This is to certify that the

thesis entitled

Derivatives of Frequency
Response Peaks

presented by

Raymond Brent Thompson

has been accepted towards fulfillment
of the requirements for

Masters degree in Mechanical Engineering

James Bernard
Major professor

Date 5/20/83

DERIVATIVES

OF

FREQUENCY RESPONSE

PEAKS

By

Raymond Brent Thompson

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

1983

ABSTRACT

DERIVATIVES OF FREQUENCY RESPONSE PEAKS

By

Raymond Brent Thompson

When a system is excited at a natural frequency, the magnitude of the response becomes large. This thesis concerns a method of redesign to reduce the magnitude of the forced response at resonance.

The method uses derivatives of the forced response to compute a first order Taylor series in the design change. This series can then be used with standard minimization techniques to select the appropriate design change to reduce the response at resonance.

ACKNOWLEDGEMENTS

I would like to express my appreciation to my major professor, Dr. James Bernard. His expert guidance, sincere friendship and encouragement greatly aided in this research.

Also, many thanks to Dr. Ronald Rosenberg and Dr. Brian Thompson whose comments concerning this thesis are greatly appreciated.

I would especially like to thank three of my colleagues at Michigan State University, Dr. John Starkey (now a professor at Purdue University), Matt Rizai and Guy Allen, who all leant a great deal of advice, as well as friendship during my graduate career.

A special thanks goes to my parents, Clifton and Lorraine Thompson, for their continuous loving support and understanding.

Also, I would like to thank the entire staff at the Case Center for Computer-Aided Design at Michigan State University for the use of the computer facilities, financial support and the friendship gained through my association with it.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

A system excited at or near one of its resonant can exibit large responses. This thesis presents a method to compute the sensitivity of this resonant response to changes in the system.

Chapter 2 presents some background information on modal analysis of damped systems. Equations of motion will be presented. From these equations an expression for the magnitude of the frequency response is obtained. This expression is then used to facilitate the derivation of the derivatives of the response at resouance.

Chapter 3 will review the derivation of the derivatives of the eigenvalues, eigenvectors, undamped natural frequencies and the damping ratios of a system.

Chapter 4 presents two methods for assesing the sensitivity of the magnitude of the frequency response. The first method reviews of a formulation which yields the sensitivity of the magnitude of the response at any frequency. The second method presents an equation for the magnitude of the response at a resonant frequency. It then finds the sensitivity of this peak to design changes in the system.

Chapter 5 introduces an optimization scheme to reduce the magnitude of a resonant peak. A penalty function formulation facilitates the selection of an appropriate change to reduce the peak response. The use of the technique is illustrated through an example.

Finally, concluding remarks summarize the thesis and discusses future work.

CHAPTER 2

MODAL ANALYSIS OF DAMPED SYSTEMS: GENERAL CASE WITH A SINOSOIDAL (EXPONENTIAL) FORCING FUNCTION

In order to lay the groundwork for the derivation of the derivatives of the magnitudes at constant and variable frequencies, this section reviews some of the fundamentals of modal analysis.

The equations of motion for a forced vibratory system with n-degrees of freedom are

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F(t)\} \tag{2.1}$$

where:

$[m]$ = mass matrix $\{\ddot{x}\}$ = Acceleration vector

$[k]$ = Stiffness matrix $\{\dot{x}\}$ = Velocity vector

$[c]$ = Viscous damping matrix $\{x\}$ = Displacement vector

$\{F(t)\}$ = Force vector

t = time

and the dot indicates differentiation with respect to time. In general, we assume the mass, stiffness and damping matricies to the positive definite and symmetric. In the case which will be discussed here $\{F(t)\}$ will be harmonic, i.e.,:

$$\{F(t)\} = \{Fo\}e^{(i\omega t)}$$

where

$\{Fo\}$ = Magnitude of the Force

i = square root of -1

ω = frequency of exitation

Our interest here is in the steady state response.

In general, the system of (2.1) cannot be uncoupled by using the eigenvalues generated from the undamped system [1]. A first order transformation of the form:

$$\{y\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix}$$

may be used. This leads to a set of 2n symmetric first order ordinary differential equations:

$$[M]\,\{\dot{y}\} + [m]\,\{y\} = \{F(t)\} \tag{2.2}$$

where:

$$[M] = \begin{bmatrix} [0] & [m] \\ [m] & [c] \end{bmatrix} \quad [K] = \begin{bmatrix} [-m] & [0] \\ [0] & [k] \end{bmatrix} \quad \{F(t)\} = \begin{Bmatrix} \{0\} \\ \{F(t)\} \end{Bmatrix}$$

This set of equations leads to a set of 2n eigenvalues (λ_i) and 2n eigenvectors of the form:

$$\{U_i\} = \begin{Bmatrix} \lambda_i\{u_i\} \\ \{u_i\} \end{Bmatrix} \tag{2.3}$$

where:

$\lambda_i\{u_i\}$ Eigenvectors corresponding to velocities of the n-degree of freedom system.

$\{u_i\}$ = Eigenvector entries corresponding to the coordinates of the n-degree of freedom system.

For non-repeated eigenvalues, the eigenvectors are [M] and [K] weighted orthogonal and therefore can decouple equation (2.2) [2]. This decoupling can be accomplished by transforming to modal coordinates, using the relation:

$$\{y\} = [U]\{q\} \tag{2.4}$$

where:

$\{q\}$ = Modal coordinates

$[U]$ Modal matrix (matrix of eigenvectors)

The pre-multiplication of equation (2.2) by $[U]^t$ (the transpose of the modal matrix) yields:

$$[U]^t[M][U]\{\dot{q}\}+[U]^t[K][U]\{q\}=[U]^t\{f(t)\} \tag{2.5}$$

Equations 2.5 are uncoupled.

Equations 2.5 are often modified by normalizing the ith element of $[U]^t\{f(t)\}$ with $(U^tMU)_{ii}$, the i^{th} diagonal element of $[U]^t[M][U]$. This yields

$$[I]\{\dot{q}\}-[\lambda]\{q\}= [U]^t\{fn(t)\} \tag{2.6}$$

where:

$[I]$ = Identity matrix

$[\lambda]$ = Diagonal matrix of eigenvalues

$\{fn(t)\}$ = The normalized force vector.

Since the forcing function has the form $\{f(t)\}=\{Fo\}e^{(i\omega t)}$. The particular solution will have the form

$$\begin{aligned} \{q\} &= \{A\}e^{(i\omega t)} \\ \{\dot{q}\} &= \{A\}iwe^{(i\omega t)} \end{aligned} \tag{2.7}$$

where:

$\{A\}$ = Modal magnitude of the response vector.

Equation (2.6) can be re-written

$$(iw[I]-[\lambda])\{A\}e^{(iwt)} = [U]^t\{fn\}e^{(i\omega t)} \tag{2.8}$$

or:

$$\{iw[I]-[\lambda]\}\{A\}=[U]^t\{fn\} \tag{2.9}$$

Pre-multiplication of equation (2.9) by $(iw[I]-[\lambda])^{-1}$ yields an expression for the modal magnitude

$$\{A\} = (iw[I]-[\lambda])^{-1}\ [U]^t\{fn\} \tag{2.10}$$

Using equation (2.10) in equation (2.7) yields

$$\{q\}=(iw[I]-[\lambda])^{-1}[U]^t\{fn\}e^{(i\omega t)}$$

which is an expression for the modal response.

Since $\{y\}=[U]\{q\}$, we have

$$\{Y\}=[U](iw[I]-[\lambda])^{-1}[U]^t\{fn\} \tag{2.11}$$

and

$$\{Y(t)\}=[U](iw[I]-[\lambda])^{-1}[U]^t\{fn\}e^{(i\omega t)} \tag{2.12}$$

where:

$\{Y\}$ = Magnitude of response.

