3.4.1 Theorem. Under Assumption 3.2.7 the following properties of a q -variate stationary stochastic process $X(t)$ continuous in mean are equivalent.

- (i) Each random vector $X(t)$, $t \in \mathbb{R}$, of the process is determined linearly by the samples $\{X(nh); n \in \mathbb{Z}\}$.
- (ii) For some irrational number ξ , $X(\xi h)$ is determined linearly by the samples.
- (iii) There exists a support Λ of the trace measure τ of the spectral distribution of the process whose translates $\{\Lambda + nh^{-1}, n \in \mathbb{Z}\}$ are mutually disjoint.

Proof. That (i) implies (ii) is clear. We show that (ii) implies (iii) and then (iii) implies (i).

Suppose $X(\xi h)$ is determined linearly by the samples i.e. $X(\xi h) \in \overline{H}_S(x)$, then $\bar{e}^{2\pi i \lambda \xi h} I$ which is the isomorph of $X(\xi h)$ in $L_{2,F}$ belongs to $L_{2,F,S}$ so is equal to its projection on $L_{2,F,S}$. Thus, by Lemmas 3.3.5 and 3.3.6 we have;

$$\bar{e}^{2\pi i \lambda \xi h} I = P \bar{e}^{2\pi i \eta \xi h} I = \left[\sum_n (\bar{e}^{2\pi i (\lambda + nh^{-1}) \xi h} I) f_n \right] \cdot \left[\sum_n f_n \right]^{-1} \text{ a.e. } (\tau).$$

Which implies,

$$\begin{aligned} \bar{e}^{2\pi i \lambda \xi h} \left(\sum_n f_n \right) &= \left(\sum_n \bar{e}^{2\pi i (\lambda + nh^{-1}) \xi h} f_n \right) \cdot \left(\sum_n f_n \right)^{-1} \cdot \left(\sum_n f_n \right) = \\ &= \left(\sum_n \bar{e}^{2\pi i (\lambda + nh^{-1}) \xi h} f_n \right) P_{R\left(\sum_n f_n\right)}, \end{aligned}$$

or

$$\bar{e}^{2\pi i \lambda \xi h} f_0 + \bar{e}^{2\pi i \lambda \xi h} \sum_{n \neq 0} f_n = \bar{e}^{2\pi i \lambda \xi h} f_0 P_{R\left(\sum_n f_n\right)} + \bar{e}^{2\pi i \lambda \xi h} \left(\sum_{n \neq 0} \bar{e}^{2\pi i n \xi} f_n \right) P_{R\left(\sum_n f_n\right)},$$

and this implies that,

$$\bar{e}^{2\pi i \lambda \xi} \left[\sum_{n \neq 0} (1 - \bar{e}^{2\pi i n \xi}) f_n \right] P_{R(\sum_n f_n)} = 0,$$

from which (since ξ is irrational) we get;

$$f_n P_{R(\sum_k f_k)} = 0 \text{ a.e. } (\tau), \quad \forall n \neq 0.$$

For $X \in R(\sum_k f_k)$ we have;

$$f_n X = f_n P_{R(\sum_k f_k)} X = 0.$$

And for $X \in R(\sum_k f_k)^\perp = N(\sum_k f_k) \subseteq N(f_n)$ we get; $f_n X = 0$. So, $f_n = 0$ a.e. (τ) , $\forall n \neq 0$, i.e. F_n 's are τ F-singular, $n \neq 0$ or what is the same $\tau_n = \tau F_n$ is τ F-singular, $n \neq 0$. Thus there exists complementary supports for τ and τ_n , $n \neq 0$. Let M_n be a support of τ such that $\tau_n(M_n) = 0$, $n \neq 0$. The intersection $N = \bigcap_{n \neq 0} M_n$ of these is a support of τ which has the property $\tau_n(N) = 0$, $n \neq 0$. From the nature of the τ_n (translates of τ) we see that $N_n = N + nh^{-1}$ is a support of τ_n which has the property $\tau_r(N_n) = 0$, $r \neq n$, $n, r \in \mathbb{Z}$, in particular, $\tau(N_n) = 0$, $n \neq 0$. Finally, the set $\Lambda = N \cup \left(\bigcup_{n \neq 0} R \setminus N_n \right)$ is a support of τ which is disjoint from each of its translates $\Lambda + nh^{-1}$, $n \neq 0$.

To show that (iii) implies (i), suppose Λ is a support of τ which is disjoint from each of its $\Lambda + nh^{-1}$, $n \neq 0$. Clearly $\Lambda + nh^{-1}$ is a support of τ_n , so that F and F_n have disjoint supports, $n \neq 0$, i.e. $f_n(\lambda) = 0$ a.e. (τ) , $n \neq 0$. Thus, by Lemma 3.3.6 for $\phi \in L_{2,F}$ we have;

$$(P\phi)(\lambda) = \phi \quad f_0 \quad f_0^- = \phi P_{R(f_0)} = \phi \text{ a.e. } (F) \text{ i.e.}$$

$$L_{2,F} = L_{2,F,S}.$$

Q.E.D.

3.4.2 Remark. Here we note that Assumption 3.2.7 or an assumption similar to that is essential for Theorem 3.4.1. For an example, consider the case $q = 2$, $X(t) = (x^1(t), x^2(t))$, where the spectral measures of $x^1(t)$ and $x^2(t)$ are supported on $[0,1]$ and $[1,2]$, respectively. By Theorem 3.1.2, $x^1(t)$ and $x^2(t)$ admit sampling theorem with $h = 1$, thus by Theorem 3.2.6, $X(t)$ admits sampling theorem with $h = 1$, but this contradicts Theorem 3.4.1 (iii) as translates of $\Lambda = [0,2]$ i.e.

