CONTRIBUTIONS TO THE THEORY OF RESTRICTED POLYNOMIAL AND RATIONAL APPROXIMATION

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THESIS

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ABSTRACT

CONTRIBUTIONS TO THE THEORY
OF RESTRICTED
POLYNOMIAL AND RATIONAL APPROXIMATION

By

Kathleen Ann Taylor

In Chapter I we consider the problem of approximating functions continuous on a compact metric space S by elements of a linear subspace V of $C(S)$ in the following manner:

1. J, K, and L are compact subsets of S.

2. Two prescribed functions ℓ and μ are given and are assumed to be continuous on L and J respectively.

3. We allow as approximants the subset V_1 of V whose elements v are such that $v(x) \leq \mu(x)$ for all $x \in J$ and $v(x) \geq \ell(x)$ for all $x \in L$.

4. For a given $f \in C(S)$, a best approximation $v_o \in V_1$ will be such that

$$\max_{x \in K} |v_o(x) - f(x)| = \min_{v \in V_1} \{\max_{x \in K} |v(x) - f(x)|\}.$$

The existence of best approximations follows from the usual compactness arguments. For functions $f \in C(S)$ such that $\ell(x) \leq f(x)$ for all $x \in L$ and $\mu(x) \geq f(x)$ for all $x \in J$, best approximations can be characterized in terms of a linear functional based on the set of critical points. There is a unique best approximation for each such

f if and only if V is a Haar subspace. A Remes-type algorithm is given to construct such best approximations.

We let V be a set of rational functions in Chapter II and consider the same problem. If we properly restrict the functions ℓ and μ and the sets L and J we obtain existence theorems, and for suitable f's we again characterize best approximations in terms of a linear functional based on the set of critical points. In special situations we have uniqueness of the best approximation.

An expository presentation of the doctoral thesis of Karl-Heinz Hoffmann is included (Chapter III) because it presents a very general theory of restricted approximations. The relationship of his work to the results presented here is discussed.

CONTRIBUTIONS TO THE THEORY OF RESTRICTED POLYNOMIAL AND RATIONAL APPROXIMATION

By

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INTRODUCTION

This paper will consider the approximation of functions in a normed linear space by functions from some subset of that space. Let $C(S)$ be the linear space of continuous real-valued functions on a space S normed with the uniform norm. Let V be a subspace of $C(S)$ and V_1 a subset of V whose elements satisfy certain prescribed conditions. We shall examine questions of existence, characterization, uniqueness, and computation of a best uniform approximation to a given function f in $C(S)$ by elements of V_1.

The problems considered in Chapters I and II are a combined generalization of work done by P.J. Laurent [13] and G.D. Taylor [21], [22], [23], [24].

In the paper by Laurent, S is the union of two compact spaces K and L and V is a finite dimensional subspace of $C(S)$. For a fixed $f \in C(S)$, V_1 is the subset of V whose elements are less than or equal to f on L. In this setting, Laurent considers the problem of approximating f by elements of V_1 where the error in the approximation of f by $v \in V_1$ is defined to be the maximum of $|f(x) - v(x)|$ on K.

In the work by Taylor, S is a compact subset of the real line and V is a finite dimensional Haar subspace (the definition of a Haar subspace will be given later) of $C(S)$.

For two fixed extended real-valued functions ℓ and μ defined on S, V_1 is the subset of V whose elements are less than or equal to μ and greater than or equal to ℓ on S. The error in approximating a given $f \in C(S)$ by $v \in V_1$ is defined to be the maximum of $|f(x) - v(x)|$ on S.

This paper will consider S to be the union of three compact spaces J, K, and L, and V a finite dimensional subspace of $C(S)$; ℓ and μ will be fixed real-valued functions with ℓ continuous on L and μ continuous on J. V_1 will consist of elements of V which are greater than or equal to ℓ on L and less than or equal to μ on J. For a function $f \in C(S)$, the error in approximation by $v \in V_1$ will be defined to be maximum $|f(x) - v(x)|$ on K. In Chapter I we shall assume $\ell(x) < \mu(x)$ for all $x \in J \cap L$; in Chapter II we shall consider what happens when $\ell(x_j) = \mu(x_j)$ for $j = 1,\dots,n$. The set V may be generalized polynomials or rational functions.

Chapter III is an expository presentation of the work of K.H. Hoffmann [7] on non-linear Chebyshev approximation with side conditions. Some remarks are made concerning the application of Hoffmann's work to the problems considered in Chapters I and II. Also some additional comments are made concerning his uniqueness results.

CHAPTER I

CHEBYSHEV APPROXIMATION WITH GENERALIZED POLYNOMIALS HAVING RESTRICTED RANGES: INEQUALITY CASE

Section 1: Basic Definitions and Existence Theorem.

Let J, K, L be three (not necessarily disjoint) compact subsets of a metric space and let $S = J \cup K \cup L$, and assume K contains at least n points. By $C(S)$ we shall mean the space of continuous real-valued functions f with the topology induced by the Chebyshev or uniform norm $\|\cdot\|_\infty$, i.e. for $f \in C(S)$

$$\|f\|_\infty = \max_{x \in S} |f(x)|.$$

We shall denote by $\|\cdot\|_K$ the seminorm on S as follows: for $f \in C(S)$

$$\|f\|_K = \max_{x \in K} |f(x)|.$$

Let $w_1, \ldots, w_n$ be linearly independent elements of $C(S)$ and let V be the subspace of $C(S)$ generated by $w_1, \ldots, w_n$. Let ℓ and μ be real-valued functions continuous on L and J respectively. In this chapter, we shall assume

$$\ell(x) < \mu(x) \quad \text{for all} \quad x \in J \cap L.$$

Let V_1 be the subset of V consisting of those elements bounded above by μ at each point of J and below by ℓ at each point of L, i.e.

$$V_1 = \{v \in V\colon v(x) \le \mu(x) \text{ for all } x \in J \text{ and } v(x) \ge \ell(x) \text{ for all } x \in L\}.$$

We shall assume V_1 is non-empty. For a given real-valued function f on S, we wish to find an element of V_1 which is closest to f in the sense of minimizing the semi-norm $\|\cdot\|_K$. That is, we want $v^* \in V_1$ such that

$$\|f - v^*\|_K = \inf_{v \in V_1} \|f - v\|_K \equiv \rho.$$

If such a v^* exists, it will be called a best restricted approximation to f on K.

In the work done by G.D. Taylor [23] concerning approximation by functions with restricted ranges on a compact subset X of the real interval $[a,b]$, the functions ℓ and μ were assumed to be extended real-valued functions with the following restrictions:

(i) ℓ may assume the value $-\infty$, but not $+\infty$.

(ii) μ may assume the value $+\infty$, but not $-\infty$.

(iii) $X_{-\infty} = \{x\colon \ell(x) = -\infty\}$ and $X_{+\infty} = \{x\colon \mu(x) = +\infty\}$ are open subsets of X.

(iv) ℓ is continuous on $X \sim X_{-\infty}$ and μ is continuous on $X \sim X_{+\infty}$.

In the present setting, no generality is lost by assuming that ℓ and μ are continuous on L and J respectively. Indeed let ℓ and μ be extended real-valued functions defined on L and J, respectively, satisfying the above conditions; define h

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$L' = L \sim X_{-\infty}$ and $J' = J \sim X_{+\infty}$. Then ℓ and μ are continuous on the compact sets L' and J'. It is clear that the subset V_1 corresponding to ℓ and μ on L and J is the same as V_1' corresponding to ℓ and μ on L' and J'. For convenience of notation we let $L = L'$ and $J = J'$.

Theorem 1.1: (Existence) Let $f \in C(S)$ be given. Then a best restricted approximation to f on K exists.

Proof: Let $v \in V_1$. Then $\|f - v\|_K = \rho_1 \geq \rho$. If w is any element of V_1 such that $\|v - w\|_K > 2\rho_1$, then

$$\|f - w\|_K \geq \|v - w\|_K - \|f - v\|_K > \rho_1 \geq \rho.$$

Therefore we need only consider approximation by elements of the set B where

$$B = \{w \in V_1: \ \|v - w\|_K \leq 2\rho_1\}.$$

That is, $\rho = \inf\limits_{w \in B} \|f - w\|_K$. But B is a closed, bounded subset of a finite dimensional normed linear space and therefore compact. Since the seminorm $\|\cdot\|_K$ is a continuous function on B, the infimum is attained and a best restricted approximation exists. ∎

Section 2: Kolmogorov-type Characterization Theorem.

In the classical problem of Chebyshev approximation of a continuous function f on a compact set X by elements of a linear subspace V of $C(X)$, the best approximation v^* is characterized by properties of the set of extreme points, i.e. the set

$$E = \{x \in X: \ |f(x) - v^*(x)| = \|f - v^*\|_\infty\}.$$

In 1948, Kolmogorov [12] proved that $v^* \in V$ was a best approximation to f if and only if

$$\min_{x \in E} (f(x) - v^*(x))v(x) \le 0$$

for all $v \in V$. By altering the set E, G.D. Taylor [21] characterized the best approximation to f in the case of restricted approximation on a compact set. And P.J. Laurent [13] was also able to characterize one-sided approximation on two compact sets by this property. We show that similar modifications in the set E make possible a characterization of the best restricted approximation considered here.

For any $v \in V_1$, define the function $e_v \in C(S)$ by

$$e_v(x) = f(x) - v(x) \quad \text{for all} \quad x \in S.$$

Now define the following sets of critical points for e_v:

$$E_v^+ = \{x \in K: \ e_v(x) = \|e_v\|_K\}.$$

$$E_v^- = \{x \in K: \ e_v(x) = -\|e_v\|_K\}.$$

$$G_v^+ = \{x \in L: \ v(x) = \ell(x)\}.$$

$$G_v^- = \{x \in J: \ v(x) = \mu(x)\}.$$

Let $E_v = E_v^+ \cup E_v^-$ and $G_v = G_v^+ \cup G_v^-$. Using $D_v = E_v \cup G_v$ as our new set of "critical points", we can obtain the following Kolmogorov-type theorem.

<u>Theorem 1.2</u>: Let $f \in C(S) \sim V_1$ and $v^* \in V_1$. Then v^* is not a best restricted approximation to f if there exists a function $v \in V$ such that

$$v(x) > 0 \quad \text{for all} \quad x \in E^+_{v^*} \cup G^+_{v^*}$$

and

$$v(x) < 0 \quad \text{for all} \quad x \in E^-_{v^*} \cup G^-_{v^*}.$$

Now suppose there exists a $v_o \in V$ with

$$v_o(x) > \ell(x) \quad \text{for all} \quad x \in L$$

and

$$v_o(x) < \mu(x) \quad \text{for all} \quad x \in J,$$

then the converse is also true.

<u>Proof</u>: If such a v exists, we shall show that there is a positive number δ such that $v^* + \delta v$ is a better restricted approximation to f. Let $\|v\|_\infty = M$. For $x \in E_{v^*}$,

$$|f(x) - v^*(x)| = \|f - v^*\|_K.$$

Let $\delta_o = \frac{\|f - v^*\|_K}{2M}$ and define the sets

$$O_1 = \{x \in S: \ f(x) - v^*(x) > \frac{\|f-v^*\|_K}{2} \ \text{and} \ v(x) > 0\},$$

$$O_2 = \{x \in S: \ f(x) - v^*(x) < -\frac{\|f-v^*\|_K}{2} \ \text{and} \ v(x) < 0\}.$$

For $\delta \le \delta_o$ and $y \in O_1$,

$$0 \le f(y) - (v^* + \delta v)(y) = f(y) - v^*(y) - \delta v(y) < \|f-v^*\|_K.$$

Similarly, for $\delta \leq \delta_o$ and $y \in O_2$

$$-\|f - v^*\|_K < f(y) - v^*(y) - \delta v(y) = f(y) - (v^* + \delta v)(y) \leq 0.$$

Now $O = O_1 \cup O_2$ is an open set and $E_{v^*} \subseteq O$. Thus $K \sim O$ is compact and there is a number $\epsilon_1 > 0$ such that for $y \in K \sim O$,

$$|f(y) - v^*(y)| \leq \|f - v^*\|_K - \epsilon_1.$$

If we choose $\delta_1 = \frac{\epsilon_1}{2M}$, then for $\delta \leq \delta_1$ and $y \in K \sim O$

$$|f(y) - v^*(y) - \delta v(y)| \leq \|f - v^*\|_K - \epsilon_1 + \frac{\epsilon_1}{2} < \|f - v^*\|_K.$$

So for $\delta \leq \frac{\epsilon_1}{2M}$,

$$\|f - (v^* + \delta v)\|_K < \|f - v^*\|_K.$$

It remains to be shown that δ can be chosen so that $v^* + \delta v \in V_1$.

For $x \in G^+_{v^*}$, $v(x) > 0$. By the compactness of $G^+_{v^*}$, there is an open set U containing $G^+_{v^*}$ such that for $y \in U$

$$v(y) > 0,$$

and

$$v^*(y) + \delta v(y) \geq \ell(y) + \delta v(y) > \ell(y).$$

Since $L \sim U$ is again compact, there is a number $\epsilon_2 > 0$ such that for $y \in L \sim U$,

$$v^*(y) \geq \ell(y) + \epsilon_2.$$

Then for $\delta < \frac{\epsilon_2}{2M}$ and $y \in L \sim U$,

$$v^*(y) + \delta v(y) \geq \ell(y) + \frac{\epsilon_2}{2} > \ell(y).$$

Thus for $y \in L$, $\delta < \frac{\epsilon_2}{2M}$ implies

$$(v^* + \delta v)(y) > \ell(y).$$

For $x \in G^-_{v^*}$, $v(x) < 0$. By the compactness of $G^-_{v^*}$, there is an open set W containing $G^-_{v^*}$ such that for $y \in W$,

$$v(y) < 0,$$

and

$$v^*(y) + \delta v(y) \leq \mu(y) + \delta v(y) < \mu(y).$$

Since $J \sim W$ is again compact, there is a number $\epsilon_3 > 0$ such that for $y \in J \sim W$,

$$v^*(y) \leq \mu(y) - \epsilon_3.$$

So for $\delta < \frac{\epsilon_3}{2M}$ and $y \in J \sim W$,

$$v^*(y) + \delta v(y) \leq \mu(y) - \frac{\epsilon_3}{2} < \mu(y).$$

By choosing δ such that

$$\delta < \min\{\epsilon_1, \epsilon_2, \epsilon_3\}/2M,$$

we obtain $v^* + \delta v \in V_1$, a better restricted approximation to f.

Conversely, if v^* is not the best restricted approximation, let w be a better approximation, i.e.

$$\|f - w\|_K < \|f - v^*\|_K.$$

Then for $x \in E_{v^*}$,

$$|f(x) - v^*(x)| = \|f - v^*\|_K > \|f - w\|_K \ge |f(x) - w(x)|.$$

Thus

$$\begin{aligned} \operatorname{sgn}(w(x) - v^*(x)) &= \operatorname{sgn}[f(x) - v^*(x) - (f(x) - w(x))] \\ &= \operatorname{sgn}[f(x) - v^*(x)] \end{aligned}$$

and $|w(x) - v^*(x)| \ge d > 0$ for some d and all $x \in E_{v^*}$.

Consider

$$w^*(x) = \frac{1}{1+\delta}(w(x) + \delta v_o(x)).$$

Let $\|v^* - v_o\|_\infty = M$. For $M \ne 0$ and $0 < \delta < \frac{d}{2M}$,

$$w^*(x) - v^*(x) = \frac{1}{1+\delta}(w(x) - v^*(x)) + \frac{\delta}{1+\delta}(v_o(x) - v^*(x))$$

is such that

$$w^*(x) - v^*(x) > 0 \quad \text{for } x \in E^+_{v^*}$$

and

$$w^*(x) - v^*(x) < 0 \quad \text{for } x \in E^-_{v^*}.$$

For $x \in G^+_{v^*}$, $v^*(x) = \ell(x)$. So

$$w(x) - v^*(x) \ge 0 \quad \text{and} \quad v_o(x) - v^*(x) > 0.$$

Thus $w^*(x) - v^*(x) > 0$.

For $x \in G^-_{v^*}$, $v^*(x) = \mu(x)$. So

$$w(x) - v^*(x) \le 0 \quad \text{and} \quad v_o(x) - v^*(x) < 0.$$

Thus $w^*(x) - v^*(x) < 0$, and $w^*(x) - v^*(x) \in V$ is the desired function v.

If $M = \|v^* - v_o\|_\infty = 0$, $G^+_{v^*} = G^-_{v^*} = \phi$ and we have shown

$$w(x) - v^*(x) > 0 \quad \text{for} \quad x \in E^+_{v^*}$$

$$w(x) - v^*(x) < 0 \quad \text{for} \quad x \in E^-_{v^*}.$$

So the function $w(x) - v^*(x) \in V$ satisfies the theorem. ∎

<u>Remark 1</u>: The extra hypothesis on V required for the converse of this theorem will be discussed in the next section.

<u>Remark 2</u>: If $(E^+_{v^*} \cup G^+_{v^*}) \cap (E^-_{v^*} \cup G^-_{v^*}) \neq \phi$ for a particular $f \in C(S)$ and $v^* \in V_1$, then the v of Theorem 1.2 cannot exist and v^* must be a best restricted approximation to f even if the extra hypothesis is not satisfied by V. The conclusions drawn here are the same as those drawn by G.D. Taylor [23] and are included here for completeness.

1. If $E^+_{v^*} \cap E^-_{v^*} \neq \phi$, then $f - v^* \equiv 0$ on K. This can occur even if $f \not\equiv v^*$ on S.

2. If $E^+_{v^*} \cap G^-_{v^*} \neq \phi$, then for some $x \in K \cap J$,

$$f(x) - v^*(x) = \|f - v^*\|_K$$

and

$$v^*(x) = \mu(x).$$

To get closer to f at this point, we would have to have $v_o(x) > \mu(x)$ thus removing v_o from V_1. So v^* must be a best restricted approximation.

3. If $E^-_{v^*} \cap G^+_{v^*} \neq \phi$, then for some $x \in K \cap L$,

$$f(x) - v^*(x) = -\|f - v^*\|_K$$

and

$$x^*(x) = \ell(x).$$

Again, to choose $v_o \in V$ closer to f, $v_o(x) < \ell(x)$ which would mean $v_o \notin V_1$. Thus v^* is a best approximation.

4. Let $\tilde{C}(S) = \{f \in C(S)\colon\ f(x) \geq \ell(x)$ for all $x \in L$, $f(x) \leq \mu(x)$ for all $x \in J$, and $\rho \equiv \inf_{v \in V_1} \|f - v\|_K > 0\}$. Then for $f \in \tilde{C}(S)$ and any $v \in V_1$,

$$(E_v^+ \cup G_v^+) \cap (E_v^- \cup G_v^-) = \phi.$$

<u>Section 3</u>: Uniqueness and Related Results.

In this section we shall use Theorem 1.2 to obtain characterization and alternation theorems which will be important in constructing an algorithm to determine the best restricted approximation. Laurent, in his one-sided approximation problem, assumed that the set V_1 contained an element strictly less than the f to be approximated. This enabled him to characterize the best approximation by means of a linear functional on $C(S)$ based on the critical points of the error function with at least one of these points a maximum point for the absolute value of the error curve. We shall make a similar assumption here in condition H and show that this assumption is satisfied in the special case that V is a Haar subspace.

Condition H: We shall say V satisfies condition H provided there is a $v \in V_1$ such that

$$v(x) < \mu(x) \quad \text{for all} \quad x \in J$$

and

$$v(x) > \ell(x) \quad \text{for all} \quad x \in L.$$

We can now characterize the best restricted approximation v^* to f from V_1 when V_1 satisfies condition H and $(E^+_{v^*} \cup G^+_{v^*}) \cap (E^-_{v^*} \cup G^-_{v^*}) = \phi$ by means of a continuous linear functional L in $C(S)^*$ whose null space contains V.

Theorem 1.3: Let V satisfy condition H. Then a necessary and sufficient condition for $v^* \in V_1$ to be a best restricted approximation to $f \in C(S)$ is that there exist k $(\leq n+1)$ critical points

$$x_1, x_2, \ldots, x_k \quad \text{in} \quad D_{v^*}$$

such that $\{x_1, x_2, \ldots, x_k\} \cap E_{v^*} \neq \phi$, and a linear functional L defined by

$$L(h) = \sum_{i=1}^{k} \lambda_i h(x_i),$$

such that L vanishes on V and

$$\lambda_i > 0 \quad \text{for} \quad x_i \in E^+_{v^*} \cup G^+_{v^*},$$

$$\lambda_i < 0 \quad \text{for} \quad x_i \in E^-_{v^*} \cup G^-_{v^*}.$$

Proof: (Sufficiency) Suppose v^* satisfies the hypotheses and that $w \in V_1$ is a better restricted approximation to f. Then

$$\|f - w\|_K < \|f - v^*\|_K ,$$

and $v = w - v^* \in V$ is such that

$$v(x_i) > 0 \quad \text{for} \quad x_i \in E^+_{v^*} \quad (\lambda_i > 0),$$

$$v(x_i) < 0 \quad \text{for} \quad x_i \in E^-_{v^*} \quad (\lambda_i < 0),$$

$$v(x_i) \le 0 \quad \text{for} \quad x_i \in G^-_{v^*} \quad (\lambda_i < 0),$$

$$v(x_i) \ge 0 \quad \text{for} \quad x_i \in G^+_{v^*} \quad (\lambda_i > 0).$$

Since at least one $x_i \in E_{v^*}$ by hypothesis, $L(v) > 0$. This is a contradiction to L vanishing on V, so v^* is a best restricted approximation.

(Necessity) Let $\{w_1,\ldots,w_n\}$ be a basis for V. Let $v^* \in V_1$ be a best restricted approximation to f with corresponding set D_{v^*}. Denote by Γ the set

$$\Gamma = \{(z_1,\ldots,z_n) \in R^n : z_i = w_i(x) \quad \text{for} \quad x \in E^+_{v^*} \cup G^+_{v^*}\}$$

$$\cup \{(z_1,\ldots,z_n) \in R^n : z_i = -w_i(x) \quad \text{for} \quad x \in E^-_{v^*} \cup G^-_{v^*}\}.$$

If $\vec{0} \in R^n$ is not in co (Γ) (the convex hull of Γ), by the theorem on linear inequalities [3 , p. 19], there exists a point $(c_1,\ldots,c_n) \in R^n$ such that $\sum_{i=1}^{n} c_i z_i > 0$ for all $(z_1,\ldots,z_n) \in \Gamma$. But then the function $v \in V$ defined by $\sum_{i=1}^{n} c_i w_i(x) = v(x)$ is such that

$$v(x) > 0 \quad \text{for} \quad x \in E^+_{v^*} \cup G^+_{v^*},$$

$$v(x) < 0 \quad \text{for} \quad x \in E^-_{v^*} \cup G^-_{v^*}.$$

This contradicts the fact that v^* is a best restricted approximation (see Theorem 1.2). So $\vec{0} \in R^n$ must be in co (Γ). Then by the theorem of Caratheodory [3 , p. 17] we can find k ($\le n+1$) points $z_1,\ldots,z_k$ in Γ, and k positive real numbers $a_1,\ldots,a_k$, such that

$$\vec{0} = \sum_{i=1}^{k} a_i z_i$$

and

$$1 = \sum_{i=1}^{k} a_i.$$

Since $z_i = \pm(w_1(x_i), w_2(x_i), \ldots, w_n(x_i))$ we have

$$0 = \sum_{i=1}^{k} a_i \epsilon_i w_j(x_i) \quad \text{for} \quad j = 1,\ldots,n,$$

with

$$\epsilon_i = \begin{cases} +1 & \text{if} \quad x_i \in E^+_{v^*} \cup G^+_{v^*}, \\ -1 & \text{if} \quad x_i \in E^-_{v^*} \cup G^-_{v^*}. \end{cases}$$

Set $\lambda_i = a_i \epsilon_i$ and

$$L(h) = \sum_{i=1}^{k} \lambda_i h(x_i).$$

L is a linear functional on $C(S)$ vanishing on V. We must show that at least one of the x_i's is in E_{v^*}. Suppose not: then for each $v \in V$ we have that

$$v(x_i) = \ell(x_i) \quad \text{for all} \quad x_i \in G^+_{v^*}$$

and

$$v(x_i) = \mu(x_i) \quad \text{for all} \quad x_i \in G^-_{v^*}.$$

Indeed, if there is $x_i \in G^+_{v^*}$ for which

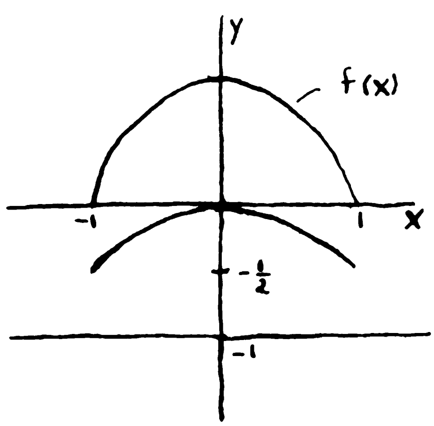
$$v(x_i) > \ell(x_i) = v^*(x_i),$$

then
$$(v(x_i) - v^*(x_i))\lambda_i > 0$$

implying $L(v - v^*) > 0$ which is a contradiction. So at least one x_i must be in E_{v^*}. ∎

The proof of the sufficiency did not require that $(E^+_{v^*} \cup G^+_{v^*}) \cap (E^-_{v^*} \cup G^-_{v^*}) = \phi$. This condition is required for the proof of the necessity as shown by the following example:

<u>Example 1.1</u>: Let $f(x) = 1 - x^2$ on the real interval $[-1,1]$. We wish to approximate f by polynomials of degree at most two which are less than or equal to $0 = \mu(x)$ and greater than or equal to $-1 = \ell(x)$ on $[-1,1]$.



Then one best restricted approximation to f is

$$v^*(x) = -\frac{1}{2}x^2.$$

The corresponding set $D_{v^*} = \{0\} = E^+_{v^*} = G^-_{v^*}$. A non-zero functional L cannot exist for v^* since $\{1, x, x^2\}$ is a basis for V and if $L(h) = \lambda h(0)$ is to vanish on V, we must have

$$L(1) = \lambda = 0.$$

An n-dimensional subspace $V \subset C(S)$ is called a Haar subspace if every non-zero element of V has at most $n-1$ zeros.

<u>Remark 1</u>: In the previous theorem, if V is a Haar subspace, then $k = n+1$. Since if $k < n+1$ and $\sum_{i=1}^{k} \lambda_i w_j(x_i) = 0$ for $j = 1,\ldots,n$, then, if $k < n$, adding points $x_{k+1},\ldots,x_n$ from S all different from $x_1,\ldots,x_k$ and setting the corresponding $\lambda_i = 0$,

$$\sum_{i=1}^{n} \lambda_i w_j(x_i) = 0 \quad \text{for} \quad j = 1,\ldots,n.$$

That is, $\det(w_j(x_i)) = 0$ so there exist real numbers $\beta_1,\ldots,\beta_n$, not all zero, such that

$$\sum_{i=1}^{n} \beta_i w_i(x_j) = 0 \quad \text{for} \quad j = 1,\ldots,n.$$

But then the function $v(x) = \sum_{i=1}^{n} \beta_i w_i(x) \in V$ has n zeros, which contradicts the Haar condition since $v \not\equiv 0$. So $k = n+1$.

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<u>Remark 2</u>: If S is a compact subset of the real line, V is a Haar subspace of dimension n on a closed interval $[a,b]$ properly containing S, and V_1 contains at least two distinct elements, then V satisfies condition H.

<u>Proof</u>: Let $v_1(x)$ and $v_2(x) \in V_1$ with $v_1 \neq v_2$. If either v_1 or v_2 satisfies condition H, we are done. Suppose not. Let $v_o = (v_1 + v_2)/2$, $v_o \in V_1$. If $x \in G^-_{v_o} = \{x \in J: \ v_o(x) = \mu(x)\}$, then

$$v_1(x) = v_2(x) = \mu(x).$$

Similarly, if $x \in G^+_{v_o} = \{x \in L: \ v_o(x) = \ell(x)\}$, then

$$v_1(x) = v_2(x) = \ell(x).$$

So v_o must meet μ and ℓ at most $n-1$ times since V is a Haar subspace and $v_1 - v_2$ can have at most $n-1$ zeros. If $J \cap L = \phi$, construct $v \in V$ such that for some $\delta > 0$,

$$v(x) = \begin{cases} +\delta & \text{for} \quad x \in G^+_{v_o}, \\ \\ -\delta & \text{for} \quad x \in G^-_{v_o}. \end{cases}$$

Then there is an open set U containing $G^+_{v_o}$ on which $v(x)$ is positive. $L \sim U$ compact implies there is an $\epsilon_1 > 0$ such that

$$v_o(x) - \ell(x) \geq \epsilon_1 \quad \text{for all} \quad x \in L \sim U.$$

Thus for $\eta < \dfrac{\epsilon_1}{2\|v\|_\infty}$,

$$v_o(x) + \eta v(x) > \ell(x) \quad \text{for all} \quad x \in L.$$

Also there is an open set W containing $G^-_{v_o}$ on which $v(x)$ is negative. $J \sim W$ compact implies there is an $\epsilon_2 > 0$ such that

$$\mu(x) - v_o(x) \geq \epsilon_2 \quad \text{for all} \quad x \in J \sim W.$$

Thus for $\eta > \frac{\epsilon_2}{2\|v\|_\infty}$,

$$v_o(x) + \eta v(x) < \mu(x) \quad \text{for all} \quad x \in J.$$

Then $\eta < \min\{\epsilon_1, \epsilon_2\}/2\|v\|_\infty$, implies $v_o(x) + \eta v(x) \in V_1$ and does not intersect either $\ell(x)$ or $\mu(x)$.

If $J \cap L \neq \phi$, order the points in $G^+_{v_o} \cup G^-_{v_o}$ and label them

$$x_1 < x_2 < \ldots < x_{n-1}.$$

Without loss of generality, assume $x_1 \in G^+_{v_o}$. Group these points so that

$$\begin{array}{lll} x_1 < \ldots < x_{k_1} & \text{are } G^+_{v_o} & \text{points,} \\ x_{k_1+1} < \ldots < x_{k_2} & \text{are } G^-_{v_o} & \text{points,} \\ \vdots & & \\ x_{k_m+1} < \ldots < x_{k_{m+1}} = x_{n-1} & \text{are } G^{(-1)^m}_{v_o} & \text{points} \quad (m+1 \leq n-1). \end{array}$$

The finite interval $[a,b]$ properly contains S. Let $y_o = a$ $(< x_1)$. Choose y_1 such that

$$x_{k_1} < y_1 < x_{k_1+1}.$$

Similarly choose y_i, $i = 2,\ldots,m$, and let $y_{m+1} = b$ ($> x_{n-1}$). Construct $v \in V$ [8, p. 28] such that

$$
\begin{aligned}
v(y_i) &= 0 \quad \text{for} \quad i = 1,\ldots,m, \\
v(x) &\geq 0 \quad \text{on} \quad [y_o, y_1], \\
v(x) &\leq 0 \quad \text{on} \quad [y_1, y_2], \\
&\vdots \\
(-1)^m v(x) &\geq 0 \quad \text{on} \quad [y_m, y_{m+1}],
\end{aligned}
$$

and $v(x) \neq 0$ for $x \in (a,b) \sim \{y_1,\ldots,y_m\}$.

Let

$$
\begin{aligned}
\delta_1 &= \min_{x\in[a,y_1]\cap J} (\mu(x) - v_o(x)), \\
\delta_2 &= \min_{x\in[y_1,y_2]\cap L} (v_o(x) - \ell(x)), \\
&\vdots \\
\delta_{m+1} &= \begin{cases} \displaystyle\min_{x\in[y_m,b]\cap J} (\mu(x) - v_o(x)) & \text{if } m \text{ is even,} \\[2ex] \displaystyle\min_{x\in[y_m,b]\cap L} (v_o(x) - \ell(x)) & \text{if } m \text{ is odd.} \end{cases}
\end{aligned}
$$

Let $\delta = \displaystyle\min_{i=1,\ldots,m+1} \delta_i$. $\delta > 0$ since each δ_i is positive by construction. Multiply the function v described above by an appropriate positive number η such that

$$\|\eta v\|_\infty < \delta/2.$$

Then $v_o(x) + \eta v(x) \in V_1$ does not intersect either $\ell(x)$ or $\mu(x)$. Hence V_1 satisfies condition H. ∎

The Haar condition on V also assures uniqueness of the best approximation. The following theorem is analogous to

the Haar uniqueness theorem [3 , p. 81] in the standard Chebyshev approximation theorem. Let $C_1(S) = \{f \in C(S): (E_v^+ \cup G_v^+) \cap (E_v^- \cup G_v^-) = \phi$ for all $v \in V_1\}$.

Theorem 1.4: Assume condition H is satisfied for V an n-dimensional subspace of $C(S)$. Then a necessary and sufficient condition for a best restricted approximation v^* to $f \in C_1(S)$ to be unique is that there does not exist a linear functional

$$L(h) = \sum_{i=1}^{k} \lambda_i h(x_i)$$

on $C(S)$ such that $k \leq n$, L vanishes on V, and at least one $x_i \in E_{v^*}$.

Proof: (Sufficiency) Suppose no such functional L exists and f has two best restricted approximations v_1, v_2. Then $v_o = (v_1 + v_2)/2$ is also a best approximation and its characterizing functional must be based on $n+1$ points. Let these be $x_1,\ldots,x_{n+1}$, and the functional

$$L(h) = \sum_{i=1}^{n+1} \alpha_i h(x_i).$$

The x_i's must be critical points for v_1 and v_2 and

$$E_{v_1}^+ \supseteq E_{v_o}^+ ,\ E_{v_1}^- \supseteq E_{v_o}^- ,\ G_{v_1}^+ \supseteq G_{v_o}^+ ,\ G_{v_1}^- \supseteq G_{v_o}^- ;$$

$$E_{v_2}^+ \supseteq E_{v_o}^+ ,\ E_{v_2}^- \supseteq E_{v_o}^- ,\ G_{v_2}^+ \supseteq G_{v_o}^+ ,\ G_{v_2}^- \supseteq G_{v_o}^- .$$

Now consider the function

$$v = v_1 - v_2 = \sum_{i=1}^{n} \beta_i w_i ;$$

$$v(x_j) = 0 \quad \text{for} \quad j = 1,\ldots,n+1.$$

By assumption, one of the x_j's, say x_{jo}, is an element of E_{v_o}. Then let

$$D = \det \{w_i(x_j): \ i = 1,\dots,n; \ j = 1,\dots,n+1; \ j \neq j_o\}.$$

Since $L(w_i) = 0$ for $i = 1,\dots,n$,

$$\sum_{\substack{j=1 \\ j \neq jo}}^{n+1} \alpha_j w_i(x_j) = -\alpha_{jo} w_i(x_{jo}).$$

If $D = 0$, the solution $\{\alpha_j\}_{j=1, j\neq jo}^{n+1}$ is not unique. So for α_{jo} fixed, we can find another solution $\{\alpha_j'\}_{j=1, j\neq jo}^{n+1}$, with $\alpha_{j1}' = 0$ for some j_1. But this gives a functional L' such that

$$L'(h) = \sum_{\substack{j=1 \\ j \neq j_1}}^{n+1} \alpha_j' h(x_j), \quad (\alpha_{jo}' = \alpha_{jo})$$

which vanishes on V. This is a contradiction and we conclude $D \neq 0$. Since

$$\sum_{i=1}^{n} \beta_i w_i(x_j) = 0,$$

we must have $\beta_i = 0$ for $i = 1,\dots,n$; so $v_1 \equiv v_2$.

