MEASURE RIESZ SPACES AND THE EGOROFF THEOREM

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ABSTRACT

MEASURE RIESZ SPACES
AND THE EGOROFF THEOREM

By

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Throughout the thesis we assume that we are working with vector lattices (Riesz spaces) over the real numbers.

In the first chapter, we study a large number of order convergence properties and their relationships. Of special interest are the following properties the first of which was introduced by W. A. J. Luxemburg and A. C. Zaanen;

<u>Definition</u> 1. Let L be a Riesz space. An element $f \in L$ is said to have the <u>Egoroff property</u> if given any double sequence $\{0 \leq b_{nk};\ n,k = 1,2,\ldots,\}$ in L with $b_{nk} \nearrow_k |f|$ for $n = 1,2,\ldots,$ there exists a sequence $0 \leq b_m \nearrow |f|$ such that, for any $m,n,$ there exists a $k = j(m,n)$ such that $b_m \leq b_{n,j(m,n)}$.

If every element of L has the Egoroff property, then we say that L has the Egoroff property.

<u>Definition</u> 2. We will say that an element $0 \leq e$ of the Riesz space L has <u>property</u> E.T.*, if given any $x_n \searrow 0$ with $x_n \leq e$ for $n = 1,2,\ldots,$ there exists a

sequence of components $\{e_m\}$ of e with $e_m \nearrow e$ such that $e_m \wedge x_n \to_n 0(\text{e-r.u.})$ for $m = 1,2,\ldots,$. (the notation $e_m \wedge x_n \to_n 0(\text{e-r.u.})$ means that there exist sequences of real numbers $\{\lambda_{mn}\}$ with $\lambda_{mn} \searrow_n 0$ for $m = 1,2,\ldots,$ such that $e_m \wedge x_n \leq \lambda_{mn} e$).

If every positive element of L has property E.T.*, then we say that L has <u>property</u> E.T.*.

<u>Definition</u> 3. We will say that an element $0 \leq e$ of the Riesz space L has <u>property</u> E.T., if given any sequence $x_n \searrow 0$ with $x_n \leq e$ for $n = 1,2,\ldots,$ there exists a sequence $\{0 \leq e_m\} \subset L$ with $e_m \nearrow e$ such that $e_m \wedge x_n \to_n 0(\text{e-r.u.})$ for $m = 1,2,\ldots,$.

If every positive element of L has property E.T., then we say that L has <u>property</u> E.T. .

The second and third properties above are abstractions of some theorems (discussed in the thesis) which are analogues of Egoroff's well known result for measure spaces.

Some interesting results involving these properties are summed up in the following:

<u>Theorem</u> 1. Let L be an order separable Archimedean Riesz space. Then, the following are equivalent:

(1) L has property E.T. .

(2) $\hat{L}$ (the Dedekind completion of L) has property E.T.* .

(3) $\hat{L}$ has the Egoroff property.

(4) L has the Egoroff property.

The results of chapter I are used extensively throughout chapter II.

We study, in chapter II, the use of spaces of equivalence classes of measurable functions to represent certain Archimedean Riesz spaces. The first section of chapter II gives the basic information necessary for this investigation. We also obtain in section 1 some interesting characterization theorems which are based on a property related to the Egoroff property. In section 2, we discuss a general embedding problem. In particular, we obtain necessary and sufficient conditions for a Dedekind-σ-complete Riesz space of extended type to be embeddable as a Riesz subspace of some space of measurable functions. In section 3 of chapter II, we consider a number of topological properties which will guarantee that the Riesz space we are dealing with can be considered as an order dense Riesz subspace of some space of measurable functions. In section 4, we show that every locally convex, separable, metrizable, topological vector lattice with the Egoroff property can be so considered. We end chapter II with a discussion of some open problems related to the material discussed chapters I and II.

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Introduction

A real vector space equipped with an order structure that is compatible with its linear structure is called an ordered vector space. If the order structure endows the space with a lattice order, then we call the space a vector lattice or a Riesz space. Throughout this exposition we will be concerned only with Riesz spaces. For the basic definitions and lattice theoretic formulae, we refer the reader to [(25) Chapter I].

In chapter I, we study a large number of order convergence properties and their relationships. For the definition of order convergence see [(25) Chapter I §5]. In the first section of chapter I, we introduce the basic definitions of many of the order convergence properties which we wish to study. Most of these properties were first studied in [(11) Chapter 5], although one, the Egoroff property, was introduced in [(18) Note VI]. Some new results are obtained in section 1, and in general our development is different from any other. Section 2 of chapter I begins a study of the Egoroff theorem for Riesz spaces, so named since it is an analogue of Egoroff's well known result for measure spaces. The first result along these lines is found in [(11) Chapter 5 2.21 p.181]. A more general result than that in [(11)] was obtained in [(19) Thm. 42.2]. Our result, see theorem (2.1), is more general still. In particular we show that it applies to arbitrary sequence spaces. In section 3 we apply the results of section 2 to a class of spaces known as the universally complete spaces. Section 4 begins with a study of a property which we call sub-order separability, and which is related to order separability. We obtain a classification of Archimedean Riesz spaces, see theorem (4.3), in terms of sub-order separability. We go on to obtain an Egoroff type theorem which applies to arbitrary order separable Archimedean Riesz spaces with the Egoroff property. Finally, we abstract the results of our Egoroff type theorems and consider them as properties. We end section 4 by showing that an order separable Archimedean Riesz space has the Egoroff theorem if and only if it has the Egoroff property, see theorem (4.10). The results of chapter I are used extensively throughout chapter II.

We study, in chapter II, the use of spaces of equivalence classes of measurable functions to represent certain Archimedean Riesz spaces. The first section of chapter II gives the basic information necessary for this investigation. We also obtain in section 1 some interesting characterization theorems which are based on a property related to the Egoroff property. In section 2, we discuss a general embedding problem. In particular, we obtain necessary and sufficient conditions for a certain class of

Riesz spaces to be embeddable as Riesz subspaces of some space of measurable functions. In section 3 of chapter II, we consider a number of topological properties which will guarantee that the Riesz space we are dealing with can be considered as an order dense Riesz subspace of some space of measurable functions. In section 4, we show that every locally convex, separable, metrizable topological vector lattice can be so represented. We end chapter II with a discussion of some open problems related to the material discussed in chapters I and II.

We wish to make some comments concerning a number of concepts which we will be making frequent use of. If L is a Riesz space, and $\{x_\alpha\}$ is a net in L, then the notation $x_\alpha \nearrow$ means that the net $\{x_\alpha\}$ is directed up. Similarly, the notation $x_\alpha \searrow$ means that the net $\{x_\alpha\}$ is directed down. The notation $x_\alpha \nearrow x$, where $\{x_\alpha, x\} \subset L$, means that the net $\{x_\alpha\}$ is directed up and $\sup \{x_\alpha\} = x$. Similarly, $x_\alpha \searrow x$ means that the net $\{x_\alpha\}$ is directed down and $\inf \{x_\alpha\} = x$. The notation $x_\alpha \to x$ will mean that the net $\{x_\alpha\}$ order converges to the element x (where by order converges we mean that there exists a net $\{y_\alpha\} \subset L$ with $y_\alpha \searrow 0$ such that for each α we have that $|x_\alpha - x| \leq y_\alpha$). If we wish to talk about convergence with respect to a topology τ on L, then we will write $x_\alpha \xrightarrow{\tau} x$. A Riesz space L is said to be <u>Dedekind complete</u> if given any set $A \subset L$ such that A is bounded above in L (i.e. there exists an element $y \in L$ such that for any $x \in A$ we have that $x \leq y$) then the supremum over A exists in L. Finally, we will say that a set $I \subseteq L$ is <u>order dense</u> in L if given any $x \in L^+$ there exists a net $\{x_\alpha\} \subset I$ such that $\sup \{x_\alpha\} = x$.

For general references to the theory of Riesz spaces, we recommend [(18) Note VI, (19), (25), and (27)].

Chapter I

Certain Order Convergence Properties and Their Relationships

Section 1. Basic Definitions and Relationships

The results of this section will be used throughout this exposition.

We begin with a property that was introduced in [(18) Note VI].

Definition 1.1. Let L be a Riesz space. An element $f \in L$ is said to have the Egoroff property if given any double sequence $\{0 \leq b_{nk};\ n,k = 1,2,\ldots,\}$ in L with $b_{nk} \nearrow_k |f|$ for $n = 1,2,\ldots,$ then there exists a sequence $0 \leq b_m \nearrow |f|$ such that, for any $m,n,$ there exists a $j(m,n)$ such that $b_m \leq b_{n,j(m,n)}$.

Luxemburg and Zaanen have shown, in [(19), Thm. 40.5], the following equivalence:

Lemma 1.A. Let L be a Riesz space. An element $f \in L$ has the Egoroff property if and only if given any double sequence $\{0 \leq b_{nk};\ n,k = 1,2,\ldots,\}$ in L with $b_{nk} \nearrow_k |f|$ for $n = 1,2,\ldots,$ there exists a sequence

$0 \leq b_m \nearrow |f|$ such that, for any m, we have $b_m \leq b_{m,k(m)}$ for an appropriate $k = k(m)$.

As a convenience to the reader we will indicate the proof.

Proof: That the Egoroff property implies this condition is obvious. To see the other direction we replace the original double sequence $\{b_{nk}\}$ by the double sequence $\{v_{nk}\}$ where $v_{nk} = \inf\{b_{1,k}, b_{2,k}, \ldots, b_{n,k}\}$. Applying the if part of the lemma to the double sequence v_{nk}, we obtain a $0 \leq v_m \nearrow |f|$ such that, for any m, we have $v_m \leq v_{m,k(m)}$ for an appropriate $k = k(m)$. Taking $j(m,n) = \begin{cases} k(m) & \text{for } n \leq m, \\ k(n) & \text{for } n \geq m \end{cases}$ it is easy to verify that $v_m \leq b_{n,j(m,n)}$ for any m,n. The proof is complete.

Remark 1.1. It is obvious from the above lemma that an element f of the Riesz space L has the Egoroff property if and only if given any double sequence $\{0 \leq b_{nk};\ n,k = 1,2,\ldots,\}$ in L with $b_{nk} \nearrow_k |f|$ for $n = 1,2,\ldots,$ there exists a diagonal sequence $\{b_{n,k(n)}\}$ such that $b_{n,k(n)} \rightarrow |f|$ for $n = 1,2,\ldots,$ (by diagonal we mean that for each $n = 1,2,\ldots,$ an appropriate $k = k(n)$ can be chosen so that $k(n) < k(n+1)$).

We now introduce a few more definitions of other order convergence properties.

Definition 1.2. We will say that order convergence is stable in the Riesz space L if given any sequence $\{x_n\} \subset L$ with $x_n \searrow 0$ there exists a sequence of real numbers $\{a_n\}$, satisfying $0 \leq a_n \nearrow \infty$, such that $a_n x_n \to 0$, where $n = 1,2,\ldots,$.

Definition 1.3. We will say that the Archimedean Riesz space L is regular if order convergence is stable in L, and in addition we have the property that for any sequence $\{y_n\} \subset L^+$ (where L^+ denotes the positive cone of the Riesz space L) there is a sequence of real numbers $\{0 < a_n\}$ such that the sequence $\{a_n y_n\}$ is bounded in L, where $n = 1,2,\ldots,$.

We will refer to the last property in definition (1.3) as property P^*.

Definition 1.4. A Riesz space L will be said to be diagonalizable (or to have the diagonalization property) if whenever f_{nk}, f_n, f in L are such that $f_{nk} \to_k f_n$ for $n = 1,2,\ldots,$ and $f_n \to f$ there exists a diagonal sequence $\{f_{n,k(n)}\}$ such that $f_{n,k(n)} \to f$, where $n,k = 1,2,\ldots,$.

Definition 1.5. A Riesz space L will be said to be diagonalizable on intervals if whenever f_{nk}, f_n, f in L are such that:

1) there exists a $g \in L^+$ such that $-g \leq f_{nk} \leq g$,

2) $f_{nk} \to_k f_n$ for $n = 1,2,\ldots,$ and $f_n \to f$, then there exists a diagonal sequence $\{f_{n,k(n)}\}$ such that $f_{n,k(n)} \to f$ where $n,k = 1,2,\ldots,$.

Definition 1.6. We will say that the Riesz space L has the Egoroff property if every element of L has the Egoroff property.

Theorem 1.1. The Riesz space L is diagonalizable on intervals if and only if for every double sequence $\{b_{n,k};\ n,k = 1,2,\ldots,\}$ in L satisfying:

1) $b_{nk} \searrow_k 0$ for $n = 1,2,\ldots,$,
2) there exists a $g \in L^+$ such that $b_{nk} \leq g$ for all $n,k = 1,2,\ldots,$

there exists a diagonal sequence $\{b_{n,k(n)}\}$ such that $b_{n,k(n)} \to 0$.

Proof. The necessity is obvious. Now, suppose that f_{nk}, f_n, f in L are such that $\{f_{nk};\ n,k = 1,2,\ldots,\} \subset [-g,g]$ for some $g \in L^+$, $f_{nk} \to_k f_n$ for $n = 1,2,\ldots,$ and $f_n \to f$. Note that $|f_n - f_{nk}| \leq |f_n| + |f_{nk}| \leq 2g$ and $|f_n - f_{nk}| \to 0$ for each $n = 1,2,\ldots,$. Hence there exists for each n a sequence $x_{nk} \searrow_k 0$ such that $|f_n - f_{nk}| \leq x_{nk}$. Taking $z_{nk} = x_{nk} \wedge 2g$ we have $z_{nk} \leq 2g$, $|f_n - f_{nk}| \leq z_{nk}$, and $z_{nk} \searrow_k 0$ for each $n = 1,2,\ldots,$ where $n,k = 1,2,\ldots,$.

By assumption there exists a diagonal sequence $\{z_{n,k(n)}\}$ such that $z_{n,k(n)} \to 0$, but then $|f_n - f_{n,k(n)}| \to 0$, and since $|f - f_{n,k(n)}| \leq |f - f_n| + |f_n - f_{n,k(n)}|$, we have $f_{n,k(n)} \to f$. This completes the proof.

We will refer to the condition in the last part of theorem (1.1) as the <u>weak diagonalization property on intervals</u>.

<u>Remark</u> 1.2. It is obvious that the same proof, only simplified by the fact that we would not have to concern ourselves with bounding the sequences $\{x_{nk}\}$, will prove the analogous result for diagonalizability. We thus have the following corollary.

<u>Corollary</u> 1.1. The Riesz space L is diagonalizable if and only if for every double sequence $\{b_{nk}; n,k = 1,2,...,\}$ in L satisfying $b_{nk} \searrow_k 0$ for each $n = 1,2,...,$ there exists a diagonal sequence $\{b_{n,k(n)}\}$ such that $b_{n,k(n)} \to 0$.

We will refer to the condition in the last part of corollary (1.1) as the <u>weak diagonalization property</u>.

Clearly, the property, given any $f \in L$ and double sequence $\{b_{nk}; n,k = 1,2,...,\}$ in L with $b_{nk} \nearrow_k |f|$ for $n = 1,2,...,$ there exists a diagonal sequence $\{b_{n,k(n)}\}$ such that $b_{n,k(n)} \to |f|$, is equivalent to the weak diagonalization property on intervals. By remark (1.1) the first property is also equivalent to L having the Egoroff property. Hence, we have the following corollary.

Corollary 1.2. Let L be a Riesz space. Then the following are equivalent.

1) L has the Egoroff property.

2) L has the weak diagonalization property on intervals.

3) L has the diagonalization property on intervals.

The implication (1) $\Leftrightarrow$ 3)) of corollary (1.2) was proved in [(19) Thm. (41.2)]. The proof contained there does not use the weak diagonalization property on intervals. In fact, this property was not considered there.

Theorem 1.2. An Archimedean Riesz space L has the weak diagonalization property if and only if L is regular.

Proof. Suppose $\{y_n\} \subset L^+$ and let $z_{nk} = k^{-1}y_n$ $(k,n = 1,2,\ldots,)$. Then $z_{nk} \searrow_k 0$ for each $n = 1,2,\ldots,$ since L is Archimedean, and hence there exists a diagonal sequence $\{z_{n,k(n)}\}$ such that $z_{n,k(n)} \to 0$. But then the set $\{k(n)^{-1}y_n\}$ is bounded in L and we have property P^*. Furthermore, if $y_k \searrow 0$ in L then $ny_k \searrow_k 0$ for each $n = 1,2,\ldots,$ and there exists a diagonal sequence

$\{ny_{k(n)}\}$ such that $ny_{k(n)} \to 0$ where $n,k = 1,2,\ldots,$.
Taking $\lambda_m = 1$ for $m = 1,2,\ldots,k(1)$

$$\lambda_{m+k(1)} = 2 \quad \text{for} \quad m = 1,2,\ldots, k(2)-k(1)$$

$$\ldots = \ldots\ldots\ldots\ldots$$

$$\vdots = \ldots\ldots\ldots\ldots$$

$$\lambda_{m+k(n)} = n \quad \text{for} \quad m = 1,2,\ldots,k(n+1) - k(n).$$

We have that $\lambda_n \nearrow \infty$ and $\lambda_n y_n \to 0$. Hence L is regular.

On the other hand suppose L is regular and let $b_{nk} \searrow_k 0$ in L for each $(n = 1,2,\ldots)$, where $n,k = 1,2,\ldots,$. Then, for each n, there exists a sequence of reals $\{0 \leq \lambda_{nk}\}$ such that $\lambda_{nk} b_{nk} \to_k 0$ and $\lambda_{nk} \nearrow_k \infty$. This implies that for each n there exists a $y_n \in L^+$ such that $\lambda_{nk} b_{nk} \leq y_n$. By property P^* there exists a sequence of real numbers $\{\beta_n > 0\}$ such that $\beta_n y_n \leq z$ for all $n = 1,2,\ldots,$ where $z \in L^+$. We have $\beta_n \lambda_{nk} b_{nk} \leq z$ for all $n,k = 1,2,\ldots,$. For each n we can find a $k(n)$ such that $k(n) < k(n+1)$ and $\frac{n}{\beta_n} \leq \lambda_{n,k(n)}$. But then $nb_{n,k(n)} \leq z$ and, since L is Archimedean, this implies that $b_{n,k(n)} \to 0$. The theorem is proved.

As a result of corollary (1.1) and the above theorem (1.2) we have the following corollary.

Corollary 1.3. Let L be an Archimedean Riesz space, then L is regular if and only if L is diagonalizable.

Corollary (1.3) above was proved in [(28) Thm. (6.2)]. The weak diagonalization property was not mentioned or used in the proof found there.

Definition 1.7. Let L be a Riesz space. We will say that a sequence $\{x_n\} \subset L$ converges relatively uniformly to an element $x \in L$, written $x_n \to x$ (r.u.), if there exists an element $e \in L^+$ and a sequence $\{\lambda_n\}$ of real numbers decreasing to zero such that $|x - x_n| \leq \lambda_n e$ for each n, where $n = 1,2,\ldots,$. The element e is called the regulator of convergence for the sequence $\{x_n\}$.

If $e \in L^+$ is the regulator of convergence for a sequence $\{x_n\}$ converging relatively uniformly in L to an element $x \in L$, we will write $x_n \to x$ (e - r.u.).

For arbitrary Riesz spaces the concepts of relatively uniform convergence and order convergence, for sequences, need not be related. However, if the Riesz space is Archimedean then quite obviously relatively uniform convergence implies order convergence, for sequences.

Remark 1.2. In Archimedean Riesz spaces, order convergence, for sequences, and relatively uniform convergence are equivalent if and only if order convergence is stable. For the proof we refer the reader to [(28) Thm. (2.1)]. The proof is not difficult and we will not produce it here.

A linear isomorphism φ taking a vector lattice L into a vector lattice E is called a *lattice* isomorphism if given any $x_1, x_2 \in L$ we have $\varphi(x_1 \wedge x_2) = \varphi(x_1) \wedge \varphi(x_2)$ - in - E. If φ is a lattice isomorphism taking a Riesz space L into a Riesz space E, we say that L is embedded in E as a Riesz subspace.

We recall now that a complete Riesz space $\hat{L}$ is said to be a *Dedekind completion* of a Riesz space L if it has the following properties:

a) there exists a lattice isomorphism φ taking L into $\hat{L}$,

b) given any element $\hat{x} \in \hat{L}$,
$\sup\{\varphi(x) : x \in L \text{ and } \varphi(x) \leq \hat{x}\} = \hat{x} =$
$\inf\{\varphi(x) : x \in L \text{ and } \varphi(x) \geq \hat{x}\}$.

It is well known that a Riesz space L has a Dedekind completion if and only if L is Archimedean. [c.f. (27) Thm. (IV.11.1) p.109].

It was shown in [(16) Lemma (32.3)] that the condition (b) above can be replaced by the following condition.

b') For every $\hat{x} \in \hat{L}^+$, $\hat{x} > 0$, there exist $x, y \in L^+$ such that $0 < \varphi(x) \leq \hat{x} \leq \varphi(y)$.

Another well known result is that all Dedekind completions of Archimedean Riesz spaces are lattice isomorphic. Hence, it makes sense to talk about <u>the</u> <u>Dedekind completion</u> of an Archimedean Riesz space. In the sequel, we will make no distinction between L and its lattice isomorphic image in its Dedekind completion.

Another important property which a Riesz space can possess is introduced in the following definition.

<u>Definition</u> 1.8. A Riesz space L is said to be <u>order separable</u> if given any set $A \subseteq L$ such that $\sup A$ exists in L, there exists a countable set $A' \subseteq A$ such that $\sup A' = \sup A$.

<u>Remark</u> 1.3. Masterson and Crofts have shown in [(21)] that an Archimedean Riesz space L is order separable if and only if its Dedekind completion $\hat{L}$ is order separable. They also showed that, providing L is Archimedean and order separable, if $\hat{x}_n \searrow 0$-in-$\hat{L}$ then there exists $x_n \searrow 0$-in-L such that $x_n \geq \hat{x}_n$ for each $n = 1,2,\ldots,$.

Using remark (1.3), the following proposition follows easily.

<u>Proposition</u> 1.1. Let L be an Archimedean order separable Riesz space. Then, the following implications are valid.

a) L has property P^* if and only if $\hat{L}$ has property P^*.

b) L is stable if and only if $\hat{L}$ is stable.

c) L is regular if and only if $\hat{L}$ is regular.

d) L is diagonalizable if and only if $\hat{L}$ is diagonalizable.

e) Order convergence in L is equivalent to relatively uniform convergence in L if and only if order convergence in $\hat{L}$ is equivalent to relatively uniform convergence in $\hat{L}$.

Proof. Implication a) is trivially valid for Archimedean Riesz spaces, without the assumption of order separability. By virtue of corollary (1.3) the validity of d) will follow from the validity of c). By remark (1.2) the validity of e) is equivalent to the validity of b). Since a) is valid, the validity of c) is seen to rest upon the validity of b). It remains only to show that b) is valid, but this is trivially so in view of remark (1.3).

Parts d) and e) of the above proposition were proved in [(21)]. The other parts of the proposition were not considered there.

We will now establish the corresponding result for the Egoroff property. This result will be used frequently throughout this exposition.

<u>Theorem</u> 1.3. Let L be an Archimedean order separable Riesz space, then L has the Egoroff property if and only if $\hat{L}$ has the Egoroff property

<u>Proof</u>. By virtue of corollary (1.2) proving this theorem is equivalent to showing that L has the weak diagonalization property on intervals if and only if $\hat{L}$ does.

Suppose $\hat{L}$ has the weak diagonalization property on intervals. Let $\{b_{nk};\ n,k = 1,2,\ldots,\}$ be a double sequence in L with $0 \leq b_{nk} \leq b \in L$ for all $n,k = 1,2,\ldots,$ and such that $b_{nk} \downarrow_k 0$ (in L) for each $n = 1,2,\ldots,$. Then $b_{nk} \downarrow_k 0$ (in $\hat{L}$) for each $n = 1,2,\ldots,$ and hence there exists a diagonal sequence $\{b_{n,k(n)}\}$ such that $b_{n,k(n)} \to 0$ (in $\hat{L}$). By virtue of remark (1.3) this implies that $b_{n,k(n)} \to 0$ (in L) and we obtain the result that L has the weak diagonalization property on intervals.

Suppose now that L has the weak diagonalization property on intervals. Let $\{\hat{b}_{nk};\ n,k = 1,2,\ldots,\}$ be a double sequence in $\hat{L}$ with $0 \leq \hat{b}_{nk} \leq \hat{b} \in \hat{L}$ for all $n,k = 1,2,\ldots,$ and such that $\hat{b}_{nk} \downarrow_k 0$ (in $\hat{L}$) for each $n = 1,2,\ldots,$. Choose a $b \in L$ such that $b \geq \hat{b}$. Using remark (1.3), there exist sequences $\{b_{nk}\} \subset L$ such that $b_{nk} \downarrow_k 0$ (in L) for each $n = 1,2,\ldots,$ and $b_{nk} \geq \hat{b}_{nk}$

for all $n,k = 1,2,\ldots,$. Taking $g_{nk} = b_{nk} \wedge b$, we see that $g_{nk} \searrow_k 0$ (in L) for each $n = 1,2,\ldots,$ $g_{nk} \le b$ for all $n,k = 1,2,\ldots,$ and $g_{nk} \ge \hat{b}_{nk}$ for all $n,k = 1,2,\ldots,$. But then there exists a diagonal sequence $\{g_{n,k(n)}\}$ such that $g_{n,k(n)} \to 0$ (in L) which implies that $g_{n,k(n)} \to 0$ (in $\hat{L}$). Finally, this implies that $\hat{b}_{n,k(n)} \to 0$ (in $\hat{L}$). The proof is complete.

Our next results indicate in what manner some of the simplifications contained in this section can be put to use. We introduce another apparently weaker diagonalization property.

Definition 1.9. We will say that the Riesz space L has the sub-diagonalization property if given any double sequence $\{x_{nk};\ n,k = 1,2,\ldots,\}$ in L with $x_{nk} \searrow_k 0$ for each $n = 1,2,\ldots,$ there exists a sub double sequence $\{x_{n(j)k};\ j,k = 1,2,\ldots,\}$, where the $n = n(j)$ can be chosen so that $n(j) < n(j+1)$, for which $\{x_{n(j)k}\}$ is diagonalizable.

Theorem 1.4. Let L be an Archimedean Riesz space. Then L is sub-diagonalizable if and only if L is diagonalizable.

Proof. The sufficiency is obvious. For the necessity, by virtue of corollary (1.1) and theorem (1.2), it is sufficient to show that L being sub-diagonalizable implies that L is regular. Let $\{x_n; n = 1,2,\ldots,\}$ be any sequence in L^+. Consider the sequence $\{y_n\}$ where $y_n = \sup_{1 \leq i \leq n} \{x_i\}$ for $n = 1,2,\ldots,$. Clearly y_n is directed up. For each $n = 1,2,\ldots, k^{-1}y_n \searrow_k 0$. So, there exist $n(j)$, $n(j) < n(j+1)$, such that $\{k^{-1}y_{n(j)}\}$ is diagonalizable. Let $k(j)^{-1}y_{n(j)}$ be the appropriate diagonal sequence such that $k(j)^{-1}y_{n(j)} \to 0$. The set $\{k(j)^{-1}y_{n(j)}\}$ is bounded in L. For each $n = 1,2,\ldots,$ we define $\lambda_n = k(1)^{-1}$ for $1 \leq n \leq n(1),\ldots,\lambda_n = k(j)^{-1}$ for $n(j-1) < n \leq n(j),\ldots,$. Then $\{\lambda_n y_n\}$ is bounded in L. Since $x_n \leq y_n$ for each $n = 1,2,\ldots,$ we have $\{\lambda_n x_n\}$ is bounded in L and hence L has property P^*.

Now let $y_k \searrow 0$ where $k = 1,2,\ldots,$. Then $ny_k \searrow_k 0$ for each $n = 1,2,\ldots,$. Thus, there exist $n(j)$, $n(j) < n(j+1)$, such that $\{n(j)y_k\}$ is diagonalizable, where $j = 1,2,\ldots,$. Proceding exactly as in the proof of theorem (1.2) we obtain that order convergence is stable in L. The theorem is proved.

A sequence $\{x_n; n = 1,2,\ldots,\}$ in a Riesz space L is said to be <u>order</u> <u>*-convergent</u> to an element $x \in L$ if every subsequence of $\{x_n\}$ has a subsequence which order converges to x in L.

Order $*$-convergence has been shown to be the convergence associated with one of the intrinsically defined topologies on certain vector lattices [c.f. (27) §3 Chapter VI]. Peressini has shown [(25) Prop. (5.6) p. 47] that if E is a Riesz space with the diagonalization property then the set mapping $A \to \bar{A}$ defined for subsets A of E by $\bar{A} = \{x \in E : \{x_n\}$ order converges to x for some $\{x_n\} \subset A\}$ is a closure operator on E. He further showed that if J is the unique topology on E determined by this closure operator then J-convergence is equivalent to order $*$-convergence; for sequences. Our next result establishes the converse of this result for Archimedean Riesz spaces.

<u>Theorem</u> 1.5. Let E be an Archimedean Riesz space. If the set mapping $A \to \bar{A}$ defined for subsets A of E by $\bar{A} = \{x \in E : \{x_n\}$ order converges to x for some $\{x_n\} \subset A\}$ is a closure operator on E, then E is diagonalizable.