$\{Y(t)\}$ = The response of the n-degree of freedom system with displacements in the second n rows and velocities in the first n rows.

The relationship between Y and ω is the so-called frequency response. Peaks on the frequency response plot indicate resonant frequencies of the system. This occurs when ω takes on the value of the imaginary part of λ_i, causing A_j to become large.

The goal of this thesis is to be able to deduce changes in the magnitude of the resonant response of the system which may result from changes in the system. This will be done through differentiation of the magnitude of the response at a resonant frequency with respect to a system change.

The next chapter discusses the techniques involved in obtaining these derivatives.

CHAPTER 3

DERIVATIVE OF EIGENVALUES AND EIGENVECTORS

3.1 Derivatives of Eigenvalues

An important step in the derivation of the derivative of the frequency response is the ability to find the derivative of the eigenvalues and eigenvectors of the system.

This section is based on a paper published by Rogers on derivatives of eigenvectors and eigenvalues [3].

Consider the homogeneous set of equations of the form:

$$[M]\{\dot{y}\}+[K]\{y\}=\{0\} \tag{3.1}$$

Assume the solution

$$\{y\}=\{U_j\}e^{(\lambda jt)}$$

where:

$\{U_j\}$=jth eigenvector of the system.
j=jth mode

Substitution placed into equation (3.1) leads to:

$$\lambda_j[M]\{U_j\}+[K]\{U_j\}=0 \tag{3.2}$$

If the [M] and [K] matricies are symmetric, pre-multiplying through by $\{U_j\}^t$ produces the Rayleigh Quotient.

$$\lambda_j\{U_j\}^t[M]\{U_j\} + \{U_j\}^t[K]\{U_j\}=0 \tag{3.3}$$

Taking the partial derivative of equatin (3.3) with respect to some parameter e yields

$$\lambda_j{,e}\ \{U_j\}^t[M]\{U_j\} + \lambda_j\{U_j\}^t{,e}\ [M]\{Uj\} + \lambda_j\{Uj\}^t[M]{,e}\ \{Uj\}+$$
$$\lambda_j\{U_j\}^t[M]\{U_j\}{,e} + \{U_j\}^t{,e}\ [M]\{U_j\} + \{U_j\}^t[M]{,e}\ \{U_j\}+$$
$$\{U_j\}^t[K]\{U_j\}{,e}=0 \qquad (3.4a)$$

where the comma indicates differentiation with respect to e. Collecting terms yields,

$$\lambda_j{,e}\ \{U_j\}^t[M]\{U_j\} + \{U_j\}^t{,e}(\lambda_j[M]\{U_j\} + [K]\{U_j\}) +$$
$$\{U_j\}^t(\lambda_j[M]{,e} + [K]{,e})\{Uj\} + (\lambda_j\{U_j\}^t[M] + \{U_j\}^t[K])\{U_j\}{,e}$$
$$= 0 \qquad (3.4b)$$

In view of equation (3.2) and the fact that for symmetric [M] and [K] equation (3.2) is also valid for the transpose $\lambda_j[M]^t\{Uj\}+[K]^t\{Uj\}$, equation (3.4b) reduces to:

$$\lambda_j{,e}=-(\{U_j\}^t[\lambda_j[M]{,e}+[K]{,e}]\{U_j\}/(\{\{U_j\}^t[M]\{U_j\}) \qquad (3.5)$$

Equation (3.5) is an expression for the derivative of an eigenvalue with a desired parameter change.

3.2 Derivatives of Damping Ratios and Undamped Natural Frequencies

To find the derivative of the frequency response, it will be necessary to find the derivatives of the damping ratios (ζ_j) and the undamped natural frequencies (ω_j) of the system. These derivatives can be obtained through term by term differentiation of the eigenvalues.

The eigenvalue can be written as:

$$\lambda_j=-\zeta_j\omega_j+i\omega_j(1-\zeta_j^2)^{1/2} \tag{3.6}$$

Take the partial of λ_j with respect to a parameter e:

$$\lambda_{j,e} = -(\zeta_j\omega_j)_{,e} + (i\omega_j(1-\zeta_j^2)^{1/2})_{,e} \tag{3.7}$$

Equate the real and imaginary parts on each side of (3.7)

$$-\mathrm{Re}(\lambda je) = \zeta_{j,e}\,\omega_j + \omega_{j,e}\,\zeta_j \tag{3.8}$$

$$\mathrm{Im}(\lambda je) = (1-\zeta_j^2)^{1/2}{}_{,e}\,\omega_j + \omega_{j,e}\,(1-\zeta_j^2)^{1/2} \tag{3.9}$$

where:

$\mathrm{Re}(\lambda j)$ = The real part of λ_j.

$\mathrm{Im}(\lambda_j)$ = The imaginary part of λ_j.

ζ_j and ω_j can be determined from the eigenvalue.

Equations 3.8 and 3.9 yield $\zeta j,e$ and w_j,e

$$\zeta_{j,e} = -(1-\zeta_j^2)^{1/2}((1-\zeta_j^2)^{1/2}\,\mathrm{Re}(\lambda_j)_{,e} - \zeta_j\mathrm{Im}(\lambda_j)_{,e})/\omega_j \tag{3.10}$$

$$w_{j,e} = (-\zeta_j\,\omega_j\,\mathrm{Re}(\lambda_j)_{,e} + \omega_j(1-\zeta_j^2)^{1/2}\,\mathrm{Im}(\lambda_j)_{,e})/\omega_j \tag{3.11}$$

3.3 Derivatives of Eigenvectors

To find the derivative of the frequency response, it will also be necessary to find the derivative of the eigenvectors. This derivative can be obtained by taking the partial derivative of equation (3.2) with respect to e.

$$(\lambda_{j,e}[M]+\lambda_j[M]_{,e}+[K]_{,e})\{U_j\}+(\lambda_j[M]+[K])\{U_j\}_{,e} = 0 \tag{3.12}$$

Since the eigenvectors are independent, derivative of an eigenvector can be written as a linear combination of the eigenvectors

$$\{U_j\},_e = \sum_{k=1}^{2n} a_{jk}\,\{U_k\} \tag{3.13}$$

If equation (3.13) is substituted into equation (3.12) and equation (3.12) is pre-multiplied through by $\{U_g\}^t$ $(g \neq j)$ then we have

$$\begin{aligned}&\{U_g\}^t(\lambda_j,_e[M] + \lambda_j[M],_e + [K],_e)\{U_j\} + \\ &\{U_g\}^t(\lambda_j[M] + [K]) \sum_{k=1}^{2n} a_{jk}\{U_k\} = 0\end{aligned} \tag{3.14}$$

Observe that, for non repeated eigenvalues, the orthogonality relation $\{U_g\}^t[M]\{U_j\}=\{U_g\}^t[K]\{U_j\}=0$ for $g \neq j$ leads to

$$\begin{aligned}&-\lambda_j,_e\{U_g\}^t[M]\{U_j\} + \{U_g\}^t([K],_e - \lambda_j[M],_e)\{U_j\} + \\ &a_{jg}\{U_g\}^t([K]-\lambda_j[M])\{U_g\}\end{aligned} \tag{3.15}$$

$$a_{jk} = -\frac{\{U_k\}^t([K],_e - \lambda_j[M],_e)\{U_j\}}{\{U_k\}^t([K] - \lambda_j[M])\{U_k\}} \tag{3.16}$$

This is an expression for all of the coefficients, except k=j. In order to obtain the k=j coefficient, assume that the largest element in the jth eigenvector has been normalized to 1. Then denote this largest element of the jth eigenvector by U_{gj}, where the normalization of the eigenvector should be the same before and after the increment in parameter. This means that:

$U_{gj} = 1$

and

$$U_{gj,e} = \sum_{k=1}^{2n} a_{jk}\, U_{gk} = 0 \qquad (3.17)$$

$$a_{jj}\, U_{gj} = -\sum_{k=1,\ k \neq j}^{2n} a_{jk}\, U_{gk}$$

or

$$a_{jj} = -\sum_{k=1,\ k \neq j}^{2n} a_{jk}\, U_{gk} \qquad (3.18)$$

Equation (3.16), (3.18) along with equation (3.13), gives an expression for the derivative of the eigenvectors.

