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On the Realization of Time Domain Models
of Real Linear Bielement Systems

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ABSTRACT

ON THE REALIZATION OF TIME DOMAIN MODELS
OF REAL LINEAR BIELEMENT SYSTEMS

by Donald John Rauch

In recent years there has been a trend in modern engineering analysis to formulate mathematical models of physical systems as a set of linear, first derivative-explicit differential equations. The question now arises as to whether a system graph can be synthesized from this same set of equations. This thesis deals with the realization of time-domain models of real linear bielelement systems.

The problems that are considered in this thesis can be classified as follows:

1. The characterization of time-domain models of real linear bielelement systems,
2. The recognition of an arbitrary time domain model to be a real linear bielelement system, and
3. The synthesis of a graph from an acceptable time-domain model.

From the time-domain analysis, real linear bielelement systems are characterized by their associated matrices. These systems are shown to be reducible to a canonical graph. The canonical associated matrix corresponding to a maximum order star tree of a canonical graph is best described as the product of two symmetric matrices. Each of the symmetric matrices has a distinctive sign pattern and exhibits the property of diagonal dominance. All other associated matrices corresponding to arbitrary maximum order trees of a graph can be made to exhibit these same properties by the use of a similarity transformation.

A necessary condition for the recognition of time-domain models of real linear bielelement systems is that the associated matrix has real eigenvalues. A sufficient condition is that the eigenvalues be real distinct. Three techniques are derived for the decomposition of an arbitrary matrix into the product of two symmetric matrices. A test is also provided to determine if one of the matrices is diagonal.

The synthesis of a graph is accomplished by interrelating the decomposed matrix to the canonical associated matrix. Necessary and sufficient conditions are given for the synthesis of a graph with positive, negative and zero elements and for the synthesis of a graph with non-negative elements. Flow charts are developed to illustrate how an arbitrary coefficient matrix can be recognized and realized as the associated matrix of a real linear bielement system.

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OF REAL LINEAR BIELEMENT SYSTEMS

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CONTENTS

	Page
LIST OF FIGURES	iv
LIST OF APPENDICES	v
LIST OF SYMBOLS	vi
I. INTRODUCTION AND ANALYSIS	1
1.0 Introduction	1
1.1 Time Domain Analysis	2
II. CHARACTERIZATION OF REAL LINEAR BIELEMENT SYSTEMS	7
2.0 Introduction	7
2.1 The Complete Graph	7
2.2 The Classification of Real Linear Bielement Systems	10
2.3 Characterization of Class 1 Systems	12
2.4 Conclusion	26
III. SYNTHESIS OF A CLASS OF REAL LINEAR BIELEMENT SYSTEMS	28
3.0 Introduction	28
3.1 Decomposition of a Square Matrix	28
3.2 Synthesis of RC Graphs	32
3.3 Synthesis of RL Graphs	36
3.4 Conclusion	39
IV. THE SYNTHESIS METHOD	41
4.0 Introduction	41
4.1 Determination of Realizable Coefficient Matrices	41
4.2 The Synthesis Method	45
4.3 Conclusion	64
V. CONCLUSION	65
5.0 Discussion of Results	65
5.1 Additional Problems	67
BIBLIOGRAPHY	68

LIST OF FIGURES

Figure		Page
2.1.0	Complete graph Q_{m+1}^e of c-elements with m+1 vertices	9
2.1.1	Star tree T_s of m+1 vertices	9
4.1.0	Flow chart for the determination of acceptable coefficient matrices	42
4.2.0	Flow chart of RC graph synthesis	47
4.2.1	Flow chart of RL graph synthesis	47
4.2.2	Synthesized RC graph G_3 of Ex. 1	48
4.2.3	Synthesized RC graph G_4 of Exs. 2 and 3	51
4.2.4	Path Tree T_j of G_4	57
4.2.5	Synthesized RL graph G_3 of Exs. 4 and 5	61
4.2.6	Synthesized graph G_4 of Ex. 5	63
5.0.1.	Elemental adaptive system	66

LIST OF APPENDICES

Appendix	Page
A. Theorems and Definition from References	71

LIST OF SYMBOLS

Symbol	Description
A	Associated or coefficient matrix
A_i, A_j, A_s	Associated matrices corresponding to trees T_i, T_j, T_s
A_j^i, A_s^i	Associated matrix of the i^{th} part of a separable graph for the subtrees T_j, T_s .
A_{LC}, A_{RC}, A_{RL}	Associated matrices of bielement systems
a_{ij}	Element of A
a, a_s	Incidence matrix
$a_c, a_T,$ a_{ci}, a_{si}	Submatrices of a
a_{sj}^i	Submatrices of the incidence matrix for the i^{th} part of a separable graph
A'_{ji}	Similarity transformation between A_i and A_j
B_1, B_2	Coefficient matrices
B_f	f circuit matrix
B_{ij}	Submatrix of B_f
b	Number of branches of a tree
b^*	Neighborhood
C	C matrix
C_1	Diagonal matrix of branch c -elements
C_2	Diagonal matrix of chord c -elements
C_i, C_s	Diagonal matrix of all c -elements of a graph corresponding to tree T_i, T_s
C_{ij}	c -element incident to vertices i and j
C_{ii}	c -element incident to vertex i and the reference node
c_{ij}	element of C
c_{ijo}	Known value of c_{ij}

Symbol	Description
D_1, D_2	Diagonal matrices
d_{ij}	Non-zero element of diagonal matrix
Det	Determinate of matrix
diag ()	Diagonal matrix
E_{ii}, E_{ij}	Elements of a complete graph
e	Number of elements of a graph
$f_{\alpha\beta}(\)$	Function
G	Graph
G_1	Diagonal matrix of the branch g-elements
G_2	Diagonal matrix of the chord g-elements
G_i, G_s	Diagonal matrices of all g-elements of a graph corresponding to trees T_i, T_s
G_v, G_{n+1}	Graphs having v, n+1 vertices
G_{ij}	g-element incident to vertices i and j
G_{ii}	g-element incident to vertex i and the reference node
g_{ij}	Element of R
g_{ijo}	Known value of g_{ij}
I	Unit or identity matrix
J	Jacobian
L	L matrix
L_1	Diagonal matrix of branch l-elements
L_2	Diagonal matrix of chord l-elements
L_{ij}	l-element incident to vertices i and j
L_{ii}	l-element incident to vertex i and the reference node
$\bar{I}$	Vector
l_{ij}	Element of L

Symbol	Description
P	Jordan similarity transformation
P	Known value of P
P_1, P_2, P_j	Parts of a graph
P_{n+1}	A part with $n+1$ vertices
P_{ij}, P_{js}	Permutation matrices
P_{11}, P_{22}	Submatrices of P_{js}
$p(\lambda)$	Polynomial in λ
$Q_{n+1}^c, Q_{n+1}^g, Q_{n+1}^l$	Complete graphs of $n+1$ vertices composed entirely of c -, g -, l -elements
R	R matrix
R	Ratio matrix
R_{ij}	Submatrix of R
r	Next largest integer to r^*
r^*	Real number
S	Submatrix of S_f
S_1, S_j, S_s	f seg matrices corresponding to trees T_1, T_j, T_s
S_f	f seg matrix
S_{ij}	Submatrix of S_f
$S_{v,n}$	The set of all σ_v
S_{11}, S_{12}	The submatrix S_{11}, S_{22} augmented by a unit matrix in the leading position
$S_{11}, S_{12}, S_{122}, S_{123}$	Submatrices of S_1
$S_{s11}, S_{s12}, S_{s22}, S_{s23}$	Submatrices of S_s
S_v^*	Subset of f seg matrix of a complete graph corresponding to a star tree

Symbol	Description
T	Transformation matrix
T_i, T_j	Maximum order trees of graph
T_s	Maximum order star tree
t_{ij}	coefficient
U	Union of graphs
V	Unitary matrix
v_j	j^{th} column of V
X	Vector
x_{ij}	Variable
x_{ijo}	Known value of x_{ij}
Y	Vector
δ	Diagonal matrix with entries of ± 1
Λ	Diagonal matrix of eigenvalues
λ_i	Eigenvalue
μ	Diagonal matrix
$\phi ()$	Function
σ_v	Submatrix of order $(v-1, n)$ of S_v^*
ϵ	Belongs to or is an element of

I. INTRODUCTION AND ANALYSIS

1.0 Introduction

In the last few years there has been an accelerated interest in time domain modeling of linear physical systems ^(1,2,3,4,5). Mathematically, these models are systems of first derivative-explicit differential equations

$$\frac{d}{dt}X = AX \quad 1.0.0$$

where X is a vector, sometimes referred to as the state vector, and A is a real matrix.

Two questions now arise:

1. Under what conditions can the matrix A of Eq. 1.0.0 be realized by a linear physical system? and
2. If A is realizable as a linear physical system, what are the element values and their interconnection pattern?

These questions are a generalized form of the classical synthesis problem. However, the concept of realizing a real time model of a system is strikingly different from the classical concepts of systems synthesis whereby driving point impedances, admittances or transfer functions are realized ^(6,7,8,9).

In Section 1.1 a complete time domain analysis of a RLC system is given and the associated matrix A of this system is introduced. Some general properties of associated matrices are also derived.

In Chapter II the class of graphs considered is restricted to real linear bielement systems, that is, systems composed of g -elements ⁽¹⁰⁾ and lossy c - or l -elements ⁽¹⁰⁾. Koenig, Tokad and Bacon have already considered the synthesis of LC graphs from the state model ⁽¹¹⁾. Real linear bielement systems, hereafter referred to specifically as RC and RL graphs, are classified by the subgraph of the system from which the formulation tree is selected. Class 1 graphs are shown to be reducible to a canonical form. The associated matrices of a class 1 graph are then shown to be interrelated by a similarity transformation.

In Chapter III the necessary and sufficient conditions for realizing A

of Eq. 1.0.0 as a real linear bielement system are given. Three techniques are developed for decomposing A into a bisymmetric form, that is, the product of two symmetric matrices. A test is also developed to determine if one of these symmetric matrices is diagonal. For matrices that are realizable, a method is given for determining element values and the f seg matrix of the class 1 graph to be synthesized. From this information a graph is constructed. Necessary and sufficient conditions are also given on the bisymmetric form of A to guarantee that the synthesized graph will have no negative elements.

In Chapter IV the synthesis method is described in detail and flow charts are developed to indicate the operations in the realization of an A matrix as a real linear bielement system. Examples of the synthesis method for both RC and RL graphs are also given.

1.1 Time Domain Analysis

The class of graphs considered is restricted as follows:

1. All components of the graph are to be g-elements, c-elements or l-elements.
2. The graph contains no drivers.
3. All initial conditions of the element variables are assumed to be zero.

The time domain analysis of a less restrictive class of graphs has been rigorously carried out by Brown ⁽²⁾ and others ^(1,3,4,5). The results pertinent to this dissertation follow.

As a direct consequence of Theorem A.1 the following definition is made.

Definition 1.1.0: A tree T of a connected graph G is said to be a fundamental tree if

1. As many c-elements as possible are branches of T, and
2. As many l-elements as possible are chords of T.

Lemma 1.1.0: The f circuit and f seg matrices ⁽¹⁰⁾ corresponding to any fundamental tree T of a connected RLC graph G are

$$B_f = \begin{bmatrix} B_{11} & 0 & 0 & I & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 & I & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 & I \end{bmatrix}, \quad S_f = \begin{bmatrix} I & 0 & 0 & S_{11} & S_{12} & S_{13} \\ 0 & I & 0 & 0 & S_{22} & S_{23} \\ 0 & 0 & I & 0 & 0 & S_{33} \end{bmatrix} \quad 1.1.0$$

where the columns of the unit matrices I of B_f correspond respectively to the c-, g-, and l-elements of the cotree and the columns of the unit matrices I of S_f correspond respectively to the c-, g-, and l-elements of the tree.

Proof: Follows directly from Theorem A.2. ¹

Definition 1.1.1: The branch (chord) l-, g-, and c-element matrices ² of a connected graph G are the diagonal element value matrices L_1, G_1, C_1 (L_2, G_2, C_2) where the subscript 1 (2) indicates the elements are branches (chords) of a tree (cotree).

Lemma 1.1.1: Corresponding to any fundamental tree T of any connected RLC graph G , there exists a system of equations

$$\frac{d}{dt}X = AX$$

such that

$$A = - \begin{bmatrix} (C_1 + S_{11}C_2S'_{11})^{-1} & 0 \\ 0 & (L_2 + S'_{33}L_1S_{33})^{-1} \end{bmatrix} \left\{ \begin{bmatrix} 0 & S_{13} \\ -S'_{13} & 0 \end{bmatrix} + \begin{bmatrix} S_{12}(G_2^{-1} + S'_{22}G_1^{-1}S_{22})^{-1}S'_{12} & -S_{12}G_2S'_{22}(G_1 + S_{22}G_2S'_{22})^{-1}S_{23} \\ -S'_{23}(G_1 + S_{22}G_2S'_{22})^{-1}S_{22}G_2S'_{12} & S'_{23}(G_1 + S_{22}G_2S'_{22})^{-1}S_{23} \end{bmatrix} \right\}, \quad 1.1.1$$

where

1. G_1, L_1, C_1 and G_2, L_2, C_2 are branch element and chord element matrices of Definition 1.1.2 respectively.
2. S_{ij} and B_{ij} for $i, j = 1, 2, 3$ are given in Lemma 1.1.1.
3. The prime and -1 superscript indicate the transpose and inverse of the indicated submatrix respectively.

Proof: The lemma results directly from Eq. 9 of reference 2.

-
1. The matrices B_{ij} and S_{ij} of Lemma 1.1.0 are not necessarily in the same positions as the B_{ij} and S_{ij} of Theorem A.2.
 2. See Eq. 2 of reference 2.

where the columns of the unit matrices I of B_f correspond respectively to the c-, g-, and l-elements of the cotree and the columns of the unit matrices I of S_f correspond respectively to the c-, g-, and l-elements of the tree.

Proof: Follows directly from Theorem A.2. ¹

Definition 1.1.1: The branch (chord) l-, g-, and c-element matrices ² of a connected graph G are the diagonal element value matrices L_1, G_1, C_1 (L_2, G_2, C_2) where the subscript 1 (2) indicates the elements are branches (chords) of a tree (cotree).

Lemma 1.1.1: Corresponding to any fundamental tree T of any connected RLC graph G , there exists a system of equations

$$\frac{d}{dt}X = AX$$

such that

$$A = - \begin{bmatrix} (C_1 + S_{11}C_2S'_{11})^{-1} & 0 \\ 0 & (L_2 + S'_{33}L_1S_{33})^{-1} \end{bmatrix} \left\{ \begin{bmatrix} 0 & S_{13} \\ -S'_{13} & 0 \end{bmatrix} + \begin{bmatrix} S_{12}(G_2^{-1} + S'_{22}G_1^{-1}S_{22})^{-1}S'_{12} & -S_{12}G_2S'_{22}(G_1 + S_{22}G_2S'_{22})^{-1}S_{23} \\ -S'_{23}(G_1 + S_{22}G_2S'_{22})^{-1}S_{22}G_2S'_{12} & S'_{23}(G_1 + S_{22}G_2S'_{22})^{-1}S_{23} \end{bmatrix} \right\}, \quad 1.1.1$$

where

1. G_1, L_1, C_1 and G_2, L_2, C_2 are branch element and chord element matrices of Definition 1.1.2 respectively.
2. S_{ij} and B_{ij} for $i, j = 1, 2, 3$ are given in Lemma 1.1.1.
3. The prime and -1 superscript indicate the transpose and inverse of the indicated submatrix respectively.

Proof: The lemma results directly from Eq. 9 of reference 2.

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1. The matrices B_{ij} and S_{ij} of Lemma 1.1.0 are not necessarily in the same positions as the B_{ij} and S_{ij} of Theorem A.2.
 2. See Eq. 2 of reference 2.

Definition 1.1.2: The matrix A given by Eq. 1.1.1 is the associated matrix of the connected RLC graph G.

Definition 1.1.3: Let T be any tree of a graph G. If the associated matrix A corresponding to the tree T exists, then T is said to be a maximum order tree.

Corollary 1.1.1: Every fundamental tree is a maximum order tree.

Proof: Follows directly from Lemma 1.1.1.

From Lemma 1.1.1 it is convenient to introduce the following notation.

Let

$$A = - \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & R_{22} \end{bmatrix} + \begin{bmatrix} 0 & S_{13} \\ -S_{13}' & 0 \end{bmatrix} \right\}, \quad 1.1.2$$

where

$$\begin{aligned} C &= C_1 + S_{11} C_2 S_{11}', \\ L &= L_2 + S_{33}' L_1 S_{33}, \\ R &= \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} S_{12}(G_2^{-1} + S_{22}' G_1^{-1} S_{22})^{-1} S_{12}' \\ -S_{23}'(G_1 + S_{22} G_2 S_{22}')^{-1} S_{22} G_2 S_{12}' \\ -S_{12} G_2 S_{22}'(G_1 + S_{22} G_2 S_{22}')^{-1} S_{23} \\ S_{23}'(G_1 + S_{22} G_2 S_{22}')^{-1} S_{23} \end{bmatrix}. \end{aligned} \quad 1.1.3$$

Definition 1.1.4: The C matrix, L matrix, and R matrix of a connected RLC graph are defined by Eq. 1.1.3.

Lemma 1.1.2: The associated matrices corresponding to the fundamental trees of bielement type connected graphs are:

1. RC Graphs

$$A_{RC} = -C^{-1} R_{11}$$

2. RL Graphs

$$A_{RL} = -L^{-1} R_{22}$$

3. LC Graphs

$$A_{LC} = - \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} 0 & S_{13} \\ -S'_{13} & 0 \end{bmatrix}$$

where C , L , R_{11} and R_{22} are given in Eq. 1.1.3 and S_{13} is given in Eq. 1.1.0.

Proof: 1., 2., and 3. are obtained from Lemma 1.1.1 by allowing the appropriate S_{ij} terms of Lemma 1.1.0 to be null.

Lemma 1.1.3: Let D_1 and D_2 be diagonal matrices of order n with real positive diagonal entries. Let S_{ij} be any conformable matrix, then $D_1 + S_{ij} D_2 S'_{ij}$ is positive definite ⁽¹²⁾.

Proof: Consider the matrix identity

$$D_1 + S_{ij} D_2 S'_{ij} = \begin{bmatrix} I & S_{ij} \end{bmatrix} \text{diag} (D_1, D_2) \begin{bmatrix} I \\ S'_{ij} \end{bmatrix} \quad 1.1.4$$

By Theorem A.3, $\text{diag} (D_1, D_2)$ is positive definite. Since $\begin{bmatrix} I & S_{ij} \end{bmatrix}$ is of maximum rank, the conclusion follows from Theorem A.4.

Theorem 1.1.0: If the branch and chord element matrices C_1 , C_2 , L_1 , L_2 , G_1 and G_2 have positive diagonal entries then

1. $(C_1 + S_{11} C_2 S'_{11})$, $(L_2 + S'_{33} L_1 S_{33})$, $(G_2^{-1} + S'_{22} G_1^{-1} S_{22})$ and $(G_1 + S_{22} G_2 S'_{22})$ are positive definite, and
2. $S_{12} (G_2^{-1} + S'_{22} G_1^{-1} S_{22})^{-1} S'_{12}$ is positive semidefinite.

Proof: Part 1. follows directly from Lemma 1.1.3. By Theorem A.5., $(G_2^{-1} + S'_{22} G_1^{-1} S_{22})^{-1}$ is positive definite. Part 2. of the theorem now follows from Theorem A.4.

Corollary 1.1.0: If S_{12} is of maximum rank, then $S_{12} (G_2^{-1} + S'_{22} G_1^{-1} S_{22})^{-1} S'_{12}$ is positive definite.

Proof: By Theorems 1.1.1 and A.5 $(G_2^{-1} + S'_{22} G_1^{-1} S_{22})^{-1}$ is positive definite. The conclusion follows from Theorem A.4.