(Necessity) Suppose condition H is satisfied and there is a non-zero linear functional L based on n points $x_1,\dots,x_n$ which vanishes on V. Then

$$L(h) = \sum_{i=1}^{n} \lambda_i h(x_i)$$

and the $\{\lambda_i\}_{i=1}^{n}$ are a non-identically zero solution of

$$\sum_{i=1}^{n} \lambda_i w_j(x_i) = 0 \quad \text{for } j = 1,\dots,n$$

which implies $\det (w_j(x_i)) = 0$, and we can obtain a function

$$v(x) = \sum_{j=1}^{n} \alpha_j w_j(x) \not\equiv 0$$

and

$$\|v\|_\infty = 1,$$

such that $v(x_i) = 0$ for $i = 1,\ldots,n$. By condition H, there is a function $v_o \in V_1$ such that

$$v_o(x) > \ell(x) \quad \text{for all} \quad x \in L,$$

$$v_o(x) < \mu(x) \quad \text{for all} \quad x \in J.$$

Let $2\delta = \min\{\min_{x\in L} (v_o(x) - \ell(x)), \min_{x\in J} (\mu(x) - v_o(x))\}$. Let $\delta' = \min\{1,\delta\}$. Choose $\tilde{f}(x) \in C(S)$ such that

$$\|\tilde{f}\|_\infty = |\tilde{f}(x_i)| = \delta' > 0$$

and

$$\operatorname{sgn} \tilde{f}(x_i) = \operatorname{sgn}^* \lambda_i.$$

$$(\operatorname{sgn}^* \lambda_i = \begin{cases} \operatorname{sgn} \lambda_i & \text{if} \quad \lambda_i \neq 0 \\ +1 & \text{if} \quad \lambda_i = 0 \end{cases}).$$

Set $f(x) = \tilde{f}(x)(1 - |v(x)|)$. Consider

$$F(x) = v_o(x) + f(x).$$

Since $|f(x)| \le \delta'$, $F(x) \ge \ell(x)$ for all $x \in L$ and $F(x) \le \mu(x)$ for all $x \in J$.

We claim: for any $w \in V_1$,

$$\|F - w\|_\infty \ge \delta'.$$

If $w \in V_1$ is such that

$$\|F - w\|_\infty < \delta',$$

then $w = v_o + v^*$ with $v^* \in V$ so

$$F - w = f - v^*,$$

and

$$|f(x_i)| = \delta'.$$

Thus $\operatorname{sgn} v^*(x_i) = \operatorname{sgn} f(x_i) = \operatorname{sgn}^* \lambda_i$, $i = 1,\ldots,n$. But then $\sum_{i=1}^{n} \lambda_i v^*(x_i) > 0$ which is a contradiction. Now consider

$$v_o(x) + \lambda v(x).$$

If $0 \le \lambda \le \delta'$, $v_o + \lambda v \in V_1$. Moreover,

$$\begin{aligned}
\delta' &\le |F(x) - (v_o + \lambda v)(x)| \\
&= |f(x) - \lambda v(x)| \\
&\le |\tilde{f}(x)|(1 - |v(x)|) + \lambda|v(x)| \\
&\le \delta'\cdot(1 - |v(x)|) + \lambda|v(x)| \\
&= \delta' - (\delta' - \lambda)|v(x)| \le \delta' \quad \text{for} \quad 0 \le \lambda \le \delta'.
\end{aligned}$$

Thus, if we choose any λ such that $0 \le \lambda \le \delta'$, the function $v_o + \lambda v$ is a best restricted approximation to F and we have constructed a function whose best restricted approximation is not unique. ∎

For the case that $J = K = L$, G.D. Taylor [23] proved that if V is a Haar subspace, then best approximations are unique for each $f \in C(S)$ which lies between the bounds. The following corollary characterizes those subspaces which yield best approximations for all $f \in C_1(S)$.

Corollary: Assume V satisfies condition H. Then each $f \in C_1(S)$, has a unique best approximation from V_1 if and only if V is a Haar subspace.

Proof: (Sufficiency) Suppose $f \in C_1(S)$ has two best restricted approximations. Then, by Theorem 1.4, there is a continuous linear functional L vanishing on V based on $k \leq n$ points. But by Remark 1 following Theorem 1.3, this contradicts the Haar condition. Thus each f has a unique best restricted approximation.

(If $f \in V_1$, then f is the unique best restricted approximation to f from V_1 and it is unique since K contains at least n points.)

(Necessity) If V is not a Haar subspace, there exists a function $v \in V$,

$$v = \sum_{i=1}^{n} \alpha_i w_i,$$

$$\|v\|_\infty = 1,$$

with distinct points $x_1,\ldots,x_n$ in S for which

$$v(x_j) = 0 \quad j = 1,\ldots,n.$$

Then, as in the proof of Theorem 1.4, we can construct a function $F \in C(S)$ which does not have a unique best restricted approximation. ∎

Alternation theorems are very useful in constructing and recognizing the best approximation in the standard Chebyshev

approximation of functions by elements of a linear space V. G.D. Taylor [23] was able to show that an alternation theorem is also valid in ordinary restricted approximation if we modify slightly the idea of alternations. A similar theorem is valid in the problem presented here. For the proof of this theorem, it will again be necessary to assume that $(E_v^+ \cup G_v^+) \cap (E_v^- \cup G_v^-) = \phi$ for the given function f and all $v \in V_1$.

<u>Theorem 1.5</u>: Let S be a compact subset of $[a,b]$, V be an n-dimensional Haar subspace of $C[a,b]$, and $f \in C_1(S)$. Then $v \in V_1$ is a best restricted approximation to f if and only if there exist $n+1$ consecutive points

$$x_1 < x_2 < \ldots < x_{n+1} \quad \text{from } E_v \cup G_v ,$$

with at least one $x_i \in E_v$, such that for

$$\sigma(x_i) = \begin{cases} +1 & \text{for } x_i \in E_v^+ \cup G_v^+ , \\ -1 & \text{for } x_i \in E_v^- \cup G_v^-, \end{cases}$$

we have $\sigma(x_i) = (-1)^{i+1}\sigma(x_1)$.

<u>Proof</u>: (Necessity) If v is a best restricted approximation, then Remark 1 following Theorem 1.3 implies that there exist $n+1$ critical points

$$x_1 < x_2 < \ldots < x_n,$$

with at least one in E_v, and a linear functional,

$$L(h) = \sum_{i=1}^{n+1} \lambda_i h(x_i),$$

vanishing on V with

$$\lambda_i > 0 \quad \text{if} \quad x_i \in E_v^+ \cup G_v^+,$$

$$\lambda_i < 0 \quad \text{if} \quad x_i \in E_v^- \cup G_v^-.$$

By a well known result for Haar systems [3 , p. 74], the λ_i's must alternate in sign. Thus $\sigma(x_i) = (-1)^{i+1}\sigma(x_1)$.

(Sufficiency) Suppose such a v exists. Then consider the matrix M

$$M = \begin{bmatrix} w_1(x_1) & w_2(x_1) & \cdots & w_n(x_1) & 0 \text{ or } (-1)^1 \\ w_1(x_2) & w_2(x_2) & \cdots & w_n(x_2) & 0 \text{ or } (-1)^2 \\ \vdots & \vdots & & \vdots & \vdots \\ w_1(x_{n+1}) & w_2(x_{n+1}) & \cdots & w_n(x_{n+1}) & 0 \text{ or } (-1)^{n+1} \end{bmatrix}$$

where the element in the last column is 0 if $x_i \in G_v \sim E_v$ and $(-1)^i$ if $x_i \in E_v$, and at least one element is non-zero. The system

$$(\lambda_1, \ldots, \lambda_{n+1}) \;\; M = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

has a solution $\{\lambda_i\}_{i=1}^{n+1}$ such that not all the λ_i are zero. Construct the linear functional

$$L(h) = \sum_{i=1}^{n+1} \lambda_i h(x_i)$$

which vanishes on V and for which the λ_i's alternate in sign. So, by choosing $\{\lambda_i'\}_{i=1}^{n+1}$ to be $\{\lambda_i\}_{i=1}^{n+1}$ or $\{-\lambda_i\}_{i=1}^{n+1}$

so that

$$\lambda_i' > 0 \quad \text{for} \quad x_i \in E_v^+ \cup G_v^+$$

and

$$\lambda_i' < 0 \quad \text{for} \quad x_i \in E_v^- \cup G_v^-,$$

Theorem 1.3 implies v is a best restricted approximation for f. ■

If $f \in \tilde{C}(S)$ we can get some idea of the size of $\|f - v\|_K$ for all $v \in V_1$ in terms of $\|f - v^*\|_K$ and $\|v - v^*\|_K$ for v^* a best restricted approximation to f. E.W. Cheney [3, p. 80] proved a theorem relating these quantities in the standard Chebyshev approximation problem with V a Haar subspace, and also for V a set of generalized rational functions. Similar theorems for the case of restricted approximation were proved by G.D. Taylor [23] for V a linear Haar subspace, and by Loeb and Moursund [16] for V a restricted set of generalized rational functions. We shall assume V_1 satisfies condition H.

<u>Theorem 1.6</u>: (Strong Uniqueness Theorem) Let V be an n-dimensional Haar subspace, $f \in \tilde{C}(S)$. Further let v^* be the unique best restricted approximation to f from V_1. Then there exists a constant $\gamma > 0$ depending only on f such that for any $v \in V_1$,

$$\|f - v\|_K \geq \|f - v^*\|_K + \gamma\|v - v^*\|_K.$$

Proof: (If $\|f - v^*\|_K = 0$, we can take $\gamma = 1$ since

$$\|f - v\|_K \geq \|v^* - v\|_K + \|f - v^*\|_K = \|v^* - v\|_K.)$$

If $v \equiv v^*$, the conclusion of the theorem holds for any positive number γ, so we shall assume $v \not\equiv v^*$. Since v^* is the best restricted approximation to f, there is a characterizing linear functional L based on points $\{x_i\}_{i=1}^{n+1} \subseteq E_{v^*} \cup G_{v^*}$,

$$L(h) = \sum_{i=1}^{n+1} \beta_i h(x_i),$$

with

$$\operatorname{sgn} \beta_i = \sigma_i = +1 \quad \text{for} \quad x_i \in E^+_{v^*} \cup G^+_{v^*},$$

$$\operatorname{sgn} \beta_i = \sigma_i = -1 \quad \text{for} \quad x_i \in E^-_{v^*} \cup G^-_{v^*},$$

and $\{x_i\}_{i=1}^{n+1} \cap E_{v^*} \neq \phi$. We shall define the function $\operatorname{sgn}^* (\cdot)$ as follows:

$$\operatorname{sgn}^* (y) = \begin{cases} \operatorname{sgn} y & \text{if } y \neq 0 \\ +1 & \text{if } y = 0. \end{cases}$$

Then for $x_i \in G_{v^*}$ and $v \in V_1$,

$$\operatorname{sgn}^* (v - v^*)(x_i) = \operatorname{sgn}^* (f - v^*)(x_i),$$

$$\sigma_i (v - v^*)(x_i) \geq 0.$$

Consider the set

$$U = \{v \in V: \ \sigma_i (v - v^*)(x_i) \geq 0 \ \text{ for all } \ x_i \in G_{v^*}\}.$$

Notice that U is closed and we have shown $V_1 \subseteq U$. For any

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any $v \in U \sim \{v^*\}$,

$$\max_{x_i \in E_{v^*}} \sigma_i(v^* - v)(x_i) > 0$$

since $L(v - v^*) = 0$ and $v - v^*$ cannot have more than $n-1$ zeros. Thus if

$$\sigma_i(v - v^*)(x_i) = 0 \quad \text{for all} \quad x_i \in G_{v^*},$$

we must have

$$\sigma_i(v - v^*)(x_i) < 0 \quad \text{for some} \quad x_i \in E_{v^*}.$$

And if

$$\sigma_i(v - v^*)(x_i) > 0 \quad \text{for some} \quad x_i \in G_{v^*},$$

we must have

$$\sigma_i(v - v^*)(x_i) < 0 \quad \text{for some} \quad x_i \in E_{v^*}.$$

It follows that there is a number $\gamma > 0$ such that

$$\min_{\substack{\|v^*-v\|_K=1 \\ v\in U}} \max_{x_i \in E_{v^*}} \sigma_i(v^* - v)(x_i) = \gamma > 0$$

since it is the minimum of a positive function on a compact set. Now let $v \in V_1 \sim \{v^*\}$, then

$$\frac{(v^*-v)(x)}{\|v^*-v\|_K} = v^*(x) - \{(1 - \frac{1}{\|v^*-v\|_K})v^*(x) + \frac{1}{\|v^*-v\|_K} v(x)\}.$$

Let

$$v_o(x) = (1 - \frac{1}{\|v^* - v\|_K})v^*(x) + \frac{1}{\|v^* - v\|_K} v(x).$$

Let $x_i \in G^-_{v^*}$, that is $v^*(x_i) = \mu(x_i)$, then

$$v_o(x_i) - v^*(x_i) = - \frac{1}{\|v^* - v\|_K} v^*(x_i) + \frac{1}{\|v^* - v\|_K} v(x_i)$$

$$\leq - \frac{1}{\|v^* - v\|_K} \mu(x_i) + \frac{1}{\|v^* - v\|_K} \mu(x_i) = 0,$$

and $\sigma_i < 0$ so that

$$\sigma_i(v_o - v^*)(x_i) \geq 0.$$

Similarly if $x_i \in G^+_{v^*}$, $v^*(x_i) = \ell(x_i)$ and

$$v_o(x_i) - v^*(x_i) = - \frac{1}{\|v^* - v\|_K} v^*(x_i) + \frac{1}{\|v^* - v\|_K} v(x_i)$$

$$\geq - \frac{1}{\|v^* - v\|_K} \ell(x_i) + \frac{1}{\|v^* - v\|_K} \ell(x_i) = 0,$$

and $\sigma_i > 0$ so that

$$\sigma_i(v_o - v^*)(x_i) \geq 0.$$

We have shown that for every $v \in V_1 \sim \{v^*\}$,

$$\frac{(v-v^*)(x)}{\|v-v^*\|_K} = v_o(x) - v^*(x)$$

with $v_o \in U$ and $\|v_o - v^*\|_K = 1$. Now for $v \in V_1 \sim \{v^*\}$, let $x_{i_o} \in E_{v^*}$ be such that

$$\max_{x_i \in E_{v^*}} \sigma(x_i)(v^* - v)(x_i) = \sigma(x_{i_o})(v^* - v)(x_{i_o}).$$

Then

$$\begin{aligned}\|f - v\|_K &\geq \sigma_{i_o}(f - v)(x_{i_o}) \\ &= \sigma_{i_o}(f - v^*)(x_{i_o}) + \sigma_{i_o}(v^* - v)(x_{i_o}) \\ &\geq \|f - v^*\|_K + \gamma\|v^* - v\|_K. \qquad \blacksquare\end{aligned}$$

We define a function T on $\tilde{C}(S) = \{f \in C(S): \rho > 0$ and $f(x) \geq \ell(x)$ for all $x \in L$ and $f(x) \leq \mu(x)$ for all $x \in J\}$. T assigns to each $f \in \tilde{C}(S)$ its best restricted approximation $v \in V_1$. Theorem 1.6 easily yields a theorem which proves T is continuous. This theorem is a logical corollary to the Strong Uniqueness Theorem and has been proved by the various authors discussed previously. The original proof of the continuity of T was done by Borel [1] for the standard polynomial case.

<u>Corollary</u>: (Continuity of the Best Approximation Operator) Let $f^* \in \tilde{C}(S)$. Then there exists a number λ corresponding to f^* such that if $T(f^*) = v^* \in V_1$ and if $f \in \tilde{C}(S)$ is arbitrary with $T(f) = v$, then

$$\|v - v^*\|_K \leq \lambda\|f - f^*\|_K.$$

<u>Proof</u>: By the Strong Uniqueness Theorem there is a number $\gamma > 0$ such that

$$\|f^* - v\|_K \geq \|f^* - v^*\|_K + \gamma\|v^* - v\|_K \quad \text{for all } v \in V_1.$$

Thus for any arbitrary $f \in \tilde{C}(S)$ and corresponding $v = T(f)$,

$$\gamma\|v - v^*\|_K \le \|f^* - v\|_K - \|f^* - v^*\|_K$$
$$\le \|f^* - f\|_K + \|f - v\|_K - \|f^* - v^*\|_K$$
$$\le \|f^* - f\|_K + \|f - v^*\|_K - \|f^* - v^*\|_K$$
$$\le \|f^* - f\|_K + \|f^* - f\|_K + \|f^* - v^*\|_K - \|f^* - v^*\|_K$$
$$= 2\|f^* - f\|_K.$$

Now let $\lambda = \frac{2}{\gamma}$. ■

Section 4: Remes Algorithm for Calculating Best Restricted Approximation

In order to obtain such a best restricted approximation of a given continuous function f with bounds ℓ and μ from a subspace V_1 of a Haar space V of dimension n (≥ 1) we can modify the algorithm developed by G.D. Taylor and M.J. Winter [25]. We shall assume that J, K, L are non-empty subsets of the real line. Thus $S = J \cup K \cup L$ is contained in some finite closed interval $[a,b]$. Assume K contains at least $n+1$ points and that V_1 satisfies condition H on $[a,b]$. Let $f \in \tilde{C}(S)$ be the function to be approximated. Then

$$\inf_{v \in V_1} \|f - v\|_K = \rho > 0.$$

We shall choose $n+1$ points of K

$$x_{1,1} < x_{2,1} < \cdots < x_{n+1,1}$$

and construct a best approximation v_1 to f from the full Haar space V on these $n+1$ points. Next, we check to see if $v_1 \in V_1$ and if $\|f - v_1\|_K$ is greater than $|f(x_{i,1}) - v_1(x_{i,1})|$

for $i = 1,\ldots,n+1$. If $\|f - v_1\|_K = |f(x_{i,1}) - v_1(x_{i,1})|$ and $v_1 \in V_1$, then we are done by Theorem 1.5. If not, we replace one of the points $x_{i,1}$ with a new point from S to get a set

$$x_{1,2} < x_{2,2} < \cdots < x_{n+1,2}$$

on which we find the best restricted approximation v_2. We repeat this procedure and obtain a sequence of functions $\{v_n\} \subseteq V$ (possibly finite) which converge to the best restricted approximation $v \in V$.

Let

$$x_{1,1} < x_{2,1} < \cdots < x_{n+1,1}$$

be distinct points in K, and let

$$\{w_1(x),\ldots,w_n(x)\}$$

be a basis for V. We further assume that f cannot be interpolated on $\{x_{i,1}\}_{i=1}^{n+1}$ by any element of V. This can be done by selecting any set of n points, interpolating f on these points and then selecting $x_{n+1,1}$ such that $f(x_{n+1,1}) \neq v(x_{n+1,1})$. This can be done since $\rho > 0$. Then the system

$$\sum_{i=1}^{n} \alpha_{i,1} w_i(x_{j,1}) + (-1)^j \alpha_{n+1,1} = f(x_{j,1}) \quad \text{for } j = 1,\ldots,n+1$$

has a unique solution $\{\alpha_{i,1}\}_{i=1}^{n+1}$ since $\{w_i(x)\}$ is a Haar system. Set

$$v_1(x) = \sum_{i=1}^{n} \alpha_{i,1} w_i(x).$$

If $\|f - v_1\|_K = |\alpha_{n+1,1}| = |f(x_j) - v_1(x_{j,1})| = e_1$, and

$$v_1(x) \geq \ell(x) \quad \text{for all} \quad x \in L,$$

$$v_1(x) \leq \mu(x) \quad \text{for all} \quad x \in J,$$

then by Theorem 1.5, v_1 is the desired best restricted approximation to f from V_1. If not, we define the following quantities:

$$M_1 = \max \{v_1(x) - \mu(x): \ x \in J\},$$

$$m_1 = \max \{\ell(x) - v_1(x): \ x \in L\},$$

$$E_1 = \|f - v_1\|_K - e_1.$$

Let $\gamma_1 = \max \{E_1, M_1, m_1\}$. (In case of equality, let γ_1 be the first largest element.) Choose $y_1 \in S$ in the following manner:

If $\gamma_1 = E_1$, let $y_1 \in K$ and $|f(y_1) - v_1(y_1)| = \|f-v_1\|_K$.

If $\gamma_1 = M_1$, let $y_1 \in J$ and $v_1(y_1) - \mu(y_1) = M_1$.

If $\gamma_1 = m_1$, let $y_1 \in L$ and $\ell(y_1) - v_1(y_1) = m_1$.

We wish to exchange one of the $x_{j,1}$'s for y_1. Define: for $v \in V$,

$$\operatorname{sgn}_1 (f(x) - v(x)) = \begin{cases} +1, \text{ if } \ v(x) = f(x) = \ell(x) \text{ and } x \in L, \\ -1, \text{ if } \ v(x) = f(x) = \mu(x) \text{ and } x \in J, \\ \operatorname{sgn} (f(x) - v(x)), \text{ otherwise.} \end{cases}$$

We then choose $x_{jo,1}$ which is to be replaced as follows:

1.a. If $y_1 \leq x_{1,1}$ and

$$\mathrm{sgn}_1(f(x_{1,1}) - v_1(x_{1,1})) = \mathrm{sgn}_1(f(y_1) - v_1(y_1));$$

then replace $x_{1,1}$ by y_1,

i.e.
$$x_{1,2} = y_1$$
$$x_{i,2} = x_{i,1} \quad \text{for} \quad i = 2,\dots,n+1.$$

1.b. If $y_1 \le x_{1,1}$ and

$$\mathrm{sgn}_1(f(x_{1,1}) - v_1(x_{1,1})) \ne \mathrm{sgn}_1(f(y_1) - v_1(y_1));$$

then "replace" $x_{n+1,1}$ by y_1,

i.e.
$$x_{1,2} = y_1$$
$$x_{i,2} = x_{i-1,1} \quad \text{for} \quad i = 2,\dots,n+1.$$

2. If $x_{j-1,1} < y_1 < x_{j+1,1}$ for some $j \in \{1,2,\dots,n+1\}$ $(x_{o,1} = a,\ x_{n+2,1} = b)$ and

$$\mathrm{sgn}_1(f(x_{j,1}) - v_1(x_{j,1})) = \mathrm{sgn}_1(f(y_1) - v_1(y_1));$$

then replace $x_{j,1}$ by y_1,

i.e.
$$x_{i,2} = x_{i,1} \quad \text{for} \quad i \ne j$$
$$x_{j,2} = y_1.$$

3.a. If $y_1 \ge x_{n+1,1}$ and

$$\mathrm{sgn}_1(f(x_{n+1,1}) - v_1(x_{n+1,1})) = \mathrm{sgn}_1(f(y_1) - v_1(y_1));$$

then replace $x_{n+1,1}$ by y_1,

i.e.
$$x_{i,2} = x_{i,1} \quad \text{for} \quad i = 1,\dots,n$$
$$x_{n+1,2} = y_1.$$

3.b. If $y_1 \geq x_{n+1,1}$ and

$$sgn_1(f(x_{n+1,1}) - v_1(x_{n+1,1})) \neq sgn_1(f(y_1) - v_1(y_1));$$

then "replace" $x_{1,1}$ by y_1,

i.e. $$x_{i,2} = x_{i+1,1} \quad \text{for} \quad i = 1,\ldots,n$$

$$x_{n+1,2} = y_1.$$

Now $\{x_{1,2},\ldots,x_{n+1,2}\} = (\{x_{1,1},\ldots,x_{n+1,1}\} \sim \{x_{jo,1}\}) \cup \{y_1\}$ for some jo. We wish to partition the set $\{x_{1,2},\ldots,x_{n+1,2}\}$ into three disjoint sets K_2, L_2, J_2 (not necessarily all non-empty) in the following way:

$$x_{j,2} \in K_2 \quad \text{if} \quad x_{j,2} \neq y_1,$$

and for jo such that $x_{jo,2} = y_1$,

$$\text{let } x_{jo,2} \in K_2 \quad \text{if} \quad \gamma_1 = E_1;$$

$$\text{let } x_{jo,2} \in J_2 \quad \text{if} \quad \gamma_1 = M_1;$$

$$\text{let } x_{jo,2} \in L_2 \quad \text{if} \quad \gamma_1 = m_1.$$

We continue the process by solving the following system for $\{\alpha_{j,2}\}_{j=1}^{n+1}$:

$$\sum_{j=1}^{n} \alpha_{j,2} w_j(x_{i,2}) + (-1)^i \alpha_{n+1,2} = f(x_{i,2}) \quad \text{for} \quad x_{i,2} \in K_2,$$

$$\sum_{j=1}^{n} \alpha_{j,2} w_j(x_{i,2}) = \mu(x_{i,2}) \quad \text{for} \quad x_{i,2} \in J_2,$$

$$\sum_{j=1}^{n} \alpha_{j,2} w_j(x_{i,2}) = \ell(x_{i,2}) \quad \text{for} \quad x_{i,2} \in L_2.$$

Let $v_2(x) = \sum_{j=1}^{n} \alpha_{j,2} w_j(x)$, $e_2 = |\alpha_{n+1,2}|$. If $\|f - v_2\|_K = e_2$ and

$$v_2(x) \geq \ell(x) \quad \text{for all} \quad x \in L,$$

$$v_2(x) \leq \mu(x) \quad \text{for all} \quad x \in J,$$

then $v_2(x)$ is the desired best restricted approximation. If not, we find γ_2, y_2 in the same manner and continue the iteration. Suppose we have not obtained the best restricted approximation but we have found v_k based on the points

$$x_{1,k} < x_{2,k} < \cdots < x_{n+1,k}$$

and at least one of these is in K. Further we have $\mathrm{sgn}_1(f(x_{i,k}) - v_k(x_{i,k})) = (-1)^{i+1} \mathrm{sgn}_1(f(x_{1,k}) - v_k(x_{1,k}))$ for $i = 1,\ldots,n+1$. If, as before, v_k is not the best restricted approximation to f, then we can find

$$E_k = \|f - v_k\|_K - e_k,$$

$$M_k = \max\,\{v_k(x) - \mu(x)\colon\; x \in J\},$$

$$m_k = \max\,\{\ell(x) - v_k(x)\colon\; x \in L\},$$

$$\gamma_k = \max\,\{E_k, M_k, m_k\} \quad \text{(treat equality as before).}$$

Choose $y_k \in S$ so that

$$\text{if } \gamma_k = E_k,\; |f(y_k) - v_k(y_k)| = \|f - v_k\|_K;$$

$$\text{if } \gamma_k = M_k,\; v_k(y_k) - \mu(y_k) = M_k;$$

$$\text{if } \gamma_k = m_k,\; \ell(y_k) - v_k(y_k) = m_k.$$

Replace $x_{jo,k}$ by y_k as described before and form

$\{x_{1,k+1},\ldots,x_{n+1,k+1}\} = (\{x_{1,k},\ldots,x_{n+1,k}\} \sim \{x_{jo,k}\}) \cup \{y_k\}$

with

$$x_{1,k+1} < x_{2,k+1} <\ldots< x_{n+1,k+1}$$

and partition $\{x_{1,k+1},\ldots,x_{n+1,k+1}\}$ into three disjoint sets K_{k+1}, L_{k+1}, J_{k+1}, so that $K_{k+1} \cup L_{k+1} \cup J_{k+1} = \{x_{1,k+1},\ldots,x_{n+1,k+1}\}$, and for $x_{j,k+1} = x_{i,k} \in \{x_{1,k},\ldots,x_{n+1,k}\} \sim \{x_{jo,k}\}$ then

$$x_{j,k+1} \in K_{k+1} \quad \text{if} \quad x_{i,k} \in K_k,$$

$$x_{j,k+1} \in L_{k+1} \quad \text{if} \quad x_{i,k} \in L_k,$$

$$x_{j,k+1} \in J_{k+1} \quad \text{if} \quad x_{i,k} \in J_k.$$

For $x_{j,k+1} = y_k$,

$$x_{j,k+1} \in K_{k+1} \quad \text{if} \quad \gamma_k = E_k,$$

$$x_{j,k+1} \in J_{k+1} \quad \text{if} \quad \gamma_k = M_k,$$

$$x_{j,k+1} \in L_{k+1} \quad \text{if} \quad \gamma_k = m_k.$$

Now set $v_{k+1}(x) = \sum_{j=1}^{n} \alpha_{j,k+1} w_j(x)$ where $\{\alpha_{j,k+1}\}_{j=1}^{n+1}$ is the solution of the system

$$\sum_{j=1}^{n} \alpha_{j,k+1} w_j(x_{i,k+1}) + (-1)^i \alpha_{n+1,k+1} = f(x_{i,k+1}) \quad \text{for} \quad x_{i,k+1} \in K_{k+1},$$

$$\sum_{j=1}^{n} \alpha_{j,k+1} w_j(x_{i,k+1}) = \mu(x_{i,k+1}) \quad \text{for} \quad x_{i,k+1} \in J_{k+1},$$

$$\sum_{j=1}^{n} \alpha_{j,k+1} w_j(x_{i,k+1}) = \ell(x_{i,k+1}) \quad \text{for} \quad x_{i,k+1} \in L_{k+1}.$$

$K_{k+1} \neq \phi$. Suppose $K_{k+1} = \phi$; then by construction

$$\operatorname{sgn}_1(f(x_{i,k+1}) - v_k(x_{i,k+1})) = (-1)^{i+1}\operatorname{sgn}_1(f(x_{1,k+1}) - v_k(x_{1,k+1})).$$

Thus, if we assume, without loss of generality, that

$$\operatorname{sgn}_1(f(x_{1,k+1}) - v_k(x_{1,k+1})) = +1,$$

i.e.

$$v_k(x_{1,k+1}) \le \ell(x_{1,k+1})$$

and

$$v_k(x_{i,k+1}) \begin{cases} \le \ell(x_{i,k+1}) & \text{if } i \text{ is odd} \\ \ge \mu(x_{i,k+1}) & \text{if } i \text{ is even.} \end{cases}$$

Then if $v \in V_1$, $v \neq v_k$ implies

$$v(x) = v_k(x)$$

for some x, $x_{i,k+1} \le x \le x_{i+1,k+1}$, for each $i = 1,\ldots,n$. Thus $(v - v_k)(x)$ has n zeros. This is a contradiction since V is an n-dimensional Haar space.

Again we check to see if $v_{k+1}(x)$ is the desired best restricted approximation and if it is not we continue.

The proof of the convergence of the algorithm results from a series of lemmas.

<u>Lemma 1.1</u>: If V is an n-dimensional Haar subspace of $C[a,b]$, $A > 0$ is given, $\delta > 0$ and

$$S = \{(x_1,\ldots,x_n) \in R^n: \; a \le x_1 <\ldots< x_n \le b, \text{ with } |x_i - x_{i+1}| \ge \delta > 0 \text{ for } i = 1,\ldots,n\},$$

then there exists $C > 0$ such that for any $v \in V$ with m

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$|v(\tilde{x}_i)| \le A$ for some $(\tilde{x}_1,\ldots,\tilde{x}_n) \in S$, we have $\|v\|_\infty =$ $\max\{|v(x)|:\ x \in [a,b]\} \le C$.

In fact there exists an N such that if

$$v(x) = \sum_{i=1}^{n} \lambda_i w_i(x),$$

then $|\lambda_i| \le N$, $i = 1,\ldots,n$, where $\{w_1,\ldots,w_n\}$ is a fixed basis for V.

Proof: Suppose there is a sequence of functions $v_k(x) = \sum_{i=1}^{n} \lambda_{ik} w_i(x)$ such that for some i_o, $|\lambda_{i_o k}| \to \infty$ as $k \to \infty$. Then we can find a subsequence $\{v_j\} \subseteq \{v_k\}$ such that

$$|\lambda_{i_1 j}| \ge |\lambda_{ij}| \quad \text{for } i = 1,\ldots,n \text{ and all } j,$$

and

$$|\lambda_{i_1 j}| \to \infty \quad \text{as} \quad j \to \infty.$$

Then $\left|\frac{\lambda_{ij}}{\lambda_{i_1 j}}\right| \le 1$ and by taking additional subsequences we can obtain $\{v_\ell\} \subseteq \{v_j\}$ with $\left|\frac{\lambda_{i\ell}}{\lambda_{i_1\ell}}\right| \to \lambda_i$ a finite number.

By assumption, for each k there is an element $\vec{x}_j = (x_{ij},\ldots,x_{nj}) \in S$ with

$$|v_j(x_{ij})| \le A.$$

Since S is compact, by taking another subsequence we can obtain $\{\mu_k\} \subseteq \{v_\ell\}$ with $|\lambda_{ik}| \to \infty$, $\left|\frac{\lambda_{ik}}{\lambda_{i_1 k}}\right| \to \lambda_i$ and $\{\vec{x}_k\} \to \vec{x} = (x_1,\ldots,x_n) \in S$. Now

$$|\mu_k(x_\nu)| = \left|\sum_{i=1}^{n} \lambda_{ik} w_i(x_\nu)\right| \le A + \epsilon$$

for $\nu = 1,\ldots,n$, for any given $\varepsilon > 0$ and k sufficiently large.

Set $M_\nu = \left|\sum_{i=1}^{n} \lambda_i w_i(x_\nu)\right|$. Then consider

$$\mu_k(x_\nu) = \sum_{i=1}^{n} \lambda_{ik} w_i(x_\nu) = \lambda_{i,k} \sum_{i=1}^{n} \frac{\lambda_{ik}}{\lambda_{i_1 k}} w_i(x_\nu).$$

Now for any $\varepsilon > 0$ and k sufficiently large,

$$\left|\sum_{i=1}^{n} \frac{\lambda_{ik}}{\lambda_{i_1 k}} w_i(x_\nu)\right| > M_\nu - \varepsilon$$

since $\sum_{i=1}^{n} \frac{\lambda_{ik}}{\lambda_{i_1 k}} w_i(x_\nu) \to \sum_{i=1}^{n} \lambda_i w_i(x_\nu)$.