CHAPTER 4

FREQUENCY RESPONSE DERIVATIVES

Consider the case wherein a change in the frequency response of a system is desired which produces a lower frequency response at a given frequency. This is illustrated by Figure 4.1, which compares the frequency response of a system before and after some change to the system. The peak response of the average system at 1.8 rad/sec has been changed by ΔYc, from about 2.8 down to 0.6. However, note that the resonant peak itself has only slightly decreased, from about 2.8 down to about 2.5.

Now consider the case wherein a change in the peak response may be desired. This is illustrated by Figure 4.2, where the response at the second peak has been reduced by ΔYv, from about 2.8 down to 0.8. Observe that, while the magnitude of the peak itself has been reduced, the magnitude of the response at 1.8 rad/sec has again only been reduced to 0.6.

This chapter will present derivatives of the frequency resonse appropriate for each of these cases. First, the derivative of the frequency response at a constant frequency will be obtained This will be followed by the derivation of the derivative of the resonant response.

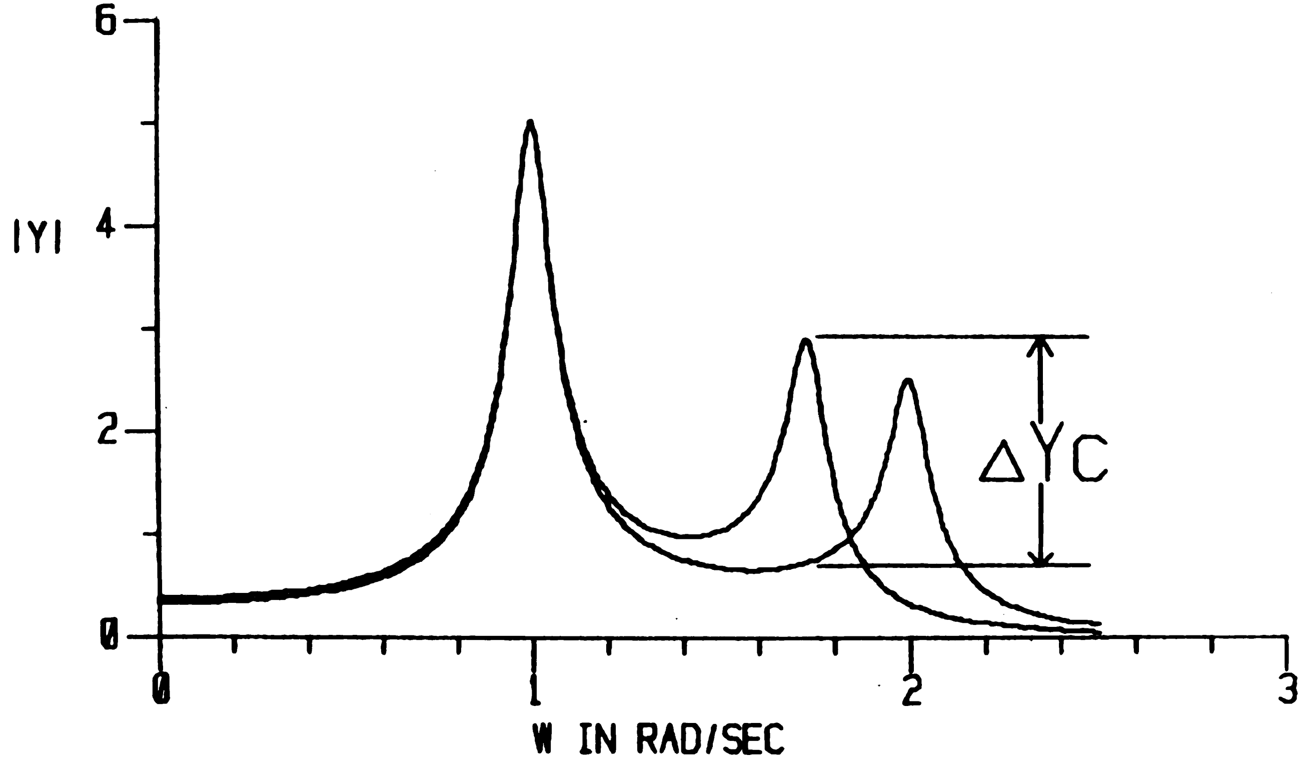

Figure 4.1 Effects of Constant Frequency Derivatives.

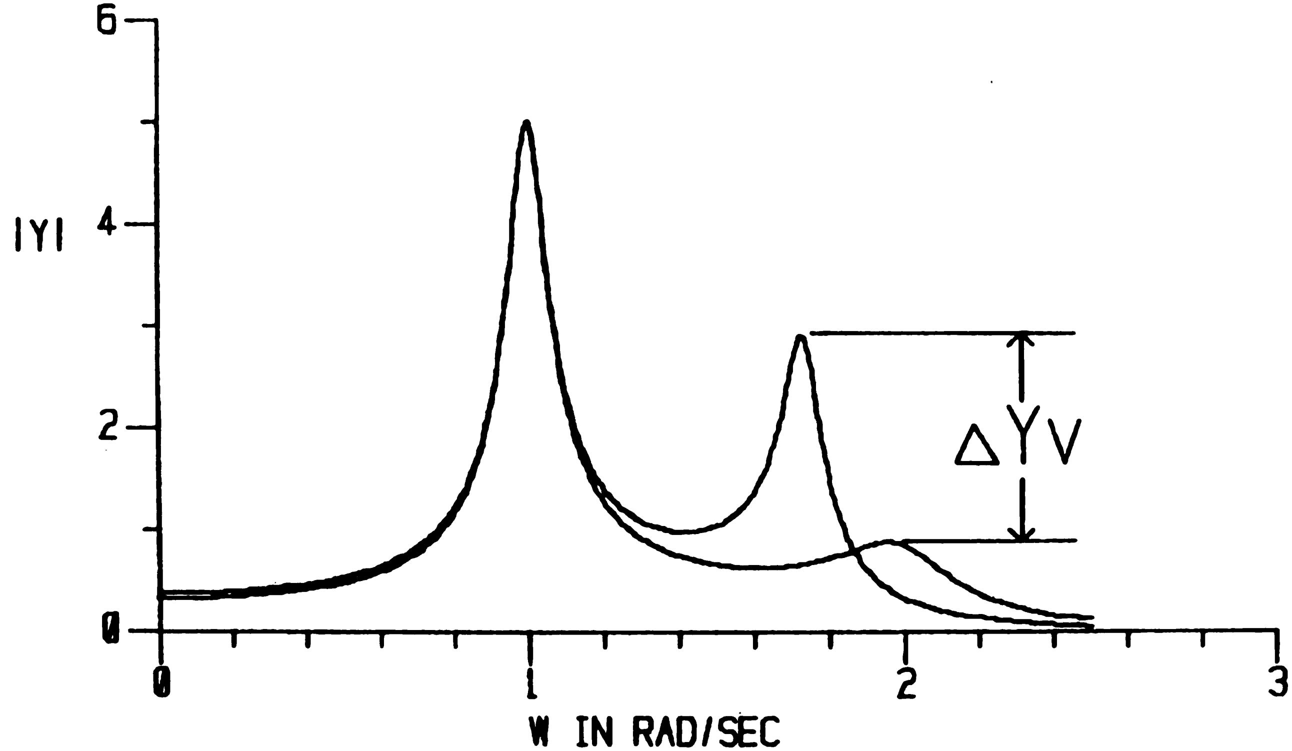

Figure 4.2 Effects of Variable Frequency Derivatives.

