

DISTRIBUTED PARAMETER SYSTEM MODELS
AND DISCRETE APPROXIMATIONS

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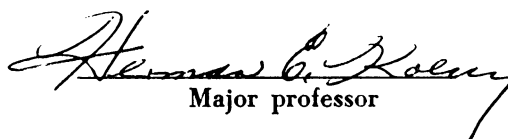
DISTRIBUTED PARAMETER SYSTEM MODELS
AND DISCRETE APPROXIMATIONS

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ABSTRACT

DISTRIBUTED PARAMETER SYSTEM MODELS AND DISCRETE APPROXIMATIONS

by Carl Henry Osterbrock

In recent years there have been rapid advances in both research and exposition in the analysis of lumped parameter physical systems. These advances have been the result, to a large degree, of the development of a formal abstract structure of lumped parameter system theory. This structure is based on the identification of the kinds of variables involved in the analysis of a physical system and on a separation of system equations into topological equations showing how components are interconnected, and component equations which show the properties of the components.

This thesis is concerned with the development of a structure for distributed parameter system theory similar to that of lumped parameter system theory, and a method for forming lumped approximations of distributed parameter physical systems. The structure is shown to be directly parallel to that of lumped parameter system theory, with the same separation of system equations into field equations, which correspond to the topological equations of the lumped parameter analysis, and constitutive equations, which correspond to the component equations.



Because of this close association between lumped and distributed parameter system analysis, it is possible to form discrete approximations of distributed parameter systems which are lumped system models of exactly the kind involved in the abstract structure of lumped parameter system theory.

The mathematical tools used in the development are those of vector differential calculus, essentially the mathematics generally referred to as vector field theory. Examples of system models and discrete approximations of these models are exhibited for electromagnetic fields, diffusion processes, fluid mechanics, and magnetohydrodynamics.



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by

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TABLE OF CONTENTS

SECTION	Page
1. INTRODUCTION	1
2. THE STRUCTURE OF LUMPED PARAMETER SYSTEM THEORY	4
3. THE STRUCTURE OF DISTRIBUTED PARAMETER SYSTEM THEORY	15
4. DISCRETE APPROXIMATIONS	26
5. EXAMPLES OF SYSTEM MODELS	30
6. CONCLUSION	52
REFERENCES	54

LIST OF FIGURES

Figure	Page
4.1.1 Microelement and associated linear graph edges . . .	27
4.1.2 Graph edges associated with microelements of different sizes	28
5.1.1 Linear graph edges for electric and magnetic variables	36
5.1.2 Magnetic graph edges and an associated electric graph edge	38
5.2.1 Microelement for diffusion process and associated graph edges	42



1. INTRODUCTION

The study of distributed parameter systems is more than two hundred years old, and many of the important and basic contributions to this study were made in the seventeenth and eighteenth centuries. Early and lasting contributions were made by such men as Euler (1707-83), Bernoulli (1700-83), d'Alembert (1717-83), Stokes (1819-1903), Helmholtz (1821-94) and Maxwell (1831-79). The mathematics used to describe distributed parameter systems, essentially vector field theory, is also old and well developed⁽¹⁾. However, very little work has been done in developing methods of analysis which are common to all kinds of systems. Such subjects as electromagnetic fields, fluid mechanics, diffusion of particles, and conduction of heat are commonly treated in separate books as separate subjects.

One method of analysis which has attracted much attention in recent years is the approximation of a distributed parameter system by a lumped parameter system, followed by the analysis of the simpler lumped system by well known methods. There is apparently no unifying principle on which these methods are based. In some cases they involve the roundabout procedure of obtaining partial differential equations by a limiting process applied to a set of ordinary differential equations, followed by an approximation of the partial differential equations by a set of ordinary differential equations⁽²⁾. The work of Gabriel Kron in this area has been directed toward replacing a distributed parameter system by an analogous lumped parameter electric network, and many results of this kind have been published with very little justification^(3,4,5).



The study of lumped parameter systems is certainly as old as the study of distributed parameter systems, and credit should probably be given to Kirchhoff (1824-87) for the origination of lumped parameter system analysis as a formal area of study. Distributed parameter systems are more fundamental than lumped parameter systems, in the sense that every lumped system model is an analyst's approximation of a physical system that actually has distributed effects. However, the analysis of lumped parameter systems is highly developed, and the methods of analysis developed by Trent⁽⁶⁾ and Koenig and Blackwell⁽⁷⁾ have been shown to have broad application and great practical usefulness.

There are two important characteristics of the methods of Trent and Koenig and Blackwell which can perhaps be claimed as reasons for the success of the methods. One is that there is a careful separation of the derivation of equations describing a physical system from the solution of these equations. Another is that topological equations, which show how components are interconnected, are carefully separated from component or constitutive equations, which have nothing to do with the geometry of the system. A measure of the success of topological methods is the fact that they can be applied to a wide variety of different kinds of physical systems including mixed systems such as electromechanical devices.

The first objective of this thesis is the description of a structure for distributed parameter system theory, similar to the structure devised by Trent and Koenig and Blackwell for lumped parameter system theory. Such a structure leads naturally to uni-



fied methods of formulating the equations which describe a broad class of distributed parameter systems. The second major objective of the thesis is the development of a systematic method for forming an approximate discrete model of a distributed parameter system. This involves dividing the system into a set of components, deriving topological equations which describe how the components are interconnected, and deriving approximate component equations which characterize the components. The two objectives of the thesis are closely interrelated, because the discrete model should represent a true approximation of the system equations. One motivation for developing the discrete approximation is that the methods of lumped parameter system analysis could be used to solve distributed parameter problems if the equations describing the discrete model are properly formulated. Digital computer programs for the solution of lumped parameter system equations could then be used for distributed parameter problems.

Section 2 of the thesis is a brief description of the structure of lumped parameter system theory, as devised by Trent, Koenig and Blackwell, and Seshu and Reed⁽⁸⁾. Section 3 describes a structure for distributed parameter system theory, which is closely parallel to the lumped parameter structure. The close relationship between the two is emphasized by similar numbering of subsections and equations in Sections 2 and 3. Section 4 of the thesis presents a method for forming a discrete approximation of the distributed parameter system equations, and Section 5 deals with four specific examples of the general concepts discussed in the earlier sections.



2. THE STRUCTURE OF LUMPED PARAMETER SYSTEM THEORY

2.1 Introduction

The analysis of a physical system is a mental process used by the analyst to help himself in thinking about the performance of the system. The analysis consists of two major parts, formulation of equations describing the system and solution of the equations. The first of these is more important than the second, in that a correct solution of equations is useless if they are improperly formulated. Lumped parameter system theory can be claimed to be a fundamental engineering discipline⁽⁹⁾, in the sense that it includes a very precise method of formulating the equations which describe a broad class of physical systems, and that the resulting equations fit into a very orderly structure.

Trent⁽¹⁰⁾ suggests that four steps are used by an analyst in the mental process of formulating the equations that describe a lumped parameter physical system: identification of physical effects, choice of variables, identification of components, and derivation of equations. Identification of physical effects essentially amounts to the elimination of physical effects which are not to be considered in the analysis. A complete analysis of a physical system should include all known phenomena, such as electrical, magnetic, gravitational and thermal effects. However, in some situations some physical phenomena are known to be completely unrelated to the problem at hand and can be ignored; in other situations an analyst chooses to ignore some physical effects on the basis of simplicity or convenience. For example, in electric network analysis it seems quite

natural to disguise the magnetic effects in inductors and the thermal effects in resistors with terminal equations involving only electrical variables. Identification of physical effects can be accomplished by naming forms in which energy occurs; an operational method is to name the kinds of meters involved in measurements on the physical system.

Success in the analysis of a lumped physical system depends upon the ability of the analyst to make a choice of variables, for each physical effect to be considered, of two different kinds. One kind of variable, called an across variable, is associated with two points or regions of space. A real or conceptual measurement of an across variable involves connecting a meter across (or "in parallel with") these two points and does not involve an internal change in the component or system under analysis. The other kind of variable, called a through variable, is associated with a single point or a single surface. Measurement of a through variable involves opening or cutting the system at this point or surface and inserting a meter in series with the system there. Success of the analysis furthermore depends on the identification of particular kinds of through and across variables, which are here called convenient variables. A convenient across variable is one that satisfies a circuit equation requiring that the sum of across variables around a closed path is zero. A convenient through variable is one that satisfies an incidence equation requiring that the sum of through variables incident at a point is zero.

An analyst makes a major decision regarding the kind of informa-



tion he seeks when he subdivides (perhaps mentally) a physical system into parts, or makes an identification of components. The analysis can provide no information regarding the internal behavior of a component, but only information about behavior of variables at the terminals of components. Part of the process of choosing components is the identification of terminals on the components.

Derivation of equations describing a lumped physical system involves the association of a linear graph with the system. Then the topological characteristics of the graph, and of the system, are expressed in terms of incidence (vertex, cutset, seg, continuity or equilibrium) equations in the through variables and circuit (mesh, loop or compatibility) equations in the across variables. Matters regarding the number of these topological equations that can be written, their form, and their independence are a well known part of linear graph theory. Constitutive equations (or component or terminal equations) interrelate through and across variables for each component. These equations are derived from existing fundamental knowledge of physical principles, from laboratory measurements, or from assumptions and inferences. An important aspect of the separation of topological and constitutive equations is that approximations and idealizations are isolated in the constitutive equations rather than being involved in all of the system equations. Furthermore a nonlinear system is characterized by nonlinear constitutive equations; the topological equations are always linear algebraic equations.

The orderly formulation technique available in the analysis of

lumped parameter systems leads to a very orderly structure of lumped parameter system theory. This structure embodies the basic variables which are necessary for the analysis of any system, a set of postulates which form the basis for the analysis of a broad class of physical systems, a set of important theorems regarding the nature of the equations which can be formulated, and formal procedures by which the sets of equations describing a lumped parameter system can be simplified. This structure is outlined in the following subsections of this section, in essentially the form given by Trent, Koenig and Blackwell, and Seshu and Reed. All statements in this summary of the structure of lumped parameter system theory are directed toward the special case of a physical system associated with a connected linear graph. In most cases an obvious generalization extends the statements to systems associated with graphs that are not connected.

2.2 The Undefined Variables

Consider a lumped parameter physical system which is isomorphic⁽⁶⁾ to a linear graph having v vertices and e edges. The isomorphism implies that there are associated with each edge of the graph two variables, a through variable and an across variable. These are the basic undefined variables in terms of which the analysis is carried out: two sets of e real-valued functions of the real variable t , time. The $e \times 1$ column matrix of through variables is designated Y and the $e \times 1$ column matrix of across variables is designated X .



2.3 Characterization of Multi-terminal Components

The terminal behavior of a two-terminal component is characterized by one pair of physical measurements, and the variables defined by these measurements are the two variables associated with the linear graph edge corresponding to the component. For a component with more than two terminals, not all of the possible pairs of measurements are necessary to completely characterize the behavior of the component. It can be shown⁽¹¹⁾ that at most, the set of through and across variables defined by a set of $n-1$ edges which form a tree of the connected graph associated with an n -terminal electrical component are independent. This leads to the following first postulate in the structure of lumped parameter system theory.