But then

$$|\lambda_{i,k}|\cdot\left|\sum_{i=1}^{n} \frac{\lambda_{ik}}{\lambda_{i_1 k}} w_i(x_\nu)\right| \to \infty \quad \text{as} \quad j \to \infty,$$

unless $M_\nu = 0$. But $\sum_{i=1}^{n} \lambda_i w_i(x) = w(x) \in V$, and if $M_\nu = 0$, for $\nu = 1,\ldots,n$, then $w(x)$ has n zeros so $\lambda_i = 0$ for $i = 1,\ldots,n$. This is a contradiction since $\lambda_i = 1$. Thus $|\lambda_{ik}|$ is bounded for each v_k. ∎

<u>Lemma 1.2</u>: If the iteration does not terminate at the $k^{\underline{th}}$ step, then

(i) $\operatorname{sgn}_1(f(x_{i,k+1}) - v_k(x_{i,k+1})) = \operatorname{sgn}_1(f(x_{i,k+1}) - v_{k+1}(x_{i,k+1}))$,

$i = 1,\ldots,n+1$,

(ii) $\operatorname{sgn}_1(f(x_{i,k+1}) - v_{k+1}(x_{i,k+1})) = (-1)^{i+1}\operatorname{sgn}_1(f(x_{1,k+1}) - v_{k+1}(x_{1,k+1}))$, $i = 1,\ldots,n+1$

(iii) $e_{k+1} \geq e_k$,

(iv) $e_{k+1} \geq \max\{f(x_{i,k+1}) - \ell(x_{i,k+1}):\ x_{i,k+1} \in L_{k+1}\}$,

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(v) $e_{k+1} \geq \max \{\mu(x_{i,k+1}) - f(x_{i,k+1}) : x_{i,k+1} \in J_{k+1}\}$,

(vi) v_{k+1} is the best approximation to f on

$\{x_{1,k+1}, \ldots, x_{n+1,k+1}\} = X_{k+1}$ with respect to V_{k+1}

where

$$V_{k+1} = \{v \in V: \ v(x) \geq \ell(x) \ \text{ for all } \ x \in (X_{k+1} \cap L)$$

$$\text{and } \ v(x) \leq \mu(x) \ \text{ for all } \ x \in (X_{k+1} \cap J)\}$$

and $\|f - v\|_{k+1} = \max\limits_{x \in X_{k+1} \cap K} |f(x) - v(x)|$.

<u>Proof</u>: This proof is as in [25] but is included here for completeness. (i) is proved by induction. First

$$\mathrm{sgn}_1(f(x_{i,2}) - v_1(x_{i,2})) = (-1)^{i+1}\mathrm{sgn}_1(f(x_{1,2}) - v_1(x_{1,2}))$$

by construction. Also $L_2 \cup J_2$ consists of at most one point. Without loss of generality, we can assume $\mathrm{sgn}_1(f(x_{1,2}) - v_1(x_{1,2}) = +1$.

1) Suppose $L_2 \cup J_2 = \phi$. If (i) does not hold then $v_1(x_{i,2}) \geq v_2(x_{i,2})$ if i is even and $v_1(x_{i,2}) \leq v_2(x_{i,2})$ if i is odd, since $\mathrm{sgn}_1(f(x_{i,2}) - v_2(x_{i,2})) = (-1)^{i+1}$ $\mathrm{sgn}_1(f(x_{1,2}) - v_2(x_{1,2}))$. So v_2 would meet v_1 at least n times and by the Haar condition $v_2 \equiv v_1$. Then since $L_2 \cup J_2 = \phi$, $e_1 = \|f - v_1\|_K$. Further, since $\gamma_1 = \max\{E_1, M_1, m_1\}$, we have $v_1 \in V_1$ and v_1 is the desired best approximation. This contradicts the assumption that the iteration does not stop. Thus (i) holds.

2) Suppose $L_2 \cup J_2 \neq \phi$. If $\gamma_1 = m_1$, $x_{i_o,2} \in L_2$. If i_o is odd, $\mathrm{sgn}_1(f(x_{i_o,2}) - v_1(x_{i_o,2})) = (-1)^{i_o+1} = 1$ and $v_1(x_{i_o,2}) \leq v_2(x_{i_o,2})$. If $\mathrm{sgn}_1(f(x_{i,2}) - v_2(x_{i,2})) =$

$-\mathrm{sgn}_1(f(x_{i,2}) - v_1(x_{i,2}))$, $i \neq i_o$, then $v_2(x_{i,2}) \geq v_1(x_{i,2})$ if i is odd and $v_2(x_{i,2}) \leq v_1(x_{i,2})$ for i even. Again, counting zeros, $v_2 \equiv v_1$ is the desired approximation contradicting the hypothesis that the iteration does not stop. Other cases follow in a similar manner.

If (i) holds for $k \geq 1$ and v_k is not the best restricted approximation, consider cases:

3) If $L_{k+1} \cup J_{k+1} = \phi$, the argument given in 1) above works.

4) If $L_{k+1} \cup J_{k+1} \neq \phi$, then for $x_{i,k+1} \in L_{k+1}$

$$v_k(x_{i,k+1}) \leq v_{k+1}(x_{i,k+1})$$

and for $x_{i,k+1} \in J_{k+1}$

$$v_k(x_{i,k+1}) \geq v_{k+1}(x_{i,k+1}).$$

Also, $\mathrm{sgn}_1(f(x_{i,k+1}) - v_k(x_{i,k+1})) = +1$ if $x_{i,k+1} \in L_{k+1}$ and $\mathrm{sgn}_1(f(x_{i,k+1}) - v_k(x_{i,k+1})) = -1$ if $x_{i,k+1} \in J_{k+1}$. If $\mathrm{sgn}_1(f(x_{i,k+1}) - v_k(x_{i,k+1})) \neq \mathrm{sgn}_1(f(x_{i,k+1}) - v_{k+1}(x_{i,k+1}))$ for $x_{i,k+1} \in K_{k+1}$, we would again have $v_{k+1} \equiv v_k$ which contradicts the hypothesis, so (i) holds.

Now (ii) is an immediate consequence of (i).

(iii). Suppose that for some k, $e_{k+1} \leq e_k$ and, without loss of generality, $\mathrm{sgn}_1(f(x_{1,k+1}) - v_k(x_{1,k+1})) = +1$. For $x_{i,k+1} \in J_{k+1}$ we must have i even and

$$v_k(x_{i,k+1}) \geq v_{k+1}(x_{i,k+1});$$

and for $x_{i,k+1} \in L_{k+1}$, we must have i odd and

$$v_k(x_{i,k+1}) \le v_{k+1}(x_{i,k+1}).$$

So for $x_{i,k+1} \in K_{k+1}$

$$|f(x_{i,k+1}) - v_k(x_{i,k+1})| \ge |f(x_{i,k+1}) - v_{k+1}(x_{i,k+1})| = e_{k+1}$$

since $|f(x_{i,k+1}) - v_k(x_{i,k+1})| \ge e_k \ge e_{k+1}$. Now by (i),

$$v_k(x_{i,k+1}) \le v_{k+1}(x_{i,k+1}) \quad \text{for } i \text{ odd}$$

and

$$v_k(x_{i,k+1}) \ge v_{k+1}(x_{i,k+1}) \quad \text{for } i \text{ even.}$$

Again counting the zeros of $v_{k+1} - v_k$ and invoking the Haar condition, we see that $v_{k+1} \equiv v_k$ which is a contradiction. So $e_{k+1} > e_k$.

(iv) and (v) are proven in essentially the same manner, so only one will be included here. We shall use induction to prove (v).

If $k = 1$ and $J_2 = \phi$, the conclusion follows. If $k = 1$ and $J_2 \neq \phi$, then $\{x_{i_o,2}\} = J_2$ for some i_o and

$$\begin{aligned} v_1(x_{i_o,2}) - \mu(x_{i_o,2}) = M_1 &> \|f - v_1\|_\infty - e_1 \\ &> -f(x_{i_o,2}) + v_1(x_{i_o,2}) - e_2 \end{aligned}$$

So $e_2 > \mu(x_{i_o,2}) - f(x_{i_o,2})$.

Suppose (v) is true for $k \ge 1$. Then if $x_{i,k+1}$ is such that $x_{i,k+1} \in J_k \cap J_{k+1}$, by (iii) and the induction hypothesis,

$$\begin{aligned} e_{k+1} > e_k &\ge \max \{\mu(x_{i,k}) - f(x_{i,k}): \; x_{i,k} \in J_k \cap J_{k+1}\} \\ &= \max \{\mu(x_{i,k+1}) - f(x_{i,k+1}) : \; x_{i,k+1} \in J_k \cap J_{k+1}\}. \end{aligned}$$

J_k and J_{k+1} can differ by at most one point. If $J_{k+1} \subseteq J_k$, then $J_{k+1} = J_k \cap J_{k+1}$ and (v) is proven by the above statements. If $J_{k+1} \sim J_k = \{x_{i_o,k+1}\}$,

$$v_k(x_{i_o,k+1}) - \mu(x_{i_o,k+1}) = M_k > \|f - v_k\| - e_k$$
$$> v_k(x_{i_o,k+1}) - f(x_{i_o,k+1}) - e_{k+1}.$$

Thus

$$e_{k+1} > \mu(x_{i_o,k+1}) - f(x_{i_o,k+1}).$$

(vi) Using Theorem 1.5 with $K' = X_{k+1} \cap K$, $L' = X_{k+1} \cap L$, $J' = X_{k+1} \cap J$, and $V_1' = V_{k+1}$, we conclude that v_{k+1} is the best restricted approximation to f on K' from V_1'. ■

For convenience of notation, let Σ' denote the sum over those $i \in \{1,\ldots,n+1\}$ for which $x_{i,k} \in K_k$; let Σ'' denote the sum over those i for which $x_{i,k} \in L_k$; let Σ''' denote the sum over those i for which $x_{i,k} \in J_k$.

<u>Lemma 1.3</u>: If the iteration does not terminate at the k^{th} step, for each $k \geq 2$ there exist constants $\lambda_{1k},\ldots,\lambda_{n+1,k}$ satisfying

(i) $e_k = \Sigma'\lambda_{ik}f(x_{ik}) + \Sigma''\lambda_{ik}\ell(x_{ik}) + \Sigma'''\lambda_{ik}\mu(x_{ik})$,

(ii) $\Sigma'|\lambda_{ik}| = 1$,

(iii) $\sum_{i=1}^{n+1} \lambda_{ik}w_j(x_{ik}) = 0$ for $j = 1,\ldots,n$,

(iv) $\lambda_{ik}\ \mathrm{sgn}_1(f(x_{ik}) - v_k(x_{ik})) > 0$ for $i = 1,\ldots,n+1$,

(v) $\sum_{i=1}^{n+1} |\lambda_{ik}| \le A < \infty$ and A is independent of k,

(vi) $e_k - e_{k-1} = \Sigma'|\lambda_{ik}|\cdot||f(x_{ik}) - v_{k-1}(x_{ik})| - e_{k-1}|$

$$+ \Sigma''|\lambda_{ik}|\cdot|\ell(x_{ik}) - v_{k-1}(x_{ik})| + \Sigma'''|\lambda_{ik}||\mu(x_{ik}) - v_{k-1}(x_{ik})|,$$

(vii) $|\lambda_{ik}| \ge \lambda > 0$, λ is independent of i and k.

Proof: By Lemma 1.2 (iv), we conclude that v_k is a best restricted approximation to f on K' (with corresponding L', J') from V_1'. Then by Theorem 1.3 there exists a linear functional

$$L(h) = \sum_{i=1}^{n+1} \beta_i h(x_{ik})$$

with $x_{i_k} \in D_{v_k}$, and

$$\beta_i > 0 \quad \text{if} \quad x_{ik} \in E^+_{v_k} \cup G^+_{v_k},$$

$$\beta_i < 0 \quad \text{if} \quad x_{ik} \in E^-_{v_k} \cup G^-_{v_k},$$

and such that L vanishes on V. Now let $\lambda_{ik} = \beta_i/\Sigma'|\beta_i|$. Then

$$\Sigma'|\lambda_{ik}| = 1,$$

$$\lambda_{ik}\, \mathrm{sgn}_1(f(x_{ik}) - v_k(x_{ik})) > 0 \quad \text{for} \quad i = 1,\dots,n,$$

and

$$\sum_{i=1}^{n+1} \lambda_{ik} w_j(x_{ik}) = 0 \quad \text{for} \quad j = 1,\dots,n.$$

Thus (ii), (iii), and (iv) are valid.

Since $K_k \ne \emptyset$ for all k and

$$v_k(x_{ik}) = \ell(x_{ik}) \quad \text{for} \quad x_{ik} \in L_k,$$

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$$v_k(x_{ik}) = \mu(x_{ik}) \quad \text{for} \quad x_{ik} \in J_k ,$$

(iii) implies

$$\begin{aligned} \Sigma'\lambda_{ik}f(x_{ik}) + \Sigma''\lambda_{ik}\ell(x_{ik}) + \Sigma'''\lambda_{ik}\mu(x_{ik}) \\ &= \Sigma'\lambda_{ik}(f(x_{ik}) - v_k(x_{ik})) \\ &= \Sigma'|\lambda_{ik}||f(x_{ik}) - v_k(x_{ik})| \\ &= e_k \, \Sigma'|\lambda_{ik}| \\ &= e_k . \end{aligned}$$

To prove (v) we shall show that the sequence $\{(x_{1k},\ldots,x_{n+1,k})\}_{k=1}^{\infty}$ is separated. (This proof is the same as in [25] except for the construction of the function which gives the contradiction.) Suppose the sequence is not separated, then we may extract a subsequence $\{(x_{1j},\ldots,x_{n+1,j})\}_{j=1}^{\infty}$ for which there exists a grouping of $(x_{1j},\ldots,x_{n+1,j})$ into $\sigma + 1$ groups $(\sigma \le n-1)$,

$$(x_{1j},\ldots,x_{i_1 j}),\ (x_{i_1+1,j},\ldots,x_{i_2 j}),\ldots,(x_{i_\sigma+1,j},\ldots,x_{n+1,j}),$$

$$\text{for all } j,$$

such that

1) there exists $\epsilon > 0$ so that for any two points x_{ij} and x_{kj} from distinct groups, we have $|x_{ij} - x_{kj}| \ge \epsilon$ for all j. (If there is only one group, this is vacuously true.)

2) Setting $i_o = 0$, $i_{\sigma+1} = n+1$

$$I_j^r = \{x \in S:\ x_{i_r+1,j} \le x \le x_{i_{r+1},j}\},\ r = 0,1,\ldots,\sigma,$$

then there exist $x_o^*,\ldots,x_\sigma^*$ such that

$$x_{i_r+1,j} \to x_r^* \text{ and } x_{i_{r+1},j} \to x_r^* \text{ for } r = 0,1,\ldots,\sigma, \text{ as } j \to \infty.$$

Due to the continuity of f, μ, ℓ and the compactness of K, J, L there exists $\eta > 0$ such that for $x \in J \cap L$ we have

$$\begin{array}{ll} \text{either} & \mu(x) - f(x) \geq \eta \\ \text{or} & f(x) - \ell(x) \geq \eta. \end{array}$$

Let $\delta = \min\{e_1,\eta\}/2$. Let $v_o \in V$ interpolate

$$f(x_r^*) \pm \delta$$

at x_r^*, $r = 0,1,\ldots,\sigma$ where "+" or "-" is chosen as follows:

choose $f(x_r^*) + \delta$ if $x_r^* \in J \cap L$ and $f(x_r^*) - \ell(x_r^*) < \eta$;
choose $f(x_r^*) - \delta$ if $x_r^* \in J \cap L$ and $f(x_r^*) - \ell(x_r^*) \geq \eta$;
choose $f(x_r^*) + \delta$ if $x_r^* \in L \sim J$;
choose $f(x_r^*) - \delta$ if $x_r^* \in J \sim L$.

Otherwise choose "+" or "-" so that

$$v_o(x_r^*) = f(x_r^*) + (-1)^r\delta.$$

Since $\sigma \leq n-1$, v_o exists. We also have

$$\begin{array}{ll} v_o(x_r^*) \geq \ell(x_r^*) & \text{for } x_r^* \in L, \\ v_o(x_r^*) \leq \mu(x_r^*) & \text{for } x_r^* \in J. \end{array}$$

There exists a j_o such that $j \geq j_o$ implies

$$v_o(x) \leq \mu(x) \quad \text{for all } x \in [(\bigcup_{r=0}^{\sigma} I_j^r) \cap J],$$

$$v_o(x) \geq \ell(x) \quad \text{for all} \quad x \in [(\bigcup_{r=0}^{\sigma} I_j^r) \cap L],$$

and $|f(x) - v_o(x)| \leq e_1$ for all $x \in [(\bigcup_{r=0}^{\sigma} I_j^r) \cap K]$.

But this contradicts the fact that $v_j(x)$ is the best restricted approximation to f on $\{x_{1j},\ldots,x_{n+1,j}\}$. Hence, the sequence is separated. Now define a family of functions $\mu_k \in V$ such that

$$\mu_k(x_{ik}) = \text{sgn}\, \lambda_{ik},\ i \in \{1,\ldots,n+1\} \sim \{i_o\},$$

where i_o is the first integer such that $x_{i_o k} \in K_k$. Now by Lemma 1.1, there exists a number $C > 0$ such that $\|\mu_k\|_\infty \leq C$. By (iii)

$$\Sigma''\lambda_{ik}\mu_k(x_{ik}) + \Sigma'''\lambda_{ik}\mu_k(x_{ik}) = -\Sigma'\lambda_{ik}\mu_k(x_{ik})$$

or

$$\Sigma''|\lambda_{ik}| + \Sigma'''|\lambda_{ik}| \leq \Sigma'|\lambda_{ik}|C,$$

and by (ii) $\sum_{i=1}^{n+1} |\lambda_{ik}| \leq C + 1$ for all k.

Now consider

$$\begin{aligned} e_k - e_{k-1} &= \Sigma'\lambda_{ik}f(x_{ik}) + \Sigma''\lambda_{ik}\ell(x_{ik}) + \Sigma'''\lambda_{ik}\mu(x_{ik}) \\ &\quad - \Sigma'|\lambda_{ik}|e_{k-1} \\ &= \Sigma'\lambda_{ik}(f(x_{ik}) - v_{k-1}(x_{ik})) + \Sigma''\lambda_{ik}(\ell(x_{ik}) - v_{k-1}(x_{ik})) \\ &\quad + \Sigma'''\lambda_{ik}(\mu(x_{ik}) - v_{k-1}(x_{ik})) - \Sigma' e_{k-1}|\lambda_{ik}| \\ &= \Sigma'|\lambda_{ik}|\Big||f(x_{ik}) - v_{k-1}(x_{ik})| - e_{k-1}\Big| \\ &\quad + \Sigma''|\lambda_{ik}||\ell(x_{ik}) - v_{k-1}(x_{ik})| \\ &\quad + \Sigma'''|\lambda_{ik}||\mu(x_{ik}) - v_{k-1}(x_{ik})|. \end{aligned}$$

If (vii) does not hold, then by taking subsequences, there is an index i_o $(1 \le i_o \le n+1)$ such that

$$|\lambda_{i_o j}| \to 0 \quad \text{as} \quad j \to \infty.$$

By (v), we may again choose subsequences so that

$$\lambda_{ij} \to \lambda_i \qquad i \neq i_o$$

$$\lambda_{i_o j} \to 0$$

and

$$x_{ij} \to x_i \qquad\qquad \text{for} \quad i = 1,\ldots,n+1.$$

We notice that $x_i \neq x_j$ for $i \neq j$, $i,j = 1,\ldots,n+1$, since $\{x_{1j},\ldots,x_{n+1,j}\}$ is separated. Applying (iii) we conclude

$$\sum_{\substack{i=1\\ i\neq i_o}}^{n+1} \lambda_i w_j(x_i) = 0 \quad \text{for} \quad j = 1,\ldots,n.$$

But, by the Haar condition, this implies $\lambda_i = 0$ for all i which contradicts (ii). Thus (vii) must be satisfied. ∎

We can now show that the algorithm described above is valid.

<u>Theorem 1.7</u>: If the iteration does not terminate after a finite number of steps, the sequence $\{v_k\}_{k=1}^{\infty}$ converges uniformly to the best restricted approximation v^* to f from V_1, and e_k converges to $\|f - v^*\|_K$.

<u>Proof</u>: (This proof is essentially the same as in [25].) We shall first show that $e_k \to e \le \|f - v^*\|_K = \rho$.

1) In Lemma 1.2, we proved v_k is the best restricted approximation to f on $\{x_{1k},\ldots,x_{n+1,k}\}$. Since this is unique (Theorem 1.4), we must have

$$e_k < \rho$$

since $v^* \in V$ and

$$v^*(x) \geq \ell(x) \quad \text{for} \quad x \in \{x_{1k},\ldots,x_{n+1,k}\} \cap L,$$

$$v^*(x) \leq \mu(x) \quad \text{for} \quad x \in \{x_{1k},\ldots,x_{n+1,k}\} \cap J.$$

$e_k < e_{k+1}$ for all k implies $\{e_k\} \to e \leq \rho$.

2) By the above bound on e_k, Lemma 1.2 (iv) and (v) and Lemma 1.1 imply the existence of a number $B > 0$ such that $\|v_k\|_\infty \leq B$ for all k. Then since V is closed and $\{v_k\}_{k=1}^\infty$ is bounded in the norm, there exists a convergent subsequence of $\{v_k\}$ converging to $v \in V$, and $\|f - v\|_\infty = e$.

3) It remains to be shown that $v \in V_1$. By Lemma 1.3 (vi) and (vii)

$$e_k - e_{k-1} \geq \lambda \max \{E_{k-1}, M_{k-1}, m_{k-1}\}.$$

Thus $\limsup_k E_k = 0$, $\limsup_k M_k \leq 0$, $\limsup_k m_k \leq 0$. That is,

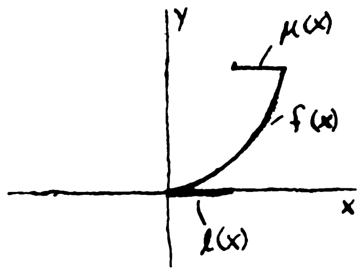
$$v(x) \geq \ell(x) \quad \text{for all} \quad x \in L,$$

$$v(x) \leq \mu(x) \quad \text{for all} \quad x \in J.$$

Since the best restricted approximation is unique, $v = v^*$, $e = \rho = \|f - v\|_K$.

<u>Example</u>: Let $S = [0,1] = K$, $J = [\frac{1}{2},1]$, $L = [0,\frac{1}{2}]$, $V =$ polynomials of degree ≤ 1. $v_1(x) \equiv 1$, $v_2(x) \equiv x$

$f(x) = x^2$ $\qquad\mu(x) = 1 \qquad \ell(x) = 0$



Let $x_{1,1} = 0$, $x_{2,1} = \frac{1}{2}$, $x_{3,1} = 1$

$$\alpha_1(1) + \alpha_2(0) - \alpha_3 = 0$$

$$\alpha_1(1) + \alpha_2(\tfrac{1}{2}) + \alpha_3 = \tfrac{1}{4}$$

$$\alpha_1(1) + \alpha_2(1) - \alpha_3 = 1$$

Solution $\qquad \alpha_1 = -\frac{1}{8} \qquad \alpha_2 = 1 \qquad \alpha_3 = -\frac{1}{8}$

Then $\quad v_1(x) = -\frac{1}{8} + x, \quad e_1 = \frac{1}{8},$

$$E_1 = \{\max_{x\in[0,1]} |x^2 - x + \tfrac{1}{8}|\} - \tfrac{1}{8} = 0$$

$$M_1 = \max_{x\in[\frac{1}{2},1]} \{(-\tfrac{1}{8} + x) - 1\} = -\tfrac{1}{8}$$

$$m_1 = \max_{x\in[0,\frac{1}{2}]} \{0 - (-\tfrac{1}{8} + x)\} = \tfrac{1}{8}$$

Thus $\quad \gamma_1 = m_1 \qquad y_1 = 0 = x_{1,1}$

$$\mathrm{sgn}^*(x_{1,1} - v_1(x_{1,1})) = +1$$

Now set $x_{1,2} = 0$, $x_{2,2} = \frac{1}{2}$, $x_{3,2} = 1$

$$K_2 = \{x_{2,2}, x_{3,2}\}$$
$$J_2 = \phi$$
$$L_2 = \{x_{1,2}\}.$$

Then solve

$$\alpha_1(1) + \alpha_2(0) = 0$$
$$\alpha_1(1) + \alpha_2(\tfrac{1}{2}) + \alpha_3 = \tfrac{1}{4}$$
$$\alpha_1(1) + \alpha_2(1) - \alpha_3 = 1$$

Solution $\alpha_1 = 0$

$$\alpha_2 = \frac{5}{6}$$
$$\alpha_3 = -\frac{1}{6}$$

Then $v_2 \;(x) = \frac{5}{6}x$, $\quad e_2 = \frac{1}{6}$

$$E_2 = \{\max_{x\in[0,1]} |x^2 - \frac{5}{6}x|\} - \frac{1}{6} = + \frac{1}{144}$$

$$M_2 = \max_{x\in[\frac{1}{2},1]} \{\frac{5}{6}x - 1\} = -\frac{1}{6}$$

$$m_2 = \max_{x\in[0,\frac{1}{2}]} \{0 - \frac{5}{6}x\} = 0$$

$\gamma_2 = E_2 \qquad\qquad y_2 = \frac{5}{12}$

$x_{1,3} = 0$, $x_{2,3} = \frac{5}{12}$, $x_{3,3} = 1$

$$\alpha_1(1) + \alpha_2(0) = 0$$
$$\alpha_1(1) + \alpha_2(\tfrac{5}{12}) + \alpha_3 = \frac{25}{144}$$
$$\alpha_1(1) + \alpha_2(1) - \alpha_3 = 1$$

$\alpha_1 = 1$, $\alpha_2 = \frac{169}{204}$, $\alpha_3 = -\frac{35}{204}$, $e_3 = \frac{35}{204} > \frac{1}{6}$

etc.

CHAPTER II

CHEBYSHEV APPROXIMATION WITH RATIONAL FUNCTIONS HAVING RESTRICTED RANGES: CONSIDERATION OF EQUALITY IN THE BOUNDS

Section 1: Introduction

In this chapter we wish to consider the problem presented in Chapter I with rational functions as the approximants. We shall assume $S = J \cup K \cup L$ is a closed interval of the real line, with J, K and L compact subsets of S. K will be assumed to have a sufficient number of points so that two approximants equal on K, are also equal on S. This will be stated more explicitly later.

Let P be the set of functions spanned by $\{w_1,\ldots,w_s\}$ where $w_1,\ldots,w_s$ are s linearly independent functions in $C(S)$, and let Q be the set of functions spanned by $\{v_1,\ldots,v_t\}$ where $v_1,\ldots,v_t$ are t linearly independent functions in $C(S)$. The set R of approximants to be considered will depend on P and Q.

For the first part of this Chapter, as in Chapter I, we shall assume ℓ and μ are given real-valued functions with ℓ continuous on L and μ continuous on J, such that

$$\ell(x) < \mu(x) \quad \text{for all} \quad x \in J \cap L.$$

In the latter part of this chapter we shall allow

$$\ell(x) \le \mu(x) \quad \text{for all} \quad x \in J \cap L,$$

where ℓ and μ are suitably controlled.

In any case we shall restrict our attention to a subset R_1 of R where

$$R_1 = \{r \in R: \ r(x) \le \mu(x) \ \text{ for all } \ x \in J, \text{ and}$$
$$r(x) \ge \ell(x) \ \text{ for all } \ x \in L\}.$$

Assuming $R_1 \neq \phi$, we wish to find $r_o \in R_1$ such that for a given $f \in C(K)$,

$$\|f - r_o\|_K = \inf_{r \in R_1} \|f - r\|_K \equiv \rho.$$

If such an r_o exists, it will be called a best restricted rational approximation to f on K.

Section 2: The Existence Problem

Let P be the set of polynomials of degree less than or equal to n and let Q be the set of polynomials of degree less than or equal to m, and let R be given by

$$R = \{r(x) = (\sum_{i=0}^{n} a_i x^i / \sum_{j=0}^{m} b_j x^j): \ a_i, b_j \ \ (i = 0,\dots,n;\ j = 0,\dots,m)$$
$$\text{are real numbers and } \sum_{j=0}^{m} b_j x^j > 0 \ \text{ for all } \ x \in S\}.$$

It is well known [3, p. 154] that a best unrestricted rational approximation from R exists for every $f \in C(S)$. The following simple example due to Loeb [14] shows that best restricted approximations do not always exist.

Example 2.1: Let $\ell(x) = -1$ for $x \in L = \{0\}$, $\mu(x) = 0$ for $x \in J = \{0\}$, $S = [0,1] = K$, $f(x) = 1$, $P = \{ax + b:\ a,b$ are real numbers$\}$, $Q = \{cx + d:\ c,d$ are real numbers$\}$.
Now let

$$r_k(x) = \frac{x}{x + \frac{1}{k}} \quad \text{for} \quad k = 1,2,\ldots$$

$r_k \in R_1$ for each k. Let $x \in (0,1]$ be fixed, then

$$\lim_{k\to\infty} r_k(x) = 1.$$

Thus the sequence $\{r_k\}$ converges point wise to the function $r(x) \equiv 1$ on $(0,1]$ and the continuous extension of $r(x)$ to $[0,1]$ is $r(x) \equiv 1$. But then $r \notin R_1$. Therefore f does not have a best restricted approximation from R_1.

To eliminate the problem in this example, for this section we shall assume $J = L$ and J contains no isolated points. Since J is a compact subset of the real line with no isolated points, it is a perfect set [18, p. 61].

The proofs of the following existence theorems are similar to the proofs in the standard rational case found in [3, pp. 154-155]. We shall assume $R_1 \neq \emptyset$ in each case.

Theorem 2.1: Let K, $J = L$ be perfect sets. Let P and Q be the polynomials of degree less than or equal to n and m respectively. For a given $f \in C(K)$, a best restricted rational approximation from R_1 to f on K exists.
Proof: Let $\{r_k\}_{k=1}^{\infty}$ be a sequence in R_1 such that

$$\lim_{k\to\infty} \|f - r_k\|_K = \rho$$

where $r_k(x) = p_k(x)/q_k(x)$ with

$$p_k(x) = \sum_{i=0}^{n} a_{ik}x^i \in P$$

and

$$q_k(x) = \sum_{i=0}^{m} b_{ik}x^i \in Q.$$

We lose no generality in assuming $\|q_k\|_\infty = 1$ for each k, thus $\|q_k\|_K \le 1$. Also, for k sufficiently large

$$\|f - r_k\|_K < \rho + \varepsilon.$$

This implies

$$\|r_k\|_K < \|f\|_K + \rho + \varepsilon$$

and

$$\|p_k\|_K \le \|r_k\|_K \cdot \|q_k\|_K < \|f\|_K + \rho + \varepsilon.$$

Then the sequences $\{|a_{ik}|\}_{k=1}^{\infty}$, $\{|b_{jk}|\}_{k=1}^{\infty}$ are bounded sequences for each $i = 0,1,\ldots,n$, $j = 0,1,\ldots,m$ [19, p. 80]. Thus there exists a subsequence of $\{r_k\}$ for which

$$a_{ik} \to a_i \quad \text{for} \quad i = 0,1,\ldots,n$$

and

$$b_{jk} \to b_j \quad \text{for} \quad j = 0,1,\ldots,m$$

with $b_j \neq 0$ for some j since $\|q\|_\infty = 1$. Then

$$p_k(x) \to p(x) = \sum_{i=0}^{n} a_i x^i$$

and

$$q_k(x) \to q(x) = \sum_{j=0}^{m} b_j x^j.$$

Now $q(x) \geq 0$ for all $x \in S$. So for $x \in S$ such that $q(x) \neq 0$, let

$$r(x) = \frac{p(x)}{q(x)} .$$

Then

$$\lim_{k\to\infty} r_k(x) = r(x)$$

for all $x \in S$ such that $q(x) \neq 0$.

$S = J \cup K$ is a perfect compact subset of the real line so q has at most m zeros in S. Since for any $\varepsilon > 0$ and k sufficiently large

$$\|r_k - f\|_K < \rho + \varepsilon,$$

we have

$$|r(x) - f(x)| \leq \rho$$

for all $x \in K$ with $q(x) \neq 0$, and

$$\ell(x) \leq r_k(x) \leq \mu(x) \qquad \text{for } x \in J = L$$

implies

$$\ell(x) \leq r(x) \leq \mu(x)$$

for all $x \in J = L$ with $q(x) \neq 0$.

So

$$|r(x)| \leq \max\{\|\ell\|_L, \|\mu\|_J, \|f\|_K + \rho\} \equiv M$$

for all $x \in S$ such that $q(x) \neq 0$. Let $z \in S$ be such that $q(z) = 0$. Since $p \in P$, $q \in Q$ are continuous functions and for all x near z but different from z

$$|p(x)| \leq M\,|q(x)|,$$

$q(z) = 0$ implies $p(z) = 0$. Since p and q are polynomials with a common zero, they have a common factor. Thus $\frac{p(x)}{q(x)} = \frac{(x-z)^j p_o(x)}{(x-z)^i q_o(x)}$ with $p_o \in P$, $q_o \in Q$, and $p_o(z) \neq 0$, $q_o(z) \neq 0$. Since r is bounded for x near z, $(j-i) \geq 0$. Repeating this argument for all the zeros of q in S, we obtain $p^* \in P$, $q^* \in Q$, $q^*(x) > 0$ for all $x \in S$ and

$$\frac{p(x)}{q(x)} = \frac{p^*(x)}{q^*(x)} \quad \text{for} \quad x \in S \text{ such that } q(x) \neq 0,$$

$$\lim_{x \to z} \frac{p(x)}{q(x)} = \frac{p^*(x)}{q^*(x)} \quad \text{for} \quad z \in S \text{ such that } q(z) = 0.$$

Thus we have $p^*(x)/q^*(x) \in R_1$ since $p^*(x)/q^*(x) \in R$ and for $x \in J = L$

$$\ell(x)q_k(x) \leq p_k(x) \leq \mu(x)q_k(x) \quad \text{for each } k,$$

so letting $k \to \infty$

$$\ell(x)q(x) \leq p(x) \leq \mu(x)q(x),$$

i.e.

$$\ell(x) \leq \frac{p(x)}{q(x)} \leq \mu(x) \quad \text{for} \quad q(x) \neq 0,$$

so by continuity

$$\ell(x) \leq \frac{p^*(x)}{q^*(x)} \leq \mu(x) \quad \text{for all } x \in J = L.$$

Also $\|\frac{p^*}{q^*} - f\|_K \leq \rho$. Thus p^*/q^* is a best restricted rational approximation to f from R_1. ∎

If we let P and Q take on a more general form we must change R slightly in order to insure the existence of best rational approximations. E.W. Cheney [3, p. 155] proved the existence of ordinary best rational approximations from the set R described below.