4.1 Constant Frequency Derivatives

The derivative of the frequency response at a constant frequency can be obtained by differenting the equation for the modal magnitude (equation 2.9). The chain rule can then be applied to obtain the constant frequency derivative. Recalling the equation for the modal magnitude,

$$\{A\}=(iw[I]-[\lambda])^{-1}[U]^t\{fn\} \tag{4.1a}$$

or in component form:

$$A_j = \sum_{k=1}^{2n} \frac{U_{jk}\ fn_k}{(iw-\lambda_j)} \tag{4.1b}$$

Differentiating A_j with respect to parameters that appear in the mass, stiffness or damping matrices gives,

$$A_{j,e} = \sum_{k=1}^{2n} \sum_{l=1}^{2n} \frac{(U^t)_{jk,e}\ fn_k}{(i\omega-\lambda_j)} + \frac{(U^t)_{kl}\ fn_k\ \lambda_{j,\ e}}{(i\omega-\lambda_j)^2} -$$

$$\frac{((U^t)_{jl,e}\ M_{lk}\ U_{kj} + (U^t)_{jl}\ M_{lk,e}\ U_{kj} + (U^t)_{jl}\ M_{lk}\ U_{kj,e})\ (U^t)_{jk}\ fn_k}{((U^t)_{jl}\ M_{lk}\ U_{kj})\ (i\omega-\lambda_j)} \tag{4.2}$$

or since [M] is symmetric,

$$A_{j,e} = \sum_{k=1}^{2n} \sum_{l=1}^{2n} \frac{(U^t)_{jk,e}\ fn_k}{(i\omega-\lambda_j)} + \frac{(U^t)_{kl}\ fn_k\ \lambda_{j,e}}{(i\omega-\lambda_j)^2} -$$

$$\frac{((U^t)_{jl}\ M_{lk,e}\ U_{kj} + 2(U^t)_{jl}\ M_{lk}\ U_{kj,e})\ (U^t)_{jk}\ fn_k}{((U^t)_{jl}\ M_{lk}\ U_{kj})(i\omega-\lambda_j)} \tag{4.3}$$

Using the relation $\{Y\}=[U]\{A\}$ and applying the chain rule yields

$$\{Y\}_{,e}=[U]_{,e}\{A\} + [U]\{A\}_{,e} \tag{4.4}$$

This derivation is presented in Reference 4.

4.2 Variable Frequency Derivatives

We have considered the derivative of the frequency response at a given frequency. This derivation is valid at any specified frequency, whether or not it corresponds to a resonant point.

Now we wish to consider the derivative of the magnitude of a resonant point, i.e., a point where the forcing frequency matches the imaginary part of one of the eigenvalues. This is different in character from the previous derivation because our frequency of interest changes as the eigenvalue of interest responds to system changes.

Recalling the values of the component terms of the eigenvalue from equation 3.6:

$$\lambda_j = -\zeta_j \omega_j + i\, \omega_j (1-\zeta_j^{\,2})^{1/2}$$

Assume the ith mode is at resonance. The equation for the ith component of the modal magnitude can be written as:

$$A_i = \sum_{k=1}^{2n} \frac{(U^t)_{ik}\ fn_k}{\zeta_i \omega_i} \tag{4.6}$$

and the other 2n-1 modal magnitudes have the form

$$\underset{j\neq i}{A_j} = \sum_{k=1}^{2n} \frac{(U^t)_{jk}\ fn_k}{(i\omega_i (1-\zeta_i^{\,2})^{1/2}-\lambda_j)} \tag{4.7}$$

Differentiating equations 4.6 and 4.7 with respect to parameters that appear in the mass, stiffness or damping produces the expressions for the rate of change of the equation of the peak. This differentiation yields (since [M] is symmetric),

$$A_{i,e} = \sum_{k=1}^{2n} \sum_{l=1}^{2n} \frac{(U^t)_{ik,e}\, fn_k}{\zeta_i \omega_i} - \frac{(U^t)_{ik}\, fn_k\, \zeta_{i,e}}{\zeta_i^{\,2} \omega_i} - \frac{(U^t)_{ik}\, fn_k\, w_{i,e}}{\zeta_i \omega_i^{\,2}} - \qquad (4.10)$$

$$\frac{(2(U^t)_{il}\, M_{lk}\, U_{ki,e} + (U^t)_{il}\, M_{lk,e}\, U_{ki})\ (U^t)_{ik}\, fn_k}{((U^t)_{il}\, M_{lk}\, U_{ki})_i\ \zeta_i \omega_i}$$

and

$$A_{j,e} = \sum_{k=1}^{2n} \sum_{l=1}^{2n} \frac{(U^t)_{jk,e}\, fn_k}{(i(1-\zeta_i^{\,2})^{1/2}\omega_i - \lambda j)} + \frac{(U^t)_{jk}\, fn_k\{(w_i\, \zeta_i/(1-\zeta_i^{\,2})^{1/2})\zeta_{i,e}\, i)}{(i(1-\zeta_i^{\,2})^{1/2}\omega_i - \lambda_j)^2}$$

$$- \frac{(2(U^t)_{jl}\, M_{lk}\, U_{kj,e} + (U^t)_{jl}\, M_{lk,e}\, U_{kj})(U^t)_{jk}\, fn_k}{((U^t)_{jl}\, M_{lk}\, U_{kj})_j\ (i(1-\zeta_i^{\,2})^{1/2}\omega_i - \lambda_j)} + \qquad (4.11)$$

$$\frac{(U^t)_{jk}\, fn_k\, \lambda_{j,e}}{(i(1-\zeta_i^{\,2})^{1/2}\ \omega_i - \lambda_j)^2} - \frac{(U^t)_{jk}\, fn_k\ (i(1-\zeta_i^{\,2})^{1/2} w_{i,e}\ )}{(i(1-\zeta_i^{\,2})^{1/2}\ \omega_i - \lambda_j)^2}$$

Equations 4.11 and 4.12 can be used in conjunction with 4.4 to obtain the derivative of the peak of a frequency response with respect to a change in the system.

CHAPTER 5

A PROCEDURE TO REDUCE RESONANT RESPONSE

This section presents an example of a procedure to lower a resonant response of a system. In particular, an optimization technique will be used to obtain changes in a system to reduce the magnitude of a peak of the frequency response.

The system to be considered here, which is shown in Figure 5.1, is made up of ten identical masses, springs and dampers. The system is subjected to an axial load on the end mass of the form,

$$\{F(t)\}=\{Fo\}\exp^{(i\omega t)} \qquad (5.1)$$

The frequency response plot of the first mass (m1) of this ten mass system is shown in Figure 5.2.

Consider the following optimization scheme. Assume the design of the system can be changed. However, a penalty will be assesed for the change.

$$P1 = \sum_{i=1}^{m} ((e_i-e_{oi})/e_{oi})^2 \qquad (5.2)$$

where,

eoi=The original value of the design parameter.
ei=The value of the design parameter after the design change.
m=The number of design variables (ei) being changed.

The ideas underlying this penalty scheme are given in Reference 5.