Postulate 2.3 Let G be the n -vertex graph defined by all possible pairs of two-terminal measurements on an n -terminal component, and let T be any tree of G . Then the $n-1$ pairs of measurements of through and across variables, associated with the edges of T , are sufficient to mathematically describe the terminal characteristics of the n -terminal component.

2.4 The Incidence Postulate

Electric currents satisfy Kirchhoff's current law, forces incident at a point sum to zero, and other kinds of through variables satisfy similar physical laws. These are embodied in the second postulate of lumped parameter system theory.

Postulate 2.4 Let Y be the $e \times 1$ column matrix of through variables associated with a linear graph of e edges and v vertices. Then

$$AY = 0, \quad (2.4.1)$$

where A is any $(v-1) \times e$ basis incidence matrix of the graph, with its rows arranged in the same order as the columns of Y .

2.5 The Circuit Postulate

As in section 2.4, the circuit postulate is a generalization of Kirchhoff's voltage law for electric networks, "compatibility" rules that require that relative displacements and velocities sum to zero around closed paths, and similar physical laws for other kinds of across variables.

Postulate 2.5 Let X be the $e \times 1$ column matrix of across variables associated with a linear graph of e edges and v vertices. Then

$$BX = 0, \quad (2.5.1)$$

where B is any $(e-v+1) \times e$ basis circuit matrix of the graph, with its rows arranged in the same order as the columns of X .

2.6 Orthogonality

The incidence and circuit matrices A and B satisfy the following fundamental theorem⁽¹²⁾ which, along with the Postulates 2.4 and 2.5, leads to many important properties of the variables X and Y .

Theorem 2.6 Let A be any incidence matrix and B any circuit matrix of the same linear graph, with their columns arranged in the same order. Then

$$AB^t = BA^t = 0. \quad (2.6.1)$$

2.7 Auxiliary Variables

The well known "mesh" and "node" methods of formulation of system equations can be justified rigorously on the basis of the



existence of auxiliary variables. The existence of these variables is shown by the following two theorems⁽¹³⁾, which depend for their proof on the Postulates 2.4 and 2.5 and on Theorem 2.6.

Theorem 2.7.1 For a graph of e edges and v vertices, there exists a $(e-v+1) \times 1$ column matrix of variables Y_m such that

$$Y = B^t Y_m, \quad (2.7.1)$$

where Y is the $e \times 1$ column matrix of through variables associated with the graph and B is any basis circuit matrix of the graph.

Theorem 2.7.2 For a graph of e edges and v vertices, there exists a $(v-1) \times 1$ column matrix of variables X_n such that

$$X = A^t X_n, \quad (2.7.2)$$

where X is the $e \times 1$ column matrix of across variables associated with the graph and A is any basis incidence matrix of the graph.

The auxiliary variables of the matrix Y_m are frequently referred to as mesh through variables, and those of X_n as node across variables.

2.8 Conservation of Energy

In many physical systems, the product of the fundamental through and across variables can be interpreted as power. In this case the following theorem⁽¹⁴⁾ shows that the total power is zero, so that the total energy in the system is constant.

Theorem 2.8 Let the variables in the column matrices X and Y associated with a graph be arranged in the same order. Then

$$Y^t X = X^t Y = 0. \quad (2.8.1)$$



This result does not depend in any way upon the nature of the relationships between through and across variables, but only upon the Postulates 2.4 and 2.5 and the orthogonality relation (2.6.1). A postulational scheme different from the one used here is possible, if conservation of energy is regarded as more fundamental than the Kirchhoff rules (2.4.1) and (2.5.1); Arsove⁽¹⁵⁾ has shown that (2.8.1) and (2.4.1) or (2.5.1) imply (2.5.1) or (2.4.1) respectively.

2.9 Component Models

By a component model is meant a mathematical description of the terminal characteristics of a component. For a two-terminal component this is a single equation in terms of the through variable and the across variable which can be measured at the component terminals. For an n-terminal component, the component model is a set of n-1 equations in terms of the n-1 through variables and n-1 across variables which, through Postulate 2.3, characterize the component. When all of the component models are linear algebraic or differential equations, the component equations can sometimes be put in one of two forms which are especially convenient for later simplification. These are a form in which the across variables are expressed as explicit functions of the through variables,

$$X = ZY + X_d, \quad (2.9.1)$$

and a form in which the through variables are expressed as explicit functions of the across variables,

$$Y = WX + Y_d, \quad (2.9.2)$$

where Z and W are square matrices of order e whose elements are in general operators in terms of the time derivative, and where X_d and Y_d are $e \times 1$ column matrices whose elements are specified functions of



time. For many systems, especially those containing multi-terminal components, the component model cannot be expressed in either of the forms (2.9.1) or (2.9.2), but takes the form

$$ZY = WX + D_d, \quad (2.9.3)$$

where D_d is a $n \times 1$ column matrix whose elements are specified functions of time.

Wirth's work⁽¹⁶⁾ on formulation of system equations in the time domain includes methods of formulation which apply to component models which include certain classes of nonlinear algebraic and differential equations. One example of such a component model is

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} X_1 \\ Y_2 \end{bmatrix} &= \begin{bmatrix} F_1(X_1, X_2, X_3, Y_1, Y_2, Y_3, t) \\ F_2(X_1, X_2, X_3, Y_1, Y_2, Y_3, t) \end{bmatrix} \\ \begin{bmatrix} X_3 \\ X_0 \\ Y_4 \end{bmatrix} &= \begin{bmatrix} F_3(Y_3, t) \\ F_0(t) \\ F_4(t) \end{bmatrix} \end{aligned} \quad (2.9.4)$$

where the system variables X and Y are subdivided into groups $X_0 \dots X_4$ and $Y_0 \dots Y_4$ according to the form of the component equations into which they enter.

2.10 System Models

A system model is a complete mathematical description of the physical system. Such a mathematical description is included in the previous sections: the 2e equations in terms of e through variables and e across variables, including e topological equations (2.4.1) and (2.5.1) and e component equations in a form such as (2.9.1), (2.9.2), (2.9.3) or (2.9.4). In most practical problems this model is cumbersome because of the large number of equations and unknowns, and the word formulation is generally meant to include some process of

reduction of the number of equations and unknowns. The particular process that can be employed and the form of the resulting system model depends upon the form of the component model that is available. Four examples are exhibited here, corresponding to each of the examples of component models discussed in section 2.9.

If the component model can be written in the form (2.9.1), equation (2.7.1) can be used in place of (2.4.1). Substitution of (2.9.1) into (2.5.1), and of (2.7.1) into this result yields the "mesh" or "chord" model,

$$BZB^t Y_m + BX_d = 0. \quad (2.10.1)$$

If the component model available is (2.9.2), equation (2.7.2) is used in place of (2.5.1). Substitution of (2.9.2) into (2.4.1), and of (2.7.2) into the result yields the "node" or "branch" model,

$$AWA^t X_n + AY_d = 0. \quad (2.10.2)$$

When the component model (2.9.3) must be used, both (2.7.1) and (2.7.2) are used instead of the Postulates 2.4 and 2.5. Substitution of (2.7.1) and (2.7.2) into (2.9.3) results in a mixed, or "branch-chord" set of equations

$$ZB^t Y_m = WA^t X_n + D_d. \quad (2.10.3)$$

Wirth⁽¹⁶⁾ has shown how a system model can be formulated in the normal form; a system of equations explicit in the first derivatives of some of the system variables. When the component model is available in the form (2.9.4) and when the equations satisfy certain properties, (2.4.1), (2.5.1) and (2.9.4) can be combined to yield the



system model

$$\frac{d}{dt} \begin{bmatrix} X_{1b} \\ Y_{2c} \end{bmatrix} = \begin{bmatrix} G_1(X_{1b}, Y_{2c}, t) \\ G_2(X_{1b}, Y_{2c}, t) \end{bmatrix} \quad (2.10.4)$$

where

$$X_1 = \begin{bmatrix} X_{1b} \\ X_{1c} \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} Y_{2b} \\ Y_{2c} \end{bmatrix}$$

and where the subscript b refers to the branches of a suitably chosen tree and the subscript c refers to the chords of the same tree.



3. THE STRUCTURE OF DISTRIBUTED PARAMETER SYSTEM THEORY

3.1 Introduction

In the analysis of distributed parameter systems, as in lumped parameter system analysis, it is important that a careful separation be made between the process of formulation of equations and the process of solution of those equations. The four steps in the process of formulating equations discussed in Section 2.1 apply also to distributed parameter systems. Truesdell⁽¹⁷⁾ considers the matter of identification of physical effects in a discussion of equipresence as one of the important aspects of the general theory of constitutive equations; he points out the fact that it is rarely possible to include more than one kind of physical effect in the analysis of a distributed parameter system problem. It is generally true that early developments in any area of study are concerned with only a very limited number of physical effects. More sophisticated methods of analysis are required for the more difficult problems involving different kinds of phenomena. At the present time the coupling between electric and magnetic fields seems to be well understood, while the coupling between thermodynamic variables and the stress and rate of strain variables of fluid mechanics is not. An area of recent interest in which coupling effects are important is magneto-hydrodynamics.

Many distributed parameter systems are characterized by vector fields, and the choice of variables in the analysis of these systems amounts to the choice of two suitable vector point functions. The association of an across variable with two points, in lumped parameter



system analysis, suggests that a vector point function should be called an across variable if its line integral along some curve is significant. Thus if

$$\alpha_{ab} = \int_a^b \bar{a} \cdot d\bar{l}$$

then $\bar{a}$ is called a vector across variable and α_{ab} is a scalar across variable associated with the two points a and b . The association in lumped parameter system analysis of a through variable with a single point or surface suggests that a vector point function should be called a through variable if its surface integral is significant. If

$$\beta_S = \int_S \bar{b} \cdot d\bar{s},$$

then $\bar{b}$ is called a vector through variable and β_S is a scalar through variable associated with the surface S . The requirement that the sum of scalar across variables around any closed path be zero is equivalent to the requirement that the line integral of the associated vector across variable around any closed path be zero. Then by Kelvin's transformation ("Stokes's theorem"),

$$\oint_C \bar{a} \cdot d\bar{l} = \int_S (\text{curl } \bar{a}) \cdot d\bar{s} = 0,$$

where C is the closed curve bounding the surface S . Since this is to hold for any closed curve C ,

$$\text{curl } \bar{a} = 0,$$

and so a convenient vector across variable is one whose curl is zero (a lamellar or irrotational vector point function). Similarly, the requirement that the sum of lumped parameter through variables incident at a point be zero is equivalent to the requirement that the surface integral of a vector through variable over any closed surface

be zero. Then by Gauss's theorem,

$$\oint_S \vec{b} \cdot d\vec{s} = \int_V (\text{div } \vec{b}) dv = 0,$$

where S is the closed surface bounding the volume V . Since this is to hold for any closed surface,

$$\text{div } \vec{b} = 0,$$

and so a convenient vector through variable is one whose divergence is zero (a solenoidal vector point function).