Theorem 1.1.1: If A_1 and A_2 are the associated matrices of the parts P_1 and P_2 respectively, then $A = \text{diag} (A_1, A_2)$ is the associated matrix of a separable RLC graph G obtained by uniting P_1 and P_2 at only one vertex. ⁽¹⁰⁾

Proof: From Eq. 1.1.2 for some maximum order tree T_1 of P_1

$$A_1 = - \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} R_{11} & R_{12} \\ R'_{12} & R_{22} \end{bmatrix} + \begin{bmatrix} 0 & S_{13} \\ -S'_{13} & 0 \end{bmatrix} \right\} \quad 1.1.5$$

and for some maximum order tree T_2 of P_2

$$A_2 = - \begin{bmatrix} C^{*-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \left\{ \begin{bmatrix} R^*_{11} & R^*_{12} \\ R^{*'}_{12} & R^*_{22} \end{bmatrix} + \begin{bmatrix} 0 & S^*_{13} \\ -S^{*'}_{13} & 0 \end{bmatrix} \right\} \quad 1.1.6$$

where C , L and $[R_{ij}]$ are the C matrix, L matrix and R matrix of P_1 and C^* , L^* and $[R^*_{ij}]$ are the C matrix, L matrix and R matrix of P_2 . S_{13} and S^*_{13} are subsets of the f seg matrices of P_1 and P_2 respectively. Consider uniting P_1 and P_2 at one vertex. $T_1 U T_2$ is a maximum order tree of $P_1 U P_2$. Correspondingly from Eq. 1.1.2 the associated matrix is

$$A = - \begin{bmatrix} C^{-1} & \bigcirc \\ & C^{*-1} \\ & & L^{-1} \\ \bigcirc & & & L^{*-1} \end{bmatrix} \left\{ \begin{bmatrix} R_{11} & 0 & R_{12} & 0 \\ 0 & R^*_{11} & 0 & R^*_{12} \\ R'_{12} & 0 & R_{22} & 0 \\ 0 & R^{*'}_{12} & 0 & R^*_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 & S_{13} & 0 \\ 0 & 0 & 0 & S^*_{13} \\ -S'_{13} & 0 & 0 & 0 \\ 0 & -S^{*'}_{13} & 0 & 0 \end{bmatrix} \right\} \quad 1.1.7$$

By interchanging the second and third rows and columns Eq. 1.1.7 becomes

$$A = - \begin{bmatrix} C^{-1} & \bigcirc \\ & L^{-1} \\ & & C^{*-1} \\ \bigcirc & & & L^{*-1} \end{bmatrix} \left\{ \begin{bmatrix} R_{11} & R_{12} & 0 & 0 \\ R'_{12} & R_{22} & 0 & 0 \\ 0 & 0 & R^*_{11} & R^*_{12} \\ 0 & 0 & R^{*'}_{12} & R^*_{22} \end{bmatrix} + \begin{bmatrix} 0 & S_{13} & 0 & 0 \\ -S'_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & S^*_{13} \\ 0 & 0 & -S^{*'}_{13} & 0 \end{bmatrix} \right\} \quad 1.1.8$$

From Eqs. 1.1.5 and 1.1.6, Eq. 1.1.8 is recognized as

$$A = \text{Diag} (A_1, A_2)$$

which completes the proof.

II. CHARACTERIZATION OF REAL LINEAR BIELEMENT SYSTEMS

2.0 Introduction

In this chapter the problem of characterizing time domain models of real linear bielement systems is considered. In Section 2.1, the complete graph⁽¹²⁾ and the star tree⁽¹³⁾ are introduced. In Section 2.2 all connected real linear bielement systems are classified by the subgraph of the system from which the formulation tree must be selected. In Section 2.3 the properties of class 1 systems are developed. These systems are shown to reduce to a canonical form and all of the associated matrices of a graph are related to the canonical associated matrix by similarity transformations. The entries of the C, L, and R matrices are also formulated for the canonical system graph of class 1.

2.1 The Complete Graph

Definition 2.1.0: A complete graph¹ Q_{n+1} is a $n+1$ vertex graph such that there is one and only one element between every pair of vertices. See Fig. 2.1.0.

The elements of a complete graph Q_{n+1} will be designated by E_{ij} in general, or specifically as G_{ij} for a g-element complete graph, C_{ij} for a c-element complete graph, and L_{ij} for a l-element complete graph, where $j \geq i$ for $i, j = 1, 2, \dots, n$. For $i \neq j$, the element subscript ij implies the element is incident to vertices i and j and is oriented from vertex i to j . For $i = j$, E_{ii} is defined to be the element incident to vertices i and $n+1$, with the orientation toward vertex $n+1$. Vertex $n+1$ is referred to as the "reference node". Furthermore, if a complete graph is composed entirely of g-elements (or c-elements, or l-elements) it will be designated by the superscript g (or c, or l). That is, Q_{n+1}^g is a complete graph of $n+1$ vertices and is composed entirely of g-elements G_{ij} , where $j \geq i$, for $i, j = 1, 2, \dots, n$.

Theorem 2.1.0: The number of elements e in a complete graph of v vertices is

$$e = \frac{v(v-1)}{2} \quad 2.1.0$$

1. This same definition was used by Brown and Reed⁽¹⁸⁾.

Proof: By induction on v . Let $v = 2$. The conclusion follows from Eq. 2.1.0 and Def. 2.1.0. Assume the theorem is true for $v = n$, for which the number of elements e_n is

$$e_n = \frac{n(n-1)}{2}. \quad 2.1.1$$

Now let

$$v = n+1. \quad 2.1.2$$

The number of elements e_{n+1} , by definition 2.1.0, will be the number of elements in the case where $v = n$ plus n new elements, one each from the n vertices to the $n+1$ vertex. Hence,

$$e_{n+1} = e_n + n, \quad 2.1.3$$

or from Eq. 2.1.1

$$e_{n+1} = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}. \quad 2.1.4$$

But from Eq. 2.1.2 $n = v-1$. Therefore, Eq. 2.1.4 becomes

$$e_{n+1} = \frac{v(v-1)}{2}.$$

Hence, the theorem follows by induction.

Property 2.1.0: Let Q_{m+1} be a complete graph of $m+1$ vertices. The f seg matrix corresponding to any star tree of Q_{m+1} is

$$S_f = \begin{bmatrix} E_{11} & E_{22} & E_{33} & \dots & E_{mn} & E_{12} & E_{13} & E_{14} & \dots & E_{1m} & E_{23} & E_{24} & E_{25} & \dots & E_{2m} & \dots & E_{m-1 m} \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & \dots & 0 \\ & & & \dots & & & & & \dots & & & & \dots & & & & \\ & & & & & & & & \dots & & & & \dots & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & -1 & \dots & -1 \end{bmatrix} \quad 2.1.5$$

where the column corresponds to the element of Q_{m+1} listed above that column.

Property 2.1.1: Let S_f be given by Eq. 2.1.5, then

$$E = [e_{ij}] = S_f \text{diag}(E_{ij}) S_f'$$

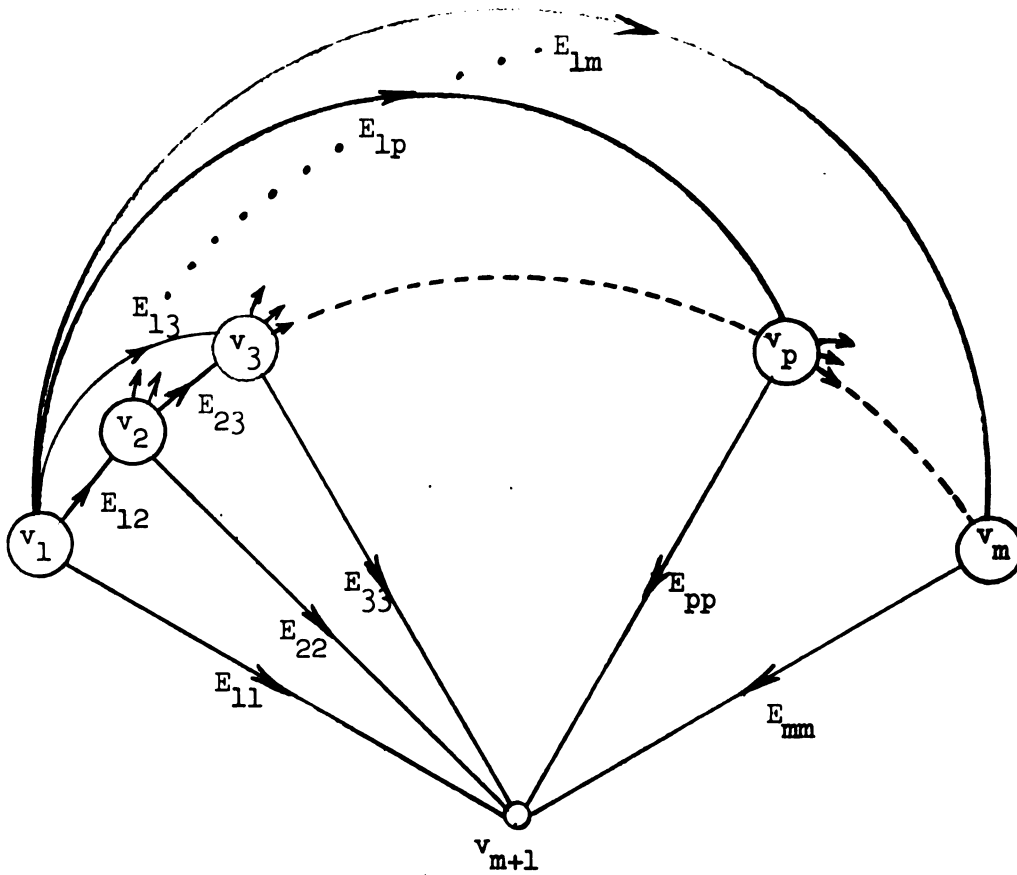


Figure 2.1.0 Complete graph Q_{m+1}^e of e elements with $m+1$ vertices.

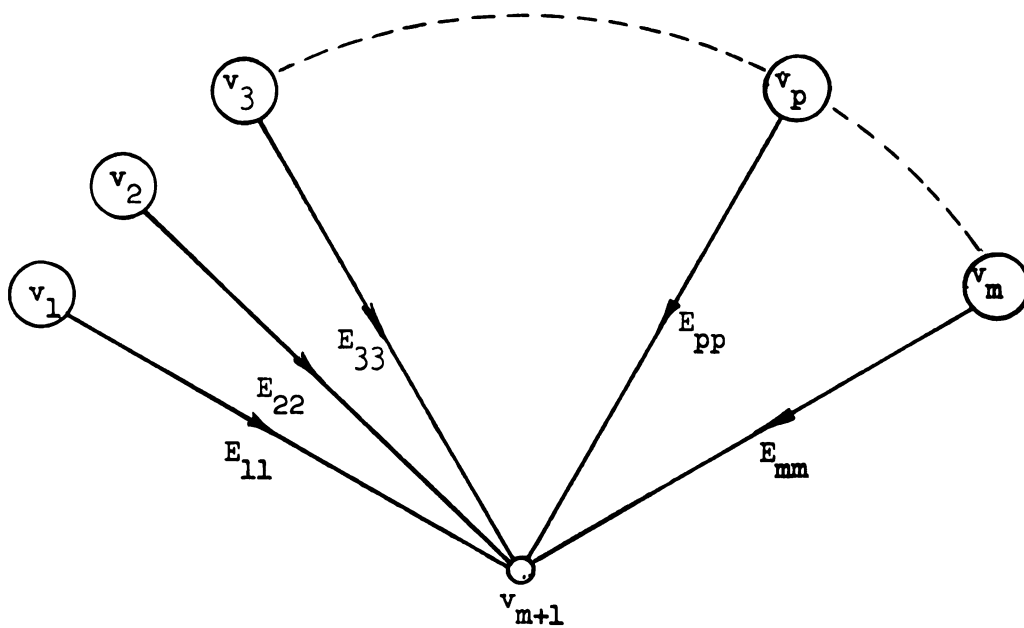


Figure 2.1.1 Star tree T_s of $m+1$ vertices.

where $\text{diag}(E_{ij}) = \text{diag}(E_{11}E_{22}E_{33}\dots E_{mm}E_{12}E_{13}E_{14}\dots E_{1m}E_{23}E_{24}E_{25}\dots E_{2m}\dots E_{m-1\ m})$
then

$$\begin{aligned} e_{ij} &= e_{ji} = -E_{ij} \text{ for } i > j & i &= 1, 2, \dots, m-1 \\ & & j &= 2, 3, \dots, m \\ e_{ii} &= \sum_{k=1}^m E_{ik}, & i &= 1, 2, \dots, m \end{aligned} \quad 2.1.6$$

where $E_{ik} = E_{ki}$ for $i > k$.

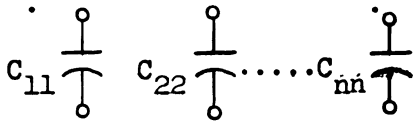
2.2 The Classification of Real Linear Bielement Systems.

From Section 1.1, all RLC systems are characterized by their associated matrices. The rank of the associated matrix, by Eq. 1.1.1, determines the number of c-elements in the tree and the number of l-elements in the cotree. Therefore, the rank of the associated matrix gives an indication as to the number of vertices of the graph and the composition of its tree and cotree. Since the reactive elements of a graph are assumed to be real, there is always a finite resistance or conductance associated with each reactive element, and hence there are at least as many g-elements as reactive elements in a given system graph. In the following, real linear bielement systems having associated matrices of order n are classified by the form of the subgraph from which the maximum order tree is selected.

RC Graphs

Let A of order n be the associated matrix of a connected RC graph $G = G^c \cup G^g$, where G^c and G^g are the subgraphs of c- and g-elements of G respectively. From Eq. 1.1.1, there are n c-elements in the tree of G . Suppose the tree of G is composed entirely of c-elements, then G^c is connected and has $v = n+1$ vertices ⁽¹⁰⁾. Suppose now that the tree contains g-elements such that G^c is composed of two disjoint but individually connected subgraphs, then the number of vertices of G^c is $v = n+2$ ⁽¹⁰⁾. Suppose now the tree contains g-elements such that G^c is composed of three disjoint but individually connected subgraphs, then the number of vertices G^c is $v = n+3$. This partitioning of G^c can be continued until G^c is composed of n parts, each of which is an individual c-element. The number of vertices in this case being $v = 2n$. This argument constitutes the proof of the following lemma.

Lemma 2.2.0: Let $G = G^c \cup G^g$ be a connected graph composed of the c-element subgraph G^c and the g-element subgraph G^g . If G has an associated matrix of order n then the number of vertices of G^c and the form of G^c are characterized as follows:

Class	# of vertices of G^c	Form of G^c	
1	$n+1$		: G^c is connected.
2	$n+2$		: $G^c = G_1^c \cup G_2^c$ where G_1^c
.	.	.	and G_2^c are individually
.	.	.	connected.
.	.	.	.
.	.	.	.
n	$2n$		: $G^c = C_{11} \cup C_{22} \dots \cup C_{nn}$

RL Graphs

Let A of order n be the associated matrix of a connected RL graph $G = G^1 \cup G^g$ where G^1 and G^g are the subgraphs of 1- and g-elements of G respectively. From Eq. 1.1.1, there are n 1-elements in the cotree of G . Suppose the tree of G is composed entirely of g-elements, then G^g is connected. Let the number of vertices of G^1 be v . Suppose further that G^1 is a complete graph. The complete graph with the least number of vertices having n elements is found by solving Eq. 2.1.0 for v . Therefore,

$$v = \frac{1 + \sqrt{1 + 8n}}{2}. \quad 2.2.0$$

In general, the solution of Eq. 2.3.0 is a real number r^* . However, since v is the number of vertices, r^* must be rounded off to the next largest integer r . The integer r , then, is the smallest number of vertices the subgraph G^1 can have if A is to be of order n . The largest number of vertices of G^1 is $v = 2n$ corresponding to each 1-element of the cotree being isolated from one another by g-elements. Therefore, the bounds on the number of vertices of G^1 are

$$r \leq v \leq 2(n + p)$$

This argument constitutes the proof of the following lemma.

Lemma 2.2.1: Let $G = G^1 \cup G^g$ be a connected graph composed of the 1-element subgraph G^1 and the g-element subgraph G^g . If G has an associated matrix of order n then the form of G^g and the bounds on the vertices of G^1 are given as follows:

Class	# of vertices of G^g		Form of G^g	
	min	max		
1	$r \leq v \leq 2n$		$\bigcirc_{G^g}$	: G^g is connected.
2	$r \leq v \leq 2n+2$		$\bigcirc_{G_1^g} \quad \bigcirc_{G_2^g}$	: $G^g = G_1^g \cup G_2^g$ where G_1^g and G_2^g are individually connected.
3	$r \leq v \leq 2n+4$		$\bigcirc_{G_1^g} \quad \bigcirc_{G_2^g} \quad \bigcirc_{G_3^g}$	: $G^g = G_1^g \cup G_2^g \cup G_3^g$ where G_1^g , G_2^g and G_3^g are individually connected.
.	.			
p	$r \leq v \leq 2(n+p)$		$\bigcirc_{G_1^g} \quad \bigcirc_{G_2^g} \quad \dots \quad \bigcirc_{G_p^g}$	: $G^g = G_1^g \cup G_2^g \dots \cup G_p^g$ where $G_1^g, G_2^g \dots G_p^g$ are individually connected.
.	.			

2.3 Characterization of Class 1 Systems.

From Section 2.2, real linear bi-element systems whose associated matrices of order n are classified by the form of the subgraph from which the maximum order tree is selected as classes 1, 2, ..., p, ... In this section, systems of class 1 are investigated and their characteristics developed in detail.

RC Graphs

Lemma 2.3.0: Let $P_{p+1} = G_{n+1}^c \cup P_{p+1}^g$ be a part such that G_{n+1}^c is a connected c-element graph of $n+1$ vertices and P_{p+1}^g is a g-element part of $p+1$ vertices where G_{n+1}^c and P_{p+1}^g are united at $p+1$ vertices, $n \leq p$. Then,

1. there exists a star tree T_s of G_{n+1}^c which is a subtree of a maximum order tree of P_{p+1} , and
2. the associated matrix is

$$A = -C^{-1}R_{11} = -C^{-1}S_{12} \left[G_2^{-1} + S_{22}' G_1^{-1} S_{22} \right]^{-1} S_{12}' \quad \text{for } n < p$$

or

$$A = -C^{-1}R_{11} = -C^{-1}S_{12} G_2 S_{12}' \quad \text{for } n = p.$$

Proof: Select any vertex of G_{n+1}^c and label it $n+1$. Label the remaining vertices $1, 2, \dots, n$. Consider the star representation T_s of the $n+1$ vertices of G_{n+1}^c given by the c-elements between vertices 1 and $n+1, 2$ and $n+1, \dots, n$ and $n+1$. Since G_{n+1}^c is connected, at least one of the c-elements, say C_{11} , of T_s is non-zero. Furthermore, since G_{n+1}^c is connected, the maximum number of c-elements in any tree of P_{p+1} is n . T_s has n c-elements. Let the tree T_{s1} of P_{p+1} be composed of T_s and T_1 where T_1 is a subgraph of P_{p+1}^g . T_{s1} is acceptable as a formulation tree by Def. 1.1.3 if only the associated matrix corresponding to T_{s1} exists. The associated matrix corresponding to any fundamental tree by Lemma 1.1.2 is

$$A = -C^{-1}R_{11} = - \left[C_1 + S_{11} C_2 S_{11}' \right]^{-1} S_{12} \left[G_2^{-1} + S_{22}' G_1^{-1} S_{22} \right]^{-1} S_{12}'. \quad 2.3.0$$

Since some of the elements of C_1 can be zero, it is only necessary to show that C^{-1} exists. By allowing the appropriate c-elements of the complete graph Q_{n+1}^c to be zero, then $G_{n+1}^c = Q_{n+1}^c$. Correspondingly, the subset S_{11} of the f seg matrix of P_{p+1} is given by Eq. 2.1.5 for $m = n$. C is nonsingular by Theorem A.6 if only C is not permutable into $\text{diag}(K_{11}, K_{22})$ where K_{11} is a $q \times q$ matrix and K_{22} is a $n - q \times n - q$ matrix. Assume C is permutable into this form. Then from Property 2.1.1, all c-elements between the vertices $1, 2, \dots, q$ and $q+1, q+2, \dots, n$ are zero. This is a contradiction since G_{n+1}^c is connected, except if these vertices are connected through vertex $n+1$. If

the vertices $1, 2, \dots, q$ are connected through $n+1$ to $q+1, q+2, \dots, n$, then K_{11} and K_{22} are to be examined individually for singularity. Applying Theorem A.6 to K_{11} and K_{22} and repeating the above argument, C is non-singular. Hence the associated matrix corresponding to T_{si} exists and T_s is a subtree of a maximum order tree which proves 1. of the conclusion. 2. of the conclusion follows directly from Lemma 1.1.2.

Lemma 2.3.1: Consider the system of linear, nonhomogeneous, algebraic equations

$$\sum_{k=1}^n t_{ik} d_{ik} = a_{ii} \quad \text{for } i = 1, 2, \dots, n$$

$$t_{ij} d_{ij} = a_{ij} \quad \text{for } i < j; i = 1, 2, \dots, n-1$$

$$j = 2, 3, \dots, n$$
2.3.1

If $t_{ij} \neq 0$ for $i, j = 1, 2, \dots, n$, then there exists a unique solution to Eq. 2.3.1 given by

$$\begin{bmatrix} d_{11} \\ d_{22} \\ \vdots \\ d_{nn} \\ d_{12} \\ d_{13} \\ \vdots \\ d_{1n} \\ \dots \\ d_{n-1 n} \end{bmatrix} = \begin{bmatrix} (a_{11} - \sum_{k \neq 1}^n a_{1k})/t_{11} \\ (a_{22} - \sum_{k \neq 2}^n a_{2k})/t_{22} \\ \vdots \\ (a_{nn} - \sum_{k=1}^{n-1} a_{nk})/t_{nn} \\ a_{12}/t_{12} \\ a_{13}/t_{13} \\ \vdots \\ a_{1n}/t_{1n} \\ \dots \\ a_{n-1 n}/t_{n-1 n} \end{bmatrix}$$
2.3.2

Proof: Since $t_{ij} \neq 0$ for all ij , $\begin{bmatrix} d_{12} & d_{13} & \dots & d_{1n} & d_{23} & \dots & d_{2n} & \dots & d_{n-1} & n \end{bmatrix}'$ can be found directly from Eq. 2.2.1 by dividing a_{ij} by t_{ij} for $i < j$; $i=1,2, \dots, n-1$ and $j=2,3, \dots, n$. Substituting this result into the remaining equations of Eq. 2.3.1 gives $d_{ii} = (a_{ii} - \sum_{k \neq i}^n a_{ik})/t_{ii}$ for $i=1,2, \dots, n$. Eq. 2.3.2 follows directly from these results.

