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ON MARGINAL SUBGROUPS
AND THEIR GENERALIZATIONS

presented by

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has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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ABSTRACT

ON MARGINAL SUBGROUPS AND THEIR GENERALIZATIONS

By

Tommy Kay Teague

In this paper several problems concerning marginal subgroups are investigated. An element a of G lies in the marginal subgroup $\phi^*(G)$ corresponding to the word ϕ if and only if $\phi(g_1, \dots, g_n) = \phi(g_1, \dots, ag_1, \dots, g_n)$ for all choices of $g_1, \dots, g_n$ in G and $i = 1, 2, \dots, n$. Let $A_0 = 1$, and define $A_{\alpha+1}$ to be the complete inverse image of $\phi^*(G/A_\alpha)$. If α is a limit ordinal, define $A_\alpha = \bigcup \{A_\beta : \beta < \alpha\}$. A group G is called ϕ -nilpotent of class n if there is a positive integer n such that $A_{n-1} \neq A_n = G$, and G is called ϕ -hypercentral if there is an ordinal β such that $A_\beta = G$.

In Chapter 2 some of the basic properties of ϕ -hypercentral groups are developed. The following theorems are proved:

Theorem. Let $G^0 = G$, and inductively let $G^{\alpha+1}$ be the subgroup generated by elements of the form $\phi(g_1, \dots, g_n)^{-1} \phi(g_1 h_1, \dots, g_n h_n)$, where each $g_i \in G$, $h_i \in G^\alpha$. Define $G^\alpha = \bigcap \{G^\beta : \beta < \alpha\}$ for α a limit ordinal. Then G is ϕ -nilpotent of class n if and only if $G^{n-1} \neq G^n = 1$.

Theorem. Assume G is ϕ -hypercentral. Then $\phi(G)$ has a descending hypercentral series.

Theorem. Let $\lambda(y_1, \dots, y_m)$, $\theta(x_1, \dots, x_n)$ be two words such that $\theta(G) \subseteq \lambda(G)$ for all G . Set $\phi(y_1, \dots, y_m, x_1, \dots, x_n) =$

$[\lambda(y_1, \dots, y_m), \theta(x_1, \dots, x_n)]$. If G is a group such that $\lambda(G)$ is nilpotent of class c , then G is ϕ -nilpotent of class no greater than c .

Theorem. Let d_n be the n -th derived word. For any G , G is d_n -nilpotent if and only if $d_{n-1}(G)$ is nilpotent.

In Chapter 3, two generalizations of the marginal subgroup of G are considered--one in G and the other in $\text{Aut}(G)$. Define $\phi^c(G) = \{a \in G: \phi(g_1, \dots, g_1^a, \dots, g_n) = \phi(g_1, \dots, g_n) \text{ for all choices of } g_1, \dots, g_n \text{ in } G \text{ and } i = 1, 2, \dots, n\}$ to be the c -marginal subgroup of G . By substituting $\alpha \in \text{Aut}(G)$ for $a \in G$ in the definition of $\phi^c(G)$, it is possible to define the automargin $\bar{\phi}(G) \subseteq \text{Aut}(G)$.

Theorem. $\phi^c(G) = \{a \in G: \phi(g_1, \dots, ag_1, \dots, g_n) = \phi(g_1, \dots, g_1a, \dots, g_n) \text{ for all choices of } g_1, \dots, g_n \text{ in } G \text{ and } i = 1, 2, \dots, n\}$.

Theorem. Define $\gamma_n = [x_1, \dots, x_n]$. Then $\gamma_n^c(G) = Z_n(G)$ for $n \geq 1$.

Theorem. Let ϕ be any outer commutator word. Then $\bar{\phi}(G) = \{\alpha \in \text{Aut}(G): [G, \alpha] \subseteq \phi^*(G)\}$. In particular, $\gamma_n(G) = \{\alpha \in \text{Aut}(G): x^\alpha \equiv x \pmod{Z_{n-1}(G)} \text{ for all } x \in G\}$.

Marginal subgroups for outer commutator words were completely characterized by R. F. Turner-Smith in 1964. In Chapter 4 the marginal subgroup for another type of commutator, the Engel word $e_2 = [x, y, y]$, is determined.

Theorem. $e_2^*(G) = \{a \in G: [x, y, a][a, y, x] = 1 \text{ for all } x, y \in G\}$.

Corollary. $e_2^*(G) \cap C_G(G') = Z_2(G)$.

Theorem. For any $a \in e_2^*(G)$, $[a, G, G]^3 = [a^3, G, G] = 1$.

Theorem. $Z_2(G) \subseteq e_2^*(G) \subseteq Z_3(G)$, and $e_2^*(G)/Z_2(G)$ is an elementary Abelian 3-group of central automorphisms on G' .

Theorem. If $Z(G') \cap Z(G)$ has no elements of order 3, or if G' has no proper subgroup of finite index, then $e_2^*(G) = Z_2(G)$.

Theorem. If $[G: e_2^*(G)] = m$ is finite, then $e_2(G)$ is finite with order which divides a power of m .

Theorem. If G is locally residually finite and e_2 is finite-valued on G , then $e_2(G)$ is finite.

Theorem. The preceding two theorems also hold for the Engel word $e_3 = [x, y, y, y]$.

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INTRODUCTION

The concept of a marginal subgroup for a word ϕ was introduced in 1940 by P. Hall [4]. It is known that the marginal subgroup of a group for the word $[x, y]$ is the center of the group. By analogy with the ascending central series of a group, we may define its ascending ϕ -series. Further, we may generalize the marginal subgroup itself by considering elements in the group which do not change the value of the word when they conjugate any of its variables. This in turn leads us to consider automorphisms which do not affect the value of the word when they are applied to any of its variables. R. F. Turner-Smith [21: page 328] has completely characterized the marginal subgroup for any outer commutator word. We conclude this paper with a characterization of the marginal subgroup for the Engel word of length two, which is not an outer commutator word.

For the sake of completeness, Chapter I contains some definitions and theorems essential to the remainder of the paper.

In Chapter II we develop some of the basic properties of ϕ -hypercentral groups. Many of the usual theorems for hypercentral groups are true in this new context. We are also able to define a lower ϕ -series for a group. We give a complete characterization of d_n -nilpotence for any derived word d_n and necessary conditions for some other words. Some possible ways to generalize the normalizer of a subgroup are also suggested.

Chapter III offers two generalizations of the marginal subgroup--the c -marginal subgroup and the automargin of a group. An alternate characterization of the c -marginal subgroup is given, and this subgroup is computed for any nilpotent word. The automargin is shown to be minimal for outer commutator words. In particular, the automargin of a group G for a nilpotent word is shown to be the group of n -normal automorphisms on G for some n .

In Chapter IV we consider the marginal subgroup E for the Engel word of length two. A complete characterization of E is given, and several interesting properties are brought to light. The relationship of the subgroup of right Engel elements of length two to the metabelian margin is considered. Some results concerning the relative size of the verbal and marginal subgroups of G for the Engel word of length two are also presented.

CHAPTER I

PRELIMINARIES

NOTATION AND DEFINITIONS. Let G be a group. The symbol 1 is used interchangeably for the identity in G and for the unit group. By $H \subseteq G$ ($H < G$) ($H \triangleleft G$) ($H \in \text{Char}(G)$) we mean that H is a subgroup of G (a proper subgroup of G) (a normal subgroup of G) (a characteristic subgroup of G). If $H \subseteq G$, $[G:H]$ is the index of H in G , $C_G(H)$ is the centralizer of H in G , and $N_G(H)$ is the normalizer of H in G . If S is a subset of G , then $\langle S \rangle$ is the subgroup of G generated by the elements of S . The order of an element g of G is written $o(g)$, and $o(S)$ denotes the cardinality of the set S . The infinite cyclic group is represented by J , the cyclic group of order n by J_n , the symmetric group on n symbols by S_n , and the alternating group on n symbols by A_n .

If G_a is a group for each a in some indexing set A , then the (unrestricted) direct product of $\{G_a : a \in A\}$ is denoted by $\prod \{G_a : a \in A\}$ and the direct sum by $\Sigma \{G_a : a \in A\}$. By $G \cong H$ ($G \subset H$) we mean G is isomorphic to (a subgroup of) H .

By a class of groups we mean a class containing the unit group as well as all isomorphic copies of any member of the class. Let Σ be a class of groups. Then:

- (a) $S\Sigma$ is the class of groups which are subgroups of Σ groups.
- (b) $Q\Sigma$ is the class of groups which are quotients of Σ groups.
- (c) $E\Sigma$ is the class of groups which are extensions of Σ groups

by Σ groups.

(d) $L\Sigma$ is the class of groups in which every finitely generated subgroup is a Σ group.

(e) DPE is the class of groups which are direct products of Σ groups.

If $P \subseteq \{S, Q, E, L, DP\}$, and if $P\Sigma \subseteq \Sigma$, we say Σ is P closed.

The center of G is $C_G(G)$, denoted by $Z(G)$. Let $Z_0(G) = 1$, $Z_1(G) = Z(G)$. The ascending central series of G is defined recursively by $Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G))$ for all ordinals α , and $Z_\alpha(G) = \bigcup \{Z_\beta(G) : \beta < \alpha\}$ for all limit ordinals α . If there is a least finite ordinal n such that $Z_n(G) = G$, then G is nilpotent of class n . If $Z_\beta(G) = G$ for some ordinal β , G is a ZA group.

If G has an ascending normal series $1 = G_0 < G_1 < \dots < G_\sigma = G$ where $G_{\alpha+1}/G_\alpha$ is Abelian for each α , then G is an SN^* group.

For $x, y \in G$ the conjugate of x by y is $x^y = y^{-1}xy$ and the commutator of x and y is $[x, y] = x^{-1}x^y$. For $x \in G$, $H \subseteq G$, $x^H = \langle x^h : h \in H \rangle$. If $H, K \subseteq G$, then $[H, K] = \langle [h, k] : h \in H, k \in K \rangle$. If $x_i \in G$, $1 \leq i \leq n$, $n \geq 3$, we define $[x_1, \dots, x_n]$ recursively to be $[[x_1, \dots, x_{n-1}], x_n]$. Similarly, if $X_i \subseteq G$, $1 \leq i \leq n$, $n \geq 3$, we define $[X_1, \dots, X_n] = [[X_1, \dots, X_{n-1}], X_n]$.

A subgroup A of $\text{Aut}(G)$ stabilizes the normal series $G = G_0 > G_1 > \dots > G_n = 1$ of G if $G_i^A = G_i$ and $[G_i, A] \subseteq G_{i+1}$, $0 \leq i \leq n-1$. We shall make frequent use of the fact that if $G_i < G$ for each i , then A is nilpotent of class $\leq n-1$ (see [7]). Also, for $A \subseteq \text{Aut}(G)$, $F(A) = \{g \in G : [g, A] = 1\}$.

Define $[x, ly] = [x, y]$ and $[x, ny]$ recursively to be $[[x, (n-1)y], y]$. An element $g \in G$ is a left Engel element if to

each $x \in G$ there is an integer n depending on x such that $[x, n\mathbf{g}] = 1$. Similarly, $g \in G$ is a right Engel element if to each $x \in G$ there is an integer n depending on x such that $[g, n\mathbf{x}] = 1$. If every element of G is a left Engel element, then G is called an Engel group. If $H, K \subseteq G$, define $[H, 1K] = [H, K]$ and $[H, nK]$ recursively to be $[[H, (n-1)K], K]$.

A group G is residually finite if for each $1 \neq x \in G$ there is a normal $N_x \triangleleft G$ with $x \notin N_x$ such that G/N_x is finite.

An element $a \in G$ has infinite height in G if the equation $x^n = a$ has a solution in G for each integer n . A group G is complete if each $a \in G$ has infinite height. A group G is Černikov complete if for each integer n G is generated by the n -th powers of all its elements.

A group G satisfies the maximum (minimum) condition if each proper ascending (descending) chain of subgroups of G is finite.

By [17: Theorem VI.7.b] every group has a unique maximum locally nilpotent normal subgroup. We call this subgroup the Hirsch-Plotkin radical of G .

A word is an element of the countably generated free group $\langle x_1, x_2, \dots \rangle$. A law in a group G is a word such that every substitution of elements from G for the variables of the word yields the identity of G . If S is a set of words, then the variety determined by S is the class of all groups G such that the elements of S are laws in G . For any word ϕ we denote by $\phi(G)$ the verbal subgroup of the group G generated by all the values of ϕ obtained by substituting elements from G for the variables of ϕ . The associated marginal subgroup $\phi^*(G)$ of G consists of all $a \in G$ such that $\phi(g_1, \dots, g_n) = \phi(g_1, \dots, ag_i, \dots, g_n)$ for every $g_i \in G$, $i = 1, 2, \dots, n$. We also refer to $\phi^*(G)$ as the ϕ -margin of G .