$\Lambda + n = [0,2] + n$, $n \in \mathbb{Z}$, are not mutually disjoint.

3.4.3 Corollary. If a q -variate stationary process, $X(t) = (x^1(t), \dots, x^q(t))^T$, admits a sampling theorem with sample spacing h , then $x^i(t)$, $i = 1, \dots, q$, admits a sampling theorem with the same sample spacing.

Proof. Since $X(t)$ admits a sampling theorem, by Theorem 3.4.1, there exists a support Λ for τ whose translates $\Lambda + nh^{-1}$ are mutually disjoint. If Λ_i is a support of the spectral measure of $x^i(t)$, then clearly $\Lambda_i \subseteq \Lambda$. Hence $\Lambda_i + nh^{-1}$, $n \in \mathbb{Z}$, are mutually disjoint, therefore by Theorem 3.1.2 (i), $x^i(t)$ admits a sampling theorem with the same sample spacing h . Q.E.D.

3.4.4 Open Problems. Here we have not studied the problem of reconstruction of the q -variate process from its samples. Sampling series similar to (ii) and (iii) in Theorem 3.1.2 with non-diagonal matrix coefficients are of considerable importance in application. In this case, for reconstruction of a particular components of a q -variate process, samples of other components of the process is used. So it is natural to ask, whether these samples from other (related) components will help the series for the reconstruction of that component to converge faster compared to the case when only samples of that particular component is used in its

reconstruction. At this time, we have no answer to these problems. Answer to these problems will be very useful in application and theory of sampling theorem for q -variate stationary processes.

3.5 Sampling Theorems for Harmonizable Processes.

The problem of sampling theorem for harmonizable processes have been studied, independently around 1967, by Z.A. Piranashivli [35] and M.M. Rao [36]. In [35] a sampling series for harmonizable processes with bounded spectra is given. Rao has extended Lloyd's theorem to the case of harmonizable processes, but Rao's condition is not necessary, as A.J. Lee [20] has shown by a counter-example that no condition on the translates of a support of the spectral measure, in this case, is necessary.

In [20], A.J. Lee obtains sampling theorem and sampling series for non-stationary second-order processes under some integrability condition on the covariance of the process along the work of Lloyd [21]. In particular, he has a sampling theorem and a sampling series for harmonizable process. But, in Lee's result the coefficients for the reconstruction of the process from its samples are not explicit as he uses the theory of distributions.

In this section, we use Abreu's theorem to obtain sampling theorems and sampling series for harmonizable processes similar to the work of Lloyd [21], Buetler [2] and Rao [36]. In our case, the coefficients in the sampling series are exactly the same as those appearing in Theorem 3.1.2.

First, we need to introduce some notions and notations. Consider the harmonizable stochastic process; (c.f. Definition 2.3.1],

$x(t) = \int_R \bar{e}^{2\pi i t \lambda} \eta(d\lambda)$, $t \in R^1$, with the covariance function

$R(s,t) = \iint_{R^2} \bar{e}^{2\pi i(s\lambda_1 - t\lambda_2)} d\mu(\lambda_1, \lambda_2)$, where η is a countably additive

H -valued measure on R^1 such that for $A, B \in \mathcal{B}$, $\mu(A, B) = (\eta(A), \eta(B))$ extends to a complex-valued measure of bounded variation on R^2 . Here, we refer to μ as the spectral measure of the process.

Let $|\mu|$ denote the total variation measure of μ , then $|\mu|$ is a positive, finite and symmetric measure on R^2 . Now, we define μ_0 a finite positive measure on R^1 , as the marginal measure of $|\mu|$ by $\mu_0(A) = |\mu|(A \times R)$. Since μ_0 is a finite positive measure on R^1 , it can be considered as the spectral measure of a stationary process $z(t)$ taking values in a Hilbert K . The following remarkable theorem of J. Abreu [1] and [26] shows that the harmonizable process $x(t)$ can be obtained by projecting $z(t)$ onto $H(x)$.

3.5.1 Theorem. If $x(t) \in H$ is a harmonizable process with spectral measure μ , then there exists a Hilbert space K containing $H(x)$ as a subspace and a stationary process $z(t) \in K$ such that if $P: K \rightarrow H(x)$ is the orthogonal projection, then $x(t) = Pz(t)$, $t \in R$. Furthermore, μ_0 the spectral measure of $z(t)$ is given by $\mu_0(A) = |\mu|(A \times R)$, $A \in \mathcal{B}$.

3.5.2 Definition. We call the stationary process $z(t)$ of Theorem 3.5.1, the associated stationary process of $x(t)$.

Abreu's theorem had been used to obtain sufficient conditions for certain properties of a harmonizable process in terms of its associated stationary process $z(t)$ i.e. in terms of the measure μ_0 . Here, we use this theorem to obtain sufficient condition for a harmonizable process to admit a sampling theorem.

In this section we introduce a more general notion of sample times than that which was used in earlier sections of this chapter. We refer to any set $\{t_n\} \subset R$ which is not dense in R as sample times. In this case, if $H(x) = H_S(x) = \sigma\{x(t_n); n \in Z\}$ we say that the process admits a sampling theorem. If $t_n = nh$, then we say that the process admits a (periodic) sampling. If $\{t_n; n \in Z\}$ is a bounded subset of R , then it is said that the process admits a non-periodic sampling theorem.

3.5.3 Theorem. A harmonizable process $x(t)$, $t \in R$ admits a sampling theorem if its associated stationary process $z(t)$ admits a sampling theorem.