Involved in the identification of components, in distributed parameter system analysis, is the choice between a continuum description of the problem and a discrete approximation to the continuum description. The equations of the classical theory of fields are partial differential equations whose independent variables are time and three space coordinates. Truesdell⁽¹⁷⁾ gives a strong justification for studying this continuum description of matter in addition to the perhaps more fundamental description in terms of the motion of particles. In the continuum description of distributed parameter systems, choice of components is not a part of the analysis; a problem is described by a set of boundary conditions on the bounding surface of some region in space, along with a set of partial differential equations which apply to the entire region. On the other hand a discrete approximation of the continuum description does involve the selection of a set of subregions, which can be called components, and a mathematical description of these subregions. The process of forming such a discrete approximation is discussed in Section 4.

Derivation of equations involves equations of two broad types. Field equations, which correspond to the topological equations of

lumped parameter system theory, have to do with the intrinsic characteristics of dynamical variables and not with the properties of the medium under analysis. For a continuum description in terms of vector point functions they have the form of a compatibility equation stating that the curl of a vector across variable is zero, and an incidence (or equilibrium or continuity) equation showing that a vector through variable is solenoidal. Constitutive equations, the equivalent of component equations in lumped parameter system theory, interrelate through variables and across variables and describe the particular characteristics of the medium under analysis. Because of the inherent complication of distributed parameter systems, constitutive equations are frequently obtained from assumptions, rationalizations or generalizations of simple experimental evidence.

The following subsections show the structure of distributed parameter system theory which follows from the formulation procedure discussed above. This structure corresponds closely to the lumped parameter system theory structure in Section 2, and the numbering of postulates, theorems and equations is similar to that used in Section 2 in order to emphasize this correspondence.

3.2 The Undefined Variables

Consider a distributed parameter physical system, confined to a closed region V with bounding surface S . Analysis of the system depends upon the identification of two complementary functions, a through variable and an across variable. These are the undefined variables of the abstract structure of distributed parameter system theory: two real-valued scalar, vector or tensor functions of time and the three space coordinates. In the case of complementary vector

point functions, the through variable is designated by the generic symbol $\bar{y}$ and the across variable by $\bar{x}$.

3.3 Characterization of Continua

Some relatively simple continuum problems are characterized by scalar point functions; for example, that of a fluid at rest. Other more complicated distributed parameter systems, especially some involving coupling between more than one kind of physical phenomenon, require a description in terms of second-order tensors. There is a large class of physical systems which are completely characterized by vector point functions. This leads to the following first postulate in the structure of distributed parameter system theory.

Postulate 3.3 Let $\bar{x}$ and $\bar{y}$ be two complementary vector point functions, which are measurable in a distributed parameter physical system confined to a closed region V with bounding surface S . Then the characteristics of the physical system are completely determined by the values of $\bar{x}$ and $\bar{y}$ at every point in V and on S .

3.4 The Divergence Postulate

Electric current density, magnetic induction and many other vector through variables satisfy the requirement that their normal surface integrals over any closed surface are zero. Gauss's theorem then requires that the volume integral of the divergence of such a vector point function over any volume be zero, and thus that the divergence itself is zero. This serves as the basis for the following postulate, equivalent to the incidence postulate of lumped parameter system theory.

Postulate 3.4 Let $\bar{y}$ be the through vector point function associated with a distributed parameter system confined to a region V with bounding surface S . Then

$$\operatorname{div} \bar{y} = 0 \quad (3.4.1)$$

throughout V .

3.5 The Curl Postulate

The following postulate is a generalization of the point form of the requirement that the line integral of many vector across variables, such as electric field intensity, around any closed path is zero. Kelvin's transformation then requires that the normal surface integral of the curl of the variables be zero for any surface; since the surface is arbitrary, the curl of the variable is zero everywhere in the region where the line integral is zero.

Postulate 3.5 Let $\bar{x}$ be the across vector point function associated with a distributed parameter system confined to a region V with bounding surface S . Then

$$\operatorname{curl} \bar{x} = 0 \quad (3.5.1)$$

throughout V .

3.6 Orthogonality

The vector differential operators curl and divergence satisfy the identities shown in the following theorem⁽¹⁸⁾ which, along with the Postulates 3.4 and 3.5, leads to important properties of the variables $\bar{x}$ and $\bar{y}$.

Theorem 3.6 Let $\bar{a} = \operatorname{curl} \bar{b}$ and $\bar{c} = \operatorname{grad} \alpha$, where $\bar{b}$ and α are any vector and scalar point functions, respectively, with continuous mixed second partial

derivatives. Then

$$\operatorname{div} \bar{a} = \operatorname{div} \operatorname{curl} \bar{b} = 0 \quad (3.6.1)$$

$$\text{and} \quad \operatorname{curl} \bar{c} = \operatorname{curl} \operatorname{grad} \alpha = 0 \quad (3.6.2)$$

3.7 Auxiliary Variables

The following two theorems⁽¹⁹⁾ are the converses of (3.6.1) and (3.6.2). They provide for the existence of auxiliary variables which are convenient in the simplification of system equations, and depend for their proof on the two Postulates 3.4 and 3.5.

Theorem 3.7.1 Let $\bar{y}$ be a vector point function which has continuous first partial derivatives and satisfies (3.4.1) in some region V . Then there exists a vector point function $\bar{u}$ such that

$$\bar{y} = \operatorname{curl} \bar{u} \quad (3.7.1)$$

in V .

Theorem 3.7.2 Let $\bar{x}$ be a vector point function which has continuous first partial derivatives and satisfies (3.5.1) in some region V . Then there exists a scalar point function ϕ such that

$$\bar{x} = \operatorname{grad} \phi \quad (3.7.2)$$

in V .

The auxiliary variable $\bar{u}$ is usually referred to as a vector potential function, and ϕ as a scalar potential function.

3.8 Conservation of Energy

It is frequently possible to make a physical interpretation of the scalar product of the fundamental through and across variables, $\bar{x} \cdot \bar{y}$, as the time derivative of energy density. In this case a volume

integral of the product $\bar{x} \cdot \bar{y}$ over a region V is the time rate of flow of energy into V , and if this volume integral is zero then the total energy contained in V is constant in time.

Consider the volume integral of $\bar{x} \cdot \bar{y}$ over a volume V with bounding surface S , where $\bar{x}$ and $\bar{y}$ satisfy the Postulates 3.4 and 3.5. Then, using (3.7.1) and (3.5.1),

$$\begin{aligned} \int_V (\bar{x} \cdot \bar{y}) dv &= \int_V (\bar{x} \cdot \text{curl } \bar{u}) dv \\ &= \int_V (\bar{u} \cdot \text{curl } \bar{x} - \text{div}(\bar{x} \times \bar{u})) dv \\ &= \int_V (\text{div}(\bar{u} \times \bar{x})) dv. \end{aligned}$$

Gauss's theorem yields

$$\int_V (\bar{x} \cdot \bar{y}) dv = \oint_S (\bar{u} \times \bar{x} \cdot \bar{n}) ds, \quad (3.8.1)$$

where $\bar{n}$ is the unit outward normal vector at the surface element ds .

Using (3.7.2) and (3.4.1), and Gauss's theorem again, in the same integral,

$$\begin{aligned} \int_V (\bar{x} \cdot \bar{y}) dv &= \int_V (\bar{y} \cdot \text{grad } \phi) dv \\ &= \int_V (\text{div } \phi \bar{y} - \phi \text{div } \bar{y}) dv \\ &= \int_V (\text{div } \phi \bar{y}) dv \\ &= \oint_S (\phi(\bar{y} \cdot \bar{n})) ds. \end{aligned} \quad (3.8.2)$$

The volume integral is zero provided the integrand in either (3.8.1) or (3.8.2) is zero, thereby proving the theorem below.

Theorem 3.8 Let $\bar{x}$ and $\bar{y}$ be vector point functions which satisfy (3.4.1) and (3.5.1) in a closed region V with

bounding surface S , and let ϕ and $\bar{u}$ be the associated scalar and vector potential functions. Then

$$\int_V (\bar{x} \cdot \bar{y}) dv = 0, \quad (3.8.3)$$

provided any one of the following conditions is satisfied:

$\bar{x}$, $\bar{y}$, ϕ or $\bar{u}$ is zero on S ;

$\bar{y}$ is normal to $\bar{n}$ on S ;

$\bar{x}$ or $\bar{u}$ is parallel to $\bar{n}$ on S ;

$\bar{x}$ is parallel to $\bar{u}$ on S .

3.9 Constitutive Equations

Constitutive equations interrelate the basic through and across vector point functions, and in general can be complicated partial differential equations; nonlinear media are characterized by nonlinear constitutive equations. When a constitutive equation is linear, it is especially convenient to express it in one of two special forms. These are a form in which the vector across variable is expressed as an explicit function of the vector through variable,

$$\bar{x} = \bar{Z} \cdot \bar{y} + \bar{x}_d, \quad (3.9.1)$$

and a form in which the through variable is expressed as an explicit function of the across variable,

$$\bar{y} = \bar{W} \cdot \bar{x} + \bar{y}_d, \quad (3.9.2)$$

where $\bar{Z}$ and $\bar{W}$ are dyadics whose components are, in general, operators in the time derivative and the gradient operator and where $\bar{x}_d$ and $\bar{y}_d$ are specified vector point functions ("source functions"). When neither (3.9.1) nor (3.9.2) can be obtained, a slightly more general form of the constitutive equation for a linear system is

$$\bar{Z} \cdot \bar{y} = \bar{W} \cdot \bar{x} + \bar{d}_d, \quad (3.9.3)$$

where $\bar{d}_d$ is a specified vector point function.

3.10 System Models

A system model, a complete mathematical description of the physical system, takes the form of a partial differential equation or a set of such equations for a distributed parameter system. One system model is the set of three equations composed of the field equations (3.4.1) and (3.5.1), along with one of the constitutive equations (3.9.1), (3.9.2) or (3.9.3). This set of equations can be reduced to a system model in terms of a single equation through the use of the auxiliary variables of Section 3.7.

When the constitutive equation (3.9.1) is available, (3.7.1) can be used in place of (3.4.1). Substitution of (3.9.1) into (3.5.1), and of (3.7.1) into the result, yields a system model in terms of a vector potential function,

$$\text{curl} (\bar{Z} \cdot \text{curl} \bar{u}) + \text{curl} \bar{x}_d = 0. \quad (3.10.1)$$

When the constitutive equation can be written in the form (3.9.2), (3.7.2) can be used in place of Postulate 3.5. Substitution of (3.9.2) into (3.4.1), and of (3.7.2) into the result, yields a system model in terms of a scalar potential function,

$$\text{div} (\bar{W} \cdot \text{grad} \phi) + \text{div} \bar{y}_d = 0. \quad (3.10.2)$$

If neither (3.9.1) nor (3.9.2) is available as a constitutive equation, but the constitutive equation can be written in the form (3.9.3), both auxiliary variables are used. Substitution of (3.7.1) and (3.7.2) into (3.9.3) yields

$$\bar{Z} \cdot \text{curl} \bar{u} = \bar{W} \cdot \text{grad} \phi + \bar{d}_d, \quad (3.10.3)$$

a system model of a single equation in terms of both a scalar potential function and a vector potential function.