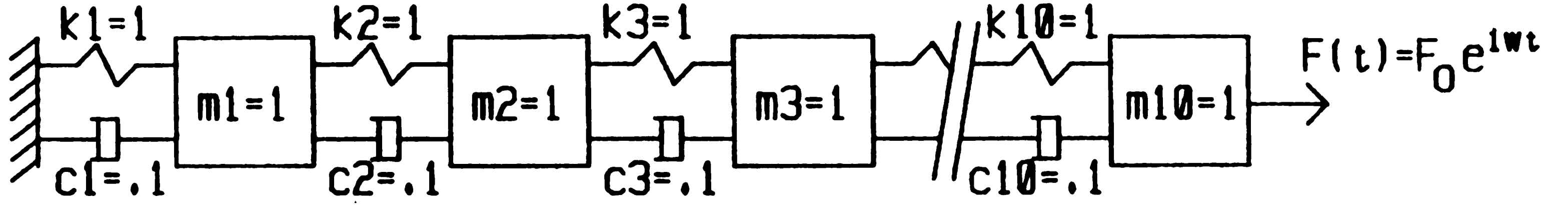

Figure 5.1 Ten Spring, Mass, Damper Example System.

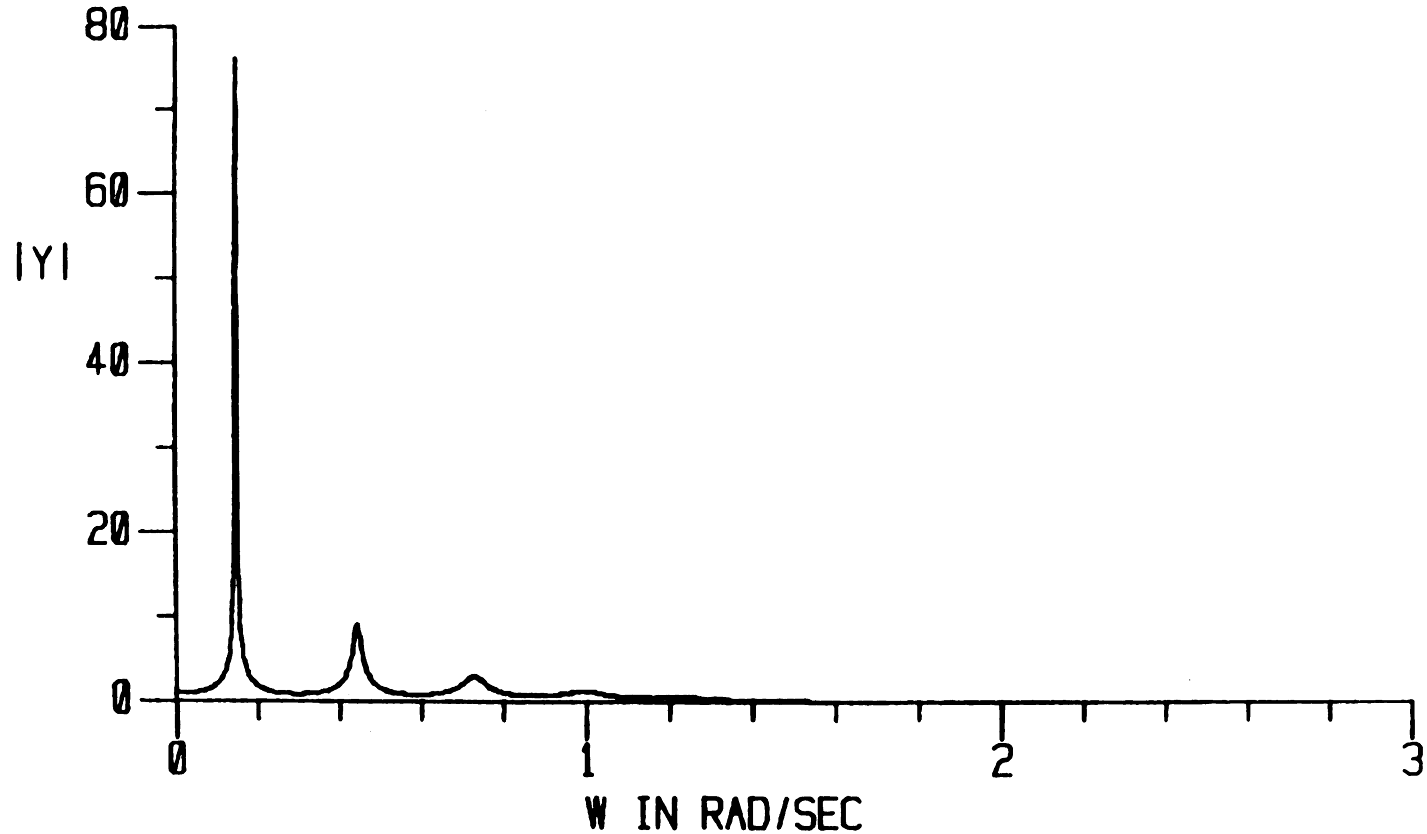

Figure 5.2 Frequency Response of the First Mass.

Another penalty will be assessed for the magnitude of the peak, namely

$$P2=|Yn|\lambda|Yo| \tag{5.3}$$

where,

$|Yn|$=The magnitude of the frequency response plot after a design change.
$|Yo|$=The magnitude of the frequency response plot before a design change.

In this section a weighted sum of the two penalties will be minimized:

$$P = bP1 + P2 \tag{5.4}$$

In this way, a lower peak will be derived while limiting the changes in the parameters to reasonable levels.

This penalty function is initially equal to one, since with no change in the system P1 equals zero. As e_i is changed, P1 increases as shown in Figure 5.3. This increase is weighted by b. The larger the value of b the greater the penalty for changing e_{0i}.