Proof. Suppose $z(t)$ admits a sampling theorem for a given sample times $\{t_n\}$, then we have $H(z) = H_S(z) = \sigma\{z(t_n); n \in Z\}$. By Theorem 3.5.1, we know that $x(t) = P z(t)$ for all $t \in R$, thus by continuity of P and the fact that $H(z) \supset H(x)$ we get;

$$\begin{aligned} H_S(x) &= \sigma\{x(t_n); n \in Z\} = \sigma\{Pz(t_n); n \in Z\} \supset P(\sigma\{z(t_n); n \in Z\}) \\ &= PH(z) = H(x). \end{aligned}$$

But, since $H_S(x) \subset H(x)$, we have $H(x) = H_S(x)$ i.e. x admits a sampling theorem with the same sample times. Q.E.D.

By combining Theorems 3.1.2, 3.5.1 and 3.5.3 we get the following sampling theorem and sampling series for a harmonizable process.

3.5.4 Theorem. Suppose $x(t)$ is a harmonizable process with the spectral measure μ and μ_0 the spectral measure of its associated stationary process (that is $\mu_0(A) = |\mu|(A \times R)$, $A \in B$).

(i) If μ_0 has a support Λ whose translates by nh^{-1} i.e. $\{\Lambda + nh^{-1}; n \in \mathbb{Z}\}$ are mutually disjoint, then $H(x) = H_S(x)$ i.e. $x(t)$ admits a sampling theorem.

(ii) If μ_0 has an open support Λ whose translates $\{\Lambda + nh^{-1}; n \in \mathbb{Z}\}$ are mutually disjoint, then we have;

$$x(t) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) x(nh)K(t - nh), \quad t \in \mathbb{R}.$$

(iii) If the Λ , from (ii), is a finite union of intervals, or, more generally, if $\sup_t |t K(t)| < \infty$, then

$$x(t) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n=-N}^N x(nh)K(t - nh), \quad t \in \mathbb{R}.$$

We note that $K(t)$ appearing in (ii) and (iii) are the same one which appears in Theorem 3.1.2.

Non-periodic sampling theorem and sampling series for stationary stochastic processes was first given by F.J. Buetler [2]. Theorem 3 in [2] gives sufficient condition for a stationary process to admit a sampling theorem, and a formula for the reconstruction of the process from its samples.

Here, again by combining Theorem 3 of [2] and Theorems 3.5.1 and 3.5.3 we obtain such a sufficient condition for a harmonizable process to admit a sampling theorem along with a formula for the reconstruction of the process.

3.5.5 Theorem. Suppose $x(t)$ is a harmonizable process with the spectral measure μ and μ_0 the spectral measure of its associated stationary process. Let t_n be a bounded subset of $\mathbb{R}$ with a limit point t_0 .

It μ_0 has the property $\int_{-\infty}^{\infty} e^{c|\lambda|} \mu_0(d\lambda) < \infty$, for every $c < \infty$, then

(i) $H(x) = H_S(x)$, i.e. x admits a non-periodic sampling theorem.

(ii) $x(t)$ has derivatives of all orders in the sense that

$$x^{(n)}(t) = \text{l.i.m.}_{t \rightarrow t'} \frac{x^{(n-1)}(t') - x^{(n-1)}(t)}{t' - t},$$

$$\text{with } x^{(0)}(t) = x(t).$$

(iii) For each $t \in \mathbb{R}$,

$$x(t) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n=0}^N \frac{(t - t_0)^n x^{(n)}(t_0)}{n!}.$$

CHAPTER IV
HELSON-SARASON THEOREM FOR DIRICHLET
ALGEBRAS AND STRONG MIXING OF MULTIPARAMETER GAUSSIAN STATIONARY PROCESSES

4.1 Introduction. Let μ be a finite nonnegative Borel measure on the unit circle $T = \{z \in \mathbb{C} ; |z| = 1\}$, let $P_0 = \sigma\{1, \bar{z}, \bar{z}^2, \dots\}$ in $L^2(d\mu)$ and for $n = 1, 2, \dots$, let $F_n = \sigma\{z^n, z^{n+1}, \dots\}$. Let ρ_n be the supremum of $|(f, g)|$ as f and g range over the unit balls of P_0 and F_n , respectively (the inner product being taken in the Hilbert space $L^2(d\mu)$).

The quantity ρ_n is a measure of the angle between the subspaces P_0 and F_n . P_0 and F_n are said to be at positive angle if $\rho_n < 1$. They are said to be asymptotically orthogonal if $\rho_n \rightarrow 0$ as $n \rightarrow \infty$.

H. Helson and G. Szego [16], H. Helson and D. Sarason [15] studied the following important problems concerning ρ_n .
Problem 1. For given integer n , find the necessary and sufficient conditions on the measure μ such that $\rho_n < 1$.

Problem 1 is of considerable importance in harmonic analysis [47], as well as probability theory [8]. Complete solution to this problem is given in [15] and [16].

Problem 2. Find the necessary and sufficient conditions on the measure μ such that $\rho_n \rightarrow 0$ as $n \rightarrow \infty$.

Problem 2, was first raised in connection with the problem of strong mixing of Gaussian stationary sequences (cf. [42], [19] and [51]). In fact, $\rho_n \rightarrow 0$ is equivalent to the strong mixing of a Gaussian stationary

sequence with spectral measure μ .

The following theorem provides an answer to Problem 2, [15] and [16].

4.1.1 Theorem (Helson-Sarason). $\lim_{n \rightarrow \infty} \rho_n = 0$ if and only if $d\mu$ is absolutely continuous with respect to dm (normalized Lebesgue measure on T), such that $d\mu = w dm$ with $w \in L^1(dm)$ and w has the form $w = |P|^2 e^{r+Cs}$, where P is an analytic polynomial, r and s are real continuous functions on the unit circle and Cs denotes the harmonic conjugate function of s .

M. Rosenblatt [44] calls attention to the importance of results similar to Theorem 4.1.1 for continuous time parameter and multi-parameter Gaussian stationary processes. Analog of Theorem 4.1.1 for one-parameter continuous time Gaussian stationary processes has been studied by E. Hayashi [13].