4. DISCRETE APPROXIMATIONS

4.1 Association of a Linear Graph with a Distributed Parameter System

It was pointed out in Section 3.1 that one step in the process of formulating system equations is not the same for distributed parameter systems as for lumped parameter systems. The analysis of a lumped parameter system always involves the subdivision of the system into a set of smaller systems, called components, which are interconnected at external terminals to form the original system; the choice of these components is always arbitrary to some degree. The formulation of the partial differential equations of a continuum description of a distributed parameter system, however, involves no such division of the system into smaller parts.

It is frequently true that although the partial differential equations describing a distributed parameter system can be formulated, no solution of these equations is known except in a limited number of very special cases; the Navier-Stokes equations of fluid mechanics are a well known example. In this situation it is desirable to direct attention toward an approximate solution of the equations. This can be done by formulating the system model and then seeking an approximate solution by numerical methods, or by formulating the system model and then replacing this partial differential equation model by a set of ordinary differential equations⁽²⁾. Instead of these methods, attention is directed here toward a method in which the approximation is made a part of the formulation procedure; a lumped parameter approximation of the distributed parameter system is made by dividing

the region of space in question into subregions, and the field and constitutive equations are approximated by the topological and component equations of lumped parameter system theory.

A discrete approximation to a three-dimensional distributed parameter physical system is formed by dividing the region of interest into a set of subregions; the name microelements was suggested for these subregions by E. C. Koenig⁽²⁰⁾, and they are the components of the resulting lumped parameter physical approximation of the distributed parameter physical system. The shape of the microelements is that of a rectangular parallelepiped with faces parallel to the coordinate system. The simplest and most convenient shape for the microelements is a rectangular parallelepiped. A linear graph can be associated with a set of microelements by associating a vertex of the graph with the center of each microelement, and a graph edge with each of the six faces of the microelement. A typical rectangular microelement and the associated graph edges are shown in Figure 4.1.1. The choice of the size and shape of the microelements is one of the

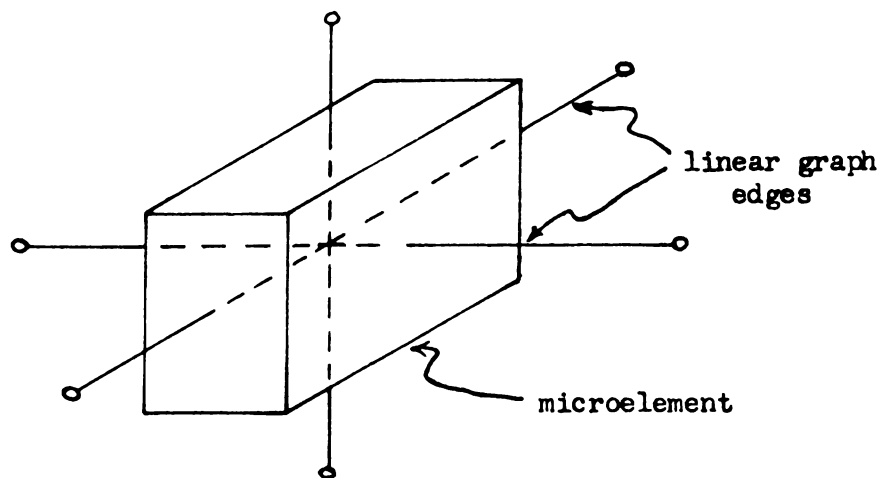


Figure 4.1.1. Microelement and associated linear graph edges

arbitrary choices made by the analyst. No difficulty is encountered in the association of the graph with the microelements when the microelements are not all chosen to be of the same size, as suggested by Figure 4.1.2.

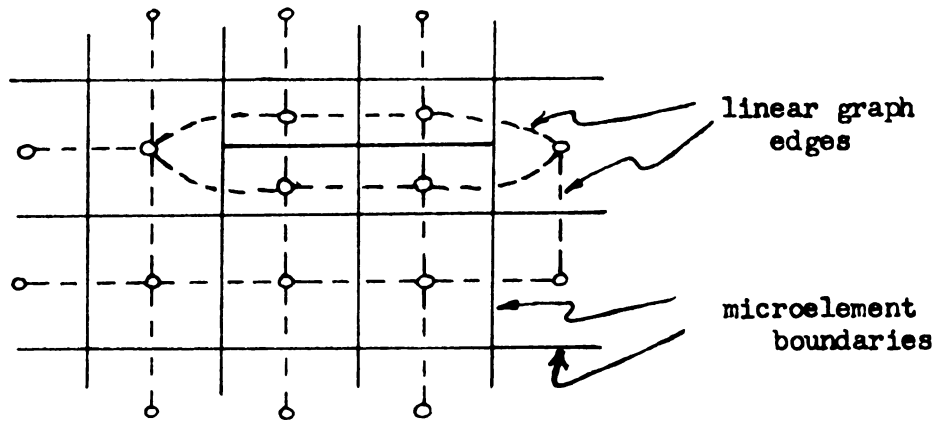


Figure 4.1.2. Graph edges associated with microelements of different sizes

4.2 Association of Variables with the Linear Graph Approximation

A scalar through variable and a scalar across variable are associated with each edge of the linear graph of the lumped parameter approximation formed by the procedure described in Section 4.1. The scalar through variable is the surface integral of the vector through variable of the continuum model, over the face of the microelement with which the edge is associated. The scalar across variable is the line integral of the vector across variable of the continuum model, along a path between the centers of the two microelements with which the vertices of the graph edge are associated. Then a volume integral of the incidence postulate (3.4.1) over the microelement, transformed with Gauss's theorem, yields the incidence relation (2.4.1) for the lumped approximation. Similarly, a surface integral of the curl

postulate (3.5.1), along with Kelvin's transformation, yields the circuit relation (2.5.1) of lumped parameter system theory. The component equations of the lumped approximation are then the line or surface integrals of the corresponding constitutive equations of the continuum model.

5. EXAMPLES OF SYSTEM MODELS

5.1 Electromagnetic Fields

Of central importance in any discussion of electromagnetic fields are the two Maxwell equations,

$$\text{curl } \bar{e}' = -\dot{\bar{b}} \quad (5.1.1)$$

$$\text{curl } \bar{h}' = \bar{i} , \quad (5.1.2)$$

where $\bar{e}'$, $\dot{\bar{b}}$, $\bar{h}'$, and $\bar{i}$ are the usual electric field intensity, time derivative of magnetic displacement density, magnetic field intensity, and electric current density, respectively. In terms of the kind of system model discussed in Section 3, the first important interpretation of these equations is that the variables $\dot{\bar{b}}$ and $\bar{i}$ are solenoidal;

$$\text{div } \dot{\bar{b}} = -\text{div curl } \bar{e}' = 0 \quad (5.1.3)$$

$$\text{div } \bar{i} = \text{div curl } \bar{h}' = 0 . \quad (5.1.4)$$

Thus $\dot{\bar{b}}$ is a convenient vector through variable for magnetic field analysis, and $\bar{i}$ is a convenient vector through variable for electric fields. The Maxwell equations also clearly show that electric and magnetic effects are interrelated, and that the two physical phenomena should be analyzed together using two sets of variables, one for each of the two physical effects. An interpretation of (5.1.4) can be made here that is slightly different from the usual one. One of Maxwell's great achievements was the substitution of (5.1.4) for the equation of continuity,

$$\text{div } \bar{i}_c + \frac{\partial \rho}{\partial t} = 0 , \quad (5.1.5)$$

and Gauss's law,

$$\rho = \text{div } \bar{d} , \quad (5.1.6)$$

where $\bar{i}_c$, ρ and $\bar{d}$ are conduction current density, charge density, and

electric displacement density, respectively. In terms of the discussion of Section 3, Maxwell's contribution was the substitution of a convenient vector through variable, "total" current density $\bar{\mathbf{i}}$, for an inconvenient vector through variable, the conduction current density $\bar{\mathbf{i}}_c$.

Equations (5.1.3) and (5.1.4), along with Theorem 3.7.1, imply that there must exist functions $\bar{\mathbf{e}}''$ and $\bar{\mathbf{h}}''$ such that

$$\dot{\bar{\mathbf{b}}} = \text{curl } \bar{\mathbf{e}}'' \quad (5.1.7)$$

$$\bar{\mathbf{i}} = -\text{curl } \bar{\mathbf{h}}'' . \quad (5.1.8)$$

Substitution of (5.1.7) and (5.1.8) into (5.1.1) and (5.1.2) yields

$$\text{curl } \bar{\mathbf{e}} = 0 \quad (5.1.9)$$

$$\text{curl } \bar{\mathbf{h}} = 0 , \quad (5.1.10)$$

where

$$\bar{\mathbf{e}} = \bar{\mathbf{e}}' + \bar{\mathbf{e}}'' \quad (5.1.11)$$

$$\bar{\mathbf{h}} = \bar{\mathbf{h}}' + \bar{\mathbf{h}}'' . \quad (5.1.12)$$

Thus $\bar{\mathbf{e}}$ and $\bar{\mathbf{h}}$ are convenient vector across variables for electric and magnetic effects respectively. The vector point function $\bar{\mathbf{e}}''$ is the time derivative of a function generally called the magnetic vector potential function and generally designated by the symbol $\bar{\mathbf{A}}$. The similar electric vector potential function $\bar{\mathbf{h}}''$ is not a part of the usual developments of electromagnetic field theory.

Equations (5.1.7) and (5.1.8) are coupling equations, showing how electric and magnetic effects are interrelated, and should properly be included among the constitutive equations. These coupling equations are of the kind called "inverse" by Firestone⁽²¹⁾ because each relates a through variable to an across variable. Other

constitutive equations are required to complete the system model, relating $\vec{I}$ to $\vec{e}'$ and $\dot{\vec{b}}$ to $\vec{h}'$. For a linear, isotropic and homogeneous medium these constitutive equations are simply

$$\vec{I} = \sigma \vec{e}' + \epsilon \frac{\partial \vec{e}'}{\partial t} \quad (5.1.13)$$

$$\dot{\vec{b}} = \mu \frac{\partial \vec{h}'}{\partial t}. \quad (5.1.14)$$

The field equations (5.1.3), (5.1.4), (5.1.9) and (5.1.10) and the constitutive equations (5.1.7), (5.1.8) and (5.1.11) through (5.1.14) agree with the general structure of Section 3. The treatment of two physical effects together and the presence of coupling equations add a minor complication. The discussion of conservation of energy in Section 3.8 is also complicated slightly by the coupling equations. Because of the very special form of these coupling equations, a development similar to that of Section 3.8 leads to the well known Poynting theorem and to the concept of radiated power.

There are several reasons for the use of the time derivative of the magnetic displacement density, $\dot{\vec{b}}$, rather than $\vec{b}$, as the magnetic vector through variable. One reason is that it is $\dot{\vec{b}}$, not $\vec{b}$, which is usually measured in any sort of physical measurement of magnetic fields. Another is that the important field equation (5.1.3) follows directly from (5.1.1), while the use of $\vec{b}$ as the magnetic vector through variable would depend on rather weak arguments in support of the assertion that this variable is solenoidal. The use of $\dot{\vec{b}}$ rather than $\vec{b}$ leads to emphasis in the constitutive equations (5.1.13) and (5.1.14) on an important difference between electric and magnetic fields: the absence of a "magnetic conduction current density" term corresponding to $\sigma \vec{e}'$. Finally, $\dot{\vec{b}}$ is the proper choice for a magnetic

through variable because the inner products $\dot{\vec{b}} \cdot \vec{h}$ and $\vec{f} \cdot \dot{\vec{e}}$ both can be interpreted as power density and can be added directly in discussions of conservation of energy.