We shall assume the functions $\{w_1,\ldots,w_s\}$ which span P and $\{v_1,\ldots,v_t\}$ which span Q are two sets of linearly independent analytic functions and that R is as follows:

$$R = \{r(x) \in C(S):\ r(x)q(x) = p(x) \ \text{ for some } \ p(x) \in P \ \text{ and } \ q(x) \in Q\}.$$

We again assume K, J, and L are perfect sets and that J = L.

<u>Theorem 2.2</u>: Let K and J = L be perfect sets and P and Q be subspaces of analytic functions of C(S) of dimensions s and t, respectively. For a given $f \in C(K)$, a best restricted rational approximation to f from R_1 exists.

<u>Proof</u>: Let $\{r_k\}_{k=1}^{\infty}$ be a sequence in R_1 such that

$$\|f - r_k\|_K \to \rho \quad \text{as} \quad k \to \infty;$$

where $p_k \in P$ and $q_k \in Q$ are such that

$$r_k(x)q_k(x) = p_k(x) \quad \text{for all} \quad x \in S.$$

We can again assume $\|q_k\|_\infty = 1$ for each k, then $\|q_k\|_K \le 1$, and for k sufficiently large

$$\|r_k\|_K < \|f\|_K + \rho + \epsilon$$

and

$$\|p_k\|_K \le \|r_k\|_K \|q_k\|_K < \|f\|_K + \rho + \epsilon.$$

Thus by the compactness of K there exist functions $p \in P$ and $q \in Q$ with

$$p_k \to p = \sum_{i=1}^{s} a_i w_i$$

$$q_k \to q = \sum_{j=1}^{t} b_j w_j$$

for a subsequence of $\{r_k\}$. Now define a function $r(x) = p(x)/q(x)$ for $x \in S$, with $q(x) \neq 0$. Then

$$r(x)q(x) = p(x),$$

and

$$\lim_{k\to\infty} r_k(x) = r(x).$$

This implies

$$|r(x) - f(x)| \le \rho \quad \text{for} \quad x \in K$$

and

$$\ell(x) \le r(x) \le \mu(x) \quad \text{for} \quad x \in J = L,$$

whenever $q(x) \neq 0$. Now let

$$M = \max \{ \|f\|_K + \rho, \|\ell\|_L, \|\mu\|_J \}.$$

Then

$$|r(x)| \le M.$$

Now let $z \in S$ be such that $q(z) = 0$. $S = K \cup J$ is a perfect set and q is an analytic function so there is a neighborhood N of z such that $N \subseteq S$ and for $x \in N$, using Taylor series, we obtain

$$q(x) = \sum_{j \geq \nu} c_j (x - z)^j$$

and

$$p(x) = \sum_{i \geq \mu} d_i (x - z)^i$$

with $c_\nu d_\mu \neq 0$. Since r is bounded by M for $x \in N \sim \{z\}$, $\nu \geq \mu$ and

$$\lim_{x \to z} \frac{\sum d_i (x-z)^i}{\sum c_j (x-z)^j} = r(z)$$

is the continuous extension of r to z, with

$$r(z)q(z) = p(z).$$

Now $\lim_{k \to \infty} r_k(x) = r(x)$ for all $x \in S$, thus

$$\ell(x) \leq r(x) \leq \mu(x) \quad \text{for all} \quad x \in J = L$$

and

$$\|r - f\|_K \leq \rho,$$

so r is a best restricted rational approximation to f from R_1. ∎

Section 3: Characterization of Best Restricted Rational Approximations

In this section we will not be concerned with the existence of best approximations but rather with characterizations. Thus we shall assume only that P and Q are s, respectively t, dimensional subspaces of $C(S)$ where J, K, L are arbitrary compact subsets of the real numbers and

$S = J \cup K \cup L$. We shall also assume $\ell(x) < \mu(x)$ for all $x \in J \cap L$. Now let R be as follows:

$$R = \{r \in C(S): \; r(x) = \sum_{i=1}^{s} a_i w_i(x) / \sum_{j=1}^{t} b_j v_j(x) \quad \text{where}$$

$$a_1,\ldots,a_s,\; b_1,\ldots,b_t \quad \text{are arbitrary real numbers}$$

$$\text{and} \quad \sum_{j=1}^{t} b_j v_j(x) > 0 \quad \text{for all} \quad x \in S\}.$$

Then

$$R_1 = \{r \in R: \; \ell(x) \le r(x) \quad \text{for all} \quad x \in L$$

$$\text{and} \quad r(x) \le \mu(x) \quad \text{for all} \quad x \in J\}$$

as before.

For a given $f \in C(K)$ and $r \in R_1$ we again denote the set of critical points as in Chapter I.

$$E_r^+ = \{x \in K: \; f(x) - r(x) = \|f - r\|_K\},$$

$$E_r^- = \{x \in K: \; f(x) - r(x) = -\|f - r\|_K\},$$

$$E_r = E_r^+ \cup E_r^- \; ;$$

$$G_r^+ = \{x \in L: \; r(x) = \ell(x)\},$$

$$G_r^- = \{x \in J: \; r(x) = \mu(x)\},$$

$$G_r = G_r^+ \cup G_r^- \; .$$

Then the following theorem holds for restricted rational approximation. It is a generalization of Theorem 1.2.

<u>Theorem 2.3</u>: Let $f \in C(K)$ and $\inf_{r \in R_1} \|f - r\| \equiv \rho > 0$. Then r_o is not a best restricted rational approximation to f if

there exists a function $\phi \in P + r_oQ$ such that

$$\phi(x) > 0 \quad \text{for all} \quad x \in E^+_{r_o} \cup G^+_{r_o},$$

$$\phi(x) < 0 \quad \text{for all} \quad x \in E^-_{r_o} \cup G^-_{r_o}.$$

The converse holds if there exists an $r_1 \in R_1$ with

$$r_1(x) > \ell(x) \quad \text{for all} \quad x \in L$$

and

$$r_1(x) < \mu(x) \quad \text{for all} \quad x \in J.$$

Proof: If $\phi(x)$ exists, let

$$\phi(x) = p(x) + r_oq(x),$$

and

$$\|\phi\|_\infty = M.$$

Consider $r_\delta(x) = \dfrac{p_o(x) + \delta p(x)}{q_o(x) - \delta q(x)}$, where $r_o(x) = p_o(x)/q_o(x)$. We wish to show that for some $\delta > 0$, r_δ is a better restricted rational approximation to f than r_o. Since $q_o(x) > 0$ on S, there is a δ_o such that $q_o(x) - \delta q(x) > 0$ on S for $\delta \le \delta_o$. Set $\sigma(x) = \operatorname{sgn}(f(x) - r_o(x))$.

Consider the following open sets:

$$O_1 = \{x \in S: \; f(x) - r_o(x) > \frac{\|f-r_o\|_K}{2} \quad \text{and} \quad \phi(x) > 0\},$$

$$O_2 = \{x \in S: \; f(x) - r_o(x) < -\frac{\|f-r_o\|_K}{2} \quad \text{and} \quad \phi(x) < 0\}.$$

$E^+_{r_o} \subseteq O_1$ and $E^-_{r_o} \subseteq O_2$. Let $O = O_1 \cup O_2$. By continuity, we can choose δ_1 small enough so that $f(x) - r_\delta(x)$ has the same sign as $f(x) - r_o(x)$ on O for all $\delta \le \min\{\delta_1, \delta_o\}$.

Now for $x \in O$ and $\delta \le \min\{\delta_1, \delta_o\}$,

$$\begin{aligned}|f(x) - r_\delta(x)| &= \sigma(x)(f(x) - r_\delta(x)) \\ &= \sigma(x)(f(x) - r_o(x)) + \sigma(x)(r_o(x) - r_\delta(x)) \\ &= |f(x) - r_o(x)| - \sigma(x)\frac{\delta\ \phi(x)}{(q_o - \delta q)(x)} \\ &< |f(x) - r_o(x)| \le \|f - r_o\|_K.\end{aligned}$$

Since O is an open set, $K \sim O$ is compact and there is an $\epsilon_1 > 0$ such that for all $x \in K \sim O$,

$$|f(x) - r_o(x)| + \epsilon_1 \le \|f - r_o\|_K.$$

Then for $x \in K \sim O$,

$$\begin{aligned}|f(x) - r_\delta(x)| &\le |f(x) - r_o(x)| + |r_o(x) - r_\delta(x)| \\ &\le \|f - r_o\|_K - \epsilon_1 + \delta\left|\frac{\phi(x)}{(q_o - \delta q)(x)}\right| \\ &< \|f - r_o\|_K\end{aligned}$$

for $\delta \le \min\{\delta_o, \delta_1, \frac{\eta\ \epsilon_1}{2M}\}$ where $\eta = \min\limits_{x \in S}(q_o - \delta_o q)(x)$.

That is $\|f - r_\delta\|_K < \|f - r_o\|_K$.

If $x \in G^+_{r_o}$, $\phi(x) > 0$ and by the compactness of $G^+_{r_o}$ there is an open set U on which $\phi(x) > 0$. Then

$$r_o(x) - r_\delta(x) = \frac{-\delta\ \phi(x)}{(q_o - \delta q)(x)} < 0 \quad \text{for} \quad \delta \le \delta_o,$$

and thus $r_\delta(x) > r_o(x) \ge \ell(x)$ for all $x \in U$. $L \sim U$ is again compact and there exists a number $\epsilon_2 > 0$ such that

$$r_o(x) \ge \ell(x) + \epsilon_2 \quad \text{for all} \quad x \in L \sim U$$

and

$$r_\delta(x) = \frac{\delta\ \phi(x)}{(q_o - \delta q)(x)} + r_o(x) > \ell(x)$$

for $\delta \le \min\{\delta_o, \frac{\epsilon_2 \eta}{2M}\}$.

Similarly for $x \in G^-_{r_o}$, $\phi(x) < 0$. So there is an open set V containing $G^-_{r_o}$ on which $\phi(x) < 0$. Then

$$r_o(x) - r_\delta(x) = -\frac{\delta\ \phi(x)}{(q_o - \delta q)(x)} > 0 \quad \text{for} \quad \delta \le \delta_o$$

and $\mu(x) \ge r_o(x) > r_\delta(x)$ for all $x \in V$. Since $J \sim V$ is compact there exists a number $\epsilon_3 > 0$ such that

$$r_o(x) \le \mu(x) - \epsilon_3 \quad \text{for all} \quad x \in J \sim V,$$

and

$$r_\delta(x) = \frac{\delta\ \phi(x)}{(q_o - \delta q)(x)} + r_o(x) < \mu(x)$$

for $\delta < \min\{\delta_o, \frac{\epsilon_3 \eta}{2M}\}$.

Thus by choosing $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $\delta < \min\{\delta_o, \delta_1, \frac{\epsilon \eta}{2M}\}$, we obtain

$$r_\delta(x) \in R_1$$

and

$$\|f - r_\delta\|_K < \|f - r_o\|_K.$$

Conversely, suppose there exists $r_1 \in R_1$ with

$$r_1(x) > \ell(x) \quad \text{for all} \quad x \in L,$$

$$r_1(x) < \mu(x) \quad \text{for all} \quad x \in J,$$

and that $r_2(x)$ is a better restricted rational approximation to f than r_o, i.e.

$$\|f - r_2\|_K < \|f - r_o\|_K.$$

Let $r_1(x) = p_1(x)/q_1(x)$ and $r_2(x) = p_2(x)/q_2(x)$.

Let $\eta = \min\limits_{x\in S} q_2(x) > 0$ since S is compact. If $r_1(x) \neq r_2(x)$, let $r_\delta = \dfrac{p_2(x) + \delta p_1(x)}{q_2(x) + \delta q_1(x)}$. For all $\delta > 0$, $q_2(x) + \delta q_1(x) > 0$ for all $x \in S$. Thus $r_\delta(x) \in R$. Let

$$\begin{aligned}\phi(x) &= [q_2(x) + \delta q_1(x)](r_\delta(x) - r_o(x)) \\ &= [p_2(x) + \delta p_1(x)] - r_o(x)[q_2(x) + \delta q_1(x)] \in P + r_oQ.\end{aligned}$$

Since $q_2(x) + \delta q_1(x) > 0$ for all $x \in S$,

$$\operatorname{sgn} \phi(x) = \operatorname{sgn} (r_\delta(x) - r_o(x)).$$

$$r_\delta(x) - r_2(x) = \frac{\delta\, q_1(x)}{(q_2+\delta q_1)(x)} (r_1(x) - r_2(x)).$$

Then for $\delta < \{\dfrac{\epsilon\, \eta}{\|q_1\|_\infty(\|r_1-r_2\|_\infty)}\}$,

$$|r_\delta(x) - r_2(x)| < \epsilon \quad \text{for any} \quad \epsilon > 0.$$

Now $\|f - r_2\|_K < \|f - r_o\|_K$ and

$$r_2(x) - r_o(x) = (f(x) - r_o(x)) - (f(x) - r_2(x))$$

implies

$$r_2(x) - r_o(x) > 0 \quad \text{for all} \quad x \in E^+_{r_o},$$

$$r_2(x) - r_o(x) < 0 \quad \text{for all} \quad x \in E^-_{r_o}.$$

The compactness of E_{r_o} implies there exists $\epsilon_1 > 0$ such that

$$|r_2(x) - r_o(x)| > \varepsilon_1 \quad \text{for all} \quad x \in E_{r_o}.$$

Choose δ such that $|r_2(x) - r_\delta(x)| < \varepsilon_1/2$. Then

$$r_\delta(x) - r_o(x) = r_\delta(x) - r_2(x) + r_2(x) - r_o(x)$$

and

$$r_\delta(x) - r_o(x) > 0 \quad \text{for all} \quad x \in E^+_{r_o},$$

$$r_\delta(x) - r_o(x) < 0 \quad \text{for all} \quad x \in E^-_{r_o}.$$

Now let $x \in G^+_{r_o}$, then

$$p_2(x) \geq \ell(x)q_2(x)$$

and

$$p_1(x) > \ell(x)q_1(x).$$

So $\quad r_\delta(x) > \dfrac{\ell(x)q_2(x) + \delta\ell(x)q_1(x)}{q_2(x) + \delta q_1(x)} = \ell(x)\,\dfrac{q_2(x) + \delta q_1(x)}{q_2(x) + \delta q_1(x)}$

$$= \ell(x).$$

Since $r_o(x) = \ell(x)$

$$r_\delta(x) - r_o(x) > 0 \quad \text{for all} \quad x \in G^+_{r_o}.$$

Similarly if $x \in G^-_{r_o}$, then

$$p_2(x) \leq \mu(x)q_2(x)$$

and

$$p_1(x) < \mu(x)q_1(x).$$

So $\quad r_\delta(x) < \dfrac{\mu(x)q_2(x) + \delta\mu(x)q_1(x)}{q_2(x) + \delta q_1(x)} = \mu(x)\,\dfrac{q_2(x) + \delta q_1(x)}{q_2(x) + \delta q_1(x)} = \mu(x),$

and since $r_o(x) = \mu(x)$,

$$r_\delta(x) - r_o(x) < 0 \quad \text{for all} \quad x \in G^-_{r_o}.$$

Thus $\phi(x) = [p_2(x) + \delta p_1(x)] - r_o(x)[q_2(x) + \delta q_1(x)]$ is the desired function.

If $r_1(x) = r_2(x)$, let $\phi(x) = p_2(x) - r_o(x)q_2(x)$ and $\operatorname{sgn} \phi(x) = \operatorname{sgn} (r_2(x) - r_o(x))$ since

$$\phi(x) = q_2(x)(r_2(x) - r_o(x)).$$

We have shown $\phi(x) > 0$ on $E^+_{r_o}$ and $\phi(x) < 0$ on $E^-_{r_o}$. Since $r_2(x) > \ell(x)$ on L and $r_2(x) < \mu(x)$ on J, it follows that $\phi(x) > 0$ on $G^+_{r_o}$ and $\phi(x) < 0$ on $G^-_{r_o}$. ∎

Let $f \in C(K)$ and $r \in R_1$. Again we can say r is a best restricted rational approximation to f if

$$(E^+_r \cup G^+_r) \cap (E^-_r \cup G^-_r) \neq \phi$$

as in Chapter I:

1. $E^+_r \cap G^-_r \neq \phi$ or $E^-_r \cap G^+_r \neq \phi$ implies that to get closer to f we would have to take $r \notin R_1$.
2. $E^+_r \cap E^-_r \neq \phi$ implies $\|f - r\|_K = 0$.

Since we shall again be primarily interested in those $f \in C(K)$ for which this intersection is empty for all $r \in R_1$, let

$$\tilde{C}(K) = \{f \in C(K): \ \rho \equiv \inf_{r \in R_1} \|f - r\|_K > 0 \ \text{ and}$$

$$(E^+_r \cup G^+_r) \cap (E^-_r \cup G^-_r) = \phi \ \text{ for all } \ r \in R_1\}.$$

Condition H for the case of rational functions becomes:

<u>Condition H</u>: The subspace R of $C(S)$ will be said to satisfy condition H if there exists an element $r_1 \in R_1$ such that

$$r(x) < \mu(x) \quad \text{for all} \quad x \in J$$

and

$$r(x) > \ell(x) \quad \text{for all} \quad x \in L.$$

<u>Remark</u>: If R_1 contains two distinct elements r_1 and r_2 and either $P + r_1Q$ or $P + r_2Q$ is a Haar subspace, then R satisfies condition H.

<u>Proof</u>: Suppose $P + r_2Q$ is a Haar subspace of dimension d. Let

$$r_1(x) = p_1(x)/q_1(x),$$

$$r_2(x) = p_2(x)/q_2(x).$$

Then

$$r_o(x) = \frac{p_1(x) + p_2(x)}{q_1(x) + q_2(x)} \in R.$$

Further, since $r_1(x), r_2(x) \geq \ell(x)$ for all $x \in L$ and $r_1(x), r_2(x) \leq \mu(x)$ for all $x \in J$,

$$r_o(x) \geq \frac{q_1(x)\ell(x) + q_2(x)\ell(x)}{q_1(x) + q_2(x)} = \ell(x) \quad \text{for all} \quad x \in L,$$

$$r_o(x) \leq \frac{q_1(x)\mu(x) + q_2(x)\mu(x)}{q_1(x) + q_2(x)} = \mu(x) \quad \text{for all} \quad x \in J,$$

with equality occurring if and only if both $r_1(x)$ and $r_2(x)$ intersect the bounding curve at that point. Since $P + r_2Q$ is a Haar subspace containing $p_1 - r_2q_1 = q_1(r_1 - r_2)$, r_o can intersect ℓ and μ in at most d-1 points. Then construct $\phi = p^* + r_2q^* \in P + r_2Q$ with $\phi(x) = +1$ for $x \in G^+_{r_o}$, $\phi(x) = -1$ for $x \in G^-_{r_o}$ [3, p. 78]. Then

$$(p^* + \ell q^*)(x) > 0 \quad \text{for} \quad x \in G^+_{r_o} \quad (\text{here } r_o = r_2 = \ell),$$

$$(p^* + \mu q^*)(x) < 0 \quad \text{for} \quad x \in G^-_{r_o} \quad (\text{here } r_o = r_2 = \mu).$$

Now there exists an open set U in L containing $G^+_{r_o}$ on which $p^* + \ell q^* > 0$ and an open set V in J containing $G^-_{r_o}$ on which $p^* + \mu q^* < 0$. Also we can find positive real numbers ϵ_1, ϵ_2 such that

$$r_o(x) \geq \ell(x) + \epsilon_1 \quad \text{for} \quad x \in L \sim U,$$

$$r_o(x) \leq \mu(x) - \epsilon_2 \quad \text{for} \quad x \in J \sim V.$$

There is a $\delta_o > 0$ such that for $\delta \leq \delta_o$, $(q_o - \delta q^*)(x) > 0$ on S.

Now assume $\delta \leq \delta_o$ and let $x \in U$. Then

$$\frac{(p_o + \delta p^*)(x)}{(q_o - \delta q^*)(x)} > \frac{(\ell q_o - \delta \ell q^*)(x)}{(q_o - \delta q^*)(x)} = \ell(x).$$

And $x \in V$ implies

$$\frac{(p_o + \delta p^*)(x)}{(q_o - \delta q^*)(x)} < \frac{(\mu q_o - \delta \mu q^*)(x)}{(q_o - \delta q^*)(x)} = \mu(x).$$

By continuity we can choose δ sufficiently small so that

$$\frac{(p_o + \delta p^*)(x)}{(q_o - \delta q^*)(x)} > \ell(x) \quad \text{on} \quad L \sim U$$

and

$$\frac{(p_o + \delta p^*)(x)}{(q_o - \delta q^*)(x)} < \mu(x) \quad \text{on} \quad J \sim V. \qquad \blacksquare$$

Now, as in Theorem 1.3, we shall characterize best approximations by means of a linear functional.

Theorem 2.4: Suppose R satisfies condition H. Then a necessary and sufficient condition for $r^* \in R_1$ to be a best restricted rational approximation to $f \in \tilde{C}(K)$ is that there exist k ($\leq \dim (P + r^*Q) + 1$) critical points

$$x_1, \ldots, x_k \quad \text{in} \quad E_{r^*} \cup G_{r^*}$$

such that $\{x_1, \ldots, x_k\} \cap E_{r^*} \neq \phi$; and a linear functional L defined by

$$L(h) = \sum_{i=1}^{k} \lambda_i h(x_i)$$

such that L vanishes on $P + r^*Q$ and

$$\lambda_i > 0 \quad \text{for} \quad x_i \in E^+_{r^*} \cup G^+_{r^*},$$

$$\lambda_i < 0 \quad \text{for} \quad x_i \in E^-_{r^*} \cup G^-_{r^*}.$$

Proof: (Sufficiency) Suppose r^* satisfies the hypotheses and r is a better restricted rational approximation to f. Then

$$\|f - r\|_K < \|f - r^*\|_K.$$

For $p(x)$ and $q(x)$ such that $r(x) = p(x)/q(x)$, consider $p(x) - r^*(x)q(x)$. Now

$$\operatorname{sgn}\,(p(x) - r^*(x)q(x)) = \operatorname{sgn}\,(r(x) - r^*(x))$$

since $q(x) > 0$ for all $x \in S$.

$$r(x_i) - r^*(x_i) > 0 \quad \text{for all} \quad x_i \in E^+_{r^*} \quad (\lambda_i > 0),$$
$$r(x_i) - r^*(x_i) < 0 \quad \text{for all} \quad x_i \in E^-_{r^*} \quad (\lambda_i < 0),$$
$$r(x_i) - r^*(x_i) \geq 0 \quad \text{for all} \quad x_i \in G^+_{r^*} \quad (\lambda_i > 0),$$
$$r(x_i) - r^*(x_i) \leq 0 \quad \text{for all} \quad x_i \in G^-_{r^*} \quad (\lambda_i < 0).$$

By hypothesis, at least one $x_i \in E_{r^*}$, so

$$L(p(x) - r^*(x)q(x)) > 0.$$

This is a contradiction to L vanishing on $P + r^*Q$, so r^* is a best restricted rational approximation.

(Necessity) Let r^* be a best restricted rational approximation to f with corresponding sets E_{r^*}, G_{r^*}. Let $\{p_1,\ldots,p_n,r^*q_1,\ldots,r^*q_m\}$ be a basis for $P + r^*Q$. Let Γ be defined by

$$\begin{aligned}\Gamma = \{(z_1,\ldots,z_{n+m}) \in R^{n+m}\colon\ & z_i = p_i(x),\ i = 1,\ldots,n;\\ & z_{n+i} = r^*(x)q_i(x),\ i = 1,\ldots,m \quad \text{for} \quad x \in E^+_{r^*} \cup G^+_{r^*}\}\\ \cup \{(z_1,\ldots,z_{n+m}) \in R^{n+m}\colon\ & z_i = -p_i(x),\ i = 1,\ldots,n;\\ & z_{n+i} = -r^*(x)q_i(x),\ i = 1,\ldots,m \quad \text{for} \quad x \in E^-_{r^*} \cup G^-_{r^*}\}.\end{aligned}$$

$0 \in co(\Gamma)$ since otherwise, by the theorem on linear inequalities [3, p. 19] there is a vector $(c_1,\dots,c_{n+m})$ such that $\sum_{i=1}^{n+m} c_i z_i > 0$ for all $\vec{z} \in \Gamma$. But then $\phi(x) = \sum_{i=1}^{n} c_i p_i(x) + \sum_{i=1}^{m} c_{n+i} r^* q_i(x)$ is positive on $E^+_{r^*} \cup G^+_{r^*}$ and negative on $E^-_{r^*} \cup G^-_{r^*}$. This contradicts Theorem 2.3. But $0 \in co(\Gamma)$ implies there exist k $(\le n+m+1)$ positive constants $\beta_1,\dots,\beta_k$ and k points $\vec{z}_1,\dots,\vec{z}_k$ of Γ such that

$$0 = \sum_{i=1}^{k} \beta_i \vec{z}_i .$$

Letting

$$\epsilon_i = \begin{cases} +1 & \text{if } \vec{z}_i = (p_1(x_i),\dots,p_n(x_i), r^* q_1(x_i),\dots,r^* q_m(x_i)) \\ -1 & \text{if } \vec{z}_i = (-p_1(x_i),\dots,-p_n(x_i),-r^* q_1(x_i),\dots,-r^* q_m(x_i)) \end{cases}$$

and $\lambda_i = \beta_i \epsilon_i$, we obtain

$$L(h) = \sum_{i=1}^{k} \lambda_i h(x_i)$$

which is a continuous linear functional vanishing on $P + r^* Q$ with

$$\lambda_i > 0 \quad \text{for} \quad x_i \in E^+_{r^*} \cup G^+_{r^*},$$

$$\lambda_i < 0 \quad \text{for} \quad x_i \in E^-_{r^*} \cup G^-_{r^*} .$$

At least one of the x_i's must be in E_{r^*}. Suppose not; then for each $r \in R_1$, $r(x) = p(x)/q(x)$, we have

$$r(x_i) = \ell(x_i) \quad \text{for} \quad x_i \in G^+_{r^*},$$

$$r(x_i) = \mu(x_i) \quad \text{for} \quad x_i \in G^-_{r^*}.$$

Indeed, if there is an $r \in R_1$ with

$$r(x_i) > \ell(x_i) \quad \text{for some} \quad x_i \in G^+_{r^*},$$

then

$$r(x_i) - r^*(x_i) > 0$$

and

$$p(x_i) - r^* q(x_i) > 0.$$

Thus

$$\lambda_i(p(x_i) - r^* q(x_i)) > 0$$

so $L(p - r^* q) > 0$ which is a contradiction. ∎

Remark: If $P + r^*Q$ is a d-dimensional Haar subspace, then $k = d + 1$. (For proof see Remark 1 following Theorem 1.3.)

Although Theorem 1.4 has no direct generalization to the case of rational approximation, the following theorem, valid in the standard case [3, p. 164] and in ordinary restricted rational approximation [15] remains valid in our case.

Theorem 2.5: Let $f \in \tilde{C}(K)$ and $r^* \in R_1$ be a best restricted rational approximation to f. If $P + r^*Q$ is a Haar subspace then r^* is unique.

Proof: Suppose $P + r^*Q$ is a Haar subspace of dimension d and that $r_o(x) = p_o(x)/q_o(x)$ is also a best restricted rational approximation to f. Then $p_o - r^* q_o \in P + r^*Q$ and

$$r_o - r^* = \frac{1}{q_o}(p_o - r^* q_o).$$

Since $q_0(x) > 0$ for all $x \in S$, we have

$$\operatorname{sgn}(p_o - r^* q_o) = \operatorname{sgn}(r_o - r^*).$$

Now the linear functional characterizing r^* as a best approximation must be based on $d + 1$ points by the Remark following Theorem 2.4. Further, $\|f - r_o\|_K = \|f - r^*\|_K$, thus for $x \in E_{r^*}$,

$$\operatorname{sgn}(r_o - r^*)(x) = \operatorname{sgn}[(f - r^*)(x) - (f - r_o)(x)],$$

i.e.,

$$(r_o - r^*)(x)\begin{cases} \geq 0 & \text{for } x \in E^+_{r^*}, \\ \leq 0 & \text{for } x \in E^-_{r^*}. \end{cases}$$

Also $x \in G^+_{r^*}$ implies $r^*(x) = \ell(x)$ so

$$(r_o - r^*)(x) \geq 0,$$

and $x \in G^-_{r^*}$ implies $r^*(x) = \mu(x)$ so

$$(r_o - r^*)(x) \leq 0.$$

But then $L(p_o - r^* q_o) \geq 0$ since $\lambda_i > 0$ for $x_i \in E^+_{r^*} \cup G^+_{r^*}$ and $\lambda_i < 0$ for $x_i \in E^-_{r^*} \cup G^-_{r^*}$. However $p_o - r^* q_o \in P + r^* Q$, thus $L(p_o - r^* q_o) = 0$ and $(p_o - r^* q_o)(x)$ must have $n + 1$ zeros. Thus $p_o \equiv r^* q_o$ and $r_o \equiv r^*$. ∎

In the case of ordinary restricted rational approximation, an alternation theorem analogous to Theorem 1.5 is valid [15]. This is also true here.

Theorem 2.6: Let $S = [a,b]$, $f \in \tilde{C}(K)$ and R satisfy condition H. Let $r \in R_1$.

1. If $e(x) = f(x) - r(x)$ has at least $2 + \nu$ alternations (i.e. there are distinct points

$$x_o < x_1 < \ldots < x_{\nu+1} \quad \text{in} \quad [a,b]$$

with at least one of the following holding at each point

a. $$|e(x_i)| = \|e\|_K, \; x_i \in K,$$

or

b. $$r(x_i) = \ell(x_i), \; x_i \in L,$$

or

c. $$r(x_i) = \mu(x_i), \; x_i \in J,$$

and for

$$\sigma(x_i) = \begin{cases} +1 & \text{if} \quad e(x_i) = \|e\|_K \quad \text{or} \quad r(x_i) = \ell(x_i), \\ -1 & \text{if} \quad e(x_i) = -\|e\|_K \quad \text{or} \quad r(x_i) = \mu(x_i), \end{cases}$$

$\sigma(x_i) = (-1)^i \sigma(x_o)$ also holds for $i = 0,1,\ldots,\nu+1$) where ν is the maximum number of zeros of elements in $P + rQ$, then r is a best restricted rational approximation to f from R_1.

2. If r is a best restricted rational approximation to f, then e has at least $1 + \eta$ alternations where η is the dimension of the largest Haar subspace of $P + rQ$.

Proof: 1. Suppose $e(x)$ has $2 + \nu$ alternations and r is not the best restricted rational approximation to f from R_1. Then by Theorem 2.3 there exists $\phi(x) \in P + rQ$ such that

$$\phi(x) > 0 \quad \text{for all} \quad x \in E_r^+ \cup G_r^+$$

and

$$\phi(x) < 0 \quad \text{for all} \quad x \in E_r^- \cup G_r^- .$$

But e alternates $2+\nu$ times, thus $\phi(x)$ must have at least $1+\nu$ zeros since it is continuous. This is a contradiction, thus r must be a best restricted rational approximation to f from R_1.

2. Let r be a best restricted rational approximation to f from R_1. Let M be a Haar subspace of $P + rQ$ of dimension η with basis $\phi_1,\ldots,\phi_\eta$. Theorem 2.3 implies there does not exist $\phi \in M$ with

$$\phi(x)\begin{cases} > 0 & \text{for all} \quad x \in E_r^+ \cup G_r^+ \\ < 0 & \text{for all} \quad x \in E_r^- \cup G_r^- . \end{cases}$$

Then by the theorem on linear inequalities [3, p. 19],

$$\vec{0} \in \text{co}\,(\{(\varphi_1(x),\ldots,\varphi_\eta(x)):\ x \in E_r^+ \cup G_r^+\}$$
$$\cup \{(-\varphi_1(x),\ldots,-\varphi_\eta(x)):\ x \in E_r^- \cup G_r^-\}).$$

So, by the theorem of Caratheodory there exist $k+1$ $(\le \eta + 1)$ points $x_o < x_1 < \ldots < x_k$ in $E_r \cup G_r$ and positive numbers $\beta_0,\ldots,\beta_k$ such that

$$\sum_{i=0}^{k} \beta_i \epsilon_i \varphi_j(x_i) = 0 \quad \text{for} \quad j = 1,\ldots,\eta$$

where $\epsilon_i = +1$ if $x_i \in E_r^+ \cup G_r^+$ and $\epsilon_i = -1$ if $x_i \in E_r^- \cup G_r^-$. Then for $\lambda_i = \beta_i \epsilon_i$,

$$\sum_{i=0}^{k} \lambda_i \varphi_j(x_i) = 0 \quad \text{for} \quad j = 1,\ldots,\eta.$$

Since $k > \eta$ contradicts the Haar condition, $k = \eta$ and by a well-known lemma for Haar systems [3, p. 74] the λ_i's alternate in sign. This means that e alternates in sign at least $\eta + 1$ times since the sign of λ_i is determined by the critical point $x_i \in E_r \cup G_r$. ∎

It was mentioned in Chapter I that a Strong Uniqueness Theorem holds for rational approximations both in the standard theory and in the restricted case. Likewise it is valid here. The proof uses the following lemma found in [3, p. 165].

<u>Lemma 2.1</u>: Let $r^* = p^*/q^* \in R$ be such that for $P + r^*Q$ as a subspace of $C(K)$ we have $\dim (P + r^*Q) = d = s + t - 1$. If $p \in P$, $q \in Q$ satisfy

$$\|p\|_K + \|q\|_K = \|p^*\|_K + \|q^*\|_K,$$

$$p = r^* q,$$

and

$$q(x) \geq 0 \quad \text{for all} \quad x \in K,$$

then $p = p^*$ and $q = q^*$ on K.