The word $\gamma_1 = \gamma_1(x) = x$ in one variable is an outer commutator word of weight $w(\gamma_1) = 1$. If $\theta = \theta(x_1, \dots, x_n)$ and $\lambda = \lambda(y_1, \dots, y_m)$ are defined outer commutator words such that $w(\theta) = n$ and $w(\lambda) = m$, then $\phi = \phi(x_1, \dots, x_{m+n}) = [\theta(x_1, \dots, x_n), \lambda(x_{n+1}, \dots, x_{m+n})]$ is an outer commutator word of weight $w(\phi) = m + n$. We write $\phi = [\theta, \lambda]$. Particular examples of outer commutator words we consider are the derived (or solvable) words, defined by $d_0 = x$, $d_n = [d_{n-1}, d_{n-1}]$, and the nilpotent (or lower central) words, defined by $\gamma_1 = x$, $\gamma_{n+1} = [\gamma_n, \gamma_1]$.

We define $e_n = e_n(x, y) = [x, ny]$ to be the Engel word of length n . For $n > 1$, we note that e_n is not an outer commutator word.

Most of the items referred to in this section are discussed in detail in [10], [13], [17], [18] or [21].

THEOREMS. We include here some known results on marginal subgroups. For the proofs, see P. Hall [5] or P. W. Stroud [20].

Theorem 1.1. For any group G and word ϕ ,

- (a) $\phi(G)$ is fully invariant in G and $\phi^*(G) \in \text{Char}(G)$.
- (b) $\phi(\phi^*(G)) = 1$.
- (c) if $K/\phi^*(G) = Z(G/\phi^*(G))$, then $[K, \phi(G)] = 1$. In particular, $[\phi^*(G), \phi(G)] = 1$.
- (d) if $H \subseteq G$ such that $G = H\phi^*(G)$, then $\phi^*(H) = H \cap \phi^*(G)$ and $\phi(G) = \phi(H)$.

Theorem 1.2. If $\{G_a : a \in A\}$ is a set of groups, ϕ a word, then $\phi^*(\pi\{G_a : a \in A\}) = \pi\{\phi^*(G_a) : a \in A\}$.

Theorem 1.3. Let $\theta = \theta(x_1, \dots, x_m)$ and $\lambda = \lambda(y_1, \dots, y_n)$ be two words and let $\phi = \phi(x_1, \dots, x_m, y_1, \dots, y_n) =$

$[\theta(x_1, \dots, x_m), \lambda(y_1, \dots, y_n)]$. Then in any group G :

(a) $\phi(G) = [\theta(G), \lambda(G)]$.

(b) if $U = C_G(\theta(G))$, $V = C_G(\lambda(G))$, $L/U = \lambda^*(G/U)$, $M/V = \theta^*(G/V)$, then $\phi^*(G) = L \cap M$.

Theorem 1.4. $\gamma_n^*(G) = Z_{n-1}(G)$ for any group G and $n \geq 1$.

Theorem 1.5. If θ and λ are words such that $\lambda(G) \subseteq \theta(G)$ for all groups G , then $\theta^*(G) \subseteq \lambda^*(G)$ for all groups G .

We shall also need the following theorems. For the proofs see [17] or [18].

Theorem 1.6. (N/C Theorem) If $H \subseteq G$, then $N_G(H)/C_G(H) \cong \text{Aut}(H)$.

Theorem 1.7. (3 Subgroups Lemma) If L , M , and N are subgroups of a group G , then $[L, M, N]^G \subseteq ([L, N, M][M, N, L])^G$.

Theorem 1.8. (Levi's Theorem) If e_2 is a law in a group G , then G is nilpotent of class at most three and $\gamma_3(G)$ has exponent dividing three.

CHAPTER II

ϕ -HYPERCENTRAL GROUPS

In this chapter we shall explore some generalizations of groups with transfinite ascending and descending central series. Unless we state otherwise, $\phi = \phi(x_1, \dots, x_n)$ is an arbitrary word in n variables.

Definition 2.1. (a) An ascending invariant series where $A_0 = 1$, $A_{\alpha+1}/A_\alpha \subseteq \phi^*(G/A_\alpha)$ for each ordinal α , and $A_\alpha = \bigcup \{A_\beta : \beta < \alpha\}$ for α a limit ordinal, is called an ascending ϕ -series for G .

(b) The upper ϕ -series for G is the ascending ϕ -series where $A_{\alpha+1}/A_\alpha = \phi^*(G/A_\alpha)$ for each ordinal α .

Theorem 2.2. Let $A_0 = 1 < A_1 < \dots$ be an ascending ϕ -series for G , and let $G_0 = 1 < G_1 < \dots$ be the upper ϕ -series for G . Then $A_\alpha \subseteq G_\alpha$ for each ordinal α .

Proof. We induct on α . Certainly $A_\alpha \subseteq G_\alpha$ for $\alpha = 0, 1$. Thus assume $A_\beta \subseteq G_\beta$ for all $1 \leq \beta < \alpha$. If α is a limit ordinal, then $A_\alpha = \bigcup \{A_\beta : \beta < \alpha\} \subseteq \bigcup \{G_\beta : \beta < \alpha\} = G_\alpha$ by the induction hypothesis. Now assume $\alpha-1$ exists. Suppose $a \in A_\alpha$ so that $aA_{\alpha-1} \in \phi^*(G/A_{\alpha-1})$. Then we see that $\phi(g_1, \dots, ag_1, \dots, g_n)A_{\alpha-1} = \phi(g_1, \dots, g_n)A_{\alpha-1}$ for every i and $g_1, \dots, g_n$ in G . Since $A_{\alpha-1} \subseteq G_{\alpha-1}$ by the induction hypothesis, we must also have that $\phi(g_1, \dots, ag_1, \dots, g_n)G_{\alpha-1} = \phi(g_1, \dots, g_n)G_{\alpha-1}$. Hence $a \in G_\alpha$ and the theorem follows.

Definition 2.3. Suppose $G_0 = 1 < G_1 < \dots$ is the upper ϕ -series for G . If $G = G_\alpha$ for some ordinal α , then G is ϕ -hypercentral. If

$G = G_n \neq G_{n-1}$ for some finite ordinal n , then G is ϕ -nilpotent of class n .

We note by Theorem 2.2 that a group G is ϕ -hypercentral if and only if every properly ascending ϕ -series for G reaches G .

Theorem 2.4. Subgroups and homomorphic images of a ϕ -hypercentral group G have upper ϕ -series of length no greater than that of the upper ϕ -series for G .

Proof. Let $G_0 = 1 < G_1 < \dots < G_\sigma = G$ be the upper ϕ -series for G , and let $H \subseteq G$. We claim that $H \cap G_{\alpha+1} / H \cap G_\alpha \subseteq \phi^*(H/G_\alpha \cap H)$ for each ordinal α . Suppose $a \in G_{\alpha+1} \cap H \setminus G_\alpha \cap H$. Then $a \in G_{\alpha+1} \setminus G_\alpha$ and $\phi(h_1, \dots, ah_1, \dots, h_n)\phi(h_1, \dots, h_1, \dots, h_n)^{-1} \in G_\alpha \cap H$ for every i and $h_1, \dots, h_n$ in H . Hence $a(G_\alpha \cap H) \in \phi^*(H/G_\alpha \cap H)$ and $H_0 = 1 < H_1 = H \cap G_1 < \dots < H_\sigma = H \cap G_\sigma = H$ is an ascending ϕ -series for H .

Now suppose $H < G$. Let $\bar{G} = G/H$, $\overline{aG_\alpha H} \in \overline{G_{\alpha+1}H/G_\alpha H}$. We may assume $a \in G_{\alpha+1}$. For every i and $g_1, \dots, g_n \in G$, $\phi(g_1, \dots, ag_1, \dots, g_n)G_\alpha = \phi(g_1, \dots, g_1, \dots, g_n)G_\alpha$ since $G_{\alpha+1}/G_\alpha = \phi^*(G/G_\alpha)$. Consequently $\overline{\phi(g_1, \dots, ag_1, \dots, g_n)G_\alpha H} = \overline{\phi(g_1, \dots, g_1, \dots, g_n)G_\alpha H}$. This implies $\overline{G_{\alpha+1}H/G_\alpha H} \subseteq \phi^*(\bar{G}/\bar{G}_\alpha H)$. Hence $1 < G_1H/H < \dots < G_\sigma H/H = G/H$ is an ascending ϕ -series for G/H .

The following two statements are due to Rhemtulla [15]:

Definition 2.5. (Rhemtulla) For $H < G$ define $\phi(H \wedge G) = \langle \phi(g_1, \dots, g_n)^{-1}\phi(g_1, \dots, hg_1, \dots, g_n) : 1 \leq i \leq n, h \in H, g_1, \dots, g_n \in G \rangle$.

Theorem 2.6. (Rhemtulla) For any G , $\phi(H \wedge G)$ is the smallest $K < G$, $K \subseteq H$ such that $H/K \subseteq \phi^*(G/K)$. Also, $\phi(H) \subseteq \phi(H \wedge G) \subseteq H \cap \phi(G)$.

Theorem 2.6 suggests a way to describe descending ϕ -hypercentral series as well as ascending ones.

Definition 2.7. (a) A descending invariant series $A_0 = G \supset A_1 \supset \dots$ of a group G where $A_\alpha/A_{\alpha+1} \subseteq \phi^*(G/A_{\alpha+1})$ for each ordinal α , and $A_\alpha = \bigcap \{A_\beta : \beta < \alpha\}$ for α a limit ordinal, is called a descending ϕ -series for G .

(b) The lower ϕ -series for G is the descending ϕ -series $G^0 = G \supset G^1 \supset \dots$ where $G^{\alpha+1} = \phi(G^\alpha \wedge G)$ for each ordinal α , and $G^\alpha = \bigcap \{G^\beta : \beta < \alpha\}$ for α a limit ordinal.

The proofs of Theorems 2.8-2.11 are identical to the proofs of the corresponding theorems for central series and will be omitted. The statements themselves are included only for the sake of completeness.

Theorem 2.8. If $G^0 \supset G^1 \supset \dots$ is the lower ϕ -series for G and $A_0 = G \supset A_1 \supset \dots$ is a descending ϕ -series for G , then $G^\alpha \subseteq A_\alpha$ for each ordinal α .

Theorem 2.9. A group G is ϕ -nilpotent of class n if and only if $G^{n-1} \neq G^n = 1$.

Theorem 2.10. Suppose G/H is ϕ -hypercentral, where $H \subseteq \phi^*(G)$. Then G is ϕ -hypercentral.

Theorem 2.11. If G is ϕ -hypercentral and $1 < H < G$, then $H \cap \phi^*(G) > 1$.

Theorem 2.12. Assume G is ϕ -nilpotent of class n . Then $\phi(G)$ is nilpotent of class less than n .

Proof. For convenience, we write the upper ϕ -series as $G = G_0 \supset G_1 \supset \dots \supset G_n = 1$, where $G_i/G_{i+1} = \phi^*(G/G_{i+1})$. Define $D_i(G) = \gamma_1(\phi(G))$ for $i \geq 1$. We claim that $D_i(G) \subseteq G_i$ for $1 \leq i \leq n$. Since $G/G_1 = \phi^*(G/G_1)$, we have that $\phi(G)G_1/G_1 = \phi(G/G_1) = 1$ and thus



that $D_1(G) = \phi(G) \subseteq G_1$. Assume then that $D_i(G) \subseteq G_i$ for $i \geq 1$.

Hence $D_i(G/G_{i+1}) = D_i(G)G_{i+1}/G_{i+1} \subseteq G_i/G_{i+1} = \phi^*(G/G_{i+1})$ and

$D_{i+1}(G) = [D_i(G), \phi(G)] \subseteq G_{i+1}$. Thus $D_i(G) \subseteq G_i$ for each $1 \leq i \leq n$.

In particular, $D_n(G) \subseteq G_n = 1$ and $\phi(G)$ is nilpotent of class less than n .

Remark 2.13. Suppose G has upper ϕ -series $G_0 = 1 < G_1 < \dots < G_\sigma = G$. Then $[G_{\alpha+1}, \phi(G)] \subseteq G_\alpha$ so that the group $A \cong \phi(G)Z(G)/Z(G)$ of automorphisms on G stabilizes the upper ϕ -series for G . By a result of Hall and Hartley [7: Theorem A1], A and consequently $\phi(G)$ have a descending hypercentral series.

Example 2.14. The converse to Theorem 2.12 is not true.

Let $G = S_3$. Since $\gamma_2^*(G) = Z(G) = 1$ by Theorem 1.4, G is not γ_2 -nilpotent.

