EMBEDDING CANTOR SETS IN MANIFOLDS

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This is to certify that the

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ABSTRACT

EMBEDDING CANTOR SETS IN MANIFOLDS

by Richard Paul Osborne

This thesis is a study of the positional properties of Cantor sets in manifolds. Chapter I is essentially a generalization to E^n of Bing's work on tame Cantor sets in E^3. Characterizations of tame Cantor sets are given in terms of neighborhoods whose boundaries do not intersect the Cantor sets. It is also proved that the countable union of tame Cantor sets is tame.

The principal result of Chapter II is that each Cantor set in E^n lies on the boundary of an n-cell in E^n.

In Chapter III a very wild Cantor set is constructed in E^4. This Cantor set is then embedded in $S^2 \times S^2$ and it is shown that it lies in no open 4-cell in $S^2 \times S^2$. This shows that there is a simple closed curve in $S^2 \times S^2$ which bounds a disk but which lies in no open 4-cell.

EMBEDDING CANTOR SETS IN MANIFOLDS

By

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CHAPTER I

TAME CANTOR SETS IN E^n

The surprising properties of the Cantor ternary set and its homemorphic images have provided the topologist with some of his most provocative examples. For instance, the "necklace" of Antoine formed the basis of the first counterexample to the Schoenfliess conjecture in dimension three. With the current interest in the topology of n-dimensional Euclidean space E^n, the positional properties of Cantor sets have assumed new importance. Many recent results concerning tame and wild imbeddings in E^n depend upon such positional properties and it is the aim of this thesis to extend the knowledge of these Cantor sets and to apply this new knowledge to problems concerning E^n.

As far as possible throughout this thesis we will use C to represent n-cells, A to represent Cantor sets or sets used in the construction of Cantor sets, and superscripts to denote dimension.

<u>Definition 1.1</u>: A set $A \subset E^n$ will be called a Cantor set if it is a homeomorphic image of the Cantor ternary set on [0, 1].

The following well known theorem [23] is the principal tool used in constructing Cantor sets in E^n and will be used freely throughout this thesis without specific reference.

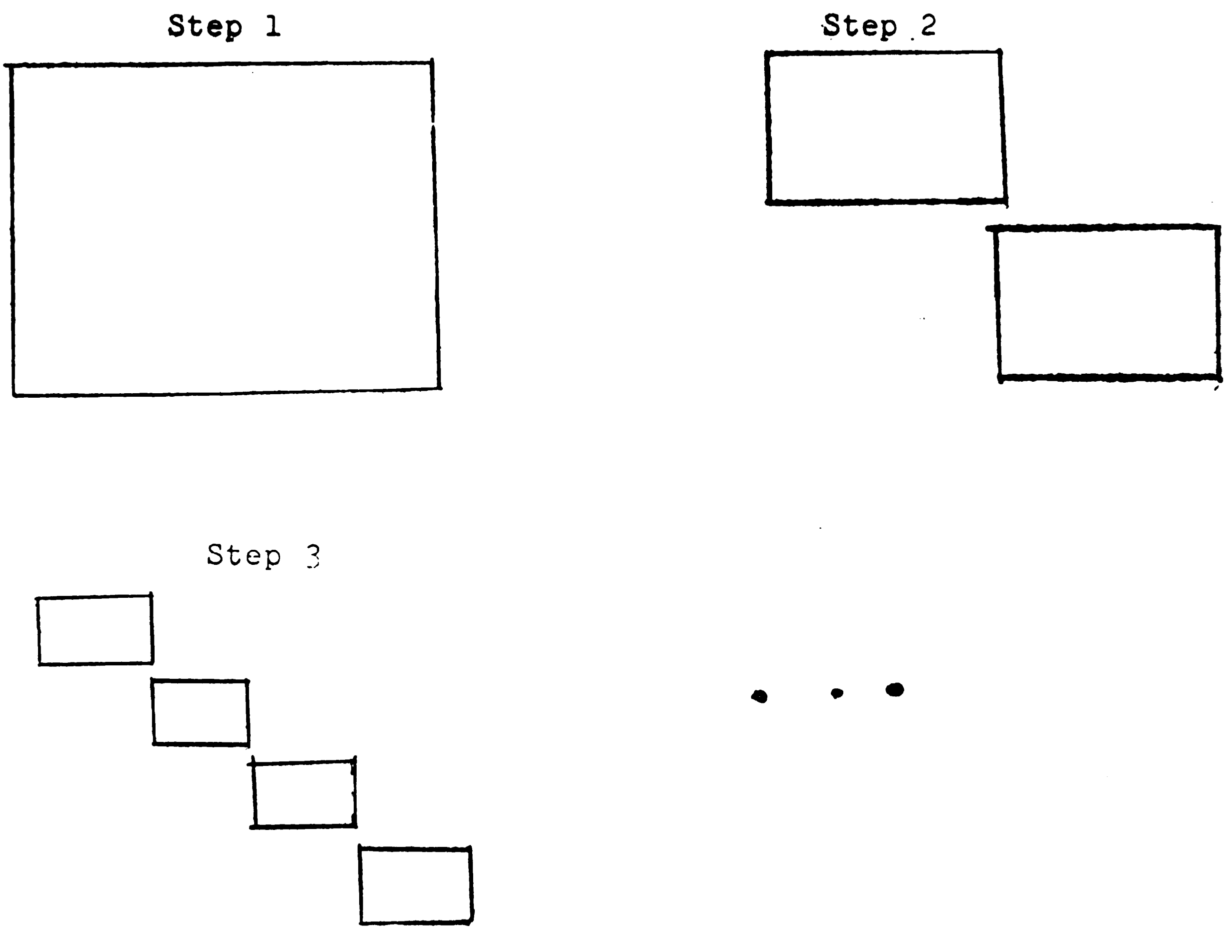
Theorem: Every 0-dimensional, compact, perfect, metric space is homeomorphic to the Cantor ternary set.

Definition 1.2: An arc $\alpha \subset E^n$ is tame if there is homeomorphism of E^n onto itself which maps α onto the unit interval on the positive X_1 axis in E^n.

Definition 1.3: (Bing) A Cantor set $A \subset E^n$ will be called tame if A lies on a tame arc in E^n.

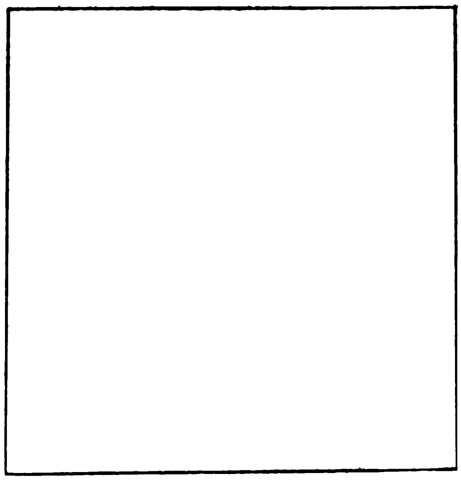
Cantor sets in E^n may have very strange properties. The following examples are all tame Cantor sets in E^n. In these examples we rely upon graphic illustrations of the first few steps in the constructions instead of attempting to write out the analytical expressions for the sets involved. In every construction of interest of a Cantor set in E^n the set is constructed as the intersection of a sequence of compact neighborhoods.

Example 1: A Cantor set in the plane (E^2) whose projection onto the x-axis covers the interval [0, 1].

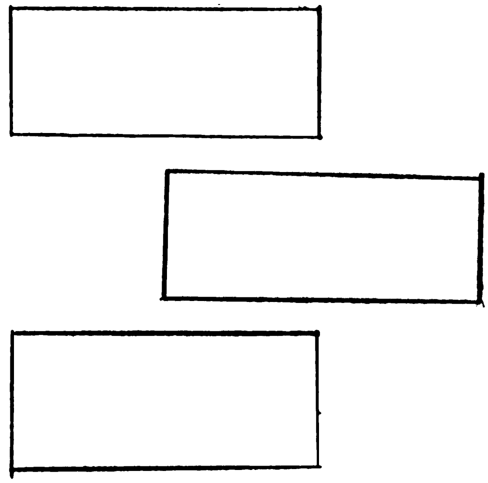


With only slight modifications we can get a Cantor set in the plane unit square which intersects every straight line passing through the top and bottom of the square. This example is essentially the one given by Bing in [7].

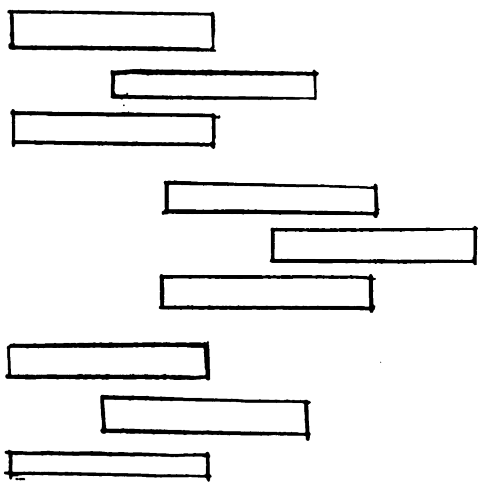
Step 1



Step 2



Step 3



Since the union of two Cantor sets is a Cantor set we can easily obtain from the above example a Cantor set in the plane which intersects every straight line cutting two opposite faces of the unit square.

This example can be generalized to the unit cube C^n in E^n. Let C_1', C_2', and C_3' be the three largest n-cubes in the unit cube C^n whose projections in the x_n, x_1 plane are the same as those in step 2 of the previous example. In each C_i' we embed 3 n-cubes $C_{i,1}'$, $C_{i,2}'$ and $C_{i,3}'$ in a fashion similar to the embedding of the C_i''s in C^n except that the roles of the X_1 and X_2 axes are interchanged. After (n-1) such steps we get 3^{n-1} n-cubes C_i, $i=1,2,---, 3^{n-1}$, in C^n whose maximum dimension in the direction of any axis is 2/3 and such that any straight line segment passing thru the faces $X_n=0$ and $X_n=1$ intersects the "top" and "bottom" of one of the 3^{n-1} embedded cubes. In each of these 3^{n-1} cubes C_1, C_2,---, $C_{3^{n-1}}$ we embed 3^{n-1} cubes in the same fashion as before to get $3^{2(n-1)}$ cubes whose maximum dimension along any of the axes is $(2/3)^2$. Continuing in this way we get a Cantor set A in C^n such that every straight line intersecting the faces $X_n=0$ and $X_n=1$ intersects A. The union of n such sets (one for each pair of faces) gives us a Cantor set in E^n which intersects every straight line meeting two opposite faces of the unit cube. As a matter of fact this example may be modified by choosing the starting cubes C_1, C_2, and C_3 properly to give a Cantor set which intersects every straight line intersecting the interior of unit cube.

Although the following theorem appears to be trivial and is well known I have not been able to find a proof given in the literature. The proof is worth presenting because the technique involved here of constructing a homeomorphism as a limit of a sequence of homeomorphisms is a powerful tool used throughout this thesis.

Theorem 1.1: If a Cantor set $A \subset E^n$ is tame then there exists a homeomorphism h of E^n onto itself mapping A onto the Cantor ternary set in the [0,1] on the X_1 axis.

Proof: If A is tame than A lies on a tame arc α and since α is tame there exists a homeomorphism g of E^n onto itself mapping α onto the interval [0,1] on the X_1-axis. We have, then, a set homeomorphic with the Cantor ternary set imbedded in [0,1]. We may assume without loss of generality that the extreme points of g(A) are 0 and 1. Since every homeomorphism of [0.1] onto itself can be extended trivially to E^n we need only show that there is a homeomorphism of [0,1] onto itself taking g(A) onto the Cantor ternary set. Choose a countable dense set $\{a_m\}$ from [0,1] -g(A) with the same order relation as the corresponding diadic rationals in (0, 1), i.e., if $m = b_1 + b_2 \cdot 2 + \dots + b_k 2^{k-1}$ then a_m corresponds to the diadic rational $b_1 . 2^{-1} + b_2 , 2^{-2} + \dots + b_k 2^{-k}$, where b_i = 0 or 1. Let I_m be the longest open interval in [0, 1] -g (A) containing a_m.