Theorem 2.3.0: Let $P_{p+1} = G_{n+1}^C \cup P_{p+1}^g$ be the graph of Lemma 2.2.0. Then there exists a graph P_{n+1} obtained by uniting G_{n+1}^C with a complete g -element graph. Q_{n+1}^{g*} at $n+1$ vertices such that the associated matrices of P_{p+1} and P_{n+1} are identical.

Proof: Let P_{p+1} be written as the union of two complete graphs Q_{p+1}^g and Q_{n+1}^C where $Q_{p+1}^g = P_{p+1}^g$ and $Q_{n+1}^C = G_{n+1}^C$. Select the tree T_{s1} which contains T_s of Lemma 2.3.0 as a subtree. The f seg matrix by Lemma 1.1.0 is

$$S_{s1} = \begin{bmatrix} I & 0 & S_{11} & S_{12} \\ 0 & I & 0 & S_{22} \end{bmatrix} \quad 2.3.3$$

The associated matrix corresponding to T_{s1} by Lemma 2.3.0 for $n < p$ is

$$A_s = -C^{-1}R_{11} = -C^{-1}S_{12} \left[G_2^{-1} + S_{22}'G_1^{-1}S_{22} \right]^{-1}S_{12}'$$

Now construct the graph P_{n+1} of the theorem where G_{n+1}^C is given and the elements of Q_{n+1}^{g*} are to be calculated. Select T_s as the tree of P_{n+1} . The f seg matrix of P_{n+1} by Lemma 1.1.0 is

$$S_{s1}^* = \begin{bmatrix} I & S_{11} & S_{12}^* \end{bmatrix} \quad 2.3.4$$

where S_{11} is the same as in Eq. 2.3.3 and S_{12}^* is identical to the first $n(n+1)/2$ columns of S_{12} of Eq. 2.3.3. The associated matrix of P_{n+1} corresponding to T_s by Lemma 2.3.0 for $n=p$ is

$$A_s^* = -C^{-1}R_{11}^* = -C^{-1}S_{12}^*G_2^*S_{12}^{*'} \quad 2.3.5$$

Therefore, multiplying Eqs. 2.3.3 and 2.3.5 by $-C$ and equating gives

$$R_{11}^* = S_{12}^*G_2^*S_{12}^{*'} = R_{11} \quad 2.3.6$$

Eq. 2.3.6 can be written as Eq. 2.3.1 and hence satisfies the hypothesis of Lemma 2.3.1. Therefore, there exists a unique set of diagonal entries $G_{ij}^* \in G_2^*$ such that the associated matrices of P_{p+1} and P_{n+1} are identical, thus proving the theorem.

Proof: Since $t_{ij} \neq 0$ for all ij , $\begin{bmatrix} d_{12} & d_{13} & \dots & d_{1n} & d_{23} & \dots & d_{2n} & \dots & d_{n-1,n} \end{bmatrix}'$ can be found directly from Eq. 2.2.1 by dividing a_{ij} by t_{ij} for $i < j$; $i=1,2, \dots, n-1$ and $j=2,3, \dots, n$. Substituting this result into the remaining equations of Eq. 2.3.1 gives $d_{ii} = (a_{ii} - \sum_{k=1}^n a_{ik})/t_{ii}$ for $i=1,2, \dots, n$. Eq. 2.3.2 follows directly from these results.

Theorem 2.3.0: Let $P_{p+1} = G_{n+1}^C \cup P_{p+1}^g$ be the graph of Lemma 2.2.0. Then there exists a graph P_{n+1} obtained by uniting G_{n+1}^C with a complete g -element graph. Q_{n+1}^{g*} at $n+1$ vertices such that the associated matrices of P_{p+1} and P_{n+1} are identical.

Proof: Let P_{p+1} be written as the union of two complete graphs Q_{p+1}^g and Q_{n+1}^C where $Q_{p+1}^g = P_{p+1}^g$ and $Q_{n+1}^C = G_{n+1}^C$. Select the tree T_{si} which contains T_s of Lemma 2.3.0 as a subtree. The f seg matrix by Lemma 1.1.0 is

$$S_{si} = \begin{bmatrix} I & 0 & S_{11} & S_{12} \\ 0 & I & 0 & S_{22} \end{bmatrix} \quad 2.3.3$$

The associated matrix corresponding to T_{si} by Lemma 2.3.0 for $n < p$ is

$$A_s = -C^{-1}R_{11} = -C^{-1}S_{12} \left[G_2^{-1} + S_{22}'G_1^{-1}S_{22} \right]^{-1}S_{12}'$$

Now construct the graph P_{n+1} of the theorem where G_{n+1}^C is given and the elements of Q_{n+1}^{g*} are to be calculated. Select T_s as the tree of P_{n+1} . The f seg matrix of P_{n+1} by Lemma 1.1.0 is

$$S_{si}^* = \begin{bmatrix} I & S_{11} & S_{12}^* \end{bmatrix} \quad 2.3.4$$

where S_{11} is the same as in Eq. 2.3.3 and S_{12}^* is identical to the first $n(n+1)/2$ columns of S_{12} of Eq. 2.3.3. The associated matrix of P_{n+1} corresponding to T_s by Lemma 2.3.0 for $n=p$ is

$$A_s^* = -C^{-1}R_{11}^* = -C^{-1}S_{12}^* G_2^* S_{12}^{*'} \quad 2.3.5$$

Therefore, multiplying Eqs. 2.3.3 and 2.3.5 by $-C$ and equating gives

$$R_{11}^* = S_{12}^* G_2^* S_{12}^{*'} = R_{11} \quad 2.3.6$$

Eq. 2.3.6 can be written as Eq. 2.3.1 and hence satisfies the hypothesis of Lemma 2.3.1. Therefore, there exists a unique set of diagonal entries $G_{ij}^* \in G_2^*$ such that the associated matrices of P_{p+1} and P_{n+1} are identical, thus proving the theorem.

Definition 2.3.0: The graph G_{p+1} is said to be equivalent to the graph G_{n+1} if and only if G_{p+1} and G_{n+1} have the same associated matrix.

Definition 2.3.1: The graph G_{p+1} is reducible to G_{n+1} for $p > n$ if the graphs G_{p+1} and G_{n+1} are equivalent.

Corollary 2.3.0: Every part P_{p+1} of a RC graph of class 1 having a c-elements subgraph of $n+1$ vertices can be reduced to a part $P_{n+1} = Q_{n+1}^c \cup Q_{n+1}^{g*}$, $n \leq p$, where the complete graphs Q_{n+1}^c and Q_{n+1}^{g*} are united at $n+1$ vertices.

Proof: From Lemma 2.2.0, class 1 graphs satisfy the hypothesis of Theorem 2.3.0 and hence are reducible to P_{n+1} . Some of the elements of the complete graphs Q_{n+1}^c and Q_{n+1}^{g*} may be zero.

Definition 2.3.2: A graph P_{n+1} is said to be a canonical RC graph of class 1 if and only if $P_{n+1} = Q_{n+1}^c \cup Q_{n+1}^{g*}$ where the two complete graphs are united at $n+1$ vertices and where Q_{n+1}^c has a connected subgraph composed of non-zero c-elements incident to all $n+1$ vertices. The associated matrix corresponding to any maximum order star tree T_s of P_{n+1} is said to be the canonical associated matrix of a RC graph of class 1.

Theorem 2.3.1: Let $S_s = \begin{bmatrix} S_{11} & S_{12} \end{bmatrix} = \begin{bmatrix} I & S_{11} \\ 0 & S_{12} \end{bmatrix}$ be the f seg matrix corresponding to a maximum order star tree T_s of a canonical RC graph G_{n+1} of class 1. Then the canonical associated matrix of G_{n+1} is

$$A = -C^{-1} R_{11}$$

such that

$$1. \quad C = [c_{ij}] = S_{11} C_s S_{11}' = \begin{bmatrix} I & S_{11} \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} I \\ S_{11}' \end{bmatrix} \quad 2.3.7$$

where $C_1 = \text{diag}(c_{11}, c_{22}, \dots, c_{nn})$ and $C_2 = \text{diag}(c_{12}, c_{13}, \dots, c_{1n}, c_{23}, \dots, c_{2n}, \dots, c_{n-1, n})$ are the branch and chord c-element matrices respectively.

$$2. \quad R_{11} = [g_{ij}] = S_{12} G_2 S_{12}' \quad 2.3.8$$

where $G_2 = \text{diag}(g_{11}, g_{22}, \dots, g_{nn}, g_{12}, g_{13}, \dots, g_{1n}, \dots, g_{n-1, n})$ is the chord g-element matrix.

$$\begin{aligned}
3. \quad c_{1j} &= -C_{1j} && \text{for } i \neq j \text{ and } i, j=1, 2, \dots, n \\
g_{1j} &= -G_{1j} \\
c_{ii} &= \sum_{j=1}^n C_{ij} && 2.3.9 \\
g_{ii} &= \sum_{j=1}^n G_{ij} && \text{for } i=1, 2, \dots, n
\end{aligned}$$

Proof: 1. and 2. follow directly from Lemma 1.1.2. S_{111} and S_{12} of the hypothesis are given by Eq. 2.1.5 where $m=n$. 3. follows from 1. and 2. and Property 2.1.1.

Lemma 2.3.2: Let $S_1, S_j, \dots$ be the f seg matrices corresponding to the maximum order trees $T_1, T_j, \dots$ respectively, of a connected graph G , then there exists nonsingular matrices A_{ij} and P_{ji} such that

$$S_i = A_{ij} S_j P_{ji}, \quad 2.3.10$$

where $A_{ij} = A_{tj}^{-1} A_{ti}$ and A_{tj} and A_{ti} are submatrices of the incidence matrix whose columns correspond to the elements of the trees T_j and T_i , respectively, and P_{ji} is a permutation matrix ⁽¹⁴⁾ which rearranges the columns of S_j to have the same element ordering as the columns S_i .

Proof: Follows directly from Theorem A.17.

Theorem 2.3.2: The associated matrix of every part P_{p+1} of a separable RC graph of class 1 is similar to the canonical associated matrix of some canonical RC graph of class 1 and conversely.

Proof: By Corollary 2.3.0, P_{p+1} can be reduced to $P_{n+1} = Q_{n+1}^c \cup Q_{n+1}^{g*}$ where Q_{n+1}^c and Q_{n+1}^{g*} are complete graphs united at $n+1$ vertices. Let A_j be the associated matrix corresponding to some maximum order tree T_j of P_{n+1} . By Lemma 2.3.0, P_{n+1} has a maximum order star tree T_s . Let S_j and S_s be the f seg matrices corresponding to the trees T_j and T_s respectively. By Lemma 2.3.2,

$$S_s = A_{sj} S_j P_{js}. \quad 2.3.11$$

Let S_s and S_j be partitioned such that the columns of the 1,1 submatrix correspond to the c -elements and the columns of the 1,2 submatrix correspond

to the g-elements of P_{n+1} . Eq. 2.3.11 becomes

$$S_s = \begin{bmatrix} S_{s11} & S_{s12} \end{bmatrix} = A_{sj} \begin{bmatrix} S_{j11} & S_{j12} \end{bmatrix} \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}, \quad 2.3.12$$

where P_{js} is partitioned to conform to the partitioning of S_j . P_{11} rearranges the columns of S_{j11} corresponding to c-elements such that the leading columns of $S_{j11}P_{11}$ correspond to the c-elements of T_s . P_{22} rearranges the columns of S_{j12} corresponding to g-elements. Since for a class 1 RC graph there are no g-elements in the tree T_s , then $P_{22}=I$. Let the branch and chord c-element matrices corresponding to the tree T_j be written

$$C_j = \text{diag} (C_1, C_2). \quad 2.3.13$$

Then the diagonal branch and chord c-element matrix C_s corresponding to the tree T_s is the same matrix as C_j with the elements rearranged on the diagonal. In fact,

$$C_s = P'_{11} C_j P_{11}. \quad 2.3.14$$

Since the g-element chord matrix G_2 is not rearranged, then G_2 is the same for both trees T_j and T_s . The associated matrix of P_{n+1} corresponding to the tree T_s by Lemma 2.3.0 is

$$A_s = -(S_{s11} C_s S'_{s11})^{-1} S_{s12} G_2 S'_{s12}. \quad 2.3.15$$

Substituting the relations of Eqs. 2.3.12 and 2.3.14 into Eq. 2.3.15 gives

$$A_s = -(A_{sj} S_{j11} P_{11} P'_{11} C_j P_{11} P'_{11} S'_{j11} A'_{sj})^{-1} A_{sj} S_{j12} G_2 S'_{j12} A'_{sj}. \quad 2.3.16$$

Since $P_{11} P'_{11} = I$ and by taking the inverse of a product of square matrices, Eq. 2.3.16 becomes

$$A_s = -A'^{-1}_{sj} (S_{j11} C_j S'_{j11})^{-1} S_{j12} G_2 S'_{j12} A'_{sj}. \quad 2.3.17$$

The associated matrix corresponding to the tree T_j by Lemma 2.3.0 is

$$A_j = -(S_{j11} C_j S'_{j11})^{-1} S_{j12} G_2 S'_{j12}. \quad 2.3.18$$

Therefore Eq. 2.3.17 is written as

$$A_s = A'^{-1}_{sj} A_j A'_{sj}. \quad 2.3.19$$

By Definition A.2, A_j is similar to A_s which proves the theorem.

Converse: The converse follows directly by solving Eq. 2.3.19 for A_j .

Corollary 2.3.2.0: The associated matrices $A_1, A_2, \dots, A_j, \dots, A_k, \dots$ of a part P_{p+1} corresponding to the trees $T_1, T_2, \dots, T_j, \dots, T_k, \dots$ of a class 1 RC graph are similar.

Proof: By Theorem 2.3.2, the associated matrices A_j and A_k are similar to the associated matrix A_s corresponding to the star tree T_s which exists by Lemma 2.3.0. Therefore, from Eq. 2.3.19

$$A_s = A'_{sj}{}^{-1} A_j A'_{sj} = A'_{sk}{}^{-1} A_k A'_{sk} \quad 2.3.20$$

for all j and k . Solving Eq. 2.3.20 for A_j in terms of A_k gives

$$A_j = A'_{sj} A'_{sk}{}^{-1} A_k A'_{sk} A'_{sj}{}^{-1}. \quad 2.3.21$$

Therefore, by letting

$$A'_{jk} = A'_{sj} A'_{sk}{}^{-1} \quad 2.3.22$$

Eq. 2.3.21 becomes

$$A_j = A'_{jk} A_k A'_{jk}{}^{-1}, \quad 2.3.23$$

for all j and k , and by Def. A.2, A_k is similar to A_j .

Corollary 2.3.2.1: The associated matrix of a connected RC graph G of class 1 is similar to the canonical associated matrix of some canonical RC graph of class 1 and conversely.

Proof: From Theorem 2.3.2, for each separable part P^i , $i=1,2,\dots,m$,

$$A_j^i = A'_{sj}{}^i A_s^i A'_{sj}{}^{i,-1}. \quad 2.3.24$$

By Theorem 1.1.1, the associated matrix of G is

$$\begin{bmatrix} A_j^1 & & & \\ & A_j^2 & & \\ & & \ddots & \\ \bigcirc & & & A_j^m \end{bmatrix} = \begin{bmatrix} a_{sj}^1 & & & \\ & a_{sj}^2 & & \\ & & \ddots & \\ \bigcirc & & & a_{sj}^m \end{bmatrix} \begin{bmatrix} A_s^1 & & & \\ & A_s^2 & & \\ & & \ddots & \\ \bigcirc & & & A_s^m \end{bmatrix} \begin{bmatrix} a_{sj}^1 & & & \\ & a_{sj}^2 & & \\ & & \ddots & \\ \bigcirc & & & a_{sj}^m \end{bmatrix}{}^{-1} \quad 2.3.25$$

where $\text{diag}(A_s^1, A_s^2, \dots, A_s^m)$ is the associated matrix of a union of m canonical RC graphs of class 1 at one vertex and hence is a canonical RC graph of class 1.

Converse: The converse follows directly by solving Eq. 2.3.25 for $\text{diag}(A_s^1, A_s^2, \dots, A_s^m)$.

RL Graphs

Lemma 2.3.2: Let $P_{p+1} = G_{p+1}^g \cup G_v^1$ be a part such that G_{p+1}^g is a connected g-element graph of p+1 vertices and G_v^1 is a forrest⁽¹⁰⁾ of 1-element graphs having v vertices, where G_{p+1}^g and G_v^1 are united at v vertices, $v \leq p$. Then,

1. there exists a star tree T_s of G_{p+1}^g which is a maximum order tree of P_{p+1} , and
2. the associated matrix of P_{p+1} is $A = -L^{-1}S'_{23}(G_1 + S_{22}G_2S'_{22})^{-1}S_{23}$.

Proof: Select any vertex of G_{p+1}^g and label it p+1. Label the remaining vertices 1, 2, ..., p. Consider the star representation T_s of the p+1 vertices of G_{p+1}^g given by the g-elements between vertices 1 and p+1, 2 and p+1, ..., p and p+1. Since G_{p+1}^g is connected, at least one of the g-elements, say G_{11} , of T_s is non-zero. Furthermore, by the connectedness of G_{p+1}^g , the maximum number of g-elements in any tree of P_{p+1} is p⁽¹⁰⁾. T_s has p g-elements. T_s is an acceptable formulation tree by Def. 1.1.3 if only the associated matrix corresponding to T_s exists. The associated matrix corresponding to T_s by Lemma 1.1.2 is

$$A = -L^{-1}S'_{23}(G_1 + S_{22}G_2S'_{22})^{-1}S_{23}.$$

Since some of the g-elements of T_s can be zero, it is only necessary to show that $(G_1 + S_{22}G_2S'_{22})$ is nonsingular. By allowing the appropriate g-elements of the complete graph Q_{p+1}^g to be zero, then $G_{p+1}^g = Q_{p+1}^g$. Correspondingly, the subset S_{22} of the f seg matrix of P_{p+1} is given by Eq. 2.1.5 for $m=p$. $G_1 + S_{22}G_2S'_{22}$ is nonsingular by Theorem A.6 if only it cannot be permuted into $\text{diag}(K_{11}, K_{22})$ where K_{11} is a q x q matrix and K_{22} is a p-q x p-q matrix. Assume $G_1 + S_{22}G_2S'_{22}$ is permutable into this form. Then from Property 2.1.1, all g-elements between the vertices 1, 2, ..., q and q+1, q+2, ..., p are zero. This is a contradiction since G_{p+1}^g is connected, except if these vertices are connected through vertex p+1. If the vertices 1, 2, ..., q and q+1, q+2, ..., p are connected through p+1, then K_{11} and K_{22} are to be examined individually for singularity. Applying Theorem A.6 to K_{11} and K_{22} and repeating the above argument, $G_1 + S_{22}G_2S'_{22}$ is nonsingular. Hence, the associated matrix corresponding to T_s exist, and hence, T_s is a maximum order tree of P_{p+1} which prove part 1. of the lemma. Part 2. of the lemma follows from Lemma 1.1.2.

Lemma 2.3.3: Let S_f be given by Eq. 2.1.5, then $S_f S_f'$ is nonsingular.

Proof: From Eq. 2.1.5,

$$S_f S_f' = \begin{bmatrix} m & -1 & . & . & . & -1 \\ -1 & m & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & -1 & . \\ -1 & . & . & . & -1 & m \end{bmatrix} \quad 2.3.26$$

is a $m \times m$ matrix. By Theorem A.6, $S_f S_f'$ is nonsingular.

Lemma 2.3.4: Let $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}' & K_{22} \end{bmatrix}$, where K_{11} and K_{22} are square submatrices,

be a positive definite matrix, then K_{11} and K_{22} are positive definite.

Proof: Since K is positive definite, then

$$\begin{bmatrix} X_1' & X_2' \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{12}' & K_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} > 0 \quad 2.3.27$$

for all vectors $X = \begin{bmatrix} X_1' & X_2' \end{bmatrix}'$. Let $X_2 = 0$, it then follows from Eq. 2.3.27 that

$$X_1' K_{11} X_1 > 0 \quad 2.3.28$$

for all $X_1 \neq 0$. By Definition A.1, K_{11} is positive definite. Similarly, by letting $X_1 = 0$, it then follows from Eq. 2.3.27, that

$$X_2' K_{22} X_2 > 0 \quad 2.3.29$$

for $X_2 \neq 0$. Hence, K_{22} is positive definite by Definition A.1 which proves the lemma.

Theorem 2.3.3: Let $P_{p+1} = G_{p+1}^g \cup G_v^1$ be the graph of Lemma 2.3.2. Then there exists a graph P_v obtained by uniting G_v^1 with a complete g -element graph Q_v^{g*} at v vertices such that the associated matrices of P_{p+1} and P_v are identical.

