END EFFECT IN A TRUNCATED SEMI-INFINITE WEDGE AND CONE

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Major professor

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ABSTRACT

END EFFECT IN A TRUNCATED SEMI-INFINITE WEDGE AND CONE

by Tommie R. Thompson

The purpose of this research is to develop the stress distribution in a two-dimensional truncated semi-infinite wedge and in a three-dimensional truncated semi-infinite cone.

Using a complex valued eigenfunction expansion for an Airy stress function formulation of the wedge problem, the stress distribution within the St. Venant boundary region is determined for several "typical" loadings.

The solution for the cone problem is formulated in terms of Papkovich-Neuber functions and the resulting stress distribution in the cone is also determined.

Eigenvalues for both problems are presented for several wedge and cone angles.

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By

Tommie R.ay Thompson

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I. INTRODUCTION AND HISTORICAL DEVELOPMENT

In 1853, Barre de Saint-Venant published his "Memoire sur la Torsion des Prismes" [1] in which he solved the problem of torsion in long prismatic bars of various shapes of cross section. In a footnote he states that the influence of forces in equilibrium acting on a small portion of the surface of a body extends very little beyond the parts upon which they act. This has been the basis for the more familiar form of St. Venant's principle, the essence of which can be stated as follows: If a system of forces acting on a small portion of the surface of an elastic body is replaced by another statically equivalent system of forces acting on the same portion of the surface, the same stress distribution and deformation are produced inside the body except in the immediate neighborhood of the region where the surface forces are applied. "Statically equivalent systems" are those which have the same resultant force and moment.

This principle is of great practical importance. Often the exact distribution of boundary stresses is unknown but the statically equivalent loading can be easily determined. For these cases, the problem may be solved with the statically equivalent system of boundary stresses and from St. Venant's principle, the solution can be taken as accurate except in the vicinity of the loading.

On the other hand, if the boundary conditions are specified according to the exact distribution of the stresses, the problem may become too complicated to solve mathematically. Frequently, by modifying the boundary conditions slightly, the solution becomes possible and gives essentially the same stress distribution in a large part of the body as does the actual loading. By means of St. Venant's principle, the solution of the problem may be simplified by altering the boundary conditions as long as the systems of applied forces are statically equivalent.

The principle agrees very well with reality as can be illustrated by simple examples but its formal mathematical proof is rather difficult in the general case.

Early applications of St. Venant's principle to justify approximations of boundary conditions include problems investigated by Thomson and Tait [2], Levy [3], Boussinesq [4,5], and Clebsch [6].

For the elastic half space, bounds have been established on the decay rates for stresses (rates at which the stresses approach zero) by Boussinesq [5] and von Mises [7]. Both investigations show that the stresses decay, as they must, but that the decay rate is a function of the type of loading applied on the surface of the body. von Mises introduced the concept of astatic equilibrium which requires surface forces to remain in equilibrium even when turned through an arbitrary angle and proposed a modification to the principle. For a more detailed discussion of astatic equilibrium, see Section 2.2. He concluded that static equilibrium was not enough to insure the maximum decay rate since astatic equilibrium may generate decay rates much faster than simple static equilibrium. In 1954, E. Sternberg [8] presented a mathematical proof for the bounds on the decay rates for the modified principle.

Recent research in this area generally follows one of the two methods:

(a) that which attempts to establish bounds on the width of the St. Venant boundary region (as done by von Mises),

(b) that which attempts to establish "exact" solutions within this boundary region for limited classes of geometry. The geometries investigated thus far include the semi-infinite strip [9,10] and the semi-infinite circular cylinder [11-14].

The purpose of this research is to further the classes of geometry for which the "exact" solution is known by determining the stress distribution in a semi-infinite two-dimensional wedge and in a semi-infinite three-dimensional cone. The cone solution should then approach that of the cylinder as the cone angle approaches zero and should approach that of the half space as the cone angle approaches $\pi/2$.

For the wedge, the solution was formulated using an Airy stress function expressed in terms of a complex valued eigenfunction expansion. All boundary conditions were taken in terms of stresses applied on the surfaces.

For the cone, the solution was formulated in terms of the Papkovich-Neuber functions with boundary conditions again being taken in terms of stresses alone.

Previous investigations of various wedge and cone problems are presented in [18] to [32].

The transcendental equations which will be developed for the wedge[Eqs. (2.23) and (2.24] agree with those developed by Williams [41] in his investigation of stress singularities resulting from extension of angular plates with free edges. However, Williams presents only the real part of the minimum root of these equations. Williams is interested in the behavior of the solution near $r = 0$ and is concerned with bounded displacements at the origin, whereas the problem outlined in Chapter II does not contain the point $r = 0$ but is concerned with solutions which are bounded as $r \to \infty$.