The classification of dynamical variables as through and across variables leads to a close association between $\vec{e}$ and $\vec{h}$ as across variables and between $\vec{f}$ and $\dot{\vec{b}}$ as through variables. This is in agreement with the early historical view in which $\vec{e}$ and $\vec{h}$ were named field intensities, and $\vec{b}$ and $\vec{d}$ were named flux densities. It is in direct disagreement with the more modern view, discussed at length by Sommerfeld⁽²²⁾, in which $\vec{e}$ and $\vec{b}$ are called variables of "intensity" and $\vec{h}$ and $\vec{d}$ are called variables of "quantity".

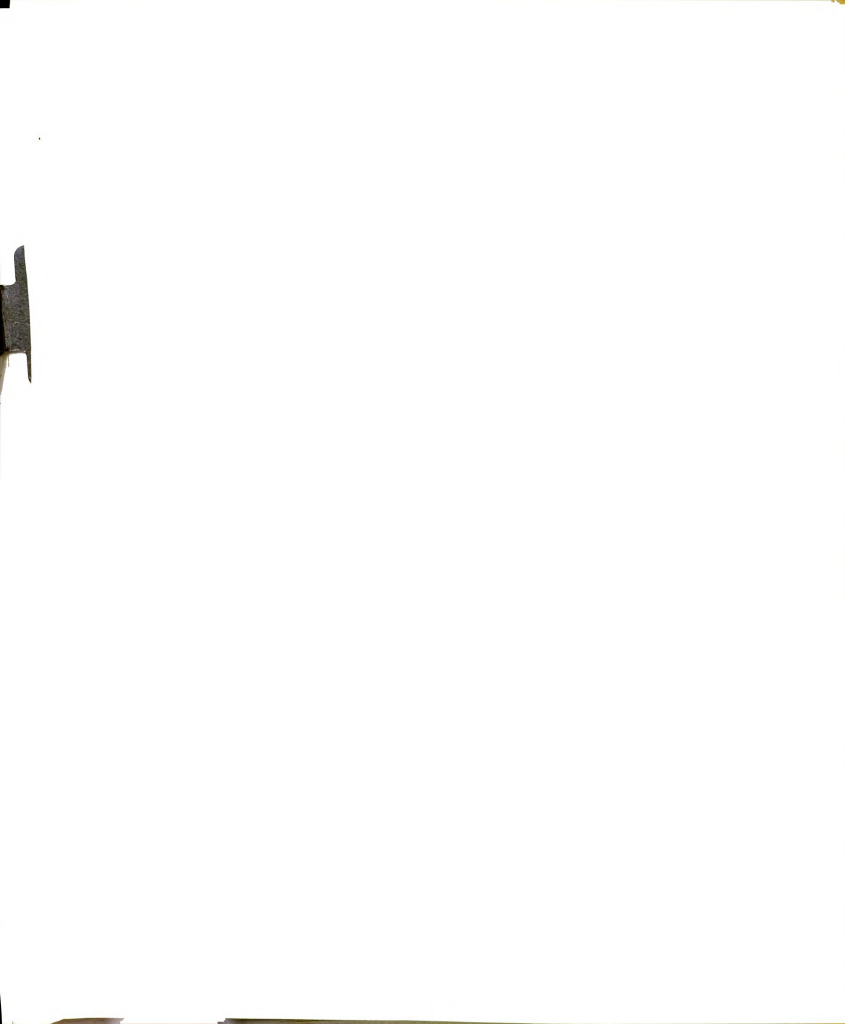
Several special cases arise in connection with electromagnetic fields when the constitutive parameters σ , ϵ and μ take on special values, or when the driving functions are of special kinds. One important case occurs when $\sigma = 0$, or when σ is small enough that the constitutive equation (5.1.13) can be approximated by

$$\vec{f} = \epsilon \frac{\partial \vec{e}}{\partial t}. \quad (5.1.15)$$

Under this condition no important change occurs in the structure of the equations describing the field. Similarly, if ϵ is small enough to allow (5.1.13) to be approximated by

$$\vec{f} = \sigma \vec{e}, \quad (5.1.16)$$

the structure of the system of equations remains essentially unchanged. In both of these special cases there is perfect coupling between electrical and magnetic effects, as in the more general case; the only simplification that occurs is that the electrical constitutive equation is changed.



More extreme changes occur in the equations when μ is small enough, or the time variations of the variables are slow enough, to allow (5.1.7) to be approximated by

$$\text{curl } \bar{e}'' = 0 . \quad (5.1.17)$$

Then the system equations reduce to

$$\text{curl } \bar{e}' = 0 \quad (5.1.18)$$

$$\text{div } \bar{i} = 0 \quad (5.1.19)$$

$$\bar{i} = (\sigma + \epsilon \frac{\partial}{\partial t}) \bar{e}' . \quad (5.1.20)$$

These equations can be combined using the scalar potential formulation shown in Section 3.10,

$$\text{div } ((\sigma + \epsilon \frac{\partial}{\partial t}) \nabla \phi) = 0 , \quad (5.1.21)$$

$$\text{or} \quad (\sigma + \epsilon \frac{\partial}{\partial t}) \nabla^2 \phi = 0 , \quad (5.1.22)$$

if σ and ϵ are not functions of position. In this special case the equations (5.1.3), (5.1.8), (5.1.10), (5.1.12), and (5.1.14) describing magnetic fields are unchanged. However, the simplified equations (5.1.18) through (5.1.20) can be analyzed by themselves, without consideration of magnetic effects. Equation (5.1.8) shows that the electric field produces a magnetic field, but the magnetic field has negligible effect on the electric field variables and can be ignored. The equations (5.1.18) through (5.1.20) characterize distributed parameter resistance-capacitance components that are currently being used, along with lumped components, in electrical networks.

A final special case is the stationary one, where all time derivatives of field variables are zero or negligible. Then the system model reduces to

$$\text{curl } \bar{e}' = 0 \quad (5.1.23)$$

$$\text{div } \bar{i} = 0 \quad (5.1.24)$$

$$\vec{I} = \sigma \vec{e}' \quad (5.1.25)$$

$$\text{curl } \vec{h} = 0 \quad (5.1.26)$$

$$\text{div } \vec{b} = 0 \quad (5.1.27)$$

$$\vec{h} = \vec{h}' + \vec{h}'' \quad (5.1.28)$$

$$\text{curl } \vec{h}'' = -\vec{I} \quad (5.1.29)$$

$$\vec{b} = \mu \vec{h}' \quad (5.1.30)$$

Here it is necessary to use the flux density $\vec{b}$ as one of the variables, rather than its time derivative $\dot{\vec{b}}$. The electrical equations (5.1.23) through (5.1.25) can be analyzed by themselves, since the magnetic field variables do not enter them. Also the magnetic equations (5.1.26) through (5.1.30) can be analyzed as a group, if the current density $\vec{I}$ is considered a known driving function. These two sets of equations are the basis for direct-current electric and magnetic network analysis.

In order to form a lumped approximation of the continuum system model of electromagnetic fields, as in Section 4, scalar variables are defined in terms of surface and line integrals of the vector variables of the continuum model. These variables are the time derivative of the magnetic flux associated with a surface S ,

$$\dot{\phi}_S = \int_S \dot{\vec{b}} \cdot d\vec{s} \quad (5.1.31)$$

the electric current associated with a surface S ,

$$\omega_S = \int_S \vec{I} \cdot d\vec{s} \quad (5.1.32)$$

the magnetic scalar potential ("magnetomotive force") associated with the two endpoints of a space curve,

$$\alpha_{ab} = \int_a^b \vec{h} \cdot d\vec{l} \quad (5.1.33)$$

and the electric scalar potential associated with the two endpoints

of a space curve,

$$\beta_{ab} = \int_a^b \bar{\mathbf{e}} \cdot d\mathbf{l} . \quad (5.1.34)$$

According to equations (5.1.11) and (5.1.12), the across variables can each be divided into two parts:

$$\begin{aligned} \alpha_{ab} &= \int_a^b \bar{\mathbf{h}} \cdot d\mathbf{l} \\ &= \int_a^b \bar{\mathbf{h}}' \cdot d\mathbf{l} + \int_a^b \bar{\mathbf{h}}'' \cdot d\mathbf{l} \\ \alpha_{ab} &= \alpha'_{ab} + \alpha''_{ab} ; \end{aligned} \quad (5.1.35)$$

$$\begin{aligned} \beta_{ab} &= \int_a^b \bar{\mathbf{e}} \cdot d\mathbf{l} \\ &= \int_a^b \bar{\mathbf{e}}' \cdot d\mathbf{l} + \int_a^b \bar{\mathbf{e}}'' \cdot d\mathbf{l} \\ \beta_{ab} &= \beta'_{ab} + \beta''_{ab} . \end{aligned} \quad (5.1.36)$$

This suggests that each graph edge in the linear graph associated with the discrete approximation should be replaced by two edges in series, as indicated in Figure 5.1.1. The through variables $\dot{\phi}$ and ω are



Figure 5.1.1. Linear graph edges for electric and magnetic variables

related to the across variables α' and β' by constitutive equations which for a linear, isotropic and homogeneous medium are derived from equations (5.1.13) and (5.1.14). For example, for a set of micro-elements with faces parallel to the coordinate planes in a Cartesian coordinate system, with side lengths Δx , Δy , and Δz , and for a graph

edge associated with a microelement face with sides Δy and Δz ,

$$\begin{aligned}
 \dot{\phi}_x &= \int_{\Delta y \Delta z} \dot{\vec{b}} \cdot d\vec{s} \\
 &= \dot{b}_x \Delta y \Delta z \\
 &= \mu \frac{\partial h'_x}{\partial t} \Delta y \Delta z \\
 &= \frac{\mu \Delta y \Delta z}{\Delta x} \frac{\partial}{\partial t} (h'_x \Delta x) \\
 \dot{\phi}_x &= \frac{\mu \Delta y \Delta z}{\Delta x} \frac{\partial}{\partial t} \alpha'_x, \quad (5.1.37)
 \end{aligned}$$

$$\begin{aligned}
 \omega_x &= \int_{\Delta y \Delta z} \dot{\vec{i}} \cdot d\vec{s} \\
 &= \dot{i}_x \Delta y \Delta z \\
 &= \left(\sigma e'_x + \epsilon \frac{\partial e'_x}{\partial t} \right) \Delta y \Delta z \\
 &= \left(\frac{\sigma \Delta y \Delta z}{\Delta x} + \frac{\epsilon \Delta y \Delta z}{\Delta x} \frac{\partial}{\partial t} \right) e'_x \Delta x \\
 \omega_x &= \left(\frac{\sigma \Delta y \Delta z}{\Delta x} + \frac{\epsilon \Delta y \Delta z}{\Delta x} \frac{\partial}{\partial t} \right) \beta'_x. \quad (5.1.38)
 \end{aligned}$$

The coefficients $\frac{\mu \Delta y \Delta z}{\Delta x}$, $\frac{\sigma \Delta y \Delta z}{\Delta x}$, and $\frac{\epsilon \Delta y \Delta z}{\Delta x}$ can be identified as the inductance, conductance and capacitance of a microelement, respectively.