Proof: Let P_{p+1} be written as the union of two complete graphs Q_{p+1}^g and Q_v^1 where $Q_{p+1}^g = G_{p+1}^g$ and $Q_v^1 = G_v^1$. Select T_g of Lemma 2.3.2 as the tree of P_{p+1} . The f seg matrix by Lemma 1.1.0 is

$$S_s = \begin{bmatrix} I & S_{22} & S_{23} \end{bmatrix}. \quad 2.3.30$$

Since there are in general more g-elements than l-elements, S_s can be partitioned as follows

$$S_s = \begin{bmatrix} I_v & 0 & S_{22}^* & S_{23}^* \\ 0 & I & S_{22}^{**} & 0 \end{bmatrix}, \quad 2.3.31$$

where I_v corresponds to the branches of T_s which are incident to G_v^1 . The associated matrix corresponding to T_s by Lemma 2.3.2 is

$$A_s = -L^{-1}R_{22} = -L^{-1}S_{23}' \left[G_1 + S_{22}G_2S_{22}' \right]^{-1}S_{23} \quad 2.3.32$$

Let $\left[G_1 + S_{22}G_2S_{22}' \right]^{-1} = K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}' & K_{22} \end{bmatrix}$ where K_{11} is a $v \times v$ matrix.

$G_1 + S_{22}G_2S_{22}'$ is positive definite by Theorem 1.1.0. K is positive definite by Theorem A.5. K_{11} is positive definite by Lemma 2.3.4. Therefore Eq. 2.3.32 is written

$$A_s = -L^{-1}R_{22} = -L^{-1} \begin{bmatrix} S_{23}' & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{12}' & K_{22} \end{bmatrix} \begin{bmatrix} S_{23}^* \\ 0 \end{bmatrix}, \quad 2.3.33$$

or

$$A_s = -L^{-1}R_{22} = -L^{-1}S_{23}'K_{11}S_{23}^*. \quad 2.3.34$$

Now construct the graph P_v of the theorem where G_v^1 is given and the elements of Q_v^{g*} are to be calculated. Select the tree T_s^* from Q_v^{g*} such that T_s^* is a subtree of T_s . The f seg matrix of P_v by Lemma 1.1.1 is

$$S_s^* = \begin{bmatrix} I_v & S_{22}^* & S_{23}^* \end{bmatrix}, \quad 2.3.35$$

where S_{22}^* and S_{23}^* are identical to the submatrices of Eq. 2.3.31. The associated matrix corresponding to T_s^* by Lemma 2.3.2 is

$$A_s^* = -L^{-1}R_{22}^* = -L^{-1}S_{23}'^* \left[G_1^* + S_{22}^*G_2^*S_{22}^{*'} \right]^{-1}S_{23}^*. \quad 2.3.36$$

Let Eqs. 2.3.34 and 2.3.36 be equated and then premultiplied by S_{23}^*L and postmultiplied by $S_{23}'^*$ which gives

$$S_{23}^*S_{23}'^*K_{11}S_{23}^*S_{23}'^* = S_{23}^*S_{23}'^* \left[G_1^* + S_{22}^*G_2^*S_{22}^{*'} \right]^{-1}S_{23}^*S_{23}'^*. \quad 2.3.37$$

S_{23}^* is given by Eq. 2.1.5 and hence by Lemma 2.3.3, $S_{23}^*S_{23}'^*$ is nonsingular. Therefore, Eq. 2.3.37 is written as

$$\begin{bmatrix} I & S_{22}^* \end{bmatrix} \begin{bmatrix} G_1^* & 0 \\ 0 & G_2^* \end{bmatrix} \begin{bmatrix} I \\ S_{22}' \end{bmatrix} = K_{11}^{-1} \quad 2.3.38$$

where K_{11}^{-1} exists since K_{11} was shown to be positive definite. Eq. 2.3.38 can be written as Eq. 2.3.1 and hence satisfies the hypothesis of Lemma 2.3.1. Therefore, there exists a unique set of diagonal entries G_{ij}^* diag (G_1^*, G_2^*) such that the associated matrices of P_{p+1} and P_v are identical, thus proving the theorem.

Corollary 2.3.3: Every part P_{p+1} of a RL graph of class 1 having a 1-element subgraph of v vertices can be reduced to a part $P_v = Q_v^1 \cup Q_v^{g*}$ $v < p+1$, where the complete graphs Q_v^1 and Q_v^{g*} are united at v vertices.

Proof: By Lemma 2.2.1, class 1 graphs satisfy the hypothesis of Theorem 2.3.3 and hence are reducible to P_v . Some of the elements of the complete graphs Q_v^1 and Q_v^{g*} may be zero.

Definition 2.3.3: A graph P_v is said to be a canonical RL graph of class 1 if and only if $P_v = Q_v^1 \cup Q_v^g$ where the two complete graphs are united at v vertices and where Q_v^g has a connected subgraph composed of non-zero g -elements incident to all v of the vertices. The associated matrix corresponding to any maximum order star tree of P_v is said to be the canonical associated matrix of a RL graph of class 1.

Theorem 2.3.4: Let $S_s = \left[S_{I22} \mid S_{23} \right] = \left[I \mid S_{22} \mid S_{23} \right]$ be the f seg matrix corresponding to a maximum order star tree T_s of a canonical RL graph P_v of class 1. Then the canonical associated matrix of P_v is

$$A_s = -L^{-1}R_{22}$$

such that

$$1. \quad L = L_2 = \text{diag} (L_{11}, L_{22}, \dots, L_{nn}, L_{12}, L_{13}, \dots, L_{1n}, \dots, L_{n-1, n}) \quad 2.3.39$$

where L_2 is the chord 1-element matrix,

$$2. \quad R_{22} = S_{23}' \left[S_{I22} \quad G_s \quad S_{I22}' \right]^{-1} S_{23} = S_{23}' \left[g_{ij} \right]^{-1} S_{23} \quad 2.3.40$$

where $G_s = \text{diag} (G_1, G_2)$, G_1 and G_2 are the branch and chord g -element matrices respectively, and

$$3. \quad \begin{aligned} g_{ij} &= -G_{ij} & \text{for } i \neq j, i, j, = 1, 2, \dots, n \\ g_{ii} &= \sum_{j=1}^n G_{ij} & \text{for } i=1, 2, \dots, n \end{aligned} \quad 2.3.41$$

$$\begin{bmatrix} I & S_{22}^* \end{bmatrix} \begin{bmatrix} G_1^* & 0 \\ 0 & G_2^* \end{bmatrix} \begin{bmatrix} I \\ S_{22}^{*'} \end{bmatrix} = K_{11}^{-1} \quad 2.3.38$$

where K_{11}^{-1} exists since K_{11} was shown to be positive definite. Eq. 2.3.38 can be written as Eq. 2.3.1 and hence satisfies the hypothesis of Lemma 2.3.1. Therefore, there exists a unique set of diagonal entries G_{ij}^* diag (G_1^*, G_2^*) such that the associated matrices of P_{p+1} and P_v are identical, thus proving the theorem.

Corollary 2.3.3: Every part P_{p+1} of a RL graph of class 1 having a 1-element subgraph of v vertices can be reduced to a part $P_v = Q_v^1 \cup Q_v^{g*}$ $v < p+1$, where the complete graphs Q_v^1 and Q_v^{g*} are united at v vertices.

Proof: By Lemma 2.2.1, class 1 graphs satisfy the hypothesis of Theorem 2.3.3 and hence are reducible to P_v . Some of the elements of the complete graphs Q_v^1 and Q_v^{g*} may be zero.

Definition 2.3.3: A graph P_v is said to be a canonical RL graph of class 1 if and only if $P_v = Q_v^1 \cup Q_v^g$ where the two complete graphs are united at v vertices and where Q_v^g has a connected subgraph composed of non-zero g -elements incident to all v of the vertices. The associated matrix corresponding to any maximum order star tree of P_v is said to be the canonical associated matrix of a RL graph of class 1.

Theorem 2.3.4: Let $S_s = \left[\begin{array}{c|c} S_{I22} & S_{23} \end{array} \right] = \left[\begin{array}{c|c} I & S_{22} \end{array} \middle| S_{23} \right]$ be the f seg matrix corresponding to a maximum order star tree T_s of a canonical RL graph P_v of class 1. Then the canonical associated matrix of P_v is

$$A_s = -L^{-1}R_{22}$$

such that

$$1. \quad L = L_2 = \text{diag} (L_{11}, L_{22}, \dots, L_{nn}, L_{12}, L_{13}, \dots, L_{1n}, \dots, L_{n-1, n}) \quad 2.3.39$$

where L_2 is the chord 1-element matrix,

$$2. \quad R_{22} = S_{23}' \left[\begin{array}{cc} S_{I22} & G_s \end{array} \right]^{-1} S_{23} = S_{23}' \left[g_{ij} \right]^{-1} S_{23} \quad 2.3.40$$

where $G_s = \text{diag} (G_1, G_2)$, G_1 and G_2 are the branch and chord g -element matrices respectively, and

$$3. \quad \begin{aligned} g_{ij} &= -G_{ij} & \text{for } i \neq j, i, j, = 1, 2, \dots, n \\ g_{ii} &= \sum_{j=1}^n G_{ij} & \text{for } i=1, 2, \dots, n \end{aligned} \quad 2.3.41$$

where $G_{ij} \in G_s$.

Proof: 1. and 2. follow directly from Lemma 1.1.2. S_{I22} and S_{23} of the hypothesis are given by Eq. 2.1.5 where $m=n$. 3. follows from 2. and Property 2.1.1.

Theorem 2.3.5: The associated matrix of every part P_{p+1} of a separable RL graph of class 1 is identical to the canonical associated matrix of some canonical RL graph of class 1 and conversely.

Proof: By Corollary 2.3.3 P_{p+1} can be reduced to $P_v = Q_v^1 U Q_v^{g*}$ where Q_v^1 and Q_v^{g*} are complete graphs united at v vertices. Let A_j be the associated matrix corresponding to some maximum order tree T_j of P_v . By Lemma 2.3.2, P_v has a maximum order star tree T_s . Let S_j and S_s be the f seg matrices corresponding to T_j and T_s respectively. By Lemma 2.3.2,

$$S_s = A_{sj} S_j P_{js}. \quad 2.3.42$$

Let S_s and S_j be partitioned such that the columns of the 1,1 submatrix correspond to the g -elements and the columns of the 1,2 submatrix correspond to the 1-elements of P_v . Eq. 2.3.42 becomes

$$\begin{bmatrix} S_{s22} & S_{s23} \end{bmatrix} = A_{sj} \begin{bmatrix} S_{j22} & S_{j23} \end{bmatrix} \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}, \quad 2.3.43$$

where P_{js} is partitioned to conform to the partitioning of S_j . P_{11} rearranges the columns of S_{j22} corresponding to g -elements such that the leading columns of $S_{j22} P_{11}$ correspond to the g -elements of T_s . P_{22} rearranges the columns of S_{j23} corresponding to 1-elements. Since for a class 1 RL graph, there are no 1-elements in T_s , then $P_{22}=I$. Let the branch and chord g -element matrices corresponding to T_j be written

$$G_j = \text{diag} (G_1, G_2). \quad 2.3.44$$

Then, the diagonal branch and chord g -element matrix G_s corresponding to the tree T_s is the same matrix as G_j with the elements rearranged on the diagonal. In fact,

$$G_s = P_{11}' G_j P_{11}. \quad 2.3.45$$

Since the 1-element chord matrix L_2 is not rearranged, then L_2 is the same for both trees T_s and T_j . The associated matrix of P_v corresponding to the tree T_s by Lemma 2.3.2 is

$$A_s = -L^{-1}S'_{s23} \left[S_{s22}G_sS'_{s22} \right]^{-1}S_{s23}. \quad 2.3.46$$

Substituting the relations of Eqs. 2.3.43 and 2.3.45 into Eq. 2.3.46 gives

$$A_s = -L^{-1}S'_{j23}Q'_{sj} \left[Q_{sj}S_{j22}P'_{11}P'_{11}G_jP'_{11}P'_{11}S'_{j22}Q'_{sj} \right]^{-1}Q_{sj}S_{j23}. \quad 2.3.47$$

Since $P'_{11}P'_{11} = I$ and by taking the inverse of a product of square matrices, Eq. 2.3.47 becomes

$$A_s = -L^{-1}S'_{j23} \left[S_{j22}G_jS'_{j22} \right]^{-1}S_{j23}. \quad 2.3.48$$

By Lemma 2.3.2, the right hand side of Eq. 2.3.48 is recognized as the associated matrix of the reduced graph P_v corresponding to the tree T_j . Hence $A_s = A_j$ which proves the theorem.

Converse: The converse follows directly from Eq. 2.3.48.

Corollary 2.3.5.0: The associated matrices $A_1, A_2, \dots, A_j, \dots, A_k, \dots$ of a part P_{p+1} corresponding to the maximum order trees $T_1, T_2, \dots, T_j, \dots, T_k, \dots$ of a class 1 RL graph are identical.

Proof: By Theorem 2.3.5, the associated matrices A_j and A_k are identical to the associated matrix A_s corresponding to the star tree T_s which exists by Lemma 2.3.2. Hence,

$$A_s = A_j = A_k \quad 2.3.49$$

for all j and k , thus proving the corollary.

Corollary 2.3.5.1: The associated matrix of a connected RL graph G of class 1 is identical to the canonical associated matrix of a canonical RL graph of class 1 and conversely.

Proof: From Corollary 2.3.5.0 the associated matrix for each separable part P^i , $i=1, 2, \dots, m$ is

$$A_j^i = A_s^i. \quad 2.3.50$$

Hence, by Theorem 1.1.1, the associated matrix of G is

$$A = \begin{bmatrix} A_j^1 & & & \\ & A_j^2 & & \\ & & \ddots & \\ & & & A_j^m \end{bmatrix} = \begin{bmatrix} A_s^1 & & & \\ & A_s^2 & & \\ & & \ddots & \\ & & & A_s^m \end{bmatrix}. \quad 2.3.51$$

$$A_s = -L^{-1}S'_{s23} \left[S_{s22}G_sS'_{s22} \right]^{-1}S_{s23}. \quad 2.3.46$$

Substituting the relations of Eqs. 2.3.43 and 2.3.45 into Eq. 2.3.46 gives

$$A_s = -L^{-1}S'_{j23}Q'_{sj} \left[Q_{sj}S_{j22}P_{11}P'_{11}G_jP_{11}P'_{11}S'_{j22}Q'_{sj} \right]^{-1}Q_{sj}S_{j23}. \quad 2.3.47$$

Since $P_{11}P'_{11} = I$ and by taking the inverse of a product of square matrices, Eq. 2.3.47 becomes

$$A_s = -L^{-1}S'_{j23} \left[S_{j22}G_jS'_{j22} \right]^{-1}S_{j23}. \quad 2.3.48$$

By Lemma 2.3.2, the right hand side of Eq. 2.3.48 is recognized as the associated matrix of the reduced graph P_v corresponding to the tree T_j . Hence $A_s = A_j$ which proves the theorem.

Converse: The converse follows directly from Eq. 2.3.48.

Corollary 2.3.5.0: The associated matrices $A_1, A_2, \dots, A_j, \dots, A_k, \dots$ of a part P_{p+1} corresponding to the maximum order trees $T_1, T_2, \dots, T_j, \dots, T_k, \dots$ of a class 1 RL graph are identical.

Proof: By Theorem 2.3.5, the associated matrices A_j and A_k are identical to the associated matrix A_s corresponding to the star tree T_s which exists by Lemma 2.3.2. Hence,

$$A_s = A_j = A_k \quad 2.3.49$$

for all j and k , thus proving the corollary.

Corollary 2.3.5.1: The associated matrix of a connected RL graph G of class 1 is identical to the canonical associated matrix of a canonical RL graph of class 1 and conversely.

Proof: From Corollary 2.3.5.0 the associated matrix for each separable part P^i , $i=1, 2, \dots, m$ is

$$A_j^i = A_s^i. \quad 2.3.50$$

Hence, by Theorem 1.1.1, the associated matrix of G is

$$A = \begin{bmatrix} A_j^1 & & & \\ & A_j^2 & & \\ & & \ddots & \\ & & & A_j^m \end{bmatrix} = \begin{bmatrix} A_s^1 & & & \\ & A_s^2 & & \\ & & \ddots & \\ & & & A_s^m \end{bmatrix}. \quad 2.3.51$$

where $\text{diag} (A_s^1, A_s^2, \dots, A_s^m)$ is the associated matrix of a union of m canonical RL graphs of class 1 at one vertex and hence, is a canonical RL graph of class 1.

Converse: The converse follows directly from Eq. 2.3.51

2.4 Conclusion.

Lemmas 2.2.0 and 2.2.1 classify all connected real linear bielement systems whose associated matrix is of order n . The classification is essentially a tabulation of all possible subgraphs from which the maximum order trees of the system graph are selected.

In Section 2.3, system graphs of class 1 are considered. From physical considerations, the g -element subgraphs of class 1 systems have at least as many vertices as the c - or l -element subgraphs. By Corollaries 2.3.0 and 2.3.3 the g -element subgraphs are reduced to an equivalent subgraph which retains only the vertices of the reactive elements of the system. The reduction of the g -element subgraph is similar to the element elimination process of Brown and Tokad⁽¹⁵⁾. Furthermore, the reduction could be accomplished by using the wye-delta transformation on the graph. Since all class 1 systems reduce to the same basic structure, Definitions 2.3.2 and 2.3.3 define this structure to be a canonical graph of class 1. The canonical graph is unique in that it always has a maximum order star tree. Corresponding to the maximum order star tree, the canonical associated matrix is defined.

By Lemma 2.3.2, there exists an equivalence relation⁽¹⁶⁾ between the f seg matrices of a graph. As a result of this equivalence, a similarity relation is derived for the associated matrices of a class 1 graph. Theorem 2.3.2 gives the similarity relationship for class 1 RC graphs. By Theorem 2.3.5 the similarity transformation for class 1 RL graphs reduces to the identity transformation. Therefore, the associated matrix of a RL graph is independent of the formulation tree.

A description of the entries in the C matrix, L matrix and R matrix for canonical RC and RL graphs is given in Theorems 2.3.1 and 2.3.4. Matrices having similar properties to the C , L and R matrix have been previously

considered by Cederbaum (17) and Brown (18). The effect of changing the orientation of elements of the graph on the canonical associated matrix is readily calculated by the use of the similarity transformation of Theorems 2.3.2 and 2.3.5.

It is postulated from the results of Section 2.3 that each of the other classes of system graphs is reducible to a canonical graph. Correspondingly, a canonical associated matrix could be defined for each class and a similarity transformation between the associated matrices derived. Finally the entries of the C, L and R matrices of each canonical graph are to be described and tabulated. This tabulation would then give all possible forms of the associated matrices of a real linear bielement system.

III. SYNTHESIS OF A CLASS OF REAL LINEAR BIELEMENT SYSTEMS

3.0 Introduction

In this chapter, the necessary and sufficient conditions on a given matrix such that it is realizable as a class 1 real linear bielement system, hereafter referred to as a RC or RL graph, are developed. In Section 3.1 the conditions for the decomposition of a square matrix into the product of two symmetric matrices are given. Three techniques are developed for this decomposition. A test is also derived to determine if the given matrix can be decomposed as the product of a diagonal matrix and a symmetric matrix. The results of Section 3.1 are used to determine sufficient conditions for the synthesis of RC and RL graphs in Sections 3.2 and 3.3 respectively. The analysis of Section 1.1 imposes the necessary conditions for RC and RL graph synthesis.

3.1 Decomposition of a Square Matrix.

From the time domain analysis of RC and RL systems of Section 1.1, the associated matrix is always written as the product of two symmetric matrices. Therefore, it is fundamental for synthesis that a given matrix be factorable into the product of two symmetric matrices.

Theorem 3.1.0: If there are n real, distinct eigenvalues of the real matrix A , then A is factorable as

$$A = -C^{-1}R \quad 3.1.0$$

where C is positive definite and R is real symmetric.

Proof: Since A is real and its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are real, distinct, there exists a nonsingular real matrix P by Theorem A.16 such that

$$A = P^{-1}\Lambda P \quad 3.1.1$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Let Eq. 3.1.1 be rewritten as follows

$$A = P^{-1}(P'^{-1}P')\Lambda P = (P'P)^{-1}P'\Lambda P. \quad 3.1.2$$

By letting

$$C = P'P \quad 3.1.3$$

and

$$R = -P' \Lambda P \quad 3.1.4$$

then A is factorable as in Eq. 3.1.0. By Theorem A.7 C is positive definite. Since P and Λ are real, R is real. Also, since $R = R'$, R is symmetric by definition.

Theorem 3.1.1: Let the matrix equation

$$\begin{bmatrix} f \\ \alpha \beta \end{bmatrix} = \begin{bmatrix} c \\ \alpha_j \end{bmatrix}^{-1} \begin{bmatrix} g \\ \beta_j \end{bmatrix} + \begin{bmatrix} a \\ \alpha \beta \end{bmatrix} = 0 \quad \text{for } \alpha, \beta, j = 1, 2, \dots, n \quad 3.1.5$$

where $c_{ij} = c_{ji}$, $g_{ij} = g_{ji}$ and where a_{ij} 's are known constants, be written as a system of n^2 nonlinear algebraic equations

$$f_{\alpha\beta}(c_{\alpha j}, g_{\beta j}) = 0 \quad \text{for } \alpha, \beta, j = 1, 2, \dots, n$$

in $n^2 + n$ variables $c_{\alpha j}, g_{\beta j}$. If A has real, distinct eigenvalues and if Jacobian J with respect to any of the n^2 variables does not vanish at some point (c_{ij0}, g_{ij0}) , then there exists n^2 unique equations $\phi_q(x_{ij})$, $q=1,2,\dots,n^2$ where x_{ij} are the remaining n variables and a n dimensional neighborhood b^* such that

1. $\phi_q(x_{ij})$ is continuous in b^* ; $q=1,2,\dots,n^2$.
2. $c_{ij0} = \phi_q^*(x_{ij0})$ where $q^*, q^{**} = 1,2,\dots$, such that
 $g_{ij0} = \phi_q^{**}(x_{ij0})$ $q^* + q^{**} = n^2$
3. $f_{\alpha\beta}(\phi_q, x_{ij}) \equiv 0$ for $|x_{ij} - x_{ij0}| < b^*$; $\alpha, \beta = 1,2,\dots,n$.
4. $\frac{\partial \phi_q}{\partial x_{ij}}$ is a continuous for $|x_{ij} - x_{ij0}| < b^*$; $q=1,2,\dots,n^2$.