<u>Proof</u>. We will show that E must be sub-diagonalizable. Suppose that $\{x_{nk}; n,k = 1,2,\ldots,\}$ is a double sequence in E with $x_{nk} \searrow_k 0$ for each $n = 1,2,\ldots,$. Let $0 < e \in E$. Consider the set $A = \{x_{nk} + \frac{1}{n} e\}$. It is evident that $\{\frac{1}{n} e\} \subset \bar{A}$ and that $0 \in \bar{\bar{A}}$. By assumption

there exists a sequence $\{y_j\} \subset A$ such that $y_j \to 0$, where $j = 1,2,\ldots,$. For each j, y_j is of the form $y_j = x_{n(j),k(j)} + \frac{1}{n(j)} e$ for appropriate $n = n(j)$ $k = k(j)$. If $n(j) \leq M < \infty$ then $x_{n(j),k(j)} + \frac{1}{n(j)} e \geq \frac{1}{M} e > 0$ for all $j = 1,2,\ldots,$ and this contradicts that $x_{n(j),k(j)} + \frac{1}{n(j)} e \to 0$. Hence, we may assume that $n(j) < n(j+1)$. Since $x_{n(j)k} \searrow_k 0$ for each j, we may assume that $k(j) < k(j+1)$. The condition $x_{n(j),k(j)} + \frac{1}{n(j)} e \to 0$ implies that for some $z_j \searrow 0$ in L we have $x_{n(j),k(j)} + \frac{1}{n(j)} e \leq z_j$. This implies $x_{n(j),k(j)} \leq z_j$ and hence $x_{n(j),k(j)} \to 0$. We have shown that the double sequence $\{x_{n(j)k};\ j,k = 1,2,\ldots,\}$ is diagonalizable. Thus, E is sub-diagonalizable. By application of theorem (1.4) we see that E is diagonalizable. The proof is complete.

<u>Examples</u> 1.1. a) Let L be the Riesz space of all finitely non-zero real sequences $\{x_n\}$. This space is easily seen to be Archimedean. Since order convergence is point wise convergence, it is not hard to check that order convergence is stable in L. The set $\{e_n;\ n = 1,2,\ldots,\}$, where $e_n = \{x_i^n\}$ and $x_i^n = 0$ for $i \neq n$, $x_i^n = 1$ for $i = n$, shows that L does not have property P^*. Thus L is not diagonalizable.

b). Let L be ℓ^∞ (i.e. the space of bounded real sequences). Since L is bounded it clearly has property P^*. The sequence $x_n \searrow 0$, where $x_n = \{y_n^m\}$ and $y_n^m = 0$ for $1 \leq m < n$, $y_n^m = 1$ for $n \leq m$, does not converge to 0 relatively uniformly. Therefore, since L is Archimedean, order convergence is not stable in L and L does not have the diagonalization property.

For $n,k,\ i = 1,2,\ldots,$ let the sequence $x_{nk} = \{y_{nk}^i\}$ and $x = \{y_i\}$ be in L and satisfy $0 \leq x_{nk} \nearrow_k x$ for each $n = 1,2,\ldots,$. For $n=1$, choose $k = k(1)$ so that $y_{1,k(1)}^1 > y_1 - \frac{1}{2} y_1$ and let $z_1 = (y_1 - \frac{1}{2} y_1, 0,\ldots,0,\ldots)$. For $n=2$, choose $k = k(2) > k(1)$ so that $y_{2,k(2)}^1 \geq y_1 - \frac{1}{2^2} y_1$, $y_{2,k(2)}^2 \geq y_2 - \frac{1}{2^2} y_2$, and let $z_2 = (y_1 - \frac{1}{2^2} y_1, y_2 - \frac{1}{2^2} y_2, 0,\ldots,0,\ldots)$. Proceding in the indicated manner we obtain a diagonal sequence $\{x_{n,k(n)}\}$ and a sequence $\{z_n\}$ such that $0 \leq z_n \leq x_{n,k(n)}$ for each $n = 1,2,\ldots,$. Clearly $z_n \nearrow x$ and hence $x_{n,k(n)} \to x$. Thus, L has the Egoroff property.

c). The Riesz space $C[0,1]$ of all real continuous functions on the interval $[0,1]$ does not have the Egoroff property. For a direct proof we refer the reader to [(7) Ex. (4.1) p. 72]. An indirect proof is given in section 1 of chapter II page 115.

d) Let L be Euclidean two space with the lexicographical ordering (i.e. $(a,b) \leq (c,d)$ if either $a \leq c$ or $a = c$ and $b \leq d$). This space is non-Archimedean. Its positive cone consists of the entire half plane with the exception of the negative y-axis. It is not hard to see that L is regular. However, the sequence $(\frac{1}{n}, \frac{1}{n})$ converges relatively uniformly to 0 but does not order converge to 0.

Section 2. The Egoroff Theorem and Relatively Uniform Convergence

A number of the properties considered in the previous section can be reduced, in certain spaces, to properties defined only on the components of fixed elements (an element f in a Riesz space L is a component of an element $g \in L^+$ if $f \wedge (g-f) = 0$).

As a result of these consideration a number of results obtained by Luxemburg and Zaanen in [(19), Chapter 7] can be shown to hold under more general hypothesis than stated there.

The results of this section require a restriction of the types of Riesz spaces considered. We in general will need some type of a projection property. We, therefore, will review now some of the pertinent definitions and results, which are associated with the theory of projections in Riesz spaces. For a complete discussion of this material we refer the reader to [(19), Chapt's. 2 and 3].

Let L be a Riesz space, and let L' be a subspace of L. L' is called a <u>Riesz subspace</u> of L if under the induced partial ordering from L we have that L' is a partially ordered vector space, and for any $x, y \in L'$ we have $x \wedge y$ and $x \vee y$ are in L'.

The Riesz subspace L' of L is said to be an <u>ideal</u> if for all $g_1, g_2 \in L'$ and $f \in L$ such that $g_1 \leq f \leq g_2$ we have $f \in L'$.

An ideal L' in L is called a <u>normal ideal</u>, <u>closed ideal</u>, or <u>band</u> if given any net $\{g_\tau\} \subset (L')^+$ with $g_\tau \nearrow$ and $\sup_\tau\{g_\tau\} = g$ exists in L we have $g \in L'$.

Let A be any subset of L, then $A^\perp = \{g \in L;\ |g| \wedge |x| = 0$ for all $x \in A\}$. It is easy to show that $A^\perp$ is in general a band in L.

A band B in L is said to be a <u>projection band</u> if given any $u \in L^+$ we have that $\sup\{v;\ v \in B \text{ and } v \leq u\}$ exists. In this case $L = B \oplus B^\perp$.

If $L = B_1 \oplus B_2$ where B_1 and B_2 are bands in L, then $B_1 = B_2^\perp = B_1^{\perp\perp} = (B_1^\perp)^\perp$ and B_1, B_2 are projection bands.

If B is a projection band in L, then for every $u \in L$ there exists $u_1, u_2 \in L$ such that $u_1 + u_2 = u$ and $u_1 \in B$, $u_2 \in B^{\perp}$. The element u_1 is called the <u>component of u in the projection band B</u>.

If P_B is the operator on L which takes every element of L to its component in the projection band B, then P_B is a Riesz homomorphism.

If $B = B_b = \{b\}^{\perp\perp}$ where $b \in L$ then B is called the <u>principal band generated by the element</u> $b \in L$.

A Riesz space L is said to have the <u>projection property</u> (P.P.) if every band in L has the projection property.

A Riesz space L is said to have the <u>principal projection property</u> (P.P.P.) if every principal band in L has the projection property.

By $A(L)$, $A_p(L)$, $P(L)$, $P_p(L)$; we will denote the set of all bands, principal bands, projection bands, principal projection bands, respectively, in the Riesz space L. $A(L)$ forms a distributive lattice with unit when ordered by inclusion. If L is Archimedean then $A(L)$ is also a Boolean Algebra. $P(L)$ always forms a Boolean algebra ordered by inclusion. An order convergence can thus be defined on $P(L)$ and $A(L)$ in a natural manner.

It is well known that if L is Dedekind complete (respectively Dedekind-σ-complete) then L has the P.P. (respectively P.P.P.).

We are now ready to proceed.

Definition 2.1. Let L be a Riesz space. We will say that the element $f \in L^+$ has the Egoroff property on its components if given any double sequence $\{f_{nk};\ n,k = 1,2,\ldots,\}$ in L with $f_{nk} \nearrow_k f$ for each $n = 1,2,\ldots,$ and $f_{nk} \wedge (f - f_{nk}) = 0$ for all $n,k = 1,2,\ldots,$ there exists an $f_m \nearrow f$ with $f_m \wedge (f - f_m) = 0$ for $m = 1,2,\ldots,$ such that $f_m \leq f_{m,k(m)}$ for some suitable choice of $k = k(m)$.

Definition 2.2. We will say that $P_p(L)$ is super order dense in $A_p(L)$, with respect to the order on $A(L)$, if given any $B \in A_p(L)$ there exists a sequence $\{B_n\} \subset P_p(L)$ with $B_n \nearrow B$, where $n = 1,2,\ldots,$.

Lemma 2.1. Let L be a Riesz space with the property that $P_p(L)$ is super order dense in $A_p(L)$. Then, the element $f \in L^+$ has the Egoroff property on its components if and only if for every double sequence $f_{nk} \nearrow_k f$ for each $n = 1,2,\ldots,$ with $f_{nk} \wedge (f - f_{nk}) = 0$ for all $n,k = 1,2,\ldots,$ there exists a diagonal sequence $\{f_{n,k(n)}\}$ such that $f_{n,k(n)} \to f$.

Proof. The necessity is obvious. We prove the sufficiency. Let $\{f_{nk};\ n,k = 1,2,\ldots,\}$ be a double sequence of components of an element f in L with $f_{nk} \nearrow_k f$ for each $n = 1,2,\ldots,$. Applying the if part of the lemma, there exists a diagonal sequence $\{f_{n,k(n)}\}$ such that $f_{n,k(n)} \to f$. Then there exists $x_n \searrow 0$ in L such that $f - f_{n,k(n)} \leq x_n$ for each $n = 1,2,\ldots,$. Consider $(f - x_n)^+ \nearrow f$, we have $(f - x_n)^+ \leq f_{n,k(n)}$ for each $n = 1,2,\ldots,$. The principal bands $B_{(f-x_n)^+} = B_n$ satisfy $B_n \nearrow B_f$. By assumption there exists for each n a sequence $\{B_{nm}\} \subset P_p(L)$ such that $B_{nm} \nearrow_m B_n$. Consider, then, the principal projection bands $G_n = \sup\limits_{1 \leq i \leq n} \{B_{i,n}\}$. We have $G_n \nearrow B_f$ and for each $n = 1,2,\ldots,$ $G_n \subseteq B_n$. Hence $P_{G_n}(f) \leq f_{n,k(n)}$, and $P_{G_n}(f) \nearrow f$. This completes the proof.

Luxemburg and Zaanen have introduced the concept of an Egoroff property for arbitrary Boolean algebras [c.f. (19) Defn. 42.2]. We will not be concerned here with this general concept, but we are interested in such a property for certain elements of the Boolean algebra $P(L)$. We make the following definition.

Definition 2.3. Let L be a Riesz space with the P.P.P. An element $B \in P_p(L)$ will be said to have the Egoroff property if for any double sequence $\{B_{nk};\ n,k = 1,2,\ldots,\}$ in $P_p(L)$ with $B_{nk} \nearrow_k B$ for each $n = 1,2,\ldots,$ there exists a sequence $\{B_m;\ m = 1,2,\ldots,\}$ in $P_p(L)$ such that $B_m \nearrow B$ and such that, for any m,n there exists a suitable $k = j(m,n)$ such that $B_m \leq B_{n,j(m,n)}$.

Remark 2.1. H. Nakano introduced in [(23), §14] the property of total continuity for a Riesz space L with the P.P.P. . His definition reduces to: "L has the P.P.P. and every $B \in P_p(L)$ has the Egoroff property."

Remark 2.2. In [(19) Thm. 42.3] it was shown that the element $B \in P_p(L)$ has the Egoroff property if and only if for any double sequence $\{B_{nk};\ n,k = 1,2,\ldots,\}$ in $P_p(L)$ with $B_{nk} \nearrow_k B$ for each $n = 1,2,\ldots,$ there exists a sequence $B_m \nearrow B$ such that, for any m, we have $B_m \subseteq B_{m,k(m)}$ for an appropriate $k = k(m)$. The proof is exactly analogous to that of lemma (1.A).

Definition 2.4. We will say that the Riesz space L has the Egoroff property on components if every $f \in L^+$ has the Egoroff property on its components.

Our next theorem is an Egoroff type theorem since it is an analogue for Riesz spaces of Egoroff's well known result for measure spaces. The first result along these lines was obtained by Kantorovich, Vulich, and Pinsker in [(11), 2.21 **p.** 181]. A more general result than that in [(11)] is found in [(19) Thm. 42.6]. The theorem we are about to prove is more general still, which we will demonstrate by example later on.

For purposes of comparison we will first state the result in [(19)].

<u>Theorem</u> A. (Luxemburg and Zaanen) Let L be a Riesz space with the P.P.P., and assume every element of $P_p(L)$ has the Egoroff property. Fix $e \in L^+$, then if $x_n \searrow 0$ there exists $e_m \nearrow e$ such that $P_{e_m}(x_n) \to_n 0$ (e-r.u.)

The greater generality of our result is derived from the relaxing of the hypothesis P.P.P. to $P_p(L)$ super order dense in $A_p(L)$. In particular, as will be shown later on, this will imply the validity of the theorem for any sequence space.

<u>Theorem</u> 2.1. Let L be a Riesz space for which $P_p(L)$ is super order dense in $A_p(L)$, and such that L has the Egoroff property on components. Then if

$x_n \searrow 0$ in L and e is a fixed element in L^+, there exists a sequence of projection elements $e_m \nearrow e$ (i.e. B_{e_m} is a projection band for each $m = 1,2,\ldots,$), $e_m \wedge (e - e_m) = 0$ for each m, such that $P_{e_m}(x_n) \to_n 0$ $(e - \text{r.u.})$.

Proof. Let $x_n \searrow 0$ in L and consider $e_{nk} = (n^{-1}e - x_k)^+$ then $0 \leq e_{nk} \nearrow_k n^{-1}e$ and $B_{e_{nk}} \nearrow_k B_e$. In the proof of lemma (2.1) we saw that if $\{B_n\} \subset A_p(L)$ and $B_n \nearrow B \in A_p(L)$ then there exists $\{G_n\} \subseteq P_p(L)$ with $G_n \subset B_n$ and $G_n \nearrow B$ provided $P_p(L)$ is super order dense in $A_p(L)$. Hence, for each n there exists a sequence $\{B_{nk}\} \subset P_p(L)$ such that $B_{nk} \subset B_{e_{nk}}$ for each $n,k = 1,2,\ldots,$ and $B_{nk} \nearrow_k B_e$ for each $n = 1,2,\ldots,$. Let $z_{nk} = P_{B_{nk}}(e)$, then $z_{nk} \wedge (e - z_{nk}) = 0$ for each $n,k = 1,2,\ldots,$ and $z_{nk} \nearrow_k e$. By virtue of the Egoroff property on components and lemma (2.1) we now have an $e_m \nearrow e$ with $e_m \wedge (e - e_m) = 0$ such that, for any m, we have $e_m \leq z_{m,k(m)}$ for an appropriate $k = k(m)$. Also we have $B_{e_m} \nearrow B_e$ and as before there exists $\{B_m\} \subset P_p(L)$ such that $B_m \subset B_{e_m}$ and $B_m \nearrow B_e$. Letting $e'_m = P_{B_m}(e)$ we have $e'_m \leq e_m$ for each $m = 1,2,\ldots,$ and $e'_m \nearrow e$. But then fixing e'_{m_0} we have that, for $m \geq m_0$, $P_{e'_{m_0}}(x_{k(m)}) \leq P_{e'_m}(x_{k(m)}) \leq n^{-1}e$ and we see that the subsequence $\{P_{e'_{m_0}}(x_{k(m)})\}$ of the monotone decreasing sequence $\{P_{e'_{m_0}}(x_k)\}$ satisfies $P_{e'_{m_0}}(x_{k(m)}) \to 0$ (e - r.u.). But then $P_{e'_{m_0}}(x_k) \to_k 0$ (e - r.u.). The theorem is proved.

Theorem 2.2. Let L be a Riesz space with the P.P.P., then an element $B \in P_p(L)$ has the Egoroff property if and only if there exists an order unit $0 < e$ for B (i.e. $e \in L^+$ and $B_e = B$) such that e has the Egoroff property on its components.

Proof. Let $B = B_e$ and assume that e has the Egoroff property on its components. Let $\{B_{nk};\ n,k = 1,2,\ldots,\}$ be a double sequence in $P_p(L)$ with $B_{nk} \nearrow_k B$ for each $n = 1,2,\ldots,$. Then the components e_{nk} of e in B_{nk} satisfy $e_{nk} \nearrow_k e$ for each $n = 1,2,\ldots,$. Since e has the Egoroff property on its components, there exists a sequence $e_m \nearrow e$ with $e_m \wedge (e - e_m) = 0$ for each $m = 1,2,\ldots,$ such that, for any m, we have $e_m \le e_{m,k(m)}$ for suitable $k = k(m)$. But then $B_{e_m} \subseteq B_{e_{m,k(m)}} = B_{m,k(m)}$ and $B_{e_m} \nearrow B = B_e$. By virtue of remark (2.2) the sufficiency is proved.

Conversely assume that the element $B \in P_p(L)$ has the Egoroff property. Let $e > 0$ be any order unit for B and suppose that $\{e_{nk};\ n,k = 1,2,\ldots,\}$ is a double sequence of components of e with $e_{nk} \nearrow_k e$ for each

$n = 1,2,\ldots,$. Then $B_{e_{nk}} \nearrow_k B_e$ and since $B = B_e$ has the Egoroff property there exists $B_m \nearrow B$ such that for any m, $B_m \subseteq B_{m,k(m)} = B_{e_{m,k(m)}}$ for suitable choice of $k = k(m)$. If e_m is the component of e in B_m, we have $e_m \nearrow e$ and for any m, $e_m \leq e_{m,k(m)}$. This completes the proof.

We can now obtain Theorem A stated above as an immediate corollary to theorems (2.1) and (2.2).

Remark 2.3. We will gather here some information concerning components of elements in a Riesz space, which will be needed in the sequel. Let L be a Riesz space and let f, e_1, and e_2 be elements of L^+. Then the following assertions are valid:

1) If e_1 and e_2 are components of f then $e_1 \vee e_2$ is a component of f.

2) If e_2 is a component of f and e_1 is a component of e_2 then e_1 is a component of f.

3) If $e_1 \leq e_2$ and e_1, e_2 are components of f then e_1 is a component of e_2.

4) If e_1, e_2 are components of f then $e_1 \wedge e_2$ is a component of f.

Proof. 1). We note first that $e_1 \vee e_2 \geq e_i$ $(i = 1,2)$, hence $f - (e_1 \vee e_2) \leq f - e_i$ $(i = 1,2)$. So $(e_1 \vee e_2) \wedge [f - (e_1 \vee e_2)] = (e_1 \wedge [f - (e_1 \vee e_2)]) \vee (e_2 \wedge [f - (e_1 \vee e_2)]) \leq (e_1 \wedge (f - e_1)) \vee (e_2 \wedge (f - e_2)) = = 0 \vee 0 = 0$. This proves part 1).

2). We note first that $f - e_1 = f - e_2 + e_2 - e_1$. Hence, $e_1 \wedge (f - e_1) = e_1 \wedge [(f - e_2) + (e_2 - e_1)]$ and by [(25), Prop. (1.2) (19) pg. 6] we have that $e_1 \wedge [(f - e_2) + (e_2 - e_1)] \leq \leq e_1 \wedge (f - e_2) + e_1 \wedge (e_2 - e_1) \leq e_2 \wedge (f - e_2) + e_1 \wedge (e_2 - e_1) = 0$. This proves part 2).

3). Since $e_1 \leq e_2$, we have $e_2 - e_1 \leq f - e_1$. Hence, $e_1 \wedge (e_2 - e_1) \leq e_1 \wedge (f - e_1) = 0$. This proves part 3).

4). We note first that $(e_1 \wedge e_2) \wedge [e_1 - (e_1 \wedge e_2)] \leq e_2 \wedge [e_1 - (e_1 \wedge e_2)]$. By a well known vector lattice identity $e_1 + e_2 = (e_1 \vee e_2) + (e_1 \wedge e_2)$ [c.f. (25) (5) pg. 4]. Hence, $e_1 - (e_1 \wedge e_2) = (e_1 \vee e_2) - e_2$ and $e_2 \wedge [e_1 - (e_1 \wedge e_2)] = e_2 \wedge [(e_1 \vee e_2) - e_2]$. By parts 1) and 3) $e_2 \wedge [(e_1 \vee e_2) - e_2] = 0$. This proves part 4). All parts of the proposition have been proved.

Remark 2.4. It is not difficult to see that if L is a Riesz space with the property that $P_p(L)$ is super order dense in $A_p(L)$ and if f is any element of L^+ which has the Egoroff property on its components, then any component e of f has the Egoroff property on its components. Indeed, if the double sequence $\{e_{nk};\ n,k = 1,2,\ldots,\}$ of components of e has the property that $e_{nk} \nearrow_k e$ for each $n = 1,2,\ldots,$ then the double sequence $\{f - e + e_{nk}\}$ of components of f (note $f - e + e_{nk} = (f - e) \vee e_{nk}$ and see remark (2.3) part 1)) has the property that $f - e + e_{nk} \nearrow_k f$ for each $n = 1,2,\ldots,$. Since f has the Egoroff property on its components, there exists a diagonal sequence $\{f - e + e_{n,k(n)}\}$ such that $f - e + e_{n,k(n)} \to f$, which implies that $e_{n,k(n)} \to e$. Applying lemma (2.1) e is seen to have the Egoroff property on its components.

<u>Remark</u> 2.5. The second part of the proof of theorem (2.2) showed that if L has the P.P.P., and $B \in P_p(L)$ has the Egoroff property, then for any order unit f of B, $f \in B^+$, we must have that f has the Egoroff property on its components. By remark (2.4) it follows that if e is any component of f then e has the Egoroff property on its components and hence B_e has the Egoroff property. Furthermore, if $g \in B^+ = B_f^+$ then, taking $f_g = P_g(f)$, we have that f_g has the Egoroff property on its components and hence $B_g = B_{f_g}$ has the Egoroff property which in turn implies that g has the Egoroff property on its components.

We have established the following corollary to theorem (2.2).

Corollary 2.1. Let L be a Riesz space with the P.P.P., and suppose $B \in P_p(L)$ has the Egoroff property. Then B, considered as a Riesz space, has the Egoroff property on components and every principal band contained in B has the Egoroff property.

Our next result establishes a relationship between the Egoroff property and the Egoroff property on components.

Theorem 2.3. Let L be a Riesz space with the property that $P_p(L)$ is super order dense in $A_p(L)$. Then L has the Egoroff property if and only if L has the Egoroff property on components.

Proof. We see from lemma (2.1) that if L has the Egoroff property then L has the Egoroff property on components.

Conversely, suppose $\{b_{nk}; n,k = 1,2,\ldots,\}$ is a double sequence in L with $b_{nk} \searrow_k 0$ for each $n = 1,2,\ldots,$ and such that for some element $b \in L$ we have that $b_{nk} \leq b$ for all $n,k = 1,2,\ldots,$. By virtue of theorem (2.1), for each n, there exists a sequence of projection elements $z_{nm} \nearrow_m b$ with

$z_{nm} \wedge (b - z_{nm}) = 0$ for any $n,m = 1,2,\ldots,$ such that $P_{z_{nm}}(b_{nk}) \to_k 0$ (b - r.u.). Hence, there exists, for each $n,m = 1,2,\ldots,$ a $k = k(n,m)$ such that $P_{z_{nm}}(b_{n,k(n,m)}) \leq n^{-1}b$, where $k(n,m)$ can be chosen to increase with increasing m (i.e. $k(n,m) < k(n,m+1)$). Since L has the Egoroff property on components we know that there exists a diagonal sequence $\{z_{n,m(n)}\}$ such that $z_{n,m(n)} \to b$. Taking $k(n) = \max\{k(i,m(i));\ 1 \leq i \leq n\}+n$. Then $k(n) < k(n+1)$ and $b_{n,k(n)} \leq P_{z_{n,m(n)}}(b_{n,k(n)}) + b - z_{n,m(n)} \leq n^{-1}b + b - z_{n,m(n)}$. Hence, $b_{n,k(n)} \to 0$. The proof is completed by application of theorem (1.1).

The above theorem is a generalization of two results due to Luxemburg and Zaanen, see [(19) Thm's. 42.5 and 42.8].

We can perhaps see this best by considering the following corollary.

<u>Corollary</u> 2.2. Let L be a Riesz space with the P.P.P. . Every element of $P_p(L)$ has the Egoroff property if and only if L has the Egoroff property.

<u>Proof</u>. By remark (2.5) we see that every element of L has the Egoroff property on its components and by theorem (2.3) we see that L has the Egoroff property. The other direction follows by application of theorem (2.3) and then theorem (2.2).

Luxemburg and Zaanen proved the "if" part of the above corollary (2.2) and the "only if" part providing L was assumed to be Dedekind-σ-complete.

Remark 2.6. In view of corollary (2.2) we see that, for Riesz spaces with the P.P.P., the concepts of totally continuous (see Remark (2.1)) and Egoroff are equivalent.

In this next corollary, we summarize a number of the results obtained so far in this section. It requires no proof.

Corollary 2.3. Let L be a Riesz space with the P.P.P., and let $B \in P_p(L)$. Then the following are equivalent:

a) B has an order unit e with the Egoroff property on its components.

b) B has the Egoroff property.

c) B, considered as a Riesz space, has the Egoroff property on components.

d) B, considered as a Riesz space, has the Egoroff property.

The following is an example of a Riesz space which has the Egoroff property, and the property that $P_p(L)$ is super order dense in $A_p(L)$, without having the principal projection property. Thus, we see that our version of the Egoroff theorem applies to a strictly larger class of spaces than does theorem A.

<u>Example</u> 2.1. Let $X = \{\ldots -2,-1,0,1,2,\ldots\}$. Let $F(X)$ be the Riesz space of all bounded real valued functions on X with the usual point wise ordering. A straight forward proof almost identical with that in Example (1.1(b)) shows that $F(X)$ has the Egoroff property.

Let L be the Riesz subspace of $F(X)$ consisting of all functions $f \in F(X)$ which satisfy the condition that for any $\epsilon > 0$ there exists an integer $n(\epsilon)$ such that for all $n \geq n(\epsilon)$ we have that $|f(n) - f(-n)| < \epsilon$.

Consider the band in L generated by the element $f_1 \in L$ where $f_1(n) = \begin{cases} \frac{1}{n+1} & ; \text{ for } n \geq 0, \\ 0 & ; \text{ for } n < 0. \end{cases}$

The element $f(n) \equiv 1$ has no projection on the band B_{f_1}. Hence L fails to have the principal projection property.

Consider now the ideal L' in L consisting of all elements $f \in L$ such that for some integer $n = n(f)$ we have that for all $n \geq n(f)$ $f(-n) = f(n) = 0$ (i.e. the finitely non-zero elements).

Each principal band in L generated by an element $f \in L'$ obviously has the projection property.

It is also clear that L' is super order dense in L (i.e. given any $f \in L^+$ there exists $\{0 \leq g_n;\ n = 1,2,\ldots,\} \subset L'$ such that $g_n \nearrow f$), and this implies that $P_p(L)$ is super order dense in $A_p(L)$.

Since $F(X) = \hat{L}$, and since $F(X)$ is order separable it follows from theorem (1.3) that L has the Egoroff property.

This next example illustrates that without some sort of a projection property the concepts of Egoroff on components and Egoroff need not be related.

Example 2.2. Let $L = C[0,1]$ be the Riesz space of all continuous real valued functions on the interval $[0,1]$ with the usual pointwise ordering. We claim that L has the Egoroff property on components.

We will show first that if $f \in L^+$ then there exists a countable system $\{f_k; k = 1,2,\ldots,\}$ of components of f satisfying the following conditions:

1) $\sup\{f_k; k = 1,2,\ldots,\} = f$

2) If g is any component of f then, for each k, $g \wedge f_k = 0$ or $g \wedge f_k = f_k$.

To see this we consider the set $U = \{x : f(x) > 0\}$. Since f is continuous, U is open and is the union of a countable collection F of disjoint intervals. The intervals in the collection F are of the form (α,β), $[0,\beta)$, $(\alpha,1]$, or $[0,1]$, where $\alpha,\beta \in [0,1]$ and $\alpha < \beta$. An interval of the form $[0,\beta)$ (respectively $(\alpha,1]$) will be in F only when $f(0) > 0$ (respectively $f(1) > 0$).

If $[0,1]$ is in F then there must exist some real number γ such that $f(x) \geq \gamma > 0$ for all $x \in [0,1]$.

For simplicity we will assume that all the intervals in F are of the form (α,β), that is $F = \{(\alpha_k,\beta_k);\ k = 1,2,\ldots,\}$. It will be clear from the proof for this situation how to handle the few exceptional situations mentioned above.

Now, for each k, we set $f_k = f\chi_{(\alpha_k,\beta_k)}$, where $\chi_{(\alpha_k,\beta_k)}$ denotes the characteristic function of the interval $(\alpha_k,\beta_k) \in F$. Fixing a $k = k'$, the continuity of $f_{k'}$ at points in any of the intervals $(\alpha_{k'}, \beta_{k'})$, $[0,\alpha_{k'})$, and $(\beta_{k'}, 1]$ is obvious. For $x = \alpha_{k'}$ (the proof for $x = \beta_{k'}$ is similar), we note first that $f(\alpha_{k'}) = 0$. Assume not. Then $f(\alpha_{k'}) > 0$ and $\alpha_{k'} \in U$. But, then, there exists a $k = k'' \neq k'$ such that $(\alpha_{k''}, \beta_{k''}) \cap (\alpha_{k'}, \beta_{k'}) \neq \emptyset$ and this contradicts that the intervals (α_k,β_k) are disjoint. So, $f(\alpha_{k'}) = 0$ and the continuity of $f_{k'}$ at $\alpha_{k'}$ is now obvious. (We note that for the one exceptional possibility where $\alpha_{k'} = 0$ and $f(\alpha_{k'}) > 0$ the continuity of $f_{k'}$ at $\alpha_{k'}$ is trivial).

To see that the f_k are components of f is easy.

Now, let $x \in [0,1]$, if $x \in \{y\colon f(y) = 0\}$ then $f_k(x) = 0$ for all k and we have

$\sup\{f_k(x); k = 1,2,\ldots,\} = f(x) = 0$. If $x \notin \{y: f(y) = 0\}$ then $x \in (\alpha_{k'}, \beta_{k'})$ for some $k = k'$. For all $k \neq k'$, we have $f_k(x) = 0$. But, then, $\sup\{f_k(x); k = 1,2,\ldots,\} = f_{k'}(x) = f(x)$. We have established that $f = \sup\{f_k; k = 1,2,\ldots,\}$.