<u>*Proof*</u>: If $r^* \equiv 0$, then $p^* \equiv 0$ and $p \equiv 0$. Furthermore $\dim Q = 1$. Since $\|q\|_K = \|q^*\|_K$ and $q(x)q^*(x) \geq 0$, it follows that $q = q^*$ on K.

If $r^* \not\equiv 0$, then $p = r^* q$ and $p^* = r^* q^*$ implies $p, p^* \in P \cap r^*Q$. However,

$$\dim (P + r^*Q) \leq \dim P + \dim Q - \dim (P \cap r^*Q).$$

Thus dim $(P \cap r^*Q) \leq 1$, and so p is a scalar multiple of p^*. Since $\|p\|_K = \|p^*\|_K$ and $p(x)p^*(x) \geq 0$, $p^* = p$ and $q^* = q$ on K. ∎

We shall assume that P, Q, K and S are such that if $p_1, p_2 \in P$, $q_1, q_2 \in Q$ are such that $p_1 \equiv p_2$, $q_1 \equiv q_2$ on K, then $p_1 \equiv p_2$ and $q_1 \equiv q_2$ on S. This will be the case if P, Q are spaces of analytic functions of $C(S)$ and K has an infinite number of points or if P, Q are Haar subspaces of $C(S)$ and K contains at least maximum $\{s,t\}$ points. We shall assume K is a perfect set (this implies K contains an infinite number of points).

<u>Theorem 2.7</u>: Let r^* be a best restricted rational approximation to $f \in \tilde{C}(K)$ from R_1. If $P + r^*Q$ is a Haar subspace of $C(S)$ of dimension $s + t - 1 = d$, then there exists a number $\gamma > 0$ such that for all $r \in R_1$,

$$\|f - r\|_K \geq \|f - r^*\|_K + \gamma\|r - r^*\|_K.$$

<u>Proof</u>: If $r \equiv r^*$ we can choose any positive number for γ. Thus we shall assume $r \not\equiv r^*$. Suppose no such γ exists. Then there is a sequence $\{r_n = p_n/q_n\} \leq R_1$ with

$$\gamma_n = \frac{\|f-r_n\|_K - \|f-r^*\|_K}{\|r^* - r_n\|_K}$$

and $\gamma_n \to 0$ as $n \to \infty$.

We may assume

$$\|p_n\|_K + \|q_n\|_K = 1.$$

Then by the compactness of P and Q, there exist $p_o \in P$, $q_o \in Q$ such that $\{p_n\}$ converges uniformly to p_o on K, $\{q_n\}$ converges uniformly to q_o on K and

$$\|p_o\|_K + \|q_o\|_K = 1.$$

Setting $r_o = p_o/q_o$ whenever $q_o \neq 0$, we have $r_n \to r_o$. Also $\gamma_n \to 0$ and

$$\gamma_n \geq \frac{\|r_n\|_K - \|f\|_K - \|f - r^*\|_K}{\|r^* - r_n\|_K}$$

implies $\|r_n\|_K$ and $\|r^* - r_n\|_K$ are bounded.

Now r^* is a best restricted rational approximation to f so there is a continuous linear functional L vanishing on $P + r^*Q$ where

$$L(h) = \sum_{i=1}^{d+1} \lambda_i h(x_i)$$

with $x_i \in E_{r^*} \cup G_{r^*}$, at least one $x_i \in E_{r^*}$, and

$$\lambda_i \begin{cases} > 0 & \text{for } x_i \in E^+_{r^*} \cup G^+_{r^*}, \\ < 0 & \text{for } x_i \in E^-_{r^*} \cup G^-_{r^*}. \end{cases}$$

Let $\sigma(x_i) = \operatorname{sgn} \lambda_i$. Then for $r = p/q \in R_1$,

$$\sigma(x_i)(p - r^*q)(x_i) \geq 0 \quad \text{for } x_i \in G_{r^*}.$$

Thus for $r_n = p_n/q_n$

$$\sigma(x_i)(p_n - r^*q_n)(x_i) \geq 0 \quad \text{for } x_i \in G_{r^*},$$

and taking limits

$$\sigma(x_i)(p_o - r^* q_o)(x_i) \geq 0 \quad \text{for} \quad x_i \in G_{r^*}.$$

Now for $x_i \in E_{r^*}$,

$$\gamma_n \|r^* - r_n\|_K = \|f - r_n\|_K - \|f - r^*\|_K$$

$$\geq \sigma(x_i)(f - r_n)(x_i) - \sigma(x_i)(f - r^*)(x_i)$$

$$= \sigma(x_i)(r^* - r_n)(x_i).$$

Thus letting $n \to \infty$,

$$0 \geq \sigma(x_i)(r^* - r_o)(x_i).$$

Since $q_o(x) \geq 0$

$$0 \geq \sigma(x_i)(r^* q_o - p_o)(x_i)$$

or

$$\sigma(x_i)(p_o - r^* q_o)(x_i) \geq 0.$$

But $L(p_o - r^* q_o) = 0$, so

$$\sigma(x_i)(p_o - r^* q_o)(x_i) = 0 \quad \text{for} \quad i = 1, \ldots, d+1$$

and since $P + r^* Q$ is a Haar subspace of dimension d, $p_o \equiv r^* q_o$. Then, using the lemma, we conclude $p_o \equiv p^*$ and $q_o \equiv q^*$. Now $q_n \to q^*$ and $q^*(x) > 0$ for all $x \in K$ implies there exists a number $\delta > 0$ such that $q_n(x) > \delta$ for all $x \in K$ and n sufficiently large.

If $x_i \in G_{r^*}$ and $r = p/q \in R_1$, we have

$$\sigma(x_i)(p - r^* q)(x_i) \geq 0$$

and $L(p - r^* q) = 0$, so

$$\max_{x_i \in E_{r^*}} \sigma(x_i)(r^* q - p)(x_i) > 0$$

since $P + r^*Q$ is a Haar subspace and no $\phi \in P + r^*Q$ can have zeros at all $x_i \in E_{r^*} \cup G_{r^*}$. Thus there is a number $c > 0$ such that

$$\inf_{\substack{\phi = p - r^* q \in T \\ \|\phi\|_K = 1}} \max_{x_i \in E_{r^*}} \sigma(x_i)(r^* q - p)(x_i) = c$$

where T is the closed set

$$T = \{p - r^* q \in P + r^* Q: \ \sigma(x_i)(p - r^* q)(x_i) \geq 0$$
$$\text{for all } x_i \in G_{r^*}\}.$$

(Notice that if $p/q \in R_1$, then $p - r^*q \in T$.) Now for r_n, let $x_{in} \in E_{r^*}$ be such that

$$\max_{x_i \in E_{r^*}} \sigma(x_i)(r^* q_n - p_n)(x_i) = \sigma(x_{in})(r^* q_n - p_n)(x_{in}).$$

Then consider the following:

$$\begin{aligned}
\gamma_n \|r^* - r_n\|_K &= \|f - r_n\|_K - \|f - r^*\|_K \\
&\geq \sigma(x_{in})(f - r_n)(x_{in}) - \sigma(x_{in})(f - r^*)(x_{in}) \\
&= \sigma(x_{in})(r^* - r_n)(x_{in}) \\
&= \sigma(x_{in})(r^* q_n - p_n)(x_{in}) \frac{1}{q_n(x_{in})} \\
&\geq \sigma(x_{in})(r^* q_n - p_n)(x_{in}) \\
&\geq c \|r^* q_n - p_n\|_K \\
&\geq c \ \delta \|r^* - r_n\|_K.
\end{aligned}$$

That is, $\gamma_n \geq c\ \delta$. But this contradicts $\gamma_n \to 0$. Thus there is a $\gamma > 0$ such that

$$\|f - r\|_K \geq \|f - r^*\|_K + \gamma\|r - r^*\|_K$$

for all $r \in R_1$. ∎

The continuity of the best approximation operator can now easily be shown as in the polynomial case.

__Corollary__: Let $f^* \in \tilde{C}(K)$ with best restricted rational approximation $r^* \in R_1$. Let $P + r^*Q$ be a Haar subsapce of $C(S)$ of dimension $s + t - 1$. Then there exists a number $\beta > 0$ such that for any $f \in \tilde{C}(K)$ with a corresponding best restricted rational approximation r,

$$\|r^* - r\|_K \leq \beta\|f^* - f\|_K.$$

__Proof__: For any $f \in \tilde{C}(K)$ with corresponding best restricted rational approximation r, the previous theorem implies

$$\gamma\|r - r^*\|_K \leq \|f^* - r\|_K - \|f^* - r^*\|_K.$$

Thus

$$\begin{aligned}\gamma\|r - r^*\|_K &\leq \|f^* - f\|_K + \|f - r\|_K - \|f^* - r^*\|_K \\ &\leq \|f^* - f\|_K + \|f - r^*\|_K - \|f^* - r^*\|_K \\ &\leq \|f^* - f\|_K + \|f - f^*\|_K .\end{aligned}$$

So

$$\|r - r^*\|_K \leq 2\gamma^{-1}\|f - f^*\|_K. \quad (\beta = 2\gamma^{-1}).$$ ∎

Section 4: Equaltiy in the Bounding Curves

In all our considerations in Chapter I and in the first two sections of this chapter, we have assumed that the functions ℓ and μ satisfy

$$\ell(x) < \mu(x) \quad \text{for all} \quad x \in J \cap L.$$

In this section we wish to investigate the problem of finding a best restricted rational approximation if we allow $\ell(x) = \mu(x)$ for some $x \in J \cap L$. Since results for the case of generalized rational approximation are valid for generalized polynomial approximation, we will consider the following problem:

Let $S = [a,b]$ be a finite interval of the real line and K, $J = L$ perfect sets, Let $\ell(x)$, $\mu(x)$ be continuous real valued functions on $J = L$ with $\ell(x) \leq \mu(x)$ for $x \in J$.

Let P be the subspace of $C(S)$ spanned by the s linearly independent elements $w_1(x),\ldots,w_s(x)$ and Q the subspace of $C(S)$ spanned by the t linearly independent elements $v_1(x),\ldots,v_t(x)$. Now for a fixed $f \in C(K)$ the existence Theorems 2.1 and 2.2 are valid for the corresponding sets R and $R_1 \leq R$, when $R_1 \neq \phi$. No special properties are required of ℓ and μ other than those imposed in the first part of this chapter.

However the characterization theorems in the second section of this chapter required

$$\ell(x) < \mu(x) \quad \text{for all} \quad x \in J.$$

G.D. Taylor [24] and L.L. Schumaker and G.D. Taylor [22] considered the problem of existence and characterizations of best ordinary restricted approximation (where $S = K = J = L = [a,b]$) to a given function $f \in C(S)$ for the equality case by extended Chebyshev polynomials, and remarked that for ordinary rational functions the same results could be obtained. The concept of an extended Chebyshev system, found in [8, p. 6], is very useful here.

<u>Definition</u>: Let U be the space spanned by n linearly independent functions $\mu_1,\ldots,\mu_n$ in $C[a,b]$. U will be called an extended Chebyshev system of order p provided $\mu_i \in C^{(p-1)}[a,b]$, $i = 1,\ldots,n$ and for all choices t_i, $i = 1,\ldots,n$,

$$a \le t_1 \le \ldots \le t_n \le b,$$

(equality can occur in groups of at most p consecutive t_i's)

$$U^*\begin{pmatrix}1,\ldots,n \\ t_1,\ldots,t_n\end{pmatrix} = \begin{vmatrix} \tilde{\mu}_1(t_1) & \tilde{\mu}_1(t_2) & \cdots & \tilde{\mu}_1(t_n) \\ \tilde{\mu}_2(t_1) & \tilde{\mu}_2(t_2) & \cdots & \tilde{\mu}_2(t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{\mu}_n(t_1) & \tilde{\mu}_n(t_2) & \cdots & \tilde{\mu}_n(t_n) \end{vmatrix} > 0$$

where $\tilde{\mu}_i(t_j) = \mu_i(t_j)$ if $t_{j-1} < t_j$; $\tilde{\mu}_i(t_j) = \mu_i^{(s)}(t_j)$ if $t_{j-s-1} < t_{j-s} = \ldots = t_j$, $1 \le i \in n$.

Let $v \in U$. We say v has a zero of order ν $(\le p-1)$ at $t_o \in [a,b]$ if $v(t_o) = v'(t_o) = \ldots = v^{(\nu-1)}(t_o) = 0$ and $v^{(\nu)}(t_o) \ne 0$. We say v has a zero of order at least p at $t_o \in [a,b]$ if $v(t_o) = v'(t_o) = \ldots = v^{(p-1)}(t_o) = 0$.

We shall assume P and Q are extended Chebyshev systems of the same order λ, and that $\ell(x)$ and $\mu(x)$ are as follows:

$$\ell(x_i) = \mu(x_i) \quad \text{for} \quad i = 1,\dots,k;$$

(let $T = \{x_1,\dots,x_k\} \subseteq J$),

$$\ell(x) < \mu(x) \quad \text{for all} \quad x \in J \sim T,$$

and there exists $\delta > 0$ such that

$$\ell(x) = \sum_{j=0}^{m_i-1} a_{ij}(x - x_i)^j - |x - x_i|^{m_i-\frac{1}{2}}$$
$$\text{for} \quad x \in [x_i - \delta, x_i + \delta] \cap J,$$

$$\mu(x) = \sum_{j=0}^{m_i-1} a_{ij}(x - x_i)^j + |x - x_i|^{m_i-\frac{1}{2}}$$
$$\text{for} \quad x \in [x_i - \delta, x_i + \delta] \cap J,$$

where $\{m_i\}_{i=1}^{k}$ is a set of positive integers, $\{a_{ij} : i = 1,\dots,k;\ j = 0,\dots,m_i-1\}$ is a set of real numbers, and let $m = \sum_{i=1}^{k} m_i$, ($m_i \le \lambda$ for $i = 1,\dots,k$).

We wish to consider the set

$$R = \{r(x) = p(x)/q(x): \ p(x) \in P,\ q(x) \in Q,\ q(x) > 0$$
$$\text{for all} \quad x \in S\},$$

and

$$R_1 = \{r \in R: \ \ell(x) \le r(x) \le \mu(x) \quad \text{for all} \quad x \in J\}.$$

We shall assume $R_1 \neq \emptyset$, $r \in R_1$ implies $r^{(j)}(x_i) = a_{ij}$ for $j = 0,\dots,m_i-1$ and $i = 1,\dots,k$. Let

$$C_1(S) = \{f \in C(S) \sim R: \ f(x_i) = a_{io}, \ i = 1,\dots,k\}.$$

For $f \in C_1(S)$ and $r \in R_1$, recall

$$E_r^+ = \{x \in K: \ f(x) - r(x) = \|f - r\|_K\},$$

$$E_r^- = \{x \in K: \ f(x) - r(x) = -\|f - r\|_K\},$$

$$G_r^+ = \{x \in J \sim T: \ r(x) = \ell(x)\},$$

$$G_r^- = \{x \in J \sim T: \ r(x) = \mu(x)\}.$$

If for a fixed $f \in C_1(S)$ and some $r \in R_1$,

$$(E_r^+ \cup G_r^+) \cap (E_r^- \cup G_r^-) \neq \phi,$$

then r is a best restricted rational approximation since our previous remarks concerning this case are still valid. Thus we shall restrict our attention to $f \in \tilde{C}(S)$ where

$$\tilde{C}(S) = \{f \in C_1(S): \ (E_r^+ \cup G_r^+) \cap (E_r^- \cup G_r^-) = \phi$$
$$\text{for all } r \in R_1\}.$$

For $r^* \in R_1$, consider the set

$$M_{r^*} = \{p + r^*q \in P + r^*Q: \ (p - r^*q)^{(j)}(x_i) = 0,$$
$$i = 1,\dots,k, \ j = 0,1,\dots,m_i-1\}.$$

(Let the dimension of $P + r^*Q$ be d. Then the dimension of M_{r^*} is d-m.) The condition $(p - r^*q)^{(j)}(x_i) = 0$ is equivalent [20] to $(p/q)^{(j)}(x_i) = r^{*(j)}(x_i)$ (the proof proceeds by induction using $p^{(j)}(x_i) = [(p/q)q]^{(j)}(x_i))$. M_{r^*} is a subspace of $C(S)$ and if $r_1 \in R_1$ with $r_1 = p_1/q_1$, then

$p_1 - r^* q_1 \in M_{r^*}$ by the form of ℓ and μ. Each element $\phi \in M_{r^*}$ has a zero of order at least m_i at x_i for $i = 1,\ldots,k$.

The following lemma is a Kolmogorov type theorem. It will be used to construct a linear functional which characterizes a best restricted rational approximation.

Lemma 2.2: Let $f \in \tilde{C}(S)$. Let $r^* \in R_1$ with $P + r^* Q$ an extended Chebyshev system of order λ and dimension d. If $m_i < \lambda$ for $i = 1,\ldots,k$, and $m \le d-1$, then r^* is not a best restricted rational approximation to f if there is an element $\phi \in M_{r^*}$ with

$$\phi(x) > 0 \quad \text{for all} \quad x \in E^+_{r^*} \cup G^+_{r^*},$$

$$\phi(x) < 0 \quad \text{for all} \quad x \in E^-_{r^*} \cup G^-_{r^*}.$$

Proof: Suppose such a $\phi = p + r^* q$ exists and let $r^* = p^*/q^*$. Consider $r_\delta = \dfrac{p^* + \delta p}{q^* - \delta p} = \dfrac{p_\delta}{q_\delta}$. Since $q^*(x) > 0$ for all $x \in S$, for sufficiently small positive δ (say $\delta \le \delta_1$),

$$(q^* - \delta q)(x) = q_\delta(x) > 0 \quad \text{for all} \quad x \in S.$$

Thus $r_\delta \in R$. $\phi \in M_{r^*}$ implies $(p + r^* q)^{(j)}(x_i) = 0$ for $i = 1,\ldots,k$, $j = 0,\ldots,m_i-1$; i.e. $(p/-q)^{(j)}(x_i) = r^{*(j)}(x_i) = a_{ij}$ for $i = 1,\ldots,k$ and $j = 0,\ldots,m_i-1$. We wish to show that there is a $\delta > 0$ such that $r_\delta \in R_1$ and r_δ is a better restricted rational approximation to f than r^*.

$$\text{Let } O = \{x \in S:\ f(x) - r^*(x) > \frac{\|f-r^*\|_K}{2} \text{ and } \phi(x) > 0\}$$

$$\cup \{x \in S:\ f(x) - r^*(x) < -\frac{\|f-r^*\|_K}{2} \text{ and } \phi(x) < 0\}.$$

Then $x \in O \cap K$ implies

$$|f(x) - r_\delta(x)| < \|f - r^*\|_K,$$

and for $x \in K \sim O$

$$|f(x) - r_\delta(x)| < \|f - r^*\|_K$$

for δ sufficiently small, say $0 < \delta \le \delta_o$, as in the proof of Theorem 2.3. So $\|f - r_\delta\|_K < \|f - r^*\|_K$. Since $r_\delta = \dfrac{p^* + \delta p}{q^* - \delta q}$ and P, Q are extended Chebyshev systems of order $\lambda \ge m+1$, $r_\delta^{(m_i)}$ is continuous in a neighborhood of x_i, $i = 1,\ldots,k$. So, using Taylor series,

$$r_\delta(x) = \sum_{j=0}^{m_i-1} a_{ij}(x - x_i)^j + r_\delta^{(m_i)}(c)\,\frac{(x-x_i)^{m_i}}{(m_i)!}$$

for some c, $x_i - \epsilon_i \le c \le x_i + \epsilon_i$ and each $x \in [x_i - \epsilon_i, x_i + \epsilon_i]$, $i = 1,\ldots,k$, we conclude that

$$\ell(x) \le r_\delta(x) \le \mu(x) \quad \text{for all } x \in ([x_i - \epsilon_i, x_i + \epsilon_i] \cap J).$$

Now setting $U = \bigcup_{i=1}^{k} ([x_i - \epsilon_i, x_i + \epsilon_i] \cap S)$, we can find δ sufficiently small, say $\delta \le \delta_2$, so that

$$\ell(x) \le r_\delta(x) \le \mu(x) \quad \text{for all } x \in (J = L) \sim U.$$

Then $\delta \le \min\{\delta_o, \delta_1, \delta_2\}$ gives $r_\delta \in R_1$ and $\|f - r_\delta\|_K < \|f - r^*\|_K$. ■

<u>Theorem 2.8</u>: Let $f \in \tilde{C}(S)$, $r^* \in R_1$ and $P + r^*Q$ be an extended Chebyshev system of order λ and dimension $d = s + t - 1$. Suppose also that $m_i < \lambda$ for $i = 1,\ldots,k$ and $m \leq d-1$. Then the following statements are equivalent:

1. r^* is a best restricted rational approximation to f.

2. The origin of Euclidean $d-m$ space belongs to the convex hull of $\{\sigma(x)\hat{x}\colon \hat{x} \in E_{r^*} \cup G_{r^*}\}$ where $\sigma(x) = +1$ if $x \in E^+_{r^*} \cup G^+_{r^*}$, and $\sigma(x) = -1$ if $x \in E^-_{r^*} \cup G^-_{r^*}$, and $\hat{x} = (\varphi_1(x),\ldots,\varphi_{d-m}(x))$ with $\varphi_1,\ldots,\varphi_{d-m}$ a basis for M_{r^*}.

3. There exists a continuous linear functional $L \in (C[a,b])^*$ based on $\mu = d - m + 1$ points $y_1,\ldots,y_\mu$ in $E_{r^*} \cup G_{r^*}$,

$$L(h) = \sum_{i=1}^{\mu} \lambda_i h(y_i)$$

$$\text{with} \quad \begin{cases} \lambda_i > 0 & \text{for } y_i \in E^+_{r^*} \cup G^+_{r^*}, \\ \lambda_i < 0 & \text{for } y_i \in E^-_{r^*} \cup G^-_{r^*}, \end{cases}$$

such that $L(\phi) = 0$ for all $\phi \in M_{r^*}$.

4. There exist $d - m + 1$ points

$$z_1 < z_2 < \ldots < z_{d-m+1} \quad \text{in } E_{r^*} \cup G_{r^*}$$

such that

$$\sigma(z_i)\pi(z_i) = (-1)^{i+1}\sigma(z_1)\pi(z_1), \; i = 1,\ldots,d-m+1$$

where $\pi(z_i) = \operatorname{sgn}\{(z_i - x_1)^{m_1} \ldots (z_i - x_k)^{m_k}\}$.

<u>Proof</u>: (1. $\Rightarrow$ 2.) Suppose 2. is not true. Then by the theorem on linear inequalities [3, p. 19], there is a $\phi \in M_{r^*}$ with $\sigma(x)\phi(x) > 0$ for all $x \in E_{r^*} \cup G_{r^*}$, i.e.

$$\phi(x) > 0 \quad \text{for} \quad x \in E^+_{r^*} \cup G^+_{r^*}$$

and

$$\phi(x) < 0 \quad \text{for} \quad x \in E^-_{r^*} \cup G^-_{r^*}.$$

But then, by Lemma 2.2, r^* is not a best restricted rational approximation.

(2. $\Rightarrow$ 3.) If $\vec{o} \in \mathrm{co}(\{(\varphi_1(y),\ldots,\varphi_{d-m}(y)): y \in E^+_{r^*} \cup G^+_{r^*}\} \cup \{(-\varphi_1(y),\ldots,-\varphi_{d-m}(y)): y \in E^-_{r^*} \cup G^-_{r^*}\})$, then by the Theorem of Caratheodory there exist positive numbers $\{\alpha_i\}_{i=1}^{\gamma}$ with $\gamma \le d - m + 1$ and

$$\sum_{i=1}^{\gamma} \alpha_i \sigma(y_i)\vec{\varphi}(y_i) = \vec{o}$$

where $\vec{\varphi}(y_i) = (\varphi_1(y_i),\ldots,\varphi_{d-m}(y_i))$. Now letting $\lambda_i = \alpha_i\sigma(y_i)$, we obtain

$$\sum_{i=1}^{\gamma} \lambda_i\varphi_j(y_i) = 0 \quad \text{for} \quad j = 1,\ldots,d-m,$$

and

$$L(h) = \sum_{i=1}^{\gamma} \lambda_i h(y_i)$$

is a continuous linear functional on $C[a,b]$ whose null space contains M_{r^*}. If $\gamma < d - m + 1$, then $\det[\varphi_j(\bar{y}_i)] = 0$ $i,j = 1,\ldots,d-m$ where $\bar{y}_i = y_i$ for $i = 1,\ldots,\gamma$ and $\{\bar{y}_i\}_{i=\gamma+1}^{d-m}$ is a set of points in $(S \sim T) \sim \{y_1,\ldots,y_\gamma\}$. Then there exist constants $\beta_1,\ldots,\beta_{d-m}$, not all zero, with

$$\sum_{i=1}^{d-m} \beta_j \varphi_j(\bar{y}_i) = 0.$$

But then $\varphi = \sum_{i=1}^{d-m} \beta_j \varphi_j \in M_{r^*}$ has $d - m$ zeros at $\{y_i\}_{i=1}^{d-m}$ and thus a total of at least d zeros. This is a contradiction since $P + r^*Q$ is an extended Chebyshev system and $M_{r^*} \subseteq P + r^*Q$. Thus $\gamma = d - m + 1$.

(3. $\Rightarrow$ 4.) We have $\sum_{i=1}^{d-m+1} \lambda_i \varphi_j(y_i) = 0$ for $j = 1,\ldots,s-m$, and $y_1 < y_2 <\ldots< y_{d-m+1}$; $\{y_i\}_{i=1}^{d-m+1} \subseteq E_{r^*} \cup G_{r^*}$. Let $\alpha_i = |\lambda_i|$, $i = 1,\ldots,d-m+1$. We can use Cramer's rule to solve

$$\sum_{i=2}^{d-m+1} \alpha_i \sigma(u_i)\varphi_j(y_i) = -\alpha_1\sigma(y_1)\varphi_j(y_1),\ j = 1,\ldots,d-m,$$

and obtain

$$\alpha_i\sigma(y_i) = (-1)^{i+1}\alpha_1\sigma(y_1)\ (\Delta_i/\Delta_1) \quad \text{for} \quad i = 2,\ldots,d-m+1,$$

where

$$\Delta_i = \begin{vmatrix} \varphi_1(y_1) & \cdots & \varphi_1(y_{i-1}) & \varphi_1(y_{i+1}) & \cdots & \varphi_1(y_{d-m+1}) \\ \vdots & & & & & \vdots \\ \varphi_{d-m}(y_1) & \cdots & \varphi_{d-m}(y_{i-1}) & \varphi_{d-m}(y_{i+1}) & \cdots & \varphi_{d-m}(y_{d-m+1}) \end{vmatrix}$$

for $i = 1,\ldots,d-m+1$.

$\Delta_i \neq 0$ for $i = 1,\ldots,d-m+1$ since $\Delta_i = 0$ implies the existence of a non-zero function $\varphi \in P + r^*Q$ with d zeros which cannot happen since $P + r^*Q$ is an extended Chebyshev system.

Now let $z_1,\ldots,z_{d-m-1}$ be an arbitrary set of $d - m - 1$ consecutive points in $S \sim T$. Construct the function $\varphi(x)$

$$\varphi(x) = \begin{vmatrix} \varphi_1(x) & \varphi_1(z_1) \cdots & \varphi_1(z_{d-m-1}) \\ \vdots & \vdots & \vdots \\ \varphi_{d-m}(x) & \varphi_{d-m}(z_1) & \varphi_{d-m}(z_{d-m-1}) \end{vmatrix} .$$

$\varphi \in M_{r^*}$ and $\varphi \not\equiv 0$ since $P + r^*Q$ is an extended Chebyshev system. φ has exactly $d-1$ zeros counting multiplicities. φ changes sign at each z_i and at x_i if and only if m_i is odd. Let $z_i = y_{i+2}$, $i = 1,\ldots,d-m-1$. Then

$$\varphi(y_1) = \Delta_2,$$

$$\varphi(y_2) = \Delta_1.$$

If $x_{i_1},\ldots,x_{i_{\ell_1}} \in (y_1,y_2)$, then

$$\begin{aligned} \operatorname{sgn} \Delta_2 = \operatorname{sgn} \varphi(y_1) &= (-1)^{m_{i_1}+\cdots+m_{i_{\ell_1}}} \operatorname{sgn} \varphi(y_2) \\ &= (-1)^{m_{i_1}+\cdots+m_{i_{\ell_1}}} \operatorname{sgn} \Delta_1 \\ &= (\pi(y_2)/\pi(y_1)) \operatorname{sgn} \Delta_1. \end{aligned}$$

Now let $z_1 = y_1$, $z_i = y_{i+2}$ $i = 2,\ldots,d-m-1$. If $x_{j_1},\ldots,x_{j\ell_2} \in (y_2,y_3)$, then

$$\varphi(y_2) = -\Delta_3,$$

$$\varphi(y_3) = -\Delta_2,$$

and

$$\begin{aligned} -\operatorname{sgn} \Delta_3 = \operatorname{sgn} \varphi(y_2) &= (-1)^{m_{j1}+\cdots m_{j\ell_2}} \operatorname{sgn} \varphi(y_3) \\ &= (-1)^{m_{j1}+\cdots m_{j\ell_2}+1} \operatorname{sgn} \Delta_2 \\ &= -(\pi(y_3)/\pi(y_2)) \operatorname{sgn} \Delta_2. \end{aligned}$$

Continuing in this manner, we obtain

$$\operatorname{sgn} \Delta_{k+1} = (\pi(y_{k+1})/\pi(y_k)) \operatorname{sgn} \Delta_k, \ k = 1,\ldots,d-m,$$

or

$$(-1)\,\sigma(y_k) = (\pi(y_{k+1})/\pi(y_k))\sigma(y_{k+1}),$$

and thus

$$\sigma(y_k)\pi(y_k) = (-1)^{k+1}\sigma(y_1)\pi(y_1).$$

(4. ⇒ 1.) Assume 1. is not true, that is, $r_o \in R_1$ and $\|f - r_o\|_K < \|f - r^*\|_K$. Let $r_o = p_o/q_o$ with $\|p_o\|_K + \|q_o\|_K = 1$, $\|p^*\|_K + \|q^*\|_K = 1$, and $p_o - r^*q_o \in M_{r^*}$. Let $x \in E_{r^*}$. Since $\|f - r_o\|_K < \|f - r^*\|_K$, and

$$r_o - r^* = f - r^* - (f - r_o),$$

we have $\operatorname{sgn}(r_o - r^*)(x) = \sigma(x)$ for all $x \in E_{r^*}$. Also $r_o \in R_1$ implies

$$(r_o - r^*)(x) \geq 0 \quad \text{for all} \quad x \in G^+_{r^*},$$

$$(r_o - r^*)(x) \leq 0 \quad \text{for all} \quad x \in G^-_{r^*}.$$

Thus, if we define

$$\operatorname{sgn}^*(r_o - r^*)(x) = \begin{cases} +1 & \text{if} \quad r_o(x) = r^*(x) = \ell(x), \\ -1 & \text{if} \quad r_o(x) = r^*(x) = \mu(x), \\ \operatorname{sgn}(r_o - r^*)(x) & \text{otherwise,} \end{cases}$$

then $\operatorname{sgn}^*(r_o - r^*)(x) = \sigma(x)$ for all $x \in E_{r^*} \cup G_{r^*}$, and by 4. there exist points $z_1,\ldots,z_{d-m+1}$ in $E_{r^*} \cup G_{r^*}$ with

$$\operatorname{sgn}^*(r_o - r^*)(z_i)\pi(z_i) = (-1)^{i+1}\operatorname{sgn}^*(r_o - r^*)(z_1)\pi(z_1)$$

for $i = 1,\ldots,d-m+1$.

If $r_o(z_i) = r^*(z_i)$ for $i = 1,\ldots,d-m+1$, then

$$p_o - r^* q_o \equiv 0$$

since M_{r^*} is an extended Chebyshev system and Lemma 2.1 yields $r_o \equiv r^*$. This is a contradiction since r_o is a better restricted rational approximation to f than r^*. So let z_i be such that $r_o(z_i) \neq r^*(z_i)$; $z_{i+1},\ldots,z_{i+t}$ such that $r_o(z_{i+\gamma}) = r^*(z_{i+\gamma})$ for $\gamma = 1,\ldots,t$ and $r_o(z_{i+t+1}) \neq r^*(z_{i+t+1})$. Assume $x_{j1},\ldots,x_{j2} \in (z_i, z_{i+t+1})$, and that $m_{j1} +\ldots+ m_{j2}$ is odd, $\sigma(z_i) = +1$ and t is odd. Then $\sigma(z_{i+t+1}) = -(-1)^{t+1}\sigma(z_i) = -1$. $\sigma(z_i) = +1$ implies $z_i \in E^+_{r^*} \cup G^+_{r^*}$. If $z_i \in E^+_{r^*}$,

$$f(z_i) - r_o(z_i) \le \|f - r_o\|_K < \|f - r^*\|_K = f(z_i) - r^*(z_i).$$

Thus $r_o(z_i) - r^*(z_i) > 0$. If $z_i \in G^+_{r^*}$, $r^*(z_i) = \ell(z_i)$ and $r_o(z_i) > \ell(z_i)$, so $r_o(z_i) - r^*(z_i) > 0$. Similarly $\sigma(z_{i+t+1}) = -1$ implies $r_o(z_i) - r^*(z_i) < 0$. Also $(r_o - r^*)^{(j)}(x_i) = 0$ for $j = 0,\ldots,m_i-1$, $i = j_1,\ldots,j_2$. Thus counting multiplicities, $r_o - r^*$ has a total of at least $m_{j1} +\ldots+ m_{j2} + t$ zeros in (z_i, z_{i+t+1}). This is an even number. If $r_o - r^*$ had no other zeros in (z_i, z_{i+t+1}) we would have $\sigma(z_i) = \sigma(z_{i+t+1})$. Since this is not true we must have at least $m_{j1} +\ldots+ m_{j2} + t + 1$ zeros in (z_i, z_{i+t+1}). Other cases must give the same result, that is, if $r_o(z_j) \neq r^*(z_j)$, $r_o(z_{j+\gamma}) = r^*(z_{j+\gamma})$ for $\gamma = 1,\ldots,w$ and $r_o(z_{j+w+1}) \neq r^*(z_{j+w+1})$, and if $x_{\ell_1},\ldots,x_{\ell_2} \in (z_j, z_{j+w+1})$ then $r_o - r^*$ has at least $m_{\ell_1} + \ldots + m_{\ell_2} + w + 1$ zeros in (z_j, z_{j+w+1}). This is also true if $w = 0$.