Because two different physical effects are under analysis, it is clear that the linear graph associated with a discrete approximation for an electromagnetic field should contain two disjoint subgraphs, one for electrical and one for magnetic variables. The electrical and magnetic microelements need not be the same, and in fact it is convenient to choose two different sets of microelements in a very special way. After choosing a set of electrical microelements, a set of magnetic microelements is chosen such that the center of each magnetic microelement is at the vertex of an electrical microelement, and vice versa. Then the two associated linear graphs are interlaced but disjoint. For example, a single electrical edge and the four magnetic

graph edges which surround it are shown in Figure 5.1.2. The coupling

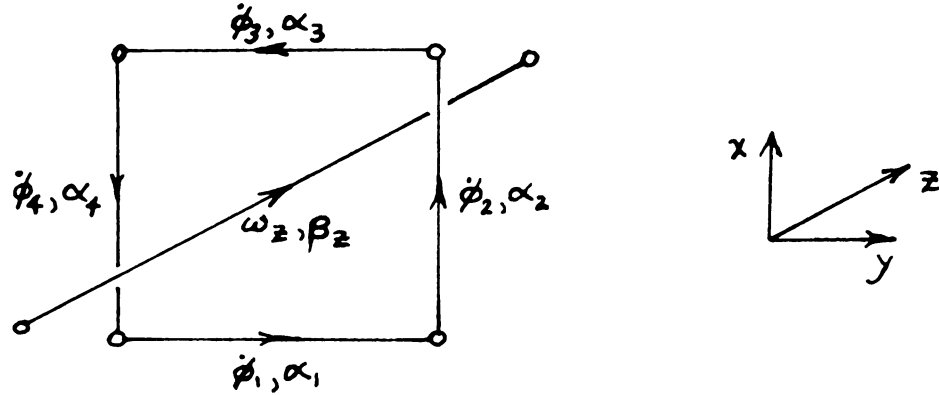


Figure 5.1.2. Magnetic graph edges and an associated electric graph edge

equations (5.1.7) and (5.1.8) provide the remaining constitutive equations for the discrete approximation:

$$\begin{aligned}
 \omega_z &= \int_{\Delta x \Delta y} \vec{f} \cdot d\vec{s} \\
 &= - \oint \text{curl } \vec{h}'' \cdot d\vec{l} \\
 &= -(h_1'' \Delta y + h_2'' \Delta x + h_3'' \Delta y + h_4'' \Delta x) \\
 \omega_z &\triangleq -\alpha_1'' - \alpha_2'' - \alpha_3'' - \alpha_4'' . \quad (5.1.39)
 \end{aligned}$$

In the same way, each magnetic through variable can be expressed in terms of four electric across variables, such as

$$\begin{aligned}
 \dot{\phi}_z &= \int_{\Delta x \Delta y} \vec{b} \cdot d\vec{s} \\
 &= \oint \text{curl } \vec{e}'' \cdot d\vec{l} \\
 &= e_1'' \Delta y + e_2'' \Delta x + e_3'' \Delta y + e_4'' \Delta x \\
 \dot{\phi}_z &\triangleq \beta_1'' + \beta_2'' + \beta_3'' + \beta_4'' . \quad (5.1.40)
 \end{aligned}$$

5.2 Diffusion Processes

Many physical processes, such as heat conduction and particle diffusion, are described or approximately described by the three equations⁽²³⁾

$$\text{div } \vec{j}' + \frac{\partial \rho}{\partial t} = 0 , \quad (5.2.1)$$

$$\vec{j}' = -k_1 \text{grad } \phi , \quad (5.2.2)$$

$$\rho = k_2 \phi . \quad (5.2.3)$$

Substitution of (5.2.2) and (5.2.3) into (5.2.1) yields

$$\text{div } (k_1 \text{grad } \phi) = k_2 \frac{\partial \phi}{\partial t}$$

or if k_1 is constant,

$$\nabla^2 \phi = \frac{k_2}{k_1} \frac{\partial \phi}{\partial t} \quad (5.2.4)$$

which is the diffusion equation. In heat conduction, for example, $\vec{j}'$ is heat flow per unit area, ρ is amount of heat per unit volume, ϕ is temperature, k_1 is specific heat capacity, and k_2 is heat capacity per unit volume.

The equations (5.2.1) through (5.2.3) are not in the general form exhibited in Section 3, but they serve as a basis for formulating a system model in that form. As a first step, it is clear that the scalar point function ϕ is a scalar across variable and that an associated vector across variable should be defined as

$$\vec{g} = -\text{grad } \phi . \quad (5.2.5)$$

In the example of heat conduction, $\vec{g}$ would be called the temperature gradient. From (5.2.5), one of the field equations is

$$\text{curl } \vec{g} = -\text{curl grad } \phi = 0 , \quad (5.2.6)$$

and (5.2.2) becomes the constitutive equation

$$\vec{j}' = k_1 \vec{g} . \quad (5.2.7)$$

The continuity equation (5.2.1) shows that the through variable $\bar{j}'$ is not a convenient variable because its divergence is not zero. A convenient through variable can be constructed by the same artifice used to define total electric current density in electromagnetic field theory, as in Section 5.1. An auxiliary variable $\bar{d}$ is defined by

$$\rho = \text{div } \bar{d} . \quad (5.2.8)$$

Then (5.2.1) becomes

$$\text{div} \left(\bar{j}' + \frac{\partial \bar{d}}{\partial t} \right) = 0 , \quad (5.2.9)$$

$$\text{or} \quad \text{div } \bar{j} = 0 , \quad (5.2.10)$$

$$\text{where} \quad \bar{j} = \bar{j}' + \bar{j}'' \quad (5.2.11)$$

$$\text{and} \quad \bar{j}'' = \frac{\partial \bar{d}}{\partial t} . \quad (5.2.12)$$

A constitutive equation for the auxiliary variable $\bar{j}''$ can be obtained from (5.2.3), (5.2.8) and (5.2.12):

$$\begin{aligned} \text{div } \bar{j}'' &= \frac{\partial \rho}{\partial t} \\ &= k_2 \text{grad } \frac{\partial \phi}{\partial t} \end{aligned} \quad (5.2.13)$$

$$\begin{aligned} \text{grad div } \bar{j}'' &= k_2 \text{grad } \frac{\partial \phi}{\partial t} \\ \text{grad div } \bar{j}'' &= -k_2 \bar{g} . \end{aligned} \quad (5.2.14)$$

The equations above constitute a system model for diffusion processes in the general form of Section 3; they consist of the field equations (5.2.6) and (5.2.10), and the constitutive equations (5.2.7), (5.2.11) and (5.2.14). The use of equation (5.2.8) as a definition of the auxiliary variable $\bar{d}$ is essentially the same as the method used by Sommerfeld⁽²²⁾ in stating a definition of the electrical variable called flux density or electric displacement density. It does not define $\bar{d}$ completely because both the divergence and curl of a vector point function must be specified in order to define it completely.



Since $\bar{d}$ is an auxiliary variable there is nothing to prevent its curl from being arbitrarily specified.

In order to form a discrete approximation of this system model, a scalar through variable can be defined as

$$\omega_S = \int_S \bar{j} \cdot d\bar{s} , \quad (5.2.15)$$

and a scalar across variable can be defined as

$$\phi_{ab} = \int_a^b \bar{g} \cdot d\bar{l} . \quad (5.2.16)$$

These variables would be interpreted physically, for a heat flow problem, as the heat flow associated with a surface S and the temperature difference between two points a and b , respectively. In order to form a discrete approximation for a diffusion process, components can be chosen and a linear graph associated with the components as described in Section 4. Then the field equations (5.2.6) and (5.2.10) become the ordinary Kirchhoff law equations of lumped systems analysis.

The constitutive equation (5.2.7) provides component equations for the discrete approximation. For a linear graph edge associated with a microelement face perpendicular to the x axis in a Cartesian coordinate system,

$$\begin{aligned} \omega'_x &= \int_{\Delta y \Delta z} \bar{j}' \cdot d\bar{s} \\ &= j'_x \Delta y \Delta z \\ &= \frac{k_1 \Delta y \Delta z}{\Delta x} g_x \Delta x \\ \omega'_x &= \frac{k_1 \Delta y \Delta z}{\Delta x} \phi'_x . \end{aligned} \quad (5.2.17)$$

Equation (5.2.13) shows the dependence of the auxiliary variable $\bar{j}$ on the scalar point function ϕ rather than on the vector across variable $\bar{g}$. Consider the volume integral of (5.2.13) over

a microelement with volume ΔV and bounding surface ΔS :

$$\begin{aligned} \int_{\Delta V} (\text{div } \mathbf{J}'') dV &= \int_{\Delta V} k_2 \frac{\partial \phi}{\partial t} dV \\ \oint_{\Delta S} \mathbf{J}'' \cdot d\bar{s} &= \int_{\Delta V} k_2 \frac{\partial \phi}{\partial t} dV \\ \omega''_{\Delta V} &\triangleq (k_2 \Delta V) \frac{\partial \phi}{\partial t}. \end{aligned} \quad (5.2.18)$$

The scalar through variable $\omega''_{\Delta V}$ and the scalar across variable ϕ must be associated with an external reference vertex in the linear graph associated with the discrete approximation. The linear graph edges associated with each microelement in the system then have the form indicated in Figure 5.2.1. The through variable $\omega''_{\Delta V}$ corresponds

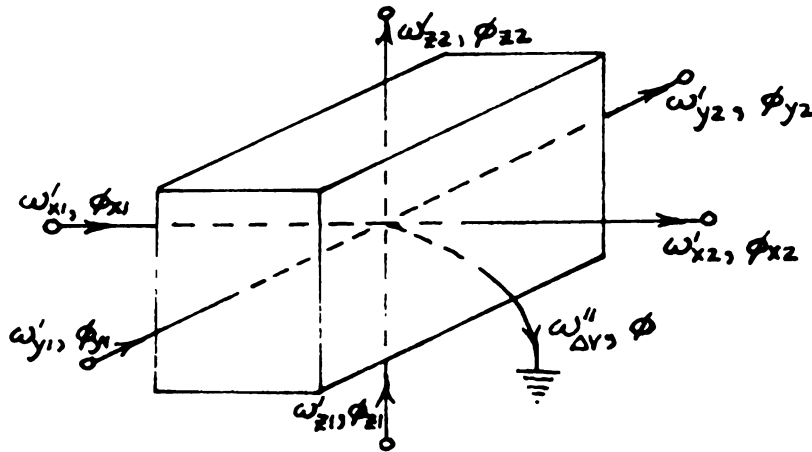


Figure 5.2.1. Microelement for diffusion process and associated graph edges

closely to the "inertial forces" in lumped mechanical systems or to capacitive currents in lumped electric systems.