For $n = 1$, A. Devinatz [7, 8], Y. Ohno [30] and S. Merrill [24] and for an arbitrary integer n , Y. Ohno [31] have studied Problem 1 in a Dirichlet algebra setting.

In section 2, we introduce some notations and preliminary results. In section 3, it is shown that analog of Theorem 4.1.1 is valid for a Dirichlet algebra setting and in section 4 we discuss its application to the problem of strong mixing of discrete time multi-parameter Gaussian stationary processes.

Our work is heavily based on Y. Ohno [31]. While doing this work, we were unaware of Y. Ohno and K. Yabuta's work on the same problem [32], but Lemma 3 of [32] is used to improve our Theorem 4.3.3 in the form given in Theorem 4.3.7. We wish to acknowledge our gratitude to Professor

Y. Ohno for sending us his papers and some of his unpublished work in this area.

Our proof of Theorem 4.3.3 has the following advantages as compared to Ohno-Yabuta's approach:

- (1) Our approach gives a purely function-algebraic proof to the extension of Helson-Sarason theorem for Dirichlet algebras. Ohno-Yabuta's approach does not give such a proof as they reduce the problem to the unit circle and then use results of [15] including a lemma on analytic continuation.
- (2) Our approach provides an essentially unified proof for Problems 1 and 2 (this can be seen by comparing proofs of Theorem 6 of [31] and Theorem 4.3.3 in this chapter). It is expected that such a unification will be of great help in other similar situations.
- (3) Our proof of Theorem 4.3.3 specialized to the unit circle gives a relatively simple and short proof to Helson-Sarason's Theorem, in this case, as compared to the one given in [15].

4.2 Notations and Preliminaries.

Let X be a compact Hausdorff space and let A be a Dirichlet algebra on X i.e. A is a uniform algebra on X such that the real parts of the functions in A are uniformly dense in the real continuous functions on X . Let m be the unique representing measure on X for a complex homomorphism of A . Let $G(m)$ be the Gleason part of m , that is, $G(m)$ is the set of all complex homomorphisms σ of A such that norm of $\sigma - m$, as a linear functional on A , is strictly smaller than 2. If $0 < p < \infty$, H^p denotes the closure of A in $L^p(dm)$ and H^∞ denotes the weak*-closure of A in $L^\infty(dm)$. We put $A_0 = \{f \in A; \int f dm = 0\}$

and $H_0^p = \{f \in H^p; \int f dm = 0\}$ ($1 \leq p \leq \infty$). We denote by $(H_0^\infty)^n$ (resp. A_0^n) the ideal generated by products of n elements in H_0^∞ (resp. A_0). We say that H_0^∞ is simply invariant if $[A_0 H_0^\infty]_\star \subsetneq H_0^\infty$, where $[B]_\star$ denotes the weak^{*}-closure of B . With notations as above the following conditions are equivalent:

- (i) H_0^∞ is simply invariant.
- (ii) There exists an inner function Z such that $H_0^\infty = Z H^\infty$ (this function Z is determined uniquely up to multiplication of constants of modulus 1 and is called "Wermer's embedding function").
- (iii) $G(m) \neq \{m\}$.

Let μ be a positive finite measure on X , on account of proposition 2 of Y. Ohno [31] it suffices to assume that $d\mu$ is absolutely continuous with respect to dm i.e. $d\mu = w dm$ and $\log w \in L^1(dm)$. The measure of the angle between the two linear manifolds $\bar{A} = \{\bar{f}; f \in A\}$ and A_0^n in $L^2(d\mu)$ is $\rho_n = \sup |\int f g w dm|$, $n = 1, 2, \dots$, where f and g range over the elements of A and A_0^n respectively, subject to the restriction,

$$(1) \quad \int |f|^2 w dm \leq 1 \quad \text{and} \quad \int |g|^2 w dm \leq 1.$$

It is easy to show that $\rho_n = \sup |\int f g w dm|$, where f and g range over the elements of H^∞ and $(H_0^\infty)^n$, respectively, subject to (1). Since $\log w \in L^1(dm)$, then $w = |h|^2$, where $h \in H^2$ is an outer function [9, Theorem 6]. Let $\phi = \text{Arg} h$, then $w = h^2 \bar{e}^{2i\phi}$ and

$$\rho_n = \sup |\int (fh)(gh) \bar{e}^{2i\phi} dm|,$$

where the supremum is taken over all $f \in H^\infty$ and $g \in (H_0^\infty)^n$ such that $\int |fh|^2 dm \leq 1$ and $\int |gh|^2 dm \leq 1$.

Throughout this section we assume that $G(m) \neq \{m\}$ i.e. the Gleason part of m is nontrivial. Thus, there exists an inner function Z in H^∞ such that $H_0^\infty = Z H^\infty$, so $(H_0^\infty)^n = Z^n H^\infty$ and we have

$$(2) \quad \rho_n = \sup |\int (fh)(gh) Z^n \bar{e}^{2i\phi} dm|$$

where f and g range over the elements of H^∞ subject to the respective restriction $\int |fh|^2 dm \leq 1$ and $\int |gh|^2 dm \leq 1$. Since h is outer in H^2 , $\{fh; f \in H^\infty\}$ is dense in H^2 and more specifically $\{fh; f \in H^\infty, \int |fh|^2 dm \leq 1\}$ is dense in the unit ball of H^2 . Thus $\{fg h^2; f, g \in H^\infty, \int |fh|^2 dm \leq 1, \int |gh|^2 dm \leq 1\}$ is dense in the unit ball of H^1 [7, Lemma 6]. Therefore (2) can be written in the form

$$(3) \quad \rho_n = \sup |\int f Z^n \bar{e}^{2i\phi} dm|$$

where f ranges over the functions in H^1 such that $\int |f| dm \leq 1$. It is clear that (3) expresses ρ_n as the norm of the linear functional on H^1 defined by $\int f Z^n \bar{e}^{2i\phi} dm$ for $f \in H^1$, thus $Z^n \bar{e}^{2i\phi} \in (H^1)^*$.