Suppose, now, that g is any component of f and that there exists a $k = k'$ such that $g \wedge f_{k'} \neq 0$ and $g \wedge f_{k'} \neq f_{k'}$. Setting $g' = f_{k'} \wedge g$, we have that g' and $f_{k'} - g'$ are non-zero components of $f_{k'}$, see remark (2.3) part 4. But, then,

$\{x : g'(x) > 0\} \cap \{x : f_{k'}(x) - g'(x) > 0\} = \emptyset$ and

$\{x : g'(x) > 0\} \cup \{x : f_{k'}(x) - g'(x) > 0\} = \{x: f_{k'}(x) > 0\}$

$= (\alpha_{k'}, \beta_{k'})$. This contradicts the connectedness of the interval $(\alpha_{k'}, \beta_{k'})$. We have established that given any component g of f either $g \wedge f_k = 0$ or $g \wedge f_k = f_k$ for all $k = 1,2,\ldots,$.

We are finally ready to establish that L has the Egoroff property on components.

Let f be an element of L^+ and let $\{f_k; k = 1,2,\ldots,\}$ be the collection of components of f which satisfy the conditions (1) and (2) above. Suppose $\{b_{nk}; n,k = 1,2,\ldots,\}$ is a double sequence of components of f with $b_{nk} \nearrow_k f$ for each $n = 1,2,\ldots,$. We define $k(n)$ in the following manner:

$$k(1) = \inf\{k : b_{1,k(1)} \geq f_1\},$$
$$k(2) = \inf\{k : k \geq k(1)+1 \text{ and } b_{2,k(2)} \geq f_1+f_2 = f_1 \vee f_2\},$$
$$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \qquad \vdots \qquad \vdots$$
$$k(n) = \inf\{k : k \geq k(n-1)+1 \text{ and } b_{n,k(n)} \geq \sup_{1\leq i\leq n} \{f_i\}\},$$
$$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \qquad \vdots \qquad \vdots$$

Taking $b_m = \sup_{1\leq i\leq m} \{f_i\}$, we have $b_m \wedge (f - b_m) = 0$, $b_m \nearrow f$, and $b_m \leq b_{m,k(m)}$. We have proved that L has the Egoroff property on components.

$C[0,1]$ does not have the Egoroff property as we mentioned earlier, see Example (1.1c). If $C[0,1]$ had the property that $P_p(L)$ was super order dense in $A_p(L)$ then the fact that it has the Egoroff property on components would imply that it had the Egoroff property. This would follow by virtue of theorem (2.3). In fact, it is well known that $C[0,1]$ has only the trivial projection bands (i.e. the whole space and the ideal consisting of the zero element).

<u>Remark</u> 2.7. Again taking $L = C[0,1]$, by virtue of theorem (1.3) and the fact that L is order separable and does not have the Egoroff property, it follows that $\hat{L}$ does not have the Egoroff property. Since $\hat{L}$ has the projection property, it follows from theorem (2.3) that $\hat{L}$

cannot have the Egoroff property on components since if it did it would have to have the Egoroff property. Hence, example (2.2) above furnishes us with an example of an order separable Archimedean Riesz space with the Egoroff property on components such that its Dedekind completion does not have this property.

We will find several applications for this next result, the first of which will follows directly after its proof. The result is due to Luxemburg and Zaanen [see (19) Thm. 40.3], and the proof is essentially theirs, although we do use some of the results in section 1 which seem to simplify the proof.

<u>Theorem</u> B. (Luxemburg and Zaanen). The set of all elements of a Riesz space L having the Egoroff property is a σ-ideal in L (where by σ-ideal is meant that if any sequence from the ideal converges to an element in L then that element is in the ideal).

<u>Proof</u>. We prove first that if $0 \leq v \leq u$, and u has the Egoroff property, then so has v. Let $0 \leq v_{nk} \nearrow_k v$ for each $n = 1,2,\ldots,$. Then $(v_{nk} + (u-v)) \nearrow_k u$ for $n = 1,2,\ldots,$ and there exists a diagonal sequence $\{v_{n,k(n)} + (u-v)\}$ such that $v_{n,k(n)} + (u-v) \rightarrow u$, which implies $v_{n,k(n)} \rightarrow v$. But, then, v has the Egoroff property (see remark (1.1)).

Now, we show that if $u,v \in L^+$ have the Egoroff property then $u+v$ has the Egoroff property. Let $0 \leq b_{nk} \nearrow_k u+v$ for $n = 1,2,\ldots,$. By the well known decomposition lemma, [c.f. (25) Cor. 1.4 p. 8], there exists v_{nk} and z_{nk} such that $v_{nk} + z_{nk} = b_{nk}$ for all $n,k = 1,2,\ldots,$ and $v_{nk} \nearrow_k u$, and $z_{nk} \nearrow_k v$. Since u and v have the Egoroff property, there exist diagonal sequences $\{v_{p,k'(p)}\}, \{z_{m,k''(m)}\}$ such that $v_{p,k'(p)} \to u$ and $z_{m,k''(m)} \to v$. For each n, taking $k(n) = \max\{k'(n),k''(n)\}$ we have that the diagonal sequence $\{b_{n,k(n)} = v_{n,k(n)} + z_{n,k(n)}\}$ satisfies $b_{n,k(n)} \to u+v$. By virtue of remark (1.1), we see that $u+v$ has the Egoroff property.

Showing, now, that the set of all elements of L with the Egoroff property is an ideal is standard.

Let L' be the ideal in L of all elements with the Egoroff property. We must show that if $0 \leq v_p \nearrow u \in L^+$, where $\{v_p;\ p = 1,2,\ldots,\} \subset L'$, then u has the Egoroff property. Suppose $0 \leq b_{nk} \nearrow_k u$ for $n = 1,2,\ldots,$. Taking $u^p_{nk} = b_{nk} \wedge v_p$, we have $u^p_{nk} \nearrow_k v_p$ for each $n = 1,2,\ldots,$. Hence, there exists, for each p, a sequence $\{u^p_m\}$ such that $0 \leq u^p_m \nearrow_m v_p$ and, for any m, $u^p_m \leq u^p_{m,k(m,p)}$ for appropriate $k = k(m,p)$. Consider $u_m = \sup\{u^i_m;\ 1 \leq i \leq m\}$. It is clear that $u_m \nearrow u$. Taking $k = k(m) = \max\{k(m,i);1 \leq i \leq m\}$

we see that $u_m \leq b_{m,k(m)} \wedge v_p \leq b_{m,k(m)}$. Applying lemma (1.A) we see that u has the Egoroff property. The theorem is proved.

We are now ready to show that theorem (2.1) is valid in every sequence space. To show this we need only show that every sequence space L has the Egoroff property and the property that $P_p(L)$ is super order dense in $A_p(L)$.

<u>Theorem</u> 2.4. Every sequence space L has the Egoroff property and the property that $P_p(L)$ is super order dense in $A_p(L)$.

<u>Proof</u>. Let L be a sequence space and let φ be the space of finitely non-zero sequences. We recall that φ is contained in any sequence space and hence $\varphi \subseteq L$. In fact, φ must be a super order dense ideal in L. Showing that φ has the Egoroff property is trivial. So, by the super order density of φ in L and theorem B above, L must have the Egoroff property.

We note, now, that if $x \in \varphi$ then the principal band generated by x in L is a projection band. It follows easily from the super order density of φ in L that $P_p(L)$ is super order dense in $A_p(L)$. The theorem is proved.

We will now introduce a property which is an abstraction of the result of theorem (2.1).

<u>Definition</u> 2.5. We will say that an element $0 < e$ of the Riesz space L has <u>property</u> E.T.*, if given any $x_n \searrow 0$ with $x_n \leq e$ for $n = 1,2,\ldots,$ there exists a sequence of components $\{e_m\}$ of e such that $e_m \nearrow e$ and $e_m \wedge x_n \to_n 0$ $(e - r.u.)$ for each $m = 1,2,\ldots,$.

<u>Definition</u> 2.6. We will say that the Riesz space L has <u>property</u> E.T.* if every $e \in L^+$ has property E.T.*.

We note that if L has the property that $P_p(L)$ is super order dense in $A_p(L)$ and in addition has the Egoroff property then by theorem (2.1) L has property E.T.*. Thus, theorem (2.4) can be restated in the following manner:

<u>Corollary</u> 2.4. Every sequence space has property E.T.*.

We are now in a position to reduce significantly another important order convergence property for Riesz spaces, namely, that of relatively uniform convergence (see Defn. (1.7)). Our next definition is designed for this purpose.

Definition 2.7. Let L be a Riesz space. An element $f \in L^+$ will be said to be relatively uniform on its components, if for every sequence of components $\{f_m\}$ of f with $f_m \nearrow f$ there exists a $u \in L^+$ and a sequence of real numbers $\{\lambda_m\}$ satisfying $\lambda_m \searrow 0$ such that $|f - f_m| \leq \lambda_m u$ for each $m = 1,2,\ldots,$.

Theorem 2.5. Let L be a Riesz space, and assume that the element $0 < e \in L$ has property E.T.*. Assume in addition that e is relatively uniform on its components. Then, if $x_n \searrow 0$ in L and $x_n \leq e$ for $n = 1,2,\ldots,$ we have $x_n \to 0$ (r.u.).

Proof. Since e has property E.T.*, there exists a sequence of components $\{e_m\}$ of e with $e_m \nearrow e$ such that, for each m, $e_m \wedge x_n \to_n 0$ $(e$ - r.u.) (i.e. there exist sequences of real numbers $\{\lambda_{mn}\}$ such that $\lambda_{mn} \searrow_n 0$ for $m = 1,2,\ldots,$ and $e_m \wedge x_n \leq \lambda_{mn} e$).

Since e is relatively uniform on its components, there exist a $u \in L^+$ and a sequence of real numbers $\{\lambda'_m\}$ with $\lambda'_m \searrow 0$ such that, for each m, $e - e_m \leq \lambda'_m u$. Hence, we have $x_n \leq x_n \wedge e_m + (e-e_m)$ $\leq \lambda_{mn} e + \lambda'_m u \leq (\lambda_{mn} + \lambda'_m)(u \vee e)$. Now, $\{\lambda_{mn}\}$ is just a double sequence of real numbers and the real numbers have the diagonalization property hence there exists a diagonal sequence $\{\lambda_{m,n(m)}\}$ such that $\lambda_{m,n(m)} \to 0$ which implies there exists a sequence of real numbers $\{\lambda''_m\}$ such that $\lambda_{m,n(m)} \leq \lambda''_m \searrow 0$. Consider, now, the subsequence $\{x_{n(m)}\} \subset \{x_n\}$. We have $x_{n(m)} \leq (\lambda_{m,n(m)} + \lambda'_m)(u \vee e) \leq (\lambda''_m + \lambda_m)(u \vee e)$ and $(\lambda''_m + \lambda_m) \searrow 0$. Hence, $x_{n(m)} \to 0$ $((u \vee e)$ - r.u.). But since the original sequence $\{x_n\}$ was monotone this implies that $x_n \to 0$ (r.u.). The theorem is proved.

<u>Definition</u> 2.8. The Riesz space L is said to be <u>relatively uniform on components</u> if every $x \in L^+$ is relatively uniform on its components.

<u>Corollary</u> 2.5. Let L be an Archimidean Riesz space, and assume that L has property E.T.*. Then the following are equivalent:

1) Order convergence and relatively uniform convergence are equivalent in L for sequences.

2) L is relatively uniform on components.

3) Given any $u \in L^+$ and sequence of components $\{u_m\}$ of u with $u_m \nearrow u$, there exists a subsequence $\{u_{m_k}\} \subset \{u_m\}$ such that the sequence $k(u-u_{m_k})$ is order bounded in L (i.e. there exists an element $e \in L^+$ such that $e \geq k(u-u_{m_k})$ for all $k = 1,2,\ldots,)$.

4) Given any $u \in L^+$ and sequence of components $\{u_m\}$ of u with $u_m \nearrow u$, there exists a subsequence $\{u_{m_k}\} \subset \{u_m\}$ such that the sequence $\{k(u_{m_{k+1}} - u_{m_k})\}$ is bounded in L.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are all trivial. That (2) $\Rightarrow$ (1) is an easy consequence of theorem (2.5). Hence, all we need to show is that (4) $\Rightarrow$ (3) $\Rightarrow$ (2). We show first that (3) $\Rightarrow$ (2).

Let $u_m \nearrow u \in L^+$ with $u_m \wedge (u - u_m) = 0$, then by (3) we have a subsequence $\{u_{m_k}\} \subset \{u_m\}$ such that for some $e \in L^+$ we have $k(u - u_{m_k}) \leq e$ for all $k = 1,2,\ldots,$. Then, $u - u_{m_k} \leq \frac{1}{k} e$ which implies that $u_{m_k} \to u$ (r.u.). But since the sequence $\{u_m\}$ was monotone this implies that $u_m \to u$ (r.u.). We have established that (3) $\Rightarrow$ (2).

We now show that (4) $\Rightarrow$ (3). To this end, let $u \in L^+$ and suppose $u_m \nearrow u$ with $u_m \wedge (u - u_m) = 0$ for $m = 1,2,\ldots,$. By (4), there exist an $e \in L^+$ and a subsequence $\{u_{m_k}\} \subset \{u_m\}$ such that $k(u_{m_{k+1}} - u_{m_k}) \leq e$ for $k = 1,2,\ldots,$. We note that $u-u_{m_k} = \sup\{(u_{m_{k+p}} - u_{m_k});\ p = 1,2,\ldots,\} = \sup\{\sum_{i=1}^{p} (u_{m_{k+i}} - u_{m_{k+i-1}});$ $p = 1,2,\ldots,\} = \sum_{i=1}^{\infty} (u_{m_{k+i}} - u_{m_{k+i-1}})$. We note that

$$\sum_{i=1}^{p} (k+i)(u_{m_{k+i}} - u_{m_{k+i-1}}) \leq e \quad \text{for} \quad p = 1,2,\ldots,.$$

Hence, $\sum_{i=1}^{p} k(u_{m_{k+i}} - u_{m_{k+i-1}}) \leq e$ for $p = 1,2,\ldots,.$

Therefore, $k \sup\{\sum_{i=1}^{p} (u_{m_{k+i}} - u_{m_{k+i-1}})\} = k(u-u_{m_k}) \leq e.$

We have shown that (4) $\Rightarrow$ (3). The theorem is proved.

Remark 2.8. As mentioned before, any Riesz space L with the property that $P_p(L)$ is super order dense in $A_p(L)$ and the Egoroff property, has property E.T.*. Hence, for all such spaces corollary (2.5) is valid.

For sequence spaces we have the following corollary:

Corollary 2.6. For every sequence space L, the following are equivalent:

(1) Order convergence and relatively uniform convergence are equivalent in L for sequences.

(2) L is relatively uniform on components.

(3) Given any element $x = \{x_i\} \in L^+$ there exists a sequence of positive real numbers $\{\lambda_i\}$ with $\lambda_i \nearrow \infty$ and an element $y = \{y_i\} \in L$ such that $\lambda_i x_i \leq y_i$ for each $i = 1,2,\ldots,.$

Proof. The equivalence of (1) and (2) follows directly from corollary (2.5). We show now that (2) $\Rightarrow$ (3). Let $x = \{x_i\} \in L^+$ and define the sequences $y_n = \{y_i^n\}$ in L by $y_i^1 = 0$ for all i, $y_i^n = x_i$ for $1 \leq i \leq n$ and $n \geq 2$, and $y_i^n = 0$ for $i \geq n+1$ and $n \geq 2$. Clearly $y_n \nearrow x$ and for each n, y_n is a component of x. Since L is relatively uniform on components, we have a sequence $\{z_i\} = z \in L$ and a sequence of real numbers $\{\lambda_n\}$ with $\lambda_n \searrow 0$ such that $x - y_n \leq \lambda_n z$ for all n. But this clearly implies that $x_i \leq \lambda_i z_i$ and hence taking $\beta_i = 1/\lambda_i$ we have $\beta_i \nearrow \infty$ and $\beta_i x_i \leq z_i$. We have established that (2) $\Rightarrow$ (3).

We will now show that (3) $\Rightarrow$ (2). Let $x = \{x_i\} \in L^+$ and suppose that $\{y_n;\ n = 1,2,\ldots,\}$ is a sequence of components of x such that $y_n \nearrow x$. We consider the sequence $z_k = \{z_i^k\}$ where for each k, $z_i^k = x_i$ for $1 \leq i \leq k$ and $z_i^k = 0$ for $i \geq k+1$. Clearly there is some subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \geq z_k$ for $k = 1,2,\ldots,$. Also, we have that $z_k \nearrow x$ and each z_k is a component of x in L. So, if we can show that $z_k \to x$ (r.u.), we will have that $y_{n_k} \to x$ (r.u.) and since $\{y_n\}$ is a monotone sequence, this will establish that $y_n \to x$ (r.u.). Applying condition (3), let $\{\lambda_i\}$ be a sequence of positive real numbers with $\lambda_i \nearrow \infty$ and $u = \{u_i\} \in L$ be such that, for each i, $\lambda_i x_i \leq u_i$. Since λ_i is monotone directed up, we have that $\lambda_k(x - z_k) \leq u$ for all $k = 1,2,\ldots,$. But then $z_k \to x$ (u - r.u.), since $1/\lambda_k \searrow 0$ and $x - z_k \leq 1/\lambda_k u$. The proof is complete.

<u>Remark</u> 2.9. For a sequence space L, corollary (2.6) reduces the question of whether or not order convergence is equivalent to relatively uniform convergence in L for sequences to the question of whether or not fixed elements in the positive cone of L satisfy condition (3) of corollary (2.6).

<u>Section</u> 3. The Egoroff Property and Universally Complete Spaces.

For a certain class of Riesz spaces many of the properties considered in the last two sections are equivalent. In this section we will discuss briefly this class of spaces.

<u>Definition</u> 3.1. A Dedekind complete Riesz space L is said to be <u>universally complete</u> (or of <u>extended type</u>) if given any collection $\{x_\alpha; \alpha \in A\}$ of elements of L^+ satisfying $x_{\alpha_1} \wedge x_{\alpha_2} = 0$ whenever $\alpha_1, \alpha_2 \in A$ and $\alpha_1 \neq \alpha_2$ (that is, the collection is disjoint) we have that $\sup\{x_\alpha\}$ exists in L.

<u>Definition</u> 3.2. Let L be an Archimidean Riesz space. A Riesz space $\bar{L}$ is said to be a <u>universal completion of</u> L if L is embeddable in $\bar{L}$ as an order dense Riesz subspace of $\bar{L}$ and $\bar{L}$ is universally complete.

<u>Remark</u> 3.1. Nakano has shown in [(22), § 34] that a Riesz space L has a universal completion if and only if L is Archimedean. He has also shown that universal completions of an Archimedean Riesz space are unique up to lattice isomorphism. Hence, it makes sense to say "the universal completion of L". Moreover, if $\bar{L}$ denotes the universal completion of L and $\hat{L}$ denotes the Dedekind completion of L, then $\bar{L}$ is the universal completion of $\hat{L}$, and given any $x \in \bar{L}^+$ there exists a disjoint collection $\{x_\alpha;\ \alpha \in A\} \subset \hat{L}^+$ such that $x = \sup\{x_\alpha;\ \alpha \in A\}$.

From now on, for an Archimedean Riesz space L, we will denote by $\bar{L}$ and $\hat{L}$ the universal completion of L and the Dedekind completion of L, respectively. Furthermore, no distinction will be made between L and its isomorphic image in $\bar{L}$ and $\hat{L}$.

Theorem 3.1. Let L be a universally complete Riesz space with the Egoroff property. Then, in L, order convergence and relatively uniform convergence, for sequences, are equivalent.

Proof. Since L is Dedekind complete, it has the projection property and hence the principal projection property. By remark (2.8), therefore, L has property E.T.*. Thus, we need only show that condition (4) of corollary (2.5) is satisfied. To this end, let $u \in L^+$ and $\{u_m\}$ be a sequence of components of u with $u_m \nearrow u$. But then the collection $\{m(u_{m+1} - u_m)\}$ is a disjoint collection of elements in L^+ and by definition $\sup\{m(u_{m+1} - u_m)\} = e$ exists in L. The theorem is proved.

Corollary 3.1. Let L be a universally complete order separable Riesz space. Then the following are equivalent:

(1) L has the Egoroff property.

(2) Order convergence and relatively uniform convergence are equivalent in L for sequences.

(3) Order convergence is stable in L.

(4) L is regular.

(5) L is diagonalizable.

Proof. That (1) ⇒ (2) follows from theorem (3.1) above. For the implication (2) ⇒ (3), see remark (1.2). For (3) ⇒ (4) we refer the reader to [(27), Thm. VI,6.1 p. 167]. For (4) ⇒ (5), see corollary (1.3). The implication (5) ⇒ (1) is obvious. The assertion is proved.

One of the most important and useful results in the theory of vector lattices is the representation of an arbitrary Archimedean Riesz space as an order dense Riesz subspace of the space of extended real valued continuous functions on some compact Hausdorff space. In fact, it is well known that every universally complete Riesz space is lattice isomorphic to $C_\infty(Q)$ where Q is some extremal compactum and $C_\infty(Q)$ denotes the set of extended real valued functions defined on Q which are finite valued and continuous on some open dense subset of Q (an extremal compactum is a compact Hausdorff space with the property that every open subset has open closure). Furthermore, every $C_\infty(Q)$, where Q is an extremal compactum, is a universally complete Riesz space. For details we refer the reader to [(27), Chapt's 5 and 6].

It is to be expected, therefore, that many order properties of Archimedean, and especially universally complete, Riesz spaces have equivalent topological formulations. It is a result in this direction, due to Z.T. Dikanova [(2)], which we are now able to strengthen with the help of theorem (3.1).

We will first state the result in [(3)].

<u>Theorem</u> C. Let Q be an extremal compactum, and let $C_\infty(Q)$ be order separable. Then the following are equivalent:

(1) Order convergence is stable in $C_\infty(Q)$.

(2) For every collection $\{F_n\}_1^\infty$, of nowhere dense closed subsets of Q there exists a closed nowhere dense G_δ set $\emptyset \subset Q$ such that $\emptyset \supset \bigcup_1^\infty F_n$.

We strengthen the result as follows:

<u>Corollary</u> 3.2. Let Q be an extremal compactum and let $C_\infty(Q)$ be order separable. Then the following are equivalent:

(1) $C(Q)$ has the Egoroff property (where $C(Q)$ is the space of continuous real valued functions).

(2) $C_\infty(Q)$ has the Egoroff property.

(3) Order convergence is stable in $C_\infty(Q)$.

(4) $C_\infty(Q)$ has the diagonalization property.

(5) For every collection $\{F_n\}_1^\infty$ of nowhere dense closed subsets of Q there exists a closed nowhere dense G_δ set $\emptyset \subset Q$ such that $\emptyset \supset \bigcup_1^\infty F_n$.

Proof. All we need to show is that (1) ⇒ (2) since (2) ⇒ (1) is obvious and the remainder of the implications are handled by application of corollary (3.1).

To see that (1) ⇒ (2) we need only note that $P_p(C_\infty(Q)) \cong P_p(C(Q))$. The result now follows by application of corollary (2.3).

Section 4. Order Separability, Sub-Order Separability, and the Egoroff Theorem.

In this section we will do four things. First we will investigate the connection between order separability and the existence in a Riesz space of a positive order basis (see Definition (4.1) below) with certain properties. Second, we will extend slightly the results of section (3). Third, we will obtain an Egoroff type theorem for arbitrary Archimedean order separable Riesz spaces, which does not depend on a projection property. Finally, we will investigate property E.T.* (see Definition (2.5)) together with a related property which we will define below. As a result of these investigations we will obtain the somewhat surprising result that if an order separable Archimedean Riesz space has the property that order convergence and relatively uniform convergence are equivalent for sequences then the space has the Egoroff property.

Definition 4.1. A positive order basis for a Riesz space L is a collection of positive elements $\{e_\alpha; \alpha \in A\} \subset L$ with the property that if x is any element of L then $|x| \wedge e_\alpha = 0$ for all α if and only if $x = 0$.

Remark 4.1. By a straight forward application of Zorn's lemma, it is easy to show that every Riesz space has a positive order basis. In fact, one can show that the basis can be chosen to consist of disjoint positive elements (i.e. if $\{e_\alpha; \alpha \in A\}$ is the basis then $e_\alpha \wedge e_\beta = 0$ whenever $\alpha, \beta \in A$ and $\alpha \neq \beta$).

Luxemburg has shown in [(14) Thm. 6(v) p. 491] that for Archimedean Riesz spaces order separability is equivalent to the following property:

K. For every disjoint system $\{f_\sigma\}$ in L, and every $g \in L$ we have $\inf\{|g|, |f_\sigma|\} \neq 0$ for at most countably many σ.

We will refer to this property as property K.

Using property K both as motivation and as a proof mechanism, we make the following definition and obtain our first theorem in this section.

<u>Definition</u> 4.2. Let L be an Archimedean Riesz space. An element $0 < e \in L$ will be said to be of <u>countable type</u> if any disjoint system $\{0 < e_\alpha\} \subset L$ with $e_\alpha \leq e$ for all α, is at most countable.

<u>Theorem</u> 4.1. An Archimedean Riesz space L is order separable if and only if it has a positive disjoint order basis $\{e_\alpha; \alpha \in A\}$, consisting of elements of countable type, such that if $f \in L$ then $|f| \wedge e_\alpha \neq 0$ for at most countably many indices α.

<u>Proof</u>. Suppose L has such a basis $\{e_\alpha\}$, and let $\{f_\sigma\}$ be any disjoint family in L and let g be any element of L. We have by assumption that $|g| \wedge e_\alpha \neq 0$ for at most countably many α, say $\{\alpha_i\}_{i=1}^{\infty}$. Since each e_α is of countable type, we have that for each α_i for $i = 1,2,\ldots,$ the number of σ such that $|f_\sigma| \wedge e_{\alpha_i} \neq 0$ is at most countable, say $\{\sigma_{ij}\}_{j=1}^{\infty}$. Now, suppose $\sigma \neq \sigma_{ij}$ for any $i,j = 1,2,\ldots,$. Consider $|g| \wedge |f_\sigma|$. We have that $|g| \wedge |f_\sigma| \leq |g|$ and $|g| \wedge |f_\sigma| \leq |f_\sigma|$. So, for each α we have that, if $\alpha \neq \alpha_i$ for $i = 1,2,\ldots,$ then $e_\alpha \wedge (|g| \wedge |f_\sigma|) \leq |g| \wedge e_\alpha = 0$,

and if $\alpha = \alpha_i$ for some $i = 1,2,\ldots,$ then $e_\alpha \wedge (|g| \wedge |f_\sigma|) \leq |f_\sigma| \wedge e_\alpha = 0$. Thus we have that, for any α, $(|g| \wedge |f_\sigma|) \wedge e_\alpha = 0$. Since $\{e_\alpha\}$ is a basis for L, it follows that $|g| \wedge |f_\sigma| = 0$ whenever $\sigma \neq \sigma_{ij}$ for any $i,j = 1,2,\ldots,$. Since the collection $\bigcup_i \{\sigma_{ij}\}_{j=1}^{\infty}$ is countable we have property K.

The converse follows once we recall that every Riesz space has an order basis consisting of disjoint positive elements. Let this basis by $\{e_\alpha\}$. That each e_α is of countable type follows directly from property K. So does the fact that if $f \in L$ then $|f| \wedge e_\alpha \neq 0$ for at most countably many α. The theorem is proved.

We are motivated by theorem (4.1) and other considerations (which will become apparent shortly) to make the following definition:

<u>Definition</u> 4.3. Let L be an Archimedean Riesz space. L is said to be <u>sub-order separable</u> if it contains an order dense ideal L' which is order separable.

<u>Theorem</u> 4.2. Let L be an Archimedean Riesz space. L is sub-order separable if and only if L has a disjoint positive order basis $\{e_\alpha;\ \alpha \in A\}$ consisting of elements of countable type.

Proof. That such a basis exists if the space is sub-order separable follows easily, since a basis of disjoint positive elements for the order dense order separable ideal $L' \subset L$ will suffice.

Now, suppose we have a basis of disjoint positive elements $\{e_\alpha; \alpha \in A\}$ such that each e_α is of countable type. Then, consider the following collection of elements:

$$L' = \{f \in L: |f| \wedge e_\alpha \neq 0 \text{ for at most countably many } \alpha\}.$$

That L' is an ideal in L is a matter of routine verification. That L' is order separable follows immediately from theorem (4.1). Now, if $f \in L$ we have $|f| \wedge e_\alpha \neq 0$, for all α unless $f \equiv 0$. So, let $f \neq 0$ be an element of L. Then, for some $\alpha = \alpha_0$, we have that $0 \neq |f| \wedge e_{\alpha_0}$. But, then, $|f| \wedge e_{\alpha_0} \in L'$ and $0 < |f| \wedge e_{\alpha_0} \leq |f|$. Since L is Archimedean, the order density of L' in L follows by application of [(18) Note IX Thm's. 29.5 and 29.10]. The theorem is proved.

A non-sub-order separable Riesz space will of course be a Riesz space which does not contain an order dense order separable ideal.

Corollary 4.1. If an Archimedean Riesz space L has the property that every non-trivial ideal in L

contains a non-trivial sub-order separable ideal, then L is sub-order seperable.

Proof. We note first that if every non-trivial ideal in L contains a non-trivial sub-order separable ideal, then every non-trivial ideal in L contains a non-trivial order separable ideal. Now, let $0 < e \in L$ and let the ideal in L generated by e be denoted by $I(e)$ (i.e. $I(e) = \{f \in L:$ for some $n = 1,2,\ldots,$ we have $|f| \leq n e\}$). $I(e)$ contains a non-trivial order dense ideal, say I'. Since for any $0 < f \in I'$ we have $e \wedge f > 0$, and since every positive element of I' is of countable type, we see now that for every element $0 < e \in L$, there exists an element $0 < f \in L$ such that $f \leq e$ and f is of countable type. A straight forward Zorn's lemma argument shows that L has a disjoint positive order basis consisting of elements of countable type. By theorem (4.2) above this implies that L is sub-order separable. The theorem is proved.