However, $\gamma_2(G) = G'$ has order 3 and is certainly nilpotent.

Theorem 2.15. If θ and λ are two words such that $\theta(G) \subseteq \lambda(G)$ for all groups G , then a λ -hypercentral group is also θ -hypercentral.

Proof. Let $1 < A_1 < \dots < A_\sigma = G$ be the upper λ -series for G . By hypothesis and Theorem 1.5 we have that $\lambda^*(H) \subseteq \theta^*(H)$ for all H . Since then $A_{\alpha+1}/A_\alpha = \lambda^*(G/A_\alpha) \subseteq \theta^*(G/A_\alpha)$ for each α , we have that $1 < A_1 < \dots < A_\sigma = G$ is an ascending θ -series for G . Hence G is θ -hypercentral.

Corollary 2.16. If G is nilpotent, then G is θ -nilpotent for any outer commutator word θ of weight greater than one.

Theorem 2.17. Let θ and λ be two words such that $\theta(G) \subseteq \lambda(G)$ for all groups G . Set $\phi = [\theta, \lambda]$. If G is a group such that $\lambda(G)$ is nilpotent of class c , then G is ϕ -nilpotent of class $\leq c$.

Proof. Define $S_1(G) = [\lambda(G), \lambda(G)]$, $S_{i+1}(G) = [S_i(G), \lambda(G)]$ for $i \geq 1$. By hypothesis, $S_c(G) = 1$, $S_{c-1}(G) \neq 1$. We induct on c .

For $c = 1$, we have that $1 = S_1(G) = [\lambda(G), \lambda(G)] = [\lambda(G), \theta(G)] = \phi(G)$.

This implies $\phi^*(G) = G$. Hence G is ϕ -nilpotent of class 1.

Now assume that for every group G such that $\lambda(G)$ is nilpotent of class $\leq c$, G is ϕ -nilpotent of class $\leq c$. Let $\lambda(G)$ be nilpotent of class $c+1$ for some G . Then $S_{c+1}(G) = [S_c(G), \lambda(G)] = 1$, $S_c(G) \neq 1$. By Theorem 1.3(b), $S_c(G) \subseteq C_G(\lambda(G)) \cap C_G(\theta(G)) \subseteq \phi^*(G)$. Let $\bar{G} = G/\phi^*(G)$. Then $S_c(\bar{G}) = \overline{S_c(G)} = \bar{1}$. By the induction hypothesis, $\bar{G}$ is ϕ -nilpotent of class $\leq c$. By the arguments used in the proof of Theorem 2.10, G is ϕ -nilpotent of class $\leq c+1$.

Example 2.18. Let $G = S_3$ and recall $d_2 = [d_1, d_1]$. Since $d_1(G) = G'$ is nilpotent, G is d_2 -nilpotent by Theorem 2.17 but not d_1 -nilpotent. Hence the converse of Theorem 2.15 does not hold.

Example 2.19. Let $\delta = [\phi, \phi]$ be a word in $2n$ variables. Note that we get the result of Theorem 2.15 for δ by Theorems 2.12 and 2.17. That is, if G is ϕ -nilpotent, then $\phi(G)$ is nilpotent by Theorem 2.12. By Theorem 2.17, G is δ -nilpotent. The advantage of Theorem 2.17 in general is that $\phi(G)$ nilpotent is a weaker condition than G ϕ -nilpotent, as was shown in Example 2.14.

Example 2.20. The hypothesis that $\theta(G) \subseteq \lambda(G)$ for all G in Theorem 2.17 is essential. Let $H = S_3$. Note $\gamma_3 = [\gamma_2, \gamma_1]$ and $\gamma_2(G) \subseteq \gamma_1(G)$ for all G . Then H is not γ_3 -nilpotent although $\gamma_2(H) = H'$ is nilpotent.

Theorem 2.21. For any nilpotent word γ_n define $\phi_n = [\gamma_n, \gamma_n]$. Then $\gamma_n(G)$ is nilpotent if and only if G is ϕ_n -nilpotent.

Proof. The necessity follows immediately from Theorem 2.17. Now assume that G is ϕ_n -nilpotent. We seek another characterization of $\phi_n^*(G)$. By Theorem 1.3(b) and Theorem 1.4,

U

$$\begin{aligned} \phi_n^*(G)/C_G(\gamma_n(G)) &= \gamma_n^*(G/C_G(\gamma_n(G))) = Z_{n-1}(G/C_G(\gamma_n(G))). \text{ Hence } \phi_n^*(G) \\ &= \{a \in G: [a, g_2, g_3, \dots, g_n] \in C_G(\gamma_n(G)) \cap \gamma_n(G) = Z(\gamma_n(G)) \text{ for} \\ &\quad \text{all } g_2, \dots, g_n \text{ in } G\} \\ &= \{a \in G: [[a, x_2, \dots, x_n], [x_{n+1}, \dots, x_{2n}]] \text{ is a law in } G\}. \end{aligned}$$

Let $1 = A_0 \triangleleft A_1 \triangleleft \dots \triangleleft A_t = G$ be the upper ϕ_n -series for G , where t is a positive integer. Since our characterization of $\phi_n^*(G)$ holds

for any G , we may conclude that $A_{i+1} = \{a \in G:$

$$[[a, g_2, \dots, g_n], [g_{n+1}, \dots, g_{2n}]] \in A_1 \text{ for all } g_2, \dots, g_{2n} \text{ in } G\}$$

for $0 \leq i \leq t-1$. In particular, $\gamma_2(\gamma_n(G)) = [\gamma_n(G), \gamma_n(G)] \subseteq A_{t-1}$.

We claim in fact that $\gamma_{(i-1)n+2}(\gamma_n(G)) \subseteq A_{t-i}$ for $1 \leq i \leq t$.

This has been shown for $i = 1$. Assume the statement holds for some

$1 \leq i < t$. Then by the induction hypothesis we have that

$$[[\gamma_{(i-1)n+2}(\gamma_n(G)), (n-1)G], \gamma_n(G)] \subseteq A_{t-i-1}. \text{ This implies that}$$

$$\gamma_{i n+2}(\gamma_n(G)) = \gamma_{(i-1)n+2+n-1+1}(\gamma_n(G)) \subseteq A_{t-(i+1)}. \text{ Hence}$$

$$\gamma_{(t-1)n+2}(\gamma_n(G)) \subseteq A_{t-t} = A_0 = 1 \text{ and } \gamma_n(G) \text{ is nilpotent of}$$

$$\text{class } \leq (t-1)n + 1.$$

Theorem 2.22. For any G and $n \geq 1$, G is d_n -nilpotent if and only if $d_{n-1}(G)$ is nilpotent.

Proof. Since $d_n = [d_{n-1}, d_{n-1}]$, the sufficiency follows from Theorem 2.17. Hence assume G is d_n -nilpotent of class t , where t is a positive integer. For each $a \in G$ define the words $w_0, w_1, \dots, w_k, \dots$ by $w_0 = w_0(a) = a$ and $w_k = w_k(a) = w_k(a, x_2, \dots, x_{2^k}) = [w_{k-1}, d_{k-1}]$. Let $M \subseteq G, H \subseteq G$. By $w_k(H, M)$ we shall mean the subgroup generated by elements of the form $w_k(h, m_2, \dots, m_{2^k})$, where $h \in H, m_i \in M$ for $2 \leq i \leq 2^k$. We note $w_0(H, M) = H$ for any M .

We claim $d_n^*(K) = \{a \in K: w_n(a) \text{ is a law in } K\}$ for any group K .

This is clear for $n = 1$, since $d_1^*(K) = Z(K)$ by Theorem 1.4 and

$a \in Z(K)$ if and only if $w_1(a, x) = [a, x]$ is a law in K . For arbitrary n we have $d_n^*(K)/C_K(d_{n-1}(K)) = d_{n-1}^*(K/C_K(d_{n-1}(K)))$ by Theorem 1.3(b). By the induction hypothesis, $a \in d_n^*(K)$ if and only if $w_{n-1}(a)$ is a law mod $C_K(d_{n-1}(K))$. This is equivalent to saying $w_n(a) = [w_{n-1}(a), d_{n-1}]$ is a law in K . Hence the claim is proved for any group K .

Let $1 = A_0 < \dots < A_t = G$ be the upper d_n -series for G . By the above claim, we know that $A_i = \{a \in G : w_n(a, G) \subseteq A_{i-1}\}$ for $1 \leq i \leq t$. Let $H = d_{n-1}(G)$. We claim that $\gamma_{jnt+1}(H) \subseteq A_{t-j}$ for $0 \leq j \leq t$. This is evident for $j = 0$, so assume it is true for some $0 \leq j < t$. We further assert that $\gamma_{jnt+s+1}(H) \subseteq w_s(A_{t-j}, G)$ for $0 \leq s \leq n$. Since $w_0(A_{t-j}, G) = A_{t-j}$, this latter conjecture holds for $s = 0$ by assumption. Hence assume it is true for some $0 \leq s < n$. Then $\gamma_{jnt+s+2}(H) = [\gamma_{jnt+s+1}(H), H] \subseteq [w_s(A_{t-j}, G), d_s(G)] = w_{s+1}(A_{t-j}, G)$ by the induction assumption and the fact that $H \subseteq d_s(G)$ for $0 \leq s \leq n-1$. Hence the latter conjecture holds. In particular, $\gamma_{(j+1)nt+1}(H) \subseteq w_n(A_{t-j}, G) \subseteq A_{t-(j+1)}$ by our characterization of A_{t-j} . Hence the first conjecture holds. Consequently, $\gamma_{tnt+1}(H) \subseteq A_{t-t} = A_0 = 1$ and $H = d_{n-1}(G)$ is nilpotent of class $\leq tn$.

Corollary 2.23. If G is a group such that $d_n(G)$ is nilpotent, $1 < H < G$, then $1 < H \cap d_{nt+1}^*(G)$.

Proof. By Theorem 2.22, G is d_{nt+1} -nilpotent. The corollary now follows from Theorem 2.11.

Remark 2.24. A group G is γ_1 -hypercentral if and only if $G = 1$, since $\gamma_1^*(G) = 1$. For $n > 1$, G is γ_n -hypercentral if and only if G is hypercentral, since the marginal subgroups for such words are in the upper central series for G by Theorem 1.4.

However, although a d_2 -hypercentral group is d_n -hypercentral for $n > 2$ by Theorem 2.15, d_n -nilpotence for some $n > 2$ does not imply d_2 -nilpotence. Let $G = S_4$. The derived series for G is $1 < V_4 < A_4 < G$, where V_4 is the four-group. Hence $d_3(G) = 1$ so that $G = d_3^*(G)$ is d_3 -nilpotent of class one. Since $G' = d_1(G) = A_4$ is not nilpotent, G is not d_2 -nilpotent by Theorem 2.22.

The following lemma is evidently well-known.

Lemma 2.25. Let $\phi = \phi(x) = x^2$. Then $\phi^*(G) = \{a \in Z(G) : a^2 = 1\}$ for any G .

Proof. Let $H = \{a \in Z(G) : a^2 = 1\}$. Clearly H is a subgroup of G contained in $\phi^*(G)$. Let $a \in \phi^*(G)$. Then $(ax)^2 = x^2$ for all $x \in G$. In particular, $a^2 = 1^2 = 1$. Furthermore, for any $x \in G$, $a^{-1}xa = axa = axax^{-1} = x^2x^{-1} = x$. Hence $a \in Z(G)$ and $H = \phi^*(G)$.

Theorem 2.26. Let $\phi = \phi(x) = x^2$. Suppose G is ϕ -nilpotent of class n . Then G is nilpotent of class $\leq n$ and has exponent dividing 2^n .

Proof. By Lemma 2.25, the upper ϕ -series for G is an ascending central series of length n . Hence G is nilpotent of class $\leq n$. Since the quotients in the upper ϕ -series for G are elementary 2-groups, G has exponent dividing 2^n .

Theorem 2.27. Assume G is e_2 -nilpotent of class n . Then

- (a) $e_2(G)$ is nilpotent of class $< n$.
- (b) there is a sequence $G_0 = 1 < G_1 < \dots < G_n = G$ of subgroups of G such that G_{i+1}/G_i is nilpotent of class at most three.
- (c) G is solvable of length $\leq 2n$.
- (d) if every element of G has odd order, then G is nilpotent of class $\leq 3n$.

Proof. Part (a) follows from Theorem 2.12. Let

$G_0 = 1 < G_1 < \dots < G_n = G$ be the upper e_2 -series for G . Since G_i/G_{i-1} , $1 \leq i \leq n$, is in the variety determined by e_2 , it is nilpotent of class at most three by Levi's Theorem and thus metabelian. Parts (b) and (c) now follow.