We define h_1 to be an order-preserving homeomorphism of $[0, 1]$ onto itself mapping I_1 onto $(1/3, 2/3)$. Suppose $m + 1 = b. + b_2 \cdot 2 + \ldots + b_k 2^{k-1}$. We choose h_{m+1} to be an order preserving homeomorphism of $[0, 1]$ onto itself mapping I_{m+1} onto the interval

$$h_m{}^{o}h_{m-1}{}^{o}\ldots{}^{o}h_1\left(\left(\frac{2b_1}{3^2} + \frac{2b_2}{3} + \ldots + \frac{2b_k}{3^k}, \frac{2b_1}{3} + \ldots + \frac{2b_k}{3^k} + \frac{2}{3^{k+1}}\right)\right).$$

and being the identity outside of

$$h_m{}^{o}h_{m-1}{}^{0}\ldots{}^{o}h_1\left(\left(\frac{2b_1}{3} + \frac{2b_2}{3^2} + \ldots + \frac{2b_k}{3^k}, \frac{2b_1}{3} + \ldots + \frac{2b_k+1}{3^k}\right)\right).$$

We define $h(x) = \lim_{m \to \infty} h_m o h_{m-1} o \ldots o h_1(x)$. h is 1-1 by construction and since h is the uniform limit of continuous functions h is continuous.

<u>Corollary 1.2</u>: If A_1 and A_2 are any two tame Cantor sets in E^n then there exists a homeomorphism of E^n onto itself mapping A_1 onto A_2.

We now wish to prove the following theorem which characterizes tame Cantor sets in E^n.

<u>Theorem 1.3</u>: A Cantor set $A \subset E^n$ is tame iff for each $\varepsilon > 0$ there exists a finite number of disjoint, tame n-cells $\{C_{\varepsilon,i}\}$ covering A such that diam $C_{\varepsilon,i} < \varepsilon$ and $B d\, C_{\varepsilon,i} \cap A = \emptyset$

To prove this theorem we shall need the following lemmas.

<u>Lemma 1.4</u>: C be an n-cell in the interior of an n-cell C^n and let $p \varepsilon$ Int C^n and U be an open neighborhood of p in Int C^n. There exists a homeomorphism h of C^n onto itself such that $h \mid \mathrm{Bd}C^n = \mathrm{id}$, $h(C) \subset U$ and $p \varepsilon$ Int $h(C)$.

<u>Proof</u>: Think of C^n as the set of points of E^n such that $\|x\| \leq 1$. If q and r are any two points of int C^n there exists a homemonphism g of C^n onto itself which is the identity on $\mathrm{Bd}C^n$ and $g(q) = r$. Now let q be the origin and let $g(p) = q$. We pick a point $r \varepsilon$ Int C and let $g'(r) = q$. We may now shrink $g'(C)$ by a homemorphism g'' of C^n onto itself into $g(U)$. Now $g^{-1}g''g'$ is the desired homeomorphism.

<u>Lemma 1.5</u>: Let C^n be a tame n-cell in E^n and let $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$ be sets of distinct points in Int C^n. There exist tame, disjoint n-cells $C_1, C_2, \ldots, C_k$ in Int C^n such that $\{p_i\} \cup \{q_i\} \subset$ Int C_i $i = 1, 2, \ldots, K$.

<u>Proof:</u> For each $i = 1, 2, \ldots, K$ we pass disjoint polyhedral arcs α_i from p_i to q_i and beyond so that p_i and q_i are not endpoints of α_i. Then we "swell up" each α_i into a tame n-cell. (This procedure of "swelling up" a polyhedral arc into an n-cell is a standard device in the literature).

Lemma 1.6: Let $C_1, C_2, \ldots, C_k$ be disjoint, tame n-cells in the interior of an n-cell C^n and let $p_1, p_2, \ldots, p_k$ be any k points in $(\text{Int } C^n - \bigcup_{i=1}^{k} C_i)$. Then there exist disjoint tame n-cells $C_1', \ldots, C_k'$ in Int C^n such $C_i \cup \{p_i\} \subset \text{Int } C_i'$ for each $i = 1, 2, \ldots, K$.

Proof: Choose k points $q_1, q_2, \ldots, q_k$ of C^n so that $q_i \varepsilon \text{ Int } C_i$. By Lemma 1.2 there exist disjoint n-cells $C_1'', C_2'', \ldots, C_k''$ such that $\{p_i\} \cup \{q_i\} \subset \text{Int } C_i''$. Since C_i is tame it is collared [11] on the outside by a collar A_i. Without loss of generality we may suppose $(C_i \cup A_i) \cap (C_j \cup A_j) = \emptyset$ for $i \neq j$. Now $C_i \cup A_i$ is an n-cell with C_i being a tame n-cell in Int $(C_i \cup A_i)$. By Lemma 1.1 we may shrink C_i by a homeomorphism h_i so that $h_i(C_i) \subset \text{Int } C_i''$ and $h_i | \text{Bd}(C_i \cup A_i) = \text{id}$. Now $h_i^{-1}(C_i'') = C_i'$ is the desired n-cell for each $i = 1, 2, \ldots, k$.

Lemma 1.7: Let $C_1, C_2, \ldots, C_k$ be tame, disjoint n-cells in Int C^n, let $p_1, p_2, \ldots, p_k$ be points of $C^n - \bigcup_{i=1}^{k} C_i$ and let $U_1, U_2, \ldots, U_k$ be disjoint neighborhoods of $p_1, p_2, \ldots, p_k$ respectively in $C^n - \bigcup_{i=1}^{k} C_i$. There exists a homeomorphism h of C^n onto itself such that $h | \text{Bd}C^n = 1$ and $p_i \varepsilon\ h(C_i) \subset U_i$ for $i = 1, 2, \ldots, k$.

Proof: By Lemma 1.5 there exist disjoint tame n-cells $C_1', C_2', \ldots, C_k'$ such that $\{p_i\} \cup C_i \subset \text{Int } C_i'$. By Lemma 1.4 there exists a homeomorphism g of C^n onto itself such that $p_i \varepsilon\ g(C_i) \subset U_i \cap C_i' \subset U_i$ and $g | (C^n - \bigcup_{i=1}^{k} C_i') = \text{id}$.

Proof of the main theorem: For each $k = 1, 2, 3 \ldots$ we choose a covering $\{C_{k,i}: i=1,\ldots N_k\}$ of A with the following properties: 1) $C_{k,i} \cap C_{k,j} = \emptyset$ for $i \neq j$, 2) $C_{k,i}$ is a tame n-cell with bicollared boundary for each k and i, 3) diam $C_{k,i} < 1/2^k$ for each $i = 1, 2, \ldots, N_k$ and 4) $C_{k+1, i} \subset \operatorname{Int} C_{k,j}$ for some $j = 1, 2, \ldots N_k$. The choice of such a sequence of coverings of A may be done inductively. For $k = 1$ choose a covering satisfying 1) thru 4) by using the hypothesis of the theorem. Suppose $\{C_{k,i}: i=1,\ldots N_k\}$ has been chosen. Since $\bigcup_{i=1}^{N_k} \operatorname{Bd} | (C_{k,i}) \cap A=\emptyset$ there exists $N > 0$ such that $d((\bigcup_{i=1}^{N_k} \operatorname{Bd} C_{k,i}), A) > N$. Now we cover A by a set of disjoint, tame n-cells of diameter less than min $(1/2^{k+1}, N)$. This then gives us the desired set $\{C_{k+1,i}: i=1,\ldots,N_{k+1}\}$ It is clear that $\bigcap_{k=1}^{\infty} (\bigcup_{i=1}^{N_k} C_{k,i}) = A$.

Next we define a set of homeomorphisms $\{h_k\}$ of E^n onto itself. Let C_1 be a tame n-cell in E^n with $\bigcup_{i=1}^{N_1} C_{1,i} \subset \operatorname{Int} C_1$ and let α be a tame arc in C_1 which intersects the interior of each $C_{1,i}$. Define $h_1 = \text{id}$. Assume now that h_k has been defined so that i) α contains points in the interior of $h_k{}^0 h_{k-1}{}^0 \ldots h_1(C_{k,i})$ for each $i = 1, 2, \ldots, N_k$, ii) $h_k | E^n - (\bigcup C_{k-1,i}) = \text{id}$, iii) h_k moves no point farther than $1/2^{k-1}$ and iv) diam $h_k{}^0 h_{k-1}{}^0 \ldots {}^0 h_1 (C_{k,i}) < 1/2^k$. Now since α intersects $\operatorname{Bd} C_{k,i}$ for each $i = 1, 2, \ldots, N_k$ it follows that in each $C_{k,i}$ there are points of α in $C_{k,i}$ not in $\bigcup_{i=1}^{N_{k+1}} C_{k+1,i}$. Now we renumber the set $\{C_{k+1,i}: i = 1, 2, \ldots, N_{k+1}\}$ using another parameter j so that $C_{k+1,j,i} \subset C_{k,j}$ for $i = 1, 2, \ldots, M_j$. Let $\{p_{k+1,j,i}: i = 1, \ldots, M_j\}$ be points of $(\alpha \cap \text{Int } C_{k,j} - \bigcup_{i=1}^{M_j} C_{k+1,j,i}$ and let $\{U_{k+1,j,i}: i = 1, 2, \ldots, M_j\}$ be a set of disjoint neighborhoods of the points $\{p_{k+1,j,i}: i - 1, 2, \ldots, M_j\}$ lying in $C_{k,j} - \bigcup_{i=1}^{M_j} C_{k+1,j,i}$. By Lemma 1.6 there exists a homeomorphism g_j of $C_{k,j}$ with the desired properties i)---iv) for each j. We define $h_{k+1} = g_1 \circ g_2 \circ \cdots \circ g_{Nk}$.

Finally we define h by $h(x) = \lim_{k\to\infty} h_k \circ h_{k-1} \circ \cdots \circ h_1(x)$. If $X \notin A$ then there exists N such that for $k > N$ $x \notin \bigcup_{i=1}^{N_k} C_{k,i}$ hence h_k leaves x fixed. We see that h is a homemorphism on $E^n - A$. Let $x, y \in A$ and $d(x,y) > 1/2^k$ then if $x \in C_{k,i}$ $y \notin C_{k,i}$ it follows that $h(x) \neq h(y)$. We must yet show that h is continuous but this follows from the fact that h is the uniform limit of a sequence of continuous functions. Clearly then, h is a homeomorphism of E^n onto itself such that $h(x) \in \alpha$ for each $x \in A$. This follows from the fact that

$$d(h_k \circ h_{k-1} \circ \cdots \circ h_1(x), \alpha) < 1/2^k$$

Note: In the hypothesis of the previous theorem we specified that the coverings of A be composed of tame n-cells. This was not necessary, for, given any n-cell C^n, C^n can be approximated from the inside by a tame n-cell. To see this let B_r^n denote the ball in E^n of radius r, let g be a homeomorphism of B_1^n into E^n and let A be a compact subset of Int $(g(B_1^n))$. There is a $\delta > 0$ such that Int $(g(B_{1-\delta}^n))$ contains A. $g(B_{1-\delta}^n))$ is a bicollared (n-1) sphere in E^n hence by the generalized Schoenflies theorem [10] $g(B_{1-\delta}^n)$ is a tame n-cell containing A in its interior.

Theorem 1.8: The union of two disjoint, tame Cantor sets in E^n is tame.

Proof: Let A_1 and A_2 be disjoint tame Cantor sets and suppose that $d(A_1,A_2) = \delta$. Let $\varepsilon > 0$ be given. Cover A_1 by a set of disjoint open n-cells of diameter less than min $(\delta/2,\varepsilon)$. Cover A_2 similarly. We then have an covering of $A_1 \cup A_2$ by disjoint n-cells. An application of Theorem 1.7 completes the proof.

Theorem 1.9: Let A be a Cantor set in E^n. A is tame iff A lies on a tame (n-1) - sphere in E^n.

Proof: If A is tame then A lies on a tame arc α. Let h be a homeomorphism of E^n onto itself mapping α onto the unit interval on the x_1-axis. The boundary S of the unit cube in E^n contains $h(\alpha)$ hence $h^{-1}(S)$ is a tame (n-1) sphere containing α.