In order to minimize the penalty function P, we need

$$dP = \sum_{i=1}^{m} (\partial P/\partial e_i)de_i=0 \tag{5.5}$$

Thus,

$$\partial P/\partial e_i=0 \tag{5.6}$$

or

$$\partial P2/\partial e_i + b\, \partial P1/\partial e_i=0 \tag{5.7}$$

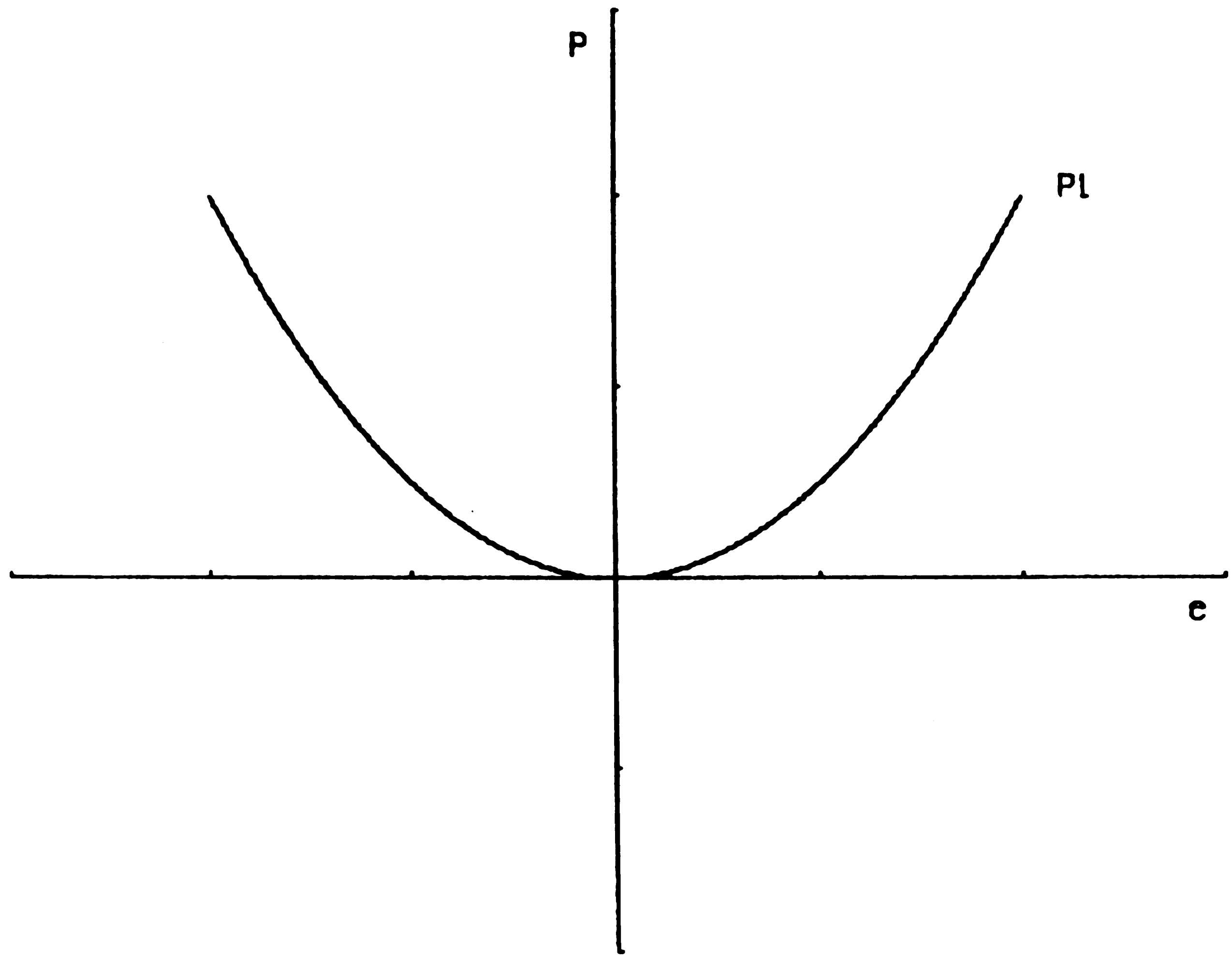

Figure 5.3 Parameter Change Penalty Function.

P2 and P1 are differentiated according to their expressions given in equations 5.2 and 5.3.

$$\partial P1/\partial e_i=2((e_i-e_{oi})/e_{oi}^2) \qquad (5.8)$$

$$\partial P2/\partial e_i=(\partial|Yn|/\partial e_i)/|Yo| \qquad (5.9)$$

When these two expressions are substututed into equation 5.7 an expression for the minimum e_i (e_{imin}) can be obtained,

$$e_{imi}=e_{oi} = ((\partial|Yn|/\partial e_i)e_o^2)/2b|Yo|) \qquad (5.10)$$

All of the terms on the right hand side of equation 5.10 are known except for the derivative of the magnitude e_i. This term can be found by considering the equation for the magnitude of the response $|Y|$,

$$|Y| = (Re(Y)^2 + Im(Y)^2)^{1/2} \qquad (5.11)$$

where

Re(Y)=The real part of the frequency response Y.
Im(y)=The imaginary part of the frequency response Y.

Differentiating the magnitude of the response $|Y|$ with respect to e_i yields,

$$\frac{\partial|Y|}{\partial e_i} = \frac{Re(Y)(\partial Re(Y)/\partial e_i) + Im(Y)(\partial Re(Y)/\partial e_i)}{|Y|} \qquad (5.12)$$

The derivative of the real and imaginary parts of the peak equation are given by (4.4).

Once the minimized parameter e_{imin} is obtained the new magnitude $|Yn|$ can be estimated with a first order expansion:

$$|Yn|=|Yo| + \sum_{i=1}^{m} (\partial|Y|/\partial e_i)\Delta e_i \tag{5.13}$$

where

$\Delta e=ei-eoi$

Since only a first order Taylor's series is being used, the procedure discussed above may be inaccurate for large changes. This can be seen more clearly in Figure 5.4, which illustrates the first order expansion of $|Y|$ vs. e about e=1. The linear expansion follows the straight line (line 1) of Figure 5.4. To deal with the inaccuracies resulting from large changes in e, after solving for the change in e, the eigenvalue roblem should be re-solved to obtain $|Y|$ an $\partial y/\partial e$. The optimization procedure can then be restarted using the new e values and the process to obtain e_{imin} can be done again. The procedure is complete when the linear expansion for $|Y|$ is satisfactorily close to the solution of the eigenvalue problem.

The example to be addressed here will obtain changes in a ten spring, mass, damper system to reduce the magnitude of a peak of the frequency response plot. Initially, each mass, spring and damper will have a value of 1., 1., and .1, respectively as shown in Figure 5.1. To illustrate the minimization process the design variables that will be optimized will be the ten dampers and ten springs. In this example the changes in the dampers will be assumed to be proportional to the changes

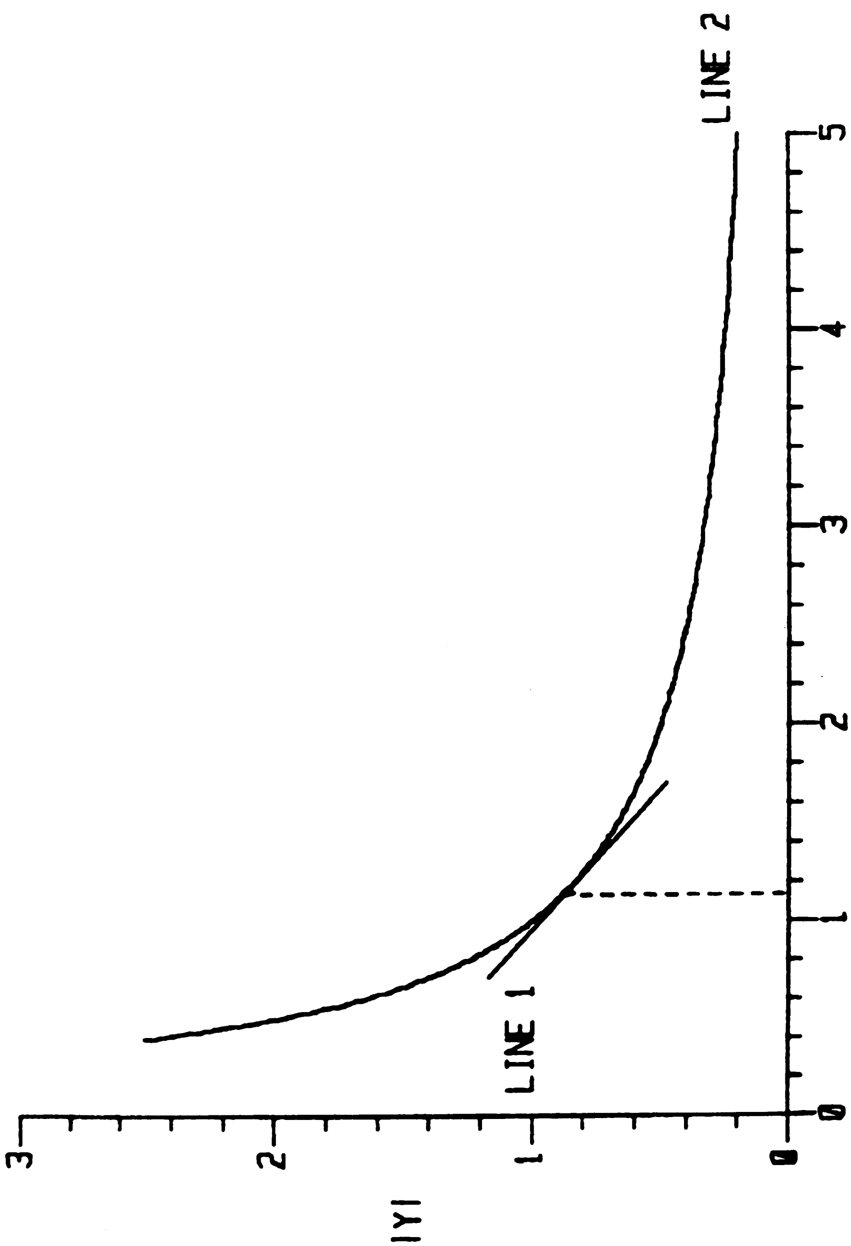

Figure 5.4 Consequenses of the Linear Assumption.

in the springs, i.e., $\Delta ci=\Delta ki/5$. In practice the design variables that can be altered are based on engineering judgement and the constraints of the system.