Kappe [9: Satz III, Teil 4] has shown that a right Engel element of length two and odd order is in the third center. Under the hypotheses of part (d), $e_2^*(G/G_i) \subseteq Z_3(G/G_i)$ for $0 \leq i \leq n-1$. Hence G is γ_4 -nilpotent and thus nilpotent.

In Theorem 4.16 we shall show that $e_2^*(G) \subseteq Z_3(G)$ for all G . Consequently the hypotheses in part (d) may be dispensed with and we may conclude that any e_2 -nilpotent group of class n is nilpotent of class $\leq 3n$.

Definition 2.28. Define $H_\phi(G) = H_\phi = \{a \in G : \phi(g_1, \dots, g_n)^{-1} \phi(g_1, \dots, ag_1, \dots, g_n) \in H \text{ for every } i \text{ and } g_1, \dots, g_n \text{ in } G\}$ to be the ϕ -normalizer of H in G .

It is not difficult to see that $H_\phi(G) \subseteq G$ and that $H < G$ implies that $H \subseteq H_\phi(G)$.

Theorem 2.29. If G is ϕ -hypercentral and $H < G$, then $H_\phi \not\subseteq H$. In particular, if also $H < G$, then $H < H_\phi$.

Proof. Let $G_0 = 1 < G_1 < \dots < G_\sigma = G$ be the upper ϕ -series for G . There is an ordinal β such that $G_\beta \subseteq H$, $G_{\beta+1} \not\subseteq H$. Choose $a \in G_{\beta+1}$, $a \notin H$. We know that $G_{\beta+1}/G_\beta = \phi^*(G/G_\beta)$. In particular, for $g_1, \dots, g_n$ in G , we have $\phi(g_1, \dots, g_n)^{-1} \phi(g_1, \dots, ag_1, \dots, g_n)$ is in $G_\beta \subseteq H$, since $a \in G_{\beta+1}$. Hence $a \in H_\phi(G)$, $a \notin H$.

Remark 2.30. Perhaps a more natural way to define a ϕ -normalizer $H(\phi, G)$ of $H \subseteq G$ is the set $a \in G$ such that

$\phi(h_1, \dots, ah_1, \dots, h_n) \in H$ for every i and $h_1, \dots, h_n \in H$. This ϕ -normalizer is $N_G(H)$ for $\phi = \gamma_2$, but it is not necessarily a subgroup for every word ϕ .

From the definitions we have that $H_\phi(G)$ is a subset of $H(\phi, G)$. Hence by Theorem 2.29 we may conclude that a proper subgroup of a ϕ -hypercentral group is a proper subset of $H(\phi, G)$.

Remark 2.31. For any word ϕ we may define a ϕ -solvable group G to be a group possessing a finite normal series $A_0 = 1 < A_1 < \dots < A_n = G$ such that $A_{i+1}/A_i = \phi^*(A_{i+1}/A_i)$. It is clear that a ϕ -nilpotent group is also ϕ -solvable. We may equivalently define a group G to be ϕ -solvable if and only if the descending series $G \supset \phi(G) \supset \dots \supset \phi^n(G) \supset \phi^{n+1}(G) = \phi(\phi^n(G)) \supset \dots$ reaches 1 in finitely many steps. A group is γ_2 -solvable if and only if it is solvable. Many of the standard theorems for solvable groups can be generalized to ϕ -solvable groups.

CHAPTER III

MARGINAL AUTOMORPHISMS

In this chapter we will consider two generalizations of the marginal subgroup of G --one in G and the other in $\text{Aut}(G)$.

Definition 3.1. Let $\phi = \phi(x_1, \dots, x_n)$ be a word in n variables, G a group. Define $\phi^c(G) = \{a \in G: \phi(g_1, \dots, g_1^a, \dots, g_n) = \phi(g_1, \dots, g_1, \dots, g_n) \text{ for all choices of } g_1, \dots, g_n \text{ in } G \text{ and } i = 1, 2, \dots, n\}$ to be the c-marginal subgroup of G corresponding to ϕ .

Theorem 3.2. In any group G ,

- (a) $\phi^c(G) \in \text{Char}(G)$.
- (b) if $K/\phi^*(G) = Z(G/\phi^*(G))$, then $Z(G)K \subseteq \phi^c(G)$. In particular, $\phi^*(G) \subseteq \phi^c(G)$.
- (c) $[\phi^c(G), \phi(G)] = 1$.
- (d) $\phi^c(G) = \{a \in G: \phi(g_1, \dots, ag_1, \dots, g_n) = \phi(g_1, \dots, g_1a, \dots, g_n) \text{ for all choices of } g_1, \dots, g_n \text{ in } G \text{ and } i = 1, 2, \dots, n\}$.

Proof. (a) Let $a, b \in \phi^c(G)$, $g_1, \dots, g_n$ in G . Then $\phi(g_1, \dots, g_1^{a^{-1}b}, \dots, g_n) = \phi(g_1, \dots, g_1^{a^{-1}}, \dots, g_n) = \phi(g_1, \dots, (g_1^{a^{-1}})^a, \dots, g_n) = \phi(g_1, \dots, g_1, \dots, g_n)$ for each i . Hence $a^{-1}b \in \phi^c(G)$ and $\phi^c(G) \subseteq G$.

Now let $\alpha \in \text{Aut}(G)$. Then $\phi(g_1^\alpha, \dots, (g_1^\alpha)^{a^\alpha}, \dots, g_n^\alpha) = \phi(g_1, \dots, g_1^a, \dots, g_n)^\alpha = \phi(g_1, \dots, g_1, \dots, g_n)^\alpha =$

$\phi(g_1^\alpha, \dots, g_n^\alpha)$. This shows that $a^\alpha \in \phi^c(G)$, since $\alpha \in \text{Aut}(G)$.

(b) It is clear that $Z(G) \subseteq \phi^c(G)$ from the definition of $\phi^c(G)$. Let $k \in K$, $g_1, \dots, g_n \in G$. Then for each i we have $g_1^k \phi^*(G) = g_1 \phi^*(G)$. Hence $[g_1, k] \in \phi^*(G)$ and $\phi(g_1, \dots, g_1^k, \dots, g_n) = \phi(g_1, \dots, g_1[g_1, k], \dots, g_n) = \phi(g_1, \dots, g_1, \dots, g_n)$. Therefore $K \subseteq \phi^c(G)$.

(c) Let $a \in \phi^c(G)$. Then $\phi(g_1, \dots, g_n)^a = \phi(g_1^a, \dots, g_n^a) = \phi(g_1, \dots, g_n)$. Hence $\phi^c(G)$ commutes elementwise with $\phi(G)$.

(d) Call the set on the right K . Let $a \in \phi^c(G)$. Then $\phi(g_1, \dots, ag_1, \dots, g_n) = \phi(g_1, \dots, (ag_1)^a, \dots, g_n) = \phi(g_1, \dots, g_1^a, \dots, g_n)$ for each i . Hence $\phi^c(G) \subseteq K$. Conversely, for $a \in K$, we have $\phi(g_1, \dots, g_1^a, \dots, g_n) = \phi(g_1, \dots, (a^{-1}g_1)a, \dots, g_n) = \phi(g_1, \dots, a(a^{-1}g_1), \dots, g_n) = \phi(g_1, \dots, g_1, \dots, g_n)$ for each i . Hence $\phi^c(G) = K$.

Example 3.3. It is not difficult to see that $\gamma_1^c(G) = Z(G)$. Furthermore, $\gamma_2^c(G) = \{a \in G: [ax, y] = [xa, y] \text{ for all } x, y \in G\} = \{a \in G: [a, x, y] = 1 \text{ for all } x, y \in G\} = Z_2(G)$ by Theorem 3.2(d). To classify $\gamma_n^c(G)$ for any $n \geq 1$, we need the following theorem. The proof is identical to that of [20; Lemma 3(b)].

Theorem 3.4. Let $\lambda = \lambda(x_1, \dots, x_m)$ and $\theta = \theta(y_1, \dots, y_n)$ be two words and define $\phi(x_1, \dots, x_m, y_1, \dots, y_n) = [\lambda(x_1, \dots, x_m), \theta(y_1, \dots, y_n)]$. If $U = C_G(\lambda(G))$, $V = C_G(\theta(G))$, $L/U = \theta^c(G/U)$, $M/V = \lambda^c(G/V)$, then $\phi^c(G) = L \cap M$.

Theorem 3.5. In any group G , $\gamma_n^c(G) = Z_n(G)$ for $n \geq 1$.

Proof. We induct on n . The cases $n = 1, 2$ were verified in Example 3.3. We recall that $\gamma_n = [\gamma_{n-1}, \gamma_1]$. Let $U = C_G(\gamma_{n-1}(G))$,

$V = C_G(\gamma_1(G)) = Z(G)$. Then define L and M by $L/U = \gamma_1^c(G/U) = Z(G/U)$ and $M/V = \gamma_{n-1}^c(G/V) = Z_{n-1}(G/V)$. We see then that $L = \{a \in G : [a, G] \subseteq C_G(\gamma_{n-1}^c(G))\}$ and $M = Z_n(G)$. By Theorem 3.4, $\gamma_n^c(G) = L \cap Z_n(G) = Z_n(G)$ and the result follows.

Since $Z(G) \subseteq \phi^c(G)$ for all words ϕ , it seems natural to replace $a \in G$ with $\alpha \in \text{Aut}(G)$ for every group G in Definition 3.1:

Definition 3.6. Let G be a group, $\phi = \phi(x_1, \dots, x_n)$ a word in n variables. Define the automargin $\bar{\phi}(G)$ of G corresponding to ϕ by $\bar{\phi}(G) = \{\alpha \in \text{Aut}(G) : \phi(g_1, \dots, g_1^\alpha, \dots, g_n) = \phi(g_1, \dots, g_1, \dots, g_n) \text{ for each } i = 1, 2, \dots, n \text{ and every } g_1, \dots, g_n \text{ in } G\}$.

Definition 3.7. Define the marginal automorphisms on G corresponding to the word ϕ by $\phi^*(G) = \{\alpha \in \text{Aut}(G) : [G, \alpha] \subseteq \phi^*(G)\}$.

Lemma 3.8. Assume $H \in \text{Char}(G)$ and $L = \{\alpha \in \text{Aut}(G) : [G, \alpha] \subseteq H\}$. Then $L \triangleleft \text{Aut}(G)$.

Proof. Let $a, b \in L, x \in G$. Then $[x, ab] = [x, b][x, a]^b \in H$ and $[x, a^{-1}] = ([x, a]^{-1})^{a^{-1}} \in H$, since $H \in \text{Char}(G)$. Hence $L \subseteq \text{Aut}(G)$.

Furthermore, for $a \in L, b \in \text{Aut}(G)$, $[G, a^b] = [G^{b^{-1}}, a]^b = [G, a]^b \subseteq H^b = H$. Therefore $L \triangleleft \text{Aut}(G)$.

Theorem 3.9. For any group G and any word ϕ ,

- (a) $\bar{\phi}(G) \triangleleft \text{Aut}(G)$, $\phi^*(G) \triangleleft \text{Aut}(G)$, $\phi^*(G) \subseteq \bar{\phi}(G)$.
- (b) $\phi(G) \subseteq F(\bar{\phi}(G)) \subseteq F(\phi^*(G))$. In particular, $\phi(G/F(\bar{\phi}(G))^G) = 1$.
- (c) if $F(\bar{\phi}(G)) \triangleleft G$, then $[G, \bar{\phi}(G)] \subseteq C_G(F(\bar{\phi}(G)))$.
- (d) if I_c is the group of automorphisms on G induced by $\phi^c(G)$, then $I_c \triangleleft \text{Aut}(G)$, $I_c \subseteq \bar{\phi}(G)$.

Proof. (a) We may show $\bar{\phi}(G) \subseteq \text{Aut}(G)$ the same way we showed $\phi^c(G) \subseteq G$ in the proof of Theorem 3.2(a). Now let $a \in \bar{\phi}(G)$, $b \in \text{Aut}(G)$. Then, for each i , $g_1, \dots, g_n \in G$,

$$\phi(g_1, \dots, g_1^{b^{-1}ab}, \dots, g_n) = \phi(g_1^{b^{-1}}, \dots, (g_1^{b^{-1}})^a, \dots, g_n^{b^{-1}})^b =$$

$$\phi(g_1^{b^{-1}}, \dots, g_1^{b^{-1}}, \dots, g_n^{b^{-1}})^b = \phi(g_1, \dots, g_n). \text{ Hence we have}$$

that $\bar{\phi}(G) \triangleleft \text{Aut}(G)$. The fact that $\phi^*(G) \triangleleft \text{Aut}(G)$ follows from Lemma 3.8. Finally, let $a \in \phi^*(G)$, $g_1, \dots, g_n \in G$. Then, for each i ,

$$\phi(g_1, \dots, g_1^a, \dots, g_n) = \phi(g_1, \dots, g_1[g_1, a], \dots, g_n) =$$

$$\phi(g_1, \dots, g_1, \dots, g_n). \text{ Therefore } \phi^*(G) \subseteq \bar{\phi}(G).$$

(b) Let $a \in \bar{\phi}(G)$. For $g_1, \dots, g_n \in G$, $\phi(g_1, \dots, g_n)^a = \phi(g_1^a, \dots, g_n^a) = \phi(g_1, \dots, g_n)$. Hence each element of $\phi(G)$ is fixed by each element of $\bar{\phi}(G)$ so that $\phi(G) \subseteq F(\bar{\phi}(G))$. By part (a), $\phi^*(G) \subseteq \bar{\phi}(G)$ so that also $F(\bar{\phi}(G)) \subseteq F(\phi^*(G))$.