Suppose now that A lies on a tame (n-1)-sphere S. Let $C \subset S$ be an (n-1)-cell such that $A \subset$ Int C. Since S is tame C is tame so there is a homeomorphism h of E^n onto itself such that $h(C)$ is a subset of the hyperplane $x_n = 0$. Let α be an arc in $h(C)$ containing A. Klee has given a homeomorphism of E^n onto itself mapping such an arc into the x_n-axis. Evidently then α is tame, hence A is tame.

The next theorem establishes that the union of two tame Cantor sets is tame. A generalization of the process used in the proof of this theorem will be used later to show that if a Cantor set is the countable union of tame Cantor sets it is tame.

The following lemmas are needed in the proof of the result mentioned above:

Lemma 1.10: Let $h':I' \to I'$ be a homeomorphism of I' leaving the endpoints of I' fixed, let I' be the unit interval on the X_1-axis in E^n and let C^n be the n-cell in E^n defined by

$$C^n = \{x\colon x = (x_1, x_2, \ldots, x_n) \text{ and } x_2^2 + x_3^2 + \ldots + x_n^2 \leq 1 \text{ and } 0 \leq x_1 \leq 1\}.$$

Then h' can be extended to a homeomorphism h of E^n which is the identity outside of C^n.

Proof: Let $S = \{x\colon x = (x_1, x_2, \ldots, x_n), x_2^2 + x_3^2 + \ldots, + x_n^2 = 1 \text{ and } x_1 = 1/2\}$. For each point $x \in S$ and each $r \in I'$ let $\ell_{x,r}$ be the line segment joining x and r. Now define

$$h(y) = \begin{cases} y \text{ if } y \notin \ell_{x,r} \text{ for same } x \in S \text{ and } r \in I' \\ y' \text{ if } y \in \ell_{x,r} \text{ where } y' \in \ell_{x,h(r)} \text{ and } d(y',I') = d(y,I'). \end{cases}$$

h is the desired homeomorphism.

Lemma 1.11: Let C^n be an n-cell in E^n, $n \neq 4$. C^n can be approximated by a polyhedral n-cell, i.e., given $\varepsilon > 0$ there exists a polyhedral n-cell, P^n, such that $d(x, BdC^n) < \varepsilon$ for all $x \varepsilon BdP^n$ and $P^n \subset C^n$.

Proof: We use the theorems due to Bing [8] and Connell [18] which say that for $n \neq 4$ stable homeomorphisms of E^n can be approximated by piecewise linear homeomorphisms. Let Δ_n be an n-simplex in Int C^n. Now Int C^n - Int Δ_n is a half open annulus [10] so there is a tame n-cell $D^n \subset C^n$ such that D^n - Int Δ_n is an annulus and $d(x, BdC^n) < \varepsilon/2$ for every $x \varepsilon BdD^n$. Since D^n-Int Δ_n is an annulus there exists a stable homeomorphism h of E^n mapping Bd Δ_n onto BdD^n. Using the aforementioned theorem we approximate h to within $\varepsilon/2$ by a piecewise linear homeomorphism f. Then $f(\Delta_n)$ is the desired piecewise linear (polyhedral) cell.

Theorem 1.12: The union of two tame Cantor sets in E^n, $n \neq 4$, is tame.

Proof: Let A_1 and A_2 be tame Cantor sets in E^n. A_1 lies on a tame arc α_1 and A_2 lies on a tame arc α_2. We may think of α_2 as being on the x_1-axis in E^n and we assume that the endpoints of α_2 are not in $A_1 \cup A_2$. Let $\varepsilon > 0$ be given and let $A_3 = \alpha_2 \cap (A_1 \cup A_2)$. A_3 is a subset of a tame Cantor set on α_2. Blow α_2 up into an n-cell C^n given by

$x{:}d(x_1, \alpha_1) \leq \varepsilon/3$. Let $x_1, x_2, \ldots, x_{k+1}$ be a finite set of points of $\alpha_2 - A_3$ such that $0 < x_{i+1} - x_i < \varepsilon/2$ for $i = 1, 2, \ldots, k$ and x_1 and x_k are the endpoints of α_2. We define $C_i' = \{y{:}y = (y_1, y_2, \ldots, y_n),\ x_i \leq y_i \leq x_{i+1},\ y \varepsilon C^n\}$ for $i = 1, 2, \ldots, k$. Evidently the C_i's are n-cells of diameter less than ε which cover α_2. Shrink each C_i' to form a new n-cell $C_i'' \subset C_i'$ so that $C_i' - \mathrm{Int}\ C_i''$ is an annulus and $A_3 \cap C_i'' = A_3 \cap C_i'$. By Lemma 1.11 there exists a polyhedral (with respect to α_1) n-cell $\mathcal{C}_i$ such that $C_i'' \subset \mathrm{Int}\ \mathcal{C}_i$ and $\mathcal{C}_i \subset \mathrm{Int}\ C_i'$. If necessary we rotate $\mathcal{C}_i$ slightly to get an n-cell C_i^* such that $C_i'' \subset \mathrm{Int}\ C_i^* \subset \mathrm{Int}\ C_i'$ and $\mathrm{Bd}\ C_i^* \cap \alpha_1$ is a finite set P_i. For each $p \varepsilon P_i$ choose Cp to be a small n-cell containing p contained Int $C_i' - C_i''$ and so that for any $q \varepsilon P_i$ $Cq \cap Cp = \emptyset$. Let $h_{i,p}'$ be a homeomorphism of $Cp \cap \alpha_1$, onto itself which leaves the endpoints of Cp $_1$ fixed and maps a point of $(Cp \cap \alpha_1) - A_1$ onto p. By Lemma 1.10 h_{ip}' can be extended to a homeomorphism h_{ip} of E^n which is the identity outside of Cp.

Finally we define $h_i = \pi_{p \varepsilon Pi}\ h_{ip}$ and set $C_i = h_i^{-1}\ (C_i^*)$. The set $\{C_i{:}i = 1, \ldots, K\}$ is a disjoint collection of n-cells which cover $\alpha_2 \cap (A_1 \cup A_2)$ such that $\mathrm{Bd}C_i \cap (A_1 \cup A_2) = \emptyset$. The remaining points of $A_1 \cup A_2 -- \bigcup_{i=1}^{k} C_i$ can be written as the union of two disjoint tame Cantor sets hence by Theorem 1.8 is a tame Cantor set. Since $A_1 \cup A_2 - \bigcup_{i=1}^{k} C_i$ is tame it may be covered by a disjoint system of n-cell of diameter less than ε whose intersection with $\bigcup_{i=1}^{k} C_i$ is void.

A study of local properties of embeddings of Cantor sets in E^n is of some interest in its own right and will enable us to prove the global theorem on the union of tame Cantor sets which is a generalization of Theorem 1.12.

Definition 1.4: A Cantor set $A \subset E^n$ is said to be locally tame at $x \epsilon A$ if there exists a neighborhood N_x of x such that $A \cap N_x$ is a tame Cantor set.

Theorem 1.13: A Cantor set $A \subset E^n$ is tame iff it is locally tame at each of its points.

Proof: Suppose A is locally tame at each of its points. It follows from the definition of local tameness that A may be covered by a finite set N_1, N_2, ..., N_k of open subsets of E^n such that $N_i \cap A$ is a tame Cantor set and Bd $N_i \cap A = \emptyset$. Define $N_1' = N_1$ and $N_i' = N_i - \bigcup_{j=1}^{k} N_j$. Since $N_i' \subset N_i$ and $N_i \cap A$ is tame, it follows that $N_i' \cap A$ is tame (or empty). The collection $\{N_i' : i = 1, 2, \ldots, k\}$ is an open cover of A such that $N_i' \cap N_j' = \emptyset$ for $i \neq j$. Thus $A = \bigcup_{i=1}^{k} (N_i' \cap A)$ is a decomposition of A into a finite number of disjoint tame Cantor sets which by Theorem 1.8 is tame. The proof of the converse is trivial.

Definition 1.5: An arc $\alpha \subset E^n$ will be called locally tame at $x \epsilon \alpha$ if there exists an open n-cell neighborhood N_x of x and a homeomorphism $h: N_x \to E^n$ such that $h(N_x) = E^n$ and $h(N_x \cap \alpha)$ is the x_1 axis in E^n.

One might have expected that local tameness of a Cantor set would be defined in terms of local tameness of arcs. That such a definition is equivalent to that given is shown by the following theorem.

Theorem 1.14: A Cantor set $A \subset E^n$ is locally tame at $x \varepsilon A$ iff A lies on an arc which is locally tame at x.

Proof: Suppose A lies on an arc α which is locally tame at x. Let N_x be a neighborhood of x and $h: N_x \to E_n$ be a homeomorphism of N_x onto E^n such that $h(N_x \cap \alpha)$ lies on the x_1-axis. Now let N'_x be a compact neighborhood of h(x) such that $\text{Bd } N'_x \cap h(A \cap N_x) = \emptyset$. Then $h(A) \cap N'_x$ is a tame Cantor set. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $d(x,y) < \delta$ implies $d(h^{-1}(x), h^{-1}(y)) < \varepsilon$ for each x and y in N'_x. Cover $h(A \cap N_x) \cap N'_x$ by a disjoint family $\{C_k\}$ of n-cells of diameter less than δ such that $(\bigcup \text{Bd } C_k) \cap h(A \cap N_x) = \emptyset$ The family $\{h^{-1}(C_k)\}$ is a covering of $A \cap h^{-1}(N'_x)$ by a disjoint collection of n-cells of diameter less than ε such that $(\bigcup \text{Bd } h^{-1}(C_k)) \cap A = \emptyset$. By Theorem 1.3 $A \cap h^{-1}(N'_x)$ is a tame Cantor set.

Conversely suppose A is tame at x. Let N_x be a neighborhood of x such that $N_x \cap A$ is a tame Cantor set. We may suppose that N_x is an n-cell such that $\text{Bd } N_x \cap A = \emptyset$. Let α be a tame arc containing $N_x \cap A$. Let α' be an arc in $E^n - N_x$ containing $A - N_x$. α and α' are disjoint arcs so they may be connected to get an arc α''. α'' contains A and is locally tame at x.

In recent papers by Cantrell and Edwards [16] and by Cantrell [14] it has been shown that if an arc in E^n, $n \geq 4$, is wild it must fail to be locally tame at an entire Cantor set of points. Papers [12], [13] and [15] have been written by Cantrell in which a principal objective is to establish the analogous result for S^{n-1} in E^n, $n \geq 4$, i.e., if an (n-1)-sphere S^{n-1} in E^n, $n \geq 4$, is wild then S^{n-1} fails to be locally flat on a Cantor set of points (see [11] for the definition of local flatness for spheres) Although this statement has not yet been proved it has been shown to be related to a generalized annulus conjecture [15]. One might wonder what sort of wildness properties a Cantor set in E^n could have. Could a Cantor set be wild at just one point? The following set of theorems is aimed at answering such questions. Although these theorems are the same in statement to those established by Bing [7] for E^3 the proofs used by Bing could not be generalized to the case of E^n, $n>3$.

Theorem 1.15: If A is a Cantor set in E^n which is locally tame at each of its points with the possible exception of a single point $x_0 \in A$, then A is tame.

Proof: In [7] Bing established this theorem for $n = 3$; consequently we assume that $n \geq 4$. Let $\{N_i\}$ be a decreasing sequence of open neighborhoods of x_0 such that $\text{Bd } N_i \cap A = \emptyset$

and diam $N_i < 1/i$ for each i = 1, 2, 3, ... Now since $A_i = (N_i - N_{i+1}) \cap A$ is locally tame at each of its points A_i is a tame Cantor set for each i. For each i let $C_{i,1}$, $C_{i,2}$, ..., C_{i,k_i} be a disjoint collection of n-cells such that Bd $C_{i,j} \cap A = \emptyset$, $C_{i,j} \subset N_i - N_{i+1}$ and $A_i \subset \bigcup_{j=1}^{k_i} C_{i,j}$. It is easily verified that the n-cells may be so chosen. For each $C_{i,j}$ let $\alpha_{i,j}$ be a tame arc in C_{ij} containing $C_{ij} \cap A$. Define $B_k = \{x : x \in E^n$ and $d(x, x_o) \leq 1/k\}$, let $>$ be an ordering on pairs of integers defined by $(i, j) > (m, \ell)$ if $i > m$ or $i = m$ and $j > \ell$, and choose arcs β_{ij} as follows: let β_{11} be a tame arc in $E^n - \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{k_i}$ joining an endpoint of α_{11} to an endpoint of α_{12}. Suppose now that arcs $\beta_{i,j}$ have been chosen for each $i < m$, let $\beta_{m,\ell}$ be a tame arc in $B_{m-1} - \left(\bigcup_{(i,j) > (m,\ell+1)} C_{ij} \right) - \bigcup_{(i,j) < (m,\ )} (\alpha_{i,j} \cup \beta_{ij})$ joing the free endpoint of $\alpha_{m,\ell}$ with an endpoint of $\alpha_{m,\ell+1}$ for $\ell = 1, 2, \ldots, k_m - 1$ and let β_{m,k_m} be a tame arc in $B_m - \bigcup_{(i,j) > (m+1,1)} C_{i,j} - \bigcup_{(i,j) < (m,km)} (\alpha_{ij} \cup \beta_{ij})$ joining the free endpoint of α_{m,k_m} to a free endpoint of $\alpha_{m+1,1}$. (See Figure 3.)