Proof: The conclusion follows immediately from the Implicit Function Theorem, Theorem A.8, if only there exists a point such that $\begin{bmatrix} c_{ij0} \end{bmatrix}^{-1} \begin{bmatrix} g_{ij0} \end{bmatrix} + \begin{bmatrix} a_{ij} \end{bmatrix} = 0$. However, A satisfies the hypothesis of Theorem 3.1.1 and hence the point known to satisfy Eq. 3.1.5 is given by Eqs. 3.1.3 and 3.1.4.

Theorem 3.1.1 indicates there are infinitely many points that will allow A to be factored into the product of two symmetric matrices, provided of course, the Jacobian of the hypothesis does not vanish. Since the solution of a system of nonlinear algebraic equations is at best difficult to find, an

easier method of generating the symmetric C and R matrices is desirable. Such a method is readily obtained after the matrix equation $A + C^{-1}R = 0$ is premultiplied by C giving $CA + R = 0$. This new matrix equation is then written as a system of n^2 linear algebraic equations in $n^2 + n$ unknowns. As in the nonlinear case, the solution to the linear system is found as functions of n of the unknowns. Therefore, by selecting arbitrary values for the n unknowns, a symmetric C and R matrix is calculated. The C matrix must be checked to insure that it is nonsingular, since a singular C matrix will not give the desired factoring of A. This method of finding the C and R matrix will be much easier than the direct solution of Eq. 3.1.5. The results of this discussion is given by the following theorem.

Theorem 3.1.2: Let the matrix equation

$$\begin{bmatrix} f \\ \alpha \beta \end{bmatrix} = \begin{bmatrix} c \\ \alpha_j \end{bmatrix} \begin{bmatrix} a \\ \beta_j \end{bmatrix} + \begin{bmatrix} g \\ \alpha \beta \end{bmatrix} = 0 \quad 3.1.7$$

where $c_{ij} = c_{ji}$, $g_{ij} = g_{ji}$ and where a_{β_j} 's are known constants, be written as a system of n^2 linear algebraic equations

$$f_{\alpha\beta}(g_{\alpha\beta}, c_{\alpha_j}) = \sum_{k=1}^n a_{\beta_k} c_{\alpha_k} + g_{\alpha\beta} = 0 \text{ for } \alpha, \beta, j=1, 2, \dots, n \quad 3.1.8$$

in $n^2 + n$ variables c_{ij} , g_{ij} . If the Jacobian J of $f_{\alpha\beta}$ with respect to any n^2 of the variables does not vanish then there exists a solution of n^2 of the variables in terms of the remaining n variables.

Proof: Write the system of equations $f_{\alpha\beta}(g_{\alpha\beta}, c_{\beta_j}) = 0$ $\alpha, \beta, j = 1, 2, \dots, n$ in matrix form.

$$B_1 X = B_2 Y \quad 3.1.9$$

where X is a column matrix composed of the n^2 variable for which the Jacobian $J \neq 0$, and Y is a column vector of the remaining n variables. B_1 and B_2 are the corresponding coefficient matrices. For the linear system $\text{Det } B_1 = J \neq 0$. Hence B_1 is nonsingular and the system of Eq. 3.1.9 has the solution

$$X = B_1^{-1} B_2 Y \quad 3.1.10$$

which proves the theorem.

Definition 3.1.0: A real matrix A of order n is said to be bisymmetric

if and only if A can be factored into the product of two real symmetric matrices of order n .

Theorems 3.1.0, 3.1.1 and 3.1.2 give alternate methods of factoring a real matrix into the product of two real symmetric matrices. Let the real matrix A be symmetric, then $C = I$ and $R = -A$ is a satisfactory decomposition of A . Since a diagonal matrix is a special form of a symmetric matrix, it is convenient to introduce the following definition.

Definition 3.1.1: A real matrix A of order n is said to be quasisymmetric if and only if A can be factored into the product $A = -L^{-1}R$ (or $-RL^{-1}$) where L is a real nonsingular diagonal matrix and R is a real symmetric matrix. If $A = -L^{-1}R$ the A is said to be left quasisymmetric. If $A = -RL^{-1}$ then A is said to be right quasisymmetric.

Since every quasisymmetric matrix is by definition bisymmetric but not conversely, it is convenient to develop a method whereby a given bisymmetric matrix can be tested for quasisymmetry.

Theorem 3.1.3: If a real matrix A is quasisymmetric, then the position of the zero entries of A are identical to the position of the zero entries of A' .

Proof: By hypothesis, A is quasisymmetric. By Def. 3.1.1, A can either be left or right quasisymmetric. Without loss of generality, assume A to be left quasisymmetric. Therefore $A = -L^{-1}R$ where L is real, nonsingular and diagonal and R is real symmetric. Since L is nonsingular, L has no zero entries on the diagonal. Therefore, the zero entries of A are determined entirely by the zero entries of R . Since R is symmetric, the zero entries of R and R' are identical. Correspondingly, the zero entries of $A = -L^{-1}R$ and $A' = -RL^{-1}$ are identical.

Definition 3.1.2: A real square matrix A is said to be zero-symmetric if and only if the zero entries of A are identical to the zero entries of A' . If A has no zero entries, then A is zero-symmetric by definition.

Definition 3.1.3: Let $A = [a_{ij}]$ for $i, j = 1, \dots, n$ be a zero-symmetric, bisymmetric matrix. Then the ratio matrix $\mathbb{R}$ is defined as

always equal to the number of c-elements in the tree. Consequently, the number of vertices v of a canonical RC graph is given in Lemma 2.2.0. In the synthesis of class 1 RC graphs, the coefficient matrix A is assumed to be the associated matrix corresponding to some maximum order tree. Hence, if A is of order n , then the number of vertices of the synthesized class 1 RC graph G_v will be $v = n + 1$. Before proceeding to the synthesis of RC graphs, the relationship between the matrices characterized in Section 2.3 and those obtained by the decomposition techniques of Section 3.1 is derived.

Lemma 3.2.0: Let C be a real symmetric matrix, then C is factorable into the triple product of Eq. 2.3.7.

Proof: By Theorem A.10 there exists a unitary matrix $V = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ where v_j is the j th column of V such that

$$C = V' \mu V \quad 3.2.0$$

where μ is a diagonal matrix. Consider the maximum rank matrix T constructed from the columns of V as follows.

$$T = \begin{bmatrix} v_1 & v_2 & \dots & v_n & (v_1 - v_2) & (v_1 - v_3) & \dots & (v_1 - v_n) \\ & & & & (v_2 - v_3) & (v_2 - v_4) & \dots & (v_2 - v_n) \\ & & & & & & \dots & (v_{n-1} - v_n) \end{bmatrix} \quad 3.2.1$$

Since the columns of a unitary matrix are orthogonal, it is easily shown that

$$S_{Ill} = V' T \quad 3.2.2$$

where S_{Ill} is a subset of the f seg matrix of a canonical RC graph as given in Theorem 2.3.1. Let C of the hypothesis be equated to C of Eq. 2.3.7

$$C = S_{Ill} C_s S'_{Ill} \quad 3.2.3$$

Eq. 3.2.3 is a system of $(n^2 + n)/2$ linear nonhomogeneous algebraic equations in $(n^2 + n)/2$ unknowns $C_{ij} \in C_s$ for $i \leq j$, $i, j = 1, 2, \dots, n$, which satisfies the hypothesis of Lemma 2.3.1. Hence, C is factorable into the triple product of Eq. 2.3.7.

Lemma 3.2.1: Let R be a real symmetric matrix. Then R is factorable into the triple product of Eq. 2.3.8.

Proof: The proof is identical to the proof of Lemma 3.2.2 with C replaced by R , S_{sll} replaced by S_{12} and Eq. 2.3.7 replaced by Eq. 2.3.8.

Theorem 3.2.0: The associated matrix A of order n of a connected RC (or RL) graph has n real eigenvalues.

Proof: Consider a canonical RC (or RL) graph whose f seg matrix S_s corresponding to a maximum order star tree T_s is given in Theorem 2.3.1 (or Theorem 2.3.4) for $p=n$. The associated matrix $A_s = -C^{-1}R_{11}$ (or $A_s = -L^{-1}R_{22}$) corresponding to T_s is also given in Theorem 2.3.1 (or Theorem 2.3.4). The characteristic equation^(14, 16, 17) for the system of differential equations whose associated matrix is A_s is given by

$$p(\lambda) = 0 = \text{Det} (A_s - \lambda I) = \text{Det} (-C^{-1}R_{11} - \lambda I) \text{ (or } \text{Det} (-L^{-1}R_{22} - \lambda I)).$$

3.2.4

Equivalently

$$p(\lambda) = \text{Det} (R_{11} + \lambda C) = 0 \text{ (or } \text{Det} (R_{22} + \lambda L)).$$

3.2.5

$p(\lambda)$ is an n^{th} order polynomial in λ , and hence has n roots. For the given f seg matrix C (or L) is positive definite by Theorem 1.1.0. From Eq. 1.1.3, R_{11} (or R_{22}) is symmetric by definition. By Theorem A.11 the eigenvalues of A are real. By Corollary 2.3.2.1 (or Corollary 2.3.5.1), the associated matrix of every connected RC (or RL) graph is similar (or identical) to some canonical RC (or RL) graph. By Theorem A.12 similar matrices have the same characteristic equations and hence the same eigenvalues. The conclusion of the theorem is now evident.

Definition 3.2.0: Let $\frac{dX}{dt} = AX$ be a system of linear first derivative-explicit differential equations. A is said to be the coefficient matrix of the system.

Theorem 3.2.1: If and only if the coefficient matrix A of order n is nonsingular bisymmetric then there exists a canonical RC graph G_{n+1} for which A is a canonical associated matrix.

Proof: Sufficiency: By hypothesis A is factorable into the product of two nonsingular matrices as $A = C^{-1}R$. By Lemma 3.2.0, C is factorable into the triple product of Eq. 2.3.7. By Lemma 3.2.1, R is factorable into the triple product of Eq. 2.3.8. The f seg matrix S_s corresponding to star tree T_s is given

$$S_s = \begin{bmatrix} S_{111} & S_{12} \end{bmatrix} \quad 3.2.6$$

where S_{111} and S_{12} are given in the proofs of Lemma 3.2.0 and 3.2.1 respectively. The element values C_{ij} and G_{ij} are found as in the proofs of Lemmas 3.2.0 and 3.2.1 respectively. Hence, by construction, there exists a canonical RC graph G_{n+1} for which A is the canonical associated matrix.

Necessity: By Theorem 2.4.0, A is bisymmetric. By Theorem 1.1.0 and Corollary 1.1.0, C and R_{11} are positive definite and hence C and R are nonsingular by Theorem A.13. By Theorem A.14, $\text{Det } A = \text{Det } C^{-1} \text{Det } R_{22} > 0$. Therefore A is nonsingular by Theorem A.15.

Corollary 3.2.1: If the coefficient matrix A has real, distinct, negative, non-zero eigenvalues, then there exists a canonical RC graph with A as a canonical associated matrix such that $A = -C^{-1}R$ where C and R are positive definite.

Proof: From Theorem 3.1.0, A is factorable as $-C^{-1}R$ where C is positive definite.. From Eq. 3.1.4

$$R = -P' \Lambda P = P' (-\Lambda) P. \quad 3.2.7$$

Since all the eigenvalues are distinct and nonzero, Λ is diagonal and nonsingular. Therefore, R is nonsingular since P is nonsingular. Since all of the eigenvalues are negative, then $(-\Lambda)$ is positive definite by Theorem A.3. Hence by Theorem A.4, R is positive definite. By Def. 3.1.0, A is bisymmetric. A is nonsingular since C and R are positive definite. Therefore from Theorem 3.2.1 the conclusion of the corollary follows.

Corollary 3.2.1 imposes one condition on the coefficient matrix A such that it can be decomposed into the product of two positive definite matrices. Theorem 1.1.0 and Corollary 1.1.0 indicate that the C and the R matrix being positive definite is a necessary condition for the RC graph to have non-negative elements. Necessary and sufficient conditions for which a coefficient matrix is realizable as a canonical RC graph with non-negative elements is given in the following theorem.

Theorem: 3.2.2: If and only if the coefficient matrix $A = [a_{ij}]$ order n is bisymmetric such that $[a_{ij}] = -[c_{ij}]^{-1} [g_{ij}]$ where

$$1. \quad \begin{aligned} c_{ij} &\leq 0 \\ g_{ij} &\leq 0 \end{aligned} \quad \text{for } i \neq j \quad i, j = 1, 2, \dots, n$$

$$2. \quad \begin{aligned} c_{ii} &> 0 \\ g_{ii} &> 0 \end{aligned} \quad \text{for } i = 1, 2, \dots, n.$$

$$3. \quad 2c_{ii} > \sum_{k=1}^n c_{ik} \quad \text{for at least one } i \text{ and}$$

$$2c_{ii} \geq \sum_{k=1}^n c_{ik} \quad \text{for all other } i.$$

$$2g_{ii} > \sum_{k=1}^n g_{ik} \quad \text{for at least one } i \text{ and}$$

$$2g_{ii} \geq \sum_{k=1}^n g_{ik} \quad \text{for all other } i.$$

and such that neither $[c_{ij}]$ nor $[g_{ij}]$ are permutable into $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where

A and B are square, then there exists a canonical RC graph G_{n+1} with non-negative elements for which A is a canonical associated matrix.

Proof: Sufficiency: From 1., 2., and 3. of the hypothesis and by Theorem 2.6 $[c_{ij}]$ and $[g_{ij}]$ are nonsingular. Therefore, A is nonsingular. By hypothesis, A is bisymmetric, and hence, from Theorem 3.2.1, there exists a canonical RC graph G_{n+1} . The element values are calculated by writing Eq. 3.2.3 as a system of n^2 equations. This system of equations satisfies the hypothesis of Lemma 2.3.1 in such a way that all element values are non-negative.

Necessity: Follows directly from Theorem 2.3.1.

3.3 Synthesis of RL Graphs.

In the synthesis of RC graphs, it was determined that the order of the associated matrix uniquely gives the number of vertices for each class of graphs. This is not the case for the RL graph. The order of the associated matrix gives only the bounds on the number of vertices v of the 1-element subgraph G_v^1 of the RL graph. However, since the g -element subgraph G^g of the RL graph can be reduced to have the same vertices as G_v^1 , the bounds on the vertices are to be used in the synthesis of RL graphs.

Definition 3.3.0: Let the matrix S_v^* of order $(v-1, v(v-1)/2)$ be given by S_f of Eq. 2.1.5,

$$S_v^* = S_f \quad 3.3.0$$

then $S_{v,n}$ is defined to be the set of all matrices σ_v of order $(v-1, n)$ composed of any n columns of S_v^* for $v = r, r+1, \dots, 2n$ and where r is the smallest positive integer such that $r(r+1)/2 \geq n$.

Lemma 3.3.0: $S_{v,n}$ for $v = r, r+1, \dots, 2n$ of Def. 3.3.0 is the set of all submatrices S_{23} of the f seg matrix as given in Theorem 2.3.4 for all canonical RL graphs G_v having only n nonzero 1-elements distributed on the v vertices for $r \leq v \leq 2n$ where r is the smallest integer satisfying $r(r+1)/2 \geq n$.

Proof: Consider S_v^* of Eq. 3.3.0. $S_v^* = S_{23}$ of the canonical graph G_v of the hypothesis for all $v(v-1)/2$ 1-elements nonzero. Since only n of the 1-elements are nonzero, then S_{23} is composed of n of the columns of S_v^* . But $S_{v,n}$ is the set of all matrices σ_v composed of n columns from S_v^* . By Lemma 2.2.1, the number of vertices v is bounded by r and $2n$. Therefore, the conclusion follows for all G_v where $r \leq v \leq 2n$.

Lemma 3.3.1: Let R be a real symmetric matrix of order n . If $\text{Det}(\sigma_v^R \sigma_v') \neq 0$ for some $\sigma_v \in S_{v,n}$ then R is factorable into the triple product of Eq. 2.3.40.

Proof: Let R be equated to R_{22} of Eq. 2.3.40.

$$R = S_{23}' \left[S_{I22} G_s S_{I22}' \right]^{-1} S_{23} \quad 3.3.1$$

Let S_{23} of Eq. 3.3.1 be equal to S_v of the hypothesis. This can always be done since by Lemma 3.3.0 $S_{v,n}$ is the set of all S_{23} for a canonical RL graph G_v and since by Corollary 2.3.5.1 the associated matrix of the graph is independent of the formulation tree. Let Eq. 3.3.1 be premultiplied by σ_v and postmultiplied by σ_v' . Therefore

$$\sigma_v^R \sigma_v' = \sigma_v \sigma_v' \left[S_{I22} G_s S_{I22}' \right]^{-1} \sigma_v \sigma_v'. \quad 3.3.2$$

By hypothesis, $\text{Det}(\sigma_v^R \sigma_v') \neq 0$, and hence Eq. 3.3.2 is solved as follows

$$S_{I22} G_s S_{I22}' = \sigma_v \sigma_v' \left[\sigma_v^R \sigma_v' \right]^{-1} \sigma_v \sigma_v'. \quad 3.3.3$$

From Eq. 3.3.3, $S_{I22} G_s S'_{22}$ is a real, symmetric matrix. Hence, by Theorem A.10, there exists a unitary matrix $V = [v_1 \ v_2 \ \dots \ v_j \ \dots]$ where v_j is the j^{th} column of V such that

$$S_{I22} G_s S'_{I22} = V' U V \quad 3.3.4$$

where U is a diagonal matrix. Let T be the transformation of Eq. 3.2.1. Therefore,

$$S_{I22} = V' T \quad 3.3.5$$

where S_{I22} is a subset of the f seg matrix of a canonical RL graph as given in Theorem 2.3.4. Eq. 3.3.4 is a system of linear nonhomogeneous algebraic equations which satisfy the hypothesis of Lemma 2.3.1. Hence R is factorable into the triple product of Eq. 2.3.40.

Theorem 3.3.0: If and only if the coefficient matrix A of order n is left quasisymmetric where $A = -L^{-1}R$ and such that $\text{Det} (\sigma_v R \sigma_v') \neq 0$ for some $\sigma_v \in S_{v,n}$ then there exists a canonical RL graph G_v for which A is the associated matrix.

Proof: Sufficiency: By hypothesis A is left quasisymmetric, therefore factorable as $-L^{-1}R$ where L is diagonal and R is symmetric. By Lemma 3.3.1 R is factorable into the triple product of Eq. 2.3.40. The f seg matrix S_s corresponding to a star tree T_s is given by

$$S_s = \begin{bmatrix} S_{I22} & S_{23} \end{bmatrix} \quad 3.3.6$$

where S_{I22} is given by Eq. 3.3.5 and $S_{23} = \sigma_v$ of the hypothesis. The g -element values G_{1j} are calculated as in the proof of Lemma 3.3.1. The l -element values are obtained directly by setting $L = L_2$ of Eq. 2.3.39. Hence by construction, there exists a canonical RL graph G_v for which A is the associated matrix.

Necessity: By Theorem 2.3.4, A is left quasisymmetric. Let $\sigma_v = S_{23}$. This is always possible by Lemma 3.3.0. From Eq. 2.3.40, calculate $\text{Det} (S_{23} R S'_{23}) = \text{Det} (\sigma_v R \sigma_v')$. By Lemma 2.3.3, $S_{23} S'_{23}$ is nonsingular. Hence, $\text{Det} (S_{23} R S'_{23}) \neq 0$ which completes the proof.

Theorem 3.3.1: If and only if the coefficient matrix A of order n is left quasisymmetric where $A = -L^{-1}R$ such that

1. $\text{Det} (\sigma_v R \sigma_v') \neq 0$ for some $\sigma_v \in S_{v,n}$
2. L is diagonal and positive definite, and
3. For $[g_{ij}] = \sigma_v \sigma_v' (\sigma_v R \sigma_v')^{-1} \sigma_v \sigma_v'$

$$g_{ij} < 0 \quad \text{for } i \neq j$$

$$g_{ii} > 0 \quad \text{for all } i \text{ and}$$

$$2g_{ii} \geq \sum_{k=1}^n g_{ik} \quad \text{for all } i$$

where $i, j = 1, 2, \dots, n$, then there exists a canonical RL graph G_v with $r \leq v \leq 2n$ having non-negative elements for which A is the associated matrix.

Proof: Sufficiency: The hypothesis satisfies the conditions of Theorem 3.3.0, hence there exists a canonical RL graph G_v for which A is a canonical associated matrix. By Theorem A.3, the l -elements calculated from L are positive since L is diagonal and positive definite. All other l -elements of the G_v are zero. The g -element values G_{jk} are calculated by writing Eq. 3.3.4 as a system of n^2 linear nonhomogeneous algebraic equations. This system of equations satisfies the hypothesis of Lemma 2.3.1 in such a way that all element values are non-negative.