(c) From $[F(\bar{\phi}(G)), \bar{\phi}(G)] = 1$ it follows that $[F(\bar{\phi}(G)), \bar{\phi}(G), G] = 1$. Since $F(\bar{\phi}(G)) \triangleleft G$ by hypothesis, we have that $[G, F(\bar{\phi}(G)), \bar{\phi}(G)] \subseteq [F(\bar{\phi}(G)), \bar{\phi}(G)] = 1$. Therefore, by the Three Subgroups Lemma applied to the holomorph of G by $\bar{\phi}(G)$, we have that $[G, \bar{\phi}(G), F(\bar{\phi}(G))] = 1$.

(d) It follows from the definitions that $I_c \subseteq \bar{\phi}(G)$. By Theorem 3.2(a), $\phi^c(G) \in \text{Char}(G)$. Thus $\phi^c(G)/Z(G) \in \text{Char}(G/Z(G))$. This implies that $I_c \in \text{Char}(\text{Inn}(G))$, where $\text{Inn}(G)$ is the group of inner automorphisms on G . Since $\text{Inn}(G) \triangleleft \text{Aut}(G)$, we have also that I_c is normal in $\text{Aut}(G)$.

Theorem 3.10. Suppose $\phi(G) = F(\phi^*(G))$. Then

(a) if $\phi^*(G) = 1$, $G = \phi(G)$.

(b) if $\phi^*(G) \subseteq Z(G)$ and $\phi^*(G)$ lies in a class $\Sigma = \{S, DP\}\Sigma$,

$G/\phi(G) \in \Sigma$.

(c) if $\phi^*(G) \subseteq Z(G)$ and both $\phi'(G)$ and $\phi^*(G)$ are finite, then $G/\phi(G)$ is finite.

Proof. If $\phi^*(G) = 1$, then for $\alpha \in \text{Aut}(G)$, $[G, \alpha] \subseteq \phi^*(G)$ if and only if $\alpha = 1$. Hence $\phi'(G) = 1$, $\phi(G) = F(\phi'(G)) = F(1) = G$ and (a) follows.

By hypothesis, $\phi'(G) \subseteq \gamma_2^*(G)$. Hence $F(\alpha) \triangleleft G$ for each $\alpha \in \phi'(G)$ and $G/\phi(G) = G/F(\phi'(G)) = G/\cap\{F(\alpha) : \alpha \in \phi'(G)\} \subseteq \pi\{G/F(\alpha) : \alpha \in \phi'(G)\} \cong \pi\{[G, \alpha] : \alpha \in \phi'(G)\} \subseteq \pi\{\phi^*(G)_\alpha : \alpha \in \phi'(G)\}$. Parts (b) and (c) now follow.

Theorem 3.11. Assume $\phi(G) \subseteq \theta(G)$ for all groups G . Then

(a) $\theta'(G) \subseteq \phi'(G)$ for all G .

(b) $\phi'(G)/\theta'(G) \subseteq \phi'(G/\theta^*(G))$ for all G .

Proof. (a) By Theorem 1.5 we have that $\theta^*(G) \subseteq \phi^*(G)$ for all G . Hence $[G, \theta'(G)] \subseteq \theta^*(G) \subseteq \phi^*(G)$ and $\theta'(G) \subseteq \phi'(G)$.

(b) By part (a) and Theorem 3.9(a) we know that $\theta'(G) \triangleleft \phi'(G)$. Let $f \in \phi'(G)$. Since $\theta^*(G) \in \text{Char}(G)$, we may take f to be an automorphism on $G/\theta^*(G)$. Then, for any $g \in G$, $(g\theta^*(G))^f = g^f\theta^*(G)$. But $f \in \phi'(G)$ implies that $[g, f] \in \phi^*(G)$ so that $[g, f]\theta^*(G) \in \phi^*(G)/\theta^*(G) \subseteq \phi^*(G/\theta^*(G))$. Hence $f \in \phi'(G/\theta^*(G))$. Also $(g\theta^*(G))^f = g^f\theta^*(G) = g\theta^*(G)$ if and only if $[g, f] \in \theta^*(G)$. Thus $f \in \phi'(G)$ induces the identity on $G/\theta^*(G)$ if and only if $f \in \theta'(G)$.

The proof of the following is similar to the proof of Theorem 3.11(b):

Theorem 3.12. For any G and word ϕ , $\overline{\phi}(G)/\phi'(G) \subseteq \text{Aut}(G/\phi^*(G))$.

Remark 3.13. Let $G = J_4$, $\phi = \phi(x) = x^2$. Since $\phi^*(G) = \{x \in G : x^2 = 1\}$ has index two, we see that $\text{Aut}(G/\phi^*(G)) = 1$ and

$$\bar{\phi}(G) = \phi^*(G) = \text{Aut}(G).$$

Remark 3.14. We now ask: for what words θ can $\theta^*(G)$ have index two in G or, more generally, be maximal in G ? If $\theta^*(G)$ is maximal in G , then there is a cyclic subgroup H of G such that $G = H\theta^*(G)$. By Theorem 1.1(d), we see that $\theta(G) = \theta(H)$. If cyclic groups are in the variety determined by θ --for example, when θ is an outer commutator word of weight greater than one--then $\theta(G) = 1$ and $G = \theta^*(G)$, contrary to the maximality of $\theta^*(G)$.

Theorem 3.15. Let $\lambda = \lambda(x_1, \dots, x_m)$ and $\theta = \theta(y_1, \dots, y_n)$ be two words and define $\phi(x_1, \dots, x_m, y_1, \dots, y_n) = [\lambda(x_1, \dots, x_m), \theta(y_1, \dots, y_n)]$. If $\bar{\lambda}(G) = \lambda^*(G)$ and $\bar{\theta}(G) = \theta^*(G)$ for all G , then $\bar{\phi}(G) = \phi^*(G)$ for all G .

Proof. Suppose $f \in \bar{\phi}(G)$, $U = C_G(\lambda(G))$, $V = C_G(\theta(G))$. Since verbal subgroups are fully invariant, U and V are characteristic in G . Thus we may assume that f is an automorphism on G/U and on G/V .

Since $f \in \bar{\phi}(G)$, we have

$$[\lambda(x_1, \dots, x_1^f, \dots, x_m), \theta(y_1, \dots, y_n)] =$$

$$[\lambda(x_1, \dots, x_1, \dots, x_m), \theta(y_1, \dots, y_n)] \text{ for all } x_1, \dots, x_m,$$

$$y_1, \dots, y_n \text{ in } G. \text{ This is equivalent to saying that } 1 =$$

$$[\lambda(x_1, \dots, x_1^f, \dots, x_m)\lambda(x_1, \dots, x_1, \dots, x_m)^{-1}, \theta(y_1, \dots, y_n)]$$

or $\lambda(x_1, \dots, x_1^f, \dots, x_m)V = \lambda(x_1, \dots, x_1, \dots, x_m)V$. By hypothesis, $f \in \bar{\lambda}(G/V) = \lambda^*(G/V)$ so that $[G, f]V/V \subseteq \lambda^*(G/V)$. By a similar argument, we may conclude that $[G, f]U/U \subseteq \theta^*(G/U)$. By Theorem 1.3(b), $[G, f] \subseteq \phi^*(G)$. Therefore $f \in \phi^*(G)$. Consequently, by Theorem 3.9(a) we have that $\bar{\phi}(G) = \phi^*(G)$.

Corollary 3.16. Let ϕ be any outer commutator word. Then $\bar{\phi}(G) = \phi^*(G)$ for all G .

Proof. We induct on the weight $w(\phi)$ of ϕ . The word $\gamma_1 = x$ is the only outer commutator word of weight one. Then, for $f \in \text{Aut}(G)$, $g^f = g$ for all g if and only if $f = 1$. Hence $\gamma_1^*(G) = \bar{\gamma}_1(G) = 1$.

Now assume that $w(\phi) > 1$, $\phi = [\lambda, \theta]$, where $w(\lambda), w(\theta)$ are less than $w(\phi)$. By the induction hypothesis, $\bar{\lambda}(G) = \lambda^*(G)$, $\bar{\theta}(G) = \theta^*(G)$ for all G . Consequently, $\bar{\phi}(G) = \phi^*(G)$ for all G by Theorem 3.15.

Example 3.17. The conclusion of Corollary 3.16 does not hold for $\phi = \phi(x) = x^2$. By Lemma 2.25, $\phi^*(G) = \{a \in Z(G) : a^2 = 1\}$ for any G . Let $G = S_3$. The set H of squares of elements of G is the derived group of G , the normal subgroup of order three. Since G is centerless, we may consider G to be a group of automorphisms on itself. Certainly H as a subgroup of $\text{Aut}(G)$ fixes each element of H . Hence $1 \neq H \subseteq \bar{\phi}(G)$. But $\phi^*(G) = \{a \in \text{Aut}(G) : [G, a] \subseteq \phi^*(G) = 1\} = 1$.

We recall that $a \in \gamma_2^*(G) = \bar{\gamma}_2(G)$ if and only if a is a normal automorphism; that is, a commutes with the inner automorphisms.

Franklin Haimo [3] and W. E. Deskins [1] have considered higher normal automorphisms. In particular, define $T_1(G) = \gamma_2^*(G)$ and $T_n(G) = \{a \in \text{Aut}(G) : x^a \equiv x \pmod{Z_n(G)} \text{ for all } x \in G\}$ for $n > 1$ to be the n -normal automorphisms of G . By Corollary 3.16 we may now add that $T_n(G) = \gamma_{n+1}^*(G) = \bar{\gamma}_{n+1}(G)$ for $n \geq 1$.

Theorem 3.18. Let $\phi = \phi(x) = x^n$, $n \geq 1$. Then $\bar{\gamma}_2(G) \cap \bar{\phi}(G) = \bar{\gamma}_2(G) \cap \phi^*(G)$. If $n = 2$, then also $\phi^*(G) \subseteq \bar{\gamma}_2(G)$ so that $\bar{\gamma}_2(G) \cap \bar{\phi}(G) = \phi^*(G)$.

Proof. Let $a \in \bar{\gamma}_2(G) \cap \bar{\phi}(G)$. Then $(x^n)^a = x^n$ for all $x \in G$. This implies $1 = (x^n)^{-1}(x^n)^a = [x^n, a] = [x, a]^n$, since $[x, a] \in Z(G)$. Hence $a \in \phi^*(G)$. By Theorem 3.9(a), we have $\bar{\gamma}_2(G) \cap \bar{\phi}(G) = \phi^*(G) \cap \bar{\gamma}_2(G)$.

If $n = 2$, $\phi^*(G) = \{x \in Z(G) : x^2 = 1\}$. Hence $\phi^*(G) \subseteq \bar{\gamma}_2(G)$.

Corollary 3.19. Assume $G = S + T$, T is torsionfree Abelian and $S = \Sigma \{ \langle a_\alpha \rangle : \alpha \in I \}$, where $o(a_\alpha)$ is a prime power. Let $\phi = \phi(x) = x^n$. Assume further that if a_α has order p_α^m , then $m_\alpha \geq 2s$, where p_α^s is the p_α -share of n . Then $\phi^*(G) = \Sigma \{ \langle a_\alpha^{p_\alpha^{m_\alpha-s}} \rangle : \alpha \in I \}$ and $\phi'(G) = \overline{\phi}(G)$ is Abelian.

Proof. Since G is Abelian, $\phi'(G) = \overline{\phi}(G)$ by Theorem 3.18. Also, $\phi^*(G) = \phi^*(T) + \Sigma \{ \phi^*(\langle a_\alpha \rangle) : \alpha \in I \} = \Sigma \{ \phi^*(\langle a_\alpha \rangle) : \alpha \in I \}$ by Theorem 1.2 and the fact that T is torsionfree.