5.3 Fluid Mechanics

Variables of central importance in the study of the motion of fluids are the pressure p and the velocity of the fluid motion $\bar{v}$. The

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$$\begin{aligned} \int_{\Delta V} (\text{div } \mathbf{j}'') dv &= \int_{\Delta V} k_2 \frac{\partial \phi}{\partial t} dv \\ \oint_{\Delta S} \mathbf{j}'' \cdot d\mathbf{s} &= \int_{\Delta V} k_2 \frac{\partial \phi}{\partial t} dv \\ \omega''_{\Delta V} &\triangleq (k_2 \Delta V) \frac{\partial \phi}{\partial t}. \end{aligned} \quad (5.2.18)$$

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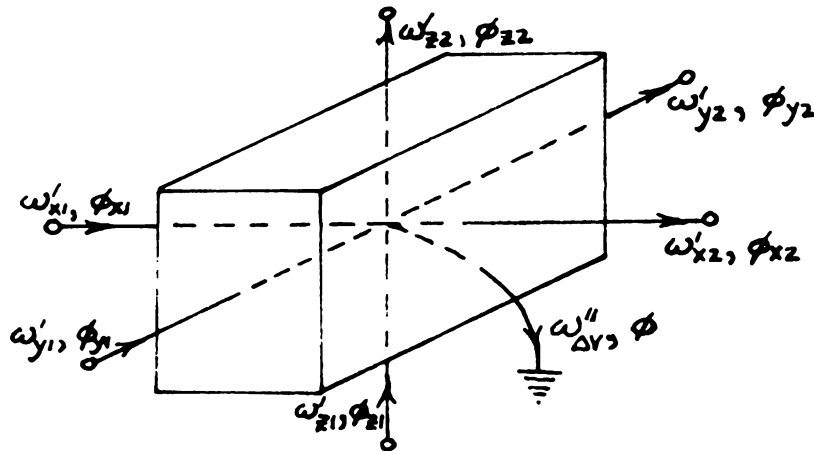


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5.3 Fluid Mechanics

Variables of central importance in the study of the motion of fluids are the pressure p and the velocity of the fluid motion $\bar{v}$. The

analysis of fluid motion is usually directed toward finding the velocity of a fluid with specified constitutive characteristics, under the influence of specified body forces and boundary conditions. Of the many equations used to describe fluid motion, a few can be considered especially important and basic. The principle of conservation of mass leads to the equation of continuity⁽²⁴⁾,

$$\frac{d\rho}{dt} + \rho \operatorname{div} \bar{v} = 0$$

or
$$\frac{\partial \rho}{\partial t} + \bar{v} \cdot \operatorname{grad} \rho + \rho \operatorname{div} \bar{v} = 0 \quad (5.3.1)$$

where ρ is the fluid density. Conservation of linear momentum leads to Cauchy's equation of motion,

$$\nabla \cdot \bar{T} + \rho \bar{f} = \rho \frac{d\bar{v}}{dt}, \quad (5.3.2)$$

where $\bar{T}$ is the second order stress tensor and $\bar{f}$ is the external body force per unit mass. Furthermore, conservation of angular momentum requires that the stress tensor be symmetric. A constitutive equation for a fluid is usually considered to be an equation that relates the stress tensor and the rate of deformation tensor $\bar{V}$, the symmetric part of the velocity gradient tensor $\operatorname{grad} \bar{v}$. A large class of fluids are characterized by the constitutive equation of Stokes,

$$\bar{T} = -p\bar{I} + \bar{P} \quad (5.3.3)$$

$$\bar{P} = \beta \bar{V} + \gamma \bar{V} \cdot \bar{V}, \quad (5.3.4)$$

where $\bar{I}$ is an identity tensor, $\bar{P}$ is the viscous stress tensor, and β and γ are scalars which in general are functions of the components of $\bar{V}$. A Newtonian fluid is a linear Stokesian fluid, for which (5.3.4) becomes

$$\bar{P} = \lambda (\operatorname{tr} \bar{V}) \bar{I} + 2\mu \bar{V} \quad (5.3.5)$$

where $(\operatorname{tr} \bar{V})$ is the trace of the rate of deformation tensor, and λ and μ are constants. Substitution of (5.3.3) into (5.3.2) leads to

a microelement with volume ΔV and bounding surface ΔS :

$$\begin{aligned} \int_{\Delta V} (\text{div } \mathbf{j}'') dv &= \int_{\Delta V} k_2 \frac{\partial \phi}{\partial t} dv \\ \oint_{\Delta S} \mathbf{j}'' \cdot d\mathbf{s} &= \int_{\Delta V} k_2 \frac{\partial \phi}{\partial t} dv \\ \omega''_{\Delta V} &\triangleq (k_2 \Delta V) \frac{\partial \phi}{\partial t}. \end{aligned} \quad (5.2.18)$$

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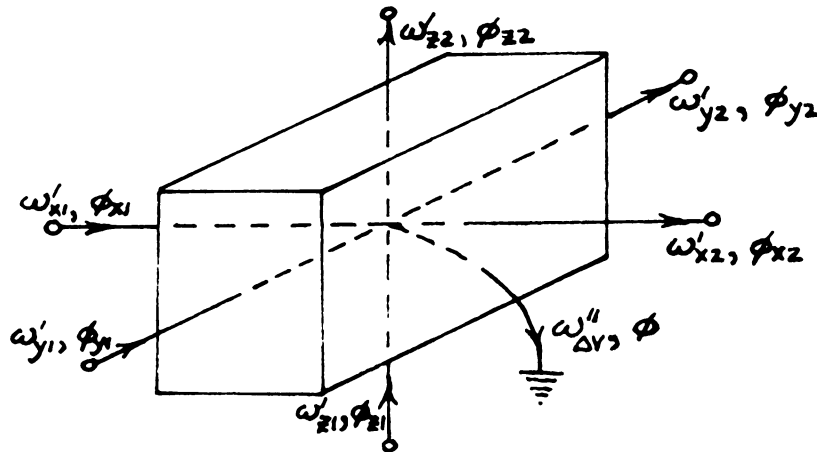


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where $\bar{I}$ is an identity tensor, $\bar{P}$ is the viscous stress tensor, and β and γ are scalars which in general are functions of the components of $\bar{V}$. A Newtonian fluid is a linear Stokesian fluid, for which (5.3.4) becomes

$$\bar{P} = \lambda (\operatorname{tr} \bar{V}) \bar{I} + 2\mu \bar{V} \quad (5.3.5)$$

where $(\operatorname{tr} \bar{V})$ is the trace of the rate of deformation tensor, and λ and μ are constants. Substitution of (5.3.3) into (5.3.2) leads to



$$\rho \frac{d\bar{\mathbf{v}}}{dt} = \rho \bar{\mathbf{f}} - \text{grad } \pi + \nabla \cdot \bar{\mathbf{P}} . \quad (5.3.6)$$

In the special case of a Newtonian fluid, (5.3.6) becomes the Navier-Stokes equation,

$$\rho \frac{d\bar{\mathbf{v}}}{dt} = \rho \bar{\mathbf{f}} - \text{grad } \pi + (\lambda + \mu) \text{grad div } \bar{\mathbf{v}} + \mu \nabla^2 \bar{\mathbf{v}} . \quad (5.3.7)$$

In order to cast the equations of fluid mechanics in the form described in Section 3, basic variables must be chosen from among the scalars ρ and π , the vector $\bar{\mathbf{v}}$ and the tensors $\bar{\mathbf{T}}$, $\bar{\mathbf{V}}$, $\bar{\mathbf{P}}$ and $\text{grad } \bar{\mathbf{v}}$. Attention is directed toward the pressure π and the velocity $\bar{\mathbf{v}}$, because these variables are of particular interest and because they are the variables commonly used in describing lumped approximations of fluid systems. The appearance of $\text{grad } \pi$ in (5.3.6) suggests the use of the pressure gradient

$$\bar{\mathbf{p}} = -\text{grad } \pi \quad (5.3.8)$$

as the vector across variable; $\bar{\mathbf{p}}$ automatically satisfies the relation

$$\text{curl } \bar{\mathbf{p}} = -\text{curl grad } \pi = 0 . \quad (5.3.9)$$

In the special case of an incompressible fluid, (5.3.1) reduces to

$$\text{div } \bar{\mathbf{v}} = 0 , \quad (5.3.10)$$

and this suggests that the velocity be used as the vector through variable. In order to account for compressibility, a device similar to that used in Section 5.2 can be employed to define a convenient through variable. Let

$$\bar{\mathbf{u}} = \bar{\mathbf{v}} + \bar{\mathbf{w}} \quad (5.3.11)$$

$$\text{where} \quad \text{div } \bar{\mathbf{w}} = \frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{1}{\rho} \bar{\mathbf{v}} \cdot \text{grad } \rho . \quad (5.3.12)$$

$$\begin{aligned} \text{Then} \quad \text{div } \bar{\mathbf{u}} &= \text{div } \bar{\mathbf{v}} + \text{div } \bar{\mathbf{w}} \\ &= \text{div } \bar{\mathbf{v}} + \frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{1}{\rho} \bar{\mathbf{v}} \cdot \text{grad } \rho \\ \text{div } \bar{\mathbf{u}} &= 0 \end{aligned} \quad (5.3.13)$$

for both compressible and incompressible fluids. The choice of

velocity and pressure gradient as variables leads to a classification of the equations describing fluid motion in the form of Section 3.

The equations consist of the field equations (5.3.9) and (5.3.13), and the constitutive equations (5.3.11), (5.3.12) and (5.3.7). For a compressible fluid, (5.3.12) must be supplemented by an equation of state, which relates the scalars ρ and π . The constitutive equation (5.3.7) can be written in the form

$$\begin{aligned}\bar{p} &= \rho \frac{d\bar{v}}{dt} - \rho \bar{f} - (\lambda + \mu) \text{grad div } \bar{v} - \mu \nabla^2 \bar{v} \\ \bar{p} &= \rho \frac{\partial \bar{v}}{\partial t} + \rho \bar{v} \cdot \text{grad } \bar{v} - \rho \bar{f} - (\lambda + \mu) \text{grad div } \bar{v} - \mu \nabla^2 \bar{v} .\end{aligned}\quad (5.3.14)$$

The term $\rho \bar{f}$ in (5.3.14) must be interpreted as a forcing function.

The presence of the convective component $\rho \bar{v} \cdot \text{grad } \bar{v}$ is responsible for the inherent nonlinearity of the equations, even in the incompressible case where ρ is a constant.

In a lumped approximation of the continuum model, the scalar variables associated with the vector variables are the across variable, pressure difference,

$$\pi_{ab} = \int_a^b \bar{p} \cdot d\bar{l} , \quad (5.3.15)$$

and the through variable, volume flow rate,

$$\phi_S = \int_S \bar{u} \cdot d\bar{s} . \quad (5.3.16)$$

Consider, for simplicity, the case of a compressible inviscid fluid in which body forces are negligible and in which pressure variations about a mean value π_0 are small enough so that nonlinear terms can be neglected. This set of conditions closely approximates the problem of acoustics. The system equations then become

$$\text{curl } \bar{p} = 0 \quad (5.3.17)$$

$$\text{div } \bar{u} = 0 \quad (5.3.18)$$



$$\bar{u} = \bar{v} + \bar{w} \quad (5.3.19)$$

$$\text{div } \bar{w} = \frac{1}{\bar{\rho}} \frac{\partial \rho}{\partial t} \quad (5.3.20)$$

$$\bar{p} = \bar{\rho} \frac{\partial \bar{v}}{\partial t} . \quad (5.3.21)$$

From (5.3.21), the pressure difference associated with a linear graph edge corresponding to a microelement face normal to the x-axis is

$$\begin{aligned} \pi_x &= \int_x^{x+\Delta x} \bar{p} \cdot d\bar{l} \\ &\approx p_x \Delta x \\ &\approx \bar{\rho} \frac{\partial v_x}{\partial t} \Delta x \\ &\approx \frac{\rho_0 \Delta x}{\Delta y \Delta z} \frac{d(v_x \Delta y \Delta z)}{dt} \\ \pi_x &\approx \frac{\rho_0 \Delta x}{\Delta y \Delta z} \frac{d\phi_x}{dt}, \end{aligned} \quad (5.3.22)$$

where ρ_0 is the mean density corresponding to the mean pressure π_0 .