Necessity: Follows directly from Theorem 2.3.4 when Equation 2.3.40 is solved for $\begin{bmatrix} S_{I22} & G_s & S_{I22}' \end{bmatrix}$. This can always be done by premultiplying the equation by S_{23} and postmultiplying it by S_{23}' .

3.4 Conclusion.

In Section 3.1, sufficient conditions for the decomposition of a real square matrix into a bisymmetric form were given. Theorems 3.1.0, 3.1.1 and 3.1.2 give three different techniques for generating the bisymmetric form. Theorem 3.1.3 allows a given bisymmetric matrix to be tested for quasisymmetry. In Sections 3.2 and 3.3, the necessary and sufficient conditions for the synthesis of class 1 RC and RL graphs are given. Theorem 3.2.0 gives a necessary condition for any coefficient matrix to be realized as an associated matrix. The synthesis technique is given in the sufficiency proof of Theorem 3.2.1 for the RC graph and Theorem 3.3.0 for the RL graph. In Theorems 3.2.2 and 3.3.1, diagonal dominance first discussed by Burlington (20) gives the

necessary and sufficient conditions for the synthesis of non-negative element RC and RL graphs respectively.

IV. THE SYNTHESIS METHOD

4.0 Introduction.

In this chapter the results of Chapter 3 are developed into a synthesis method whereby an arbitrary coefficient matrix is examined for acceptance in the class of realizable matrices and acceptable coefficient matrices are realized as real linear bielelement systems. Flow charts are then developed for the determination of acceptable coefficient matrices and for the synthesis of RC and RL graphs. To illustrate the synthesis method, five examples are carried out in detail.

4.1 Determination of Realizable Coefficient Matrices.

Before proceeding to the actual synthesis technique, the class of coefficient matrices that can be realized as real linear bielelement systems is determined. An outline of the determination procedure follows:

1. Theorem 3.2.0 imposes a necessary condition for the decomposition of the coefficient matrix A . Therefore, the first step is to test A for real eigenvalues. If the eigenvalues of A are real distinct, then Theorem 3.1.0 gives a sufficient condition for the decomposition. If the eigenvalues of A are not distinct, then the decomposition technique of Theorem 3.1.2 is applied to determine if A is bisymmetric.
2. Theorems 3.2.1 and 3.3.0 imply that if A is to be realized as a RC or RL graph then A must be
 - a. nonsingular bisymmetric,
 - b. nonsingular left quasisymmetric, or
 - c. singular left quasisymmetric.

Therefore, A must be tested for singularity. Theorems 3.1.3 and 3.1.4 give the necessary and sufficient conditions for A to be quasisymmetric. Therefore, A must be tested for zero-symmetry and the rank of the ratio matrix must be calculated. Finally, A must be checked for left or right quasisymmetry.

The procedure for determining if an arbitrary coefficient matrix is realizable as a real linear bielelement system is illustrated in the flow chart of Fig. 4.1.0.

Example 1: Consider the coefficient matrix A given by

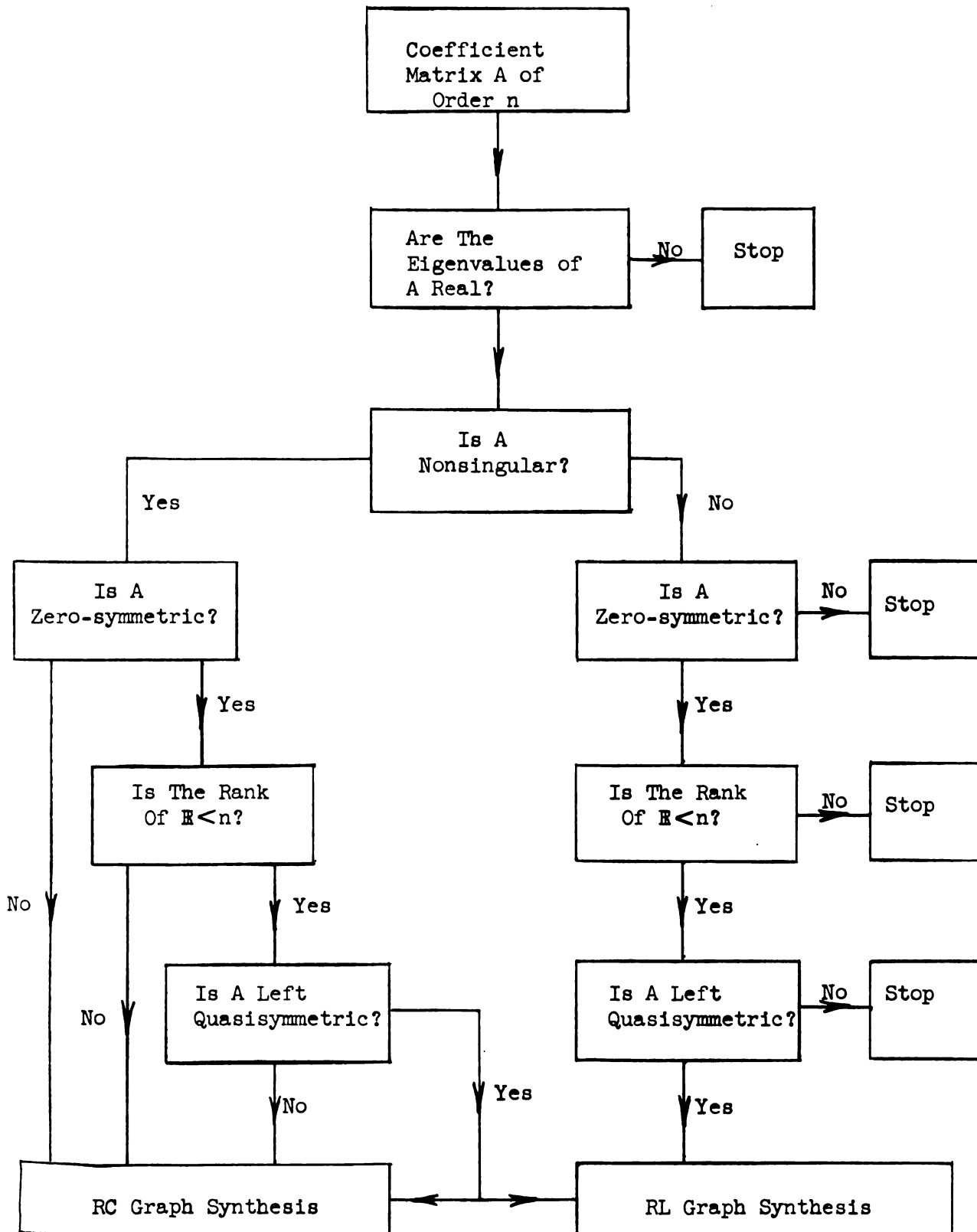


Figure 4.1.0 Flow Chart for the Determination of Acceptable Coefficient Matrices

$$A = \begin{bmatrix} -1 & -0.2 \\ 0 & -1.4 \end{bmatrix}. \quad 4.1.0$$

The eigenvalues of A are calculated as the roots of $\text{Det } (A - \lambda I) = 0$.
Hence,

$$\begin{aligned} \lambda_1 &= -1.0 \\ \lambda_2 &= -1.4 \end{aligned} \quad 4.1.1$$

Therefore A is nonsingular. A is not zero-symmetric. Hence, from Fig. 4.1.0, A can only be realized as a RC graph.

Example 2: Consider the coefficient matrix given by

$$A = \begin{bmatrix} -8 & 2 & -2 \\ 2 & -2 & -10 \\ 5 & 1 & -22 \end{bmatrix}. \quad 4.1.2$$

The eigenvalues of A are

$$\begin{aligned} \lambda_1 &= -2.715 \\ \lambda_2 &= -8.170 \\ \lambda_3 &= -21.116 \end{aligned} \quad 4.1.3$$

Therefore A is nonsingular. Since A has no zero entries, A is zero-symmetric by definition. The ratio matrix from Def. 3.1.2 is

$$R = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -2/5 \\ 0 & 1 & -10 \end{bmatrix}. \quad 4.1.4$$

The rank of R is 3, and hence, A can be realized only as a RC graph.

Example 3: Consider the coefficient matrix given by

$$A = \begin{bmatrix} -8 & 4 & 4 \\ 0 & -12 & -4 \\ 0 & -4 & -12 \end{bmatrix}. \quad 4.1.5$$

The eigenvalues of A are

$$\lambda_1 = -8$$

$$\lambda_2 = -8$$

4.1.6

$$\lambda_3 = -16$$

Therefore A is nonsingular. Since A is not zero-symmetric, then A can only be realized as a RC graph. In this example, there is no guarantee that A can be decomposed since the eigenvalues of A are not distinct.

Example 4: Consider the coefficient matrix given by

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}. \quad 4.1.7$$

The eigenvalues of A are

$$\lambda_1 = 0$$

$$\lambda_2 = -3$$

4.1.8

$$\lambda_3 = -5$$

Therefore A is singular. By inspection, A is symmetric and hence left-quasisymmetric. Hence, from Fig. 4.1.0, A can be realized only as the associated matrix of an RL graph.

Example 5: Consider the coefficient matrix given by

$$A = - \begin{bmatrix} 24 & 7 & 11 \\ 14 & 46 & 14 \\ 33 & 21 & 72 \end{bmatrix}. \quad 4.1.9$$

The eigenvalues of A are

$$\lambda_1 = -16.765$$

$$\lambda_2 = -37.485$$

4.1.10

$$\lambda_3 = -87.755$$

Therefore, A is nonsingular. Since A has no zero entries, A is zero-symmetric

by definition. The ratio matrix from Def. 3.1.2 is

$$\mathbb{R} = \begin{bmatrix} 1 & -1/2 & 0 \\ 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \end{bmatrix} \quad 4.1.11$$

From Eq. 4.1.11, $\text{Det } \mathbb{R} = 0$, and therefore the rank of $\mathbb{R}$ is less than the order of A . Hence, A is quasisymmetric. A must now be checked for left or right quasisymmetry. If A is right quasisymmetric, then A can only be realized by a RC graph. If A is left quasisymmetric, then A can be realized as a RC graph or a RL graph.

4.2 The Synthesis Method.

The technique for synthesizing RC and RL graphs is described below and summarized in the flow charts of Figures 4.2.0 and 4.2.1.

RC Graphs

If the coefficient matrix A of order n is nonsingular bisymmetric, then A can be factored into the product of two real symmetric matrices. Any of the methods of Section 3.1 can be used to accomplish this decomposition. If the eigenvalues of A are not all negative, then from physical considerations it is useless to look for a positive element graph. However if all eigenvalues of A are negative, then all possible decompositions of A should be examined and compared with 1., 2., and 3. of Theorem 3.2.2 to determine if all of the element values are non-negative. If none of these decompositions of A gives an all non-negative element graph, then A is assumed to be the associated matrix corresponding to some other maximum order tree rather than a star tree of the graph and the similarity transformation of Theorem 2.3.2 is made on A for every other tree of the graph. In each case this gives a new A matrix. Each new A matrix must now be decomposed and examined for non-negative elements. If none of these decompositions gives a non-negative element graph, then it must be concluded that no positive element class 1 graph exists for which A is the associated matrix, and hence, only a RC graph with some negative elements can be synthesized.

RL Graphs

If the coefficient matrix A of order n is left quasisymmetric then A is factorable as $-L^{-1}R$ where L is real, nonsingular diagonal and R is real symmetric. $\sigma_v R \sigma_v'$ must now be tested for singularity where $\sigma_v \in S_{v,n}$. If for some σ_v , $\sigma_v R \sigma_v'$ is nonsingular then A can be realized as a RL graph G_v . Here again, from physical considerations, it is useless to search for a positive element graph if all of the eigenvalues are not negative. However, if all of the eigenvalues are negative, then $\sigma_v R \sigma_v'$ for all $\sigma_v \in S_{v,n}$ should be tested for singularity. Corresponding to each σ_v for which $\sigma_v R \sigma_v'$ is nonsingular there exists a RL graph G_v . The element values must be checked for each graph to determine if they are negative. If all of the graphs have negative elements, then it can be concluded that no all positive element class 1 RL graph exists for which A is the associated matrix, and hence, only a RL graph with some negative elements can be synthesized.

The examples of Section 4.1 are now synthesized as RC or RL graphs. These examples are to illustrate the Synthesis Method and to point out some of the more subtle properties not directly developed in the preceding chapters.

Example 1: From Example 1 of Section 4.1, A is realizable as a RC graph. Since A satisfies the hypothesis of Theorem 3.1.0, then $A = -C^{-1}R$ where C is given by Eq. 3.1.3 and R is given by Eq. 3.1.4. The P matrix can be found by calculating the eigenvectors ⁽¹⁹⁾ of A . Therefore

$$P = \begin{bmatrix} -2k_1 & k_2 \\ 0 & -2k_2 \end{bmatrix} \quad 4.2.0$$

where k_1 and k_2 are arbitrary constants. Letting $k_1 = k_2$, then the C matrix of Equation 3.1.3 is

$$C = P'P = k^2 \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}. \quad 4.2.1$$

The R matrix of Eq. 3.1.4 is

1. A technique for calculating the eigenvectors is given in § 4, Chapter IV.

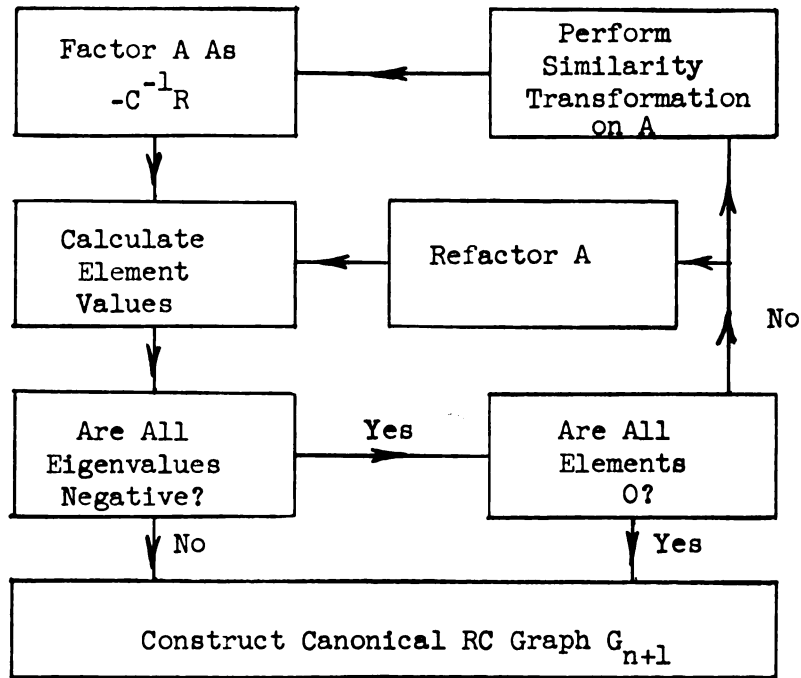


Figure 4.2.0 Flow Chart for RC Graph Synthesis.

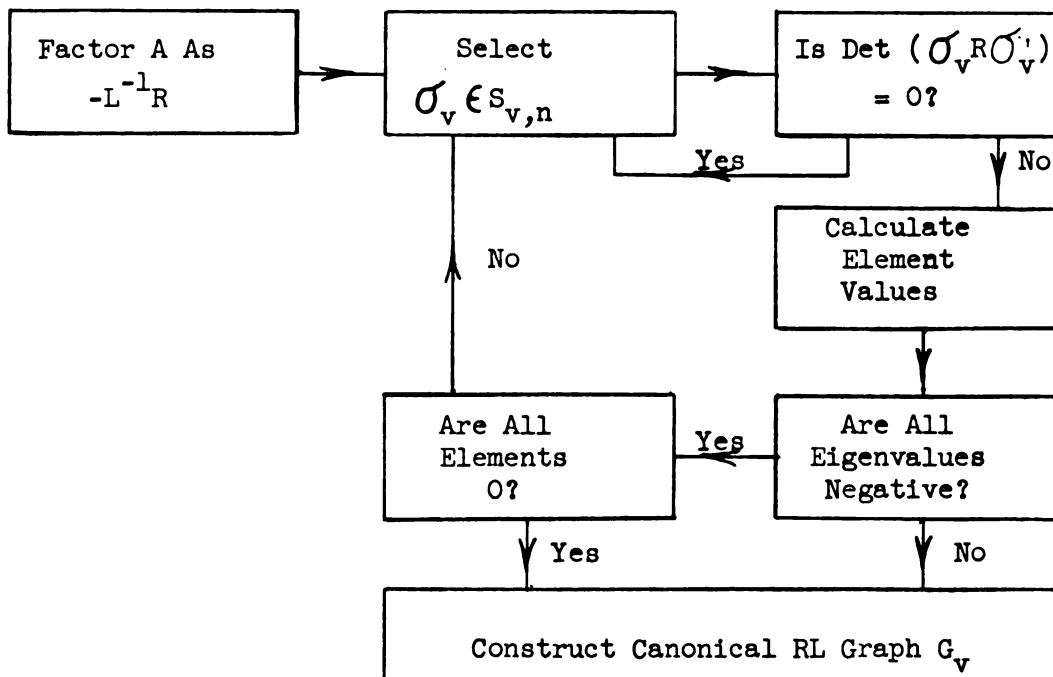


Figure 4.2.1 Flow Chart for RL Graph Synthesis.

$$R = -P' \bigwedge P = k^2 \begin{bmatrix} 4 & -2 \\ -2 & 6.6 \end{bmatrix}. \quad 4.2.2$$

The f seg submatrices are calculated as in Lemmas 3.2.0 and 3.2.1. Hence

$$S_s = \left[\begin{array}{c|c} S_{111} & S_{12} \end{array} \right] = \left[\begin{array}{ccc|ccc} C_{11} & C_{22} & C_{12} & G_{11} & G_{22} & G_{12} \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{array} \right]. \quad 4.2.3$$

Substituting Eqs. 4.2.1 and 4.2.3 into Eq. 3.2.3 and applying the conclusion of Lemma 2.2.1, the c-element values are

$$\begin{bmatrix} C_{11} \\ C_{22} \\ C_{12} \end{bmatrix} = \begin{bmatrix} c_{11} + c_{12} \\ c_{22} + c_{12} \\ -c_{12} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad 4.2.4$$

where k was assumed to be unity. From Eqs. 4.2.2 and 4.2.3,

$$R = S_{12} G_2 S_{12}'. \quad 4.2.5$$

By applying the conclusion of Lemma 2.2.1, the g-element values are

$$\begin{bmatrix} G_{11} \\ G_{22} \\ G_{12} \end{bmatrix} = \begin{bmatrix} g_{11} + g_{12} \\ g_{22} + g_{12} \\ -g_{12} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4.6 \end{bmatrix}. \quad 4.2.6$$

Therefore, from Eq. 4.2.3, 4.2.4 and 4.2.6, the canonical RC graph G_3 for which A is the corresponding canonical associated matrix is constructed in Figure 4.2.2.

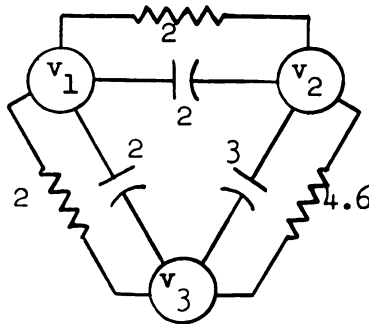


Figure 4.2.2 Synthesized RC Graph G_3 of Example 1.

An interesting result of this method of factoring the A matrix is that the signs of the entries in the P, C and R matrices are somewhat arbitrary. This result follows directly from the fact that p_i which is the i^{th} column of P is an eigenvector of A corresponding to $\lambda_i^{(19)}$. But any constant times an eigenvector is also an eigenvector $^{(19)}$. Therefore, the P matrix can be written as

$$P^* = \mathbb{E} D_1 \quad 4.2.7$$

where the columns of $\mathbb{E}$ are made up from a particular set of eigenvectors and D_1 is a diagonal matrix with arbitrary, nonzero constants on the diagonal. When the eigenvalues are distinct, as is the case here, the A matrix is similar to a diagonal matrix. That is,

$$A = P^{-1} \Lambda P \quad 4.2.8$$

Let δ be a diagonal matrix with only +1 or -1 on the diagonal. Therefore,

$$\delta \delta = I = \delta \delta^{-1}.$$

Since diagonal matrices commute, Eq. 4.2.8 is written as

$$A = P^{-1} \Lambda P = P^{-1} \delta^{-1} \Lambda \delta P. \quad 4.2.9$$

By letting P of Eq. 4.2.9 equal P^* of Eq. 4.2.7, then

$$A = D_1^{-1} \mathbb{E}^{-1} \delta^{-1} \Lambda \delta \mathbb{E} D_1. \quad 4.2.10$$

Hence, from Eqs. 4.2.8 and 4.2.10, a general P matrix is given by

$$P = \delta \mathbb{E} D_1 \quad 4.2.11$$

where D_1 , δ and $\mathbb{E}$ are defined in the above discussion. Therefore, the signs of the entries in the columns of P are changed by changing the signs of the arbitrary constants in D_1 and the signs of the entries in the rows of P are changed by changing the signs of the diagonal entries of δ . Correspondingly, from Eq. 4.2.11 and Eq. 3.1.3

$$C = D_1 \mathbb{E}' \delta \delta \mathbb{E} D_1 = D_1 \mathbb{E}' \mathbb{E} D_1 \quad 4.2.12$$

and from Eq. 4.2.11 and Eq. 3.1.4

$$R = -\mathbb{E}' \delta \Lambda \delta \mathbb{E} D_1 = -\mathbb{E}' \Lambda \mathbb{E} D_1. \quad 4.2.13$$

Therefore, by changing the signs of the entries of D_1 , the sign pattern of

the C matrix and the R matrix are also changed. Although Eqs. 4.2.11, 4.2.12 and 4.2.13 give a technique for changing the sign patterns of the P, C and R matrices, it is clear that the sign of each entry of these matrices can not be selected arbitrarily. The ability to change certain signs is illustrated in the next example.