Table 5.1 summarizes the first example. In this case, the change penalty is weighted by b=1. The columns of the table show the value of each of the ten dampers and springs after each iteration. These values are followed by the magnitude and predicted penalty function values, predicted using the first order Taylor series, and the new magnitudes and penalties obtained after resolving the eigenvalue problem.

Figure 5.5 compares the frequency response of the original system and the final system. As the table indicates, the peak has dropped from 84 to about 72. Note that the frequency response at the frequency of the original peak has dropped far below 72, to about 30. Thus, if the minimization was performed at a constant frequency rather than traking the peak amplitude, a deceptively low value would have been calculated which is not at all descriptive of the peak magnitude.

Table 5.2 summarizes a closely related example, this time with b=0.5. In this case, it took five interactions to converge to within one percent. Figure 5.6 compares the frequency response of the original system and the final system. Clearly the frequency of the peak has changed, with the magnitude of the peak of the peak reduced from about 84 down to about 66. The magnitude of the frequency response at the initial resonant frequency now has a magnitude of approximately 13, again illustrating that the magnitude at the initial resonant frequency has been changed by an amount much different than the change in the peak values.

TABLE 5.1 Optimization Iteration Results

b=1.

Iteration Number

	Initial	1	2	3	4
C1	.1	.1318687	.1210132	.1239234	.1230873
C2	.1	.1062208	.1047288	.1051104	.1050050
C3	.1	.1055331	.1042626	.1045870	.1044975
C4	.1	.1048260	.1037732	.1040424	.1039681
C5	.1	.1041132	.1032662	.1034842	.1034238
C6	.1	.1033996	.1027418	.1029133	.1028658
C7	.1	.1026861	.1021988	.1023284	.1022923
C8	.1	.1019750	.1016385	.1017304	.1017048
C9	.1	.1012752	.1010698	.1011275	.1011114
C10	.1	.1006061	.1005126	.1005397	.1005321
K1	1.	1.159344	1.105066	1.119617	1.115436
K2	1.	1.031104	1.023644	1.025552	1.025025
K3	1.	1.027666	1.021313	1.022935	1.022487
K4	1.	1.024130	1.018866	1.020212	1.019840
K5	1.	1.020566	1.016331	1.017421	1.017119
K6	1.	1.016998	1.013709	1.014567	1.014329
K7	1.	1.013431	1.010994	1.011642	1.011462
K8	1.	1.009875	1.008193	1.008652	1.008524
K9	1.	1.006376	1.005349	1.005638	1.005557
K10	1.	1.003030	1.002563	1.002698	1.002660
|Y|pre.	-------	65.29032	73.04958	72.03468	72.41298
Ppre.	-------	.8850195	.9138961	.9162259	.9164101
|Y|	84.7883	68.79299	73.32258	72.05652	72.41472
P	1.0	.9263303	.9171158	.9164833	.9164307

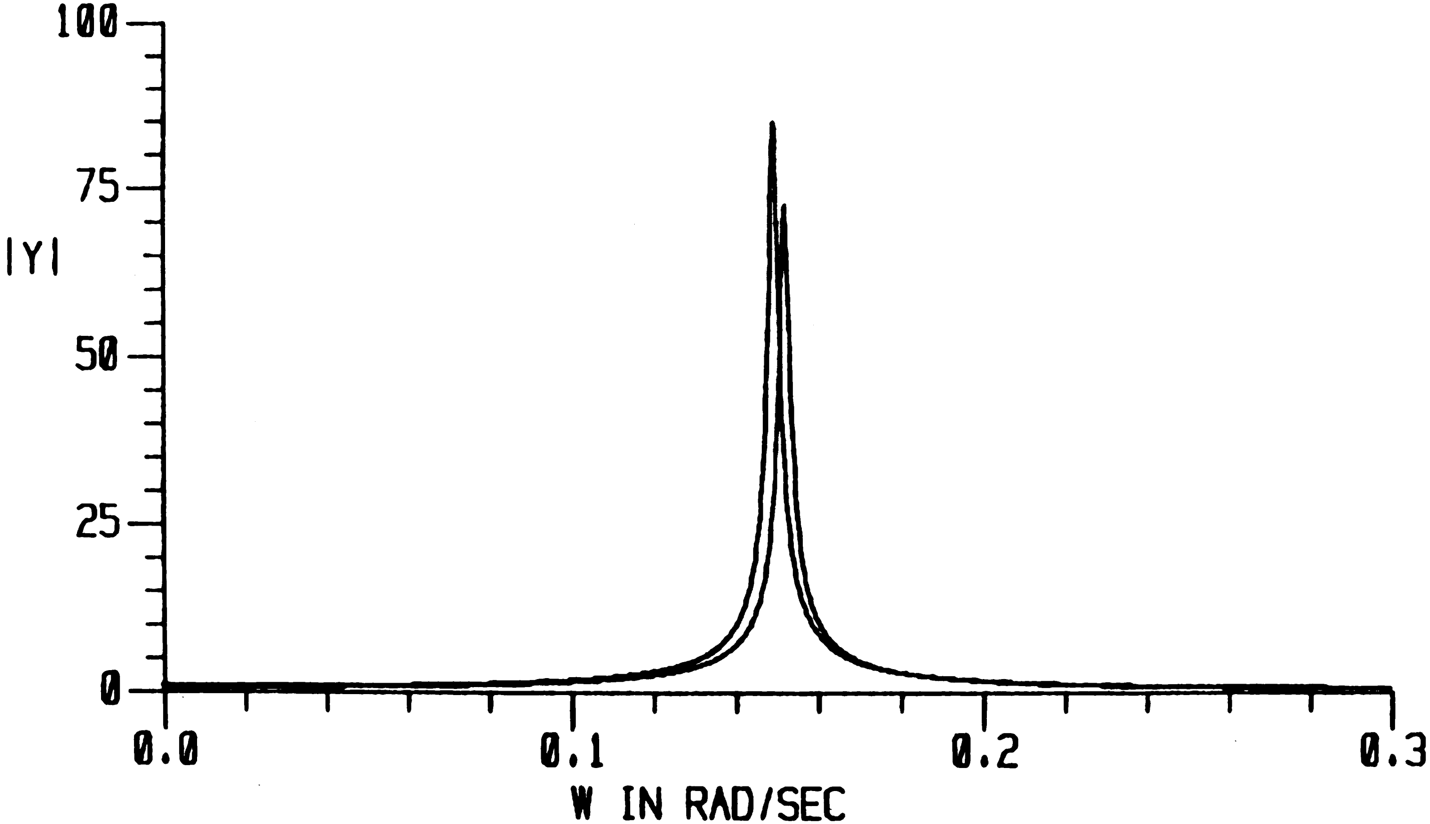

Figure 5.5 Original System vs. Optimized System with b=1.