II. THE WEDGE PROBLEM

2.1. Formulation of the Wedge Problem

Consider the wedge shown in Fig. 2.1. Formulating the problem in terms of an Airy stress function, ψ, the stresses in polar coordinates can be expressed as:

$$\sigma_{rr} = \frac{1}{r}\,\psi_{,r} + \frac{1}{r^2}\,\psi_{,\theta\theta} \tag{2.1}$$

$$\sigma_{\theta\theta} = \psi_{,rr} \tag{2.2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\,\psi_{,\theta} - \frac{1}{r}\,\psi_{,r\theta} \tag{2.3}$$

where $\psi_{,r}$ denotes the partial derivative of ψ with respect to r. Assuming plane stress conditions, the equilibrium equations of elasticity are satisfied and the defining equation for ψ becomes

$$\nabla^2\nabla^2\psi = 0 \tag{2.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad .$$

The boundary conditions to be satisfied are:

$$\text{Solution} \to 0 \text{ as } r \to \infty \tag{2.5}$$

$$\sigma_{\theta\theta}(r, \pm\beta) = 0 \tag{2.6}$$

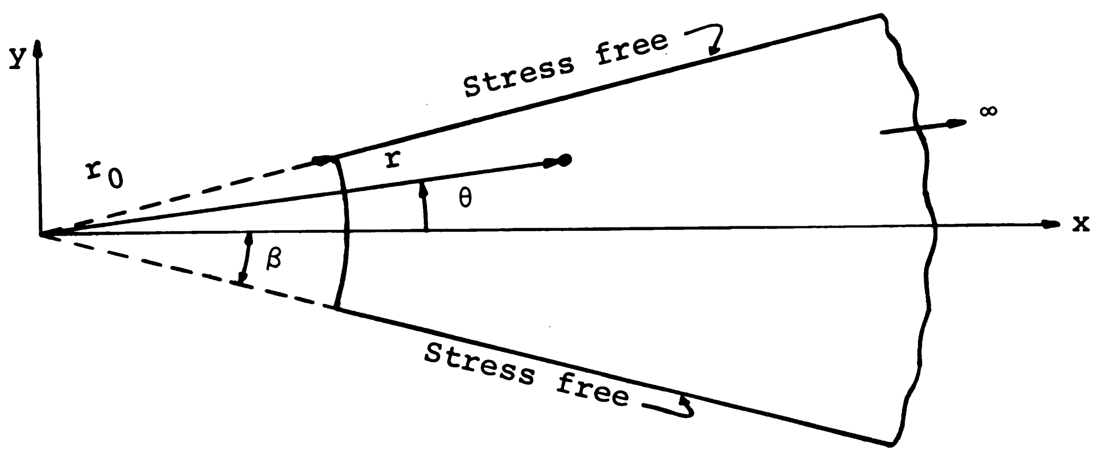

Fig. 2.1. Truncated semi-infinite two-dimensional wedge.

$$\tau_{r\theta}(r, \pm \beta) = 0 \tag{2.7}$$

$$\sigma_{rr}(r_0, \theta) = \sigma^0_{rr}(\theta) \tag{2.8}$$

$$\tau_{r\theta}(r_0, \theta) = \tau^0_{r\theta}(\theta) \tag{2.9}$$

where $\sigma^0_{rr}(\theta)$ and $\tau^0_{r\theta}(\theta)$ are the specified loading functions.

Assume the solution for ψ in Eq. (2.4) to be of the form:

$$\psi = \sum_n r^{\alpha_n} f_n(\theta) + Cr\theta \sin \theta + Dr\theta \cos \theta \tag{2.10}$$

where the last two terms are included in order to incorporate that portion of the solution corresponding to $\alpha_n = 1$. The necessity of these terms will be more apparent after the next few steps. Substituting the assumed expression for ψ into the biharmonic equation yields the defining equation for $f_n(\theta)$ as:

$$f_n^{IV}(\theta) + [\alpha_n^2 + (\alpha_n - 2)^2] f_n''(\theta) + \alpha_n^2 (\alpha_n - 2)^2 f_n(\theta) = 0 \tag{2.11}$$

where the primes denote differentiation with respect to θ.