Remark 4.2. It is obvious from the above corollary (4.1), that a non-sub-order separable Archimedean Riesz space must contain an ideal such that every non-trivial ideal in this ideal is non-sub-order separable.

Perhaps the best description of the situation is given in this next theorem.

Theorem 4.3. Let L be an Archimedean Riesz space. Then there exist bands L_1 and L_2 in L with $L_1 = L_2^{\perp}$ and satisfying:

(1) L_1 is sub-order separable.

(2) L_2 has the property that no ideal $\{0\} \neq I \subseteq L_2$ is order separable.

Proof. We will say that an element $f \in L$ has property NS, if f satisfies the following conditions:

a) $f \neq 0$.

b) For all $0 < g \leq |f|$, there exists a disjoint uncountable collection $\{g_\sigma\}$ with $0 < g_\sigma \leq g$ for all σ.

We let $L_2 = \{f \in L\colon f \text{ has property NS}\} \cup \{0\}$. We wish to show that L_2 is a band in L. Clearly, if $g \in [-f, f]$ and $f \in L_2$ then by construction $g \in L_2$. Also, if a is a real number and $f \in L_2$ then $af \in L_2$. Since $|f \vee g|$, $|f \wedge g|$, and $|f + g|$ are all less than or equal to $|f| + |g|$ we need only show that if $f, g \in L_2$ then $|f| + |g| \in L_2$, in order to see that L_2 is an ideal. To this end, let $0 < g' \leq |f| + |g|$. Then, $g' \wedge |f| \neq 0$, or $g' \wedge |g| \neq 0$, or both. Assume

$g'' = g' \wedge |f| \neq 0$. Then, $0 < g'' \leq |f|$ and there exists an uncountable disjoint collection $\{g''_\sigma\}$ such that $0 < g''_\sigma \leq g''$ for all σ. But, then, $o < g''_\sigma \leq g'$ for all σ. Hence, $|f| + |g| \in L_2$. We have established that L_2 is an ideal. To show that L_2 is a band, it remains to show that if $\{f_\tau\} \subset L_2^+$ and $\sup\{f_\tau\} = f$ exists in L, then $f \in L_2$. Let $g \in L$ and suppose that $0 < g \leq f$. We must have that, for some $\tau = \tau_0$, $0 < g \wedge f_{\tau_0} = g'$. But, then, since $f_{\tau_0} \in L_2$ and $g' \leq f_{\tau_0}$, we can find a disjoint uncountable collection $\{g'_\sigma\}$ such that $0 < g'_\sigma \leq g' \leq g$. Hence, f has property NS and $f \in L_2$. We have established that L_2 is a band.

It is clear that no non-zero positive element of L_2 is of countable type. It follows easily, therefore, from theorem (4.1) that no non-trivial ideal in L_2 is order separable.

Let $L_1 = L_2^\perp$. Given any $f \in L_1$ with $|f| > 0$, there exists a $g \in L$ with $0 < g \leq |f|$ such that if $\{g_\sigma\}$ is any disjoint collection of elements of L such that $0 < g_\sigma \leq g$ for all σ, then the collection $\{g_\sigma\}$ is at most countable. If not we would have that $f \in L_2$ and therefore certainly not in $L_1 = L_2^\perp$. This merely says that if $f \in L_1$, then there exists a $g \in L_1$ such that $0 < g \leq |f|$ and g is of countable type. By a straight forward application of Zorn's lemma, we see that L_1 has a positive disjoint order basis consisting of elements of countable type. By application of theorem (4.2), it follows that L_1 is sub-order separable. That L_1 is a band follows immediately from its definition. The theorem is proved.

Remark 4.3. It follows from the fact that L is Archimedean and from [(18) Note IX Thm's. 29.5 and 29.10] that the ideals L_1 and L_2, discussed in the above theorem, have the property that $L_1 \oplus L_2$ is order dense in L. Furthermore, if L has the projection property, in particular if L is Dedekind complete, then $L_1 \oplus L_2 = L$.

There are examples, as we will have occasion to see later, of non-sub-order separable Riesz spaces in the literature. They have not occurred as a result of considerations of the type we have been indulging in and they are in general rather complicated. We give now an example which is on the one hand easily constructed and on the other exhibits some properties we will need to consider in the next chapter.

Example 4.1. For $i = 1,2,\ldots,$ let X_i be the set of real numbers $r \in [0,1]$, and place on each X_i the discrete topology.

Let $Y = \Pi_{i=1}^{\infty} X_i$ with the product topology. Y is a locally compact Hausdorff space and is totally disconnected (i.e. has a basis of closed open sets).

Let $L = C(Y)$ be the Riesz space of all continuous bounded functions from Y into the reals with the pointwise ordering. L is an Archimedean Riesz space and we will now show that it is a non-sub-order separable Riesz space. To see this it will suffice to show that L has no non-zero positive elements of countable type. To this end, let $0 < f \in L$. Then, $f \geq \alpha > 0$ on some open set $U \subset Y$ and hence on some basic closed open set $V = V_{i_1} \times V_{i_2} \times \ldots \times V_{i_n} \times \Pi_{i \neq i_j} X_i \subset U$, where $j = 1,2,\ldots,n$.

Let $\{y_{i_j}\}$ be a collection of reals such that $y_{i_j} \in V_{i_j}$ for $j = 1,2,\ldots,n$. Then $B = \{y_{i_1}\} \times \{y_{i_2}\} \times \ldots \times \{y_{i_n}\} \times \Pi_{i \neq i_j} X_i \subseteq V$, and is closed and open.

Choose an $i_{n+1} = i$ so that $i_{n+1} \neq i_j$ for any $j = 1,2,\ldots,n$. For each real number $v_{n+1} \in X_{i_{n+1}}$, consider the sets $G_{v_{n+1}} = \{y_{i_1}\} \times \ldots \times \{y_{i_n}\} \times \{v_{n+1}\} \times \Pi_{i \neq i_j} X_i$, where $j = 1,2,\ldots,n,n+1$. Each $G_{v_{n+1}}$ is closed and open, and if $v'_{n+1} \neq v''_{n+1}$ then $G_{v'_{n+1}} \cap G_{v''_{n+1}} = \emptyset$.

Consider now the functions $\alpha\chi_{G_{v_{n+1}}}$ where $\chi_{G_{v_{n+1}}}$ denotes the characteristic function of $G_{v_{n+1}}$ and v_{n+1} is allowed to range over all reals in $X_{i_{n+1}}$. Each $\alpha\chi_{G_{v_{n+1}}} > 0$ and if $v'_{n+1} \neq v''_{n+1}$ then $\alpha\chi_{G_{v'_{n+1}}} \wedge \alpha\chi_{G_{v''_{n+1}}} = 0$. Hence, the collection $\{\alpha\chi_{G_{v_{n+1}}}\}$ is an uncountable disjoint collection of strictly positive elements of L and since $G_{v_{n+1}} \subset B$ for each $v_{n+1} \in X_{i_{n+1}}$ we have $f \geq \alpha\chi_B \geq \alpha\chi_{G_{v_{n+1}}}$ for all $v_{n+1} \in X_{i_{n+1}}$. This shows that L has no non-zero positive elements of countable type and therefore is non-sub-order separable.

<u>Remark</u> 4.4. The reader should note that this example has the property that every point $y \in Y$ is a G_δ set. Indeed, let $y = (y_1, y_2, \ldots, y_n, \ldots)$ then if we take $G_j = \{y_1\} \times \{y_2\} \times \ldots \times \{y_j\} \times \prod_{i>j} X_i$ we have that $\bigcap_{j=1}^{\infty} G_j = \{y\}$.

Since for Archimedean Riesz spaces it is true that L is order separable if and only if $\hat{L}$ is order separable, we have easily that L is sub-order separable if and only if $\hat{L}$ is sub-order separable. Indeed, if $I \subseteq L$ is an order separable order dense ideal in L then $\hat{I}$ is also an order separable order dense ideal in $\hat{L}$. Furthermore, since any ideal which is order dense in $\hat{L}$ is also order dense in the universal completion $\bar{L}$ of L we have that, for Archimedean Riesz spaces, L is sub-order separable if and only if $\bar{L}$ is sub-order separable. It is not true, however, that if L is order separable then $\bar{L}$ is order separable. For the purpose of extending the results of section (3), we can remedy this situation by the introduction of an intermediate space. Before we can do this properly, a few preliminaries are necessary.

Let L be a universally complete sub-order separable Riesz space. By theorem (4.2), L must have a disjoint positive order basis consisting of elements of countable type. To any such basis $\{e_\alpha; \alpha \in A\}$ we associate the following ideal:

$$\{e_\alpha\}L(\sigma) = \{f \in L: |f| \wedge e_\alpha \neq 0 \text{ for at most countably many } \alpha \in A\}.$$

Clearly, by theorem (4.1), any such $\{e_\alpha\}L(\sigma)$ is order separable.

<u>Lemma</u> 4.1. Let L be a universally complete sub-order separable Riesz space. Let $\{e_\alpha; \alpha \in A\}$ and $\{f_\beta; \beta \in B\}$ be any two positive order basis for L consisting of elements of countable type. Then,

$$\{e_\alpha\}L(\sigma) = \{f_\beta\}L(\sigma).$$

<u>Proof</u>. Let $g \in \{e_\alpha\}L(\sigma)$. Then $|g| \wedge e_\alpha \neq 0$ for at most countably many α, say $\{\alpha_i\}_{i=1}^{\infty}$. Consider now $\{f_\beta \wedge e_{\alpha_i}\}$, we must have that for each $i = 1,2,\ldots,$ $f_\beta \wedge e_{\alpha_i} \neq 0$ for at most countably many β, otherwise we would contradict that e_{α_i} is an element of countable type. For each i, denote the set of all β for which $f_\beta \wedge e_{\alpha_i} \neq 0$ by $\{\beta_{ij}\}_{j=1}^{\infty}$.

Now, if $\beta \neq \beta_{ij}$ for any $i,j = 1,2,\ldots,$ we have that $(|g| \wedge f_\beta) \wedge e_\alpha = 0$ for every $\alpha \in A$ which implies that $|g| \wedge f_\beta = 0$. Hence, the set of all β such that $|g| \wedge f_\beta \neq 0$ is contained in $\bigcup_{i=1}^{\infty} \{\beta_{ij}\}_{j=1}^{\infty}$, which is a countable set. This implies that $g \in \{f_\sigma\}L(\sigma)$ and establishes that

$$\{e_\alpha\}L(\sigma) \subseteq \{f_\beta\}L(\sigma).$$

The proof that $\{f_\beta\}L(\sigma) \subseteq \{e_\alpha\}L(\sigma)$ is exactly the same with the roles of f_β and e_α reversed. The lemma is proved.

The above lemma establishes that the formation of the ideals $\{e_\alpha\}L(\sigma)$ is independent of the countable type basis chosen.

Let L be a universally complete sub-order separable Riesz space. By <u>$L(\sigma)$</u> we will mean the unique ideal in L associated with any disjoint positive order basis for L consisting of elements of countable type.

<u>Lemma</u> 4.2. Let L be a universally complete sub-order separable Riesz space. Let L' be any order dense order separable ideal contained in L. Then, $L' \subseteq L(\sigma)$.

<u>Proof</u>. By theorem (4.1), we have that L' has an order basis consisting of disjoint positive elements of countable type, say $\{e_\alpha;\ \alpha \in A\}$, such that if $f \in L'$ then $|f| \wedge e_\alpha \neq 0$ for at most countably many α. Since L' is order dense in L we have that $\{e_\alpha;\ \alpha \in A\}$ is a basis for L. But, then $L(\sigma) = \{e_\alpha\}L(\sigma)$ and it is now clear that $L' \subseteq L(\sigma)$.

<u>Corollary</u> 4.2. Let L be a universally complete sub-order separable Riesz space. Then, $L(\sigma)$ is the largest order separable ideal contained in L.

The relation of $L(\sigma)$ to an order dense order separable ideal $L' \subseteq L$ is made clearer by the following lemma.

<u>Lemma</u> 4.3. Let L be a universally complete sub-order separable Riesz space, and let L_1 be an order dense order separable ideal in L. For an ideal L_2 in L, the following are equivalent:

(1) $L_2 = L(\sigma)$.

(2) $L_2 = \{f \in L:$ for any disjoint collection of elements in L_1, say $\{f_\beta\}$, we have $|f| \wedge |f_\beta| \neq 0$ for at most countably many $\beta\}$.

(3) L_2 is order separable and given any disjoint countable collection $\{f_i;\ i = 1,2,\ldots,\} \subset L_1^+$ we have $\sup\{f_i;\ i = 1,2,\ldots,\} \in L_2$.

<u>Proof</u>. Since L_1 is order separable, it has a basis of positive disjoint elements of countable type. We choose and fix one such basis $\{e_\alpha;\ \alpha \in A\}$. We know that $\{e_\alpha\}L(\sigma) = L(\sigma)$. Now, we will say an element $f \in L$ has property KL_1 if for any disjoint collection of elements in L_1 $\{f_\beta\}$ we have $|f| \wedge |f_\beta| \neq 0$ for at most countably many β. It is clear that to show (1) $\Leftrightarrow$ (2) we must show that f has property KL_1 if and only if $|f| \wedge e_\alpha \neq 0$ for at most countably many α. We clearly have that f has property KL_1 implies

$|f| \wedge e_\alpha \neq 0$ for at most countably many α. Suppose now that $f \in L$ and $|f| \wedge e_\alpha \neq 0$ for at most countably many α, say $\{\alpha_i\}_{i=1}^\infty$. Now, if $\{f_\beta\}$ is any disjoint collection in L_1 we have that, for each i, $|f_\beta| \wedge e_{\alpha_i} \neq 0$ for at most countably many β, say $\{\beta_{ij}\}_{j=1}^\infty$. Proceeding exactly as before, we see that if $\beta \neq \beta_{ij}$ for any i,j, then $|f| \wedge f_\beta = 0$. Hence, f has property KL_1 and we have established (1) $\Leftrightarrow$ (2).

We show now that (1) $\Leftrightarrow$ (3). We need only show that if $\{f_i;\ i = 1,2,\ldots,\} \subset L_1^+$ is a disjoint collection, then $\sup\{f_i\} = f \in L(\sigma)$ (the $\sup\{f_i\}$ exists since L is universally complete). Since $f_i \in L_1$ for each i, we have $f_i \wedge e_\alpha \neq 0$ for at most countably many α, say $\{\alpha_{ij}\}_{j=1}^\infty$. Suppose $\alpha \neq \alpha_{ij}$ for any i,j, then $f \wedge e_\alpha \neq 0$ implies that for some f_i we have $f_i \wedge e_\alpha \neq 0$ and this is a contradiction. Hence, $f \wedge e_\alpha = 0$ for all $\alpha \neq \alpha_{ij}$ for some i,j, and therefore $f \in L(\sigma)$. We have established that (1) $\Rightarrow$ (3).

It only remains to show that (3) $\Rightarrow$ (1). We must show that if $f \in L(\sigma)^+$, then $f = \sup\{f_i;\ i = 1,2,\ldots,\}$ for some countable disjoint collection of positive elements in L_1. Since f is also in L and $L = \bar{L}_1$, we must have that $f = \sup\{f_\beta;\ \beta \in B\}$ for some disjoint collection of strictly positive elements in

L_1 (see remark (3.1)). Now, if $\{f_\beta\}$ is not a countable collection then $\{\alpha : e_\alpha \wedge f_\beta \neq 0 \text{ for all } \beta\}$ is uncountable and hence $\{\alpha : f \wedge e_\alpha \neq 0\}$ is uncountable. Hence, $f \notin L(\sigma)$, which is a contradiction. The lemma is proved.

<u>Remark</u> 4.5. Suppose that L_1 and L_2 are sub-order separable universally complete Riesz spaces. Suppose in addition that there exists a lattice isomorphism φ from L_1 onto L_2 (see p. 13). Then, it is easy to see that φ restricted to $L_1(\sigma)$ takes $L_1(\sigma)$ onto $L_2(\sigma)$. Indeed, if $L_1(\sigma) = \{e_\alpha\}L_1(\sigma)$ then $\{\varphi(e_\alpha)\}$ is an order basis for L_2 consisting of disjoint positive elements of countable type, and it is trivial to show that if $f \in \{e_\alpha\}L_1(\sigma)$ then $\varphi(f) \in \{\varphi(e_\alpha)\}L_2(\sigma) = L_2(\sigma)$. Therefore, $\varphi(L_1(\sigma)) \subseteq L_2(\sigma)$. The same argument applied to φ^{-1} shows that $\varphi^{-1}(L_2(\sigma)) \subseteq L_1(\sigma)$ and hence that $L_2(\sigma) \subseteq \varphi(L_1(\sigma))$. Thus, $\varphi(L_1(\sigma)) = L_2(\sigma)$. This establishes that $L_1(\sigma)$ and $L_2(\sigma)$ are lattice isomorphic.

We can now make the following definition:

<u>Definition</u> 4.4. Let L be an Archimedean order separable Riesz space. By the σ-<u>universal</u> <u>completion</u> of

L we will mean $\bar{L}(\sigma)$ where $\bar{L}$ is the universal completion of L.

Remark 4.6. By lemma (4.2), and the fact that L order separable implies that $\hat{L}$ is order separable, we have that $L \subseteq \hat{L} \subseteq \bar{L}(\sigma)$. L is clearly order dense in $\bar{L}$ and hence in $\bar{L}(\sigma)$. In fact, it follows from lemma (4.3)(3) that $\hat{L}$ is super order dense in $\bar{L}(\sigma)$ and hence L itself is super order dense in $\bar{L}(\sigma)$. It also follows from lemma (4.3)(3) that if $\{f_i\}_{i=1}^{\infty} \subset \bar{L}(\sigma)^+$ and $f_{i_1} \wedge f_{i_2} = 0$ whenever $i_1 \neq i_2$ then $\sup\{f_i\} = f$ exists in $\bar{L}(\sigma)$. From remark (4.5) it follows that $\bar{L}(\sigma)$ is uniquely determined up to lattice isomorphism since $\bar{L}$ is uniquely determined up to lattice isomorphism.

It turns out that the properties discussed in remark (4.6) actually characterize $\bar{L}(\sigma)$.

Theorem 4.4. Let L be an Archimedean order separable Riesz space. Suppose that L' is any Riesz space satisfying the following conditions:

(1) L' is Dedekind complete.

(2) L is a super order dense Riesz subspace of L'.

(3) If $\{f_i;\ i = 1,2,\ldots,\}$ is any countable disjoint collection of elements in $(L')^+$, then $\sup\{f_i\}$ exists in L'.

Then, $L' = \bar{L}(\sigma)$.

<u>Proof</u>. Since L' contains L as an order dense Riesz subspace, it follows that the universal completion of L' is the universal completion of L. In addition, L' being Dedekind complete implies that L' is an ideal in $\overline{L'} = \bar{L}$. The super order density of L in L' implies that L' is order separable. It now follows from condition (3), by virtue of lemma (4.3)(3), that $L' = \bar{L}(\sigma)$. The theorem is proved.

<u>Definition</u> 4.5. Let L be a Riesz space. L will be said to be a σ-universally complete space if it satisfies the following conditions:

(1) L is order separable and Dedekind complete.

(2) If $\{f_i;\ i = 1,2,\ldots,\}$ is any countable disjoint collection of positive elements in L, then $\sup\{f_i\}$ exists in L.

From the above discussion it follows that every Archimedean order separable Riesz space has a σ-universal completion which is unique up to lattice isomorphism.

We need one more result before we can extend the results of section (3). It is contained in our next lemma.

Lemma 4.4. If L is a σ-universally complete Riesz space with an order unit, then L is universally complete.

Proof. Let e be the order unit for L. To show that L is universally complete we must show that every disjoint collection of positive elements in L has a supremum. To this end, let $\{f_\beta; \beta \in B\}$ be a disjoint collection of strictly positive elements in L. The result will follow if we can show that the collection $\{f_\beta\}$ is at most countable. But, since L is order separable, we must have that e is an element of countable type. Therefore, $\{\beta : f_\beta \wedge e \neq 0\}$ is at most countable since the f_β are disjoint. But, since $f_\beta > 0$ for all β and e is a unit for L, we have $\{\beta : f_\beta \wedge e \neq 0\} = B$. Therefore, the collection $\{f_\beta\}$ is at most countable and the lemma is proved.

We now suppose that order convergence is stable in the σ-universally complete Riesz space L (see Definition (1.2)). Let $\{x_n; n = 1,2,\ldots,\}$ be any countable collection in L^+ and consider $L_1 = \{x_n\}^{\perp\perp}$-in-L. We claim that L_1 is universally complete. To see this we need only show that it has an order unit. To this end, let $\{e_\alpha; \alpha \in A\}$ be a positive disjoint order basis for L consisting of elements of countable type. For each n, we have $x_n \wedge e_\alpha \neq 0$ for at most countably many α, say $\{\alpha_{ni}\}_{i=1}^{\infty}$. Now, $\{e_\alpha\colon$ for some n, $x_n \wedge e_\alpha \neq 0\} = \bigcup_{n=1}^{\infty} \{e_{\alpha_{ni}}\}_{i=1}^{\infty}$ which is an at most countable set. Let $\{e_{\alpha_j}; j = 1,2,\ldots,\} = \{e_\alpha\colon$ for some n, $x_n \wedge e_\alpha \neq 0\}$. Since L is σ-universally complete, we have that $\sup\{e_{\alpha_j}\} = e$ exists in L. Clearly $\{e\}^{\perp\perp} \supseteq \{x_n\}^{\perp\perp} = B$. Since L is Dedekind complete, we have that the projection $P_B(e)$ of e onto B exists and it is clearly an order unit for $B = \{x_n\}^{\perp\perp}$. Therefore, B is a universally complete space.

Since order convergence is stable in L implies that order convergence is stable in any band in L, we have that order convergence is stable in B. But, by corollary (3.1), this implies that B is regular, and thus there exists a sequence of real numbers $\{\lambda_n > 0\}$ such that the set $\{\lambda_n x_n\}$ is bounded in B and hence in L. Thus L has property P^* and is itself regular. This implies that L is diagonalizable and hence that L has the Egoroff property.

Now, suppose that the σ-universally complete Riesz space L has the Egoroff property. By remark (2.8) it follows that L has property E.T.*. Obviously, L is relatively uniform on components. Hence, by corollary (2.5), order convergence and relatively uniform convergence are equivalent, for sequences, in L which implies that order convergence is stable in L.

We have established the following result:

Theorem 4.5. Let L be a σ-universally complete Riesz space, then the following are equivalent:

(1) Order convergence is stable in L.

(2) L is diagonalizable.

(3) L has the Egoroff property.

Lemma 4.5. Let L be an Archimedean order separable Riesz space. Then L has the Egoroff property if and only if $\bar{L}(\sigma)$ has the Egoroff property.

Proof. We have already shown that, if L is order separable and Archimedean, L has the Egoroff property if and only if $\hat{L}$ has the Egoroff property. Now, since $\hat{L}$ is a super order dense ideal in $\bar{L}(\sigma)$ (see remark (4.6)), we have, by virtue of Theorem B in section (2), that $\hat{L}$ has the Egoroff property if and only if $\bar{L}(\sigma)$ has the Egoroff property. The assertion is proved.

This next result provides us with a version of the Egoroff theorem which holds for arbitrary Archimedean order separable Riesz spaces with the Egoroff property.

<u>Theorem</u> 4.6. Let L be an Archimedean order separable Riesz space with the Egoroff property. Fix $e \in L^+$, and suppose that $x_n \searrow 0$ in L. Then, there exists a sequence $\{u_m\} \subset L^+$ with $u_m \nearrow e$ such that $x_n \wedge u_m \to_n 0$ (e - r.u.).

<u>Proof</u>. L has the Egoroff property implies that $\hat{L}$ has the Egoroff property and hence there exists, by theorem (2.1), a sequence $e_m \nearrow e$ (in $\hat{L}$) such that $P_{e_m}(x_n) \to_n 0$ (e - r.u.). By remark (1.3), there exists a sequence $u_m \nearrow e$ with $\{u_m\} \subset L^+$ and $u_m \leq e_m$ for $m = 1,2,\ldots,$. Clearly, $u_m \wedge x_n \leq P_{u_m}(x_n) \leq P_{e_m}(x_n)$ (where the projections are taken in $\hat{L}$). Hence if for each m, $\{\lambda_{mn}\}$ is a sequence of reals with $\lambda_{mn} \searrow_n 0$ and such that $P_{e_m}(x_n) \leq \lambda_{mn}\, e$, we also have that $u_m \wedge x_n \leq \lambda_{mn}\, e$. This says that, for each m, $u_m \wedge x_n \to_n 0$ (e - r.u.). The theorem is proved.

In this next definition we introduce a property which is motivated by theorem (4.6). As should be expected, since property E.T.* (see Definition (2.5)) was motivated by theorem (2.1), this property is related to property E.T.*.

Definition 4.6. We will say that an element $0 \leq e$ of the Riesz space L has property E.T. if given any sequence $x_n \searrow 0$ with $x_n \leq e$ for $n = 1,2,\ldots,$ there exists a sequence $\{0 \leq e_m\} \subset L$ with $e_m \nearrow e$ such that $e_m \wedge x_n \to_n 0$ $(e - \text{r.u.})$.

Definition 4.7. We will say that the Riesz space L has property E.T. if every $e \in L^+$ has property E.T. .

We note immediately that property E.T.* implies property E.T. .

We will now establish some more relationships between properties E.T. and E.T.*. We will begin with a somewhat large collection of lemmas which will result in a large characterization of the properties involved.

Lemma 4.6. Let L be a Riesz space. If f and g are elements of L^+ satisfying $f \wedge g = 0$, and if both f and g have property E.T.*, then $f + g = f \vee g$ has property E.T.*.

Proof. Suppose $\{x_n\}$ is a sequence in L with $x_n \searrow 0$ and $x_n \leq f+g$ for $n = 1,2,\ldots,$. By the decomposition property [c.f. (25) Corollary 1.4 p. 8], we have that $x_n = f_n + g_n$ where $f_n \leq f$ and $g_n \leq g$ for $n = 1,2,\ldots,$.

Since f and g have property E.T.*, we have sequences $e_m \nearrow f$ and $z_m \nearrow g$ such that, for each m, $e_m \wedge (f - e_m) = 0$ and $z_m \wedge (g - z_m) = 0$, and such that $e_m \wedge f_n \to_n 0$ $(f - \text{r.u.})$ and $z_m \wedge g_n \to_n 0$ $(g - \text{r.u.})$. That is, there exist sequences of positive real numbers $\{\lambda_{mn}\}$ and $\{\alpha_{mn}\}$ satisfying $\lambda_{mn} \searrow_n 0$ and $\alpha_{mn} \searrow_n 0$ for $m = 1,2,\ldots,$ and such that $e_m \wedge f_n \leq \lambda_{mn} f$ and $z_m \wedge g_n \leq \alpha_{mn} g$ for $m,n = 1,2,\ldots,$.

Since $f \wedge g = 0$, we clearly have that f and g are components of $f + g$. Therefore, by remark (2.3)(2), we have that, for each m, e_m and z_m are components of $f + g$. Furthermore, since for each m, $e_m \leq f$ and $z_m \leq g$, we have $e_m \wedge z_m = 0$ and therefore $e_m + z_m = e_m \vee z_m$. But then by remark (2.3)(1) $e_m + z_m$ is a component of $f + g$ for each m. Clearly, $e_m + z_m \nearrow f + g$. Also, by [(25) Prop. 1.2 (19) p. 6], we have that the following chain of inequalities holds:

$$(e_m + z_m) \wedge x_n = (e_m + z_m) \wedge (f_n + g_n) \leq$$
$$\leq e_m \wedge f_n + z_m \wedge g_n \leq \lambda_{mn} f + \alpha_{mn} g \leq (\lambda_{mn} + \alpha_{mn})(f + g).$$

This completes the proof.

<u>Remark</u> 4.7. It is not difficult to see that if an element $f > 0$ of the Riesz space L has property E.T.* and if g is any component of f, then g has property E.T.*.

Lemma 4.7. Let L be a Riesz space. Let $f \in L^+$ and assume that $f = \sup\{f_n;\ n = 1,2,\ldots,\}$ where $f_n \nearrow$, $f_n \wedge (f - f_n) = 0$ for each n, and each f_n has property E.T.*. Then, f has property E.T.*.

Proof. Suppose the sequence $x_m \searrow 0$ is such that $x_m \leq f$ for $m = 1,2,\ldots,$. Now, each $f_n \wedge x_m \searrow_m 0$ for $n = 1,2,\ldots,$ and satisfies $f_n \wedge x_m \leq f_n$. Hence, since for each n, f_n has property E.T.*, there exist components f_{nk} of the f_n such that $f_{nk} \nearrow_k f_n$, and sequences of positive real numbers $\{\lambda_{nkm}\}$ with $\lambda_{nkm} \searrow_m 0$ for $n,k = 1,2,\ldots,$ such that $f_{nk} \wedge x_m \leq \lambda_{nkm} f_n \leq \lambda_{nkm} f$ for $n,k,m = 1,2,\ldots,$.

Now, take $f_k^* = \sup\{f_{ik};\ 1 \leq i \leq k\}$. Then, $f_k^* \wedge (f - f_k^*) = 0$, and $f_k^* \nearrow f$ for $k = 1,2,\ldots,$.

We have now the inequality

$$f_k^* \wedge x_m \leq \sup\{\lambda_{ikm} f;\ 1 \leq i \leq k\} = \alpha_{km} f$$

where $\alpha_{km} = \max\{\lambda_{ikm};\ 1 \leq i \leq k\}$ for $k,m = 1,2,\ldots,$. Clearly $\alpha_{km} \searrow_m 0$ for $k = 1,2,\ldots,$ and the proof is complete.

Lemma 4.8. Let L be a Riesz space with the P.P.P., and let $f,g \in L$ with $0 \leq g \leq f$. Then, if f has property E.T.*, g has property E.T.*.