By the Hahn-Banach theorem we have

$$(4) \quad \rho_n = \inf_{\substack{g \in H_0^\infty \\ 0}} \|Z^n \bar{e}^{2i\phi} - g\|_\infty = \inf_{F \in H^\infty} \|1 - Z^{1-n} \bar{e}^{2i\phi} F\|_\infty.$$

4.3 Main results. In this section by using (4) we get necessary and sufficient condition on μ such that $\rho_n \rightarrow 0$ as $n \rightarrow \infty$, when $G(m) \neq \{m\}$. Complete characterization of μ when $G(m) = \{m\}$ i.e. when the Gleason part of m is trivial is given in [32].

First, we prove the following theorem, which is an analog of Theorem 3 of [15], and plays a major role in the proof of our main theorem.

From definition of ρ_n it is clear that ρ_n is non-increasing in n , so that $\lim_{n \rightarrow \infty} \rho_n$ exists.

4.3.1 Theorem. $\lim_{n \rightarrow \infty} \rho_n = 0$ if and only if for each $\varepsilon > 0$ there exists $F \in H^\infty$ and a positive integer n such that $|\text{Arg}(Fh^2Z^{1-n})| < \varepsilon$ and $|\log|F|| < \varepsilon$ a.e. (m) on X .

Proof. $\lim_{n \rightarrow \infty} \rho_n = \lim_{n \rightarrow \infty} \inf_{F \in H^\infty} \|1 - Z^{1-n} e^{2i\phi} F\|_\infty = 0$, if and only if for every $\varepsilon > 0$ there exists a positive integer n such that;

$$\inf_{F \in H^\infty} \|1 - Z^{1-n} e^{2i\phi} F\|_\infty < \varepsilon.$$

This holds if and only if, there exists an $F \in H^\infty$ such that

$\|1 - Z^{1-n} e^{2i\phi} F\|_\infty < \varepsilon$. And this in turn holds if and only if, $|\text{Arg}(Fh^2Z^{1-n})| < \varepsilon$ and $|\log|F|| < \varepsilon$ a.e. (m) on X (In this proof ε may not be the same throughout).

Next, we quote the following result from [25]. Let H^p denote the closure in $L^p(dm)$ of the set of polynomials in Z and L^p the closure in $L^p(dm)$ of the set of polynomials in Z and $\bar{Z}$. For $1 \leq p \leq \infty$, we put

$$I^p = \{f \in H^p; \int f \bar{Z}^k dm = 0, k = 0, 1, 2, \dots\}.$$

4.3.2 Lemma [25]. If $1 \leq p \leq \infty$, then

$$H^p = H^p \oplus I^p$$

$$L^p = L^p \oplus N^p$$

where $\oplus$ denotes the algebraic direct sum and N^P denotes the closure of $\bar{I}^P + I^P$ in $L^P(dm)$.

Here is our main theorem, whose proof is essentially the same as that of Theorem 6 in [31]. For the sake of completeness and comparison we present its proof in detail.

4.3.3 Theorem. $\lim_{n \rightarrow \infty} \rho_n = 0$ if and only if, for every positive $\varepsilon < \frac{\pi}{2}$ there exist real functions $r, s \in L^\infty(dm)$ with $\|r\|_\infty < \varepsilon, \|s\|_\infty < \varepsilon$ such that $w = |p|^2 e^{r+Cs}$, where P is a function in H^∞ so that $P \perp A_0^n$ in $L^2(dm)$ for some n and Cs denotes the conjugate function of s .

Proof. Assume $\lim_{n \rightarrow \infty} \rho_n = 0$, then by Theorem 4.3.1, for each $0 < \varepsilon < \frac{\pi}{2}$ there exists a positive integer n and $F \in H^\infty$ such that $|\operatorname{Log}|F|| < \varepsilon$ and $|\operatorname{Arg}(Fh^2 Z^{1-n})| < \varepsilon$.

Let s be the real function bounded by ε such that;

$$(5) \quad s + \operatorname{Arg}(Fh^2 Z^{1-n}) = 0.$$

From here on we proceed as the proof of Theorem 6 of [31]. We put,

$$(6) \quad S = Fh^2 Z^{1-n} e^{-Cs+is}.$$

Then by (5) $S \geq 0$. From Theorem 10 of [7], we conclude that $e^{-Cs+is} e^{H^1}$ is outer. By proposition 4 of [31] and the fact that $|\operatorname{Arg}(Fh^2 Z^{1-n})| < \varepsilon < \frac{\pi}{2}$, we may write $Fh^2 = Z^m B$, where $B \in H^1$, $\int B dm \neq 0$ and $0 \leq m \leq n-1$. Therefore

$$(7) \quad S = BZ^{-k} e^{-Cs+is} \geq 0$$

and so

$$(8) \quad Z^k S = B \bar{e}^{Cs+is} \in H^{\frac{1}{2}}$$

where $k = n-m-1$. Furthermore, by Jensen's inequality,

$$(9) \quad \begin{aligned} \int \log |Z^k S| dm &= \int \log |B| dm + \int \log |\bar{e}^{Cs+is}| dm \\ &\geq \log |\int B dm| + \log |\int \bar{e}^{Cs+is} dm| > -\infty. \end{aligned}$$