TABLE 5.2 Optimization Iteration Results

b=.5

Iteration Number

	Initial	1	2	3	4	5
C1	.1	.1637374	.1297519	.1427908	.1369241	.1393969
C2	.1	.1124415	.1074316	.1093442	.1085300	.1088748
C3	.1	.1110663	.1067751	.1084135	.1077186	.1080134
C4	.1	.1096520	.1060750	.1074448	.1068654	.1071119
C5	.1	.1082264	.1053329	.1064487	.1059772	.1061784
C6	.1	.1067992	.1045414	.1054232	.1050506	.1052103
C7	.1	.1053722	.1036933	.1043619	.1040792	.1042008
C8	.1	.1039500	.1027881	.1032628	.1030620	.1031486
C9	.1	.1025503	.1018406	.1021395	.1020131	.1020677
C10	.1	.1012121	.1008894	.1010294	.1009703	.1009959
K1	1.	1.318687	1.148760	1.213954	1.184620	1.196985
K2	1.	1.062208	1.037158	1.046721	1.042650	1.044374
K3	1.	1.055331	1.033875	1.042068	1.038593	1.040067
K4	1.	1.048260	1.030377	1.037224	1.034327	1.035560
K5	1.	1.041132	1.026665	1.032243	1.029886	1.030892
K6	1.	1.033996	1.022707	1.027116	1.025253	1.026051
K7	1.	1.026861	1.018467	1.021809	1.020396	1.021004
K8	1.	1.019750	1.013940	1.016314	1.015310	1.015743
K9	1.	1.012751	1.009203	1.010697	1.010066	1.010339
K10	1.	1.006060	1.004447	1.005147	1.004851	1.004979
|Y|pre.	-------	45.79242	67.27395	63.86480	66.25365	65.40542
Ppre.	-------	.7700397	.8483277	.8607706	.8631559	.8635876
|Y|	84.7883	57.64297	69.09643	64.22137	66.31769	65.80454
P	1.0	.9098061	.8698223	.8649759	.8639112	.8636940

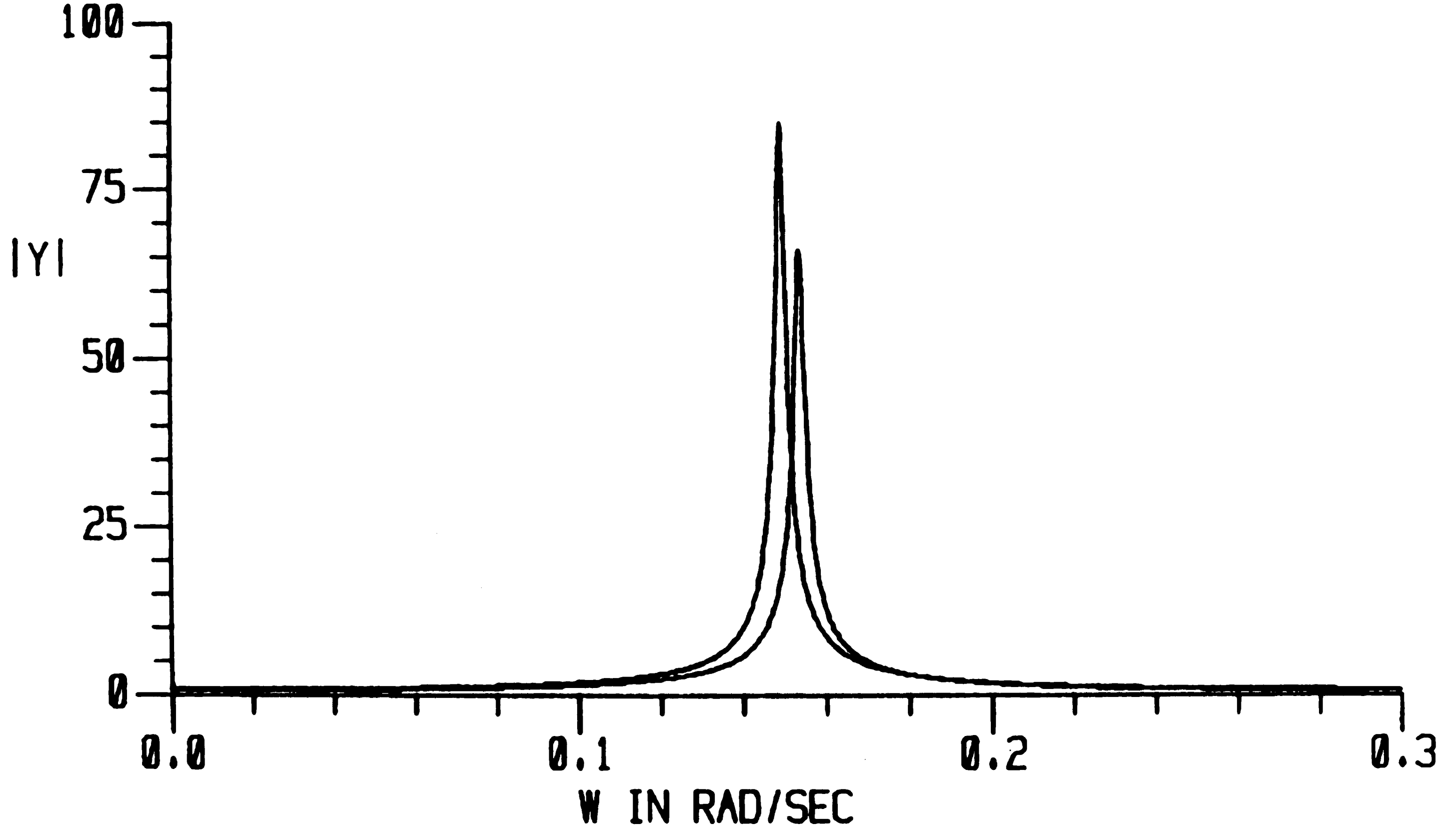

Figure 5.6 Original System vs. Optimized System with b=.5

CHAPTER 6

CONCLUSIONS

This thesis presents a formulation which can determine the sensitivity of the resonant response of the vibratory system to changes in the system. The thesis then illustrated through an example, that this sensitivity could be used to determine changes which are useful in lowering the peak response.

Since the sensitivity is only computed to the first order the optimization scheme which was used to obtain the desired design changes is iterative. The number of iteractions involved in this procedure could be reduced if a more sophisticated optimization technique was employed. This technique would consider higher order derivatives in order to facilitate the determination of an improved design.

A point that was not considered here is the effect of design changes in a system on other resonant peaks of a system. This could be of importance when the resonant frequency of a system are closely spaced. In this case, the optimization technique should be extended to include all the peaks of interest.

REFERENCES

1. Meirovitch, L. "Analytical Methods in Vibrations", MacMillan Company, New York, 1967.

2. Caughey, T. K., and O'Kelly, M. E. J., "Classical Normal Modes in Damped Linear Dynamic Systems," Journal of Applied Mechanics, Vol. 32, pp. 583-588, 1965.

3. Rogers, L. C., "Derivatives of Eigenvalues and Eigenvectors," Technical Mote, AIAA Journal, Vol. 8, No. 5, pp. 943-994, May 1970.

4. Chrostowski, J. D., Evensen, D. A., and Hasselman, T. K., "Model Verification of Mixed Dynamic Systems," Journal of Mechanical Design, Vol. 100 pp. 266-273, April 1978.

5. Starkey, J. M. and Bernard, J. E., "Optimal Redesign Based on Modal Data," "Proceedings," First International Modal Analysis Conference, Orlando, Florida, November 1982.