Example 2: From Ex. 2 of Section 4.1, A is realizable as an RC graph. Using the decomposition technique of Theorem 3.1.0, a specific P matrix is first found to be

$$P = \begin{bmatrix} 2.618 & 2.387 & 1.000 \\ 0.778 & 7.308 & 6.975 \\ 1.000 & 1.000 & 13.533 \end{bmatrix} \quad 4.2.14$$

By letting $D_1 = \delta = I$ in Eq. 4.2.11, then $P = P$. Therefore, Eq. 4.2.12 becomes

$$C = P'P = \begin{bmatrix} 8.39 & 12.92 & 21.57 \\ 12.92 & 60.10 & 66.89 \\ 21.57 & 66.89 & 232.80 \end{bmatrix} \quad 4.2.15$$

and Eq. 4.2.13 becomes

$$R = -P'\Lambda P = \begin{bmatrix} 68.8 & 87.5 & 321.8 \\ 87.5 & 212.7 & 443.5 \\ 321.8 & 443.5 & 4007.5 \end{bmatrix} \quad 4.2.16$$

where $\Lambda = \text{diag} (-8.170, -2.715, -21.116)$. The subsets of the f seg matrix S_{I11} and S_{12} are calculated as in Lemmas 3.2.0 and 3.2.1. Hence

$$S_s = \left[S_{I11} \mid S_{12} \right] = \begin{bmatrix} c_{11} & c_{22} & c_{33} & c_{12} & c_{13} & c_{23} & | & g_{11} & g_{22} & g_{33} & g_{12} & g_{13} & g_{23} \\ 1 & 0 & 0 & 1 & 1 & 0 & | & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & | & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & | & 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \quad 4.2.17$$

Substituting Eq. 4.2.15 and S_{I11} of Eq. 4.2.17 into Eq. 3.2.3 and applying

the conclusion of Lemma 2.2.1, the c-elements are

$$\begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix} = \begin{bmatrix} c_{11}+c_{12}+c_{13} \\ c_{22}+c_{12}+c_{23} \\ c_{33}+c_{13}+c_{23} \\ -c_{12} \\ -c_{13} \\ -c_{23} \end{bmatrix} = \begin{bmatrix} 42.88 \\ 139.91 \\ 321.26 \\ -12.96 \\ -21.57 \\ -66.89 \end{bmatrix}. \quad 4.2.18$$

Substituting Eq. 4.2.16 and S_{12} of Eq. 4.2.17 into Eq. 4.2.5, and applying the conclusion of Lemma 2.2.1, the g-elements are

$$\begin{bmatrix} G_{11} \\ G_{22} \\ G_{33} \\ G_{12} \\ G_{13} \\ G_{23} \end{bmatrix} = \begin{bmatrix} g_{11}+g_{12}+g_{13} \\ g_{22}+g_{12}+g_{23} \\ g_{33}+g_{13}+g_{23} \\ -g_{12} \\ -g_{13} \\ -g_{23} \end{bmatrix} = \begin{bmatrix} 478.1 \\ 743.7 \\ 4772.8 \\ -87.5 \\ -321.8 \\ -443.5 \end{bmatrix}. \quad 4.2.19$$

Therefore, the canonical RC graph G_4 for which A is the associated matrix is constructed in Fig. 4.2.3 and the element values are given in Eq. 4.2.18 and Eq. 4.2.19.

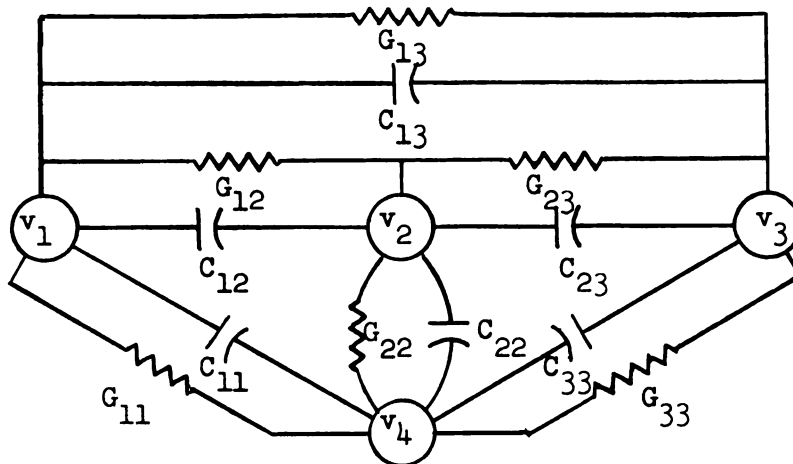


Figure 4.2.3 Synthesized RC graph G_4 of Examples 2 and 3.

Using the same method, a new set of element values can be calculated by letting

$$D_1 = \delta = \text{diag} (-1, 1, -1). \quad 4.2.20$$

Therefore Eq. 4.2.12 becomes

$$C = D_1 P' E D_1 = \begin{bmatrix} 8.39 & -12.92 & 21.57 \\ -12.92 & 60.10 & -66.89 \\ 21.57 & -66.89 & 232.80 \end{bmatrix} \quad 4.2.21$$

and Eq. 4.2.13 becomes

$$R = -D_1 P' \bigwedge E D_1 = \begin{bmatrix} 68.8 & -87.5 & 321.8 \\ -87.5 & 212.7 & -443.5 \\ 321.8 & -443.5 & 4007.5 \end{bmatrix}. \quad 4.2.22$$

The f seg matrix is given by Eq. 4.2.17, and the element values are calculated as in Eqs. 4.2.18 and 4.2.19. Correspondingly, the element values are

$$\begin{bmatrix} C_{11} \\ C_{22} \\ C_{33} \\ C_{12} \\ C_{13} \\ C_{23} \end{bmatrix} = \begin{bmatrix} 17.04 \\ -19.71 \\ 187.48 \\ 12.92 \\ -21.57 \\ 66.89 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} G_{11} \\ G_{22} \\ G_{33} \\ G_{12} \\ G_{13} \\ G_{23} \end{bmatrix} = \begin{bmatrix} 303.1 \\ -318.3 \\ 3885.8 \\ 87.5 \\ -321.8 \\ 443.5 \end{bmatrix}. \quad 4.2.23$$

Eq. 4.2.23 gives another set of element values for the graph G_4 of Fig. 4.2.3. Therefore, by comparing Eq. 4.2.23 with Eqs. 4.2.18 and 4.2.19, the number of negative elements in G_4 has been changed from six to four, and hence, changing the sign pattern of the P matrix has reduced the number of negative elements in the synthesized graph.

From the last two calculations, it is evident that the technique of Theorem 3.1.0 for decomposing A will never give a positive element graph. Hence, let A be factored by the method of Theorem 3.1.3. Therefore, $CA + R = 0$ is conveniently written as

$$\begin{bmatrix} f_{1j} \end{bmatrix} = \begin{bmatrix} c_{11} & -c_{12} & -c_{13} \\ -c_{12} & c_{22} & -c_{23} \\ -c_{13} & -c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 10 \\ 5 & 1 & -22 \end{bmatrix} + \begin{bmatrix} g_{11} & -g_{12} & -g_{13} \\ -g_{12} & g_{22} & -g_{23} \\ -g_{13} & -g_{23} & g_{33} \end{bmatrix} = 0 \quad 4.2.24$$

Therefore, from Eq. 4.2.24

$$\begin{bmatrix} f_{11} \\ f_{22} \\ f_{33} \\ f_{21} \\ f_{12} \\ f_{31} \\ f_{13} \\ f_{32} \\ f_{23} \end{bmatrix} = \begin{bmatrix} -8 & 0 & 0 & -2 & -5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -22 & 0 & 2 & 10 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 8 & -5 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 8 & -2 & 0 & 0 & 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & 10 & 22 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -10 & 0 & 2 & 22 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \\ g_{11} \\ g_{22} \\ g_{33} \\ g_{12} \\ g_{13} \\ g_{23} \end{bmatrix} = 0. \quad 4.2.25$$

By subtracting f_{21} from f_{12} , f_{31} from f_{13} and f_{32} from f_{23} , the variables g_{12} , g_{13} and g_{23} can be eliminated. Hence, Eq. 4.2.25 reduces to six equations in nine unknowns. Rewriting Eq. 4.2.25 gives

$$\begin{bmatrix} -8 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -22 & 0 & 0 & 1 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 & 0 & 0 \\ 0 & 10 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ g_{11} \\ g_{22} \\ g_{33} \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & -10 \\ -6 & -1 & 5 \\ 10 & 14 & 2 \\ 2 & 2 & 20 \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{13} \\ c_{23} \end{bmatrix}. \quad 4.2.26$$

Solving Eq. 4.2.26 gives

$$\begin{bmatrix} f_{1j} \end{bmatrix} = \begin{bmatrix} c_{11} & -c_{12} & -c_{13} \\ -c_{12} & c_{22} & -c_{23} \\ -c_{13} & -c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 10 \\ 5 & 1 & -22 \end{bmatrix} + \begin{bmatrix} g_{11} & -g_{12} & -g_{13} \\ -g_{12} & g_{22} & -g_{23} \\ -g_{13} & -g_{23} & g_{33} \end{bmatrix} = 0 \quad 4.2.24$$

Therefore, from Eq. 4.2.24

$$\begin{bmatrix} f_{11} \\ f_{22} \\ f_{33} \\ f_{21} \\ f_{12} \\ f_{31} \\ f_{13} \\ f_{32} \\ f_{23} \end{bmatrix} = \begin{bmatrix} -8 & 0 & 0 & -2 & -5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -22 & 0 & 2 & 10 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 8 & -5 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 5 & 0 & 8 & -2 & 0 & 0 & 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & 10 & 22 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -10 & 0 & 2 & 22 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \\ g_{11} \\ g_{22} \\ g_{33} \\ g_{12} \\ g_{13} \\ g_{23} \end{bmatrix} = 0. \quad 4.2.25$$

By subtracting f_{21} from f_{12} , f_{31} from f_{13} and f_{32} from f_{23} , the variables g_{12} , g_{13} and g_{23} can be eliminated. Hence, Eq. 4.2.25 reduces to six equations in nine unknowns. Rewriting Eq. 4.2.25 gives

$$\begin{bmatrix} -8 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -22 & 0 & 0 & 1 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 & 0 & 0 \\ 0 & 10 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ g_{11} \\ g_{22} \\ g_{33} \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & -10 \\ -6 & -1 & 5 \\ 10 & 14 & 2 \\ 2 & 2 & 20 \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{13} \\ c_{23} \end{bmatrix}. \quad 4.2.26$$

Solving Eq. 4.2.26 gives

$$\begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ g_{11} \\ g_{22} \\ g_{33} \end{bmatrix} = \frac{1}{48} \begin{bmatrix} 150 & 21 & -27 \\ 6 & -3 & 93 \\ 36 & 126 & 30 \\ 1296 & 408 & -216 \\ 108 & -6 & 242 \\ 792 & 2676 & 180 \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{13} \\ c_{23} \end{bmatrix}. \quad 4.2.27$$

By letting

$$\begin{bmatrix} c_{12} \\ c_{13} \\ c_{23} \end{bmatrix} = 48 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad 4.2.28$$

Eq. 4.2.27 becomes

$$\begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ g_{11} \\ g_{22} \\ g_{33} \end{bmatrix} = \begin{bmatrix} 96 \\ 192 \\ 96 \\ 864 \\ 592 \\ 1152 \end{bmatrix}. \quad 4.2.29$$

Solving f_{12} , f_{13} and f_{23} of Eq. 4.2.25 for g_{12} , g_{13} and g_{23} and evaluating with Eqs. 4.2.28 and 4.2.29 gives

$$\begin{bmatrix} g_{12} \\ g_{13} \\ g_{14} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 & -1 & 0 \\ -2 & 0 & 0 & 10 & 22 & 0 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix} = 48 \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}. \quad 4.2.30$$

From Eqs. 4.2.28, 4.2.29 and 4.2.30, the C and R matrix can be constructed. The f seg matrix is given by Eq. 4.2.17. The element values are calculated

as in Eqs. 4.2.18 and 4.2.19.¹ Therefore,

$$\begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix} = \begin{bmatrix} c_{11}-c_{12}-c_{13} \\ c_{22}-c_{12}-c_{23} \\ c_{33}-c_{13}-c_{23} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 48 \\ 0 \\ 48 \\ 0 \\ 96 \end{bmatrix}, \quad 4.2.31$$

and

$$\begin{bmatrix} G_{11} \\ G_{22} \\ G_{33} \\ G_{12} \\ G_{13} \\ G_{23} \end{bmatrix} = \begin{bmatrix} g_{11}-g_{12}-g_{13} \\ g_{22}-g_{12}-g_{23} \\ g_{33}-g_{13}-g_{23} \\ g_{12} \\ g_{13} \\ g_{23} \end{bmatrix} = \begin{bmatrix} 288 \\ 16 \\ 576 \\ 288 \\ 288 \\ 288 \end{bmatrix} \quad 4.2.32$$

are the element values of the RC graph G_4 of Fig. 4.2.3. Hence, from the decomposition technique of Theorem 3.1.2 a non-negative element graph is synthesized.

Example 3: In the previous examples, the coefficient matrix A was factored such that it could be realized as a non-negative element graph. In this example, a positive element RC graph cannot be synthesized directly from the coefficient matrix. However, if A is assumed to be the coefficient matrix corresponding to a maximum order path tree, a non-negative element RC graph is synthesized for which A is realized as the associated matrix.

Applying the decomposition technique of Theorem 3.1.2 to Example 3 of Sec. 4.1 gives

1. The sign differences between Eqs. 4.2.31, 4.2.32 and Eqs. 4.2.18, 4.2.19 occur because of the assumed sign pattern of Eq. 4.2.24.

$$\begin{bmatrix} f_{1j} \end{bmatrix} = \begin{bmatrix} c_{11} & -c_{12} & -c_{13} \\ -c_{12} & c_{22} & -c_{23} \\ -c_{13} & -c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} -8 & 4 & 4 \\ 0 & -12 & -4 \\ 0 & -4 & -12 \end{bmatrix} + \begin{bmatrix} g_{11} & -g_{12} & -g_{13} \\ -g_{12} & g_{22} & -g_{23} \\ -g_{13} & -g_{23} & g_{33} \end{bmatrix} = 0. \quad 4.2.33$$

Eq. 4.2.33 is rewritten as

$$\begin{bmatrix} f_{11} \\ f_{22} \\ f_{33} \\ f_{21} \\ f_{12} \\ f_{31} \\ f_{13} \\ f_{32} \\ f_{23} \end{bmatrix} = \begin{bmatrix} -8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & -4 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 & -4 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & 12 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 4 & 12 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -4 & 0 & -4 & 12 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -4 & 0 & -4 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \\ g_{11} \\ g_{22} \\ g_{33} \\ g_{12} \\ g_{23} \end{bmatrix} = 0. \quad 4.2.34$$

From f_{11} , f_{21} and f_{31} of Eq. 4.2.34

$$\begin{bmatrix} g_{11} \\ g_{12} \\ g_{13} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix}. \quad 4.2.35$$

Substituting Eq. 4.2.35 into the remaining five equations of Eq. 4.2.34 and solving for five of the variables in terms of the remaining four variables gives

$$\begin{bmatrix} c_{11} \\ c_{22} \\ g_{22} \\ g_{33} \\ g_{23} \end{bmatrix} = - \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 1 \\ 0 & -12 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & -4 & 0 & 12 \\ 0 & -4 & 0 & 4 \\ -12 & 0 & -4 & 4 \\ -4 & 0 & -4 & 12 \end{bmatrix} \begin{bmatrix} c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix}. \quad 4.2.36$$

After performing the indicated operations, Eq. 4.2.36 becomes

$$\begin{bmatrix} c_{11} \\ c_{22} \\ g_{22} \\ g_{33} \\ g_{23} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 16 & -8 & 12 & 32 \\ 12 & 0 & 4 & -4 \\ 4 & 0 & 4 & -12 \end{bmatrix} \begin{bmatrix} c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix} \quad 4.2.37$$

Eqs. 4.2.35 and 4.2.37 determine all of the entries in the C and the R matrices. Correspondingly, the element values can be calculated. However, from the first row of Eq. 4.2.37, there exists no positive values of c_{12} and c_{13} which will make c_{11} positive. Furthermore, if $c_{12} = c_{13} = 0$, then $c_{11} = 0$. This implies the C matrix is singular and hence, is not a satisfactory solution. It can then be concluded that no positive element solution exists for which A is realized as the associated matrix corresponding to a maximum order star tree. Therefore, assume A is the associated matrix of G_4 of Fig. 4.2.3 corresponding to the maximum order path tree of Fig. 4.2.4.

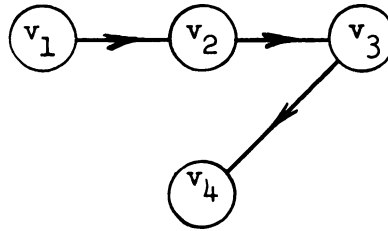


Figure 4.2.4 Path Tree T_j of G_4 .

The relationship between A and the associated matrix for a star tree is given by Eq. 2.3.19. Therefore,

$$A_s = Q_{sj}^{-1} A Q_{sj}' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -8 & 4 & 4 \\ 0 & -12 & -4 \\ 0 & -4 & -12 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & -4 & 0 \\ 0 & -16 & 0 \\ 0 & -4 & -8 \end{bmatrix} \quad 4.2.38$$

where Q_{sj} is given in Lemma 2.3.2.

Factoring A_s by the method of Theorem 3.1.3 gives

$$\begin{bmatrix} f_{1j} \end{bmatrix} = \begin{bmatrix} c_{11} & -c_{12} & -c_{13} \\ -c_{12} & c_{22} & -c_{23} \\ -c_{13} & -c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} -8 & -4 & 0 \\ 0 & -16 & 0 \\ 0 & -4 & -8 \end{bmatrix} + \begin{bmatrix} g_{11} & -g_{12} & -g_{13} \\ -g_{12} & g_{22} & -g_{23} \\ -g_{13} & -g_{23} & g_{33} \end{bmatrix} = 0 \quad 4.2.39$$

from which

$$\begin{bmatrix} f_{11} \\ f_{22} \\ f_{33} \\ f_{21} \\ f_{12} \\ f_{31} \\ f_{13} \\ f_{32} \\ f_{23} \end{bmatrix} = \begin{bmatrix} -8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -16 & 0 & 4 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -4 & 0 & 0 & 16 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -4 & 0 & 4 & 16 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \\ g_{11} \\ g_{22} \\ g_{33} \\ g_{12} \\ g_{13} \\ g_{23} \end{bmatrix} = 0. \quad 4.2.40$$

From f_{11} , f_{33} , f_{21} , f_{31} , f_{13} and f_{32} of Eq. 4.2.40

$$\begin{bmatrix} g_{11} \\ g_{33} \\ g_{12} \\ g_{13} \\ g_{23} \end{bmatrix} = 8 \begin{bmatrix} c_{11} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix}. \quad 4.2.41$$

Substituting Eq. 4.2.41 into the remaining three equations of Eq. 4.2.39 and solving for three variables in terms of the remaining four variables gives

$$\begin{bmatrix} c_{11} \\ c_{33} \\ g_{22} \end{bmatrix} = - \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 8 & 4 & 0 \\ 0 & 0 & 4 & 8 \\ -16 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} c_{22} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix}. \quad 4.2.42$$

After performing the indicated operations, Eq. 4.2.42 becomes

$$\begin{bmatrix} c_{11} \\ c_{33} \\ g_{22} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 16 & -4 & -4 & 0 \end{bmatrix} \begin{bmatrix} c_{22} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix} \quad 4.2.43$$

By letting

$$\begin{bmatrix} c_{22} \\ c_{12} \\ c_{13} \\ c_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad 4.2.44$$

Eq. 4.2.43 becomes

$$\begin{bmatrix} c_{11} \\ c_{33} \\ g_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 12 \end{bmatrix}, \quad 4.2.45$$

and Eq. 4.2.41 becomes

$$\begin{bmatrix} g_{11} \\ g_{33} \\ g_{12} \\ g_{13} \\ g_{23} \end{bmatrix} = \begin{bmatrix} 8 \\ 24 \\ 0 \\ 8 \\ 8 \end{bmatrix}. \quad 4.2.46$$

Eqs. 4.2.44, 4.2.45 and 4.2.46 give the entries in the C and R matrices. The element values of the RC graph G_4 of Fig. 4.2.3 are calculated symbolically in Eqs. 4.2.31 and 4.2.32. Hence,

$$\begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{12} \\ c_{13} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} G_{11} \\ G_{22} \\ G_{33} \\ G_{12} \\ G_{13} \\ G_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 8 \\ 0 \\ 8 \\ 8 \end{bmatrix}. \quad 4.2.47$$

From Eq. 4.2.47, all elements are either positive or zero. Therefore, by the use of a similarity transformation of A, a positive element graph is synthesized.