Let $\langle a \rangle$ be a summand of S , where $o(a) = p^m$, $m \geq 2s$. We know $\phi^*(\langle a \rangle) = \{ x \in \langle a \rangle : x^n = 1 \} = \{ x \in \langle a \rangle : x^{p^s} = 1 \} = \langle a^{p^{m-s}} \rangle$. Write $n = p^s t$, where $(p, t) = 1$. Since also $(o(a), t) = 1$, there is a $b \in \langle a \rangle$ such that $a = b^t$. Hence $a^{p^s} = b^{tp^s} = b^n$ and $\phi^*(\langle a \rangle) = \langle (a^{p^s})^{p^{m-2s}} \rangle = \langle (b^{p^{m-2s}})^n \rangle$. Then $\phi'(G)$ stabilizes the series $1 \triangleleft \phi^*(G) \triangleleft G$. Consequently $\phi'(G)$ is Abelian.

Remark 3.20. Let $G = \Sigma \{ (J_4)_\alpha : \alpha \in I \}$, $\phi = \phi(x) = x^2$. By Corollary 3.19, $\phi'(G) = \overline{\phi}(G)$ is Abelian. Since $f \in \text{Aut}(G)$ defined by $x^f = x^{-1}$ is in $\phi'(G)$, we have that $\phi'(G) \neq 1$. We also note that if $o(I) = 1$, then $\phi'(G) = \overline{\phi}(G) = \text{Aut}(G) \cong J_2$.

Also, for $\phi = x^2$, if G is torsionfree Abelian or periodic where each element has odd order, then $\phi'(G) = \overline{\phi}(G) = 1$ by Corollary 3.19. Furthermore, if G is Černikov complete, it is generated by its n -th powers so that $\phi'(G) = \overline{\phi}(G) = 1$ for $\phi = \phi(x) = x^n$.

CHAPTER IV

THE ENGEL MARGIN

In this chapter we shall investigate the marginal subgroup for the Engel word e_2 of length two. We note that Theorem 1.3(b) does not apply, since e_2 is a commutator with a repeated variable. The metabelian margin $d_2^*(G)$ will play a role with $e_2^*(G)$ for each G , so we will also derive some results for d_2 and, where possible, extend them to any solvable word d_n .

By "Engel word" we will mean "Engel word of length two". For any G we will write $M = M(G) = d_2^*(G)$ and $E = E(G) = e_2^*(G)$ for the metabelian and Engel margins of G respectively.

Lemma 4.1. In any group G ,

- (a) $d_n^*(G)/C_G(d_{n-1}(G)) = d_{n-1}^*(G/C_G(d_{n-1}(G)))$. In particular, $M(G) = \{a \in G: [[a, x], [y, z]] \text{ is a law in } G\}$.
- (b) $Z_{n(n+1)/2}(G) \subseteq d_n^*(G)$. In particular, $Z_3 \subseteq M$.
- (c) $[M, d_n(G)] \subseteq Z(d_n(G))$ for $n \geq 1$ and $[M, G] \subseteq Z(G')$.

Proof. (a) This follows from Theorem 1.3(b), since d_n is an outer commutator word for each n . We note in particular that $M/C_G(G') = Z(G/C_G(G'))$. The second statement was verified in the proof of Corollary 2.22.

(b) We induct on n . For $n = 1$, $Z_1(G) \subseteq d_1^*(G) = Z(G)$. For $n > 1$, let $\bar{G} = G/C_G(d_{n-1}(G))$. Then $\overline{d_n^*(G)} = d_{n-1}^*(\bar{G}) \subseteq$

$Z_{n(n-1)/2}(\bar{G})$ by part (a) and the induction hypothesis. Furthermore,

$$[Z_{n(n+1)/2}(G), n(n-1)/2(G)] \subseteq Z_{n(n+1)/2 - n(n-1)/2}(G) = Z_n(G) \text{ and}$$

$$[Z_n(G), d_{n-1}(G)] \subseteq [Z_n(G), \gamma_n(G)] = 1 \text{ so that}$$

$$[Z_{n(n+1)/2}(G), n(n-1)/2(G)] \subseteq C_G(d_{n-1}(G)). \text{ Consequently,}$$

$$\overline{Z_{n(n+1)/2}(G)} \subseteq Z_{n(n-1)/2}(\bar{G}) \subseteq \overline{d_n^*(G)} \text{ and } Z_{n(n+1)/2}(G) \subseteq$$

$$d_n^*(G)C_G(d_{n-1}(G)) = d_n^*(G), \text{ as desired.}$$

$$(c) \text{ By part (a), } [M, d_n(G)] \subseteq d_n(G) \cap C_G(G') \subseteq$$

$$d_n(G) \cap C_G(d_n(G)) = Z(d_n(G)). \text{ Similarly, } [M, G] \subseteq$$

$$G' \cap C_G(G') = Z(G').$$

Lemma 4.2. For each $m \in M$, $x \in d_n(G)$, $n \geq 1$, define $f_m(x) = [x, m]$. Then

$$(a) \quad f_m \in \text{Hom}(d_n(G), Z(d_n(G))).$$

$$(b) \quad \text{Ker } f_m = d_n(G) \text{ if and only if } M \subseteq C_G(d_n(G)).$$

$$(c) \quad d_{n+1}(G) \subseteq \text{Ker } f_m.$$

Proof. (a) By Lemma 4.1(c), $f_m(d_n(G)) = [d_n(G), m] \subseteq Z(d_n(G))$. Let $x, y \in d_n(G)$, $m \in M$. Then $f_m(xy) = [xy, m] = [x, m]^y[y, m] = [x, m][y, m] = f_m(x)f_m(y)$.

(b) This follows immediately from Lemma 4.1(c). We note for $n = 1$ that $\text{Ker } f_m = G'$ if and only if $M = C_G(G')$, since $C_G(G') \subseteq M$ by Lemma 4.1(a).

(c) From the fact that $d_{n+1}(G) \subseteq d_2(G)$ for all G , we have that $d_2^*(G) \subseteq d_{n+1}^*(G)$ for all G by Theorem 1.5. Then $f_m(d_{n+1}(G)) = [d_{n+1}(G), m] \subseteq [d_{n+1}(G), d_2^*(G)] \subseteq [d_{n+1}(G), d_{n+1}^*(G)] = 1$ by

Theorem 1.1(c). Certainly $d_{n+1}(G) \subseteq d_n(G)$.

Corollary 4.3. For each $m \in M$ and all $g_1, \dots, g_{2n}$ in G ,
 $[d_n(g_1, \dots, g_{2n-1})^m, d_n(g_{2n-1+1}, \dots, g_{2n})] =$
 $[d_n(g_1, \dots, g_{2n-1}), d_n(g_{2n-1+1}, \dots, g_{2n})]$.

Proof. Define $f_m \in \text{Hom}(d_n(G), Z(d_n(G)))$ as in Lemma 4.2.
 Let $a = d_n(g_1, \dots, g_{2n-1})$, $b = d_n(g_{2n-1+1}, \dots, g_{2n})$. Then
 $[a^m, b] = [af_m(a), b] = [a, b]^{f_m(a)}[f_m(a), b] = [a, b]$, since
 $f_m(a) \in Z(d_n(G))$.

Remark 4.4. By the N/C Theorem, $M/C_G(G') \subseteq \text{Aut}(G')$. By
 Lemma 4.1(c), we also know that $[M, G'] \subseteq Z(G')$. Thus $M/C_G(G') \subseteq$
 $\gamma_2'(G')$.

Theorem 4.5. If $d_{n+1}(G) \neq 1$, then for each $m \in M$ there is an
 $x \in d_n(G)$, $x \neq 1$, such that $m \in C_G(x)$. In particular, $M \subseteq$
 $\langle C_G(x) : x \in d_n(G), x \neq 1 \rangle$.

Proof. Deny. Then there is an $m \in M$ such that $x^m \neq x$ for all
 $x \in d_n(G)$, $x \neq 1$. Define f_m as in Lemma 4.2. For $x, y \in d_n(G)$,
 $f_m(x) = f_m(y)$ implies that $x^{-1}x^m = y^{-1}y^m$ or that $yx^{-1} = (yx^{-1})^m$.
 By assumption, $x = y$ and f_m is an isomorphism from $d_n(G)$ onto a
 subgroup of $Z(d_n(G))$. Hence $d_n(G) = Z(d_n(G))$ and $d_{n+1}(G) = 1$,
 a contradiction.

We now turn to the Engel margin E . For convenience we define
 $E_1 = \{a \in G : [ax, y, y] = [x, y, y] \text{ for all } x, y \in G\}$ and $L(G) =$
 $\{a \in G : [a, x, x] = 1 \text{ for all } x \in G\}$ to be the subgroup of right
 Engel elements of length two. It is not difficult to show that
 $E \subseteq E_1 \in \text{Char}(G)$. We will need the following lemma. For the proofs
 of the various parts, see [9].

Lemma 4.6. (Kappe) In any group G , where $a \in L(G)$, $g, h \in G$,

- (a) $L(G) \in \text{Char}(G)$.
- (b) $[a, g, h] = [a, h, g]^{-1}$.
- (c) $[a, [g, h]] = [a, g, h]^2$.
- (d) $[a, g, [h, g]] = 1$.
- (e) if a has odd order, then $a \in Z_3(G)$.

Lemma 4.7. Every $a \in E(G)$ is both a right and a left Engel element. In particular, $Z_2(G) \subseteq E(G) \subseteq L(G)$.

Proof. Let $x \in G$. Then $[x, a, a] = [x, 1, 1] = 1$ and $[a, x, x] = [1, x, x] = 1$. Hence a is both a left and a right Engel element of length two. Furthermore, $e_2(G) \subseteq \gamma_3(G)$ for all G implies that $\gamma_3^*(G) \subseteq e_2^*(G)$ for all G by Theorem 1.5. By Theorem 1.4, $\gamma_3^*(G) = Z_2(G)$ and the result follows.

Theorem 4.8. In any group G ,

- (a) $E_1 = \{a: [a, x] \in C_G(x^G) \text{ for all } x \in G\} = L(G)$.
- (b) $[a, x] \in C_G(x^G) \cap C_G(a)$ for all $a \in E_1, x \in G$. Furthermore, $[a, x]^{rs} = [a^r, x^s]$ for all integers r and s .
- (c) a^G and x^{E_1} are Abelian for $a \in E_1, x \in G$.
- (d) $E_1 \subseteq I$, where $I = \bigcap \{C_G((x^G)^\circ); x \in G\} \triangleleft G$.

Proof. (a) Let $a \in E_1$. Then $[ay, x, x] = [y, x, x]$ for all $x, y \in G$. This is equivalent to saying that $1 = [[ay, x][y, x]^{-1}, x] = [[a, x]^y[y, x][y, x]^{-1}, x] = [[a, x]^y, x]$ for all $x, y \in G$. Since x and y are independent, we may conclude that $a \in E_1$ if and only if $1 = [a, x, x^y]$ for all $x, y \in G$ or, equivalently, $[a, x] \in C_G(x^G)$ for all x .

That $E_1 \subseteq L(G)$ follows from $[a, x, x^y] = 1$ by letting $y = 1$. Conversely, let $a \in L(G)$. We have, for $x, y \in G$, $[a, x, x^y] =$

$[a, x, x[x, y]] = [a, x, [x, y]][a, x, x]^{[x, y]}$. By the definition of $L(G)$ we see that $[a, x, x] = 1$. By Lemma 4.6(d) we also have that $[a, x, [x, y]] = 1$. Hence $[a, x, x^y] = 1$ and $a \in E_1$.

(b) Since a is a right Engel element, we have $[a, x] \in C_G(a)$ by [9: Lemma 2.1]. Part (a) says that $[a, x] \in C_G(x^G)$ for all $x \in G$. The remainder of part (b) follows from [18: Theorem 3.4.4].

(c) From part (b) we see that $a^x = a[a, x] \in C_G(a)$, since a and $[a, x]$ are in $C_G(a)$. This implies that a^G is Abelian.

The proof that x^{E_1} is Abelian follows similarly from $x^a = x[x, a]$, $[x, a] \in C_G(x^G) \subseteq C_G(x)$.

(d) By part (b) we may conclude that $[a, x^y] \in C_G((x^y)^G) = C_G(x^G)$ for all $a \in E_1$, $x, y \in G$.