Now let i_o be the least positive integer such that $r_o(z_{i_o}) \neq r^*(z_{i_o})$ and let i_1 be the greatest positive integer such that $r_o(z_{i_1}) \neq r^*(z_{i_1})$. Then if $x_{j1},\ldots,x_{j2} \in (z_{i_o}, z_{i_1})$ and the rest of T is exterior to (z_{i_o}, z_{i_1}), looking at subintervals as above if necessary, $m_{j1} + \ldots + m_{j2} + (i_1 - i_o)$ zeros are interior to (z_{i_o}, z_{i_1}). This means that $r_o - r^*$ has a total of d zeros, counting multiplicities, in $[a,b]$. Now consider $p_o - r^*q_o \in M_{r^*}$. We have shown that $p_o - r^*q_o$ has a zero of multiplicity m_i at each x_i and from the above discussion, counting other zeros as simple zeros, $(p_o - r^*q_o)(y) = 0$ whenever $(r_o - r^*)(y) = 0$ since $q_o(y) > 0$ for all $y \in S$. Thus $p_o - r^*q_o$ has d zeros. But $p_o - r^*q_o \in P + r^*Q$ an extended Chebyshev system of dimension d which implies $p_o - r^*q_o \equiv 0$. We have already shown this to be a contradiction. ∎

Theorems concerning the uniqueness of the best restricted rational approximation in the equality case described here differ only slightly from the same theorems in the inequality case. The simple modifications needed for their proofs will be mentioned but the details will not be carried out. (For uniqueness results in a more general setting where the forms of ℓ and μ are not specified, see L.L. Schumaker and G.D. Taylor [22].)

Theorem 2.9: Let $f \in \tilde{C}(S)$ and $r^* \in R_1$ be a best restricted rational approximation to f. If $P + r^*Q$ is an extended Chebyshev system of dimension d and order λ with $m_i \leq \lambda$ for $i = 1,\ldots,k$ and $m \leq d-1$, then r^* is unique.

Proof: This follows in the same manner as the proof of Theorem 2.5. It is necessary only to note that if $r_o = p_o/q_o \in R_1$, then $p_o + r^*q_o \in M_{r^*} \subseteq P + r^*Q$, and that the linear functional L whose existence is given by Theorem 2.8 vanishes on M_{r^*}. ∎

Both in Chapter I and Section 3 of Chapter II, we found it necessary to add another restriction, condition H, to the set of approximants in order to say that the set of points y_i, on which the characterizing functional L depends, included a point of E_{r^*}. So far in this section we have not made such an assumption but neither have we any guarantee that one of the y_i's described above is in E_{r^*}. This will be necessary for the proof of the Strong Uniqueness Theorem given here, so we introduce condition H'.

Condition H': The set R will be said to satisfy condition H' if there exists an $r \in R_1$ with

$$\ell(x) < r(x) < \mu(x) \quad \text{for all} \quad x \in (J = L) \sim T.$$

Lemma 2.3: Given the hypotheses of Theorem 2.9 and R satisfying condition H', then the set of points in 3. of Theorem 2.8 on which L is based must contain at least one element of E_{r^*}.

Proof: If all the y_i's are in G_{r^*}, then for each $r \in R_1$, $r(x) = p(x)/q(x)$, we have

$$r(y_i) = \ell(y_i) \quad \text{for} \quad y_i \in G^+_{r^*}$$

and

$$r(y_i) = \mu(y_i) \quad \text{for} \quad y_i \in G^-_{r^*}.$$

Since if for some $y_i \in G^-_{r^*}$

$$r(y_i) < \mu(y_i) = r^*(y_i),$$

then

$$r(y_i) - r^*(y_i) < 0$$

and

$$p(y_i) - r^* q(y_i) < 0,$$

so

$$\lambda_i (p - r^* q)(y_i) > 0$$

which gives $L(p - r^* q) > 0$, but this is a contradiction since $p - r^* q \in M_{r^*}$ and thus

$$L(p - r^* q) = 0.$$

But since H' is satisfied, there must be at least one $y_i \in E_{r^*}$. ∎

<u>Theorem 2.10</u>: Let r^* be a best restricted rational approximation to $f \in \tilde{C}(S)$ from R_1 and let R satisfy condition H'. If $P + r^* Q$ is an extended Chebyshev system of dimension $s + t - 1 = d$ and order λ and $m_i < \lambda$ for $i = 1,\ldots,k$ and $m \le d-1$, then there exists a number $\gamma > 0$ such that for all $r \in R_1$,

$$\|f - r\|_K \ge \|f - r^*\|_K + \gamma \|r - r^*\|_K.$$

The proof is the same as in the inequality case, Theorem 2.7, and again we obtain the continuity of the best restricted rational approximation operator.

Corollary: Let $f^* \in \tilde{C}(S)$ with best restricted rational approximation $r^* \in R_1$. Let dimension $(P + r^*Q) = s + t - 1 = d$ and $P + r^*Q$ be an extended Chebyshev system of order λ, $m_i < \lambda$ for $i = 1,\ldots,k$ and $m \le d-1$. Then there exists a number $\beta > 0$ such that for any $f \in \tilde{C}(S)$ with a corresponding best restricted rational approximation r,

$$\|r^* - r\|_K \le \beta\|f^* - f\|_K.$$

Comments: 1. In Chapter I we can consider ordinary unrestricted approximation by choosing $L = J = \phi$, or regular restricted approximation by letting $J = K = L$. One-sided approximation can also be considered by choosing either ℓ or μ equal to the function to be approximated and the appropriate J or $L = K$ and the other to be the empty set.

2. In Chapter II the assumption $J = L$ was used to show the existence of best restricted rational approximations by bounding the sequence $\{r_n\}$ for which $\|f - r_n\|_K \to \rho$. The same result is obtained if we assume $J \subseteq K$, $L \subseteq K$ or $J \sim K = L \sim K$. In this way we could consider usual unrestricted rational approximation or one-sided rational approximation. However if we do not assume J, L are perfect sets we cannot guarantee existence (see example 2.1).

3. The results of Section 4 of Chapter II can be obtained with arbitrary compact subsets J, K, L of the real line if $Q = \text{span}\,\{1\}$ since existence of best restricted

approximations follows from compactness considerations in this case. We would assume P to be an extended Chebyshev system of order λ ($\geq m_i$ for $i = 1,\ldots,k$) and dimension d ($\geq m + 1$). Interpolation and approximation can then be considered.

CHAPTER III

NON-LINEAR CHEBYSHEV APPROXIMATION WITH SIDE CONDITIONS

A very general treatment of Chebyshev approximation with side conditions was given by Karl-Heinz Hoffmann in his doctoral thesis [7]. In this chapter we shall present an expository discussion of his work. Some of the results of Chapters I and II can be obtained using the theory presented here, namely the Kolmogorov theorems and the characterizations of best approximations by continuous linear functionals. However, in obtaining results applicable to so many different problems, some practicality is lost. For example the uniqueness theorem presented in this chapter is difficult to apply to any specific problem and the uniqueness theorems obtained in Chapters I and II are not results of this work.

Any unreferenced result in this chapter is taken from the thesis of Karl-Heinz Hoffmann.

Section 1: Definitions and Statement of the Problem and Standard Theory

We wish to consider approximating a continuous function f which maps a compact Hausdorff space Q into a Hilbert space H. Let $C[Q,H]$ denote the set of continuous functions from Q to H with the topology induced by the uniform norm,

$$\|f\| = \max_{x \in Q} \|f(x)\|_H,$$

and let E be a Banach space. We shall assume that there is an open subset P of E and a continuous function F such that

$$F : P \to C[Q,H],$$

and for $\mathfrak{A} \in P$ we shall denote

$$F(\mathfrak{A}) = v(\cdot,\mathfrak{A}) \in C[Q,H].$$

Now let $V = \{v(\cdot,\mathfrak{A}): \mathfrak{A} \in P\}$ be the set of approximating functions.

We may further restrict the set of admissible approximants to a subset of V whose elements satisfy given side conditions. Let K be the scalar field for the Hilbert space H. We shall assume that K is either the reals or the complex numbers. Two kinds of side conditions are considered. Let

$$\text{(S)} \qquad \begin{aligned} &f_i: \ P \to K && \text{for } i = 1,\ldots,k, \\ &g_j: \ Q_j \times P \to K && \text{for } j = 1,\ldots,k', \end{aligned}$$

where the sets Q_j are compact Hausdorff sets. Define

$$\begin{aligned} V_{1,0} &= \{v(\cdot,\mathfrak{A}) \in V: \ f_i(\mathfrak{A}) = 0, \ i = 1,\ldots,k\}, \\ V_{0,1} &= \{v(\cdot,\mathfrak{A}) \in V: \ \operatorname{Re} g_j(t^{(j)},\mathfrak{A}) \geq 0; \text{ for all } \\ &\qquad\qquad t^{(j)} \in Q_j, \text{ and } j = 1,\ldots,k'\}, \\ V_{1,1} &= V_{1,0} \cap V_{0,1} \\ V_{0,0} &= V. \end{aligned}$$

When we wish to refer to any one of these sets without specifying which one we shall write $V_{\alpha,\beta}$.

The problem to be considered in this chapter is the following:

(T) For a given function $f \in C[Q,H]$, we wish to find $v_o = v(\cdot,\mathfrak{A}_o) \in V_{\alpha,\beta}$ such that

$$\|f - v_o\| \le \|f - v\| \quad \text{for all} \quad v \in V_{\alpha,\beta},$$

that is, v_o satisfies

$$\|f - v_o\| = E(f,V_{\alpha,\beta}) = \inf_{v \in V_{\alpha,\beta}} \|f - v\|.$$

The concept of extremal signatures will play an important role in the characterization of the v_o described above. Since we have not required Q to be a metric space, the definitions used here differ slightly from the standard definitions given by B. Brosowski [2].

Let $\mathcal{S}$ be a non-empty set of ordered pairs (ε,M) where $\varepsilon \in C[Q,H]$ with $\|\varepsilon\| \le 1$ and $M \subseteq Q$ is closed and non-empty, and $\varepsilon|_M$ (the restriction of ε to M) maps M into the unit sphere of H. We define an equivalence relation on $\mathcal{S}$ as follows:

(ε_1,M_1), $(\varepsilon_2,M_2) \in \mathcal{S}$ are equivalent if

$$M_1 = M_2,$$

and

$$\varepsilon_1|_{M_1} = \varepsilon_2|_{M_2}.$$

The following definitions explain precisely the concept of signatures used in this work.

Definition 3.1:

1. Let $\Sigma = \overline{(\varepsilon,M)}$ be an equivalence class of $\mathcal{S}$. Σ is called a signature.

2. If $\Sigma_1 = \overline{(\varepsilon_1,M_1)}$ and $\Sigma_2 = \overline{(\varepsilon_2,M_2)}$ are two signatures, we say $\underline{\Sigma_1 \subseteq \Sigma_2}$ if

$$M_1 \subset M_2$$

and

$$\varepsilon_1|_{M_1} = \varepsilon_2|_{M_2}$$

for any arbitrary members $(\varepsilon_1,M_1) \in \Sigma_1$ and $(\varepsilon_2,M_2) \in \Sigma_2$.

3. Σ is called an extremal signature for $v(\cdot,\mathfrak{A}_o) \in V_{\alpha,\beta}$ with respect to $V_{\alpha,\beta}$ if for any representative $(\varepsilon,M) \in \Sigma$

$$\min_{x \in M} \operatorname{Re}\, (\varepsilon(x),\ v(x,\mathfrak{A}) - v(x,\mathfrak{A}_o)) \leq 0$$

for all $v(\cdot,\mathfrak{A}) \in V_{\alpha,\beta}$.

4. If a signature Σ is extremal for every element $v(\cdot,\mathfrak{A}) \in V_{\alpha,\beta}$ with respect to $V_{\alpha,\beta}$ it is called extremal for $V_{\alpha,\beta}$.

When it is clear that we mean Σ is extremal for $v(\cdot,\mathfrak{A}_o) \in V_{\alpha,\beta}$ with respect to $V_{\alpha,\beta}$ we shall just say Σ is extremal for $v(\cdot,\mathfrak{A}) \in V_{\alpha,\beta}$.

The following examples will help to clarify the above definitions.

Example 3.1: Let Q be the interval $[a,b]$ of the real line and V be the polynomials of degree less than or equal to n. For $f \in C[a,b]$, let $v_o = v(\cdot,\mathfrak{A}_o)$ be the best approximation to f in the uniform norm. Then by the Chebyshev alternation theorem, there exist $n + 2$ points

$$a \leq x_1 < x_2 < \ldots < x_{n+2} \leq b$$

such that for $\gamma = 0$ or 1 (fixed)

$$f(x_i) - v_o(x_i) = (-1)^{\gamma+1} \|f - v_o\|.$$

Now let $M = \{x_1, \ldots, x_{n+2}\}$ and $\epsilon(x_i) = (-1)^{\gamma+i}$. Then $\overline{(\epsilon,M)}$ is an extremal signature for $v_o \in V$, since if

$$\min_{x \in M} \epsilon(x)(v(x) - v_o(x)) > 0,$$

$v(x) - v_o(x)$ must change sign at least $n + 2$ times. But since $v(x) - v_o(x)$ cannot have more than n zeros this is a contradiction.

The next example, due to B. Brosowski [2], shows that extremal signatures do not always exist.

<u>Example 3.2</u>: Let $V = C[Q,H]$ and $\Sigma = \overline{(\epsilon,M)}$ any signature. Now for $(\epsilon,M) \in \Sigma$ we have $\epsilon \in C[Q,H] = V$. Since $\epsilon \not\equiv 0$,

$$\text{Re } (\epsilon(x), \epsilon(x)) > 0.$$

Thus the inequality

$$\min_{x \in M} \text{Re } (\epsilon(x), v(x)) \leq 0$$

is invalid for $v(x) = \epsilon(x)$ and therefore no signature can be extremal.

The following inclusion theorem makes use of extremal signatures.

<u>Theorem 3.1</u>: Let $f \in C[Q,H]$ with Σ extremal for $v_o = v(\cdot,\mathfrak{A}_o) \in V_{\alpha,\beta}$. If $(\epsilon,M) \in \Sigma$ and

$$f(x) - v(x,\mathfrak{A}_o) = \epsilon(x)\|f(x) - v(x,\mathfrak{A}_o)\|_H \quad \text{for all } x \in M,$$

then

$$\min_{x\in M} \|f(x) - v(x,\mathfrak{A}_o)\|_H \le E(f,V_{\alpha,\beta}) \le \|f - v_o\|.$$

Meinardus and Schwedt [17] proved a similar inclusion theorem for the approximation of real or complex valued functions and this can easily be generalized to the case of approximation with side conditions.

Let the signature $\Sigma[f]$ be defined as follows: $\Sigma[f] = \overline{(\epsilon,M[f])}$ where

$$M[f] = \{x \in Q: \ \|f(x)\|_H = \|f\|\},$$

$$\epsilon \in \mathcal{E}_f = \{\epsilon \in C[Q,H]: \ \|\epsilon\| \le 1, \ \epsilon(x) = \frac{f(x)}{\|f\|}, \ x \in M[f]\}.$$

Using this signature, the Kolmogorov criterion can be stated as:

<u>Theorem 3.2</u>: Let $f \in C[Q,H]$, $V_{\alpha,\beta} \subseteq C[Q,H]$. If $\Sigma[f - v_o]$ is extremal for $v_o = v(\cdot, \mathfrak{A}_o) \in V_{\alpha,\beta}$ then v_o is a solution of the problem (T) for f.

This theorem gives a sufficient condition for v_o to be an absolute or <u>global minimal solution</u> to the problem (T), i.e., if v_o satisfies the hypotheses of Theorem 3.2, then

$$\|f - v_o\| \le \|f - v\| \quad \text{for all} \quad v \in V_{\alpha,\beta}.$$

We shall call v_o a local minimal solution of the problem (T) if there is a neighborhood (in the relative topology on $V_{\alpha,\beta}$) U_o of v_o such that

$$\|f - v_o\| \le \|f - v\|, \text{ for all } v \in U_o.$$

Throughout this chapter we shall not be concerned with the existence of a solution to the problem (T) but rather with the characterization of solutions whenever they do exist.

Section 2: Structure of V and Properties of the Side Conditions

We wish to assume the Frechet differentiability of the functions v, f_i, g_j with respect to the parameter $\mathfrak{A}$. Thus the following well known definition is in order [5, p. 92].

Definition 3.2: Let X, Y be normed linear spaces and Z an open set in X. A function h mapping Z into Y is said to be Frechet differentiable at a point $\mathfrak{A} \in Z$ if there exists a bounded linear operator $Dh(\mathfrak{A})(\cdot)$ (called the Frechet derivative) mapping X into Y such that for all $\mathfrak{b} \in X$

$$\|h(\mathfrak{A} + \mathfrak{b}) - h(\mathfrak{A}) - Dh(\mathfrak{A})(\mathfrak{b})\|_Y = o(\|\mathfrak{b}\|_X)$$

for $\|\mathfrak{b}\|_X \to 0$.

Consider the following properties:

(D1) The elements $v(\cdot,\mathfrak{A}) \in V$ are Frechet differentiable with respect to the parameter $\mathfrak{A}$ at every point $\mathfrak{A} \in P$. In this case, for each point $\mathfrak{b} \in E$, $Dv(\cdot,\mathfrak{A})\mathfrak{b} \in C[Q,H]$. Let $\mathcal{L}[\mathfrak{A}] = \{Dv(\cdot,\mathfrak{A})\mathfrak{b}: \mathfrak{b} \in E\}$ and denote the Hamel dimension of $\mathcal{L}[\mathfrak{A}]$ by $d[\mathfrak{A}]$.

(D2) The functions f_i $(i = 1,\ldots,k)$ are Frechet differentiable at every point $\mathfrak{A} \in P$ and

$$Df_i(\mathfrak{A})(\cdot): \quad E \to K, \quad i = 1,\ldots,k.$$

(D3) The functions g_j $(j = 1,\ldots,k')$ are Frechet differentiable at each point $\mathfrak{A} \in P$ and

$$Dg_j(\cdot,\mathfrak{A})(\cdot): \quad E \to C[Q_j,K], \text{ for } j = 1,\ldots,k',$$

where the topology on $C[Q_j,K]$ is that induced by the uniform norm.

Assuming one or more of these properties we wish to find necessary conditions for a local minimal solution of (T). The regularity conditions given below will enable us to construct functions in $V_{\alpha,\beta}$.

For $v(\cdot,\mathfrak{A}_o) \in V_{1,1}$ we define

$$M_j = \{t^{(j)} \in Q_j: \quad g_j(t^{(j)},\mathfrak{A}_o) = 0\},$$

$$J = \{1,\ldots,k'\},$$

$$J_o = \{j \in J: \quad M_j \neq \emptyset\}.$$

<u>Definition 3.3</u>:

1. The side conditions (S) are said to be <u>(R1)-regular</u> at $\mathfrak{A}_o$ if they satisfy (D2) and for every $\mathfrak{b} \in E$ such that

$$Df_i(\mathfrak{A}_o)b = 0, \quad i = 1,\ldots,k,$$

there exists a curve $\mathfrak{A}(s)$ in P ($\mathfrak{A}(\cdot)$: $[0,1] \to P$, continuous), Frechet differentiable at the point $s = 0$ with Frechet derivative $\mathfrak{A}'(0)$ and a real number $s_o \in (0,1]$ such that

$$f_i(\mathfrak{A}(s)) = 0, \text{ for } s \in [0,s_o] \text{ and } i = 1,\ldots,k,$$

$$\mathfrak{A}(0) = \mathfrak{A}_o$$

and there exists a real number $\lambda > 0$ with

$$\mathfrak{A}'(0) = \lambda b.$$

2. The side conditions (S) are said to be <u>(R2)-regular</u> at $\mathfrak{A}_o$ if they satisfy (D3) and there exists a $b \in E$ such that

$$Df_i(\mathfrak{A}_o)b = 0 \quad \text{for} \quad i = 1,\ldots,k$$

and

$$\text{Re } Dg_j(t^{(j)},\mathfrak{A}_o)b > 0 \quad \text{for all} \quad j \in J_o,\ t^{(j)} \in M_j.$$

3. The side conditions are called <u>regular</u> at $\mathfrak{A}_o$ if (D2), (D3), (R1) and (R2) are satisfied.

The following example will illustrate these definitions.

<u>Example 3.3</u>: Suppose E is Euclidean (n+1)-space, $Q = [0,1]$, H = reals, V is the set of polynomials of degree less than or equal to n, and for $\mathfrak{A} = (a_o,\ldots,a_n) \in E$, $v(x,\mathfrak{A}) = \sum_{i=0}^{n} a_i x^i$. For a fixed $f \in C[Q,H]$, we require $v(x,\mathfrak{A})$ to interpolate

f at $x_1,\ldots,x_k$. Then let

$$f_j(\mathfrak{A}) = \sum_{i=0}^{n} a_i x_j^i - f(x_j), \quad \text{for} \quad j = 1,\ldots,k,$$

and

$$Df_j(\mathfrak{A})\mathfrak{b} = \sum_{i=0}^{n} b_i x_j^i \ , \quad \text{for} \quad j = 1,\ldots,k,$$

for $\mathfrak{b} = (b_o,\ldots,b_n) \in E$.

Now suppose $\mathfrak{A}_o \in E$ is such that

$$f_j(\mathfrak{A}_o) = 0, \quad \text{for} \quad j = 1,\ldots,k.$$

Then if $Df_j(\mathfrak{A}_o)\mathfrak{b} = 0$, let

$$\mathfrak{A}(s) = \mathfrak{A}_o + s\mathfrak{b}$$

and (R1) is satisfied with $s_o = 1$.

Now suppose we further require $v(x,\mathfrak{A}) \geq f(x)$ on $[0,1]$. Then let $Q_1 = Q = [0,1]$ and

$$g_1(x,\mathfrak{A}) = \sum_{i=0}^{n} a_i x^i - f(x),$$

$$Dg_1(x,\mathfrak{A})\mathfrak{b} = \sum_{i=0}^{n} b_i x^i.$$

In this case (R2) cannot be satisfied since

$$Dg_1(x,\mathfrak{A}_o)\mathfrak{b} = \sum_{i=0}^{n} b_i x^i > 0$$

and

$$f_j(\mathfrak{A}_o)\mathfrak{b} = \sum_{i=0}^{n} b_i x_j^i = 0$$

are incompatible. However, if we let $Q_1 \subseteq [0,1] \sim \{x_1,\ldots,x_k\}$ and $k \leq [\frac{n}{2}] - 1$ then (R2) will be satisfied for some $\mathfrak{b}$ [8, p. 30].

The following lemma guarantees the existence of functions in $V_{1,1}$ "close" to a given function $v(\cdot,\mathfrak{A}) \in V_{1,1}$.

Lemma 3.1: Let the side conditions (S) satisfy (D2), (D3) and (R1) at $\mathfrak{A}_o$. Then for each $b \in E$ such that

$$Df_i(\mathfrak{A}_o)b = 0 \quad \text{for} \quad i = 1,\ldots,k$$

and

$$\operatorname{Re} Dg_j(t^{(j)},\mathfrak{A}_o)b > 0 \quad \text{for} \quad j \in J_o,\ t^{(j)} \in M_j,$$

there exists a curve $\mathfrak{A}(\cdot)$ in P, Frechet differentiable at the point $s = 0$, and a real number $s_1 \in (0,1]$ such that

$$f_i(\mathfrak{A}(s)) = 0 \quad \text{for} \quad s \in [0,s_1],\ i = 1,\ldots,k,$$

$$\operatorname{Re} g_j(t^{(j)},\mathfrak{A}(s)) \geq 0 \quad \text{for} \quad s \in [0,s_1],\ j \in J,\ t^{(j)} \in Q_j,$$

$$\mathfrak{A}(0) = \mathfrak{A}_o,$$

and

$$\mathfrak{A}'(0) = \lambda b \quad \text{for some} \quad \lambda > 0.$$

Proof: By the (R1) regularity, there exist a curve $\mathfrak{A}(s)$ and $s_o \in (0,1]$ such that

$$f_i(\mathfrak{A}(s)) = 0 \quad \text{for} \quad s \in [0,s_o] \quad i = 1,\ldots,k,$$

$$\mathfrak{A}(0) = \mathfrak{A}_o,$$

and

$$\mathfrak{A}'(0) = \lambda b \quad \text{for some} \quad \lambda > 0.$$

Since $\mathfrak{A}(s)$ is continuous and $\mathfrak{A}(0) = \mathfrak{A}_o$,

$$\mathfrak{A}(s) - \mathfrak{A}(0) = o(s).$$

Also $g_j(t^{(j)},\cdot)$ is continuous for each $t^{(j)} \in Q_j$, so

$$g_j(t^{(j)},\mathfrak{A}(s)) - g_j(t^{(j)},\mathfrak{A}_o + s\lambda b)$$

$$= [g_j(t^{(j)},\mathfrak{A}(s)) - g_j(t^{(j)},\mathfrak{A}_o)] + [g_j(t^{(j)},\mathfrak{A}_o) - g_j(t^{(j)},\mathfrak{A}_o + s\lambda b)]$$

$$= o(s) + o(s).$$

So $g_j(t^{(j)},\mathfrak{A}(s)) = g_j(t^{(j)},\mathfrak{A}_o + s\lambda b) + o(s)$, and

$\mathrm{Re}\; g_j(t^{(j)},\mathfrak{A}(s)) = \mathrm{Re}\; g_j(t^{(j)},\mathfrak{A}_o) + s\lambda\, \mathrm{Re}\; Dg_j(t^{(j)},\mathfrak{A}_o)b + o(s)$ by the Frechet differentiability of $g_j(t^{(j)},\cdot)$. Then consider cases:

1. $j \in J \sim J_o$. This means $\mathrm{Re}\; g_j(t^{(j)},\mathfrak{A}_o) > 0$ on Q_j which is a compact set. Thus for some sufficiently small s_1,

$$\mathrm{Re}\; g_j(t^{(j)},\mathfrak{A}(s)) \geq 0 \quad \text{for} \quad s \in [0,s_1].$$

2. $j \in J_o$. M_j is compact, so there is an open set $U_j \supseteq M_j$ on which

$$\mathrm{Re}\; Dg_j(t^{(j)},\mathfrak{A}_o)b \geq d > 0,$$

and for some $s_2 \in (0,s_1]$,

$$\mathrm{Re}\; g_j(t^{(j)},\mathfrak{A}(s)) \geq 0 \quad \text{for} \quad t^{(j)} \in U_j \quad \text{and} \quad s \in [0,s_2].$$

Now $Q_j \sim U_j$ is again compact and $t^{(j)} \in Q_j \sim U_j$ implies

$$\mathrm{Re}\; g_j(t^{(j)},\mathfrak{A}(s)) \geq 0 \quad \text{for} \quad s \in [0,s_3],$$

where $s_3 \in (0,s_2]$ by the same argument used in part 1. ∎

Using this lemma we obtain the following Kolmogorov type theorem:

Theorem 3.3: Suppose V satisfies (D1) and the side conditions (S) satisfy (D2), (D3), (R1) and (R2). If $v_o = v(\cdot,\mathfrak{A}_o) \in V_{1,1}$ is a local minimal solution of (T) for $f \in C[Q,H]$, then for all $b \in E$ such that

$$Df_i(\mathfrak{A}_o)b = 0 \quad \text{for} \quad i = 1,\dots,k,$$

and

$$\operatorname{Re} Dg_j(t^{(j)},\mathfrak{A}_o)b \geq 0 \quad \text{for} \quad j \in J_o,\ t^{(j)} \in M_j,$$

we have

$$\min_{x\in M[f-v_o]} \operatorname{Re}\ (f(x) - v_o(x),\ Dv(x,\mathfrak{A}_o)b) \leq 0.$$

Proof: The proof proceeds as in the standard case, i.e. we assume there is a $b_1 \in E$ satisfying the hypothesis and such that

$$\operatorname{Re}\ (f(x) - v_o(x),\ Dv(x,\mathfrak{A}_o)b_1) > 0$$

for all $x \in M[f - v_o]$. Then we construct a better approximation to f using v_o, b_1, and lemma 3.1. First, since (R2) is satisfied there is a $b_o \in E$ with

$$Df_i(\mathfrak{A}_o)b_o = 0 \quad \text{for} \quad i = 1,\dots,k,$$

and

$$\operatorname{Re} Dg_j(t^{(j)},\mathfrak{A}_o)b_o > 0 \quad \text{for} \quad j \in J_o,\ t^{(j)} \in M_j.$$

Then, for $\alpha > 0$ and sufficiently small,

$$b = b_1 + \alpha\, b_o \in E,$$

$$Df_i(\mathfrak{A}_o)\mathfrak{b} = 0 \quad \text{for} \quad i = 1,\ldots,k ,$$

$$\text{Re } Dg_j(t^{(j)},\mathfrak{A}_o)\mathfrak{b} > 0 \quad \text{for} \quad j \in J_o,\ t^{(j)} \in M_j,$$

and $\text{Re } (f(x) - v_o(x), Dv(x,\mathfrak{A}_o)\mathfrak{b}) > 0$ for all $x \in M[f - v_o]$ by the linearity of $Dv(x,\mathfrak{A}_o)(\cdot)$. Let U be an open set containing $M[f - v_o]$ and $a > 0$ such that

$$\text{Re } (f(x) - v_o(x), Dv(x,\mathfrak{A}_o)\mathfrak{b}) \geq 2a > 0 \quad \text{for all} \quad x \in U.$$

By lemma 3.1, there is a curve $\mathfrak{A}$ in P such that

$$v(\cdot,\mathfrak{A}(s)) \in V_{1,1},$$

$$\mathfrak{A}(0) = \mathfrak{A}_o,$$

and

$$\mathfrak{A}'(0) \quad = \lambda\mathfrak{b}, \text{ with } \lambda > 0.$$

By (D1),

$$\|v(x,\mathfrak{A}(s)) - v(x,\mathfrak{A}_o) - Dv(x,\mathfrak{A}_o)(\mathfrak{A}(s) - \mathfrak{A}_o)\|_E$$
$$= o(\|\mathfrak{A}(s) - \mathfrak{A}_o\|_E),$$

and since for any inner product $\text{Re } (a,b) \geq -(\|a\|)(\|b\|)$, we have

$$\begin{aligned}\text{Re}(f(x)-v_o(x),v(x,\mathfrak{A}(s))-v(x,\mathfrak{A}_o)) &= \text{Re}(f(x)-v_o(x),Dv(x,\mathfrak{A}_o)(\mathfrak{A}(s)-\mathfrak{A}_o)) \\ &\quad + \text{Re}(f(x)-v_o(x),v(x,\mathfrak{A}(s))-v(x,\mathfrak{A}_o)-Dv(x,\mathfrak{A}_o)(\mathfrak{A}(s)-\mathfrak{A}_o)) \\ &\geq \text{Re}(f(x)-v_o(x),Dv(x,\mathfrak{A}_o)(\mathfrak{A}(s)-\mathfrak{A}_o))-o(\|\mathfrak{A}(s) - \mathfrak{A}_o\|_E).\end{aligned}$$

$Dv(\cdot,\mathfrak{A}_o)\mathfrak{b} \in C[Q,H]$ for each $\mathfrak{b} \in E$, and $Dv(x,\mathfrak{A}_o)(\cdot)$ is a continuous linear operator from E into H for each $x \in Q$. Thus

$$\|Dv(x,\mathfrak{U}_o)(\cdot)\| = \sup_{\|c\|_E=1} \|Dv(x,\mathfrak{U}_o)(c)\|$$

$$\le \sup_{\|c\|_E=1} \max_{x\in Q} \|Dv(x,\mathfrak{U}_o)(c)\| = \sup_{\|c\|_E=1} \|Dv(\cdot,\mathfrak{U}_o)(c)\|$$

$$= \|Dv(\cdot,\mathfrak{U}_o)(\cdot)\|.$$

This implies

$$\mathrm{Re}(f(x)-v_o(x),v(x,\mathfrak{U}(s))-v(x,\mathfrak{U}_o)) \ge \lambda s\ \mathrm{Re}(f(x)-v_o(x),Dv(x,\mathfrak{U}_o)\mathfrak{b})$$

$$+ \mathrm{Re}(f(x)-v_o(x),Dv(x,\mathfrak{U}_o)(\mathfrak{U}(s)-\mathfrak{U}_o-\lambda s\mathfrak{b})) - o(\|\mathfrak{U}(s) - \mathfrak{U}_o\|_E)$$

$$\ge \lambda s\ \mathrm{Re}(f(x)-v_o(x),Dv(x,\mathfrak{U}_o)\mathfrak{b}) - \|f-v_o\|\cdot\|Dv(\cdot,\mathfrak{U}_o)(\cdot)\|o(s) - o(s)$$

$$\ge 2a\lambda s - o(s), \quad \text{for all } x \in U.$$

Thus there exists a real number $s_2 > 0$ such that $s \in [0,s_2]$ implies

$$\mathrm{Re}\ (f(x) - v_o(x),\ v(x,\mathfrak{U}(s)) - v(x,\mathfrak{U}_o)) \ge \lambda as \quad \text{for } x \in U.$$

Now let $h \equiv \|f - v_o\| - \max_{x\in Q\sim U} \|f(x) - v_o(x)\|_H$. $h > 0$ since $Q \sim U$ is compact and $M[f - v_o] \subseteq U$. For $s \to 0$,

$$\|v(x,\mathfrak{U}(s)) - v(x,\mathfrak{U}_o)\|_H \le \|Dv(x,\mathfrak{U}_o)(\mathfrak{U}(s) - \mathfrak{U}_o)\|_H + o(s)$$

$$\le \lambda s\ \|Dv(\cdot,\mathfrak{U}_o)\|\cdot\|\mathfrak{b}\|_E + o(s).$$

So for some $0 < s_3 \le s_2$ and $s \in [0,s_3]$

$$\|v(x,\mathfrak{U}(s)) - v(x,\mathfrak{U}_o)\|_H \le 2\lambda s\ \|Dv(\cdot,\mathfrak{U}_o)\|\cdot\|\mathfrak{b}\|_E.$$

Choose s_o such that

$$0 < s_o < \min\left\{s_3, \frac{a}{4\lambda\|Dv(\cdot,\mathfrak{U}_o)\|^2\|\mathfrak{b}\|_E^2}, \frac{h}{4\lambda\|Dv(\cdot,\mathfrak{U}_o)\|\ \|\mathfrak{b}\|_E}\right\}.$$

Then for $x \in U$,

$$\|f(x)-v(x,\mathfrak{A}(s_o))\|_H^2 = \|f(x)-v(x,\mathfrak{A}_o)\|_H^2 - 2\mathrm{Re}(f(x)-v_o(x), v(x,\mathfrak{A}(s_o))$$

$$- v(x,\mathfrak{A}_o)) + \|v(x,\mathfrak{A}(s_o)) - v(x,\mathfrak{A}_o)\|_H^2$$

$$\leq \|f - v_o\|^2 - 2\, a\lambda s_o + 4\, \lambda^2 s_o^2\, \|Dv(\cdot,\mathfrak{A}_o)\|_H^2$$

$$< \|f - v_o\|^2.$$

For $x \in Q \sim U$,

$$\|f(x)-v(x,\mathfrak{A}(s_o))\|_H \leq \|f(x)-v_o(x)\|_H + \|v(x,\mathfrak{A}_o)-v(x,\mathfrak{A}(s))\|_H$$

$$\leq \|f - v_o\| - h + \frac{h}{2} < \|f - v_o\|.$$

Therefore $v(\cdot,\mathfrak{A}(s_o))$ is a better approximation to f than $v(\cdot,\mathfrak{A}_o)$. This is a contradiction, so

$$\min_{x\in M[f-v_o]} (f(x) - v_o(x),\, Dv(x,\mathfrak{A}_o)b) \leq 0. \quad \blacksquare$$

The following lemma will be used to prove a generalization of the "zero in the convex hull" property of the set of extreme points in standard Chebyshev approximation.