<u>Proof</u>. Consider $g_p = P_{(pg-f)^+}(g)$ for $p = 1,2,\ldots,$. It is trivial to see that $g_p \nearrow g$, $g_p \wedge (g - g_p) = 0$, and $P_{g_p}(f) \leq pg_p \leq pg$.

Now, let $x_n \searrow 0$ with $x_n \leq g_p$ for $n = 1,2,\ldots,$ and for fixed p. At the same time we have $x_n \leq f$ for all n, and hence there exist a sequence of components $\{f_m\}$ of f with $f_m \nearrow f$ and appropriate sequences of real numbers $\{\lambda_{mn}\}$ with $\lambda_{mn} \searrow_n 0$ for $m = 1,2,\ldots,$ such that $f_m \wedge x_n \leq \lambda_{mn} f$. Taking $z_{pm} = f_m \wedge g_p$, we see easily that $z_{pm} \searrow_m g_p$, and $z_{pm} \wedge (g_p - z_{pm}) = 0$.

Taking projections on both sides of the inequality $x_n \wedge f_m \leq \lambda_{mn} f$ we obtain

$$z_{pm} \wedge x_n = P_{g_p}(x_n \wedge f_m) \leq \lambda_{mn} P_{g_p}(f) \leq p\, \lambda_{mn}\, g_p.$$

This establishes that, for each p, the element g_p has property E.T.*, and hence by application of lemma (4.7) the proof is completed.

<u>Lemma</u> 4.9. Let L be a Riesz space with the P.P.P. . Let $f \in L^+$, and suppose there exists a sequence $\{u_n\} \subset L^+$ such that $u_n \nearrow f$ and each u_n satisfies property E.T.*, where $n = 1,2,\ldots,$. Then, f has property E.T.*.

Proof. Let $f_n = P_{u_n}(f)$, then $f_n \nearrow f$ and $f_n \wedge (f - f_n) = 0$ for $n = 1,2,\ldots,$.

For each n, consider $f_{nm} = P_{(mu_n - f_n)^+}(f_n)$, where $m = 1,2,\ldots,$. Clearly $f_{nm} \nearrow_m f_n$, and $f_{nm} \wedge (f_n - f_{nm}) = 0$ for $m = 1,2,\ldots,$.

We also have that $f_{nm} \leq mu_n$ and hence by lemma (4.8) each f_{nm} has property E.T.*.

Applying lemma (4.7) twice then completes the proof.

Lemma 4.10. Let L be a Riesz space with the P.P.P. . Suppose the elements $f,g \in L^+$ have property E.T.*. Then $f + g$ has property E.T.*.

Proof. Let $z = P_f(g)$. Then $f + g = (z+f) + (g-z)$. Now, $P_f(z+f) = z + f = \sup_n (nf \wedge (z+f))$ and hence by application of lemmas (4.8) and (4.9) $z + f$ has property E.T.*. Since $0 \leq g - z \leq g$, by lemma (4.8), $g - z$ has property E.T.*. We also have, though, that $(g-z) \wedge (z+f) = 0$ and by application of lemma (4.6) the proof is completed.

All of these results combine to give this next theorem.

<u>Theorem</u> 4.7. Let L be a Riesz space with the P.P.P. . Then the set of all $f \in L$ such that $|f|$ has property E.T.* is a σ-closed ideal in L.

<u>Proof</u>. Let $I = \{f \in L: |f| \text{ has property E.T.}^*\}$. If $f,g \in I$, by lemma (4.10), we have that $|f| + |g|$ is an element of I and by lemma (4.8) we have that $(f+g) \in I$. In fact, since $|f \vee g| \leq |f| + |g|$ and $|f \wedge g| \leq |f| + |g|$, we have, by lemma (4.8), that if $f,g \in I$ then $f \vee g$ and $f \wedge g$ are elements of I. It is trivial to see that if λ is a real number and $f \in I$ then $\lambda f \in I$. We thus have that I is an ideal. The fact that I is σ-closed follows from lemma (4.9). The theorem is proved.

We now obtain a connection between property E.T. and property E.T.*.

<u>Theorem</u> 4.8. Let L be a Riesz space with the P.P.P. . Then an element $e \in L^+$ has property E.T. if and only if e has property E.T.*.

<u>Proof</u>. Property E.T.* always implies property E.T. . We prove the necessity. Suppose $e \in L^+$ has property E.T. and let the sequence $x_n \searrow 0$ with $x_n \leq e$ for $n = 1,2,\ldots,$. By property E.T. there exist a sequence $\{e_m\} \subset L^+$ with $e_m \nearrow e$, and corresponding

sequences of positive real numbers $\{\lambda_{mn}\}$ with $\lambda_{mn} \searrow_n 0$ for $m = 1,2,\ldots,$ such that $e_m \wedge x_n \leq \lambda_{mn} e$ for $m,n = 1,2,\ldots,$. Fix m for the time being and consider the component z_m of e, where $z_m = P_{e_m}(e)$. Since $P_{e_m}(P_{e_m}(e)) = P_{e_m}(e)$, we have $z_m = \sup\{pe_m \wedge z_m;$ $p = 1,2,\ldots,\}$. Consider $v_p^m = P_{(pe_m - z_m)}(z_m)$, we have $v_p^m \nearrow_p z_m$ and $v_p^m \wedge (z_m - v_p^m) = 0$ for $p = 1,2,\ldots,$. Also, since $v_p^m \leq pe_m$, we have that $x_n \wedge v_p^m \leq (pe_m) \wedge x_n \leq p(e_m \wedge x_n) \leq p\lambda_{mn}\, e$.

Now, take $u_p = \sup\{v_p^i;\ 1 \leq i \leq p\}$, we clearly have that $u_p \nearrow e$ and satisfies $u_p \wedge (e - u_p) = 0$, for $p = 1,2,\ldots,$. Also,

$$u_p \wedge x_n \leq (x_n \wedge v_p^1) \vee (x_n \wedge v_p^2) \vee \ldots \vee (x_n \wedge v_p^p) \leq$$

$$\leq (x_n \wedge (pe_1)) \vee (x_n \wedge (pe_2)) \vee \ldots \vee (x_n \wedge (pe_p)) \leq$$

$$\leq p[\lambda_{1n} \vee \lambda_{2n} \vee \ldots \vee \lambda_{pn}]e.$$

But $p[\lambda_{1n} \vee \lambda_{2n} \vee \ldots \vee \lambda_{pn}] \searrow_n 0$ and this shows that $u_p \wedge x_n \to_n 0$ (e - r.u.). The theorem is proved.

As an immediate consequence we obtain

<u>Corollary</u> 4.3. Let L be a Riesz space with the P.P.P. . Then, L has property E.T. if and only if L has property E.T.*.

<u>Theorem</u> 4.9. If L is order separable and Archimedean, then L has property E.T. if and only if $\hat{L}$ has property E.T.* .

Proof. The proof that $\hat{L}$ has property E.T.* implies L has property E.T. is contained clearly in the proof of theorem (4.6). We show, therefore, the necessity. In view of lemma (4.8) it will suffice to show that every element $e \in L^+ \subseteq \hat{L}^+$ has property E.T.* relative to $\hat{L}$. Since $\hat{L}$ is Dedekind complete and hence has the P.P.P., we see by theorem (4.8) that it will suffice to show that every $e \in L^+$ has property E.T. relative to $\hat{L}$. To this end, let $\{\hat{x}_n\}$ be a sequence in $\hat{L}$ with $\hat{x}_n \searrow 0$, and such that $\hat{x}_n \leq e$ for $n = 1,2,\ldots,$. By remark (1.3), there exists a sequence $\{y_n\} \subset L$ with $y_n \searrow 0$ such that $y_n \geq \hat{x}_n$ for $n = 1,2,\ldots,$. Taking $x_n = y_n \wedge e$ we have $x_n \searrow 0$, $x_n \geq \hat{x}_n$, $\{x_n\} \subset L$, and $x_n \leq e$ for $n = 1,2,\ldots,$. Since L has property E.T., there exists a sequence $\{e_m\}$ in L^+ with $e_m \nearrow e$ such that for appropriate sequences of positive real numbers $\{\lambda_{mn}\}$ with $\lambda_{mn} \searrow_n 0$ we have $e_m \wedge x_n \leq \lambda_{mn}\, e$ for $m,n = 1,2,\ldots,$. But $\hat{x}_n \leq x_n$ implies that $e_m \wedge \hat{x}_n \leq e_m \wedge x_n \leq \lambda_{mn}\, e$, and this says that e has property E.T. relative to $\hat{L}$. The proof is complete.

This next theorem summarizes much of what we have accomplished in this chapter.

<u>Theorem</u> 4.10. If L is an Archimedean order separable Riesz space, then the following are equivalent:

(1) L has property E.T. .

(2) $\hat{L}$ has property E.T.*.

(3) $\bar{L}(\sigma)$ has property E.T.*.

(4) Order convergence is stable in $\bar{L}(\sigma)$.

(5) $\bar{L}(\sigma)$ is diagonalizable.

(6) $\bar{L}(\sigma)$ has the Egoroff property.

(7) L has the Egoroff property.

<u>Proof</u>. (1) $\Leftrightarrow$(2) was theorem (4.9).

(2) $\Rightarrow$ (3) follows from the fact that $\hat{L}$ is super order dense in $\bar{L}(\sigma)$ combined with theorem (4.7).

(3) $\Rightarrow$ (4) follows as before by application of corollary (2.5)(4).

(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) is theorem (4.5).

(6) $\Leftrightarrow$ (7) is lemma (4.5).

(7) $\Rightarrow$ (1) follows from theorem (4.6).

The theorem is proved.

A somewhat surprising application of theorem (4.10) is the following:

<u>Corollary</u> 4.4. Let L be an Archimedean order separable Riesz space. If in L order convergence and relatively uniform convergence are equivalent for sequences, then L has the Egoroff property.

<u>Proof</u>. Since, in this situation, L has the property that order convergence and relatively uniform convergence are equivalent for sequences if and only if $\hat{L}$ does (see Proposition (1.1)(e)), we may as well assume that L is Dedekind complete.

Now, let $e \in L^+$ and let the sequence $\{x_n\}$ in L be such that $x_n \searrow 0$ and $x_n \leq e$ for $n = 1,2,\ldots,$. We have, by assumption, a $u \in L$ and a sequence of positive real numbers $\{\lambda_n\}$ with $\lambda_n \searrow 0$ such that $x_n \leq \lambda_n u$ for $n = 1,2,\ldots,$.

Since $x_n \leq \lambda_n u$ implies that $x_n \leq \lambda_n (u \vee e)$, we may assume that $u \geq e$.

Consider now $e_m = P_{(me-u)^+}(e)$, where $m = 1,2,\ldots,$. We have $e_m \wedge (e - e_m) = 0$ for each m, $e_m \nearrow e$, and $P_{e_m}(u) \leq me_m$.

Applying the projection operator P_{e_m} to both sides of the inequality $x_n \leq \lambda_n u$ we obtain

$$e_m \wedge x_n = P_{e_m}(x_n) \leq \lambda_n P_{e_m}(u) \leq m\lambda_n e_m \leq m\lambda_n e.$$

From this we conclude that $e_m \wedge x_n \to_n 0$ $(e - \text{r.u.})$. We have established that L has property E.T.*, and the conclusion follows from theorem (4.10).

CHAPTER II

MEASURE RIESZ SPACES

In the theory of Riesz spaces, an important role has been played by obtaining concrete representations of certain abstractly defined Riesz spaces. The first example of such is Kakutani's representation of an abstract (L)-space (Banach lattice such that $\|u + v\| = \|u\| + \|v\|$ for $u \geq 0$ and $v \geq 0$) as a space $L_1(S, \Sigma, \mu)$ of equivalence classes of μ-summable functions where μ is a completely additive measure on a σ-algebra of subsets of some set S; [see (9)].

A more general result is the realization of arbitrary Archimedean Riesz spaces as order dense Riesz subspaces of $C_\infty(Q)$ where Q is some extremal compactum and $C_\infty(Q)$ is the Riesz space of all extended real valued functions on Q which are finite valued and continuous on some open dense subset of Q (see Chapter I §3). One of the features of this representation is that every order dense Riesz subspace of an Archimedean Riesz space determines the same $C_\infty(Q)$ up to homeomorphism of the compactum Q. Until recently, however, one unfortunate gap in the theory was that there was no definitive means for distinguishing between a $C_\infty(Q)$ which represents the equivalence classes of all measurable finite almost every functions on some completely additive measure space and say a $C_\infty(Q)$ which represents the universal completion of $C[0,1]$.

We might reason, however, that one plausible criterion would be to demand that the $C_\infty(Q)$ possess an order dense abstract (L) Riesz subspace, say D. By Kakutuni's result, we would have that $D = L_1(S,\Sigma,\mu)$. But, if $M(S,\Sigma,\mu)$ is the space of equivalence classes of all μ-measurable finite almost everywhere functions, then $M(S,\Sigma,\mu)$ is clearly a universal completion of $L_1(S,\Sigma,\mu) = D$ and hence we would have that $M(S,\Sigma,\mu) = C_\infty(Q)$.

J.J. Masterson has recently obtained a characterization of spaces $M(S,\Sigma,\mu)$ (the set of equivalence classes of almost everywhere finite valued μ-measurable functions on the localizable measure space (S,Σ,μ) with the finite subset property and with a completely additive measure μ) in [(20)]. One of the essential concepts involved in his investigation is the one mentioned in the preceding paragraph. We mention this here, because doing so focuses attention on one important concept, that being that the universal completion of an Archimedean Riesz space L is completely determined by any order dense Riesz subspace of L.

The purpose of this chapter is to investigate the use of order dense Riesz subspaces of spaces of measurable functions to represent Archimedean Riesz spaces. In the course of the investigation a number of questions concerning general embeddings of certain Riesz spaces into spaces of measurable functions will be considered.

Section 1. Basic Definitions and Results.

In this section we begin an investigation into the use of spaces of measurable functions to represent Archimedean Riesz spaces. Throughout the remainder of this chapter, unless specified otherwise, we will mean by the term "space of measurable functions", or by the notation $M(S,\Sigma,\mu)$, the Riesz space of all equivalence classes of finite almost everywhere measurable functions on a localizable measure space (S,Σ,μ) with the finite subset property and with a completely additive measure μ. For a complete discussion of such spaces we refer the reader to [(29)].

For certain Riesz spaces, embeddings into spaces of measurable functions exist which are sufficiently nice so that the Riesz spaces are seen to possess the essential properties of spaces of measurable functions. This will be used as justification for calling this class of Riesz spaces the "measure Riesz spaces".

We will then introduce a variant of the Egoroff property and obtain some characterization theorems for measure Riesz spaces where this property yields significant reductions in the hypothesis of theorems we will consider early in the discussion.

We begin with a brief discussion of some concepts and results we will be needing in our investigations.

Let L_1 and L_2 be Riesz spaces. A lattice isomorphism $\varphi: L_1 \to L_2$ (from L_1 into L_2) is said to be <u>continuous</u> if for any net $0 \leq u_\tau \downarrow 0$-in-$L_1$ we have $\inf\{|\varphi(u_\tau)|\} = 0$-in-$L_2$.

The real linear functional φ on the Riesz space L is said to be <u>order bounded</u> if, for every $u \in L^+$, the number $\sup\{|\varphi(f)| : |f| \leq u\}$ is finite. The set of all order bounded linear functionals will be denoted by $L^\sim$.

A linear functional φ on the Riesz space L is said to be <u>positive</u> if for any $0 \leq f \in L$ we have $\varphi(f) \geq 0$.

We say that $\varphi_1 \leq \varphi_2$ for φ_1 and φ_2 order bounded linear functionals on the Riesz space L whenever $\varphi_2 - \varphi_1$ is positive.

It is shown in [(18) Note VI Thm. 18.4] that $L^{\sim}$ is a Dedekind complete Riesz space with respect to the partial ordering just introduced.

The linear functional φ on the Riesz space L is said to be an <u>integral</u> if whenever $u_n \downarrow 0$-in-L we have $\inf\{|\varphi(u_n)|\} = 0$. The set of all integrals will be denoted by $L_c^{\sim}$.

The element $\varphi \in L^{\sim}$ will be called a <u>singular functional</u> if $\inf\{|\varphi|, |\psi|\} = 0$ for every integral $\psi \in L_c^{\sim}$. The set of all singular functionals will be denoted by $L_s^{\sim}$.

It is shown in [(18) Note VI Thm. 20.3] that both $L_c^{\sim}$ and $L_s^{\sim}$ are bands in $L^{\sim}$ and that $L^{\sim} = L_c^{\sim} \oplus L_s^{\sim}$.

The element $\varphi \in L^{\sim}$ will be called a <u>normal integral</u> if whenever $\{u_\tau\}$ is any net in L with

$u_\tau \searrow 0$ we have $\inf\{|\varphi(u_\tau)|\} = 0$. The set of all normal integrals on the Riesz space L will be denoted by $L_n^\sim$.

The element $\varphi \in L^\sim$ will be said to be <u>singular with respect to the normal integrals</u> on L if $\inf\{|\varphi|, |\psi|\} = 0$ for every $\psi \in L_n^\sim$. The set of all such φ will be denoted by $L_{sn}^\sim$.

Again we have that both $L_n^\sim$ and $L_{sn}^\sim$ are bands in $L^\sim$ and that $L^\sim = L_n^\sim \oplus L_{sn}^\sim$.

For any $\varphi \in L^\sim$, we set $N_\varphi = \{f : f \in L, |\varphi|(|f|) = 0\}$. Then N_φ is an ideal in L, and we call N_φ the <u>null ideal</u> of φ. The ideal $C_\varphi = N_\varphi^\perp$-in-$L$ is called the <u>carrier</u>, or <u>support</u>, of φ. If φ is a normal integral then N_φ is a band and hence for normal integrals $C_\varphi = \{0\}$ if and only if $\varphi \equiv 0$.

For a complete discussion of the concepts introduced above we refer the reader to [(18) Notes VI, VIII].

Now, let L be an Archimedean Riesz space. By F we denote the family of all order dense ideals contained in L. F is a filter basis. Let $\Phi = \cup\{I_n^\sim : I \in F\}$. If $\phi \in \Phi$, we denote by I_ϕ its domain of definition.

We define the following relation on Φ:

$\emptyset_1 \equiv F\emptyset_2$ whenever $\{f : \emptyset_1(f) = \emptyset_2(f),\ f \in L\}$ contains an order dense ideal. Since F is a filter basis, it follows immediately that the relation $\equiv F$ is an equivalence relation. The set of classes of equivalent elements will be denoted by $\Gamma(L)$. $\Gamma(L)$ turns out to be a Riesz space under the partial ordering induced by defining $[\emptyset] \geq 0$ if there exists a $\emptyset' \in [\emptyset]$ such that on $I_{\emptyset'}$ we have if $0 \leq f \in I_{\emptyset'}$ then $\emptyset'(f) \geq 0$.

The set $\{I_{\emptyset}: \emptyset \in [\emptyset]\}$ can be partially ordered by inclusion and can be shown to contain a maximal element which we denote by $D_{\bar{\emptyset}}$. It can be further shown that the $\bar{\emptyset}$ corresponding to $D_{\bar{\emptyset}}$ is a uniquely defined element of $[\emptyset]$. Thus, we can now identify $\Gamma(L)$ with the set of its maximal elements.

Furthermore, it can be shown that $\Gamma(L)$ is universally complete and that $L_n^{\sim}$ embeds in $\Gamma(L)$ as an ideal.

The space $\Gamma(L)$ was first introduced by J.J. Masterson in his thesis (A Generalization of the Concept of the Order Dual of a Vector Lattice; Purdue University 1965). For a more accessible reference, we refer the reader to [(17)].

Remark 1.1. We list here a few more results concerning $\Gamma(L)$ which we will find of particular interest to our discussion. These results can also be found in [(17)].

(1) Let L be an Archimedean Riesz space and $M \subseteq L$ be an order dense Riesz subspace of L, then $\Gamma(M) = \Gamma(L)$.

(2) Let L be an Archimedean Riesz space, then the following are equivalent:

(a) $\Gamma(L)$ is separating on L.

(b) There exists an order dense ideal $I \subseteq L$ such that $I_n^\sim$ is separating.

(c) There exists an order dense ideal $I \subseteq L$ which possesses a strictly positive normal integral.

Masterson also proved in [(20)] the following theorem:

Theorem D. (Masterson) Let L be a Riesz space. The following conditions are necessary and sufficient that there exist a completely additive localizable measure μ with the finite subset property such that $L = M(S, \Sigma, \mu)$:

(1) $\Gamma(L)$ is separating,

(2) L is universally complete.

Remark 1.2. If we combine theorem D above with remark (1.1) we obtain the following:

If L is an Archimedean Riesz space, in order that there exist a space of measurable functions $M(S,\Sigma,\mu)$ such that $\bar{L} = M(S,\Sigma,\mu)$ it is both necessary and sufficient that $\Gamma(L)$ is separating.

This says that L is lattice isomorphic to an order dense Riesz subspace of a space of measurable functions if and only if $\Gamma(L)$ is separating.

Remark 1.3. In the above remark (1.2), we may assume that the measure μ is σ-finite if and only if L has a countable disjoint basis of positive elements of countable type (for definitions see Chapter I §4).

We will find this next result useful.

Lemma 1.1. Let L be an Archimedean Riesz space. Suppose that given any $f \in L^+$ we have that if $f \neq 0$ then $I(f)_n^\sim \neq \{0\}$ (where $I(f) = \{g \in L: |g| \leq f\}$ i.e. is the ideal generated by f). Then $\Gamma(L)$ is separating.

Proof. Since $\varphi \in I(f)_n^\sim$ implies that φ^+ and φ^- are in $I(f)_n^\sim$, we may assume that φ is positive and hence that $\varphi(f) \neq 0$. Consider now the ideal $B = I(f) \oplus (I(f)^{\perp}\text{-in-}L)$. By [(18) Note IX, Thm. 29.10 and Thm. 29.5], we have that B is an order dense ideal in L. If $g \in B$, we will denote its components in $I(f)$ and $I(f)^{\perp}$-in-L by g_1 and g_2 respectively. Define φ_0 on B by taking $\varphi_0(g) = \varphi_0(g_1 + g_2) = \varphi(g_1)$ for $g \in B$. The verification that φ_0 is a positive linear functional on B is trivial. To see that φ_0 is a normal integral, we note first that if $\{g_\tau = g_{1\tau} + g_{2\tau}\}$ is any net in B, then $g_\tau \searrow 0$ if and only if $g_{1\tau} \searrow 0$ and $g_{2\tau} \searrow 0$. After noting this, the normality of φ_0 follows trivially from the normality of φ.

Obviously, the unique $\hat{\varphi}_0 \in \Gamma(L)$ determined by φ_0 has the property that $\hat{\varphi}_0(f) \neq 0$.

Now, for an arbitrary $f \in L$, we can apply the lemma to either f^+ or f^-, say f^-; and since f^- will then be in $I(f^+)^{\perp}$-in-L, the same construction we have given above will guarantee the existence of a $\varphi \in \Gamma(L)$ such that $\varphi(f) \neq 0$. The assertion is proved.

It is not difficult to see that a lattice embedding φ from a Riesz space L_1 onto an order dense Riesz subspace I of a Riesz space L_2 must be continuous. Therefore, by remark (1.2), we see that if

$\Gamma(L)$ is separating then there exists an order continuous embedding of L into a Riesz space of measurable functions.

It is natural to ask whether the existence of an order continuous embedding φ from a Riesz space L onto a Riesz subspace of a space of measurable functions (not necessarily an order dense Riesz subspace) is enough to guarantee that $\Gamma(L)$ is separating.

The answer turns out to be affirmative.

Corollary 1.1. Let L be an Archimedean Riesz space. In order that there exist a space of measurable functions $M(S,\Sigma,\mu)$ and an order continuous embedding φ from L onto a Riesz subspace of $M(S,\Sigma,\mu)$ (not necessarily an order dense Riesz subspace), it is both necessary and sufficient that $\Gamma(L)$ is separating.

Proof. We need only show the necessity. Suppose φ is an order continuous embedding of L into $M(S,\Sigma,\mu)$. If $f \in M(S,\Sigma,\mu)$, it is trivial to show that $I(f)_n^{\sim} \neq \{0\}$. Now, let $g \in L^+$. We have that $\varphi(I(g)) \subseteq I(\varphi(g))$ and $\varphi(g) \in M(S,\Sigma,\mu)^+$. Let $0 < \psi \in [I(\varphi(g))_n^{\sim}]^+$. Then $\psi_0 = \psi \circ \varphi \in I(g)_n^{\sim}$ and $\psi_0(g) = \psi(\varphi(g)) \neq 0$. By application of lemma (1.1), we have that $\Gamma(L)$ is separating. The assertion is proved.

We have established the validity of this next proposition.

Proposition 1.1. The following conditions on an Archimedean Riesz space L are equivalent:

(1) There exists an order continuous embedding of L into some space of measurable functions.

(2) There exists an embedding of L onto an order dense Riesz subspace of a space of measurable functions.

(3) $\bar{L}$ is a space of measurable functions.

(4) $\Gamma(L)$ is separating.

(5) There exists an order dense ideal in L, say I, such that $I_n^\sim$ is separating.

(6) There exists an order dense ideal $I \subseteq L$ which possesses a strictly positive normal integral.

(7) Given any $f \in L^+$, $I(f)_n^\sim \neq \{0\}$ $(f \neq 0)$.

Of the equivalent conditions in proposition (1.1) above, the most descriptive is condition (2). For this reason, we will use this condition to make a definition.

Definition 1.1. An Archimedean Riesz space L will be said to be a measure Riesz space if it is embeddable as an order dense Riesz subspace of some space of measurable function $M(\mathbf{S},\Sigma,\mu)$ where μ is a completely additive, localizable measure with the finite subset property.

We will find that this next lemma helps to illucidate the role of measure Riesz spaces in the general theory of Riesz spaces.

Lemma 1.2. Let L be an Archimedean Riesz space, then there exist bands L_1 and L_2 in L satisfying the following conditions:

(1) $L_1 = L_2^{\perp}$.

(2) $\Gamma(L_1) = \Gamma(L)$ and $\Gamma(L_1)$ is separating.

(3) $\Gamma(L_2) = \{0\}$.

Proof. Let $\emptyset \in \Gamma(L)$, $D_{\emptyset}$ its domain of definition, and $N_{\emptyset}$ its null ideal in $D_{\emptyset}$. It is easy to see that, for each $\emptyset \in \Gamma(L)$, $N_{\emptyset}$ is a band in L. Let $L_2 = \cap\{N_{\emptyset};\ \emptyset \in \Gamma(L)\}$. L_2 is a band in L since each $N_{\emptyset}$ is a band. We show first that $L_2 = \{0\}$ if and only if $\Gamma(L)$ is separating. Then sufficiency is obvious. So, suppose that $L_2 = \{0\}$. Then, given any $f \in L^+$, there exists a $\emptyset \in \Gamma(L)^+$ such that $\emptyset(f) \neq 0$. If $\emptyset(f) < \infty$ we are done. If $\emptyset(f) = \infty$, we are done by virtue of [(17) Thm. 2.1]. Therefore, if $L_2 = \{0\}$, we have that $\Gamma(L)$ is separating and by taking $L_1 = L$ we see that conditions (1), (2), and (3) hold.

Now, suppose that $L_2 \neq \{0\}$. Consider then $L_2^{\perp} = L_1$. We have that L_1 is a band by its definition. We will show now that $\Gamma(L_1) = \Gamma(L)$. To this end, consider the map $\phi \to \phi'$ where $\phi \in \Gamma(L)$ and $\phi' = \phi | D_{\phi} \cap L_1$. It is clear that $\phi' \in \Gamma(L_1)$, and that the mapping $\phi \to \phi'$ is a lattice homomorphism. We wish to show that it is one-to-one and onto. To show that the mapping $\phi \to \phi'$ is one-to-one, we recall first that if $\phi_1, \phi_2 \in \Gamma(L)^+$ and $\phi_1 \neq \phi_2$, then for any order dense ideal $I \subseteq D_{\phi_1} \cap D_{\phi_2}$ there must exist an $f \in I^+$ such that $\phi_1(f) - \phi_2(f) \neq 0$. But, if $\phi_1, \phi_2 \in \Gamma(L)^+$ and $\phi_1 \neq \phi_2$ then, since $I = D_{\phi_1} \cap D_{\phi_2} \cap (L_1 \oplus L_2)$ is order dense in L and since $I \subseteq D_{\phi_1} \cap D_{\phi_2}$, we must have an element $0 < f \in I$ such that $\phi_1(f) - \phi_2(f) \neq 0$. But, $f = f_1 + f_2$ where $f_1 \in L_1$ and $f_2 \in L_2$ and hence $\phi_1(f) - \phi_2(f) = \phi_1(f_1) - \phi_2(f_1) \neq 0$. We have established that the mapping $\phi \to \phi'$ is one-to-one. To see that the mapping $\phi \to \phi'$ is onto, we let $\phi' \in \Gamma(L_1)$ and $D_{\phi'}$ be its domain of definition. Consider $D = D_{\phi'} \oplus (D_{\phi'}^{\perp}\text{-in-}L)$. Each $f \in D$ has a representation of the form $f = f_1 + f_2$ where $f_1 \in D_{\phi'}$ and $f_2 \in D_{\phi'}^{\perp}\text{-in-}L$. We define $\bar{\phi}$ by $\bar{\phi}(f) = \bar{\phi}(f_1 + f_2) = \phi(f_1)$ for $f \in D$. It is clear that $\bar{\emptyset}$ is a normal integral on D which is an order dense ideal in L. It is clear that the element $\emptyset_1 \in \Gamma(L)$ determined by $\bar{\emptyset}$ satisfies the condition that $\emptyset_1 \to \emptyset'$. Hence the mapping $\emptyset \to \emptyset'$ is onto. We have established that $\Gamma(L_1) = \Gamma(L)$.

Seeing that $\Gamma(L_2) = \{0\}$ is trivial.

The fact that $\Gamma(L_1)$ is separating follows from [(17) Thm. 2.1]. The proof is completed.