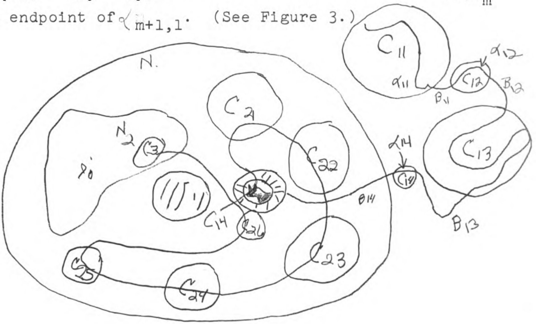


Figure 3

Let $\alpha = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{Ki} (\alpha_{ij} \cup B_{ij}) \cup \{x_0\}$. Then α is a metric continuum with exactly two non-cut points, hence is an arc. By the method of construction α is locally flat except possibly on a countable set. By the result of Cantrell [14] such an arc in E^n, $n \geq 4$ must be tame.

Corollary 1.16: The set of points at which a Cantor set $A \subset E^n$ is wild can contain no isolated point.

Proof: Suppose a Cantor set A is wild at the point x_0 and suppose there is a neighborhood N_{x_0} of x_0 such that $A \cap N_{x_0}$ is locally tame at each point except x_0. The Cantor set $A \cap N_{x_0}$ contradicts the previous theorem.

Corollary 1.17: The set of points at which a wild Cantor set $A \subset E^n$ fails to be locally tame is a Cantor set.

Proof: Let W denote the set of points of A at which A fails to be locally tame. From the definition of local tameness it is clear that W is a closed subset of A and by Corollary 1.16, W contains no isolated points. It follows that W is closed and dense-in-itself hence W is a Cantor set.

Theorem 1.18: If a Cantor set $A \subset E^n$ is locally tame at each of its points with the possible exception of the points of a tame Cantor set B in A, then A is tame.

Proof: Let $\varepsilon > 0$ be given. Using Theorem 1.7 we shall show that A is tame by covering it by a set of disjoint n-cells of diameter less than ε whose boundaries do not intersect A. Cover B by disjoint n-cells $C_1, C_2, \ldots, C_k$ of diameter less than $\varepsilon/2$ such that $B \cap (\bigcup_{i=1}^{k} \mathrm{Bd}\, C_i) = \emptyset$. Let $0 < 2\delta < \min_{i \neq j} d(C_i, C_j)$. We restrict our attention now to a particular C_i. Let $0 < \delta_i < d(B, \mathrm{Bd}C_i)$, let $\eta_i = \min(\varepsilon/2, \delta, \delta_i)$ and let $N_i = \{x : d(\mathrm{Bd}C_i, x) \leq \eta_i\}$. The Cantor set $A \cap N_i$ is tame at each of its points hence it is tame by Theorem 1.13. Cover $A \cap N_i$ by disjoint n-cells of diameter less than η_i whose boundaries do not intersect $A \cap N_i$. Let $C_{i,1}, C_{i,2}, \ldots, C_{i,m}$ be the set of all such n-cells containing a point of $\mathrm{Bd}C_i \cap A$. Next we choose homeomorphisms $h_{i,1}, h_{i,2}, \ldots, h_{i,m}$ with the properties: 1) $h_{i,j} | E^n - C_{i,j} = \mathrm{id}$ and 2) $h_{i,j}(A \cap C_{ij}) \cap \mathrm{Bd}C_i = \emptyset$ Define $h_i = h_{i,1} \circ h_{i,2} \circ \cdots h_{i,m}$. Now $h_i^{-1}(C_i)$ is an n-cell of diameter less than ε such that $\mathrm{Bd}\, h_i^{-1}(C_i) \cap A = \emptyset$. If we have defined h_i for each i and we define $h = h_1 \circ h_2 \circ \cdots \circ h_k$, then $h^{-1}(C_1), h^{-1}(C_2), \ldots, h^{-1}(C_k)$ is a covering of B by disjoint n-cells of diameter less than ε such that $A \cap (\bigcup_{i=1}^{k} \mathrm{Bd}h^{-1}(C_i)) = \emptyset$. Since $A - (\bigcup_{i=1}^{k} h^{-1}(C_i))$ is tame it may be covered by disjoint n-cells of diameter less than ε which do not intersect $\bigcup_{i=1}^{k} h^{-1}(C_i)$.

Corollary 1.19: Each wild Cantor set in E^n contains a Cantor set which is wild at each of its points.

Proof: Let A be a wild Cantor set in E^n and suppose A fails to be locally tame on a Cantor set W. Then W must be wild at each of its points, for if W were locally tame at $x \varepsilon W$ then there would be a neighborhood N_x of x such that $N_x \cap W$ is tame. But then $A \cap N_x$ is locally tame except for the points of a tame Cantor set, contradicting Theorem 1.17.

Corollary 1.20: The set of points at which a Cantor set is wild is empty or is a Cantor set which is not locally tame anywhere.

Using the previous results we may prove the following theorem on the union of tame Cantor sets in E^n. This theorem was given by Bing [7] for E^3 and the proof now generalizes easily. It is repeated here for the sake of completeness.

Theorem 1.21: If the Cantor set $A \subset E^n$ is the countable union of tame Cantor sets A_1, A_2, ... then A is tame.

Proof: If A were wild then by Corollary 1.18, A would contain a Cantor set A' which is wild at each of its points. The Baire-Moore theorem tells us that no compact Hausdorff space is the union of a countable number of closed subsets, no one of which contains an open subset of the space (for a proof see[23]). So A' must contain a Cantor set A" which is open in A' and which lies in one A_i. But the A' is not locally tame at any of its points. This contradicts the fact that A_i is locally tame at each of its points.

CHAPTER II

AN EXTENSION THEOREM FOR HOMEOMORPHISMS ON CANTOR SETS

In 1921 L. Antoine [3] gave an example of a Cantor set in E^3 whose complement was not simply connected. This then was the first known example of a wild embedding of a Cantor set in E^n. Shortly thereafter (1924) J. W. Alexander [1] showed that the Cantor set of Antoine, often called Antoine's necklace, was contained in a 2-sphere in E^3 disproving the Schoenflies theorem for E^3. Concurrently Alexander [2] gave an example of a 2-sphere in E^3 which was wild at a tame Cantor set of points. In 1949 Artin and Fox [5] constructed 2-spheres in E^3 which were wild at a single point. Shortly thereafter (1951) Blankinship [9], a student of Fox, published a paper in which he generalized the construction of Antoine's necklace to E^n for any $n \geq 3$, i.e. he constructed Cantor sets in E^n whose complements were not simply connected. In this same paper he showed that these generalized necklaces must lie on the boundary of a K-cell, $0 < K \leq n$; thus giving a method for constructing wild K-cells and spheres in E^n.

In this chapter we shall show that every Cantor set in E^n, $n \geq 2$, lies on the boundary of a K-cell in E^n. This theorem is a direct extension for E^n of the well known theorem [25]: Any 0-dimensional, compact subset of a Peano space lies on an arc.

We shall need the following lemmas.

Lemma 2.1: Let U be a component of the set V in a locally connected space X. Then $BdU \subset BdV$ and if V is open then U is open.

Proof: Let $x \varepsilon BdU$; then for each neighborhood N_x of x in X N_x contains points of U and U' (the complement in X of U). If x were not a boundary point of V then there would be an open connected neighborhood N_x of x which was contained in V. Then $U \cup N_x$ is a connected subset of V properly containing U, contrary to the assumption that U was a component of V. If V is open then $BdV \cap V = \emptyset$ so $BdU \cap U = \emptyset$ and U is open.

Lemma 2.2: Let U be a bounded, connected, open subset of E^n and let A be a compact subset of U. Then there exists a polyhedron $P \subseteq U$ such that $A \subset Int P$ and Int P is connected.

Proof: Let $\delta = \min_{x \varepsilon A} d(x, BdU)$. Triangulate E^n by a triangulation T of mesh less than $\delta/2$ and let P' be the polyhedron composed of all simplexes of T contained in the star of a simplex containing a point of A. Let P_1, P_2,---, P_k be the closures of the components of the interior of P'. Since U is connected there is an arc α joining each of the polyhedra $P_1, P_2, \cdots, P_k$.

Let $\delta' = \min_{x\in\alpha} d(x, \mathrm{Bd}U)$, let T' be a refinement of T of mesh less than $\delta'/2$ and define P" to be the set of all simplexes of T' which are contained in the star of a simplex which contains a point of α. Finally define $P = P' \cup P''$. P is then the desired polyhedron.

Lemma 2.3: Let $\varepsilon>0$ be given and let A be a compact, 0-dimensional subset of E^n. Then there exists a finite collection of disjoint, open, connected subsets $\{U_i: i = 1,2,\cdots,K\}$ of E^n which cover A and such that 1) diam $U_i < \varepsilon$, 2) $\overline{U}_i$ is a polyhedron and 3) $E^n - \overline{U}_i$ is connected.

Proof: First we select an open neighborhood N_x of each point x of A of diameter less than ε such that $\mathrm{Bd}\, N_x \cap A = \emptyset$. Select from this cover of A a finite subcover $N_1, N_2, \cdots, N_p$. Now let $N = \bigcup_{\ell=1}^{P} N_\ell - \bigcup_{\ell=1}^{P} (\mathrm{Bd}N_\ell)$. By Lemma 2.1 each component V_j of N is an open set whose boundary is in $\mathrm{Bd}N = \bigcup_{\ell=1}^{P} \mathrm{Bd}N_\ell$; thus $\mathrm{Bd}V_j \cap A = \emptyset$. The set $\{V_j\}$ is an open cover of A hence there is a finite subcover $\{V_i: i=1,2,\cdots,m\}$ of A by sets of $\{V_j\}$. We now have a finite covering $\{V_i: j=1,2.\cdots,m\}$ of A by disjoint, open, connected subsets of E^n, each of diameter less than ε such that $\mathrm{Bd}V_i \cap A = \emptyset$. Now for each i apply Lemma 2.2 to get a polyhedron in V_i whose interior W_i is connected and contains $V_i \cap A$. Finally for each i let U_i be the complement of the unbounded component of $E^n - W_i$. If $U_i \subset U_j$ for $i \neq j$ drop this set from the list of polyhedra covering A. We now have the desired covering of A.

Lemma 2.4: Let P and Q be disjoint compact polyhedra in E^n both of which are the union of n-simplexes and let α be a polyhedral arc with endpoints p and q such that $\alpha \cap P = \{p\}$ and $\alpha \cap Q = \{q\}$ and q is in the interior of an (n-1)-simplex σ of Q. Then α can be "blown up" into a polyhedral n-cell C such that $C \cap P$ and $C \cap Q$ are (n-1)-cells in BdP and BdQ respectively and for a given $\varepsilon > 0$ $d(x, \alpha) < \varepsilon$ for any $x \varepsilon C$.

Proof: Let α_1 be a line segment in E^n and let P_1 and P_2 be (n-1)-dimensional hyperplanes in E^n intersecting α_1 at its endpoints a_1 and a_2 respectively. Let σ_1 be an (n-1)-simplex in P_1 containing p and not intersecting P_2. Then the set C_1 consisting of all points lying on line segments parallel to α_1 with one endpoint in σ_1 and the other endpoint on P_2 is a polyhedral n-cell. To see that C_1 is, indeed, an n-cell is a straightforward but messy computation in analytic geometry.