Using Theorem 2 of [9], it follows from (8) and (9) that there exists an outer function P in H^1 and an inner function q in H^∞ such that

$$(10) \quad Z^k S = q P^2.$$

Since $S = |S|$ and $|S| = |P|^2$, we have from (10) that

$$(11) \quad q P^2 = Z^k |P|^2.$$

Since P is outer, it follows that P is not zero. Thus we may divide (11) by P and obtain

$$(12) \quad q P = Z^k \bar{P}.$$

By Lemma 4.3.2, we can write

$$P = \sum_{j=0}^{\infty} a_j Z^j + \alpha_I \in H^1 \oplus I^1$$

where α_I belongs to I^1 . Now

$$(13) \quad \begin{aligned} Z^k \bar{P} &= \bar{a}_0 Z^k + \bar{a}_1 Z^{k-1} + \dots + \bar{a}_{k-1} Z + \bar{a}_k + \\ &\quad \bar{a}_{k+1} \bar{Z} + \bar{a}_{k+2} \bar{Z}^2 + \dots + Z^k \alpha_I. \end{aligned}$$

Because $a_{k+1}Z + a_{k+2} + \dots \in H_0^1$ and $\bar{Z}^k \alpha_I \in I^1 \subset H_0^1$, we have $g = a_{k+1}Z + a_{k+2}Z^2 + \dots + \bar{Z}^k \alpha_I \in H_0^1$. By (12) and $Z^k \bar{P} \in H^1$ we conclude $\bar{g} \in H^1$ by (13). Hence $g \in \bar{H}^1 \cap H_0^1$. Since $\bar{A} + A_0$ is weak*-dense in $L^\infty(dm)$, we have $g = 0$ and

$$Z^k \bar{P} = \bar{a}_0 Z^k + \bar{a}_1 Z^{k-1} + \dots + \bar{a}_{k-1} Z + \bar{a}_k.$$

Thus P has the form

$$P = a_0 + a_1 Z + \dots + a_k Z^k$$

where $0 \leq k \leq n-1$. Therefore $P \in H^\infty$ and $P \perp A_0^n$ in $L^2(dm)$. Indeed, if $G \in A_0^n \subset (H_0^\infty)^n$, then $G = Z^n K$ for some $K \in H^\infty$ and we have;

$$\begin{aligned} (P, G) &= \int \left(\sum_{j=0}^k a_j Z^j \right) \bar{Z}^n \bar{K} dm = \sum_{j=0}^k a_j \int \bar{Z}^{n-j} \bar{K} dm \\ &= \sum_{j=0}^k a_j \int \bar{Z} dm \int \bar{Z}^{n-j-1} \bar{K} dm = 0, \end{aligned}$$

since m is multiplicative on H^∞ and $n-1 \geq k$. Now by (10) and (6) we have;

$$|P|^2 = S = |S| = |F| |h|^2 e^{Cs}$$

and since $w = |h|^2$,

$$w = |P|^2 |F|^{-1} e^{Cs} = |P|^2 e^{r+Cs}$$

where $r = -\log|F|$, $\|r\|_\infty < \varepsilon$ and $\|s\|_\infty < \varepsilon$.

Conversely, suppose w satisfies conditions of the theorem.

Let $S = |P|^2$. Since $Z^{n-1} P f \in (H_0^\infty)^n$ for $f \in I^\infty$, we have

$$(14) \quad \int Z^{n-1} S f \, dm = (Z^{n-1} P f, P) = 0, \quad f \in I^\infty.$$

If $f \in I^\infty$, then it is easy to see that $\bar{Z}^{2(n-1)} f$ is also in I^∞ .

Therefore, by (14)

$$\int \bar{Z}^{n-1} S f \, dm = \int Z^{n-1} S \bar{Z}^{2(n-1)} f \, dm = 0, \quad f \in I^\infty.$$

since $S = \bar{S}$,

$$(15) \quad \int Z^{n-1} S \bar{f} \, dm = 0, \quad f \in I^\infty.$$

It follows from (14) and (15) that

$$\int Z^{n-1} S f \, dm = 0, \quad f \in \bar{I}^\infty \oplus I^\infty.$$

By Lemma 4.3.2, $Z^{n-1} S \in L^1$. Furthermore, we have

$$\int Z^{n-1} S \bar{Z}^k \, dm = \begin{cases} (Z^{n-1-k} P, P) = 0 & (n-1-k \geq n, \text{ i.e., } k = -1, -2, \dots) \\ (P, Z^{k+1-n} P) = 0 & (k+1-n \geq n, \text{ i.e., } k = 2n-1, 2n, \dots) \end{cases}$$

We conclude that $Z^{n-1} S$ has the form

$$Z^{n-1} S = a_0 + a_1 Z + \dots + a_{2n-2} Z^{2n-2}.$$

We put, $k = \max\{m; 0 \leq m \leq n-1, a_{m+n-1} \neq 0\}$. Since $S \neq 0$ and $\bar{S} = S$, such k exists. Then $Z^k S \in H^\infty$ and $\int Z^k S \, dm \neq 0$, therefore by Theorem 2 of [9], $Z^k S$ has the factoring

$$Z^k S = qG^2$$

where q is inner and G is outer in H^∞ . If we take an outer function F in H^∞ such that $|F| = \bar{e}^r$, then

$$(16) \quad Z_{qS}^{k-} e^{Cs-is} = Fh^2 ,$$

up to constant factors of modulus 1. Indeed, by Theorem 10 of [7], e^{Cs-is} is outer in H^1 , and $Z_{qS}^{k-} = G^2$ is also outer in H^∞ , so that the left hand side of (16) is outer in H^1 . Furthermore, since F is outer in H^∞ and h is outer in H^2 , the right hand side of (16) is also outer in H^1 . Now by the assumption on w

$$|Z_{qS}^{k-} e^{Cs-is}| = S e^{Cs} = |P|^2 e^{Cs} = w \bar{e}^r = |h|^2 |F| = |Fh^2|.$$

Since an outer function is determined up to a constant factor by its modulus, (16) follows. From (16), $S = Fh^2 Z_{qS}^{k-} e^{-Cs+is}$ and $S \geq 0$, it follows that

$$\text{Arg}(Fh^2 Z_{qS}^{k-} e^{-Cs+is}) = 0.$$

Hence $|\text{Arg}(Fh^2 Z_{qS}^{k-})| = |s| \leq \|s\|_\infty < \epsilon$ and $|\log|F|| = |r| \leq \|r\|_\infty < \epsilon$.