Remark 1.4. It follows easily from lemma (1.2) that if L is an Archimedean Riesz space, then there exist a space of measurable functions $M(S,\Sigma,\mu) = L_1$, and a universally complete space L_2 such that $\bar{L} = L_1 \oplus L_2$, $\Gamma(L_1) = \Gamma(L)$, and $\Gamma(L_2) = \{0\}$. This essentially says that for any question concerning normal integrals we may as well restrict our attention to spaces of measurable functions.

Remark 1.5. Another example of the usefulness of this theory is the following.

Every reflexive Banach lattice is a measure Riesz space.

To see this, we recall first the well known result [c.f. (27) p. 228] that if L is a Banach lattice then

$L' = L^{\sim}$ (where L' is the topological dual of L). Also, it follows from an equally well known result, that L' is a Dedekind complete Banach lattice [c.f. (27) Thm. IX, 4.1 p. 255]. Since L is reflexive, conbining these two results, we obtain that $L = L'' = (L^{\sim})^{\sim}$, and that L is Dedekind complete. If $\|\cdot\|$ is the norm on L, it follows from [(27) Thm. IX, 7.2 p. 269] that whenever $x_n \searrow 0$ in L we have that $\|x_n\| \to 0$. From this and the fact that L is Dedekind complete, it follows from [(25) Prop. 2.6 p. 164] that L is order separable. It is now easy to see that $L' = L^{\sim} = L_n^{\sim}$ and hence $\Gamma(L)$ is separating. We have shown that not only is L a measure Riesz space, but that if $M(S,\Sigma,\mu) = \bar{L}$, then L is an ideal in $M(S,\Sigma,\mu)$. Thus for the study of reflexive Banach lattices we can restrict attention solely to Riesz spaces which are ideals in spaces of measurable functions.

Remarks (1.4) and (1.5) were meant merely to give an early indication of the uses to which our current investigations can be put. Much more general results along similar lines will be obtained in succeeding sections.

In the following we give an example of a space of measurable functions which is continuously embedded into another space of measurable functions as a Riesz subspace which is not order dense. Showing that condition (1) of proposition (1.1) is not a vacuous consideration.

Example 1.1. Let m be the Riesz space of all real sequences. Let L be the space of equivalence classes of Lebesgue measurable functions on the real half line $[0,\infty)$. Consider the set $\{\chi_{[i,i+1]};\ i = 0,1,2,\dots,\}$ where $\chi_{[i,i+1]}$ denotes the characteristic function of the interval $[i,i+1]$ for $i = 0,1,2,\dots,$. We define the map φ from m into L by $\varphi(\{x_i\}) = \sum_{i=1}^{\infty} x_i\chi_{[i-1,i]}$. Then, φ is clearly an order continuous embedding of m into L, but $\varphi(m)$ is obviously not order dense.

Remark 1.6. The space $C[0,1]$ of all continuous real valued functions on the real interval $[0,1]$ is a Riesz subspace of the space of equivalence classes of Lebesgue measurable functions on $[0,1]$. However $C[0,1]$ cannot be embedded on an order dense Riesz subspace, or equivalently in an order continuous manner, in any space of measurable functions since $\Gamma(C[0,1])$ is known to consist only of the zero functional [c.f. (17) p. 497].

We will now begin to take advantage of some other properties that measure Riesz spaces must possess. We feel that one of the most logical choices for a condition which one could hope to profit from is the Egoroff property. A certain amount of care must be employed however. For it is known that, if the continuum hypothesis holds, the only spaces of measurable functions which have the Egoroff property are those whose measures are σ-finite. [c.f. (19) Thm. 43.6].

What is true, is that every space of measurable functions has the Egoroff property on an order dense ideal (if $M(S,\Sigma,\mu)$ is a space of measurable functions, then the $L_1(S,\Sigma,\mu)$ subspace is order dense and has the Egoroff property).

This induces us to make the following definition:

<u>Definition</u> 1.2. An Archimedean Riesz space L will be said to be <u>sub-Egoroff</u> if there exists an order dense ideal $I \subseteq L$ such that I has the Egoroff property.

The measure Riesz spaces satisfy the additional condition that they are all sub-order separable. Again, if $M(S,\Sigma,\mu)$ is a space of measurable functions, then the $L_1(S,\Sigma,\mu)$ is an order dense order separable ideal contained in $M(S,\Sigma,\mu)$. The fact that this is true takes on added importance in view of this next example. The example, although used by him for different purposes, is due to J.A.R. Holbrook [(8) §2 p. 215].

Example 1.2. Let $L = \{f = (f^0, f^1, f^2, \ldots),$ where $f^n \colon R^n \to R$, $n = 0,1,2,\ldots,$ and $R^0 = \{\emptyset\}$, R is the reals$\}$. It is clear that the algebraic operations

$$(f+g)^n = (f^n+g^n) \quad \text{and} \quad (\lambda f)^n = \lambda f^n \quad \text{make } L \text{ into}$$

a vector space, and defining $f \leq g$ if and only if $f^n \leq g^n$ for all n makes L into a Riesz space.

Define the operation $c : R^n \to R^{n-1}$ by $v \in R^n$ $(n \geq 1)$ and $v = (v_1, v_2, \ldots, v_n)$ then $c(v) = (v_1, v_2, \ldots, v_{n-1})$.

Now, let L' be the vector lattice of all $f \in L$ satisfying the following two conditions:

(1) There exists a constant $M(f)$ such that $\sup\{|f^n(v)| : v \in R^n\} \leq M(f)$ for each $n = 1,2,\ldots,$.

(2) If $E_n(f) = \{v \in R^n\colon f^n(v) \neq f^{n-1}(c(v))\}$ then $E_n(f)$ is finite for each $n = 1,2,\ldots,$.

The following were shown in [(18)]:

(1) If we define ρ on L' by

$$\rho(f) = \sup_{n \geq 0}\{(n+1)^{-1}(\sup\{|f^n(v)| : v \in R^n\})\};$$

then ρ is an integral norm on L' (i.e. if $f_m \searrow 0$ then $\rho(f_m) \to 0$).

(2) L' has the Egoroff property,

(3) Given any $u \in (L')^+$ there exists a net $u_\tau \searrow 0$ with $u_\tau \leq u$ such that $\rho(u_\tau) \not\to 0$.

We make the additional observation that L' is non-sub-order separable. For, if there were to exist any order separable ideal I in L then ρ restricted to I would be normal and this contradicts condition (3).

Thus, we have that L' is an example of a non-sub-order separable Riesz space (and hence is not a measure Riesz space) with the Egoroff property.

<u>Remark</u> 1.7. If $M(S,\Sigma,\mu)$ is a space of measurable functions, and if $f \in M^+$ is an element of countable type; then the band generated by f is a space of measurable functions over a σ-finite measure space. As such $B_f = \{f\}^{\perp\perp}$-in-M has a strictly positive linear functional defined on the order dense ideal $B_f \cap L_1(S,\Sigma,\mu)$.

<u>Lemma</u> 1.3. Let L be an Archimedean sub-order separable sub-Egoroff Riesz space, and let $e \in L^+$ be an element of countable type. Then the band generated by e, $B_e = \{e\}^{\perp\perp}$-in-L, is an order separable Riesz space with the Egoroff property.

Proof. The fact that B_e is order separable follows easily by application of theorem (4.1) of chapter I. Let I be the order dense ideal in L which has the Egoroff property. Then, $I \cap B_e$ is super order dense in B_e and is easily seen to have the Egoroff property. Since the collection of all elements of a Riesz space with the Egoroff property is a σ-closed ideal, we have that B_e has the Egoroff property. The assertion is proved.

The following theorem shows that if the Archimedean Riesz space L is assumed to be sub-order separable and sub-Egoroff then a significant reduction of the properties of the dual will guarantee that L is a measure Riesz space.

Theorem 1.1. Let L be a sub-order separable, sub-Egoroff, Archimedean Riesz space. Then the following are equivalent:

(1) L is a measure Riesz space.

(2) If $e \in L^+$ is any positive element of countable type, then $B_e = \{e\}^{\perp\perp}$-in-L has a strictly positive linear functional defined on an order dense ideal.

Proof. We show first that (1) $\Rightarrow$ (2). To this end, recall that if L is a measure Riesz space, then $\bar{L}$ is a space of measurable functions. Since if $e \in L^+$ is of countable type then e is of countable type in $\bar{L}$, we see by remark (1.7) that (1) $\Rightarrow$ (2).

We show now that (2) $\Rightarrow$ (1). First, we show that if $e \in L^+$ is of countable type then B_e is a measure Riesz space. To this end, let I be an order dense ideal in B_e which possesses a strictly positive linear functional φ. By application of lemma (1.3), B_e is order separable and has the Egoroff property. Therefore, I is also order separable and has the Egoroff property. Since $I^\sim = I_c^\sim \oplus I_s^\sim$, we have that $\varphi = \varphi_c + \varphi_s$ where $\varphi_c \in I_c^\sim$ and $\varphi_s \in I_s^\sim$. We have, by [(18) Note VI Cor. 20.7], that the null ideal N_{φ_s} of φ_s is an order dense ideal in I. This implies that φ_c must be strictly positive on the order dense ideal N_{φ_s} in I, and hence on I. The order separability of I implies that $\varphi_c \in I_n^\sim$. By proposition (1.1)(6), this implies that B_e is a measure Riesz space.

Now, let $0 < f \in L^+$ and let $\{e_\alpha\}$ be a disjoint positive order basis for L consisting of elements of countable type. Since $\{e_\alpha\}$ is a basis for L, there exists some $\alpha = \alpha_0$ such that

$0 < e_{\alpha_0} \wedge f \leq f$. Since also $0 < e_{\alpha_0} \wedge f \leq e_{\alpha_0}$, we have that $e_{\alpha_0} \wedge f$ is an element of countable type in L. By what we have proved, there exists an order dense ideal $I \subset B_{e_{\alpha_0} \wedge f}$ such that I possesses a strictly positive normal integral φ. Hence, there is some $g \in I$ with $0 < g \leq e_{\alpha_0} \wedge f \leq f$ such that $\varphi(g) > 0$. Taking $D = I \oplus I^{\perp}$, and for each $x \in D$ denoting its components in I and $I^{\perp}$ by x_1 and x_2, respectively, we know that we can extend φ to a φ' defined on D by taking $\varphi'(x) = \varphi'(x_1+x_2) = \varphi(x_1)$ for each $x \in D$. Furthermore, φ' is a normal integral on D. Letting $\bar{\varphi}$ denote the unique element of $\Gamma(L)$ defined by φ', we have that $\bar{\varphi}(f) > 0$. If $\bar{\varphi}(f) < \infty$, we are done. If $\bar{\varphi}(f) = \infty$, we apply [(17) Thm. 2.1] to obtain a $\psi \in \Gamma(L)^{+}$ such that $0 < \psi(f) < \infty$. Thus, we have that $\Gamma(L)$ is separating and L is a measure Riesz space. The theorem is proved.

Corollary 1.2. Let L be an Archimedean Riesz space which has a countable order basis consisting of elements of countable type. Suppose in addition that L has the Egoroff property. Then L is a measure Riesz space if and only if it has an order dense ideal possessing a strictly positive linear functional.

As we have seen in example (1.2), for an Archimedean Riesz space L, the Egoroff property alone is not enough to guarantee that L is a measure Riesz space. By adding the condition that L has to be sub-order separable, we see that example (1.2) no longer applies. As far as we know, the question of whether an arbitrary sub-order separable Archimedean Riesz space with the Egoroff property has to be a measure Riesz space is still open. We discuss this further in the last section of this chapter. For the present, however, this next corollary shows that at least for one reasonably large class of Riesz spaces the Egoroff property alone is enough to place us within the realm of measure Riesz spaces. The corollary also seems to indicate that if a counter example exists, it will not be easy to obtain it.

Corollary 1.3. Let L be a Riesz subspace of the space $M(S,\Sigma,\mu)$ over the σ-finite measure space (S,Σ,μ). Then, L is a measure Riesz space if and only if L has the Egoroff property.

Proof. We need only show the sufficiency. We note first that L must be order separable, since any disjoint positive order basis for L must consist of at most countably many elements each of which is of countable type; otherwise we would contradict the order separability of $M(S,\Sigma,\mu)$.

Now, let $0 < f \in L^+$ and consider the ideal generated by f in $M(S,\Sigma,\mu)$. We will denote this ideal by $I(f,\mu)$. We claim that $I(f,\mu)$ possesses a strictly positive linear functional. To see this, let $S_m \nearrow S$ and satisfy $\mu(S_m) < \infty$ for $m = 1,2,\ldots,$. Let $Y_n = \{t \in S\colon f(t) \leq n\}$ for $n = 1,2,\ldots,$. Clearly $Y_n \nearrow S$. Let $Z_{m,n} = Y_n \cap S_m$ and let

$$X_m = Z_{1,m} \cup Z_{2,m} \cup \ldots \cup Z_{m,m}.$$

We have that $X_m \nearrow S$ and $X_m \subset Y_m \cap S_m$ for $m = 1,2,\ldots,$. Hence, for $t \in X_m$ we have that $f(t) \leq m$, and $\mu(X_m) \leq \mu(S_m)$.

Now, let $B_m = X_m - X_{m-1}$ for $m = 2,\ldots,$ and let $B_1 = X_1$, then $B_m \cap B_{m+1} = \emptyset$ and $\mu(B_m) \leq \mu(X_m) \leq \mu(S_m) < \infty$ for $m = 1,2,\ldots,$. Let $g_m = (\frac{1}{m})(\frac{1}{2^m})\frac{1}{\mu(B_m)}\chi_{B_m}$. Then, since $M(S,\Sigma,\mu)$ is universally complete, there exists an element $g \in M(S,\Sigma,\mu)$ such that $\sup\{g_m;\ m = 1,2,\ldots,\} = \sum_{m=1}^{\infty} g_m = g$.

Define the functional φ on $I(f,\mu)$ as follows:

$$\varphi(g') = \int g'g\,d\mu \quad \text{where} \quad g' \in I(f,\mu).$$

To show that $\varphi \in I(f,\mu)^{\sim}$ we need only show that $\varphi(f) < \infty$. But,

$$\varphi(f) = \int f g \, d\mu = \lim_{n\to\infty} \int f(\sum_{1}^{m} g_k) d\mu =$$

$$\lim_{m\to\infty} \sum_{k=1}^{m} \int f g_k \, d\mu = \lim_{m\to\infty} \sum_{k=1}^{m} \int f \left(\frac{1}{k}\right)\left(\frac{1}{2^k}\right)\left(\frac{1}{\mu(B_k)}\right) \chi_{B_k} d\mu.$$

And $\sum_{k=1}^{m} f(\frac{1}{k})(\frac{1}{2^k})(\frac{1}{\mu(B_k)}) \chi_{B_k} d\mu \leq \sum_{k=1}^{m} \frac{1}{2^k}$

and hence, $\int f g \, d\mu \leq 1.$

It is easy to see that φ is strictly positive on $I(f,\mu)$.

Now, consider $I(f)$ the ideal generated by f in L. Clearly $I(f) \subseteq I(f,\mu)$ and $\varphi_0 = \varphi | I(f)$ is a strictly positive linear functional on $I(f)$. Since L has the Egoroff property, we certainly have that $I(f)$ has the Egoroff property. By the same reasoning we used in the proof of theorem (1.1), it follows that φ_{0c} is a strictly positive normal integral on $I(f)$. Thus, by application of lemma (1.1) it follows that $\Gamma(L)$ is separating. The assertion is proved.

<u>Remark</u> 1.8. We note now that the measure Riesz spaces have an analogue of the Egoroff theorem at least on an order dense ideal, namely, theorem (4.6) of chapter I. We also wish to mention that an analogue to the Radon Nikodym theorem exists for these spaces. We won't go into this at the present time, however.

Remark 1.9. In the proof of corollary (1.3) we actually established that if L was a Riesz subspace of a space of measurable functions on a σ-finite measure space, then every ideal in L generated by an element of L possessed a strictly positive order bounded linear functional. Applying this to $C[0,1]$, the space of real continuous functions on the real interval $[0,1]$, we see that if $C[0,1]$ were to have the Egoroff property, then it would have to possess a strictly positive normal integral, this by application of corollary (1.3) and the fact that $C[0,1]$ is a space generated by one element. But, it is well known [c.f. (10) IV p. 520] that $C[0,1]_n^\sim = \{0\}$. We conclude by this admittedly indirect method that $C[0,1]$ does not have the Egoroff property.

Section 2. Some More on Embeddings.

We showed in the preceeding section that there exists an order continuous embedding φ of a Riesz space L onto a Riesz subspace (not necessarily order dense) of a space of measurable functions $M(S,\Sigma,\mu)$ if and only if there exists an embedding φ' of L onto an order dense Riesz subspace of some space of measurable functions $M'(S',\Sigma',\mu')$ (where $M(S,\Sigma,\mu)$ and $M'(S',\Sigma',\mu')$ do not have to be equal, see example (1.1)).

A natural question to ask is whether every Archimedean Riesz space can at least be embedded into some space of measurable functions (not necessarily in an order continuous manner). We have already seen that $C[0,1]$ can be embedded as a Riesz subspace of the space of equivalence classes of Lebesgue measurable functions on $([0,1],\mu)$ but cannot be embedded as an order dense Riesz subspace of any space of measurable functions.

In fact, it is well known that any Riesz space with a separating order bounded dual $L^\sim$ can be embedded as a Riesz subspace of $(L^\sim)_n^\sim$ [c.f. (10)II (3.7) p. 334]. Furthermore, L forms a separating set of normal integrals for $L^\sim$ and therefore $\Gamma(L^\sim)$ is separating. But also $L^\sim$ is a separating set of continuous linear functionals for $(L^\sim)_n^\sim$, hence $\Gamma^2(L^\sim) = \Gamma(\Gamma(L^\sim))$ is separating and so $\Gamma(L^\sim)$ is a space of measurable functions.

So, every Riesz space L with a separating order bounded dual $L^\sim$ can be embedded as a Riesz subspace of the space of measurable functions $\Gamma(L^\sim)$.

However, as we will soon see, the answer to the question of whether an arbitrary Archimedean Riesz space can be embedded as a Riesz subspace of some space of measurable functions is negative.

In order to show this we will restrict our attention to the following class of Riesz spaces:

Definition 2.1. The Dedekind-σ-complete Riesz space X is said to be extended (or of extended type) if it contains a unit and if every countable set of pairwise disjoint elements $\{x_n\} \subset X$ is bounded.

A compactum Q is said to be quasi extremal if it has a basis of open closed sets, and the closure of every countable union of open closed sets is open.

If Q is a quasi extremal compactum, then the set of all extended real valued functions on Q which are continuous and finite valued on an open dense subset of Q can be given a Riesz space structure. We will denote such spaces by $C_\infty(Q)$. $C_\infty(Q)$, where Q is a quasi extremal compactum, is in general a Dedekind-σ-complete Riesz space of extended type.

It can be shown that every Dedekind-σ-complete Riesz space can be embedded as an order dense ideal in $C_\infty(Q)$ for some quasi extremal compactum Q. This embedding will be an isomorphism onto if and only if the Dedekind-σ-complete Riesz space is of extended type.

For details concerning Riesz spaces of extended type, we refer the reader to [(27) Chapter V].

The main result of this section is that if Q is a quasi extremal compactum, then $C_\infty(Q)$ can be embedded as a Riesz subspace of some space of measurable functions if and only if $C(Q)_n^\sim$ is separating (where $C(Q)$ is the space of continuous real valued functions on Q).

In order to obtain this result, we will have to establish some relationships between the integrals on a space of continuous functions $C(X)$ on a compact Hausdorff space and the regular Borel measures on X which represent the integrals. We will be assuming, for this reason, that the reader is familiar with the material in [(6) Chapter X].

This next result is absolutely essential to our analysis. Variations on the theme of this result have appeared in the literature often [c.f. (25) p. 111]. As far as we know, in the form it takes below, it has not been proved.

The essential part of the proof can be carried out for compact X. Once proved for this case, it is easily seen that the proof generalizes to locally compact σ-compact X. Since we will only need the result for compacta, we will prove it in this context keeping in mind that a more general result can be easily obtained.

<u>Theorem</u> 2.1. Let X be a compact Hausdorff space and let $C(X)$ be the space of continuous real valued functions on X. A linear functional φ on $C(X)$ is an integral if and only if its representing regular Borel measure vanishes on every closed G_δ nowhere dense subset contained in X.

<u>Proof</u>. Suppose that G is a G_δ closed nowhere dense subset of X and let $\{G_n\}$ be a sequence of open sets such that $\bigcap_{n=1}^{\infty} G_n = G$. Since X is normal and $(\text{com } G_n) \cap G$ (where $(\text{com } G_n) =$ complement of G_n) is empty for $n = 1,2,\ldots,$ we have, for each n, a Urysohn function f_n satisfying:

$$f_n(x) = 1 \quad \text{for} \quad x \in G,$$
$$f_n(x) = 0 \quad \text{for} \quad x \in (\text{com } G_n),$$
$$0 \leq f_n(x) \leq 1 \quad \text{for all} \quad x \in X.$$

If we take $g_n = \inf\{f_i;\ 1 \leq i \leq n\}$, we have $g_n \searrow$ and $\inf\{g_n;\ n = 1,2,\ldots,\} = 0$. For suppose that

$x \notin G$, then for some $n = n_0$, we must have that $x \notin G_{n_0}$ which implies that $g_{n_0}(x) = 0$.

Now, let $\varphi \in C(X)_c^{\sim}$ and let μ_φ be the regular Borel measure such that, for each $f \in C(X)$, $\varphi(f) = \int_X f \, d\mu_\varphi$.

We have that $|\varphi|(g_n) \to 0$ which implies that $\int g_n \, d|\mu_\varphi| \to 0$. But, $|\mu_\varphi|(G) \leq \int g_n \, d|\mu_\varphi|$, and hence $|\mu_\varphi|(G) = 0$.

Now, suppose that μ is a positive regular Borel measure on X and that for each G_δ closed nowhere dense subset G of X, we have that $\mu(G) = 0$. We wish to show that $\int_X f \, d\mu$, as a linear functional on $C(X)$, is an integral.

Let $\{f_n;\ n = 1,2,\ldots,\}$ be a sequence in $C(X)^+$ such that $f_n \searrow 0$.

Let $D_m = \{t \in X\colon f_n(t) \geq \frac{1}{m}$ for all $n = 1,2,\ldots,\}$, where $m = 1,2,\ldots,$.

Each D_m is obviously closed. We must show that each D is a nowhere dense G_δ. Suppose that for some $m = m_0$ we have that D_{m_0} contains a sphere, say B, then (com B) is closed and let $x_0 \in B$. We can find a Urysohn function g on X satisfying:

$g(x_0) = 1/m_0$,

$g(x) = 0$; for $x \in (\text{com } B)$,

$0 \leq g(x) \leq 1/m_0$ for all $x \in X$.

But, then, $g \leq f_n$ for all $n = 1,2,\ldots,$ and this implies that $\inf\{f_n; n = 1,2,\ldots,\} \neq 0$ which contradicts our hypothesis. This shows that for each m, D_m is nowhere dense.

Now, let $D_{n,m} = \{t \in X: f_n(t) \geq \frac{1}{m}\}$, where $n,m = 1,2,\ldots,.$ Each $D_{n,m}$ is a G_δ for $n,m = 1,2,\ldots,$ since if $D_{n,m,k} = \{t \in X: f_n(t) > \frac{1}{m} - \frac{1}{2^k}\}$, where $n,m = 1,2,\ldots,$ and $k = m,m+1,\ldots,;$ we have that $D_{n,m,k}$ is open and $\bigcap_{k=m}^{\infty} D_{n,m,k} = D_{n,m}$ for $n,m = 1,2,\ldots,.$ and hence each D_m is a G_δ. By assumption, we have that $\mu(D_m) = 0$ for $m = 1,2,\ldots,.$

Now, $\{t \in X: \inf\{f_n(t); n = 1,2,\ldots,\} = 0\} = \bigcup_{m=1}^{\infty} D_m$, and since μ is countably additive, this set has μ-measure zero.

Since, for each m, $D_m = \bigcap_{n=1}^{\infty} D_{n,m}$ and $\mu(D_m) = 0$, we must have that, for some $n(m) = n$, $\mu(D_{n(m),m}) \leq \frac{1}{m}$.

Consider $\int_X f_{n(m)} d\mu = \int_{D_{n(m),m}} f_{n(m)} d\mu$ $+ \int_{X-D_{n(m),m}} f_{n(m)} d\mu$. On $X - D_{n(m),m}$ we have that $f_{n(m)} \leq \frac{1}{m}$. If $M(f_1)$ is the maximum of the function f_1 we have that $f_{n(m)}(t) \leq M(f_1)$ for all $t \in X$. Hence $\int_X f_{n(m)} d\mu \leq \frac{1}{m} M(f_1) + \frac{1}{m} \mu(X - D_{n(m),m}) \leq \frac{1}{m}(M(f_1) + \mu(X))$. So, if $\epsilon > 0$ is given and we choose

$m = m'$ such that $1/m' \leq \frac{\epsilon}{M(f_1)+\mu(X)}$, then we can choose an $n = n(m', \epsilon) = \mathbf{n}(\epsilon)$ such that for all $n \geq n(\epsilon)$ we have;

$$\int_X f_n \, d\mu \leq \frac{1}{m'} (M(f_1) + M(X)) \leq \epsilon.$$

This shows that $\int_X f_n \, d\mu \to 0$.

By application of the Hahn decomposition theorem for measures we can extend this proof to signed regular Borel measures on X. The proof is completed.

Before proceding to our main result we need a couple of lemmas concerning the supports of regular Borel measures on a compact Hausdorff space. The support of a measure μ, written supp μ, is defined in the following manner:

Let $G = \{x \in X:$ there exists an open set $U \subseteq X$, with $x \in U$ and such that $\mu(U) = 0\}$ (it is trivial to show that G is an open set). Then,

$$\text{supp } \mu = (\text{com } G).$$

<u>Lemma</u> 2.1. Let μ be a regular Borel measure on the compact Hausdorff space X, and assume that $\mu > 0$. Let $B \subset X$ be a closed G_δ nowhere dense subset of X. Then supp $\mu \subseteq B$ implies that $\mu(B) > 0$. In fact $\mu(B) = \mu(X)$.

<u>Proof</u>. Let $U(B)$ be any open set containing B. Then, $\mu(X - U(B)) = 0$. To see this, we merely consider the following:

Since X is compact and Hausdorff and $X - U(B)$ is closed, we have that $X - U(B)$ is a compact subset of X contained in the complement of the supp μ. Therefore, for each $x \in X - U(B)$, there exists an open set U_x containing x such that $\mu(U_x) = 0$. The collection $U = \{U_x : x \in X \quad U(B)\}$ is an open covering of $X - U(B)$ and hence there exists a finite subcovering $\{U_{x_i}\}_{i=1}^{n}$. Since $\mu(U_{x_i}) = 0$ for each i, we have that $\mu(X - U(B)) \leq \mu(\bigcup_{i=1}^{n} U_{x_i}) \leq \sum_{i=1}^{n} \mu(U_{x_i}) = 0$.

Now, since $\mu(X) = \mu(X - U(B)) + \mu(U(B))$, we must have that $\mu(U(B)) = \mu(X)$ for all open sets $U(B)$ containing B.

The regularity of the measure μ guarantees that $\mu(B) = \inf\{\mu(U(B)) : U(B)$ is an open set containing $B\}$. Therefore, $\mu(B) = \mu(X) > 0$. The assertion is proved.

Lemma (2.1) also has an extension for locally compact, σ-compact Hausdorff spaces.

This next lemma reverses lemma (2.1).

<u>Lemma</u> 2.2. Let X be a compact Hausdorff space. Let $\mu > 0$ be a regular Borel measure defined on X. Then, if B is any closed G_δ nowhere dense subset of X such that $\mu(B) > 0$, there exists a regular Borel measure $\mu_1 > 0$ such that $\mu_1 \leq \mu$ and $\operatorname{supp} \mu_1 \subseteq B$.

<u>Proof</u>. Let φ_1 be the positive linear functional defined on $C(X)$ by taking $\varphi_1(f) = \int_B f d\mu$. If φ is the linear functional on $C(X)$ defined by taking $\varphi(f) = \int_X f d\mu$, we have clearly that $\varphi_1(f) \leq \varphi(f)$, and hence the regular Borel measure $\mu_1 > 0$ such that $\varphi_1(f) = \int_X f d\mu$, has the property that $\mu_1 \leq \mu$.

Now, let $x_0 \in X - B$. Then there exists an open set $U(x_0)$ containing x_0 such that $U(x_0) \cap B$ is empty. The complement of $U(x_0)$ is closed so choose a Urysohn function such that,

$$g(x_0) = 1,$$

$$g(x) = 0; \quad \text{for} \quad x \in X - U(B),$$

$$0 \leq g(x) \leq 1; \quad \text{for all} \quad x \in X.$$

Then $\varphi_1(g) = \int_B g d\mu = 0$. Now, let $V(x_0) = \{x \in X : g(x) > \frac{1}{2}\}$. $V(x_0)$ obviously contains x_0 and $\mu_1(V(x_0)) \leq \varphi_1(2g) = 2\varphi_1(g) = 0$. This implies that every $x \in X - B$ is contained in the complement of the support of μ_1. Therefore, $\operatorname{supp} \mu_1 \subseteq B$. The assertion is proved.

We feel that at this time it might be best to compare this next theorem, the central result of this section, with theorem D of section 1 of this chapter, a result due to J.J. Masterson.

First of all, we recall that every universally complete Riesz space is $C_\infty(Q)$ for some extremal compactum Q (see section 3 of chapter I), hence Masterson's result can be restated as:

A necessary and sufficient condition that $C_\infty(Q) = M(S,\Sigma,\mu)$, where Q is an extremal compactum, is that $\Gamma(C_\infty(Q))$ be separating.

Our result differs from this result in two respects. First of all, we are not requiring that Q be an extremal compactum, although every extremal compactum is quasi extremal. Second, we are not requiring that the embedding of $C_\infty(Q)$ into $M(S,\Sigma,\mu)$ be an order continuous embedding. We are merely asking that $C_\infty(Q)$, where Q is quasi extremal, be a Riesz subspace of $M(S,\Sigma,\mu)$, which means that only finite suprema and infima are necessarily preserved.