A volume integral of (5.3.20) leads to

$$\begin{aligned} \phi_{\Delta V} &= \int_{\Delta V} (\text{div } \bar{w}) dv \\ &= \oint_S \bar{w} \cdot d\bar{s} \\ &= \int_{\Delta V} \frac{1}{\bar{\rho}} \frac{\partial \rho}{\partial t} dv \\ &\approx \frac{1}{\bar{\rho}} \frac{\partial \rho}{\partial t} \Delta V \\ &\approx \frac{1}{\bar{\rho}_0} \frac{d\rho}{d\pi} \Delta V \frac{d\pi}{dt} \\ \phi_{\Delta V} &\approx \frac{\Delta V}{K} \frac{d\pi}{dt}, \end{aligned} \quad (5.3.24)$$

where $K = (\bar{\rho} \frac{d\pi}{d\rho})_{\pi_0}$ is the bulk modulus of the fluid. The component equations (5.3.22) and (5.3.24), along with the topological equations which are the integrals of (5.3.17) and (5.3.18), lead to a system linear graph similar to the one shown in Figure 5.2.1. The symbol $\phi_{\Delta V}$ represents a volume flow from each vertex of the graph to an external reference vertex.



The two terms due to viscosity in the constitutive equation (5.3.14) are more difficult to approximate in the lumped model. One of the viscosity terms is

$$\bar{p}_1 = -(\lambda + \mu) \text{grad div } \bar{v} . \quad (5.3.25)$$

The corresponding pressure difference, for a graph edge associated with the x coordinate direction, is

$$\begin{aligned} \pi_{1x} &= \int_x^{x+\Delta x} \bar{p}_1 \cdot d\bar{l} \\ &= (\lambda + \mu) ((\text{div } \bar{v})_x - (\text{div } \bar{v})_{x+\Delta x}) \\ &= -(\lambda + \mu) ((\text{div } \bar{w})_x - (\text{div } \bar{w})_{x+\Delta x}) \\ \pi_{1x} &= \left(\frac{\lambda + \mu}{\Delta V} \right) ((\phi_{\Delta V})_{x+\Delta x} - (\phi_{\Delta V})_x) , \end{aligned} \quad (5.3.26)$$

from equations (5.3.11), (5.3.13) and (5.3.23). Thus there is a pressure difference associated with each edge which is proportional to the difference between the values of the volume flow rates directed from the two vertices of the edge, toward the reference vertex. The second viscosity term in (5.3.14) is

$$\bar{p}_2 = -\mu \nabla^2 \bar{v} . \quad (5.3.27)$$

This leads to a pressure difference, for a graph edge associated with the x coordinate direction, of

$$\begin{aligned} \pi_{2x} &= \int_x^{x+\Delta x} \bar{p}_2 \cdot d\bar{l} \\ &= -\mu \Delta x \nabla^2 v_x \\ &= -\mu \Delta x \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) . \end{aligned}$$

The usual incremental approximations of the partial derivatives can be used:

$$\frac{\partial^2 v_x}{\partial x^2} \approx \frac{v_x(x+\Delta x) + v_x(x-\Delta x) - 2v_x}{(\Delta x)^2} .$$



$$\frac{\partial^2 v_x}{\partial x^2} = \frac{1}{\Delta y \Delta z} \frac{e_x(x+\Delta x) + e_x(x-\Delta x) - 2e_x}{(\Delta x)^2}. \quad (5.3.28)$$

Thus

$$\pi_{2x} = \frac{\mu \Delta x}{\Delta y \Delta z} \left(\frac{e_x(x+\Delta x) + e_x(x-\Delta x) - 2e_x}{(\Delta x)^2} + \frac{e_x(y+\Delta y) + e_x(y-\Delta y) - 2e_x}{(\Delta y)^2} + \frac{e_x(z+\Delta z) + e_x(z-\Delta z) - 2e_x}{(\Delta z)^2} \right). \quad (5.3.29)$$

Here e_x is the volume flow rate associated with the graph edge with which π_{2x} is associated; $e_x(x+\Delta x)$ and $e_x(x-\Delta x)$ are the flow rates associated with the two graph edges which are parallel to this but displaced from it by amounts $\pm \Delta x$ in the x coordinate direction. Thus this pressure difference due to viscosity depends on a flow rate and on the six parallel, adjacent flow rates surrounding the first.

5.4 Magnetohydrodynamics

Magnetohydrodynamics (MHD) has to do with the interaction of an electromagnetic field with a plasma, or ionized gas⁽²⁵⁾. An analysis of MHD involves the properties of the electromagnetic field, the mechanical properties of the fluid, and the coupling between the two. The motion of the fluid in a magnetic field produces a component of electric field intensity, $\vec{e}'' = -\vec{v} \times \vec{b}$, in addition to the two components discussed in Section 5.1. This coupling from fluid to electrical effects is accompanied by a coupling from electrical to fluid effects; a body force $\vec{p}_e = -\vec{i} \times \vec{b}$ must be added to the constitutive equation (5.3.14) in Section 5.3.

The equations needed to characterize MHD are the electromagnetic field equations (5.1.3), (5.1.4) and (5.1.7) through (5.1.14), and the fluid equations (5.3.9) and (5.3.11) through (5.3.14). Equation



(5.1.11) must be changed to

$$\bar{\mathbf{e}} = \bar{\mathbf{e}}' + \bar{\mathbf{e}}'' + \bar{\mathbf{e}}''' , \quad (5.4.1)$$

where

$$\bar{\mathbf{e}}''' = -\bar{\mathbf{v}} \times \bar{\mathbf{b}}. \quad (5.4.2)$$

Also, equation (5.3.14) should be changed to

$$\begin{aligned} \bar{p} = & \rho \frac{\partial \bar{v}}{\partial t} + \rho \bar{v} \cdot \text{grad } \bar{v} - p \bar{f} - (\lambda + \mu) \text{grad div } \bar{v} \\ & - \mu \nabla^2 \bar{v} - \bar{f} \times \bar{b} , \end{aligned} \quad (5.4.3)$$

where $\bar{f}$ is now the resultant of any body forces other than that due to the electromagnetic field. The coupling terms (5.4.2) and

$$\bar{p}_e = -\bar{f} \times \bar{b} \quad (5.4.4)$$

correspond to "inverse" couplers because they relate across variables to through variables. It should be noted that the coupling equations require the use of the magnetic flux density $\bar{b}$, even though the time derivative $\dot{\bar{b}}$ was found to be a more convenient variable in Section 5.1.

A discrete approximation of the continuum model can be formed for the electromagnetic field variables as in Section 5.1 and for the fluid variables as in Section 5.3. Thus three disjoint graphs are required, for the electrical, magnetic and fluid variables. Discrete approximations for the coupler equations (5.4.2) and (5.4.4) must be added to complete the equations describing the MHD process.

Consider a special case, which is the situation usually analyzed in connection with MHD power generation and in connection with solar MHD waves. Let the magnetic flux density be

$$\bar{b} = \bar{b}_0 + \bar{b}_1 \quad (5.4.5)$$

where

$$b_0 \gg b_1$$

and where $\bar{b}_0$ is constant in time and space. This amounts to linearizing the coupling equations (5.4.2) and (5.4.4), so that they

become

$$\bar{e}''' = -\bar{v} \times \bar{b}_0 \quad (5.4.6)$$

$$\bar{p}_e = -\bar{i} \times \bar{b}_0 . \quad (5.4.7)$$

For simplicity, let $\bar{b}_0$ be directed along the z coordinate axis, so that

$$\bar{b}_0 = b_0 \bar{k} . \quad (5.4.8)$$

Then the electric potential difference due to the coupling term (5.4.6), for a linear graph edge associated with the x axis, is

$$\begin{aligned} \beta_x''' &= \int_x^{x+\Delta x} \bar{e}''' \cdot d\bar{l} \\ &\doteq e_x''' \Delta x \\ &\doteq -v_y b_0 \Delta x \\ &\doteq -\frac{b_0}{\Delta z} (v_y \Delta x \Delta z) \\ \beta_x''' &\doteq -\frac{b_0}{\Delta z} e_y . \end{aligned} \quad (5.4.9)$$

Similarly,

$$\beta_y''' \doteq \frac{b_0}{\Delta z} e_x , \quad (5.4.10)$$

$$\beta_z''' = 0 . \quad (5.4.11)$$

Also, the pressure differences due to the coupling term (5.4.7) are

$$\pi_{ex} \doteq -\frac{b_0}{\Delta z} \omega_y , \quad (5.4.12)$$

$$\pi_{ey} \doteq \frac{b_0}{\Delta z} \omega_x , \quad (5.4.13)$$

$$\pi_{ez} = 0 . \quad (5.4.14)$$

Thus the potential differences and pressure differences, in the x and y coordinate directions, due to the coupling are related to the volume flow rates and electric currents in the respective normal directions. The use of equations (5.4.9), (5.4.10), (5.4.12) and (5.4.13) can be facilitated by choosing microelements for the fluid variables which

are displaced in the x and y coordinate directions from the micro-elements used for the electric variables.



6. CONCLUSION

Section 3 of this thesis presents a formal structure for distributed parameter system theory, built on a postulate concerning the variables required to characterize a physical system, two postulates regarding the nature of these variables, and four important theorems based on the postulates. These lead to convenient forms in which system models can be generated, at least when the medium under analysis is characterized by linear constitutive equations. This structure, along with the content of the postulates and theorems, is directly parallel to that of lumped parameter system theory. The important postulates and theorems which form the structure of lumped parameter system theory are outlined in Section 2.

Section 4 of the thesis presents a method by which a discrete approximation of a distributed parameter system can be formulated, based on the structure of Section 3. Because of the connection between the structure of distributed parameter system theory and that of lumped parameter system theory, the resulting discrete approximation is a lumped parameter system model which fits exactly into the structure of Section 2. The examples of Section 5 exhibit both continuum models and lumped approximations for four different kinds of physical systems which agree with the abstract prototypes of Sections 3 and 4.

Recent efforts toward the development of a formal abstract structure for lumped parameter system analysis have resulted in two important scientific advances. One is the research in the analysis



of sophisticated lumped parameter systems that has been based on the abstract theory. The second important result of the development of this structure is concerned with exposition. Lumped parameter system theory has become recognized as a fundamental engineering discipline, and the abstract structure has provided an effective teaching tool. The formal structure of distributed parameter system theory should provide the same kind of stimulus for research and teaching in this area of study.

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