If we put $B = F q Z^{n-1-k}$, then $B \in H^\infty$ and

$$|\text{Arg}(Bh^2 Z^{1-n})| = |\text{Arg}(Fh^2 Z_{qS}^{k-})| < \epsilon ,$$

$$|\log|B|| = |\log|F|| < \epsilon .$$

Thus the assertion follows from Theorem 4.3.1.

Q.E.D.

4.3.4 Corollary. $\lim_{n \rightarrow \infty} \rho_n = 0$ if and only if, for every $0 < \epsilon < \frac{\pi}{2}$,

there exists real-valued functions $r, s \in L^\infty(dm)$ with $\|r\|_\infty < \epsilon$, $\|s\|_\infty < \epsilon$ and $w = |P|^2 e^{r+Cs}$, where P is a polynomial in Z of arbitrary degree and Cs is the conjugate function of s .

4.3.5 Example. Let $X = T$, A be the disc algebra on T i.e.

$$A = \{f \in C(T); \hat{f}(n) = \int_0^{2\pi} e^{-in\lambda} f(\lambda) \frac{d\lambda}{2\pi} = 0, n = -1, -2, \dots \text{ and } m$$

normalized Lebesgue measure on T . Then A is a Dirichlet algebra on

T and it is well-known that the Gleason part of μ is non-trivial. Wermer's embedding function, in this case, is $Z = e^{i\lambda}$. Thus, by Sarason's Lemma [46], Corollary 4.3.4 reduces to Theorem 4.1.1.

4.3.6 Example. Let $X = T \times T$ and $S = \{(m,n) \in \mathbb{Z}^2; m > 0\} \cup \{(0,n) \in \mathbb{Z}^2; n \geq 0\}$. Let $A = A(S)$ be the Dirichlet algebra of continuous functions on $T \times T$ which are uniform limits of polynomials in $e^{i(mx+ny)}$, $(m,n) \in S$. Let μ be the normalized Lebesgue measure on $T \times T$ (torus). Then the Gleason part of μ can be identified with $\{(0,\alpha) \in \mathbb{D}^2; |\alpha| < 1\}$ which is non-trivial, the Wermer's embedding function is given by $Z(e^{ix}, e^{iy}) = e^{iy}$ and P of corollary 4.3.4 is a polynomial in e^{iy} [31].

Corollary 4.3.4 is similar to Theorem 5 of [15] and its form resembles that of Theorem 1 of [13]. Example 4.3.5 shows that when X is the unit circle, the characterization of μ does not depend on ε . Actually, this is the case when X is any compact Hausdorff space as is shown in [32]. Let $C(Z) = \{f(Z); f \in C(T)\}$, then $H^\infty + C(Z)$ is closed in $L^\infty(d\mu)$ [32, Lemma 3]. Thus, by using an extension of Sarason's Lemma [46]; Corollary 4.3.4 can be restated as:

4.3.7 Theorem. $\lim_{n \rightarrow \infty} \rho_n = 0$ if and only if w has the form $w = |P(Z)|^2 e^{r(Z)} + Cs(Z)$, where Z is the Wermer's embedding function, P is an analytic polynomial, r and s are real valued continuous functions on the unit circle T and Cs is the usual harmonic conjugate function of s .

4.4 Strong Mixing of Multi-parameter Gaussian Stationary Processes.

Unlike the prediction theory for stationary stochastic processes with one parameter, prediction theory for multi-parameter stationary stochastic processes is more diversified. Because there is no natural

distinction between "past" and "future" in the latter case as compared to the former one. Here, for simplicity, we only consider the two-parameter or doubly stationary stochastic processes with discrete parameters.

Let $(\Omega, \mathcal{B}, P)$ be a probability space and $x_{m,n} \in L^2(\Omega, \mathcal{B}, P)$ such that $\int x_{m,n}(\omega) dP(\omega) = 0$, $(m,n) \in \mathbb{Z}^2$. We say that $\{x_{m,n}\}$ is a two-parameter stationary stochastic process if for all integers m,n,k,l we have $(x_{m+k,n+l}, x_{m,n}) = (x_{k,l}, x_{0,0})$. In this case, we call $C(k,l) = (x_{k,l}, x_{0,0})$ the covariance of the process. It is easy to see that $C(\cdot, \cdot)$ is a positive definite function on $\mathbb{Z}^2$. Thus, by Herglotz-Bochner-Weil Theorem [45, page 19] on positive definite functions, there exists a finite non-negative measure μ on Borel sets of the torus such that $C(k,l) = \iint e^{i(kx+ly)} d\mu(x,y)$, $(k,l) \in \mathbb{Z}^2$. μ is called the spectral measure of the process.

H. Helson and D. Lowdenslager [14] developed the theory for predicting $x_{0,0}$ by linear combination of elements $x_{m,n}$ with $(m,n) \in S$, where S is a half-plane of lattice points. The fact that the proofs and some of the results of [14] are independent of the particular choice of S have been crucial in the development of abstract Hardy spaces. Also, this fact is very useful in theory and applications of two-parameter stationary stochastic processes as will be seen in this section.

Here, we adopt the following definition of half-plane of lattice points. A set S of lattice points of $\mathbb{Z}^2$ is called a half-plane if;

- 1) $(0,0) \in S$,
- 2) $(m,n) \in S$ if and only if $(-m,-n) \notin S$ unless $m=n=0$,
- 3) $(m,n) \in S$ and $(m',n') \in S$ imply $(m+m', n+n') \in S$.