<u>Lemma 3.2</u>: Let $V_{1,1}$ satisfy (D1) and g_j $(j \in J)$ satisfy (D3). Then the family of functionals

$$\mathfrak{F} = \{(f(x) - v_o(x),\, Dv(x,\mathfrak{A}_o)(\cdot)) \in C[E,K]:\ x \in Q\} \cup$$

$$[\bigcup_{j\in J_o} \{Dg_j(t^{(j)},\mathfrak{A}_o)(\cdot) \in C[E,K]:\ t^{(j)} \in M_j\}]$$

is an equicontinuous family with respect to the norm topology defined on E.

<u>Proof</u>: We shall show that $\{Dg_j(t^{(j)},\mathfrak{A}_o)(\cdot) \in C[E,K]: t^{(j)} \in M_j\}$ is equicontinuous. A similar proof shows $\{(f(x) - v_o(x), Dv(x,\mathfrak{A}_o)(\cdot)) \in C[E,K]: x \in Q\}$ is equicontinuous and the conclusion follows since a finite union of equicontinuous families is equicontinuous.

Let $\epsilon > 0$ be given and $b_o \in E$ fixed. We wish to find $\delta > 0$ such that for any b satisfying $\|b_o - b\|_E < \delta$, we have

$$\|Dg_{j_o}(t^{(j_o)},\mathfrak{A}_o)(b)\|_H < \epsilon \quad \text{for all} \quad t^{(j_o)} \in M_{j_o}.$$

Since

$$\|Dg_{j_o}(t^{(j_o)},\mathfrak{A}_o)(b_o - b)\|_H \le \max_{t^{(j_o)} \in M_{j_o}} \|Dg_{j_o}(t^{(j_o)},\mathfrak{A}_o)(b_o - b)\|_H$$

$$= \|Dg_{j_o}(\cdot,\mathfrak{A}_o)(b_o - b)\| \le \|Dg_{j_o}(\cdot,\mathfrak{A}_o)\| \, \|b_o - b\|_E,$$

if we choose $\delta_{j_o} < \dfrac{\epsilon}{\|Dg_{j_o}(\cdot,\mathfrak{A}_o)\|}$, then for $\|b_o - b\|_E < \delta_{j_o}$,

$$\|Dg_{j_o}(t^{(j_o)},\mathfrak{A}_o)(b_o - b)\|_H < \epsilon$$

and δ_{j_o} is independent of $t^{(j_o)} \in M_{j_o}$. ∎

<u>Remark</u>: The convex hull of an equicontinuous family of functions is also equicontinuous.

Theorem 3.4 is the main result of Hoffmann's thesis [7, Thm 1.10, p. 33] and gives a sufficient condition for $v_o \in V_{1,1}$ to be a local best approximation when $V_{1,1}$ is regular. It will be used later to characterize local best

approximation when $V_{1,1}$ satisfies further restrictions.

Theorem 3.4: Let $V_{1,1}$ satisfy (D1) and let the side conditions (S) satisfy (D2), (D3), (R1) and (R2). If $v_o = v(\cdot,\mathfrak{A}_o)$ is a local minimal solution from $V_{1,1}$ for $f \in C[Q,H]$, then the following are valid and equivalent:

(A) In the dual space E^*, the weak * closure of the convex hull of the set of functionals $\mathfrak{F}$,

$$\mathfrak{F} = \{(f(x) - v_o(x), Dv(x,\mathfrak{A}_o)\cdot) \in C[E,K]: \ x \in M[f - v_o]\} \cup$$

$$[\bigcup_{j\in J_o} \{Dg_j(t^{(j)},\mathfrak{A}_o) \in C[E,K]: \ t^{(j)} \in M_j\}],$$

and the linear space $\mathcal{L}$ spanned by the functionals

$$\{Df_i(\mathfrak{A}_o): \ i = 1,\dots,k\},$$

have non-empty intersection.

(B) For all $\mathfrak{b} \in E$ with the property that

$$Df_i(\mathfrak{A}_o)\mathfrak{b} = 0 \quad \text{for} \quad i = 1,\dots,k,$$

and

$$\operatorname{Re} Dg_j(t^{(j)},\mathfrak{A}_o)\mathfrak{b} \geq 0 \quad \text{for} \quad j \in J_o, \ t^{(j)} \in M_j,$$

we have

$$\min_{x\in M[f-v_o]} \operatorname{Re} (f(x) - v_o(x), Dv(x,\mathfrak{A}_o)\mathfrak{b}) \leq 0.$$

Proof: Theorem 3.3 says that (B) must be satisfied if v_o is a minimal solution.

Assume (A) is not true, that is, $\mathfrak{F} \cap \mathcal{L} = \phi$. Then Ascoli's Theorem [6, p. 64] implies that $\overline{co(\mathfrak{F})}$ (closure with respect to the $\sigma(E^*,E)$ topology) is compact in E^*, since for each $b \in E$

$$H_o(b) = \{(f(x)-v_o(x), Dv(x,\mathfrak{A}_o)b) \in K: \ x \in M[f - v_o]\}$$

and

$$H_j(b) = \{Dg_j(t^{(j)},\mathfrak{A}_o)b \in K: \ t^{(j)} \in M_j\}, \ j \in J_o$$

are compact sets, and in a finite dimensional space the convex hull of a compact set is again compact. So the convex hull of the above sets is compact for each $b \in E$.

$\mathcal{L}$ is a finite dimensional subspace of E^* and is $\sigma(E^*,E)$ closed. Then by a standard separation theorem [5, p. 147] for convex sets, there is a $\sigma(E^*,E)$ continuous functional on E^* which strictly separates $\mathfrak{Y}$ and $\mathcal{L}$. According to the representation theorem for $\sigma(E^*,E)$ continuous functionals [11, p. 140], there is an element $b \in E$ such that

$$\text{Re } Dg_j(t^{(j)},\mathfrak{A}_o)b > 0 \quad \text{for} \quad j \in J_o, \ t^{(j)} \in M_j$$

and

$$\text{Re } (f(x) - v_o(x), \ Dv(x,\mathfrak{A}_o)b) > 0 \quad \text{for} \quad x \in M[f - v_o]$$

and

$$Df_i(\mathfrak{A}_o)b = 0.$$

But then, by Theorem 3.3, v_o cannot be a local minimum. This is a contradiction, thus $\mathcal{L} \cap \mathfrak{Y} \neq \emptyset$. This proof also shows (B) $\Rightarrow$ (A).

(A) implies (B) will be shown indirectly, so we assume that there exists a $b_o \in E$ such that

$$Df_i(\mathfrak{A}_o)b_o = 0 \quad \text{for} \quad i = 1,\ldots,k,$$

$$\mathrm{Re}\ Dg_j(t^{(j)},\mathfrak{U}_o)b_o \geq 0 \quad \text{for} \quad j \in J_o,\ t^{(j)} \in M_j,$$

and

$$\mathrm{Re}\ (f(x) - v_o(x),\ Dv(x,\mathfrak{U}_o)b_o) > 0 \quad \text{for all} \quad x \in M[f - v_o].$$

By proceeding as in the proof of Theorem 3.3, we obtain $b \in E$ such that

$$Df_i(\mathfrak{U}_o)b = 0 \quad \text{for} \quad i = 1,\ldots,k,$$

$$\mathrm{Re}\ Dg_j(t^{(j)},\mathfrak{U}_o)b > 0 \quad \text{for} \quad j \in J_o,\ t^{(j)} \in M_j,$$

and

$$\mathrm{Re}\ (f(x) - v_o(x),\ Dv(x,\mathfrak{U}_o)b) > 0 \quad \text{for} \quad x \in M[f - v_o].$$

But then $\mathfrak{J} \cap \mathcal{L} = \phi$. Since the sets M_j $(j \in J_o)$, and $M[f - v_o]$ are compact, $\overline{\mathrm{co}\ \mathfrak{J}} \cap \mathcal{L} = \phi$. ∎

The usual "zero in the convex hull" theorem is a corollary to the above theorem since if there are no side conditions we can set $f_1(\mathfrak{U}) \equiv 0$. Then $\mathcal{L} = \{0\}$ and $0 \in \overline{\mathrm{co}(\mathfrak{J})}$. More particularly, if $v(\cdot,\mathfrak{U})$ is linear in $\mathfrak{U}$,

$$Dv(\cdot,\mathfrak{U}_o)b_o = v(\cdot,b_o + \mathfrak{U}_o) - v(\cdot,\mathfrak{U}_o)$$

and

$$\mathfrak{J} = \{(f(x) - v_o(x),\ v(x,b) - v(x,\mathfrak{U}_o)) \in C[E,K]:\ x \in M[f - v_o]\}.$$

We make the following definitions for convenience of notation:

$\mathcal{L}[\mathfrak{U}_o]$ will be the linear subspace of $C[Q,H]$ consisting of all elements $Dv(\cdot,\mathfrak{U}_o)b$ with $b \in E$.

$$\mathcal{L}_{1,0}[\mathfrak{U}_o] = \{Dv(\cdot,\mathfrak{U}_o)b \in \mathcal{L}[\mathfrak{U}_o]:\ Df_i(\mathfrak{U}_o)b = 0,\ i = 1,\ldots,k\}.$$

$\mathcal{L}_{1,0}[\mathfrak{U}_o]$ is a linear subspace.

$$\mathcal{L}_{0,1}[\mathfrak{A}_o] = \{Dv(\cdot,\mathfrak{A}_o)b \in \mathcal{L}[\mathfrak{A}_o]: \ \text{Re } Dg_j(t^{(j)},\mathfrak{A}_o)b \geq 0;$$

$j \in J_o$, $t^{(j)} \in M_j\}$. $\mathcal{L}_{0,1}[\mathfrak{A}_o]$ is a convex cone.

Let $\mathcal{L}_{1,1}[\mathfrak{A}_o] = \mathcal{L}_{1,0}[\mathfrak{A}_o] \cap \mathcal{L}_{0,1}[\mathfrak{A}_o]$. $\mathcal{L}_{1,1}[\mathfrak{A}_o]$ is a convex cone.

If we do not wish to specify any particular set, we will write $\mathcal{L}_{\alpha,\beta}[\mathfrak{A}_o]$. $(\alpha,\beta \in \{0,1\})$.

Assume that the Banach space E is of finite dimension n. Let the set V satisfy (D1) and the side conditions (S) satisfy (D2), (D3), (R1) and (R2). Every element in $\mathcal{L}[\mathfrak{A}_o]$ can be written in the form

$$Dv(\cdot,\mathfrak{A}_o)b = \sum_{i=1}^{n} \alpha_i Dv(\cdot,\mathfrak{A}_o)b_i$$

where $b_1,\ldots,b_n$ form a basis for E and $\alpha_1,\ldots,\alpha_n$ are elements of the scalar field K for E. The following theorem relates the minimal solution and a linear operator on $C[Q,H]$.

<u>Theorem 3.5</u>: Let $v_o = v(\cdot,\mathfrak{A}_o)$ be a local minimal solution from $V_{1,1}$ for $f \in C[Q,H]$. Then for $\overline{(\epsilon,M[f - v_o])}$ in $\Sigma[f - v_o]$ there exist points

$$x_1,\ldots,x_r \ \ (r \geq 1) \ \text{ from } M[f - v_o],$$

$$t_1^{(j)},\ldots,t_{sj}^{(j)} \ \text{ from } M_j \ \text{ for each } j \in J_o,$$

and real numbers

$$\lambda_i,\ i = 1,\ldots,r\ ;\ \mu_{ij},\ i = 1,\ldots,s_j,\ j \in J_o,$$

and

$$\gamma_i \;, \; i = 1,\ldots,k$$

such that

(i) $g_j(t_i^{(j)},\mathfrak{A}_o) = 0, \; i = 1,\ldots,s_j; \; j \in J_o,$

(ii) $\mu_{ij} > 0, \; i = 1,\ldots,s_j; \; j \in J_o,$

(iii) $\lambda_i > 0 \;, \; i = 1,\ldots,r,$

(iv) $$r + \sum_{j\in J_o} s_j \le \begin{cases} \dim \mathcal{L}_{1,0}[\mathfrak{A}_o] + 1, & \text{if } H \text{ is real,} \\ 2 \dim \mathcal{L}_{1,0}[\mathfrak{A}_o] + 1, & \text{if } H \text{ is complex,} \end{cases}$$

and

(v) $$\sum_{i=1}^{r} \lambda_i(\varepsilon(x_i),Dv(x_i,\mathfrak{A}_o)\cdot) + \sum_{j\in J_o} \sum_{i=1}^{s_j} \mu_{ij} Dg_j(t_i^{(j)},\mathfrak{A}_o)\cdot + \sum_{i=1}^{k} \gamma_i Df_i(\mathfrak{A}_o)\cdot = 0^* \in E^*.$$

<u>Proof</u>: If v_o is a local minimal solution from $V_{1,1}$ for $f \in C[Q,H]$, then by Theorem 3.4, the $\sigma(E^*,E)$ closure of $co(\mathfrak{K})$ and the linear space $\mathcal{L}$ have non-empty intersection in E^*. Thus $\bar{0}$ of the quotient space $E^*/\mathcal{L}$ lies in the convex hull of the set of elements

$$\{c + \mathcal{L}: \; c \in co(\mathfrak{K})\}.$$

The dimension of E^* is n, so $E^*/\mathcal{L}$ has dimension $n - k$. By the Theorem of Caratheodory [3, p. 17], $0^* \in E^*$ is a convex linear combination of at most $n - k + 1$ (or $2n - 2k + 1$) elements of the form

$$c + \ell, \; \ell \in \mathcal{L}.$$

It follows that the dimension of $\mathcal{L}_{1,0}[\mathfrak{U}_o] = n - k$.

Now every element of $\mathcal{L}$ can be written as a linear combination of the elements $Df_i(\mathfrak{U}_o)(i = 1,\ldots,k)$. So there exist points

$$x_1,\ldots,x_r \in M[f - v_o]$$

$$t_1^{(j)},\ldots,t_{sj}^{(j)} \in M_j,\ j \in J_o$$

and real numbers $\lambda_i, \mu_{ij}, \gamma_i$ such that

$$\lambda_i > 0,\ i = 1,\ldots,r;$$

$$\mu_{ij} > 0,\ i = 1,\ldots,s_j,\ j \in J_o;$$

with

$$r + \sum_{j\in J_o} s_j \leq \dim \mathcal{L}_{1,0}[\mathfrak{U}_o] + 1$$

(or $\leq 2 \dim \mathcal{L}_{1,0}[\mathfrak{U}_o] + 1$ in the complex case)

and

$$\sum_{i=1}^{r} \lambda_i(\epsilon(x_i),\ Dv(x_i,\mathfrak{U}_o)\cdot) + \sum_{j\in J_o} \sum_{i=1}^{s_j} \mu_{ij} Dg_j(t_i^{(j)},\mathfrak{U}_o)\cdot$$
$$+ \sum_{i=1}^{k} \gamma_i Df_i(\mathfrak{U}_o)\cdot = 0^*.$$

r must be greater than or equal to 1 since, by (R2), there is a $\mathfrak{b}_o \in E$ such that

$$Df_i(\mathfrak{U}_o)\mathfrak{b}_o = 0 \quad \text{for} \quad i = 1,\ldots,k$$

and

$$\text{Re}\ Dg_j(t^{(j)},\mathfrak{U}_o)\mathfrak{b}_o > 0 \quad \text{for} \quad j \in J_o,\ t^{(j)} \in M_j.$$

Finally, $g_j(t^{(j)},\mathfrak{U}_o) = 0$ since $t^{(j)} \in M_j$.

Section 3: A Special Class of Non-Linear Approximation Problems

In this section we shall discuss a property of V which will make the Klomogorov criterion both necessary and sufficient for a best approximation.

Definition 3.4: Let $V_{\alpha,\beta}$ satisfy (D1). Then $V_{\alpha,\beta}$ is called an equibasis system if for every element $v_o = v(\cdot,\mathfrak{A}_o) \in V_{\alpha,\beta}$, the signature Σ is extremal with respect to $V_{\alpha,\beta}$ if and only if it is extremal for the zero element with respect to $\mathcal{L}_{\alpha,\beta}[\mathfrak{A}_o]$. ($\alpha,\beta$ are the same for $\mathcal{L}$ and V.)

Not every set V is an equibasis system as the following example shows.

Example 3.4: Let the set V consist of all elements of the form

$$v(x,a) = a - 4a^2(x - \tfrac{1}{2})^2$$

where a is a real number and let v be defined on $[0,1]$. The linear space $\mathcal{L}[\tfrac{1}{2}]$ consists of all elements

$$Dv(x,\tfrac{1}{2})b = b - 4b(x - \tfrac{1}{2})^2$$

where b is any real number. The signature $\Sigma = \overline{(\varepsilon,M)}$ where

$$M = \{0,1\},$$

$$\varepsilon \in \mathcal{E}_M = \{\varepsilon \in C[0,1]: \ |\varepsilon(x)| \le 1, \ \varepsilon(0) = -1 = \varepsilon(1)\}$$

is extremal for $\mathscr{L}[\frac{1}{2}]$; that is

$$\min_{x\in\{0,1\}} \epsilon(x)[1 - 4(x - \tfrac{1}{2})^2]b \le 0$$

for all real numbers b. However it is not extremal for

$$v(x,\tfrac{1}{2}) = \tfrac{1}{2} - (x - \tfrac{1}{2})^2,$$

since for all $b \neq \frac{1}{2}$,

$$\min_{x\in\{0,1\}} \epsilon(x)[b - 4b^2(x - \tfrac{1}{2})^2 - \tfrac{1}{2} + (x - \tfrac{1}{2})^2] \not\le 0. \quad \blacksquare$$

Many familiar sets are equibasis systems.

<u>Example 3.5</u>: Let V be a linear subspace of $C[Q,H]$, i.e. the functions $v(\cdot,\mathfrak{A})$ are linear in $\mathfrak{A} \in P$ and P is a subspace of E. Let the side conditions f_i, $i = 1,\ldots,k$, be linear functionals. Then $V_{1,0}$ is an equibasis system. By definition 3.2, we have

$$v(\cdot,\mathfrak{A}_o + \mathfrak{b}) - v(\cdot,\mathfrak{A}_o) = v(\cdot,\mathfrak{b}) = Dv(\cdot,\mathfrak{A}_o)(\mathfrak{b}),$$

and

$$f_i(\mathfrak{A}_o + \mathfrak{b}) - f_i(\mathfrak{A}_o) = f_i(\mathfrak{b}) = Df_i(\mathfrak{A}_o)(\mathfrak{b}),\ i = 1,\ldots,k.$$

Now let $\Sigma = \overline{(\epsilon,M)}$ be extremal for $v(x,\mathfrak{A}_o) \in V_{1,0}$ $= \{v(\cdot,\mathfrak{A}) \in V:\ f_i(\mathfrak{A}) = 0 \text{ for } i = 1,\ldots,k\}$. Then

$$\min_{x\in M} \epsilon(x)(v(x,\mathfrak{b}_1) - v(x,\mathfrak{A}_o)) \le 0$$

for all $\mathfrak{b}_1 \in P$ such that $v(x,\mathfrak{b}_1) \in V_{1,0}$. For $Dv(\cdot,\mathfrak{A}_o)\mathfrak{b} \in \mathscr{L}_{1,0} = \{Dv(\cdot,\mathfrak{A}_o)\mathfrak{b}:\ Df_i(\mathfrak{A}_o)\mathfrak{b} = 0,\ i = 1,\ldots,k\}$, we have $Df_i(\mathfrak{A}_o)\mathfrak{b} = 0$ for $i = 1,\ldots,k$ and

$$Df_i(\mathfrak{A}_o)\mathfrak{b} = f_i(\mathfrak{A}_o + \mathfrak{b}) - f_i(\mathfrak{A}_o) = f_i(\mathfrak{A}_o + \mathfrak{b}).$$

Thus $v(x,\mathfrak{A}_o + b) \in V_{1,0}$, so

$$\min_{x \in M} \varepsilon(x)(Dv(x,\mathfrak{A}_o)b - 0) \leq 0$$

for all $Dv(x,\mathfrak{A}_o)b \in \mathcal{L}_{1,0}[\mathfrak{A}_o]$ and Σ is extremal for 0 with respect to $\mathcal{L}_{1,0}[\mathfrak{A}_o]$. Likewise, if we know

$$\min_{x \in M} \varepsilon(x)(Dv(x,\mathfrak{A}_o)b_1 - 0) \leq 0$$

for all $Dv(x,\mathfrak{A}_o)b_1 \in \mathcal{L}_{1,0}[\mathfrak{A}_o]$, and if $v(x,b) \in V_{1,0}$ then $f_i(b) = 0$, $i = 1,\ldots,k$, and

$$0 = f_i(b) - f_i(\mathfrak{A}_o) = f_i(b - \mathfrak{A}_o) = Df_i(\mathfrak{A}_o)(b - \mathfrak{A}_o), \quad \text{for } i = 1,\ldots,k.$$

Thus $Dv(x,\mathfrak{A}_o)(b - \mathfrak{A}_o) \in \mathcal{L}_{1,0}[\mathfrak{A}_o]$ and

$$Dv(x,\mathfrak{A}_o)(b - \mathfrak{A}_o) = v(x,b) - v(x,\mathfrak{A}_o)$$

so

$$\min_{x \in M} \varepsilon(x)(v(x,b) - v(x,\mathfrak{A}_o)) \leq 0$$

and Σ is extremal for $v(x,\mathfrak{A}_o) \in V_{1,0}$. ∎

Example 3.6: Let $\mu_1,\ldots,\mu_m$; $v_1,\ldots,v_n$ be two sets of linearly independent real-valued continuous functions defined on a compact metric space Q. For $\mathfrak{A} = (a_1,\ldots,a_m, b_1,\ldots,b_n) \in R^{n+m}$, let

$$r(x,\mathfrak{A}) = \frac{\sum_{i=1}^{m} a_i \mu_i(x)}{\sum_{i=1}^{n} b_i v_i(x)},$$

and set

$$V = \{r(x,\mathfrak{A}): \mathfrak{A} \in R^{n+m} \text{ and } \sum_{i=1}^{n} b_i v_i(x) > 0 \text{ for all } x \in Q\}.$$

We claim V is an equibasis system. Let $\Sigma = \overline{(\varepsilon,M)}$ be extremal for $r_o = r(x,\mathfrak{A}_o)$ in V where $\mathfrak{A} = (a_1^o,\ldots,a_m^o, b_1^o,\ldots,b_n^o)$. Then

$$\min_{x \in M} \varepsilon(x)(r(x,\mathfrak{A}) - r(x,\mathfrak{A}_o)) \le 0$$

for all $r \in V$. We must show

$$\min_{x \in M} \varepsilon(x)(Dr(x,\mathfrak{A}_o)b) \le 0$$

for all $Dr(x,\mathfrak{A}_o)b \in \mathcal{L}[\mathfrak{A}_o] = \{Dr(x,\mathfrak{A}_o)b: \ b \in R^{n+m}\}$, i.e., Σ is extremal for 0 with respect to $\mathcal{L}[\mathfrak{A}_o]$. By the extension theorem of J. Dugundji [2, p. 14], there is an element $(\varepsilon,M) \in \Sigma$ such that for all $x \in Q \sim M$,

$$|\varepsilon(x)| < 1.$$

Now,

$$\min_{x \in M} ((\varepsilon(x) + r_o(x) - r_o(x))(r(x,\mathfrak{A}_o) - f(x)) \le 0.$$

So the signature $\Sigma[(\varepsilon(x) + r_o(x)) - r_o(x)] = \Sigma$ is extremal for r_o with respect to V, and, by Theorem 3.2, r_o is a minimal solution for $\varepsilon + r_o \in C[Q]$. Thus Theorem 3.3 implies

$$\min_{x \in M} \varepsilon(x) Dr(x,\mathfrak{A}_o)b \le 0,$$

i.e.,

$$\min_{x \in M} \varepsilon(x) \frac{1}{\sum_{i=1}^{n} b_i^o v_i(x)} \left(\sum_{i=1}^{m} a_i \mu_i(x) - r_o(x) \sum_{i=1}^{n} b_i v_i(x) \right) \le 0.$$

This says Σ is extremal for 0 with respect to $\mathcal{L}[\mathfrak{A}_o]$.

Conversely, if Σ is extremal for $0 \in \mathcal{L}[\mathfrak{A}_o]$, then since $\sum_{i=1}^{n} b_i^o v_i(x) > 0$,

$$\min_{x \in M} \epsilon(x) \left(\sum_{i=1}^{m} a_i \mu_i(x) - r_o(x) \sum_{i=1}^{n} b_i v_i(x) \right) \le 0,$$

and if $\sum_{i=1}^{n} b_i v_i(x) > 0$, then for $\mathfrak{A} = (a_1, \ldots, a_m, b_1, \ldots, b_m)$, $r(x,\mathfrak{A}) \in V$ and

$$\min_{x \in M} \epsilon(x)(r(x,\mathfrak{A}) - r(x,\mathfrak{A}_o)) \le 0.$$

So Σ is extremal for v_o with respect to V. ∎

Example 3.7: Let Q be a compact metric space and V a subset of $C[Q,H]$ with the following property:

To each pair $\mathfrak{A}, \mathfrak{A}_o \in P$ and every real number $t \in [0,1]$, there is a parameter $\mathfrak{A}(t)$ and a continuous function

$$g: \ Q \times [0,1] \to R,$$

such that

1. $g(x,0) > 0$ for all $x \in Q$,
2. $(1 - tg)v(\cdot,\mathfrak{A}_o) + tgv(\cdot,\mathfrak{A}) - v(\cdot,\mathfrak{A}(t)) = o(t)$ for $t \to 0$.

Meinardus and Schwedt [17] called such a set asymptotically convex. We shall also assume that our set V satisfies (D1) and has the following two properties:

3. The function $\mathfrak{A}(t)$, given above, is Frechet-differentiable.
4. $\mathfrak{A}(0) = \mathfrak{A}_o$.

The following property was proven by B. Brosowski [2].

(F) $\quad Dv(\cdot,\mathfrak{A}_o)\mathfrak{A}'(0) = g(\cdot,0)(v(\cdot,\mathfrak{A}) - v(\cdot,\mathfrak{A}_o))$

where $\mathfrak{A}'(0)$ is the Frechet derivative of $\mathfrak{A}(t)$ at the point $t = 0$. Thus $Dv(\cdot,\mathfrak{A}_o)\mathfrak{A}'(0)$ is a function from $[0,1]$ into $C[Q,H]$.

An asymptotically convex set which satisfies 3. and 4. is an equibasis system. Let Σ be extremal for $v(\cdot,\mathfrak{A}_o) \in V$ with respect to V. Then for all $\overline{(\varepsilon,M)} \in \Sigma$,

$$\min_{x\in M} \operatorname{Re}\, (\varepsilon(x), v(x,\mathfrak{A}) - v(x,\mathfrak{A}_o)) \le 0$$

for all $v(\cdot,\mathfrak{A}) \in V$. Again using the extension theorem of J. Dugundji, we can find $\varepsilon(x)$ such that

$$|\varepsilon(x)| < 1 \quad \text{for all} \quad x \in Q \sim M.$$

And, as in the last example, $\Sigma = \Sigma[(\varepsilon(x) + v_o(x)) - v_o(x)]$. So by Theorem 3.3

$$\min_{x\in M} \operatorname{Re}\, (\varepsilon(x), Dv(x,\mathfrak{A}_o)b) \le 0$$

for all $D(\cdot,\mathfrak{A}_o)b \in \mathcal{L}[\mathfrak{A}_o]$, and Σ is extremal for 0 with respect to $\mathcal{L}[\mathfrak{A}_o]$.

Conversely, if Σ is extremal for 0, using (F) we obtain

$$\min_{x\in M} \operatorname{Re}(\varepsilon(x), v(x,\mathfrak{A}) - v(x,\mathfrak{A}_o)) \le 0$$

for all $v(\cdot,\mathfrak{A}) \in V$, that is Σ is extremal for $v_o = v(\cdot,\mathfrak{A}_o)$ with respect to V. ∎

In the previous examples, no side conditions were assumed. When side conditions are required, we must consider more localized properties. In the remainder of this section we shall assume that the side conditions are regular and that for a fixed $\mathfrak{U}_o \in P$ and each $\mathfrak{b} \in E$ satisfying (R1), there is an associated curve $\mathfrak{U}_{\mathfrak{b}}$ in P such that

$$\mathfrak{U}_{\mathfrak{b}}(s) = \mathfrak{U}_o + \lambda_{\mathfrak{b}}(s)\mathfrak{b},$$

where $\lambda_{\mathfrak{b}}$ is a continuous, real-valued, positive function, differentiable at $s = 0$ and

$$\lambda_{\mathfrak{b}}(0) = 0,$$

$$\lambda_{\mathfrak{b}}'(0) > 0.$$

The set of elements $\mathfrak{U} \in P$ which lie on any such curve is denoted by $\mathcal{A}_o$.

<u>Definition 3.5</u>: Let $v_o \in V_{\alpha,\beta}$. A set W_o is called a <u>neighborhood$^+$</u> of v_o in $V_{\alpha,\beta}$ if

$$W_o = \{v(\cdot,\mathfrak{U}) \in V_{\alpha,\beta}\colon \ \mathfrak{U} \in \mathcal{A}_o\}.$$

<u>Definition 3.6</u>: A set $V_{\alpha,\beta}$ is called a <u>local$^+$ equibasis system</u> if for every $\mathfrak{U}_o \in P$:

Whenever the signature Σ is extremal for 0 with respect to $\mathcal{L}_{\alpha,\beta}[\mathfrak{U}_o]$, there exists a local$^+$ neighborhood $W_o \subset V_{\alpha,\beta}$ of $v_o = v(\cdot,\mathfrak{U}_o)$ such that Σ is extremal for v_o with respect to W_o.

In general, the neighborhood^{+} W_o is not an open set with respect to the norm topology, but consists of separate paths in V with v_o as origin.

<u>Example 3.8</u>: Let V be a linear subspace of $C[Q,H]$ and let the side conditions satisfy (D2), (D3), (R1) and (R2). Let Σ be extremal for 0 with respect to $\mathcal{L}_{1,1}[\mathfrak{A}_o]$, that is

$$\min_{x\in M} \operatorname{Re}\,(\varepsilon(x),\, Dv(x,\mathfrak{A}_o)\mathfrak{b}) \le 0$$

for all $Dv(\cdot,\mathfrak{A}_o)\mathfrak{b} \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$. By (R1) and (R2), there is a curve

$$\mathfrak{A}_b(s) = \mathfrak{A}_o + \lambda_b(s)\mathfrak{b}$$

satisfying the side conditions and

$$\mathfrak{A}_b(0) = \mathfrak{A}_o$$

$$\lambda_b(s) \ge 0$$

$$\mathfrak{A}_b'(0) = \lambda_b'(0)\mathfrak{b} \quad \text{with} \quad \lambda_b'(0) > 0.$$

By the linearity of V,

$$\min_{x\in M} (\varepsilon(x),\, v(x,\mathfrak{A}_b(s)) - v(x,\mathfrak{A}_o)) \le 0$$

for all elements $\mathfrak{A}_b(s)$ of $\mathcal{A}_o$. Thus Σ is locally^{+} extremal for $v(\cdot,\mathfrak{A}_o) \in V_{1,1}$. Since the argument above is reversible, the set $V_{1,1}$ is a local^{+} equibasis system.

A similar argument shows that a system of rational functions or a modified asymptotically convex system with side conditions satisfying (R1) and (R2) is a local^{+} equibasis system.

<u>Definition 3.7</u>: Let $V_{\alpha,\beta}$ be a subset of $C[Q,H]$. An element $v_o \in V_{\alpha,\beta}$ is called a <u>local$^+$ minimal solution</u> for $f \in C[Q,H]$ if there is a neighborhood$^+$ $W_o \subseteq V_{\alpha,\beta}$ of v_o such that v_o is a minimal solution for f from W_o.

We can characterize a best approximation to a function $f \in C[Q,H]$ from an equibasis system.

<u>Theorem 3.6</u>: Let $V_{1,1}$ be a subset of $C[Q,H]$ and an equibasis system. The element $v(\cdot,\mathfrak{A}_o) \in V_{1,1}$ is a local minimal solution for f with respect to $V_{1,1}$ if and only if

$$\min_{x \in M[f-v_o]} \operatorname{Re}\,(f(x) - v_o(x),\ Dv(x,\mathfrak{A}_o)b) \le 0$$

for all $Dv(\cdot,\mathfrak{A}_o)b \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$.

<u>Proof</u>: This theorem is an immediate consequence of Theorems 3.2, 3.3 and the fact that $V_{1,1}$ is an equibasis system.

<u>Theorem 3.7</u>: Let $V_{1,1}$ be a subset of $C[Q,H]$ and a local$^+$ equibasis system. The element $v(\cdot,\mathfrak{A}_o) \in V_{1,1}$ is a local$^+$ minimal solution (with respect to a neighborhood$^+$ of v_o) for the function f if and only if

$$\min_{x \in M[f-v_o]} \operatorname{Re}\,(f(x) - v_o(x),\ Dv(x,\mathfrak{A}_o)b) \le 0$$

for all $Dv(\cdot,\mathfrak{A}_o)b \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$.