Theorem 2.2. Let Q be a quasi extremal compactum. Then the following are equivalent:

(1) $C_\infty(Q)$ is embeddable as a Riesz subspace (not necessarily order dense) of some space of measurable functions $M(S,\Sigma,\mu)$.

(2) $C(Q)_c^\sim$ is separating.

Proof. We show first that (2) ⇒ (1). Suppose that $f \in C_\infty(Q)$ and let $1 = \chi_Q$ be the identity for $C(Q)$. Define $1_f = |f| \vee 1$. 1_f is an order unit for $C_\infty(Q)$; and, if $I(1_f)$ is the ideal generated in $C_\infty(Q)$ by 1_f, we have immediately that $I(1_f) \supseteq I(1) = C(Q)$. Hence, $I(1_f)^\sim$ is an ideal in $C(Q)^\sim$, and $I(1_f)^\sim \cap C(Q)_c^\sim = I(1_f)_c^\sim$ is an ideal in $C(Q)_c^\sim$. We show now that $I(1_f)_c^\sim$ is actually order dense in $C(Q)_c^\sim$. To show this we need only show that if $0 < \varphi_1 \in C(Q)_c^\sim$, then there exists a $0 < \varphi_2 \in I(1_f)_c^\sim$ such that $\varphi_2 \leq \varphi_1$.

To this end, consider the set $G = \{x \in Q: |f(x)| = \infty\}$. It is easy to see that G is a closed G_δ nowhere dense subset of Q.

Suppose that $0 < \mu_{\varphi_1}$ is the regular Borel measure on Q for which $\varphi_1(g) = \int_X g\,d\mu_{\varphi_1}$ for each $g \in C(Q)$. By theorem (2.1), we see that $\mu_{\varphi_1}(G) = 0$. We now apply lemma (2.1) to see that $\operatorname{supp} \mu_{\varphi_1} \not\subseteq G$. Hence, there exists an $x_0 \in Q - G$ such that

$x_0 \in \operatorname{supp} \mu_{\varphi_1}$. Now, $x_0 \notin G$ implies that there exists an integer $n_0 > 1$ such that $x_0 \in \{x \in Q: 1_f(x) < n_0\}$ and this set is open. Letting U_{x_0} be any basic closed open set in Q such that $U_{x_0} \subseteq \{x \in Q: 1_f(x) < n_0\}$, and such that $x_0 \in U_{x_0}$, we have that the positive functional $\varphi_2(g) = \int_{U_{x_0}} g d\mu_{\varphi_1}$ is also an element of $I(1_f)^\sim_c$. For, $\int_{U_{x_0}} 1_f \, d\mu_{\varphi_1} \leq n_0 \, \mu_{\varphi_1}(U_{x_0}) < \infty$. Clearly $\varphi_2 \leq \varphi_1$, and we need only show now that $\varphi_2 > 0$ to complete our argument. But, $x_0 \in \operatorname{supp} \mu_{\varphi_1}$ implies that for every neighborhood of x_0, say U, we have that $\mu_{\varphi_1}(U) > 0$. Hence, $\varphi_2(\chi_{U_{x_0}}) = \int_{U_{x_0}} \chi_{U_{x_0}} d\mu_{\varphi_1} = \mu_{\varphi_1}(U_{x_0}) > 0$. The order density of $I(1_f)^\sim_c$ in $C(Q)^\sim_c$ is established.

Since $I(1_f)^\sim_c$ is order dense in $C(Q)^\sim_c$, we have that $I(1_f)^\sim_c$ is separating on $I(1_f)$ since $C(Q)^\sim_c$ is separating on $C(Q)$.

Now, we have that $f \in I(1_f)$, and hence $f \in (I(1_f)^\sim_c)^\sim_n$. The order density of $I(1_f)^\sim_c$ in $C(Q)^\sim_c$ implies that f determines a unique element of $\Gamma(C(Q)^\sim_c)$. We will denote the element of $\Gamma(C(Q)^\sim_c)$ associated with each $f \in C_\infty(Q)$, by the process described above, by $\bar{f}$.

To complete this direction of the proof, we need to show that the mapping $f \to \bar{f}: C_\infty(Q) \to \Gamma(C(Q)_c^\sim)$ is a vector space isomorphism and preserves the order. The proof that $(\overline{af}) = a(\bar{f})$ where a is a real number is trivial. We show first that $(\overline{f+g}) = \bar{f} + \bar{g}$. This will be obvious if all three objects can be shown to exist on a common order dense ideal in $C(Q)_c^\sim$. That this is true follows from the fact that $I(1_{f+g})_c^\sim$, $I(1_f)_c^\sim$, and $I(1_g)_c^\sim$ all contain $I(1_{|f|+|g|})_c^\sim$ and this is order dense in $C(Q)_c^\sim$.

It remains only to show that the mapping $f \to \bar{f}$ is order preserving. To prove this it is sufficient to show that, for any $f \in C_\infty(Q)$, we have that $\overline{f^+} = (\bar{f})^+$. But, $I(1_f)$ contains both f and f^+ simultaneously and since $I(1_f)_c^\sim$ is order dense in $C(Q)_c^\sim$ and separating on $I(1_f)$, we will have that $\overline{f^+} = (\bar{f})^+$ if the canonical embedding $\varphi : I(1_f) \to (I(1_f)_c^\sim)_n^\sim$ has the property that $\varphi(f^+) = (\varphi(f))^+$. This is a well known result in the theory of Riesz spaces [c.f. (1) p. 39]. We have proved that (2) $\Rightarrow$ (1).

We prove now that (1) $\Rightarrow$ (2).

So, assume that $C_\infty(Q)$ is a Riesz subspace of some space of measurable functions $M(S,\Sigma,\mu)$. We may as well assume that the identity $1 = \chi_Q$ in $C(Q)$ is a unit for $M(S,\Sigma,\mu)$. Let $B = I(1)$ be the ideal in $M(S,\Sigma,\mu)$ generated by the element 1. It is easy to see that $B_n^{\sim}$ is separating, and hence, by [(17) Thm's. 1.6 and 3.1] $B_n^{\sim}$ is an order dense ideal in $\Gamma(M(S,\Sigma,\mu))$.

According to [(17) Thm. 2.1], for any $0 < u \in M(S,\Sigma,\mu)$, if we have a $0 \leq \phi \in B_n^{\sim}$ such that $\bar{\phi}(u) = \infty$ (where
* $\bar{\phi}(u) = \sup\{\phi(v): 0 \leq v \leq u,\ \phi(v) < \infty\}$),
there exists a $\psi \in B_n^{\sim}$ such that $0 \leq \psi \leq \phi$ and $0 < \psi(u) < \infty$.

The proof for (1) $\Rightarrow$ (2) will be to assume that $C(Q)_c^{\sim}$ is not separating and obtain a contradiction to (*).

Since 1 is the strong order unit for both B and $C(Q)$, it follows that every $\varphi \in B_n^{\sim}$, when restricted to $C(Q)$ is in $C(Q)^{\sim}$. Also, if $0 < \varphi \in B_n^{\sim}$, then the restriction of φ to $C(Q)$, say φ_r, is positive and since $\varphi_r(1) = \varphi(1)$ we have that $\varphi_r > 0$.

If $0 < \varphi_2 \leq \varphi_1$, then $0 < \varphi_{2r} \leq \varphi_{1r}$.

Suppose that $0 < \varphi' \leq \varphi_r$ where $\varphi' \in C(Q)^{\sim}$. By a trivial application of [(18) Note VI Thm. 19.2], there exists a positive extension of φ', say $\bar{\varphi}$, to B such that $0 < \bar{\varphi} \leq \varphi$ and hence $\bar{\varphi} \in B_n^{\sim}$.

Now, suppose that $C(Q)_c^{\sim}$ is not separating on $C(Q)$. Since the restrictions of the elements of $B_n^{\sim}$ onto $C(Q)$ are separating, there exists a $0 < \varphi \in B_n^{\sim}$ such that φ_r is not an integral on $C(Q)$. If μ_{φ_r} is the regular Borel measure on Q such that $\varphi_r(f) = \int_Q fd\mu_{\varphi_r}$, then by theorem (2.1) there exists a closed G_δ nowhere dense subset of Q, say D, such that $\mu_{\varphi_r}(D) > 0$. By lemma (2.1), there exists a measure $0 < \mu_1 \leq \mu_{\varphi_r}$ such that $\operatorname{supp} \mu_1 \subseteq D$.

Taking $\varphi_1(f) = \int_Q fd\mu_1$, we have that $0 < \varphi_1 \leq \varphi_r$ and, by what we have shown already, there exists a positive extension of φ_1, say $\bar{\varphi}_1$, to B such that $0 < \bar{\varphi}_1 \in B_n^{\sim}$ and $\bar{\varphi}_1 \leq \varphi$.

Now, if φ' is any element of $(B_n^{\sim})^+$ such that $\varphi_1 \leq \bar{\varphi}_1$ then, $0 \leq \mu_{\varphi'_r} \leq \mu_{\bar{\varphi}_{1r}} = \mu_1$, we have that $\operatorname{supp} \mu_{\varphi'_r} \subseteq D$. If $0 < \varphi' \leq \bar{\varphi}_1$, then $\mu_{\varphi'_r} > 0$ and by lemma (2.2) we have that $\mu_{\varphi'_r}(D) > 0$.

Now, let $f > 1$ be any element of $C_\infty(Q)$ such that $\{x\colon f(x) = \infty\} = D$. That such functions exist follows easily from the fact that D is a G_δ closed nowhere dense set.

For all $\varphi \in B_n^{\sim}$ such that $0 < |\varphi| \leq n\bar{\varphi}_1$ for some $n = 1,2,\ldots,$ we must have that $|\varphi|(f) = |\varphi_r|(f) = \infty$. To see this we just note the fact that

$$|\varphi_r|(f) \geq \int_{\{x \in Q:f(x) > n\}} nd\mu|\varphi_r| \geq n\mu_{\varphi_r}(D).$$

But this is the desired contradiction to (*).

The theorem is proved.

Let Q be a quasi extremal compactum. Suppose that $L \subseteq C_\infty(Q)$ is an order dense ideal for which $L_c^{\sim}$ is separating on L. Taking $B = C(Q) \cap L$, we know that B is order dense in $C(Q)$. Since the restriction of each $\varphi \in L_c^{\sim}$ onto B is also an integral, we have immediately that $B_c^{\sim}$ is separating on B.

Now, let $0 < f \in C(Q)$. Since B is order dense in $C(Q)$, there exists a $0 < g \leq f$ such that $g \in B$. Let χ_U be the characteristic function of any closed open subset U of Q such that, for some positive real number $\alpha > 0$, we have that $U \subset \{t \in Q:g(t) > \alpha\}$.

It follows immediately, from the fact that $\chi_U \leq \frac{1}{\alpha}g$, that $\chi_U \in B$. It is easy to verify that the ideal generated by χ_U in $C(Q)$ satisfies

$$I(\chi_U) = \{\chi_U\}^{\perp\perp}\text{-in-}B = \{\chi_U\}^{\perp\perp}\text{-in-}C(Q).$$

Let $0 < \varphi$ be an element of $B_c^{\sim}$ for which $\varphi(\chi_U) > 0$. Using the fact that $C(Q)$ is Dedekind-σ-complete, and therefore has the principal projection property, we define the following positive linear functional on $C(Q)$:

$\overline{\varphi}(g) = \varphi(P_{\chi_U}(g))$ for $g \in C(Q)$ (where $P_{\chi_U}(g) = \sup\{n\chi_U \wedge g^+ ;\ n = 1,2,\cdots,\} - \sup\{n\chi_U \wedge g^- ;\ n = 1,2,\ldots,\})$.

It is clear that $\overline{\varphi} \in C(Q)_c^{\sim}$, since if $g_n \searrow 0$ (in $C(Q)$) then $P_{\chi_U}(g_n) \searrow 0$ (in B) which implies that $\overline{\varphi}(g_n) = \varphi(P_{\chi_U}(g_n)) \to 0$. Since $f \geq \alpha \chi_U$, we have that $\overline{\varphi}(f) = \varphi(P_{\chi_U}(f)) \geq \alpha\, \varphi(\chi_U) > 0$. Thus, $C(Q)_c^{\sim}$ is separating on $C(Q)$.

The above argument, when combined with theorem (2.2), establishes this next result.

<u>Theorem</u> 2.3. Let Q be a quasi extremal compactum. Then, the following are equivalent:

(1) $C_\infty(Q)$ is embeddable as a Riesz subspace of a space of measurable functions.

(2) There exists an order dense ideal $L \subseteq C_\infty(Q)$ for which $L_c^{\sim}$ is separating.

Since every universally complete Riesz space is $C_\infty(Q)$ on some extremal compactum (and hence quasi extremal), we have the following result for universally complete Riesz spaces.

Corollary 2.1. Let L be a universally complete Riesz space. Then, the following are equivalent:

(1) L is a Riesz subspace (not necessarily order dense) of some space of measurable functions.

(2) There exists an order dense ideal $I \subseteq L$ such that $I_c^{\sim}$ is separating.

This next result shows that, for sub-order separable universally complete Riesz spaces, Masterson's result (see theorem D of section 1) is the best possible in the sense that if a sub-order separable universally complete Riesz space can be embedded in anyway as a Riesz subspace of some space of measurable functions then it itself must be a space of measurable functions.

Theorem 2.4. Let L be a universally complete sub-order separable Riesz space. Then, the following are equivalent:

(1) L is a Riesz subspace of some space of measurable functions.

(2) $\Gamma(L)$ is separating.

(3) L is a space of measurable functions.

Proof. In view of theorem D of section 1, we need only show the implication (1) $\Rightarrow$ (2). To this end, let $L^1 \subseteq L$ be the order dense order separable ideal contained in L. Let L^2 be the order dense ideal contained in L for which $L_c^{2\sim}$ is separating on L^2. Taking $L^3 = L^1 \cap L^2$

we have that L^3 inherits a separating set of integrals from L^2. Since L^3 is order separable, each integral on L^3 is normal. Therefore, $L_n^{3\sim}$ is separating. The fact that L^3 is order dense finally tells us that $\Gamma(L)$ is separating. The theorem is proved.

<u>Corollary</u> 2.2. Let L be a sub-order separable Archimedean Riesz space. Let $\bar{L}$ be the universal completion of L. A necessary and sufficient condition that $\bar{L}$ is a Riesz subspace of a space of measurable functions is that $\Gamma(L)$ is separating.

<u>Example</u> 2.1. Let $L = C[0,1]$ (i.e. the Riesz space of continuous functions on the unit interval). This space is order separable, and it is known that $\Gamma(L) = \{0\}$. If $\bar{L}$ is the universal completion of L, then by corollary (2.2) above we have that $\bar{L}$ cannot be embedded as a Riesz subspace of any space of measurable functions.

This next example shows that there exist Dedekind-σ-complete Riesz spaces of extended type which can be embedded as Riesz subspaces of spaces of measurable functions but which are not measure Riesz spaces.

<u>Example</u> 2.2. Let X be a topological space. A point $x \in X$ is called a P point if for any countable collection $\{U_n; n = 1,2,\ldots,\}$ of open subsets of X containing the point x, we have that $\bigcap_{n=1}^{\infty} U_n$ is open in X.

A topological space X for which every point $x \in X$ is a P point is called a P space. Any discrete space is a P space. For our purposes such spaces are not of interest. Completely regular P spaces exist, however, which have no isolated points [c.f. (4) 13P p. 193]. So, let X be a completely regular P space for which no point $x \in X$ is isolated. Let $C(X)$ be the Riesz space of all real valued continuous functions on X. It was shown in [(16) Note XV_A Ex. 50.7 p. 420] that $C(X)$ has the following properties:

(a) $C(X)$ is Dedekind-σ-complete,

(b) $C(X)_c^{\sim} = C(X)^{\sim}$,

(c) $C(X)_c^{\sim}$ is separating,

(d) $C(X)_n^{\sim} = \{0\}$.

We note further that $C(X)$ is non-sub-order separable. To see this, let $0 < g \in C(X)$. Let U be an open set in X such that $U \subseteq \{x \in X: g(x) > \alpha\}$ where $\alpha > 0$ is some real number for which a U exists which is not void. Fix an $x \in U$ and let $\{U_\beta\}$ be a collection of open sets such that, for each β, we have that $U_\beta \subset U$ and $\bigcap_\beta U_\beta = \{x\}$. For each β, let f_β be a Urysohn function such that $f_\beta(x) = \alpha$, and for all $t \in (\text{com } U_\beta)$ (where $(\text{com } U_\beta)$ = complement of U_β) we have that $f_\beta(t) = 0$. Then, we have that $\inf \{f_\beta\} = 0$ and

$0 \leq f_\beta \leq g$ for all β. Now, if $\{f_{\beta n}\}$ is any countable subset of $\{f_\beta\}$, then, if $B_n = \{t \in X: f_{\beta n}(t) > \frac{1}{2}\alpha\}$, we have that $\bigcap_{n=1}^{\infty} B_n$ is open and contains x. Let g_o be a Urysohn function satisfying:

$$g_o(x) = \frac{1}{2}\alpha ,$$

$$g_o(t) = 0 ; \quad \text{for} \quad t \in (\text{com} \bigcap_{n=1}^{\infty} B_n) ,$$

$$0 \leq g_o(t) \leq \frac{1}{2}\alpha ; \quad \text{for } t \in X .$$

Then $0 < g_o \leq f_{\beta n}$ for all $n = 1,2,\ldots,$ and hence $\inf\{f_{\beta n} ; n = 1,2,\ldots,\} \neq 0$. This implies that the ideal $I(g)$ generated by g is not order separable and hence that g is not an element of countable type. Since the choice of $0 < g \in C(X)$ was arbitrary, this says that $C(X)$ has no elements of countable type and therefore is not sub-order separable. We learn two things from this. First, $C(X)$ cannot be a measure Riesz space. Second, $\Gamma(C(X))$ cannot be separating.

Since $C(X)$ is Dedekind-σ-complete, it can be embedded as an order dense ideal in $C_\infty(Q)$ for some quasi extremal compactum Q. By application of theorem (2.3), since $C(X)^\sim_c$ is separating, we obtain that $C_\infty(Q)$ is a Riesz subspace of some space of measurable functions.

Since $\Gamma(C_\infty(Q)) = \Gamma(C(X))$, and since $\Gamma(C(X))$ is not separating, we have that $\Gamma(C_\infty(Q))$ is not separating. Thus, we have that $C_\infty(Q)$ cannot be embedded either continuously into or as an order dense Riesz subspace of, any space of measurable functions.

In the above example, the non Dedekind completeness of the $C_\infty(Q)$ used was critical. If we could obtain a Dedekind complete example with the same properties we would at the same time have an example of a Dedekind complete Riesz space with an integral which was not normal. This would imply the existence of a measurable cardinal [see (15)]. The question of the existence of measurable cardinals is still open.

Section 3. Some Topological Considerations.

A vector space, which is at once a vector lattice and a topological vector space, can exhibit interesting relationships between the order and the topology. We wish to discover which properties, order and topological, combine to force us into the setting of measure Riesz spaces.

We begin by introducing some concepts which are basic to this investigation. The general setting follows closely Peressini's organization in [(25)]. Most of the properties which we introduce are straight forward generalizations of those used by Luxemburg and Zaanen in [(16) and (18)].

Definition 3.1. Let E be an Archimedean Riesz space. A set $B \subseteq E$ is called solid if for any $b \in B$ and any $a \in E$ such that $|a| \leq |b|$, we have that $a \in B$.

Definition 3.2. Let E be an Archimedean Riesz space. Suppose that E is also a topological vector space under a topology τ. We will say that (E, τ) is a topological vector lattice (T.V.L.) if E has a neighborhood basis at 0 consisting of solid sets which is a basis for the topology τ.

It is clear what we will mean by a convex (locally convex) T.V.L. (E, τ).

Definition 3.3. Let E be an Archimedean Riesz space. An element $x \in E$ is said to be relatively uniformly continuous if for all sequences $\{x_n\}$ in E with $x_n \searrow 0$ and such that $0 \leq x_n \leq x$, we have that $x_n \to 0$ (r.u) where $n = 1, 2, \ldots,$.

Definition 3.4. Let (E, τ) be a T.V.L. . We will say that $x \in E$ is τ-absolutely continuous if for every sequence $\{y_n\}$ in E with $y_n \searrow 0$ and such that $y_n \leq |x|$ for all $n = 1, 2, \ldots,$ we have that $y_n \xrightarrow{\tau} 0$.

Definition 3.5. Let E be an Archimedean Riesz space. We will say that an element $x \in E$ is absolutely continuous if for every sequence $\{y_n\}$ in E with $y_n \searrow 0$ and such that $y_n \leq |x|$ for all $n = 1, 2, \ldots,$ we have that, for any $\varphi \in E^{\sim}$, $\varphi(y_n) \to 0$.

Definition 3.6. Let (E, τ) be a T.V.L. . An element $x \in E$ will be said to be $(E,\tau)^*$ absolutely continuous (or just *-absolutely continuous) if for any sequence $\{y_n\}$ in E with $y_n \searrow 0$ and such that $y_n \le |x|$ for all $n = 1,2,\ldots,$ we have that, for any $\varphi \in (E,\tau)^*$, $\varphi(y_n) \to 0$ (where $(E,\tau)^*$ denotes the topological dual of (E,τ)).

We will denote the set; of all relatively uniformly continuous elements of E by $E_{r.c.}$, of all τ-absolutely continuous elements of (E,τ) by $E(\tau_a)$, of all absolutely continuous elements of E by E^a, of all *-absolutely continuous elements of (E,τ) by $E(\tau_a^*)$.

Lemma 3.1. Let E be an Archimedean Riesz space. Then E^a and $E_{r.c.}$ are ideals in E. If in addition, E is a T.V.L. with topology τ, then $E(\tau_a)$ and $E(\tau_a^*)$ are ideals in E.

Proof. This proof is straight forward, and we will not produce it here.

Lemma 3.2. Let (E,τ) be a convex T.V.L. . Then, $E(\tau_a) = E(\tau_a^*)$.

Proof. It is obvious that $E(\tau_a) \subseteq E(\tau_a^*)$. For the reverse inclusion, we will need [(12) 17.2 p. 154].

This result tells us that the weak closure of a subset A of E is a subset of the τ-closure of the convex extension of A.

Let $x \in E(\tau_a^*)$, and let $y_n \searrow 0$, $y_n \leq |x|$ for $n = 1,2,\ldots,$. Since $x \in E(\tau_a^*)$, we have that 0 is an element of the weak closure of the set $\{y_n\}$. Let $\{w_\alpha\}$ be a neighborhood basis at 0 for the topology τ consisting of solid, convex, circled, absorbing subsets of E. We have that there exists a sequence $\{z_m\}$, $z_m = \sum_{i=1}^{p_m} \alpha_i^m y_i^m$ where $\{y_i^m\} \subset \{y_n\}$, and $\sum_{i=1}^{p_m} \alpha_i^m = 1$, such that $z_m \xrightarrow{\tau} 0$ for $m = 1,2,\ldots,$.

This says that, for each α, there exists an $m = m_\alpha$ such that for all $m \geq m_\alpha$ we have that $z_m \in w_\alpha$. Let $n = n_\alpha$ be chosen so that $y_{n_\alpha} \leq y_i^{m_\alpha}$ for all $1 \leq i \leq p_{m_\alpha}$.

Then, for all $n \geq n_\alpha$, we have that

$$y_n \leq y_{n_\alpha} = \sum_{i=1}^{p_{m_\alpha}} \alpha_i^{m_\alpha} y_{n_\alpha} \leq \sum_{i=1}^{p_{m_\alpha}} \alpha_i^{m_\alpha} y_i^{m_\alpha} = z_{m_\alpha}$$

The solidity of w_α implies then that $y_n \in w_\alpha$ for all $n \geq n_\alpha$. This establishes that $y_n \xrightarrow{\tau} 0$ which implies that $x \in E(\tau_a)$. Hence, $E(\tau_a) \supseteq E(\tau_a^*)$. The assertion is proved.

This next result is a collection of relationships between the ideals $E_{r.c.}$, E^a, $E(\tau_a)$, and $E(\tau_a^*)$. It can be improved on, but we will not do so here. For our present purposes, the relationships we obtain are adequate.

Theorem 3.1. Let E be an Archimedean Riesz space. We have the following relationships:

(1) $E_{r.c.} \subseteq E^a$.

(2) If (E,τ) is a T.V.L., then $(E,\tau)^* \subseteq E^\sim$ and is an ideal in $E^\sim$.

(3) If (E,τ) is a convex T.V.L., then $E_{r.c.} \subseteq E(\tau_a) = E(\tau_a)^*$.

(4) If (E,τ) is a complete, metrizable, convex T.V.L., then $E^\sim = (E,\tau)^*$ and $E^a = E(\tau_a) = E_{r.c.}$.

Proof. The verification of (1) is trivial. Part (2) follows from [(25) Prop. 4.17 p. 108]. For part (3), the containment is trivial to see, and the equality follows from lemma (3.2). Part (4) follows easily using the well known result that for Frechet spaces $E^\sim = (E,\tau)^*$ combined with [(25) Prop. 2.4 p. 162]. The theorem is proved.

Example 3.1. Let $E = L^\infty[0,1]$ (the space of equivalence classes of essentially bounded Lebesgue measurable functions on $[0,1]$). For $f \in E$, let $\rho(f) = \rho(|f|) = \int_0^1 |f| d_\mu$ where μ is Lebesgue measure. E_ρ^* is not equal to $E^\sim$ since every element of E_ρ^* is a normal integral on E. This same example shows that $E_{r.c.}$ need not always equal $E(\tau_a)$.

We now introduce one of the topologies on a Riesz space which is totally determined by the order structure.

Definition 3.7. Let E be an Archimedean Riesz space; the order topology τ_O on E is the finest locally convex topology τ for which every order bounded set is τ-bounded.

For a complete discussion of the order topology, we refer the reader to [(25) Chapter III, (5), and (24)].

For our purposes, it is only necessary to know that if $E^\sim$ is separating on E then τ_O is the Mackey topology $\tau(E,E^\sim)$.

Definition 3.8. Let L be an Archimedean Riesz space and let I be an ideal in L. We will say that I is a weakly regular ideal in L if it possesses a positive order basis $\{e_\alpha\}$ satisfying the following condition:

For each e_α, there exists a countable collection of components $\{e_\alpha^m\}$ of e_α with $e_\alpha^m \nearrow e_\alpha$ such that for each $m = 1,2,\ldots,$ and any component e' of e_α^m, if $\{e_n\}$ is a sequence of components of e' with $e_n \nearrow e'$, then there exists a subsequence $\{e_{n_k}\} \subseteq \{e_n\}$ such that the collection $\{k(e_{n_{k+1}} - e_{n_k})\}$ is bounded in I.

This next theorem and corollary yield an intrinsic characterization of measure Riesz spaces.

Theorem 3.2. Let L be an Archimedean sub-Egoroff sub-order separable Riesz space. If L contains a weakly regular order dense ideal I, for which the order topology on I is Hausdorff, then L is a measure Riesz space.

Proof. It is not difficult to see that if L contains a weakly regular order dense ideal I for which the order topology on I is Hausdorff, then $\hat{L}$ contains a weakly regular order dense ideal for which the order topology is Hausdorff, namely $\hat{I} = \{\hat{x} \in \hat{L}$: there exist $x_1, x_2 \in I$ such that $x_1 \le \hat{x} \le x_2\}$. Thus we may assume that L is Dedekind complete. Since L, and hence I, is sub-order separable, we can take the positive order basis for I which is guaranteed by the weak regularity of I to consist of elements of countable type. If $\{e_\alpha\}$ is such an order basis, then in order to show that L is a measure Riesz space, we must show that $\Gamma(L)$ is separating which is equivalent to showing that, for each e_α, $\Gamma(B_{e_\alpha})$ is separating where $B_{e_\alpha} = \{e_\alpha\}^{\perp\perp}$-in-$L$. Hence, we may as well assume that I has an order unit of countable type satisfying the condition in definition (3.8). Let $0 < e$ be this order unit for I and let $e_m \nearrow e$ be a sequence of components of e such that, for each m, and each component e' of e_m if $\{z_n\}$ is a sequence of components of e' such that $z_n \nearrow e'$, then there exists a subsequence $\{z_{n_k}\} \subseteq \{z_n\}$ such that the collection $\{k(z_{n_{k+1}} - z_{n_k})\}$ is bounded in I. Consider the ideal generated in I by taking all $f \in I$ such that $|f| \leq \lambda e_m$ for some $m = 1,2,\ldots,$ and some real number $\lambda \geq 0$. We denote this ideal by $I(\{e_m\})$.

Now, let $\{y_n\}$ be a sequence in $I(\{e_m\})$ with $y_n \searrow 0$. By corollary (2.5) of chapter I, we have that $y_n \to 0$ (r.u.)-in-I. Therefore, $I(\{e_m\}) \subseteq I_{r.c.} \subseteq I(\tau_{0a})$. But, then, $I(\tau_{0a})$ is order dense in I and hence in L, and $I(\tau_{0a})^\sim_n \supseteq I(\tau_0)^*$ which is separating on $I(\tau_{0a})$. This implies that L is a measure Riesz space. The theorem is proved.

<u>Corollary</u> 3.1. Let L be a universally complete sub-order separable sub-Egoroff Riesz space. Then the following are equivalent:

(1) L is a measure Riesz space.

(2) There exists an order dense ideal $I \subseteq L$ with a complete Riesz norm ρ satisfying that if $x_n \searrow 0$ in I then $\rho(x_n) \to 0$ where $n = 1, 2, \ldots,$.

(3) There exists a weakly regular ideal I in L with a locally convex Hausdorff topology under which I becomes a T.V.L.

(4) There exists a weakly regular ideal I in L such that the order topology on I is Hausdorff.

<u>Proof</u>. To show that (1) ⇒ (2) we merely consider the $L_1(S,\Sigma,\mu)$ subspace of $L = M(S,\Sigma,\mu)$. To show that (2) ⇒ (3) we apply theorem (3.1) (4) to obtain that $L_\rho = L_{\rho,a} = L_{r.c.}$. The fact that (3) ⇒ (4) follows from the definition of τ_0. Finally, that (4) ⇒ (1) follows from theorem (3.2).

This next corollary is somewhat more descriptive.