Now, starting at p, number the linear segments of α in order: $\alpha_1, \alpha_2, \cdots, \alpha_k$. Let $p = a_1, a_2, \cdots, a_{k-1} = q$ be the set of endpoints of $\alpha_1, \alpha_2, \cdots, \alpha_k$ where a_i and a_{i+1} are endpoints of α_i. Let σ_1 be an (n-1)-simplex in BdP such that $\text{diam}\, \sigma_1 < \varepsilon$, $p \varepsilon \sigma$, and α_1 does not lie in the (n-1)-dimensional hyperplane determined by σ_1. Let P_2 be a (n-1) dimensional hyperplane intersecting α_1 and α_2 at a_2. Applying the remarks of the above paragraph blow α_1 up into an n-cell C_1 such that $BdC_1 \cap P_1$ and $BdC_2 \cap P_2$ are (n-1)-simplexes. Now apply the entire process again to α_2 and $BdC_1 \cap P_2$. We continue expanding in this fashion until we come to α_k. Choose $\sigma_{k+1} \subset \sigma$ so that $q \varepsilon \sigma_{k+1}$ and so the n-cell generated by lines parallel to α_k with endpoints in σ_{k+1} intersects C_{k-1} in an (n-1) = simplex. $C = \bigcup_{i=1}^{k+1} C_i$ is the desired n-cell.

Theorem 2.5: Let A be a Cantor set in E^n. Then A lies on the boundary of an n-cell $C^n \subset E^n$. Furthermore C^n can be so chosen that A is tamely imbedded in BdC^n. (Note that C^n itself may well be wild in E^n and in fact C^n must be wild if A is wild.)

Proof: Let C_0 be a polyhedral n-cell in E^n whose distance from A is 1. Let $P_{11}, P_{1,2}, \cdots, P_{1,k_1}$ be disjoint polyhedra of diameter less than 1/2 such that Int $P_{1,i}$ is connected and $A \subset \bigcup_{i=1}^{k_1}$ Int $P_{1,i}$ (Lemma 2.3). Let α be a polygonal arc in BdC_0, let $x_{1,1}, x_{1,2}, \cdots, x_{1,k1}$ be k_1 distinct points of α and let $y_{1,1}, y_{1,2}, \cdots, y_{1,k_1}$ be k_1 points such that $y_{1,i}$ lies in the interior of an (n-1)-simplex on the boundary of $P_{1,i}$. Choose disjoint polyhedral arcs $\alpha_{1,1}, \alpha_{1,2}, \cdots, \alpha_{1,k_1}$ so that the endpoints of $\alpha_{1,i}$ are $x_{1,i}$ and $y_{1,i}$ and so that $\alpha_{1,i} \cap C_0 = \{x_{1,i}\}$ and $\alpha_{1,i} \cap \bigcup_{j=1}^{k} P_{1,i}) = \{Y_{1,i}\}$. Applying Lemma 2.4 blow each $\alpha_{1,i}$ up into a polyhedral n-cell $C_{1,i}$ such that $C_{1,i} \cap (\bigcup_{j \neq i} C_{1,j}) = \emptyset$, $C_{1,i}$ C_o is a polyhedral (n-1)-cell and $C_{1,i} \cap P_{1,i}$ is a polyhedral (n-1)-cell. Let $T_o = C_o$. Let h_1 be a homeomorphism of C_o onto $C_1 = C_0(\bigcup_{i=1}^{k_1} C_{1,i})$ such that $h_1\ (X_{1,i}) = Y_{1,i}$.

Suppose now that the sets $P_{m,1}, P_{m,2}, \cdots, P_{m,k_m}$, $C_{n,1}, C_{m,2}, \cdots, C_{m,k_m}$ and C_m together with T_m and h_m have been defined. For the moment we restrict our attention to a polyhedron $P_{m,i}$. Applying Lemma 2.3 we get disjoint polyhedra $P^i_{m+1,1}, P^i_{m+1,2}, \cdots, P^i_{m+1,\ell_i}$ in Int $P_{m,i}$ of diameter less than $1/2^{m+1}$ whose interiors cover $P_{m,i} \cap A$. Let $f_m = h_m \circ h_{m-1} \circ \cdots \circ h_1$ and choose distinct points $X^i_{m+1,1}$, $X^i_{m+1,2}, \cdots X^i_{m+1,\ell_i}$ of $f_m\ (\alpha) \cap \text{Bd}\ P_{m,i}$. For each $j = 1,2, \cdots, \ell_i$ let $Y^i_{m+1,i}$ be a point in the interior of an (n-1)-simplex on the boundary of $P^i_{m+1,j}$. At this point, in order to avoid ever increasing numbers of subscripts or superscripts, we shall reorder the $P^i_{m+1,j}$'s, $j = 1,2, \ldots, \ell_i$, $i = 1,2, ---, k_m$ lexicographically in (i,j) (and correspondingly the $X^i_{m+1,j}$'s and $Y^i_{m+1,j}$'s). So we now have an ordering using two subscripts $P_{m+1,1}, P_{m+1,2}, --- P_{m+1,\ k_{m+1}}$. Now let $\alpha_{m+1,1}, \alpha_{m+1,2}, ---, \alpha_{m+1,\ k_{mh}}$ be disjoint polyhedral arcs such that 1) $x_{m+1,i}$ and $y_{m+1,i}$ are the endpoints of $\alpha_{m+1,i}$, 2) $\alpha_{m+1,i} \subset P_{m,j_i}$ for some j_i, 3) $\alpha_{m+1,i} \cap \text{Bd}\ P_{m,j_i} = \{x_{m+1,i}\}$, 4) $\alpha_{m+1,i} \cap \text{Bd}(P_{m+1,i}) = \{y_{m+1,i}\}$ and 5) $\alpha_{m+1,i} \cap (\bigcup_{\ell \neq i} P_{m+1,\ell}) = \emptyset$ (See Figure 4). Next we wish to define the set T_{m+1}. Let $S_{\varepsilon,x} = \{y\colon y\varepsilon C_o$ and $d(x,y) \leq \varepsilon\}$ and let $Z_{m+1,i} =$

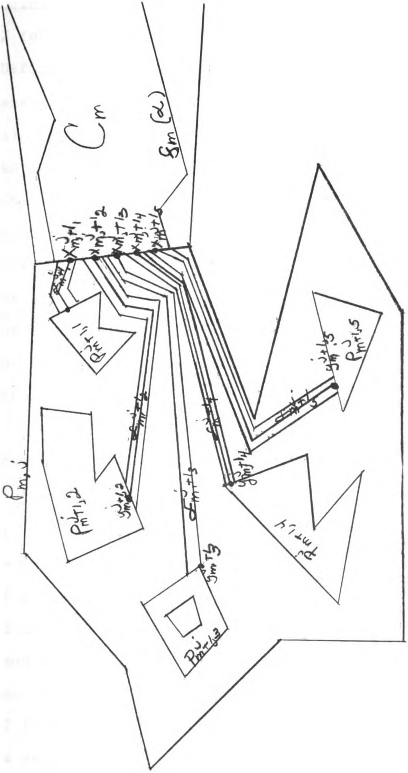


Figure 4.

$f_m^{-1}(x_{m+1,i})$. Choose $\varepsilon_{m+1} > 0$ small enough so that $S_{\varepsilon_{m+1}}, Z_{m+1,i} \subset \text{Int } T_m$, $\varepsilon_{m+1} < 1/2^{m+1}$ and $S_{\varepsilon_{m+1}}, Z_{m+1,i} \cap S_{m+1,j} = \emptyset$ for $i \neq j$.

Define $T_{m+1} = \bigcup_{i=1}^{k_{m+1}} S_{\varepsilon_{m+1}}, Z_{m+1,i}$. Applying Lemma 2.4 blow each $\alpha_{m+1,i}$ up into a polyhedral n=cell $C_{m+1,i}$ such that 1) $C_{m+1,i} \cap C_{m+1,j} = \emptyset$ for $i \neq j$, 2) $C_{m+1,i} \cap C_{m,j_i}$ is an (n-1)-cell in $f_m(T_{m+1})$, 3) $C_{m+1,i} \cap P_{m+1,i}$ is an (n-1)-cell. Let $C_{m+1} = C_m \cup (\bigcup_{i=1}^{k_{m+1}} C_{m+1,i})$ and choose a homeomorphism $h_{m+1}: C_m \twoheadrightarrow C_{m+1}$ of C_m onto C_{m+1} such that $h(x_{m+1,i}) = y_{m+1,i}$ and $h_{m+1} | C_m - f(T_{m+1}) = \text{id}$. Finally we define $f(x) = \lim_{m\to\infty} f_m(x)$. Since f is the uniform limit of a sequence of continuous functions f is continuous. Because the domain of f is C_o, a compact set, we need only show that f is 1-1 to establish that f is a homeomorphism. Clearly $T_{m+1} \subset T_m$ and $T = \bigcap_{m=1}^{\infty} T_m$ is a Cantor set in BdC_o. For any point $x \in C_o = T$ there exists an N such that for m>N $x \notin T_m$ thus for all m>N $f_m(x) = h_m(f_{m-1}(x)) = f_{m-1}(x)$ so $f = f_N$ in a neighborhood of x and f is a homeomorphism in a neighborhood of x. We see that f is 1-1 on $C_o - T$. Since f is continuous $f(BdC_o)$ is compact. Now for any point $a \in A$ $d(a, f_m(BdC_o)) < 1/2^m$ hence $d(a, f(BdC\ )) = 0$ so $a \in f(BdC)$ and $A \subset f(BdC_o)$. Because $f_m(BdC_o) \cap A = \emptyset$ for each m and h_m, m=1,2,--- is eventually the identity on each $z \notin T$ it follows that $A \subset f(t)$. Since for each $z \in T$ there exists a sequence $\{z_m\}$ of points from the set $\{z_{m,i} : i=1,2,---, X_m\}$ such that $d(z_m, z) < 1/2^m$ and $d(f_m(z_m), A) < 1/2^m$. Let $\varepsilon > 0$ be given, by uniform continuity of f there is a $\delta>0$ such that for $d(x,y)<\delta$ $d(f(x),f(y))<\varepsilon/2$. Choose m large enough so that $1/2^m<\delta$ and $1/2^{m-2}<\varepsilon/2$. We have

$$\begin{aligned} d(f(z),A) &\leq d(f(z), f(z_m)) + d(f(z_m), f_m(z_m)) \\ &\quad + d(F_m(z_m), A) \\ &\leq \varepsilon/2 + 1/2^{m-1} + 1/2m. \end{aligned}$$

It follows that $f(z)\subset A$ hence $f(T)\subset A$: $f(T)=A$. Now let $y\neq z$ be another point of T and let $\{Y_m;\ m=1,2,\text{---}\}$ be a sequence of points from $\{z_{m,i}:\ i=1,2,\text{---}k_m,\ m=1,2,\text{---}\}$ such that $d(y_m,y)<1/2^m$. There exists N such that for $m>N$ $d(z_m, y_m)>\gamma>0$. Now since $\lim_{m\to\infty} f_m(z_m)=f(z)$ and $\lim_{m\to\infty} f_m(y_m)=f(y)$ and from the fact that $f_m(y_m)$ and $f_m(z_m)$ are eventually in distinct, disjoint polyhedral neighborhoods it follows that $f(y)\neq f(z)$.

Finally we want to show that $A\subset f(\alpha)$. This follows from the fact that $d(f_m(\alpha), a)<1/2^m$ for each $a\varepsilon A$.

Note that $f(C_o)$ is an n-cell which is polyhedral except at the points of A.

At first glance the above theorem may not so appear but it is an extension theorem which may be stated thus:

Corollary 2.6: Let f^1 be a homeomorphism mapping the Cantor ternary set on the x_1-axis in E^n into E^n. f^1 can be extended to a homeomorphism f of the unit cube C^n in E^n into E^n.

If f^1 can be extended to C^n it can surely be extended to any face of C^n, thus:

Corollary 2.7: Each Cantor set in E^n is tamely imbedded in the boundary of a K-cell in E^n for $0<K\leq n$.