We claim now that $(x^G)' = S$, where $S = \langle [x^w, x^z]; w, z \in G \rangle$. Clearly $S \subseteq (x^G)'$. A generator a of $(x^G)'$ may be written $a = [(x^{\pm 1})^{y_1} \dots (x^{\pm 1})^{y_m}, (x^{\pm 1})^{y_{m+1}} \dots (x^{\pm 1})^{y_n}]$ for elements $y_i \in G$. By [18: Theorem 3.4.2] we may write $a = \prod [(x^{\pm 1})^{y_i}, (x^{\pm 1})^{y_j}]^{z_{i,j}}$ for $z_{i,j} \in x^G$. But $x^G \triangleleft G$, so we may assume $(x^G)'$ is generated by elements of the form $[(x^{\pm 1})^w, (x^{\pm 1})^z]$ for various $w, z \in G$. Then $[(x^{-1})^w, x^z] = ([x^w, x^z]^{-1})(x^{-1})^w = ([x^z, x^w])(x^{-1})^w$ is in S , since $S \triangleleft G$. Similarly, we may show that $[x^w, (x^{-1})^z]$ and $[(x^{-1})^w, (x^{-1})^z]$ are in S . Hence the claim follows.

Let $a \in E_1$. By Lemma 4.6(c), we have $[a, [x^w, x^z]] = [[a, x^w], x^z]^2 = 1$. By the claim this implies that $a \in C_G((x^G)')$. Furthermore, $x^G \triangleleft G$, $(x^G)' \in \text{Char}(x^G)$ for each x imply $(x^G)' \triangleleft G$. By the N/C Theorem, $C_G((x^G)') \triangleleft G$ for each $x \in G$. Therefore $I \triangleleft G$.

Theorem 4.9. In any group G , $E = E(G) = \{a \in G : 1 = [x, a, y][x, y, a] \text{ for all } x, y \in G\}$.

Proof. Set $E_2 = \{a \in G : [x, ay, ay] = [x, y, y] \text{ for all } x, y \in G\}$. We know $E_1 = \{a \in G : [a, x] \in C_G(x^G) \text{ for all } x \in G\}$ by Theorem 4.8(a) and $E = E_1 \cap E_2$. Let S be the set given on the right in the statement of the theorem. Suppose $a \in S$, $x \in G$. Then $1 = [x, a, x][x, x, a] = [x, a, x]$. This implies that $a \in E_1 = L(G)$. Since also $E \subseteq E_1$, it suffices to show that $E \cap E_1 = E_1 \cap E_2 = E_1 \cap S$. Then, for $x, y \in G$, $a \in E_1 \cap E_2$ if and only if

$$\begin{aligned} [x, y, y] &= [x, ay, ay] \\ &= [x, ay, y][x, ay, a]^y \\ &= [[x, y][x, a]^y, y][[x, y][x, a]^y, a]^y \\ &= [x, y, y]^{[x, a]^y} [[x, a]^y, y][x, y, a]^{[x, a]^y} [[x, a]^y, a]^y. \end{aligned}$$

By assumption, $[a, x] \in C_G(x^G)$. Since $C_G(x^G) \triangleleft G$ by the N/C Theorem, we also have that $[a, x]^y \in C_G(x^G)$. Consequently, conjugation by $[x, a]^y$ is irrelevant in the last statement above because all the commutators are in x^G . Therefore the above is equivalent to $[x, y, y] = [x, y, y][x, a]^y, y][x, y, a]^y[x, a]^y, a]^y$ or $1 = [x, a, y][x, y, a][x, a]^y, a]$ for all $x, y \in G$, $a \in E(G)$.

Now a and $[x, a]^y$ are elements of a^G . By Theorem 4.8(c), a^G is Abelian. This implies that $[[x, a]^y, a] = 1$. Therefore $E(G)$ is contained in the set S .

We have already shown that S is a subset of $E_1 = L(G)$. Consequently, all the above arguments are reversible and we may conclude that $S = E(G)$.

Corollary 4.10. In any group G , $E(G) \cap C_G(G') = Z_2(G)$.

Proof. We need only verify that $E(G) \cap C_G(G') \subseteq Z_2(G)$ by Lemma 4.7. Let $a \in E(G) \cap C_G(G')$. By Theorem 4.9, $1 = [x, a, y][x, y, a]$ for all $x, y \in G$. But $a \in C_G(G')$ implies that $[x, y, a] = 1$ and thus that $[x, a, y] = 1$ for all $x, y \in G$. Hence $a \in Z_2(G)$.

Corollary 4.11. (a) Suppose $a \in L(G)$, $x, y \in G$. Then $[x, a, y] = [a, y, x]$.

(b) $E(G) = \{a \in G : [x, y, a][a, y, x] = 1 \text{ for all } x, y \in G\}$.

Proof. (a) $[a, y, x] = [a, x, y]^{-1}$ by Lemma 4.6(b),

$$= [[x, a]^{-1}, y]^{-1}$$

$$= (([x, a, y]^{-1})^{-1})^{[a, x]}$$

$$= [x, a, y], \text{ since } [a, x] \in C_G(x^G)$$

by Theorem 4.8(a).

(b) Let S be the set given on the right in the statement of part (b). By part (a) and Theorem 4.9, we have that $E(G)$ is a subset of S .

To prove the opposite inclusion, we need only show that S is a subset of $L(G)$ and then use part (a). Suppose $a \in S$, $x \in G$. Then $[x, x, a][a, x, x] = [a, x, x] = 1$. Hence $a \in L(G)$ and the theorem follows.

Corollary 4.12. In any group G , $[M, G] \cap E(G) \subseteq Z_2(G)$.

Proof. By Lemma 4.1(a), we have $M/C_G(G') = Z(G/C_G(G'))$. Hence $[M, G] \cap E \subseteq C_G(G') \cap E = Z_2(G)$ by Corollary 4.10.

Theorem 4.13. Let $a \in E(G)$. Then $[a, G, G]^3 = [a^3, G, G] = 1$.

Proof. Let $x, y \in G$. By Theorem 4.11(b), $[x, y, a][a, y, x] =$

$$\begin{aligned} 1. \text{ Then } [x, y, a] &= [a, [x, y]]^{-1} = ([a, x, y]^2)^{-1} \text{ by Lemma 4.6(c),} \\ &= [a, y, x]^2 \text{ by Lemma 4.6(b).} \end{aligned}$$

Hence $1 = [x, y, a][a, y, x] = [a, y, x]^2[a, y, x] = [a, y, x]^3$. By Theorem 4.8(c) we have that a^G is Abelian. Hence $[a, x, y]^3 = 1$ for all $x, y \in G$ implies $[a, G, G]$ has exponent dividing three, and $[a, x, y]^3 = [a^3, x, y] = 1$.

The following two corollaries are immediate from Theorem 4.13:

Corollary 4.14. For any group G , E/Z_2 has exponent three.

Corollary 4.15. If E has no elements of order three, then $E = Z_2$.

Theorem 4.16. In any group G , $E(G) \subseteq Z_3(G) \subseteq M(G)$. In addition, E/Z_2 is an elementary Abelian 3-group.

Proof. We need a slightly stronger result than our Lemma 4.6(e). In his proof, Kappe [9: Satz III, Teil 4] shows that for $a \in L(G)$, $x, y, z \in G$, $[a, x, y, z]^4 = 1$. Since a^G is Abelian by Theorem 4.8(c), we may assert that $[a^4, x, y, z] = 1$ or $a^4 \in Z_3(G)$. By Corollary 4.14, for $a \in E \subseteq E_1 = L(G)$, we also have $a^3 \in Z_2(G) \subseteq Z_3(G)$. Hence $a = a^4(a^3)^{-1} \in Z_3(G)$ and $E(G) \subseteq Z_3(G)$. It follows now from Lemma 4.1(b) that $Z_3(G) \subseteq M(G)$. The second statement then follows from Corollary 4.14.

Corollary 4.17. In any group G , $E(G)$ is nilpotent of class no greater than three and metabelian, and $[E, E, E]$ has exponent three. Furthermore, if $C_G(G') \subseteq E(G)$, then $M(G) = Z_3(G)$.

Proof. The first statement follows immediately from Levi's Theorem, since e_2 is a law in $E(G)$. Alternatively, we may conclude that $E(G)$ has nilpotence class no greater than three and is metabelian

from Theorem 4.16. By the same theorem, we know that $[E, E, E] \subseteq Z(E)$ is Abelian. Hence Theorem 4.13 implies that $[E, E, E]$ has exponent three.

Suppose $C_G(G') \subseteq E$. By Corollary 4.10 this implies that $C_G(G') = Z_2(G)$. From Lemma 4.1(a), $M/C_G(G') = Z(G/C_G(G'))$. Hence $M(G) = Z_3(G)$.

Theorem 4.18. (a) $[G', M, E_1] = [G', E_1, M] = [M, G, G'] = 1$.

(b) $[G, M', E_1] = [M', E_1, G] = [G, G, M'] = 1$. In particular, $[M', E_1] \subseteq Z(G)$.

Proof. (a) By Lemma 4.1(c), $[M, G] \subseteq Z(G')$ so that $1 = [M, G, G']$. Now let $a \in E_1, m \in M, x \in G'$. By Lemma 4.6(c), $[a, [m, x]] = [a, m, x]^2 = 1$. This implies $[G', M, E_1] = 1$. By the Three Subgroups Lemma, we also have that $[G', E_1, M] = 1$.

(b) As in the proof of part (a), $M' \subseteq Z(G')$ so that $1 = [G, G, M']$. Let $a \in E_1, x \in M', g \in G$. We have $[a, [g, x]] = [a, g, x]^2 = 1$. Hence $[M', G, E_1] = 1$ and, as above, $[M', E_1, G] = 1$.

Corollary 4.19. Suppose $M'/F(A)$ is divisible, where $A \subseteq \text{Aut}(M')$, $A \cong E_1 C_G(M')/C_G(M')$ and $F(A) = \{x \in M' : x^a = x \text{ for all } a \in A\}$ is the set of points fixed by A . Then

(a) $[M', a]$ is divisible for each $a \in A$. In particular, $[M', E_1] = [M', A]$ is divisible.

(b) for all $e \in E_1, x \in M', f_e(x) = [x, e]$ defines a homomorphism from M' onto a direct divisible summand of $Z(G)$.

(c) for all $e \in E_1, m \in M$, there is a homomorphism $t = t(m, e)$ from G' into a direct divisible summand of $Z(G)$ such that $\{x \in G' : [x, m] \in M'\} \subseteq \text{Ker } t$.

]

Proof. By Theorem 4.18(b), $[M', E_1] \subseteq Z(G)$. For $e \in E_1$, let $a \in A$ be the automorphism on M' corresponding to $eC_G(M')$. Then $M'/F(a) \cong [M', a] = [M', e] \subseteq Z(G)$, where $F(a) = \text{Ker } f_a$ and $[M', a] = f_a(M')$. Since $M'/F(a) \cong (M'/F(A))/(F(a)/F(A))$ is divisible, we have that $[M', e]$ is a divisible subgroup of $Z(G)$ and hence a summand of $Z(G)$. Furthermore, $[M', E_1] = \langle [M', e] : e \in E_1 \rangle$ is divisible because it is generated by divisible subgroups of $Z(G)$. Parts (a) and (b) now follow.

Let $e \in E_1$, $m \in M$. Define the homomorphism f_e from M' onto $[M', e]$ as in part (b). We know $M' \subseteq Z(G')$ and $[M', e]$ is divisible. Since then $[M', e]$ is an injective Z -module, f_e may be extended to $f_e^* \in \text{Hom}(Z(G'), [M', e])$. Define $f_m \in \text{Hom}(G', Z(G'))$ as in Lemma 4.2(a). Then $t = f_e^* f_m \in \text{Hom}(G', [M', e])$. If $x \in G'$ such that $[x, m] \in M'$, then $t(x) = f_e^* f_m(x) = f_e^*([x, m]) = f_e([x, m]) = [x, m, e] \in [G', M, E_1] = 1$ by Theorem 4.18(a).

Remark 4.20. By Theorem 4.16, $E \subseteq Z_3$ so that $[G', E] \subseteq Z(G)$. Then for all $a \in E$, $x \in G'$, $f_a(x) = [x, a]$ defines a homomorphism from G' into $Z(G)$.

We shall investigate the action of E and E_1 on G' . By Remark 4.4 we know that $M/C_G(G')$ acts as a group of Abelian central automorphisms on G' . Then $(E_1 \cap M)/(E_1 \cap C_G(G')) \subseteq M/C_G(G')$ is also such a group. Let $A_2 \subseteq \text{Aut}(G')$ be the corresponding group of automorphisms. Furthermore, $E/Z_2 = (E \cap M)/(E \cap C_G(G')) \subseteq A_2$ by Corollary 4.10 and Theorem 4.16. Let $A_1 \subseteq A_2$ be the corresponding group of automorphisms on G' .

Theorem 4.21. (a) If $\text{Exp}(Z(G')) = n < \infty$, then $\text{Exp}(A_2) \mid n$.

(b) If G' is a p -group, $A \subseteq A_2$ is periodic, then A is a p -group.

(c) If G' is polycyclic, then $A_1 \cong E/Z_2$ is finite.