<u>Corollary</u> 3.2. Let Q be an extremal compactum such that $C_\infty(Q)$ is order separable and Egoroff. Then, the following are equivalent:

(1) $C_\infty(Q) = M(S,\Sigma,\mu)$, where (S,Σ,μ) is a σ-finite measure space.

(2) There exists an ideal I with $C(Q) \subseteq I \subseteq C_\infty(Q)$ for which the order topology is Hausdorff, and such that given any closed nowhere dense G_δ subset G of Q, there exists a function $f \in I^+$ such that $\{x \in Q: f(x) = \infty\} \supseteq G$.

Proof. We show that (1) $\Rightarrow$ (2). We note first that if (S,Σ,μ) is a σ-finite measure space, then there exists a finite measure space (S,Σ,μ') such that $\mu'(S) = 1$ and $M(S,\Sigma,\mu) = M'(S,\Sigma,\mu')$. Hence, we may assume that $M(S,\Sigma,\mu)$ is a finite measure space. It follows easily, from [(27) p. 133] that we may take $C(Q) = L^\infty(S,\Sigma,\mu)$. In fact, we may take $1 = \chi_Q = \chi_S$.

If G is a closed nowhere dense G_δ subset of Q and if $\{E_n; n = 1,2,\ldots\}$ is the sequence of closed open subsets of Q such that $\bigcap_{n=1}^{\infty} E_n = G$ then $\chi_{E_n} \searrow 0$ in $C(Q)$. If, for each n, g_n is the element of $L^\infty(S,\Sigma,\mu)$ such that $\chi_{E_n} = g_n$ then $g_n \le \chi_S$ for all n, and $g_n \searrow 0$ which implies that $\int_S g_n \, d\mu \to 0$. Let $n = n_k$ be such that $\int_S g_{n_k} \, d\mu \le \frac{1}{2^k}\left(\frac{1}{k}\right)$. Consider now the function $\sum_{k=2}^{\infty} k(g_{n_k} - g_{n_{k+1}}) = g$. Clearly, $\int_S g \, d\mu < \infty$. But, this corresponds to the function $\sum_{k=2}^{\infty} \chi_{(E_{n_k} - E_{n_{k+1}})} = g' \in C_\infty(Q)$. It is easy to see that

$\{x \in Q: g'(x) = \infty\} \supseteq G$. So we see that if we take $I = L_1(S,\Sigma,\mu)$, then I satisfies condition (2). We have proved that (1) $\Rightarrow$ (2).

To show that (2) $\Rightarrow$ (1) it is sufficient to show that I is a weakly regular ideal. To this end, let $1 = \chi_Q$ and let $\{U_n\}$ be a sequence of closed open subsets of Q such that $\chi_{U_n} \nearrow \chi_Q$. We then have that $\bigcup_{n=1}^{\infty} U_n = Q - F$ where F is a G_δ closed nowhere dense subset of Q. In fact, $F = \bigcap_{n=1}^{\infty} Q\text{-}U_n$.

Let f be a function in I such that $\{x: f(x) = \infty\} \supseteq F$. Then, if $F_k = \{x: f(x) \geq k\}$, we have that $\bigcap_{k=1}^{\infty} F_k \supseteq F$.

Let $n = n_k$ be the first n such that $Q - U_{n_k} \subset F_k$. That such an n exists follows from the compactness of the $Q - U_n$ and the F_k.

Then, $\{k(\chi_{U_{n_{k+1}}} - \chi_{U_{n_k}})\}$ is bounded above by $f(x)$. Hence I is weakly regular and by corollary (3.1), $C_\infty(Q)$ is a space of measurable functions. The assertion is proved.

The above theorem and corollaries seem to indicate that in some sense, the universally complete sub-order separable sub-Egoroff Riesz spaces which are not measure Riesz spaces have a rather thin lattice of ideals with locally convex Hausdorff topologies defined on them.

Remark 3.1. If (E, τ) is a T.V.L. for which $E^{\sim}$ is separating and such that E is sub-order separable, then any of the conditions;

(1) $E_{r.c.}$ is order dense,

(2) E^a is order dense,

(3) $E(\tau_a)$ is order dense,

(4) $E(\tau_a^*)$ is order dense

implies that $E_n^{\prime\sim}$ is separating on some order dense ideal $E' \subseteq E$, and, therefore, that E is a measure Riesz space.

Section 4. Separability and Measure Riesz Spaces.

The object of this section is to show that the separable, metrizable topological vector lattices possessing a separating continuous (topological) dual are measure Riesz spaces if and only if, as lattices, they have the Egoroff property.

We must first obtain a few relationships between the order dual and the topological dual.

Remark 4.1. Let (E, τ) be a topological vector lattice and let E^* denote the topological dual of (E, τ). We mentioned in the previous section that E^* is an ideal in $E^{\sim}$. Since $E^{\sim}$ is a Dedekind complete Riesz space, we have that E^* is a Dedekind complete Riesz space.

From the decomposition $E^{\sim} = E_c^{\sim} \oplus E_s^{\sim}$, it follows that $E^* = E_c^* \oplus E_s^*$ where $E_c^* = E^* \cap E_c^{\sim}$ and $E_s^* = E^* \cap E_s^{\sim}$.

Using [(18) Note VI Thm. 20.4], it is trivial to show that $E(\tau_a^*) = \bigcap_{\varphi \in E_s^*} N_\varphi$ (where N_φ denotes the null ideal of the functional φ).

In the case that (E,τ) is locally convex, we have that $E(\tau_a) = E(\tau_a^*) = \bigcap_{\varphi \in E_s^*} N_\varphi$.

Remark 4.2. Let (E,τ) be a T.V.L.. Let $\{W_\alpha : \alpha \in A\}$ be a neighborhood basis of zero for the topology τ consisting of solid, absorbing, circled subsets of E. Recall that the polar of W_α in E^* is just the set $\{\varphi \in E^* : |\varphi(f)| \le 1$ for all $f \in W_\alpha\}$. As is standard, we denote the polar of W_α by W_α^O. Taking $E_\alpha = \bigcap_{\varphi \in E_s^* \cap W_\alpha^O} N_\varphi$, we claim that;

$$* \quad \bigcap_\alpha E_\alpha = \bigcap_{\varphi \in E_s^*} N_\varphi = E(\tau_a^*) .$$

The inclusion $\bigcap_\alpha E_\alpha \supseteq \bigcap_{\varphi \in E_s^*} N_\varphi$ is obvious. So, let $f \in E$ and suppose that $f \notin E(\tau_a^*)$. Then, there exists an element $\varphi \in E_s^*$ such that $|\varphi|(f) > 0$. Since $\varphi \in E^*$, the set $\{g \in E : |\varphi(g)| \le 1\}$ contains a neighborhood of zero, say $W_{\alpha'}$. Then $\varphi \in W_{\alpha'}^O \cap E_s^*$ and hence $f \notin E_{\alpha'}$.

This shows that $\bigcap_{\alpha} E_{\alpha} \subseteq \bigcap_{\varphi \in E_s^*} N_{\varphi}$. We have proved (*).

We now need to recall some facts from the theory of topological vector spaces. Let (E, τ) be a separable topological vector space. Let $\{W_{\alpha}\}$ be a neighborhood basis of zero for the topology τ on E consisting of circled absorbing sets. By the Banach-Alaoglu theorem [c.f. (12) Thm. 17.4 p. 155], we know that W_{α}^{o} -in-E^* is a compact Hausdorff space under the weak * topology.

Let $\{f_n;\ n = 1,2,\ldots,\}$ be a topologically dense collection of elements from E. For each $\varphi,\ \psi \in W_{\alpha}^{o}$ -in-E^*, define

$$d(\varphi, \psi) = \sum_{1}^{\infty} \frac{1}{2^n} \frac{|(\varphi-\psi) f_n|}{1 + |(\varphi-\psi) f_n|} .$$

This is a metric on W_{α}^{o} -in-E^*, and every open set in this topology is easily seen to be weak * open. Since both of these topologies are Hausdorff, and since the weak * topology is a compact topology, we must have the equivalence of these two topologies. Hence, W_{α}^{o} -in-E^* with the weak * topology is a compact metric space and is therefore separable.

The next two theorems will yield the result which we are seeking.

<u>Theorem</u> 4.1. Let (E, τ) be a metrizable, separable topological vector lattice. Then E, as a vector lattice, is order separable. In fact, given any disjoint collection of elements in E^+, say $\{f_\sigma\}$, the collection is at most countable.

<u>Proof</u>. Let $\{W_n;\ n = 1,2,\ldots,\}$ be a neighborhood basis of zero for the topology τ on E consisting of solid absorbing circled sets satisfying $W_{n+1} + W_{n+1} \subseteq W_n$ for $n = 1,2,\ldots,$. Let $\{f_\sigma;\ \sigma \in \Gamma\}$ be any collection of disjoint strictly positive elements in E. To each $\sigma \in \Gamma$, assign the integer n_σ as follows:

$$n_\sigma = \begin{cases} 1; & \text{if } f_\sigma \notin W_1 \\ \sup\{n : f_\sigma \in W_n\} \end{cases}$$

Clearly, $\{f_\sigma;\ \sigma \in \Gamma\} = \bigcup_{n=1}^{\infty} \{f_\sigma : \frac{1}{n_\sigma} \geq \frac{1}{n}\}$. If, for each $n = 1,2,\ldots,$ we have that $\{f_\sigma : \frac{1}{n_\sigma} \geq \frac{1}{n}\}$ is countable, then so is $\{f_\sigma : \sigma \in \Gamma\}$. Hence, if $\{f_\sigma : \sigma \in \Gamma\}$ is not countable, there exists an $n = n'$ such that $\{f_\sigma : \frac{1}{n_\sigma} \geq \frac{1}{n'}\}$ is not countable. Denote this set by $\{f_\beta;\ \beta \in B\} = G$.

Now, if f_{β_1} and f_{β_2} are elements of G with $\beta_1 \neq \beta_2$, then $f_{\beta_1} - f_{\beta_2} \notin W_{n'+1}$ since otherwise we would have that $f_{\beta_1} \leq |f_{\beta_1} - f_{\beta_2}| = f_{\beta_1} + f_{\beta_2} \in W_{n'+1}$ and this would imply that $f_{\beta_1} \in W_{n'+1}$ which contradicts that $f_{\beta_1} \in G$.

It follows easily that the system $\{f_\beta + W_{n'+2};\ \beta \in B\}$ is a collection of disjoint open subsets of (E, τ). If this collection is not countable, then we would have that (E, τ) is not second countable which contradicts the separability of (E, τ). Therefore, $\{f_\sigma;\ \sigma \in \Gamma\}$ is countable and this implies the order separability of E (see section 4 of Chapter I). The theorem is proved.

<u>Theorem</u> 4.2. Let (E, τ) be a metrizable, separable topological vector lattice. Let $\{W_n;\ n = 1, 2, \ldots,\}$ be a neighborhood basis at zero for the topology τ on E satisfying $W_{n+1} + W_{n+1} \subseteq W_n$ for $n = 1, 2, \ldots,$ and consisting of solid, absorbing, circled subsets of E. Assume also that E has the Egoroff property. Let $E_n = \bigcap_{\varphi \in E_s^* \cap W_n^O} N_\varphi$ for $n = 1, 2, \ldots,$. Then, E_n is a super order dense ideal in E for each $n = 1, 2, \ldots,$.

<u>Proof</u>. We have already noted that each W_n^O is a separable metric space in the weak * topology. Therefore, $E_s^* \cap W_n^O$ is also separable in the weak * topology.

Let $\{\varphi_m: m = 1,2,\ldots,\}$ be a dense subset of $E_s^* \cap W_n^O$. We claim first that $\bigcap_{m=1}^{\infty} N_{\varphi_m} = \bigcap_{\varphi \in E_s^* \cap W_n^O} N_\varphi$.

The inclusion $\bigcap_{m=1}^{\infty} N_{\varphi_m} \supseteq \bigcap_{\varphi \in E_s^* \cap W_n^O} N_\varphi$ is trivial.

Now, let $o < u \in N_{\varphi_m}$ for all $m = 1,2,\ldots,$. Let $\varphi \in E_s^* \cap W_n^O$. Since the polar of a solid subset of E is solid (this follows easily from [(25) Equation 6 p. 22]), we have that $|\varphi| \in E_s^* \cap W_n^O$. Let $\epsilon > 0$ be chosen. Since $\{\varphi_m; m = 1,2,\ldots,\}$ is weak * dense in W_n^O, there exists an $m = m_O$ such that $\big|\, |\varphi|(u) - \varphi_{m_O}(u) \big| < \epsilon$. Hence, $|\varphi|(u) < \epsilon$. This holds for all $\epsilon > 0$. So, $|\varphi|(u) = 0$ and $u \in N_\varphi$. This establishes that

$$\bigcap_{m=1}^{\infty} N_{\varphi_m} = \bigcap_{\varphi \in E_s^* \cap W_n^O} N_\varphi .$$

It follows from [(18) Note VI Cor. 20.7], that N_{φ_m} is a super order dense ideal in E for each $m = 1,2,\ldots,$. By [(16) Note XIV_A Lemma 44.1], we have that the countable intersection of super order dense ideals in a space with the Egoroff property is again a super order dense ideal. Hence,

$$E_n = \bigcap_{m=1}^{\infty} N_{\varphi_m} \quad \text{is a super order dense}$$

ideal in E for each $n = 1,2,\ldots,$. The theorem is proved.

<u>Theorem</u> 4.3. Let (E, τ) be a separable, metrizable topological vector lattice. Suppose in addition that the topological dual E^* of (E, τ) is separating on E. Then the following are equivalent:

(1) There exists a σ-finite measure space (S, Σ, μ) such that E is an order dense Riesz subspace of $M(S, \Sigma, \mu)$.

(2) E has the Egoroff property.

<u>Proof</u>. We need only show that (2) $\Rightarrow$ (1). If E has the Egoroff property, then theorem (4.2) above together with [(16) Note XIV_A Lemma 44.1] imply that $E(\tau_a^*)$ is an order dense (actually super order dense) ideal in E. The fact that E^* is separating implies that E^* is also separating on $E(\tau_a^*)$. From theorem (4.1), we have that E is order separable and hence that $E(\tau_a^*)$ is order separable. This implies that every $\varphi \in E^*$ is a normal integral on $E(\tau_a^*)$ and hence that E is a measure Riesz space. Since, by theorem (4.1), we saw that E actually has a countable disjoint basis of positive elements of countable type, we see that the universal completion $\overline{E}$ of E is order separable and thus $\overline{E} = M(S, \Sigma, \mu)$ where (S, Σ, μ) is a σ-finite measure space.

Corollary 4.1. Let E be a separable, metrizable, locally convex topological vector lattice. Then the following are equivalent:

(1) There exists a σ-finite measure space (S,Σ,μ) such that E is an order dense Riesz subspace of $M(S,\Sigma,\mu)$.

(2) E has the Egoroff property.

Remark 4.3. Although we will not pursue it here, it should be possible, in view of some of the results found in [(13)], to show that the σ-finite measure space (S,Σ,μ) obtained in theorem (4.3) and corollary (4.1) of this section may actually be assumed to be a separable measure space.

Section 5. Some More on Non-Sub-Order Separable Spaces.

Throughout this exposition, we have had occasion to investigate three examples of non-sub-order separable Riesz spaces. It can be easily verified that the space of continuous real valued functions on $\beta N-N$ (where βN is the Stone-Čech compactification of $N = \{1,2,\ldots,\}$) is another example of a non-sub-order separable Riesz space. From a historical point of view, this space was probably the first Riesz space of this type to be investigated. S. Kaplan used it, for instance, as an example of a $C(X)$

such that every radon measure on X has first category support [(10) IV p. 520]. It was also studied by Dixmier in [(3)]. So far, these spaces have been most useful in providing examples of pathological behavior from the vector lattice point of view.

All of our examples have actually been of non-sub-order separable Riesz spaces with the property that no non-trivial ideal in them is sub-order separable.

One thing all such spaces have in common is the following:

<u>Theorem</u> 5.1. Let L be an Archimedean Riesz space with the property that no non-trivial ideal in L is sub-order separable. Let I be any ideal in L such that $I^{\sim}$ is not just the zero functional. If $\varphi \in I^{\sim}$, then N_{φ} (the null ideal of φ) is order dense in I.

<u>Proof</u>. Suppose that for some $\varphi \in I^{\sim}$ we have that $N_{\varphi}^{\perp}$ -in-I $\neq \{0\}$. We may as well assume that $\varphi > 0$ since $N_{\varphi} = N_{|\varphi|}$. Letting $C_{\varphi} = N_{\varphi}^{\perp}$ -in-I, we have that φ is strictly positive on C_{φ}. Let $0 < f \in C_{\varphi}$. Since no element of L and hence of I is of countable type, we must have an uncountable collection $\{0 < f_{\alpha} \leq f;\ \alpha \in A\}$ of positive disjoint elements in C_{φ}.

Clearly, since $\{f_{\alpha}\} = \bigcup_{n=1}^{\infty} \{f_{\alpha}: \varphi(f_{\alpha}) \geq \frac{1}{n}\}$, we must have that, for some $n = n_0$, the set $\{f_{\alpha}: \varphi(f_{\alpha}) \geq \frac{1}{n_0}\}$

is uncountable. Let $k \in \{1,2,\ldots,\}$ be such that $\frac{k}{n_0} > \varphi(f)$. Then, choosing $f_{\alpha_1}, f_{\alpha_2}, \ldots, f_{\alpha_k}$ from $\{f_\alpha\colon \varphi(f_\alpha) \geq \frac{1}{n_0}\}$, we have that

$\sup\{f_{\alpha_i};\ 1 \leq i \leq k\} = \sum_{i=1}^{k} f_{\alpha_i} \leq f$ and

$$\varphi\Big(\sum_{i=1}^{k} f_{\alpha_i}\Big) = \sum_{i=1}^{k} \varphi(f_{\alpha_i}) \geq \frac{k}{n_0} > \varphi(f) .$$

This is a contradiction. The theorem is proved.

<u>Corollary</u> 5.1. Let L be an Archimedean Riesz space with the property that no non-trivial ideal in L is sub-order separable. Let I be any ideal in L. Then, $I^{\sim} = I_{sn}^{\sim}$.

<u>Proof</u>. If there exists a $0 \neq \varphi \in I_n^{\sim}$, then $N_\varphi^{\perp}$-in-I $\neq \{0\}$ and this contradicts theorem (5.1). The assertion is proved.

We wish to note that in view of corollary (5.1) above, if L is a Dedekind complete Riesz space with the property that no non-trivial ideal in L is sub-order separable, and if I is any ideal in L such that $I_c^{\sim} \neq \{0\}$ then by Luxemburg's result [(15)], there exists a real valued measurable cardinal. We thus have another example of a Riesz space condition which would imply the existence of a measurable cardinal.

Remark 5.1. If X is a compact Hausdorff space for which $C(X)$ has the property that no non-trivial ideal contained in $C(X)$ is sub-order separable, then every regular Borel measure on X has nowhere dense support. Otherwise, we would be able to obtain a contradiction to theorem (5.1) of this section. If X is the Stone-Čech compactification of the space of Example (4.1) of Chapter I, then every regular Borel measure on X has nowhere dense support. However, by Remark (4.4) of Chapter I, we see that X does not have the property that the intersection of a descending sequence of open sets has nonempty interior. As far as we know, this is the first example of such a space in the literature.

Section 6. Miscellaneous Problems.

A number of problems arise naturally from the research done in this exposition. Our purpose in this section is to briefly state and discuss some of these problems.

The first problem is motivated by theorem (5.1) of chapter II.

Problem A. Let L be an Archimedean Riesz space possessing a positive countable order basis consisting of elements of countable type. Does L then contain an order dense ideal $L_1 \subseteq L$ such that L_1 possesses a strictly positive linear functional?

It is quite obvious that a positive answer to problem A would prove the converse of theorem (5.1) of this chapter. Thus, if the answer to problem A is positive, we would have an interesting characterization of Archimedean Riesz spaces, which possess no non-trivial sub-order separable ideals, in terms of dual space elements.

The statement of problem A is somewhat awkward. It has a much nicer formulation. Unfortunately the equivalence of the two formulations takes significant effort to see. We will put forth the effort, however, since we believe the techniques involved to be interesting in themselves.

Remark 6.1. If L is Archimedean and possesses a strictly positive linear functional, a straight forward application of [(18) Note VI Thm. 19.2] proves that $\hat{L}$ (the Dedekind completion of L) also has a strictly positive linear functional. Clearly, if $\hat{L}$ has a strictly positive linear functional, then, by restriction, so also does L. Problem A is now easily seen to be equivalent to the question;

> Does every order separable universally complete Riesz space L contain an order dense ideal L, such that L_1 possesses a strictly positive linear functional?

In order to simplify further the formulation of problem A, we need a method for obtaining extensions of positive linear functionals defined on ideals in a Riesz space to larger ideals. The method is described as follows:

Let L be an Archimedean Riesz space. Let φ be a positive linear functional defined on an ideal I_φ contained in L.

Consider

$\varphi_1(u) = \sup\{\varphi(v): v \in I_\varphi,\ 0 \le v \le u\}$, where $u \in (I_\varphi^{\perp\perp}\text{-in-}L)^+$.

We have easily that

(a) $0 \le \varphi_1(u) \le \infty$ for $u \in (I_\varphi^{\perp\perp}\text{-in-}L)^+$,

(b) If $a \ge 0$ is a real number, then

$\varphi_1(a\,u) = a\,\varphi_1(u)$.

We claim in addition that if $u_1, u_2 \in (I_\varphi^{\perp\perp}\text{-in-}L)^+$ then (c) $\varphi_1(u_1) + \varphi_1(u_2) = \varphi_1(u_1+u_2)$.

To see that $\varphi_1(u_1+u_2) \ge \varphi_1(u_1) + \varphi_1(u_2)$ is a standard argument which we will not produce here.

To show that $\varphi_1(u_1+u_2) \le \varphi_1(u_1) + \varphi_1(u_2)$, we consider first the case when $\varphi_1(u_1+u_2) = \infty$. If $\varphi_1(u_1+u_2) = \infty$, then, for each $N \in \{1,2,\dots,\}$, there exists a $v \in I_\varphi$ such that $0 \le v \le u_1 + u_2$ and $\varphi(v) > N$.

Suppose that $\varphi_1(u_1) < \infty$. Since $v = v_1 + v_2$, where $v_1, v_2 \in I_\varphi$ and $0 \le v_1 \le u_1$, $0 \le v_2 \le u_2$, we must have that $\varphi(v_2) \ge N - \varphi_1(u_1)$. Hence, $\varphi_1(u_2) = \infty$ and we are done for this case.

We now need only consider the case when $\varphi_1(u_1+u_2) < \infty$.

In this case, given $\epsilon > 0$ there exists a $v = v_1 + v_2$ with $v_1, v_2 \in I_\varphi$, $0 \le v_1 \le u_1$, $0 \le v_2 \le u_2$ and such that $\varphi(v) + \epsilon = \varphi(v_1+v_2) + \epsilon = \varphi(v_1) + \varphi(v_2) + \epsilon \ge \varphi_1(u_1+u_2)$. This implies that $\varphi_1(u_1) + \varphi_1(u_2) + \epsilon \ge \varphi_1(u_1+u_2)$. Since the choice of ϵ was arbitrary, we have that $\varphi_1(u_1) + \varphi_1(u_2) \ge \varphi_1(u_1+u_2)$ and our claim is proved.

Now, let $I_{\varphi_1} = \{u \in (I_\varphi^{\perp\perp}\text{-in-}L)^+ : \varphi_1(u) < \infty\}$. Taking $I_{\hat{\varphi}} = I_{\varphi_1} - I_{\varphi_1}$ we obtain easily that $I_{\hat{\varphi}}$ is an ideal in L containing I_φ.

For each $f \in I_{\hat{\varphi}}$, define $\hat{\varphi}(f) = \varphi_1(f^+) - \varphi_1(f^-)$. Then, $\hat{\varphi}$ is an extension of φ to $I_{\hat{\varphi}}$.

It is obvious that if I_φ is order dense in L, then $I_{\hat{\varphi}}$ is order dense in L. Also, if φ is strictly positive on I_φ, then $\hat{\varphi}$ is strictly positive on $I_{\hat{\varphi}}$.

<u>Definition</u> 6.1. Let L be an order separable universally complete Riesz space. An element $f \in L^+$ is called an <u>almost unit</u> for an ideal $I \subseteq L$, if there exists a sequence of components of f, $\{f_n; n = 1,2,...,\}$, with $f_n \nearrow f$ such that if $g \in I$ then there exists an integer $k \in \{1,2,...,\}$ such that $|g| \le f_k$.

<u>Theorem</u> 6.1. Let φ be a positive linear functional defined on an order dense ideal $I_\varphi \subseteq L$, where L is a universally complete order separable Riesz space. Suppose that $f \in L^+$ is an almost unit for I_φ and let $\{f_n; n = 1,2,...,\}$ be the sequence of components of f in I_φ such that $f_n \nearrow f$ and such that, for any $g \in I_\varphi$, there exists a $k \in \{1,2,...,\}$ such that $|g| \le f_k$. Suppose in addition that $\varphi(f_n) \le M < \infty$ for all $n = 1,2,...,$. Then, we have that $f \in I_{\hat{\varphi}}$ (where $I_{\hat{\varphi}}$ is the ideal obtained by applying the extension process described previously in this section).

<u>Proof</u>. Since for any $v \in I_\varphi$ such that $0 \le v \le f$, we must have a $k \in \{1,2,...,\}$ such that $v \le f_k$, we see that $\varphi_1(f) \le \sup_n \{\varphi(f_n)\} \le M < \infty$. The theorem is proved.

<u>Theorem</u> 6.2. Let L be a universally complete order separable Riesz space. Suppose there exists an order dense ideal $L_1 \subseteq L$ such that L_1 possesses a strictly

positive linear functional. Let $0 < f \in L$ be any order unit in L. Then, the ideal generated by the element f, $I(f)$, has a strictly positive linear functional.

Proof. Let $0 < f \in L^+$ be an order unit for L. Let $L_1 \subseteq L$ be an order dense ideal in L possessing a strictly positive linear functional φ. It is easy to see that there exists a sequence $\{f_n;\ n = 1,2,\ldots,\}$ of components of f with $f_n \nearrow f$ and such that $f_n \in L_1$ for $n = 1,2,\ldots,$.

Let $I(\{f_n\})$ be the ideal generated in L by taking $g \in I(\{f_n\})$ if and only if, for some $k \in \{1,2,\ldots,\}$, we have that $|g| \leq f_k$.

Obviously, $I(\{f_n\})$ is order dense in L and the restriction of φ to $I(\{f_n\})$ is a strictly positive linear functional on $I(\{f_n\})$.

Let $\{g_n\}$ be the sequence of elements of $I(\{f_n\})$ defined by; $g_1 = f_1$, $g_2 = f_2 - f_1, \ldots, g_n = f_n - f_{n-1}$ for $n = 1,2,\ldots,$.

Let $\alpha_n = \varphi(g_n)$, for each n. Define φ' on $I(\{f_n\})$ by $\varphi'(g) = \varphi\left(\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n}\right)\left(\frac{1}{2^n}\right) P_{g_n}(g)\right)$ for $g \in I(\{f_n\})$.

It follows from the additivity and homogeneity of the projection operator and from the definition of $I(\{f_n\})$

that φ' is a strictly positive linear functional on $I(\{f_n\})$.

We note that f is an almost unit for $I(\{f_n\})$, and that $\varphi'(f_n) \leq 1$ for all $n = 1,2,\ldots,$.

By application of theorem (6.1), $f \in I_{\hat{\varphi}'}$ (where again $I_{\hat{\varphi}'}$ is the ideal obtained by applying the extension process described earlier in this section).

Then, $\hat{\varphi}'$ is strictly positive on $I_{\hat{\varphi}'} \supseteq I(f)$. Therefore, the restriction of $\hat{\varphi}'$ to $I(f)$ is strictly positive on $I(f)$. The theorem is proved.

Combining remark (6.1), the fact that every universally complete Riesz space is a $C_\infty(Q)$ where Q is some extremal compactum, the fact that $C_\infty(Q)$ is order separable if and only if $C(Q)$ is order separable, and theorem (6.2) above, we obtain that problem A is equivalent to the following problem.

<u>Problem</u> A' Let Q be an extremal compactum. If $C(Q)$ is order separable, does it then possess a strictly positive linear functional?

Or equivalently, does every order separable $C(X)$, where X is a compact Hausdorff space, possess a strictly positive linear functional?

The next problem is motivated by theorem (1.1) of this chapter.

Problem B. Is every sub-order separable sub-Egoroff Archimedean Riesz space a measure Riesz space?

In view of theorem (1.1) of this chapter, we see immediately that a positive answer to problem A would imply a positive answer to problem B.

In fact, problem B has the following equivalent formulation:

Problem B$'$. Does every order separable $C(X)$, where X is a compact Hausdorff space, with the Egoroff property possess a strictly positive linear functional?

Although we have not been able to solve problem B above, we feel that the results of this chapter give sufficient indication of the value of pursueing this next problem.

Problem C. The theory of Banach function spaces has been extensively covered in [(18) Notes I-V]. The existing theory assumes one to be working on a σ-finite measure space. The first part of this problem then is to extend the theory of Banach function spaces to arbitrary measure spaces with the finite subset property and with a completely additive measure.

Our investigations have shown that many of the topologized Riesz spaces which possess "nice" relationships between the topology and the order are measure Riesz spaces. The second part of this problem then is to extend the theory of Banach function spaces to just order dense Riesz subspaces of spaces of measurable functions.

The results we have obtained hold for locally convex topological vector lattices. The third part of this problem then is to obtain a theory for locally convex topological vector lattice order dense Riesz subspaces of spaces of measurable functions analogous to the theory of Banach function spaces.

In section 2 of this chapter, we gave necessary and sufficient conditions for a Dedekind-σ-complete Riesz space of extended type to be a Riesz subspace (not necessarily order dense) of a space of measurable functions. It would be better to have conditions for an arbitrary Archimedean Riesz space.

Problem D. Find necessary and sufficient conditions for an arbitrary Archimedean Riesz space to be a Riesz subspace of some space of measurable functions.

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