Note that if f^1 could be extended farther to a neighborhood of the unit interval on the x_1- axis then A would be tame.

It is not difficult to see that Theorem 2.5 may be generalized to piecewise linear manifolds. The analogs of the Lemmas 2.2, 2.3, and 2.4 are easily established and the proof of Theorem 2.5 follows as before. Thus we have

Theorem 2.8: Let M^n be a piecewise linear n-manifold. Then each Cantor set in M^n lies on the boundary of an n-cell in M^n.

It might have been tempting to assert that each Cantor set in an n-manifold lies in some open n-cell. However the arguments of [21] show that this is not the case.

Without the piecewise linear structure and Lemmas 2.2, 2.3, and 2.4 are meaningless so a proof of Theorem 2.5 for manifolds without piecewise linear structure would involve a proof quite different from the one used for piecewise linear manifolds. Perhaps the local linear structure could be used.

Corollary 2.9: Let A be the Cantor ternary set on the real line and let $f:A \longrightarrow M^n$ be a homeomorphism of A into a piecewise linear manifold M^n. Then f is homotopically trivial.

Proof: f may be extended to a map F of the unit square D^2 to M^n. Since $F(BdD^2)$ is homotopically trivial f is.

A generalization of Corollary 2.9 to continuous maps of Cantor sets into Peano continua is possible. One need only show that each map of the Cantor set into a Peano continuum can be extended to the cone over the Cantor set.

CHAPTER III

A TAME CANTOR SET WHICH LIES IN NO OPEN N-CELL

Characterizing spaces by certain properties of the set of homeomorphisms of the space onto itself is not new in topology. Such properties as homogeneity and near-homogeneity have long been used to characterize simple closed curves in the plane. In 1960 Hocking and Doyle characterized the n-sphere [19] by a property called invertibility. (A space S is invertible if for every open set $U \subset S$ there is a homeomorphism h of S onto itself such that $h(S - U) \subset U$.) They showed that an invertible n-manifold is an n-sphere and that a weakly invertible open n-manifold is E^n. (A space S is weakly invertible if for each open set $U \subset S$ and each compact set $C \subset S$ there is a homeomorphism h of S onto itself such that $h(C) \subset U$). As a natural generalization of weak invertibility Hocking and Doyle undertook the study of what was called weak dimensional invertibility [21]. (An n-manifold M^n is weakly k-invertible if every compact subset of M^n of dimension k lies in an open n-cell in M^n). By the use of a theorem of Stallings [26] it is easily shown that if $k > [n/2]$ then a k-invertible, compact, combinatorial n-manifold is an n-sphere. A very surprising theorem proved in [21] states that a 0-invertible 3-manifold is S^3. This theorem may be restated as follows: Let M^3 be a compact 3-manifold such that each compact, 0-dimensional set in M^3 lies in an open 3-cell. Then M^3. is a 3-sphere. In [21] it was observed that in all of the decided cases an $(n - 3)$-invertible, compact, combinatorial n-manifold is an n-sphere, the only undecided case being $n = 4$.

It is natural then to attempt to find an example of a compact, combinatorial 4-manifold M^4 with the property that every 0-dimensional, compact subset of M^4 lies in an open 4-cell in M^4. In [21] Hocking and Doyle indicated that M^4 would have to be simply connected.

With these facts in mind it is natural to conjecture that each compact, 0-dimensional subset of $S^2 \times S^2$, the topological product of 2-spheres, lies in an open 4-cell. It is the purpose of this chapter to show that this is not the case, i.e. that $S^2 \times S^2$ contains a Cantor set which lies in no open 4-cell.

<u>Definition 3.1:</u> In the space $S^n \times E^m$ any set of the form $\{x\} \times E^m$ where $x \varepsilon S^n$ will be called a parameter m-plane. If P is a parameter m-plane in $S^n \times E^m$ and $\{h_t\}$ is an isotopy of $S^n \times E^2$ onto itself then $h_1(P)$ will be called a curved parameter m-plane. In $S^n \times S^m$ a parameter m-sphere and a curved parameter m-sphere are similarly defined.

<u>Theorem 3.1</u>: Let $A \subset S^n \times S^m$ be compact. If A intersects every curved parameter m-sphere then A lies in no open $(n + m)$-cell in $S^n \times S^m$.

<u>Proof</u>: Suppose A lies in an open $(m + n)$-cell C, then given $\varepsilon > 0$ there is an isotopy $\{h_t\}$ of $S^n \times S^m$ onto itself such that $h_1(A)$ has diameter less than ε and $h_t | S^n \times S^m - C = id$. Let S^n and S^m be metrized in the metric which they inherit as unit spheres in E^{n+1} and E^{m+1} respectively. Metrize $S^p \times S^q$ by the standard product metric i.e. $d((x,y), (X',y')) = ([d_n(x,x')]^2 + [d_m(y,y')]^2)^{1/2}$ where d_n and d_m are the metrics for S^n and S^m respectively. Now $d(\{x\} \times S^m, \{x'\} \times S^m) = d_m(x,x')$. If we choose x' so that $d_n(x,s') > \varepsilon$ then $\{x\} \times S^m$ and $\{x'\} \times S^m$ cannot intersect the same set of diameter less than ε, i.e. they cannot both intersect $h_1(A)$. Suppose $\{x\} \times S^m$ does not intersect $h_1(A)$ then $h_1^{-1}(\{x\} \times S^m)$ is a curved parameter m-sphere which does not intersect A.

A similar construction will establish the following:

<u>Theorem 3.2</u>: Let $A \subset S^n \times E^m$ be compact. If A intersects every curved parameter m-plane then A lies in no open $(p + q)$-cell in $S^n \times E^m$.

The above theorem makes it clear that if an example could be given of a Cantor set which intersects every curved parameter plane in $S^n \times E^2$ then such a Cantor set could not lie in an open $(n + 2)$-cell. Such a Cantor set "approximating" S^2 in $S^2 \times E^2$ will be constructed.

In giving the construction of a Cantor set in E^4, which will be used in "approximating" a 2-sphere and in verifying the desired properties of it we shall use the following lemmas, the first is due to Blankinship [9], the second is a generalization of Artin's work in [4].

Lemma 3.3: Let d_r, d_s, D_s be arbitrary real numbers with $0<d_s \leq D_s$. Let S be a compact set in E^n contained in the set defined by $x_r = d_r$ and $0<d_x \leq x_s \leq D_s$. Let $\tilde{S}$ be the set generated by rotating S about the (n-2)-plane defined by $x_r = d_r$, $x_s = 0$, or more explicitly

$\tilde{S} = \{x \mid x \in E^n$ and there exists $y \in S$ and there exists Θ such that $x_i = y_i$ if $i \neq r$ or s and $x_r = d_r + y_s \sin\Theta$, $x_s = y_s \cos\Theta\}$

Then

a) for each $(y,\Theta) \in S \times E'$ (mod 2π), the correspondence $(y,\Theta) \longrightarrow x$ where $x_i = y_i$, $i \neq r$ or s, $x_r = d_r + y_s \sin\Theta$, $x_s = y_s \cos\Theta$ is a homeomorphism onto $\tilde{S}$. We can therefore use the pair (y,Θ) as a set of coordinates for $\tilde{S}$.

b) if $\tilde{U}$ is the set in $\tilde{S}$ consisting of all points with representations (u,Θ) for which $u \in U \subset S, \alpha \leq \Theta \leq \beta, \beta - \alpha \leq 2\pi$ where $U \neq \emptyset$ then $\max\{\operatorname{diam} U, d_s\, p(\beta - \alpha)\} \leq \operatorname{diam} \tilde{U} \leq \operatorname{diam} U + D_s\, p(\beta-\alpha)$

where

$$p(\Theta) = \begin{cases} 2\sin 1/2\, \Theta & \text{if } 0 \leq \Theta \leq \pi \\ 2 & \text{if } \pi \leq \Theta \leq 2\pi \end{cases}$$

c) if $x \varepsilon \tilde{S}$ then $d_r - D_s \leq x_r \leq d_r + D_s$.

Lemma 3.4: Let E^n be the hyperplane in E^{n+1} defined by $x_{n+1} = 0$. Let $\bar{E}^n_+$ be the half space in E^n defined by $x_n \geq 0$, let S be a set in $\bar{E}^n_+$ and let C be a simple closed curve in the complement of S. Let $\tilde{S}$ be the set in E^{n+1} which we get by rotating S about the hyperplane defined by $x_n = x_{n+1} = 0$, i.e.

$\tilde{S} = \{x: \exists\, y \in S$ and $\exists \theta \ni 0 \leq \theta < 2\pi$ and $x_i = y_i$ for $i \neq n$ or $n+1$, $x_n = y_n \cos\theta$, $x_{n+1} = y_n \sin\theta\}$

Then C is null homotopic in $E^{n+1} - \tilde{S}$ iff C is null homotopic in $\bar{E}^n_+ - S$.

Proof: If C is null homotopic in $\bar{E}^n_+ - S$ then certainly C is null homotopic in $E^{n+1}_+ - \tilde{S}$. Conversely we define the continuous map $f: E^{n+1} \longrightarrow \bar{E}^n_+$ by $f(x) = y$ if x is the image of y under a rotation of $\bar{E}^n_+$ about the hyperplane $x_n = x_{n+1} = 0$. It is clear from the definition that $x \varepsilon \tilde{S}$ iff $f(x) \varepsilon S$. Suppose C is null homotopic in $E^{n+1} - \tilde{S}$, i.e. that C bounds a singular disk in $E^{n+1} - \tilde{S}$. Let $g: D^2 \longrightarrow E^{n+1} - \tilde{S}$ be a continuous map such that $g|BdD^2$ is a homeomorphism of BdD^2 onto C. Then because f is the identity on C, $fg: D^2 \longrightarrow \bar{E}^n_+ - S$ is a continuous map such that $fg|BdD^2$ is a homeomorphism of BdD^2 onto C. Thus C bounds a singular disk in $\bar{E}^n_+ - S$ and hence C is null homotopic in $\bar{E}^n_+ - S$.

It should be remarked that the properties of linked sets established in [24] and [9] will be heavily relied upon in verifying the properties of the Cantor sets constructed. In a sense the Cantor set constructed will be a generalization of Antoine's construction and indeed Antoine's necklace is the Cantor set used in E^3 to approximate the one sphere. The Cantor set to be constructed here is not, however, the same as that constructed by Blankinship in [9]. The Cantor sets of Blankinship could be used to "approximate" a surface homeomorphic to $S^1 \times S^1 \times - - - \times S^1$ (n - 2 factors) in E^n.

The Construction

Let S^1 be the unit circle in the x_1,x_2-plane in E^4. Rotate S^1 about the 2-dimensional hyperplane defined by $x_2 = x_4 = 0$ to get a 2-sphere in the x_1, x_2, x_4-hyperplane. Direct calculation using Lemma 3.3 shows that we get the 2-sphere whose equations are $x_1^2 + x_2^2 + x_4^2 = 1$ and $x_3 = 0$. This will be the 2-sphere which is approximated by the Cantor set to be constructed.

Expand S^1 into a solid torus T^3 homeomorphic to $S^1 \times D^2$ where D^2 is the two dimensional disk, T^3 lying in the x_1, x_2, x_3-hyperplane. We shall use only the half of T^3 with $x_2 \geq 0$ and we shall refer to it as $\overline{T}^3$.