For $X = T^2$ and S a fixed half-plane of lattice points, it is easy to show that $A = A(S) = \{f \in C(X); \hat{f}(m,n) = \frac{1}{4\pi^2} \int_0^2 \int_0^2 f(x,y) e^{-i(mx+ny)} dx dy = 0, (m,n) \notin S\}$ is a Dirichlet algebra on the torus.

Let $S_k = \{(m,n) \in Z^2; e^{-i(mx+ny)} \in A_0^k\}$ and $B(S_k)$ the σ -algebra generated by the collection of random variables $\{x_{m,n}; (m,n) \in S_k\}$.

We say that the process is strongly mixing if,

$$\sup_{A,B} |P(A \cap B) - P(A)P(B)| = \alpha(n) \rightarrow 0$$

as $n \rightarrow \infty$, where A and B range over $B(S)$ and $B(S_n)$, respectively.

By using a remarkable result of Kolmogorov and Rosanov [19] it can be shown that a Gaussian stationary process $\{x_{m,n}; (m,n) \in Z^2\}$ is strongly mixing if and only if $\bar{A} = \{\bar{f}; f \in A\}$ and A_0^n are asymptotically orthogonal in $L^2(d\mu)$, that is, if and only if $\rho_n \rightarrow 0$ as $n \rightarrow \infty$.

Therefore, necessary and sufficient conditions for strong mixing of such processes is obtained by specializing Theorem 4.3.7 to the case when X is the torus and S is any half-plane of lattice points. Thus, the problem of strong mixing of two-parameter Gaussian stationary processes is solved in the spirit of [14].

A slightly different notion of strong mixing and a sufficient condition for strong mixing of such processes is given in [43].

4.4.1 An Open Problem. In this special case i.e. when $X = T^2$, S a fixed half-plane and m a complex homomorphism of $A(S)$ whose Gleason part $G(m)$ is non-trivial, it is important to know whether there exists a complex homomorphism in $G(m)$ such that its corresponding Wermer's embedding function Z shifts the exponentials $e^{i(mx+ny)}$, $(m,n) \in S$, "properly". To make this problem more clear, in Example 4.3.6, the

Wermer's embedding function $Z(e^{ix}, e^{iy}) = e^{iy}$ corresponding to m (the normalized Lebesgue measure on the torus) shifts the desired exponentials along the y -axis, in this case from viewpoint of application to strong mixing problem, it would be more meaningful if we could find a complex homomorphism in $G(m)$ such that its corresponding Wermer's embedding function would shift exponentials along the x -axis or along the line $y = x$.

APPENDIX

APPENDIX

Here, we explain in more detail some of the terminologies related to a Dirichlet algebra.

Throughout this appendix, X will denote a compact Hausdorff space, $C(X)$ will denote the linear space of all continuous complex-valued functions on X . It is well-known that this linear space is a Banach space (Banach algebra) under the sup norm $\|f\| = \sup_{x \in X} |f(x)|$. By a measure on X we mean a finite complex measure on X .

A uniform algebra on X is a complex linear subalgebra A of $C(X)$ which satisfies:

- (i) A is uniformly closed;
- (ii) The constant functions are in A ;
- (iii) A separates the points of X , i.e. if x and y are distinct points of X , there is an f in A with $f(x) \neq f(y)$.

If A is a uniform algebra on X , then a complex homomorphism of A is an algebra homomorphism from A onto the field of complex numbers. Since the uniform algebra A is closed, it is a Banach space (Banach algebra) under the sup norm, it can be shown that each complex homomorphism ϕ is a bounded linear functional on that Banach space.

A representing measure for ϕ is a positive measure m on X such that

$$\phi(f) = \int_X f \, dm, \quad f \in A.$$

Since $\phi(1) = 1$, we have $\int_X dm = 1$, therefore a representing measure for ϕ is a probability measure on X .

For a uniform algebra A , we denote by $M(A)$ the set of all complex homomorphisms of A . With each f in A we associate a complex-valued function $\hat{f}$ (called Gelfand transform of f) on $M(A)$ by

$$\hat{f}(\phi) = \phi(f), \quad \phi \in M(A).$$

If we topologize $M(A)$ with the weakest topology which makes all these functions $\hat{f}$ continuous, then it can be shown that $M(A)$ is a compact Hausdorff space. This space $M(A)$ is known as the space of complex homomorphisms of A or the maximal ideal space of A or the space of multiplicative linear functionals on A .

By Riesz representation theorem, it can be shown that for each complex homomorphism of A , there exists at least one representing measure m on X . To show that this measure m is unique it is necessary to impose more restrictions on A .

A uniform algebra A is called a Dirichlet algebra on X if the real parts of the functions in A are uniformly dense in the space of real continuous functions on X . It can be shown that A is a Dirichlet algebra on X if, and only if $A + \bar{A}$ is uniformly dense in $C(X)$, or, if, and only if, there is no non-zero real measure on X which is orthogonal to A .

For a Dirichlet algebra A , it can be shown that the relation $\phi_1 \sim \phi_2$ defined by $\|\phi_1 - \phi_2\| < 2$ is an equivalence relation on

$M(A)$. The equivalence classes for this relation is called the Gleason parts of $M(A)$. For ϕ a complex homomorphism of A with the unique representing measure m , $G(m)$ the Gleason part of ϕ is defined by

$$G(m) = \{\psi \in M(A); \psi \sim \phi\}.$$

For more information on this subject and proof of the statements made earlier the following paper of K. Hoffman may be consulted (Analytic functions and logmodular Banach algebras, Acta Math., 108 (1962), 271-317).

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