Example 4: From Ex. 4 of Sect. 4.1, A can only be realized as the associated matrix of an RL graph. Since A is symmetric, a satisfactory factoring of A is

$$L = I \text{ and } R = -A. \quad 4.2.48$$

The number of vertices of a canonical RL graph for which A can be the associated matrix is given by Lemma 2.2.1 as

$$3 \leq v \leq 6. \quad 4.2.49$$

Therefore, from Eq. 3.3.0

$$S_3^* = \begin{matrix} & \begin{matrix} L_{11} & L_{22} & L_{12} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \end{matrix} \quad 4.2.50$$

and

$$S_4^* = \begin{matrix} & \begin{matrix} L_{11} & L_{22} & L_{33} & L_{12} & L_{13} & L_{23} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \end{matrix}. \quad 4.2.51$$

By Def. 3.3.0, σ_v is a matrix composed of any three columns of either S_3^* , S_4^* , S_5^* or S_6^* . Let $\sigma_v = S_3^*$ and consider

$$\sigma_v R \sigma_v' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -5 & 10 \end{bmatrix}. \quad 4.2.52$$

The determinate of Eq. 4.3.4 does not vanish, and hence by Lemma 3.3.1, R is factorable into the triple product of Eq. 2.3.40. Therefore, carrying out the operations of Eq. 3.3.3 gives

$$S_{I22} G_s S_{I22}' = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \quad 4.2.52$$

where S_{I22} is given by Eq. 3.3.5. From Eqs. 4.2.48 and 4.2.52, A satisfies the hypothesis of Theorem 3.3.1 and hence there exists a canonical RL graph G_3 with non-negative elements for which A is the associated matrix. The f seg matrix is

$$S_s = \left[S_{I22} \mid \sigma_v \right] = \left[\begin{array}{ccc|ccc} G_{11} & G_{22} & G_{12} & L_{11} & L_{22} & L_{12} \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{array} \right]. \quad 4.2.53$$

The g -elements are calculated from Eq. 4.2.42 by applying the conclusion of Lemma 2.2.1. Hence

$$\begin{bmatrix} G_{11} \\ G_{22} \\ G_{12} \end{bmatrix} = \begin{bmatrix} g_{11} + g_{12} \\ g_{22} + g_{12} \\ -g_{12} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}. \quad 4.2.54$$

The l -elements are obtained directly from Eq. 4.2.48. Therefore, from Eqs. 4.2.48, 4.2.53 and 4.2.54, a canonical RL graph G_3 of Fig. 4.2.5 is constructed

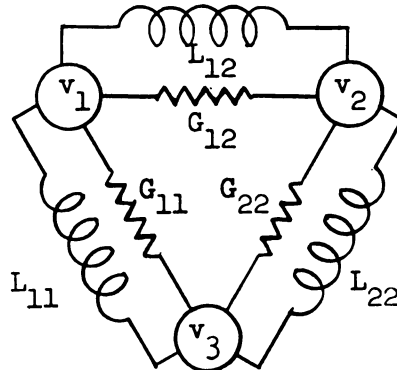


Figure 4.2.5 Synthesized RL Graph G_3 of Exs. 4 and 5.

Other solutions to this example can be attempted by returning to Eq. 4.2.52 and using different σ_v 's obtained from S_4^* of Eq. 4.3.4. However, there are no other solutions since any σ_v composed of the columns of S_4^* Eq. 4.2.52 becomes a singular matrix and hence, Eq. 4.2.52 cannot be solved for $S_{I22} G_s S_{I22}'$. A similar statement is true for all $\sigma_v \in S_5^*$ and $\sigma_v \in S_6^*$.

Example 5: From Ex. 5 of Sect 4.1, A is quasisymmetric. Using the factoring technique of Theorem 3.1.2, a solution to $LA + R = 0$ is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, R = \begin{bmatrix} 24 & 7 & 11 \\ 7 & 23 & 7 \\ 11 & 7 & 24 \end{bmatrix}. \quad 4.2.55$$

Since the order of A is three, as in Ex. 4, then the bounds on the number of vertices of the RL graph G_v is given by Eq. 4.2.49 and correspondingly S_3^* and S_4^* are given by Eqs. 4.2.50 and 4.2.51 respectively. Let $\sigma_v = S_3^*$ and consider

$$\sigma_v R \sigma_v' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 24 & 7 & 11 \\ 7 & 23 & 7 \\ 11 & 7 & 24 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 70 & -21 \\ -21 & 30 \end{bmatrix}. \quad 4.2.56$$

The determinate of Eq. 4.2.56 does not vanish, hence by Lemma 3.3.1, R is factorable into the triple product of Eq. 2.3.40. Therefore carrying out the operations of Eq. 3.3.3 gives

$$S_{I22} G_s S_{I22}' = \frac{1}{1659} \begin{bmatrix} 107 & -97 \\ -97 & 227 \end{bmatrix} \quad 4.2.57$$

where S_{I22} is given in Eq. 3.3.5. The f seg matrix is the same as in Ex. 4 and is given by Eq. 4.2.53. The g-element values are calculated symbolically in Eq. 4.2.54 and the l-elements are known from Eq. 4.2.55. Hence,

$$\begin{bmatrix} G_{11} \\ G_{22} \\ G_{12} \end{bmatrix} = \frac{1}{1659} \begin{bmatrix} 10 \\ 110 \\ 97 \end{bmatrix}, \begin{bmatrix} L_{11} \\ L_{22} \\ L_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \quad 4.2.58$$

Eq. 4.2.58 gives the element values for the canonical RL graph G_3 of Fig. 4.2.5.

Another solution of Ex. 5 is obtained by returning to Eq. 4.2.56 and using a new $\sigma_v \in S_4^*$. Therefore, let

$$\sigma_v = I,$$

and without repeating all of the details, there exists a canonical RL graph G_4 such that A is the associated matrix. The element values for G_4 of Fig. 4.2.6 are

$$\begin{bmatrix} G_{11} \\ G_{22} \\ G_{33} \\ G_{12} \\ G_{13} \\ G_{23} \end{bmatrix} = 102 \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} L_{11} \\ L_{22} \\ L_{33} \\ L_{12} \\ L_{13} \\ L_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad 4.2.59$$

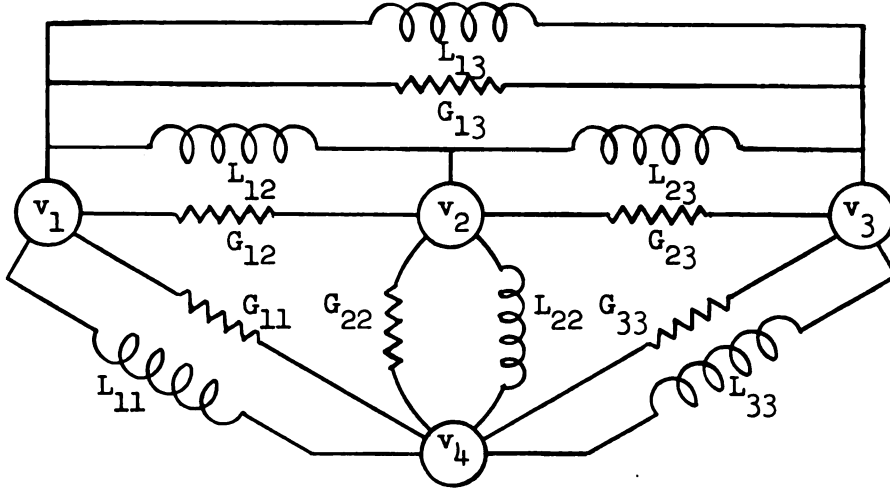


Figure 4.2.6 Synthesized Graph G_4 of Example 5.

Other RL graphs can be synthesized by considering a new $\sigma_v \in S_4^*$, or S_5^* or S_6^* . This procedure, of course, can be repeated for all $\sigma_v \in S_{4,3} \cup S_{5,3} \cup S_{6,3}$. Furthermore, if desired, A is realizable as the associated matrix of a canonical RC graph G_4 since A is nonsingular bisymmetric.

4.3 Conclusion.

In this chapter, the synthesis method is described and illustrated in the flow charts of Figs. 4.1.0, 4.2.0 and 4.2.1. Section 4.1 gives the conditions under which a coefficient matrix can be realized as the associated matrix of a real linear bielement system. Section 4.2 describes the synthesis procedure whereby an acceptable coefficient matrix is decomposed and a RC or RL graph is constructed. Five examples are also given.

In Ex. 1, the decomposition technique of Theorem 3.1.0 is illustrated. The method immediately synthesized a positive element graph. The same decomposition technique is used in Ex. 2. However, it is shown that only graphs with some negative elements can be synthesized. The linear decomposition technique of Theorem 3.1.2 is then applied to Ex. 2, with the results that a positive element graph is synthesized. In Ex. 3 the eigenvalues of A are negative and nondistinct. A is shown to be bisymmetric by the decomposition technique of Theorem 3.1.2. However, a positive element graph cannot be synthesized from A . A is then assumed to be the associated matrix corresponding to a maximum order path tree. The appropriate similarity transformation is made on A and the new matrix is realized as the associated matrix corresponding to a star tree of a positive element graph. In Ex. 4, a singular matrix is realized as the associated matrix of a RL graph. In Ex. 5, two RL graphs are synthesized from a nonsingular left quasisymmetric coefficient matrix.

Ideally, in the decomposition of a given matrix into the product of two symmetric matrices, it would be desirable to have a closed form solution to the nonlinear algebraic system of Eq. 3.1.5 in terms of n of the variables. Then by using conditions 1., 2. and 3. of Theorem 3.2.2 or conditions 2. and 3. of Theorem 3.3.1 as bounds on the solutions, an optimum decomposition should be attainable. Because of the inherent difficulty in the solution of nonlinear algebraic equations, the decomposition technique of Theorem 3.1.1 was avoided in all of the examples.

V. CONCLUSION

5.0 Discussion of Results

In the preceding chapters a new concept of the classical synthesis problem is developed. The problem involves the realization of time domain models of the form of Eq. 1.0.0 as real, linear, bielement systems. In this sense, the synthesized graph can be thought of as a real time model of the process described by Eq. 1.0.0.

Almost all classical synthesis techniques employ the same basic approach in solving the synthesis problem. That is, each technique assumes a fundamental topology for the graph and then generates conditions on the mathematics such that the impedance, admittance or transfer function is realizable. Examples of this method are the Foster, Brune, Darlington, Bott-Duffin, image parameter and etc. methods (6, 7, 8). Guillemin⁽²⁵⁾ has stated that one of the shortcomings of this approach is the rigidity of the assumed topology of the graph. It is felt that the synthesis technique of Chapter III has overcome this objection in part, without abandoning the basic approach entirely. This is accomplished by assuming the canonical topology for the graph to be the union of complete graphs, each complete graph being composed entirely of one type of element. The use of the complete graph gives a generalized topology to the synthesis problem since all other topological structures such as paths, pis, tees, ladders and lattices are but special cases of the complete graph.

In Chapter II, arbitrary real linear bielement systems are classified according to a subgraph of the system from which the formulation tree is selected. The fundamental properties of class 1 systems are then investigated in detail. Class 1 systems are found to be reducible to a canonical form. The associated matrix corresponding to any maximum order star tree of a canonical graph is defined to be the canonical associated matrix. The sign pattern and the magnitude of the entries in the canonical associated matrix are then calculated. All other associated matrices of a class 1 system graph are related to the canonical associated matrix by a similarity transformation. For the RC graph, this similarity transformation is found from the incidence matrix. For the RL graph, the similarity transformation

reduces to the identity transformation and hence, the associated matrix is independent of the formulation tree.

In Chapter III, a necessary condition and three techniques are given for the decomposition of an arbitrary real square matrix into its bisymmetric form. A test is also developed to determine when a bisymmetric matrix is quasisymmetric. From these results necessary and sufficient conditions are also developed to guarantee the synthesized graph has only non-negative elements. Theorem 3.2.0 points out the well known fact that the eigenvalues of real linear bielement systems are real, correspondingly, if the eigenvalues are negative, these systems can be classified by their transient solutions as overdamped.

In Chapter IV, a synthesis method is developed from which a digital computer program can be written. It is conceivable with the use of dynamic programming techniques ⁽²³⁾, that an optimum solution to the synthesis problem can be obtained.

Since the synthesized graph can be thought of as a real time model of the process described by Eq. 1.0.0, one application of this technique is in the area of adaptive control systems.

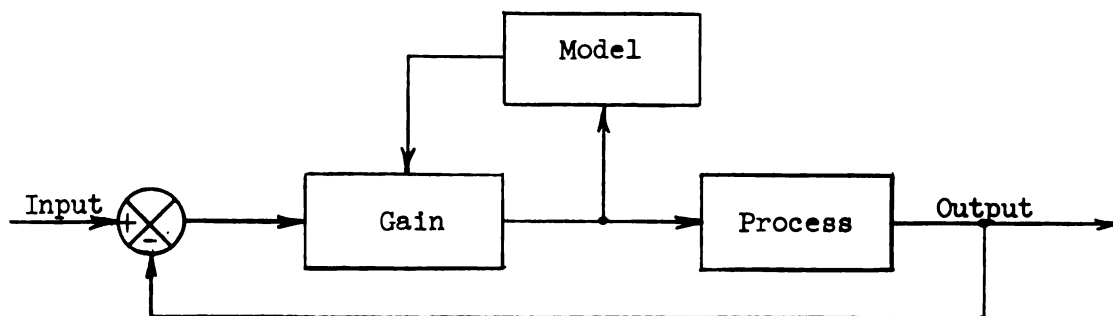


Figure 5.0.1 Elemental Adaptive System.

Consider the adaptive control system of Figure 5.0.0 ⁽²⁴⁾. The dynamics of the process are adaptively controlled by the use of a "model" of the system. One method of obtaining this "model" for suitable processes is as follows. First, the process is analyzed and a time domain model of the system in the form of Eq. 1.0.0 is obtained. Then this time domain model is realized by the synthesis techniques of Chapter III. If the process is

nonlinear, then a "model" can be synthesized by several linear approximations over the operating range of the process.

5.1 Additional Problems.

As with any research many additional problems arise which warrant further investigation. The following are five problems which fit in this category.

In Chapter I, restrictions were made on the graph to exclude all drivers and to assume all initial conditions to be zero. These restrictions should be removed and their effect on the synthesis procedure investigated. Brown ⁽²⁾ has given the preliminary analysis for such an investigation.

In Chapter II, only class 1 systems were considered. It was postulated in the conclusion of Chapter II that all other classes of RC and RL systems have similar properties to class 1 systems. Further investigation is needed in this area. Ideally, it would be convenient to have a canonical graph and associated matrix for each class of systems.

The flow chart of Fig. 4.1.0 shows that only coefficient matrices that are nonsingular left quasisymmetric can be realized as either a RC or RL network. Therefore a closer investigation of nonsingular left quasisymmetric matrices should give a new dimension to the duality concept of graphs.

Another interesting problem is the relationship between classical synthesis and the synthesis of graphs from time domain models. Some research in the area has already been done ⁽¹¹⁾.

Finally, for the "modeling" of adaptive control systems it would be extremely useful to develop a synthesis technique for realizing time domain models of RLC systems.

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APPENDIX A

THEOREMS AND DEFINITIONS FROM

REFERENCES

Definition A.1: (14, p. 256)¹ A real symmetric matrix A is called a positive definite matrix if and only if the corresponding quadratic form $X'AX$ is positive definite.

Definition A.2: (16, p. 261) A is similar to B if and only if there exists a matrix P such that $B = PAP^{-1}$, with P nonsingular.

Theorem A.1: (2, p. 505) Let G be a connected graph containing R -, L -, C -, N_e - and N_h -elements such that there are no all N_e -element circuits and no all N_h -element segs. Then there exists a tree T of G such that

1. all N_e -elements plus as many C -elements as possible are branches of T , and
2. all N_h -elements plus as many L -elements as possible are chords of T .

Theorem A.2 : (2, p. 506) For any graph G as indicated in Theorem 2 and tree T satisfying (1) and (2) of Theorem 2, the f -circuit and f -seg equations, respectively, are:

$$\begin{bmatrix} B_{11} & B_{12} & 0 & B_{14} & U & 0 & 0 & 0 \\ B_{21} & B_{22} & B_{23} & B_{24} & 0 & U & 0 & 0 \\ B_{31} & 0 & 0 & B_{34} & 0 & 0 & U & 0 \\ B_{41} & B_{42} & B_{43} & B_{44} & 0 & 0 & 0 & U \end{bmatrix} \begin{bmatrix} v_e \\ v_{R1} \\ v_{L1} \\ v_{C1} \\ v_{R2} \\ v_{L2} \\ v_{C2} \\ v_h \end{bmatrix} = 0$$

1. Parentheses give the reference and page number.

$$\begin{bmatrix}
 U & 0 & 0 & 0 & S_{11} & S_{12} & S_{13} & S_{14} \\
 0 & U & 0 & 0 & S_{21} & S_{22} & 0 & S_{24} \\
 0 & 0 & U & 0 & 0 & S_{32} & 0 & S_{34} \\
 0 & 0 & 0 & U & S_{41} & S_{42} & S_{43} & S_{44}
 \end{bmatrix}
 \begin{bmatrix}
 I_e \\
 I_{R1} \\
 I_{L1} \\
 I_{C1} \\
 I_{R2} \\
 I_{L2} \\
 I_{C2} \\
 I_h
 \end{bmatrix}
 = 0$$

where the columns of the unit matrix of the f-circuit (f-seg) equations correspond to the elements of the cotree (tree) of G and the columns of the non-unit matrix of the f-circuit (f-seg) equations correspond to the elements of the tree (cotree) of G. The subscripts e, R1 or R2, L1 or L2, C1 or C2, h indicate the V or I variables associated with N_e -, R-, L-, C-, N_h -elements, respectively.

Theorem A.3: (14, p. 257) The diagonal matrix $D(\lambda_1, \lambda_2, \dots, \lambda_n)$ is positive definite if and only if all the λ 's are positive.

Theorem A.4: (21, p. 142) Let P be a real matrix of order (r,n) and rank r ($\leq n$, naturally). Then TPT' is positive definite. If P is positive semidefinite, then TPT' is positive definite or semidefinite.

Theorem A.5: (14, p.257) The real matrix A is symmetric and positive definite if and only if A^{-1} exists and is symmetric and positive definite.

Theorem A.6: (27, p. 148) Let $A = [a_{ij}]$ is an arbitrary matrix. If A cannot be transformed by a symmetrical permutation into the form $\begin{bmatrix} P & Q \\ 0 & R \end{bmatrix}$ where P and R are square and 0 is zero matrix and if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \text{for all } i$$

and $|a_{ii}| > \sum_{j=1} |a_{ij}| \quad \text{for at least one } i$

then A is nonsingular.

Theorem A.7: (16, p. 273) A real symmetric matrix A is positive definite if and only if there exists a real nonsingular matrix P such that $A = PP'$.

Theorem A.8: (23, p. 147) (Implicit function theorem). Let $\bar{f} = (f_1, \dots, f_n)$ be a vector valued function defined on an open set S in E_{n+k} with values in E_n . Suppose $\bar{f} \in C'$ on S . Let $(\bar{x}_0, \bar{t}_0) = 0$ and for which the $n \times n$ determinate $\det [D_j f_i(\bar{x}_0; \bar{t}_0)] \neq 0$. Then there exists a k -dimensional neighborhood T_0 of $\bar{t}_0$ and one, and only one, vector-valued function $\bar{g}$, defined on T_0 and having values in E_n , such that

- i. $\bar{g} \in C'$ on T_0 ,
- ii. $\bar{g}(\bar{t}_0) = \bar{x}_0$
- iii. $\bar{f}(\bar{g}(\bar{t}); \bar{t}) = 0$ for every $\bar{t}$ in T_0 .

Theorem A.9: (14, p.118) A necessary and sufficient condition that the system of m homogeneous linear equations in n unknowns,

$$\sum_{j=1}^n a_{ij} x_j = 0, \quad i = 1, 2, \dots, m$$

have a nontrivial solution is that its coefficient matrix have a rank less than the number of unknowns.

Theorem A.10: (14, p. 228) If A is a real symmetric matrix, there exists an orthogonal matrix U such that $U'AU$ is a diagonal matrix whose diagonal elements are the characteristic roots of A .

Theorem A.11: (14, p. 266) The roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of the equation $\det [A - \lambda B] = 0$, where A and B are symmetric and B is positive definite, are all real.

Theorem A.12: (16, p. 313) Similar matrices have the same characteristic polynomial.

Theorem A.13: (14, p. 255) If $X'AX$ is positive definite, then $\det A > 0$.

Theorem A.14: (14, p. 58) The determinant of the product of two square matrices of order n is equal to the product of their determinants.



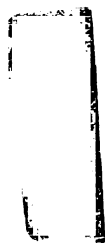
Theorem A.15: (14, p. 65) A square matrix A has an inverse if and only if $\det A \neq 0$.

Theorem A.16: (22, p. 133) If the characteristic roots of a matrix A are distinct, then A is similar to a diagonal matrix.

Theorem A.17: (4, p. 4-50) Let A_1 and A_2 represent any two cut-set matrices of a given connected graph G . Then there exists nonsingular transformations relating A_1 and A_2 .

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