In T^3 embed four cyclically linked solid tori T_1^3, - - -, T_4^3 (figure 5) so that the diameter of T_1^3 and T_2^3 is less than 2/3 the diameter of T^3. Rotate $\overline{T}^3$ about the plane defined by $x_2 = x_4 = 0$ to get T^4 which is homeomorphic to $S^2 \times D^2$. In T^4 the rotated images of $\overline{T}_1^3 = T_1^3 \cap \overline{T}_2^3$, $\overline{T}_2^3 = T_2^3 \cap \overline{T}$ and $\overline{T}_3^3$ will be T_1^4, T_2^4 and T_3^4 respectively where T_1^4 and T_2^4 are

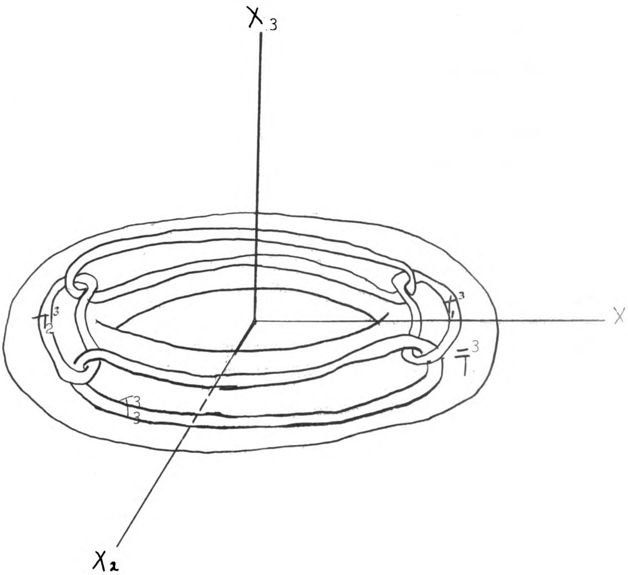


Figure 5.

homeomorphic to T^4 and T_3^4 is homeomorphic to $S' \times S' \times D^2$.

In T_3^4 construct a generalized Antoine's necklace $\tilde{A}$, as done by Blankinship [9]. Let g be the linear map of E^4 which shrinks T^4 to the size of T_1^4 and T_2^4 and leaves the center of the image at the origin. Let f_1 and f_2 be Euclidean motions such that $f_1 g$ maps T^4 onto T_1^4 and $f_2 g$ maps T^4 onto T_2^4. Define $g^k = gogo\ldots og$ (k factors) and let $f_{k,i} = g^k f_i (g^{-1})^k$. Now set

$$A_1' = \tilde{A}_1$$

$$A_2' = A_1' \cup f_{o,1} g(\tilde{A}_1) \cup f_{o,2} g(\tilde{A}_1)$$

$$A_3' = A_2' \cup f_{1,1} f_{o,1} g^2(\tilde{A}_1) \cup f_{1,2} f_{o,1} g^2(\tilde{A}_1) \cup f_{1,1} f_{o,2} g^2(\tilde{A}_1) \cup f_{1,2} f_{o,2} g^2(\tilde{A}_1)$$

$$A_4' = A_3' \cup \bigcup_{i,j,k=1,2} f_{2,i} f_{1,j} f_{o,k} g^3(\tilde{A}_1)$$

.

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Define the sequence $\{A_\alpha : \alpha = 1,2,3, \ldots\}$ by

$$A_o = T^4$$

$$A_1 = A_1' \cup T_1^4 \cup T_2^4$$

$$A_2 = A_2' \cup f_{o,1} g(T_1^4) \cup f_{o,2} g(T_1^4) \cup f_{o,1} g(T_2^4) \cup f_{o,2} g(T_2^4)$$

$$A_3 = A_3' \cup \left(\bigcup_{i,j,k=1,2} f_{1,i} f_{o,j} g^2(T_k^4) \right)$$

.

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Finally define $A = \bigcap_{\alpha=1}^{\infty} A_\alpha$. A is the desired Cantor set. Note that A is the union of a countable union of linked generalized necklaces together with limit points. These limit points constitute a tame Cantor set in the x_1, x_4-hyperplane.

Compactness of A follows from compactness of A_α for each α and from the fact that $A_\alpha \subset A_{\alpha - 1}$.

A is certainly 0-dimensional at each point of $\bigcup_{i=1}^{\infty} A_i'$ and for each point x of $A - \bigcup_{i=1}^{\infty} A_i'$ there is a neighborhood of x of the form $h_{i_1} h_{i_2} \cdots h_{i_k} (T^4)$ which has diameter less than $(2/3)^k$ times the diameter of T^4 and whose boundary does not intersect A. It follows that A is 0-dimensional at each of its points; hence A is 0-dimensional.

That A is perfect is not difficult to establish although it is of no interest to us in the arguments to follow.

Let C be the circle in the x_1, x_3-plane which is the boundary of the intersection of this plane with T^3.

<u>Lemma 3.5</u>: C is not null homotopic in $E^4 - A$.

Proof: By the theorems of [17] we see that C is not null homotopic in $\bar{E}^3 - \bigcup_{i=1}^{3} T_i^3$. Lemma 3.4 then assures us that C is not null homotopic in $E^4 - \bigcup_{i=1}^{3} T_i^4$.

Let $B_1^n = T_1^n \cup T_2^n \cup T_3^n$

$$B_2^n = T_1^n \cup f_{o,1} g(B_1^n) \cup f_{o,2}\, g(B_1^n)$$

$$B_3^n = T_1^n \cup (\bigcup_{i=1,2} f_{o,i} g(T_3^n)\ (\bigcup_{i,j=1,2} f_{1,i} f_{o,j} g^2(B_1^n))$$

.

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. $\qquad n = 3$ or 4

Set $B^n = \bigcap_{\alpha=1}^{\infty} B_\alpha^n$. Note that B^4 is like A except that the generalized Antoine's necklaces of Blankinship have not been substituted for the 4-tubes.

The sets B_α^4, $\alpha = 1, 2, 3, ---$ and B^4 can be constructed in yet another way: by rotating B_α^3 and B^3 about the x_1,x_3-plane. Now by Theorem 1 of [17] we see that C is not null homotopic in $\bar{E}^3 - B_\alpha^3$ for each . It follows that C is not null homotopic in $\bar{E}^3 - B^3$. For if it were then it would bound a singular disk D in $\bar{E}^3 - B^3$. But such a disk would lie a positive distance from B^3 hence it would not intersect $\bar{E}^3 - B_\alpha^3$ for a sufficiently large α. This contradicts the fact that C is not null homotopic in $\bar{E}^3 - B_\alpha^3$. Applying Lemma 3.4 we see that C is not null homotopic in $E^4 - B^4$.

The homotopy relations computed in [9] assure us that replacing T_3^4 in B^4 by the generalized Antoine's necklace does not change the homotopic non-triviality of C in $E^4 - B^4$. (This can be proven by an argument similar to that used in Lemma 3.10.)

Define:

$$H_1 = B_1^4$$
$$H_2 = (T_3^4 \cap A) \cup (\bigcup_{i=1,2} f_{o,i} g)$$
$$H_3 = (T_3^4 \cup (\bigcup_{i=1,2} f_{o,i} g(T_3^4)) \cap A) \cup (\bigcup_{i,j,=1,2} f_{i,j} f_{o,1} g^2(B_1^4))$$

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By repeated use of the above remark we can conclude that C is not null homotopic in $E^4 - H_\alpha$, $\alpha = 1, 2, 3, ---$.

Suppose now that C is null homotopic in $E^4 - A$, i.e. that C bounds a singular disk D in $E^4 - A$. Let the distance from D to A be greater than diam $T^4 \cdot (2/3)^k$. Then D cannot intersect H_k, a contradiction. This proves the lemma.

Definition 3.2: Let $h : T^4 \longrightarrow S^2 \times D^2$ be a surjective homeomorphism $n = 3,4$, let $A \subset T^4$ be the Cantor set constructed above and let $o \varepsilon$ Int D^2. Then h(A) will be said to approximate $S^n \times \{o\}$ in $S^n \times D^2$. h(A) also approximates $S^n \times \{o\}$ in $S^n \times \text{Int } D^2 = S^n \times E^2$.

As an immediate consequence of the construction of approximating Cantor sets we get the following theorem.

Theorem 3.6: $S^2 \times E^2$ contains a Cantor set which lies in no open 4-cell.

Proof: Let A approximate $S^2 \times \{o\}$ in $S^2 \times D^2$. Then the one-sphere C which is the boundary of $\{p\} \times D^2$ for $p \varepsilon S^n$ is not null homotopic in $(S^2 \times D^2) - A$, hence neither C not any of its homotopic images bounds a disk in the complement of A. Thus each parameter disk in $S^2 \times D^2$ intersects A. Applying Theorem 3.1 we get the desired result.

Corollary 3.7: $S^2 \times E^2$ is not 0-invertible.

Bing [5] has given an example of a simple closed curve in $S^1 \times E^2$ which bounds a 2-cell but lies in no 3-cell. We may now prove the following:

Theorem 3.8: There is a simple closed curve in $S^2 \times E^2$ which bounds a disk but lies in no open 4-cell.

Proof: Let A be a Cantor set in $S^2 \times E^2$ approximating a parameter S^2. Using Theorem 2.8 construct a disk D^2 whose boundary C contains A. Since A lies in no 4 cell C lies in no 4-cell.

The following theorem provides a negative answer to the question which was the genesis of this paper, namely, is $S^2 \times S^2$ 0-invertible?

Theorem 3.9: In the 4-manifold $S^2 \times S^2$ there exists a Cantor set which lies in no open 4-cell.

Proof: Let $S_1^{\ 2}$ and $S_2^{\ 2}$ be 2-spheres, let K be an annular region about the equator of S_2^2, let D_1 and D_2 be the closures of the complementary domains of K and let S_1 and S_2 be the boundaries of D_1 and D_2 respectively. Let A_1 be a Cantor set in $S_1^{\ 2} \times D_1$ which approximates $S_1^{\ 2} \times \{p_1\}$, $p_1 \in \operatorname{Int} D_1$, and let A_2 be a Cantor set in $S_1^{\ 2} \times D_2$ which approximates $S_1^{\ 2} \times \{p_2\}$, $p_2 \in \operatorname{Int} D_2$. Finally let $A \subset S_1^{\ 2} \times S_2^{\ 2}$ be given by $A = A_1 \cup A_2$. We shall show that A lies in no open 4-cell in $S_1^{\ 2} \times S_2^{\ 2}$. As a first step in establishing this we need the following lemmas.

Lemma 3.10: Let $f:D^2 \longrightarrow M$ be a continuous map of a disk D^2 into a space M and let C be a simple closed curve in D^2 bounding the disk B in D^2. If f(C) is null homotopic in subspace N of M then there is a map $g:D^2 \longrightarrow M$ such that $g(x) = f(x)$ for $x \in D^2 - \operatorname{Int} B$ and $g(B) \subset N$.

Proof: Assume that $f|C:C \to N$ is null homotopic. By a well known result of Borsuk (see for example [23]) $f|C$ can be extended to a map f' on pC, the join of C with a point, so that $f'(pC) \subset N$. Since such a join is homeomorphic to B we define $g: D^2 \longrightarrow M$ to be

$$g(x) = \begin{cases} f(x) \text{ for } x \ (D^2 - \text{Int } B) \\ f'(x) \text{ for } x \ B \end{cases}$$

Since the two definitions agree on C, g is continuous.

Lemma 3.11: The simple closed curve $C = \{q\} \times S_1$, $q \varepsilon S_1^{\ 2}$ is not null homotopic in $(S_1^{\ 2} \times S_2^{\ 2}) - A$.