Proof. (a) Suppose $Z(G')$ has exponent n . Then, for $x \in G'$, $\alpha \in A_2$, $1 = [x, \alpha]^n = [x, \alpha^n]$ by Theorem 4.8(b). Consequently, $\alpha^n = 1$ and A_2 has exponent dividing n .

(b) Now assume A is periodic. By Theorem 4.18(a) we may conclude that $[G', M, E_1] = [G', A, A] = 1$. Thus A stabilizes the normal series $1 \triangleleft [G', A] \triangleleft G'$ of G' . By the arguments used in [2; Corollary 5.33], we have that A is a p -group.

(c) Smirnov [19] has shown that a solvable group of automorphisms of a polycyclic group is polycyclic. By Theorem 4.16 we have that A_1 is a finitely generated periodic Abelian group. Hence A_1 is finite.

Theorem 4.22. If $Z(G) \cap Z(G')$ has no elements of order three, then $E = Z_2$.

Proof. We shall show that $A_1 = 1$. Let $\alpha \in A_1$. By Theorem 4.16, $E \subseteq Z_3$. Hence $[G', E] = [G', A_1] \subseteq Z(G) \cap Z(G')$. Then, by Corollary 4.14, $1 = [x, \alpha^3] = [x, \alpha]^3$. By hypothesis, this implies that $1 = [x, \alpha]$. Consequently $\alpha = 1$.

Theorem 4.23. Suppose $A_2 \neq 1$ is not torsionfree. Then G' has a proper subgroup of finite index.

Proof. For $1 \neq \alpha \in A_2$, the homomorphism from G' into $Z(G')$ defined by $f_\alpha(x) = [x, \alpha]$ for each $x \in G'$ is nontrivial. We may find an $a \in E_1 \cap M \setminus E_1 \cap C_G(G')$ such that $[x, \alpha] = [x, a]$ for all $x \in G'$. If α has finite order, then there is an integer n such that $a^n \in C_G(G')$.



Thus $1 = [x, \alpha]^n = [x, a^n]$ and $G'/\text{Ker } f_\alpha \cong f_\alpha(G') \subseteq Z(G')$ is a non-trivial direct sum of cyclic groups each of order bounded by n . In particular, there are subgroups H and C of G' such that $G'/\text{Ker } f_\alpha = H/\text{Ker } f_\alpha + C/\text{Ker } f_\alpha$ and $C/\text{Ker } f_\alpha$ is nontrivial and finite. Consequently $H < G'$ and $G'/H \cong C/\text{Ker } f_\alpha$ is finite.

Corollary 4.24. If $E > Z_2$, then G' has a proper subgroup of finite index.

Proof. If $E > Z_2$, then A_1 is a nontrivial torsion subgroup of A_2 by Theorem 4.16. Hence $A_2 \neq 1$ is not torsionfree and the theorem applies.

It is known that no complete, or even Černikov complete, group can have a proper subgroup of finite index (see [10: p. 234]). The following two corollaries come directly from this fact.

Corollary 4.25. If G' is Černikov complete, then $E = Z_2$.

Corollary 4.26. If G is metabelian and G' is divisible, then $E = Z_2$.

Theorem 4.27. If $G'/(G' \cap C_G(E_1 \cap M))$ is periodic, then A_2 has no elements of infinite height.

Proof. Suppose $a \in E_1 \cap M \setminus E_1 \cap C_G(G')$ is such that the induced automorphism $\alpha \in A_2$ has infinite height. Let $x \in G'$, and assume there is an integer n such that $x^n \in C_G(E_1 \cap M)$. By hypothesis, there is a $\beta \in A_2$ such that $\alpha = \beta^n$. Thus we may find $b \in E_1 \cap M \setminus E_1 \cap C_G(G')$, $y \in E_1 \cap C_G(G')$ such that $a = b^n y$. Then $[x, a] = [x, b^n y] = [x, b^n] = [x^n, b] = 1$ by Theorem 4.8(b) and $a \in E_1 \cap C_G(G')$, contrary to assumption.

We note that since A_1 has bounded exponent, it can have no elements of infinite height; certainly no nontrivial element of A_1

can be divided by $3 = \text{Exp}(A_1)$. Similarly we may show that A_2 has no elements of infinite height if $Z(G')$ has bounded exponent by Theorem 4.21(a).

Theorem 4.28. Assume $Z(G')$ is torsionfree. Then A_2 is torsionfree.

Proof. Let $1 \neq a \in A_2$, $o(a) = n < \infty$. Then there is an $x \in G'$ such that $1 \neq [x, a] \in Z(G')$. But $[x, a]^n = [x, a^n] = 1$ so that $o([x, a]) \mid n$. Hence a has infinite order. Since $A_1 \subseteq A_2$ is torsion, we must have that $A_1 = 1$ or $E = Z_2$.

The proof of Theorem 4.29 is an improvement on a result of Turner-Smith [22; Lemma 3.2], who has shown that for any automorphism group A on G , A has to have bounded exponent whenever the set $\{[x, a] : x \in G, a \in A\}$ is finite.

Theorem 4.29. If $a \in E_1$ is such that $S_a = \{[x, a] : x \in G'\}$ is finite, then $aZ(G) \in E_1/Z(G)$ has finite order. The orders of all such elements are bounded if the cardinalities of the sets are bounded.

Proof. Assume that S_a has n elements, $x \in G'$. Then at least two of the elements $[x, a], [x^2, a], \dots, [x^{n+1}, a]$ are the same. Hence there is an integer $k = k(x) \leq n$ depending on x such that $[x, a^k] = [x^k, a] = 1$ by Theorem 4.8(b). Since the k 's are bounded by n , we may assert that $[x, a^{n!}] = 1$ for all x . Consequently $a^{n!} \in Z(G)$.

Remark 4.30. We have proved that if $Z(G) \cap Z(G')$ has no elements of order three, or if G' has no proper subgroup of finite index, then $E(G) = Z_2(G)$. We shall now show that there exists a group G such that $Z_2(G) < E(G) < Z_3(G)$.

Let $H = \langle a_1, a_2, a_3; x^3 \rangle$. Levi and van der Waerden [11] have shown that H has nilpotence class exactly three and satisfies the law e_2 . Hence $E(H) = H = Z_3(H) > Z_2(H)$. Let K be any group of nilpotence class at least three having no elements of order three (see for example [17: p. 198]). By Theorem 4.22, $E(K) = Z_2(K) < Z_3(K) \subseteq K$. Let $G = H \times K$. By Theorem 1.2, we have $E(G) = E(H) \times E(K) = H \times Z_2(K)$. Consequently $Z_2(G) < E(G) < Z_3(G)$.

We have also shown that $E \subseteq E_1 = L(G)$. Define $N_A(G) = \bigcap \{N_G(H) : H \text{ maximal Abelian subgroup of } G\}$ to be the A-Norm (or Abelian-Norm) of G . Kappe [9] introduces this concept and proves that $a \in N_A(G)$ if and only if $[g, h] = 1$ for $g, h \in G$ implies that $[a, g, h] = 1$. By Corollary 4.10 it follows immediately that $E \subseteq N_A(G) \subseteq E_1$.

Definition 4.31. We shall say that a word ϕ satisfies the Schur-Baer property if $[G; \phi^*(G)] = m$ finite implies $\phi(G)$ finite with order which divides a power of m for all groups G .

Schur showed that γ_2 satisfies the Schur-Baer property; Baer extended this result to any outer commutator word ϕ (see [20]). We shall need the following theorem. For a proof (due to P. Hall), see [20: Theorem 2].

Theorem 4.32. If ϕ generates a locally residually finite variety, then ϕ satisfies the Schur-Baer property.

Theorem 4.33. If $\phi \in \{e_2, e_3\}$, then ϕ satisfies the Schur-Baer property.

Proof. Suppose $\phi = e_2$. A group in the variety generated by ϕ is nilpotent by Levi's Theorem. A finitely generated nilpotent group

is residually finite by P. Hall [6]. Consequently a finitely generated group in the variety generated by ϕ is residually finite and Theorem 4.33 applies.

Let $\phi = e_3$. Heineken [8] has shown that a group in the variety generated by ϕ is locally nilpotent. Hence a finitely generated group in this variety is also residually finite and the theorem follows as above.

Remark 4.34. We note that in P. Hall's proof of Theorem 4.33 we may sharpen the result somewhat if we put some restrictions on G itself. That is, if $\phi^*(G)$ is locally residually finite for all G in some class Σ such that $\Sigma = \{Q, S\}\Sigma$, then ϕ satisfies the Schur-Baer property for all $G \in \Sigma$. In particular, we have the following:

Theorem 4.35. If G satisfies the maximum or the minimum condition, or if G is an SN^* group, then e_n satisfies the Schur-Baer property for G .

Proof. Suppose G satisfies the maximum condition. Then, by [17: Theorem VI.8.j], we have that the set of left Engel elements (of all lengths) is the Hirsch-Plotkin radical R . Since then $e_n^*(G) \subseteq R$ is locally nilpotent, it is locally residually finite. By Remark 4.34, e_n satisfies the Schur-Baer property for G .

Vilyacer [23] has shown that an Engel group satisfying the minimum condition is locally nilpotent. Plotkin [14] has proved that an Engel group which is also an SN^* group is locally nilpotent. Hence the remainder of the theorem follows as above.

P. Hall has made the following three conjectures concerning arbitrary words ϕ and groups G (see Turner-Smith [21]):

- I. If ϕ is finite-valued in G , then $\phi(G)$ is finite.
- II. The word ϕ satisfies the Schur-Baer property.
- III. If G has the maximum condition on subgroups and $\phi(G)$ is finite, then $G/\phi^*(G)$ is finite.

It is not known whether these conjectures are universally true. We have shown that Conjecture II is satisfied for $\phi \in \{e_2, e_3\}$. Our results are more limited for these words and the other two conjectures.

We shall need the following lemma. The arguments follow those used in [22; Proposition 1].

Lemma 4.36. Suppose G is in a class of groups in which Conjecture II is satisfied locally for ϕ . If G is also locally residually finite, then ϕ and G satisfy Conjecture I.

Proof. Assume ϕ is finite-valued on G . Then there is a finitely generated subgroup H of G such that $\phi(H) = \phi(G)$. Since ϕ is finite-valued on H , we have that the set of elements of the form $t = \phi(h_1, \dots, ah_1, \dots, h_n)^{-1}\phi(h_1, \dots, h_1, \dots, h_n)$ for $a, h_1 \in H$, $1 \leq i \leq n$, is finite. Let $t_1, \dots, t_k$ be the nontrivial values. By hypothesis, H is residually finite. Hence we may find $L < H$ such that H/L is finite and $t_i \notin L$, $1 \leq i \leq k$. Let $a \in L$, $h_1, \dots, h_n \in H$. Then $w = \phi(h_1, \dots, ah_1, \dots, h_n)^{-1}\phi(h_1, \dots, h_1, \dots, h_n) \in L$. Since no such nontrivial element can be in L , we must have that $w = 1$ and $L \subseteq \phi^*(H)$. Hence $H/\phi^*(H)$ is finite and, since Conjecture II holds in H , we have that $\phi(H) = \phi(G)$ is finite.

Theorem 4.37. If $\phi \in \{e_2, e_3\}$ and G is locally residually finite, then ϕ and G satisfy Conjecture I.

Proof. By Theorem 4.33 we know that ϕ satisfies the Schur-Baer property. Hence Conjecture II is satisfied for ϕ in any group and the result follows from the lemma.

Remark 4.38. Conjectures I and III seem quite difficult to verify for Engel words. Conjecture I has been substantiated for nilpotent and solvable words but not for outer commutator words in general (see [16] and [21]). Conjecture III was proved for any outer commutator word by P. Hall (see [21]). We note that Theorem 4.36 verifies Merzljakov's [12] variant of the last conjecture for finitely generated residually finite groups G and $\phi \in \{e_2, e_3\}$:

IV. If ϕ is finite-valued on G (where G does not necessarily satisfy the maximum condition), then $G/\phi^*(G)$ is finite.

Merzljakov [12] proves that for an arbitrary word and an arbitrary linear group over a field all four conjectures are true. We note also that Turner-Smith [22] has shown that all three conjectures hold for every word ϕ and every group G in the class of groups whose homomorphic images are all residually finite—for example, the class of polycyclic groups.

Theorem 4.39. If G satisfies the maximum condition and $\gamma_n(G)$ is finite for some n , then $G/e_{n-1}^*(G)$ is finite.

Proof. By Remark 4.38 we know that Conjecture III holds for γ_n . Hence $G/\gamma_n^*(G)$ is finite. That $\gamma_n^*(G) \subseteq e_{n-1}^*(G)$ follows from the fact that $e_{n-1}^*(G) \subseteq \gamma_n(G)$ and Theorem 1.5. Hence $G/e_{n-1}^*(G)$ is finite.

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