<u>Theorem 3.8</u>: Let $V_{1,1}$ be a (local$^+$) equibasis system. Then $v(\cdot,\mathfrak{A}_o) \in V_{1,1}$ is a local (local$^+$) minimal solution for $f \in C[Q,H]$ if and only if the $\sigma(E^*,E)$ closure (in E^*) of the convex hull of the set $\mathfrak{J}$,

$$\mathfrak{J} = \{(f(x) - v_o(x),\ Dv(x,\mathfrak{A}_o)\cdot) \in C[E,K]:\ x \in M[f - v_o]\}$$

$$\cup\ [\bigcup_{j\in J_o} \{Dg_j(t^{(j)},\mathfrak{A}_o)\cdot \in C[E,K]:\ j \in J_o,\ t^{(j)} \in M_j\}],$$

and the linear space $\mathcal{L}$ spanned by the functionals

$$Df_i(\mathfrak{A}_o),\ i = 1,\ldots,k,$$

have non-empty intersection.

Proof: The necessity follows from Theorem 3.4.

(Sufficiency) We assume that v_o is not a local (local$^+$) minimal solution for f. Then the signature Σ cannot be extremal for 0 with respect to $\mathcal{L}_{1,1}[\mathfrak{A}_o]$. So there is an element $Dv(\cdot,\mathfrak{A}_o)b \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$ such that for all $x \in M[f - v_o]$,

$$\mathrm{Re}\ (f(x) - v_o(x),\ Dv(x,\mathfrak{A}_o)b) > 0.$$

But then the intersection of the $\sigma(E^*,E)$ closure of the convex hull of the set $\mathfrak{J}$ and the linear space $\mathcal{L}$ must be empty. This is a contradiction. ■

As a corollary to this theorem we have the following result of Cheney and Loeb [4].

Corollary: Let V be the set of generalized rational functions as in example 3.6 and $C[Q]$ the linear space of real-valued functions defined on a compact metric space Q. Then a necessary and sufficient condition for $r_o \in V$ to be a minimal solution for $f \in C[Q]$ is that the zero of Euclidean $(n+m)$-space lie in the convex hull of the set

$$\{(f(x) - r_o(x))[\mu_1(x),\ldots,\mu_m(x), -v_1(x)r_o(x),\ldots,-v_n(x)r_o(x)]:$$
$$x \in M[f - r_o]\}.$$

Section 4: Results Concerning Uniqueness

The following theorem characterizes those equibasis systems which yield unique approximations.

Theorem 3.9: Let $V_{1,1} \subset C[Q,H]$ be an equibasis system. Then the following assertions are equivalent:

(A) Every $f \in C[Q,H]$ has at most one minimal solution in $V_{1,1}$.

(B) For all $v_o = v(\cdot,\mathfrak{A}_o) \in V_{1,1}$ and every signature Σ extremal for v_o, the difference $v - v_o$ with $v(\cdot,\mathfrak{A}) \in V_{1,1}$ and $v - v_o \not\equiv 0$ is non-zero in at least one point $x \in M$ of the signature Σ.

(C) For each $f \in C[Q,H] \sim V_{1,1}$ and for every best approximation $v_o \in V_{1,1}$ for f

$$\|v - v_o\| < 2\|f - v_o\| \quad \text{for all} \quad v \in V_{1,1}.$$

(D) For every pair of functions $f_1, f_2 \in C[Q,H] \sim V_{1,1}$ and for every pair of best approximations $v_1 = v(\cdot,\mathfrak{A}_1) \in V_{1,1}$ for f_1 and $v_2 = v(\cdot,\mathfrak{A}_2)$ for f_2,

$$\|v_2 - v_1\| < \|f_1 - v_2\| + \|f_2 - v_1\|.$$

Proof: (A) $\Rightarrow$ (B). Suppose there exists $v_o \in V_{1,1}$ and an extremal signature Σ and an element $v \in V_{1,1}$ with

$$v - v_o \not\equiv 0$$

and

$$v(x) - v_o(x) = 0 \quad \text{for all} \quad x \in M.$$

Then, we claim, there is a function $f \in C[Q,H] \sim V_{1,1}$ for which v_o and v are both minimal solutions with respect to $V_{1,1}$. Let

$$m(x) = \|v(x) - v_o(x)\|_H \quad \text{for all} \quad x \in Q,$$

$$m(x) \in C[Q,H],$$

$$\overline{K} = \max_{x \in Q} m(x).$$

Let $(\varepsilon, M) \in \Sigma$. Then define

$$h(x) = \varepsilon(x)(\overline{K} - m(x)).$$

$$\|h(x)\|_H \begin{cases} \leq \overline{K} & \text{for} \quad x \in Q \sim M, \\ = \overline{K} & \text{for} \quad x \in M. \end{cases}$$

Set $f(x) = h(x) + v_o(x)$. Then v_o is a minimal solution for f with respect to $V_{1,1}$ with

$$\|f - v_o\| = \overline{K}.$$

Since $\Sigma \subset \Sigma[f - v_o]$, Σ is extremal for v_o with respect to $V_{1,1}$ and therefore so is $\Sigma[f - v_o]$. Then by the Theorem 3.2, v_o is a minimal solution. But, $v \in V_{1,1}$ is also a minimal solution since for all $x \in Q$,

$$\|f(x) - v(x)\|_H = \|h(x) + v_o(x) - v(x)\|_H$$

$$\leq \|h(x)\|_H + \|v_o(x) - v(x)\|_H$$

$$\leq \|\epsilon(x)\|_H(\bar{K} - m(x)) + m(x)$$

$$\leq \bar{K}.$$

(B) ⇒ (A). Suppose there is an element $f \in C[Q,H]$ with two best approximations $v_o, v \in V_{1,1}$. Then the signature

$$\Delta = \Sigma[f - v_o] \cap \Sigma[f - v]$$

is extremal for v_o and v with respect to $V_{1,1}$ [2, p. 105]. For every point $x \in M_\Delta$,

$$f(x) - v_o(x) = f(x) - v(x).$$

So

$$v(x) - v_o(x) = 0 \quad \text{for all} \quad x \in M_\Delta.$$

But this contradicts (B) unless $v(x) \equiv v_o(x)$.

(A) ⇒ (C). Let $v_o \in V_{1,1}$ be the unique minimal solution for $f \in C[Q,H] \sim V_{1,1}$; that is

$$\|f - v_o\| < \|f - v\| \quad \text{for all} \quad v \in V_{1,1}.$$

Then

$$\|v_o - v\| - \|f - v\| < \|f - v\|,$$

so

$$\|v_o - v\| < 2\|f - v\|.$$

(C) ⇒ (B). If (B) is not true, we wish to show there is a function $f \in C[Q,H] \sim V_{1,1}$ with a best approximation $v_o \in V_{1,1}$ such that for some $v \in V_{1,1}$,

$$\|v_o - v\| = 2\|f - v\|.$$

Note: $\|v_o - v\| - \|f - v\| \le \|f - v_o\| \le \|f - v\|$ for all $v \in V_{1,1}$. Thus $\|v_o - v\| \le 2\|f - v\|$.

Now let v_o, v give a contradiction to (B) with the signature Σ. Let $m, \bar{K}, \varepsilon$ be as in the proof of (A) $\Rightarrow$ (B). Set

$$h(x) = \tfrac{1}{2}\,\varepsilon(x)(\bar{K} - m(x)) + \tfrac{1}{2}\,\varepsilon(x)(v(x) - v_o(x)).$$

For $x \in M$, $\|h(x)\|_H = \dfrac{\bar{K}}{2}$ since $v(x) - v_o(x) = 0$. For $x \in Q \sim M$, $\|h(x)\|_H \le \tfrac{1}{2}(\bar{K} - m(x)) + \tfrac{1}{2}\|v(x) - v_o(x)\|_H \le \dfrac{\bar{K}}{2}$. Now define $f(x) = h(x) + v_o(x)$. Then the following are true:

1. $\|v_o - v\| = 2\|f - v\|$ since $\|f(x) - v(x)\|_H$

$$= \tfrac{1}{2}(\bar{K} - m(x))\|\varepsilon(x)\|_H \begin{cases} = \bar{K}/2 & \text{for } x \in M, \\ \le \bar{K}/2 & \text{for } x \in Q \sim M. \end{cases}$$

2. v_o is a minimal solution for f with respect to $V_{1,1}$ since

$$\|f - v_o\| = \|h\| = \frac{\bar{K}}{2}$$

and $\Sigma \subset \Sigma[f - v_o]$, so, by Theorem 3.2, v_o is a best approximation for f.

(C) $\Rightarrow$ (D). Let v_1 be a best approximation to f_1 and v_2 a best approximation to f_2. Then, by (C),

$$\|v_2 - v_1\| < 2\|f_1 - v_2\|,$$

$$\|v_1 - v_2\| < 2\|f_2 - v_1\|.$$

Thus

$$\|v_1 - v_2\| < \|f_1 - v_2\| + \|f_2 - v_1\|.$$

(D) $\Rightarrow$ (B). Again suppose that there exist a v_1 and v_o and a signature Σ extremal for v_o contradicting (B). Define

$$h_o(x) = \tfrac{1}{2}\,\epsilon(x)(\overline{K} - m(x)) + \tfrac{1}{2}(v_1(x) - v_o(x)),$$

$$h_1(x) = \tfrac{1}{2}\,\epsilon(x)(\overline{K} - m(x)) + \tfrac{1}{2}(v_o(x) - v_1(x)).$$

Then setting

$$f_o(x) = h_o(x) + v_o(x)$$

and

$$f_1(x) = h_1(x) + v_1(x)$$

will give the desired contradiction to (D).

1. $\|v_o - v_1\| = \|f_1 - v_o\| + \|f_o - v_1\|$, since

$$\|f_1(x) - v_o(x)\|_H = \|f_o(x) - v_1(x)\|_H \begin{cases} = \overline{K}/2 & \text{for } x \in M, \\ \leq \overline{K}/2 & \text{for } x \in Q \sim M. \end{cases}$$

2. v_o is a minimal solution for f_o.

$$\|f_o(x) - v_o(x)\|_H \begin{cases} = \overline{K}/2 & \text{for } x \in M, \\ \leq \overline{K}/2 & \text{for } x \in Q \sim M, \end{cases}$$

and $\Sigma \subset \Sigma[f - v_o]$. Since Σ was extremal for v_o, v_o is a best approximation to f_o.

3. $\|f_1(x) - v_1(x)\|_H = \overline{K}/2$ as above. So $\Sigma \subset \Sigma[f_1 - v_1]$. It remains to be shown that Σ is extremal for v_1. We know

$$\mathrm{Re}\ (\varepsilon(x),\ v_o(x) - v_1(x)) = 0 \quad \text{for all} \quad x \in M$$

and

$$\min_{x \in M} \mathrm{Re}\ (\varepsilon(x),\ v(x) - v_o(x)) \leq 0 \quad \text{for all} \quad v \in V_{1,1}.$$

But these imply

$$\min_{x \in M} \mathrm{Re}\ (\varepsilon(x),\ v(x) - v_o(x)) \leq 0$$

for all $v \in V_{1,1}$. Therefore Σ is extremal for v_1. ∎

Remark: Suppose $V_{1,1}$ is a finite dimensional linear subspace of $C[Q,H]$. The existence of a best approximation for any $f \in C[Q,H]$ is well known. Theorem 3.9 says that every $f \in C[Q,H]$ has exactly one best approximation $v_o = v(\cdot,\mathfrak{A}_o) \in V_{1,1}$ if and only if

$$\|v_o\| < 2\|f\|.$$

By (C), each $f \not\equiv 0$ has a unique best approximation v_o if

$$\|v_o - v\| < 2\|f - v\|$$

for all $v \in V_{1,1}$. Since $V_{1,1}$ is linear, $v \equiv 0 \in V_{1,1}$, so

$$\|v_o\| < 2\|f\|.$$

Conversely, if v_o is a best approximation for f, it follows that $v_o - v$ is a best approximation for $f - v$ for each $v \in V_{1,1}$. Thus

$$\|v_o - v\| < 2\|f - v\|.$$

By Theorem 3.9, v_o must be unique.

Section 5: A Special Case

We wish to consider approximation of functions $f \in C[Q,H]$ by elements of a subset $V_{1,1}$ of $C[Q,H]$ when the elements of $V_{1,1}$ are determined by a finite dimensional parameter space E.

Theorem 3.10: Let the subset $V_{1,1}$ of $C[Q,H]$ be an equi-basis system and let the parameter space E of the elements of $V_{1,1}$ have finite dimension n. Then $v_o = v(\cdot,\mathfrak{A}_o)$ is a minimal solution from $V_{1,1}$ for $f \in C[Q,H]$ if and only if there exist points

$$x_1,\ldots,x_r \text{ from } M[f - v_o] \quad (r \geq 1),$$

$$t_1^{(j)},\ldots,t_{s_j}^{(j)} \text{ from } M_j,\ j \in J_o,$$

and real numbers $\{\lambda_i\}_1^r$, $\{\mu_{ij}\}$, $\{\gamma_i\}_1^k$ such that

$$g_j(t_i^{(j)},\mathfrak{A}_o) = 0 \quad \text{for} \quad i = 1,\ldots,s_j;\ j \in J_o,$$

$$\mu_{ij} > 0 \quad \text{for} \quad i = 1,\ldots,s_j;\ j \in J_o,$$

$$\lambda_i > 0 \quad \text{for} \quad i = 1,\ldots,r,$$

with

$$r + \sum_{j\in J_o} s_j \leq \begin{cases} \dim \mathcal{L}_{1,0}[\mathfrak{A}_o] + 1 & \text{if } H \text{ is real,} \\ 2\dim \mathcal{L}_{1,0}[\mathfrak{A}_o] + 1 & \text{if } H \text{ is complex,} \end{cases}$$

and

$$\sum_{i=1}^{r} \lambda_i(\epsilon(x_i),Dv(x_i,\mathfrak{A}_o)\cdot) + \sum_{j\in J_o}\sum_{i=1}^{s_j} \mu_{ij}Dg_j(t_i^{(j)},\mathfrak{A}_o)\cdot$$

$$+ \sum_{i=1}^{k} \lambda_i Df_i(\mathfrak{A}_o)\cdot = 0^*.$$

<u>Proof</u>: The necessity has been established in Theorem 3.5.

(Sufficiency) Assume

$$\sum_{i=1}^{r} \lambda_i(\varepsilon(x_i), Dv(x_i,\mathfrak{A}_o)b) + \sum_{j\in J_o} \sum_{i=1}^{s_j} \mu_{ij} Dg_j(t_i^{(j)},\mathfrak{A}_o)b$$

$$+ \sum_{i=1}^{k} \gamma_i Df_i(\mathfrak{A}_o)b = 0$$

for all b such that $Dv(\cdot,\mathfrak{A}_o)b \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$. By the definition of $\mathcal{L}_{1,1}[\mathfrak{A}_o]$ we conclude

$$\sum_{i=1}^{r} \lambda_i(\varepsilon(x_i), Dv(x_i,\mathfrak{A}_o)b) \le 0,$$

i.e.
$$\min_{x\in M} \mathrm{Re}\ (\varepsilon(x), Dv(x,\mathfrak{A}_o)b) \le 0$$

for all $Dv(x,\mathfrak{A}_o)b \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$. This means $\Sigma[f - v_o]$ is extremal for 0 with respect to $\mathcal{L}_{1,1}[\mathfrak{A}_o]$. Since $V_{1,1}$ is an equibasis system, $\Sigma[f - v_o]$ is extremal for $v(\cdot,\mathfrak{A}_o)$ with respect to $V_{1,1}$. Thus $v(\cdot,\mathfrak{A}_o)$ is a minimal solution for $f \in C[Q,H]$. ■

An analogous theorem holds in the case that $V_{1,1}$ is a local$^+$ equibasis system which characterizes a local$^+$ minimal solution.

<u>Section 6</u>: Relation of Chapter III to Chapters I and II.

Some of the results of Chapters I and II can be obtained from the theory of Chapter III. Consider the problem presented in Chapter I. Here $Q = K$ and H = real numbers. The Banach space $E \equiv P$ is Euclidean n-space R^n, and the function $F : P \to C[Q,H]$ is defined by

$$F(\mathfrak{A}) = \sum_{i=1}^{n} a_i w_i(x)$$

where $\mathfrak{A} = (a_1,\ldots,a_n) \in P = R^n$ and $w_1,\ldots,w_n$ are linearly independent elements of $C[Q,H] = C(K)$.

$$V = \{v(x,\mathfrak{A}) = F(\mathfrak{A}): \ \mathfrak{A} \in R^n\},$$

$$V_{1,1} = \{v(x,\mathfrak{A}): \ v(x,\mathfrak{A}) \geq \ell(x) \ \text{ for all } \ x \in L$$

$$\text{and } \ v(x,\mathfrak{A}) \leq \mu(x) \ \text{ for all } \ x \in J\}.$$

$V_{1,1}$ is the subset of V obtained by the restrictions:

$$f_1: \ R^n \to R^1,$$

$$f_1(\mathfrak{A}) \equiv 0;$$

$$g_1: \ L \times R^n \to R^1,$$

$$g_1(x,\mathfrak{A}) = v(x,\mathfrak{A}) - \ell(x);$$

$$g_2: \ J \times R^n \to R^1,$$

$$g_2(x,\mathfrak{A}) = \mu(x) - v(x,\mathfrak{A}).$$

(The restriction $f_1(\mathfrak{A}) \equiv 0$ is stated for convenience and does not limit the set of approximants, i.e., $V_{1,0} = V$.) The functions v, f_1, g_1, g_2 are all Frechet-differentiable with respect to the parameter $\mathfrak{A}$ with the following Frechet derivatives:

Let $\mathfrak{b} = (b_1,\ldots,b_n) \in R^n$ be arbitrary,

$$Dv(x,\mathfrak{A})(\mathfrak{b}) = \sum_{i=1}^{n} b_i w_i(x),$$

$$Df_1(\mathfrak{A})(\mathfrak{b}) \equiv 0,$$

$$Dg_1(x,\mathfrak{A})(b) = \sum_{i=1}^{n} b_i w_i(x),$$

$$Dg_2(x,\mathfrak{A})b = -\sum_{i=1}^{n} b_i w_i(x).$$

<u>Remark 1</u>: The side conditions $S = \{f_1, g_1, g_2\}$ are (R1)-regular for all $\mathfrak{A} \in R^n$ since

$$Df_1(\mathfrak{A})(b) \equiv 0 \quad \text{for all} \quad b \in R^n.$$

Set $\mathfrak{A}(t) = \mathfrak{A} + tb$, for $t \in [0,1]$. Then

$$f_1(\mathfrak{A}(t)) = 0 \quad \text{for all} \quad t \in [0,1]$$

and

$$\mathfrak{A}(0) = \mathfrak{A}$$

and

$$\mathfrak{A}'(0) = (1)b.$$

<u>Remark 2</u>: The side conditions $S = \{f_1, g_1, g_2\}$ are (R2)-regular for all $\mathfrak{A} \in R^n$ if and only if V satisfies condition H, i.e. if and only if there is an element $v(x,\mathfrak{A}_1) \in V_{1,1}$ such that

$$v(x,\mathfrak{A}_1) > \ell(x) \quad \text{for all} \quad x \in L$$

and

$$v(x,\mathfrak{A}_1) < \mu(x) \quad \text{for all} \quad x \in J.$$

<u>Proof</u>: Suppose the side conditions are R2 regular for all $\mathfrak{A} \in R^n$. Fix $\mathfrak{A}_o \in R^n$ and let $v_o = v(x,\mathfrak{A}_o)$, then there is a $b \in R^n$ such that

$$Dg_i(t^{(i)},\mathfrak{A}_o)b > 0$$

for all $t^{(i)} \in M_i$, $i = 1,2$. Now

$$M_1 = \{x \in L: \; g_1(x,\mathfrak{U}_o) = 0\} = G^+_{v_o}$$

and

$$M_2 = \{x \in J: \; g_2(x,\mathfrak{U}_o) = 0\} = G^-_{v_o} .$$

If $M_1 = M_2 = \phi$, then v_o satisfies condition H. If $M_1 \cup M_2 \neq \phi$, then $Dg_i(t^{(i)},\mathfrak{U}_o)\mathfrak{b} = (-1)^{i+1}v(t^{(i)},\mathfrak{b})$ and by choosing δ sufficiently small we obtain $v(x,\mathfrak{U}_o + \delta\mathfrak{b})$ satisfying condition H as follows. Without loss of generality, assume $M_1 \neq \phi$. Now by the compactness of M_1 there is an open set U containing M_1 such that for all $x \in U$

$$v(x,\mathfrak{b}) > 0.$$

And again by the compactness of $L \sim U$ there is a positive number $\epsilon_1 > 0$ such that

$$v(x,\mathfrak{U}_o) - \ell(x) \geq \epsilon_1$$

for all $x \in L \sim U$. Then for $\delta_1 > 0$ such that $\delta_1 \leq \frac{\epsilon_1}{2\eta}$ where $\eta = \|v(x,\mathfrak{b})\|$, we have

$$v(x,\mathfrak{U}_o + \delta_1\mathfrak{b}) > \ell(x) \quad \text{for all} \quad x \in L.$$

Similarly choosing δ_2 suitably small, we have

$$v(x,\mathfrak{U}_o + \delta_2\mathfrak{b}) < \mu(x) \quad \text{for all} \quad x \in J,$$

thus $\delta \leq \min(\delta_1,\delta_2)$ implies $v(x,\mathfrak{U}_o + \delta\mathfrak{b})$ satisfies condition H.

If condition H is satisfied by $v(x,\mathfrak{A}_1)$ and for $\mathfrak{A}_o \in R^n$, $M_1 \cup M_2 \neq \phi$, let $\mathfrak{b} = \mathfrak{A}_1 - \mathfrak{A}_o$. Then

$$Dg_i(t^{(i)},\mathfrak{A}_o)\mathfrak{b} > 0$$

for all $t^{(i)} \in M_i$, $i = 1,2$. ∎

Remark 3: $V_{1,1}$ is an equibasis system.

Proof: Suppose $\Sigma = \overline{(\varepsilon,M)}$ is extremal for $v(x,\mathfrak{A}_o) \in V_{1,1}$, and that there exists $Dv(x,\mathfrak{A}_o)\mathfrak{b} \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$ such that

$$\min_{x \in M} \varepsilon(x) Dv(x,\mathfrak{A}_o)\mathfrak{b} > 0.$$

Now $Dv(x,\mathfrak{A}_o)\mathfrak{b} \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$ implies

$$v(t^{(1)},\mathfrak{b}) = Dg_1(t^{(1)},\mathfrak{A}_o)\mathfrak{b} \geq 0 \quad \text{for all} \quad t^{(1)} \in M_1,$$

and

$$-v(t^{(2)},\mathfrak{b}) = Dg_2(t^{(2)},\mathfrak{A}_o)\mathfrak{b} \geq 0 \quad \text{for all} \quad t^{(2)} \in M_2.$$

But then for some sufficiently small δ,

$$v(x,\mathfrak{A}_o + \delta\mathfrak{b}) = v(x,\mathfrak{A}_o) + \delta v(x,\mathfrak{b}) \in V_{1,1}$$

and $Dv(x,\mathfrak{A}_o)\delta\mathfrak{b} \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$ with

$$\min_{x \in M} \varepsilon(x) Dv(x,\mathfrak{A}_o)\delta\mathfrak{b} > 0$$

implies

$$\min_{x \in M} \varepsilon(x)(v(x,\mathfrak{A}_o + \delta\mathfrak{b}) - v(x,\mathfrak{A}_o)) > 0$$

Which is a contradiction.

Suppose Σ is extremal for $0 \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$ and there exists $\mathfrak{b} \in R^n$ such that $v(x,\mathfrak{b}) \in V_{1,1}$ and

$$\min_{x \in M} \varepsilon(x)(v(x,\mathfrak{b}) - v(x,\mathfrak{A}_o)) > 0.$$

Since $v(x,\mathfrak{b}) - v(x,\mathfrak{A}_o) = Dv(x,\mathfrak{A}_o)(\mathfrak{b} - \mathfrak{A}_o)$ and

$$Dg_1(t^{(1)},\mathfrak{A}_o)(\mathfrak{b} - \mathfrak{A}_o) = v(t^{(1)},\mathfrak{b}) - v(t^{(1)},\mathfrak{A}_o) \geq 0 \quad \text{for all } t^{(1)} \in M_1,$$

$$-Dg_2(t^{(2)},\mathfrak{A}_o)(\mathfrak{b} - \mathfrak{A}_o) = v(t^{(2)},\mathfrak{b}) - v(t^{(2)},\mathfrak{A}_o) \leq 0 \quad \text{for all } t^{(2)} \in M_2,$$

$Dv(x,\mathfrak{A}_o)(\mathfrak{b} - \mathfrak{A}_o) \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$. But

$$\min_{x \in M} \varepsilon(x) Dv(x,\mathfrak{A}_o)(\mathfrak{b} - \mathfrak{A}_o) > 0$$

is a contradiction.

We also have the signature $\Sigma[f - v_o] = \overline{(\varepsilon, M[f - v_o])}$ where

$$M[f - v_o] = \{x \in K: \ |f(x) - v_o(x)| = \|f - v_o\|_K\} = E_{v_o}$$

and

$$\varepsilon(x) = \frac{f(x) - v_o(x)}{\|f - v_o\|_K} \quad \text{for } x \in M[f - v_o].$$

Then Theorem 3.6 is equivalent to Theorem 1.2 and Theorem 3.10 is equivalent to Theorem 1.3.

For Chapter II, let $E = R^{s+t}$, and for $\mathfrak{A} = (a_1,\ldots,a_s, b_1,\ldots,b_t) \in R^{s+t}$,

$$F(a_1,\ldots,a_s, b_1,\ldots,b_t) = r(x,\mathfrak{A}) = \frac{\sum_{i=1}^{s} a_i \mu_i(x)}{\sum_{j=1}^{t} b_j w_j(x)}$$

for $\mathfrak{A}$ such that $\sum_{j=1}^{t} b_j w_j(x) > 0$ for all $x \in S = J \cup K \cup L$.

The side conditions for the case

$$\ell(x) < \mu(x) \quad \text{for all} \quad x \in J = L$$

are:

$$f_1(\mathfrak{A}) = 0,$$

$$g_1(x,\mathfrak{A}) = r(x,\mathfrak{A}) - \ell(x) \quad \text{for} \quad x \in L,$$

$$g_2(x,\mathfrak{A}) = \mu(x) - r(x,\mathfrak{A}) \quad \text{for} \quad x \in J.$$

Then for $\mathfrak{A}_o = (a_1^o, a_2^o, \ldots, a_s^o, b_1^o, \ldots, b_t^o)$, $\mathfrak{b} = (a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t)$, $r(x,\mathfrak{A}_o) = r_o(x)$,

$$Df_1(\mathfrak{A}_o)\mathfrak{b} \equiv 0,$$

$$Dr(x,\mathfrak{A}_o)\mathfrak{b} = \frac{1}{\sum_{j=1}^{t} b_j^o w_j(x)} \left(\sum_{i=1}^{s} a_i \mu_i(x) - r_o(x) \sum_{j=1}^{t} b_j w_j(x) \right),$$

$$Dg_1(x,\mathfrak{A}_o)\mathfrak{b} = Dr(x,\mathfrak{A}_o)\mathfrak{b},$$

$$Dg_2(x,\mathfrak{A}_o)\mathfrak{b} = -Dr(x,\mathfrak{A}_o)\mathfrak{b}.$$

Then the side conditions are (R1)-regular as before and (R2)-regular at $\mathfrak{A}_o$ if and only if there exists $\phi \in P + r_o Q$ such that

$$\phi(x) > 0 \quad \text{for} \quad x \in M_1$$

and

$$\phi(x) < 0 \quad \text{for} \quad x \in M_2.$$

<u>Remark 4</u>: $V_{1,1}$ is again an equibasis system.

<u>**Proof**</u>: Suppose Σ is extremal for $r(x,\mathfrak{A}_o) \in V_{1,1}$ and there exists $\mathfrak{b} \in E$ such that $Dr(x,\mathfrak{A}_o)\mathfrak{b} \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$ and

$$\min_{x \in M} \varepsilon(x) Dr(x,\mathfrak{A}_o)\mathfrak{b} > 0.$$

δ can be chosen sufficiently small so that

$$\frac{\sum_{i=1}^{s} (a_i^o + \delta a_i)\mu_i(x)}{\sum_{j=1}^{t} (b_j^o - \delta b_j)w_j(x)} = r_\delta(x) \in V_{1,1}.$$ Then

$$r_\delta(x) - r_o(x) = \frac{\delta}{\sum_{j=1}^{t} (b_j^o - \delta b_j)w_j(x)} \Big(\sum_{i=1}^{s} a_i\mu_i(x) + r_o \sum_{j=1}^{t} b_j w_j(x) \Big)$$

is such that

$$\min_{x \in M} \epsilon(x)(r_\delta(x) - r_o(x)) > 0$$

which is a contradiction.

Conversely if Σ is extremal for $0 \in \mathcal{L}_{1,1}[\mathfrak{A}_o]$, and there exists $r(x,b) \in V_{1,1}$ such that

$$\min_{x \in M} \epsilon(x)(r(x,b) - r(x,\mathfrak{A}_o)) > 0,$$

then

$$r(x,b) - r(x,\mathfrak{A}_o) = \frac{1}{\sum_{j=1}^{t} b_j w_j(x)} \Big[\sum_{i=1}^{s} a_i\mu_i(x) - r_o \sum_{j=1}^{t} b_j w_j(x) \Big].$$

Thus

$$\min_{x \in M} \epsilon(x) Dr(x,\mathfrak{A}_o)b > 0,$$

which is a contradiction.

Then Theorem 3.6 is equivalent to Theorem 2.3 and Theorem 3.10 is equivalent to Theorem 2.4.

For the case $\ell(x) \le \mu(x)$, the side conditions are:

$$f_1(\mathfrak{A}) = \sum_{i=1}^{s} a_i w_i(x_1) - a_{10} \sum_{j=1}^{t} b_j w_j(x_1),$$

$$f_2(\mathfrak{A}) = \sum_{i=1}^{s} a_i w_i'(x_1) - a_{11} \sum_{j=1}^{t} b_j v_j(x_1) - a_{10} \sum_{j=1}^{t} b_j v_j'(x_1),$$

$$\vdots$$

$$f_{m_1}(\mathfrak{A}) = \sum_{i=1}^{s} a_i w_i^{(m_1-1)}(x_1) - \sum_{\lambda=1}^{m_1-1} \{ \binom{m_1-1}{\lambda} a_{1,\lambda-1} \sum_{j=1}^{t} b_j v_j^{(m_1-1-\lambda)}(x_1) \},$$

$$f_{m_1+1}(\mathfrak{A}) = \sum_{i=1}^{s} a_i w_i(x_2) - a_{20} \sum_{j=1}^{t} b_j v_j(x_2),$$

$$\vdots$$

$$f_{\sum_{i=1}^{k} m_i}(\mathfrak{A}) = \sum_{i=1}^{s} a_i w_i^{(m_k-1)}(x_k) - \sum_{\lambda=1}^{m_k-1} \{ \binom{m_k-1}{\lambda} a_{k,\lambda-1} \sum_{j=1}^{t} b_j v_j^{(m_k-1-\lambda)}(x_k) \},$$

$$g_1(x,\mathfrak{A}) = r(x,\mathfrak{A}) - \ell(x) \quad \text{for} \quad x \in L,$$

$$g_2(x,\mathfrak{A}) = \mu(x) - r(x,\mathfrak{A}) \quad \text{for} \quad x \in J.$$

<u>Remark 5</u>: These side conditions are (R1)-regular.

Let $\mathfrak{b} = (c_1,\ldots,c_s, d_1,\ldots,d_t) \in R^{s+t}$ and

$$Df_i(\mathfrak{A})\mathfrak{b} = 0 \quad \text{for} \quad i = 1,\ldots,m = \sum_{i=1}^{k} m_i .$$

Set $\mathfrak{A}(t) = \mathfrak{A} + t\mathfrak{b}$, for $t \in [0,1]$. Since the f_i's are linear in $\mathfrak{A}$, $Df_i(\mathfrak{A})\mathfrak{b} = f_i(\mathfrak{b})$, and $Df_i(\mathfrak{A})\mathfrak{b} = 0$ for $i = 1,\ldots,m$ implies

$$f_i(\mathfrak{A}(t)) = f_i(\mathfrak{A}) + t\, f_i(\mathfrak{b}) = 0$$

for all $t \in [0,1]$. Also $\mathfrak{A}'(0) = \mathfrak{b}$.

Remark 6: (R2)-regularity is impossible, since if $\mathfrak{b} \in R^{s+t}$ is such that

$$Df_i(\mathfrak{A}_o)\mathfrak{b} = 0 \quad \text{for} \quad i = 1,\dots,m$$

then $f_i(\mathfrak{A})\mathfrak{b} = 0$ for $i = 1,\dots,m$. That is

$$r(x_j,\mathfrak{b}) = \ell(x_j) = \mu(x_j) \quad \text{for} \quad j = 1,\dots,k,$$

and thus

$$Dg_1(x_j,\mathfrak{A})\mathfrak{b} = Dg_2(x_j,\mathfrak{A})\mathfrak{b} = 0 \quad \text{for} \quad j = 1,\dots,k.$$

However if we redefine Q_1 to be $L \sim U$ and Q_2 to be $J \sim U$ where U is any open set containing T, then (R2)-regularity follows from the existence of an element $\phi \in M_r$ (where $r = r(x,\mathfrak{A})$) such that

$$\phi(x) > 0 \quad \text{for all} \quad x \in L \sim T$$

and

$$\phi(x) < 0 \quad \text{for all} \quad x \in J \sim T.$$

$V_{1,1}$ is again an equibasis system.

The results concerning uniqueness obtained by Hoffmann deal with uniqueness of best restricted approximation to every continuous function f.

Example 1 of Chapter I shows that Hoffmann's uniqueness theorem cannot apply to the work of Chapters I and II. In this example we let $f(x) - 1 - x^2$, $K = [0,1]$, $J = [0,1]$, $L = [0,1]$ with $\mu(x) = 0$, $\ell(x) = -1$. Then we considered best restricted approximations to f from

$V_{1,1} = \{ax^2 + bx + c\colon\ \ell(x) \le ax^2 + bx + c \le \mu(x)$ for all

$x \in [0,1]$ where a, b, c are real numbers$\}$.

Now $v_1(x) \equiv 0 \in V_{1,1}$ and v_1 is a best restricted approximation also $v_2(x) = -\frac{1}{2} x^2$ is a best restricted approximation. Thus f does not have a unique best restricted approximation.

The results of Chapters I and II beyond the characterization theorems do not follow from the material presented in Chapter III.

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