Proof: If C were null homotopic in $(S_1^{\ 2} \times S_2^{\ 2}) - A$ then C would bound a singular disk D' in $S_1^{\ 2} \times S_2^{\ 2} - A$. Give $S_1^{\ 2} \times S_2^{\ 2}$ a polyhedral structure so that $S_1^{\ 2} \times D_1$, $S_1^{\ 2} \times D_2$ and $S_1^{\ 2} \times K$ are polyhedral. Let $d(A, D') > \Sigma$ and using the analog of Lemma 2.4 for manifolds let N_1 and N_2 be polyhedral neighborhoods of A_1 and A_2 respectively such that $d(x, A_1) < \Sigma$ for each $x \varepsilon N_1$ and $d(x, A_2) < \Sigma$ for each $x \varepsilon N_2$. Then $S_1^{\ 2} \times D_1 - \text{Int } N_1$ and $[(S_1^{\ 2} \times D_2) - \text{Int } (N_2)] \cup (S_1^{\ 2} \times K)$ are polyhedra in which C fails to be null homotopic. An application of the simplicial approximation theorem produces a singular polyhedral disk in $S_1^{\ 2} \times S_2^{!} - \text{Int } (N_1 \cup N_2)$ whose boundary is C, i.e. we get a simplicial mapping s of a polyhedral disk D into $S_1^{\ 2} \times S_2^{\ 2} - \text{Int } (N_1 \cup N_2)$ such that $s(\text{Bd } D) = C$. Let $C_1, C_2, \ldots, C_K$ be polyhedral simple closed curves in D which bound disks $B_1, B_2, \ldots, B_k$ in D respectively such that $\overset{k}{C_1} \subset \overset{k}{\text{Bd}} s^{-1} (s(D) \cap (S_1^{\ 2} \times D_2))$ and $s^{-1} (s(D) \cap \text{Int } (S_1^{\ 2} \times D_2)) \subset {}_i\bigcup_{i=1} B_i$. If for each $i = 1, 2, \ldots, k$, $s\ (C_i)$ is homotopically trivial in $S_1^{\ 2} \times K$ then k

applications of Lemma 3.9 produces a singular disk in $(S_1^2 \times D_1 \cup S_1^2 \times K) - N_1$ whose boundary is C, contradicting known properties of A. Assume that for some i, say i = j, $s(C_j)$ is not null homotopic in $S_1^2 \times K$. Then $s(C_j)$ is not null homotopic in $S_1^2 \times D_2 - \text{Int } N_2$. Let $C_{j,1}, C_{j,2}, \ldots, C_{j,l}$ be polyhedral simple closed curves in B_j which bound disks $B_{j,1}, B_{j,2}, \ldots, B_{j,l}$ in B_j respectively such that $C_{j,i} \subset \text{Bd}[S^{-1}(s(B_j) \cap (S^2 \times D_1))]$ and $s^{-1}(s(B_j)$ Int $(S_1^2 \times D_2)) \subset \bigcup_{i=1}^{\ell} B_{j,i}$. If for each i = 1,2 . . ., ℓ, $s(C_{j,i})$ is null homotopic in $S_1^2 \times K$ then we have a contradiction. If some $C_{j,i}$ is not null homotopic in $S_1^2 \times D$ we continue as before using $C_{j,i}$ and $B_{j,i}$. This sets up an infinite regression, which is impossible due to the polyhedral structure of D. Hence we have a contradiction. This completes the proof of the lemma.

Now suppose $A \subset S_1^2 \times S_2^2$ were contained in an open 4-cell. It is an easy exercise to show that A would lie in a collared 4-cell C^4 in $S_1^2 \times S_2^2$. If we could find a curve C' in $S_1^2 \times K - C^4$ which is not null homotopic in $S_1^2 \times K$ then we would have a contradiction. For we would have a curve C' in $S_1^2 \times S_2^2 - A$ which is homotopic to C or a multiple of C in $S_1^2 \times S_2^2 - A$: but since C' would lie in the complement of C^4 C' would be homotopically trivial in the complement of C^4. Hence C' would be homotopically trivial in the complement of A. We now proceed to show that $(S_1^2 \times K) - C^4$ does, indeed, contain a closed curve which is not homotopically trivial in $S_1^2 \times K$.

Let S^2 be a polyhedral 2-sphere, let f be a homeomorphism of S^2 onto a curved parameter sphere S' in $S_1^2 \times S_2^2 - C^4$ and let $s : S^2 \longrightarrow (S_1^2 \times S_2^2) - C^4$ be a simplicial map which is homotopic to f.

Lemma 3.12: The 1-skeleton of $s(S^2) \cap (S_1^2 \times K_1)$ contains a closed curve which is not null homotopic in $S_1^2 \times S_1$.

Proof: Suppose to the contrary that each closed curve in the 1-skeleton of $s(S^2) \cap (S^2 \times K))$ is null homotopic in $S_1^2 \times K$. Let U_1 be a component of $s^{-1}(s(S^2) \cap (\text{Int}(S_1^2 \times D_1))$ and let C_1 be a simple closed curve in $\text{Bd}U_1$ such that U_1 lies entirely in one component B_{11} of $S^2 - C_1$. Let B_{12} be the other component of $S^2 - C_1$. Since $s(C_1)$ is homotopically trivial in $S_1^2 \times K$ there exist simplicial maps s_{11} and s_{12} of S^2 into $S_1^2 \times S_2^2$ such that $s_{11}(B_{11}) \subset S_1^2 \times K$, $s_{11}|(B_{12} \cup C_1) = s|(B_{12} \cup C_1) = s|(B_{12} \cup C_1)$, $s_{12}(B_{12}) \subset S_1^2 \times K$ and $s_{12}(B_{11} \cup C_1) = s|(B_{11} \cup C_1$

Let $\pi_2(S_1^2 \times S_2^2)$ be the 2-dimensional homotopy group of $S_1^2 \times S_2^2$. Let $t: S^2 \longrightarrow S_1^2 \times \{p\}$, $p \varepsilon S_2^2$, be a homeomorphism. Then $\pi_2(S_1^2 \times S_2^2)$ is the abelian group generated by [s] and [t], where [ ] denotes homotopy class. Since $[s_{11}] * [s_{12}] = [s]$ (* is homotopy juxtaposition) it follows that either $[s_{11}]$ or $[s_{12}]$ contains a non-zero multiple of [s], i.e. either $[s_{11}]$ or $[s_{12}]$ can be written as $p[t] * q[s]$ where $q \neq 0$, p and q integers.

If $[s_{11}]$ contains a non-zero multiple of [s] we continue as above with s_{11} and another component U_2 of $s_{11}^{-1}(s_{11}(S^2) \cap (\text{Int}\ S_1^2 \times D_1))$. Notice that $s_{11}(B_{11}) \subset S_1^2 \times K$. If $[s_{11}]$

contains no non-zero multiple of [s] we proceed as follows: let U_2 be a component of $s_{12}^{-1}(s_{12}(S^2) \cap \text{Int}\,(S_1^{\ 2} \times D_2))$, let C_2 be a simple closed curve in Bd U_2 such that U_2 lies entirely in one component B_{21} of $S^2 - C_2$, and let B_{22} be the other component of $S^2 - C_2$. We should note that $C_2 \subset \overline{B}_{11}$ and that $s_{12}(\overline{B}_{12}) \subset S_1^{\ 2} \times K$. Since $s_{12}(C_2) = s(C_2) \subset S_1^{\ 2} \times K$, $s_{12}(C_2)$ is homotopically trivial in $S_1^{\ 2} \times K$. Hence there exist simplicial maps s_{21} and s_{22} of S^2 into $S_1^{\ 2} \times S_2^{\ 2}$ such that $s_{21}(B_{21}) \subset S_1^{\ 2} \times K$, $s_{21}\,(B_{22} \cup C_2) = s_{12}|(B_{22} \cup C_2)$. Thus we have $[s_{21}] * [s_{22}] = [s_{12}]$. Hence either $[s_{21}]$ or $[s_{22}]$ is a non-zero multiple of $[s_{12}]$. Continuing these arguments in a finite number of such steps we must arrive at a simplicial map $s_{k,i}: S^2 \longrightarrow S_1^{\ 2} \times S_2^{\ 2}$ which is a non-zero multiple of [s] and $s_{k,i}(S^2)$ is a subset of either $S_1^{\ 2} \times (D_1 \cup K)$ or $S_1^{\ 2} \times (S_2 \cup K)$. In either case we have a contradiction because either of these spaces can be continuously deformed onto $S_1^{\ 2} \times \{p\}$, p a point of D_1 or D_2, hence any map of S^2 into one of these subspaces is homotopic to a multiple of [t] only.

This completes the proof of Theorem 3.9.

We should observe that the Cantor set A defined in the proof of the previous theorem may be considered to lie in one of the half spaces $S_1^{\ 2} \times D_1$. To verify this we need only note that there is a parameter 2-sphere in $S_1^{\ 2} \times S_2^{\ 2}$ which does not intersect A, so A is a subset of $S_1^{\ 2} \times (S_2^{\ 2} - p) = S_1^{\ 2} \times E^2$. It is easily seen that there is a homeomorphism mapping A into $S_1^{\ 2} \times D_1$.

Theorem 3.14: Let A be a Compact set in Int(S^k x D^m), let S be a parameter k-sphere in S^k x D^m and let h be a homeomorphism of S^k x D^m into an n-manifold M^n, n = k+m. If h(S) lies in an open n-cell in M^n then h(A) lies in an open n-cell in M^n.

Proof: Let C^n be an open n-cell containing S. Then $h^{-1}(C^n)$ is a neighborhood of S in S^k x D^m. There exists $\Sigma >$ 0 such that if d(x, S) $< \Sigma$ then x ε $h^{-1}(C^n)$. Let $p:S^k \times D^m \longrightarrow D^m$ be the natural projection and let p(S) = x ε D^m. Let B be an m-cell in Int D^m such that p(A) $\subset$ B. There is a homeomorphism g_1 of D^m onto itself such that g_1(B) is in an Σ-neighborhood of x and $g_1|$Bd D^m = id. Let (x,y) be the coordinates of a point of S^k x D^m. Define the homeomorphism $g:S^k \times D^m \longrightarrow S^k \times D^m$ by $g((x,y)) = (x,g_1(y))$. We can easily see that g(A) $\subset$ $h^{-1}(C^n)$ and $g|Bd(S^k \times D^m)$ = id. Finally we define the surjective homeomorphism $f:M^n \longrightarrow M^n$ by

$$f(t) = \begin{cases} h\,g\,h^{-1}(t) & \text{for } t \,\varepsilon\, h\,(S^k \times D^m) \\ t & \text{otherwise} \end{cases}$$

f maps h(A) into C^n: so $h^{-1}(C^n)$ is an n-cell containing A.

Corollary 3.15: Let A be a Cantor set in S^k x D^m, let M^n be an n-manifold with or without boundary, n = k+m, and let $h:S^k \times D^m \longrightarrow M^n$ be a homeomorphism. If A $\subset$ Int (S^k x D^m) is a Cantor set and if h(S^k x {p}), p ε D^m, lies in an open n-cell in M^n then A lies in an open n-cell.

Although it appears likely the converse of the above corollary has not yet been established for S^2 x S^2. A proof of this theorem would seem to depend on establishing the analogs of Lemmas 3.11 and 3.12 for non-trivial embeddings of $S^2 \times D^2$.

Theorem 3.9 could easily be generalized to $S^k \times S^2$ if one could construct a Cantor set in $S^k \times D^2$ with properties analogous to those of the Cantor set constructed in $S^2 \times D^2$. The proofs of the remaining lemmas do not depend on the dimension of S^k and could easily be generalized. I strongly suspect that the Cantor set of Blankinship could be used for this purpose in the following way: let $S_1{}^1 \times S_2{}^1 \times \ldots \times S_k{}^1$ be the Cartesian product of k 1-spheres where each 1-sphere is parameterized by the real numbers modulo 2π, let $f' : S_1{}^1 \times S_2{}^1 \times \ldots \times S_k{}^1 \longrightarrow S^k$ be the map defined by $f'(\theta_1, \theta_2, \ldots, \theta_k) = x \in S^k$ where x is the point on S^k with polar coordinates $\theta_1, \theta_2, \ldots, \theta_k$ and let $f: S_1{}^1 \times S_2{}^1 \times \ldots \times S_k{}^1 \times D^2 \longrightarrow S^k \times D^2$ be the natural extension of f'. It is not difficult to see that if A is the Cantor set of Blankinship in $S_1{}^1 \times S_2{}^1 \times \ldots \times S_k{}^1 \times D^2$ then $f(A)$ is a Cantor set in $S^k \times D^2$. The generalization of Theorem 3.9 then depends on proving that $f(A)$ has the desired properties in $S^k \times D^2$.

The foregoing results lead quite naturally to another conjecture. If one can approximate a k-sphere in a campact, piecewise linear n-manifold M^n, $n \geq k + 2$, then why not approximate the k-skeleton of M^n by linking many Cantor sets together? If it could be shown that the k-skeleton lies in an open n-cell iff the Cantor set does then, by a theorem of Stallings [26], M^n is the n-sphere, $n > 2$. Thus we are led to the following:

Conjecture: The only compact, piecewise linear n-manifold, $n > 2$, which is 0-invertible is the n-sphere.

A proof of this conjecture would lead to a generalization of the characterization by Bing [6] of the 3-sphere, namely: If M^n is a compact piecewise linear n-manifold such that each simple closed curve in M^n lies in an open n-cell then M^n is an n-sphere. Bing's result was originally proposed as a weakened form of the Poincare conjecture for 3-manifolds. If one looks at the above conjecture from this point of view it is indeed surprising!

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