Taking the solution for $f_n(\theta)$ in the form

$$f_n = k_n e^{m\theta} \quad ,$$

the characteristic equation for allowable values of m is:

$$m^4 + \left[\alpha_n^2 + (\alpha_n - 2)^2\right] m^2 + \alpha_n^2 (\alpha_n - 2)^2 = 0 \tag{2.12}$$

$$\left[m^2 + \alpha_n^2\right]\left[m^2 + (\alpha_n - 2)^2\right] = 0$$

Thus,

$$m = \pm i\alpha_n \tag{2.13}$$

$$m = \pm i(\alpha_n - 2) \quad . \tag{2.14}$$

Thus, the general solution for $f_n(\theta)$ can be written as:

$$f_n(\theta) = A_n' \sin \alpha_n\theta + B_n' \cos \alpha_n\theta + C_n' \sin (\alpha_n - 2)\theta$$
$$+ D_n' \cos (\alpha_n - 2)\theta \quad . \tag{2.15}$$

If $\alpha_n = 1$ in Eq. (2.12) above, then repeated roots exist and the solution corresponding to these roots satisfies all the boundary conditions. These are the terms included in ψ as $Cr\theta \sin \theta$ and $Dr\theta \cos \theta$.

In terms of the stress function assumed initially, the non-zero stresses are:

$$\sigma_{rr} = \sum_n r^{\alpha_n-2}\left[\alpha_n f_n(\theta) + f_n''(\theta)\right] + \frac{2}{r} C \cos \theta - \frac{2}{r} D \sin \theta \tag{2.16}$$

$$\sigma_{\theta\theta} = \sum_n r^{\alpha_n-2}\left[\alpha_n(\alpha_n - 1)f_n(\theta)\right] \tag{2.17}$$

$$\tau_{r\theta} = -\sum_n r^{\alpha_n-2}\left[(\alpha_n - 1)f_n'(\theta)\right] \quad . \tag{2.18}$$

For stress free boundary conditions at $\theta = \pm\beta$:

$$\sigma_{\theta\theta}\Big|_{\theta = \pm\beta} = 0 \Rightarrow f_n(\pm\beta) = 0$$

$$\tau_{r\theta}\Big|_{\theta = \pm\beta} = 0 \Rightarrow f_n'(\pm\beta) = 0 \quad .$$

Thus, the boundary conditions at $\theta = \pm\beta$ reduce to specifying:

$$f_n(\pm\beta) = 0 \tag{2.19}$$

$$f_n'(\pm\beta) = 0 \quad . \tag{2.20}$$

However, since any linear elasticity problem can be solved by the method of superposition, separate $f_n(\theta)$ into its even and odd parts in θ. This will facilitate the solution for the eigenvalues, α_n, and in general will make the problem more tractable. From Eq. (2.15),

$$f_n^{(o)}(\theta) = A_n' \sin \alpha_n^{(o)}\theta + C_n' \sin\left(\alpha_n^{(o)} - 2\right)\theta \tag{2.21}$$

$$f_n^{(e)}(\theta) = B_n' \cos \alpha_n^{(e)}\theta + D_n' \cos\left(\alpha_n^{(e)} - 2\right)\theta \quad . \tag{2.22}$$

Reference to an even problem implies one in which the stress function, ψ, and the σ_{rr} and $\sigma_{\theta\theta}$ stresses are even functions of θ and reference to an odd problem implies one in which these are odd functions of θ. The even problem will be indicated with a superscript (e); the odd problem will be indicated with a superscript (o).

The boundary conditions at $\theta = \pm\beta$ can then be expressed as:

$$\left.\begin{aligned} f_n^{(o)}(\beta) &= 0 \\ f_n^{(o)\,'}(\beta) &= 0 \end{aligned}\right\} \text{for an odd problem} \qquad \left.\begin{aligned} f_n^{(e)}(\beta) &= 0 \\ f_n^{(e)\,'}(\beta) &= 0 \end{aligned}\right\} \text{for an even problem}$$

Applying the above boundary conditions to the odd problem yields the following transcendental equation for the odd eigenvalues:

$$\alpha_n^{(o)} \sin 2\beta - 2 \sin \left(\alpha_n^{(o)}\beta\right) \cos\left[\left(\alpha_n^{(o)} - 2\right)\beta\right] = 0 \quad . \tag{2.23}$$

For the even problem, the corresponding transcendental equation for the even eigenvalues is:

$$\alpha_n^{(e)} \sin 2\beta + 2 \cos \left(\alpha_n^{(e)}\beta\right) \sin\left[\left(\alpha_n^{(e)} - 2\right)\beta\right] = 0 \quad . \tag{2.24}$$

The eigenfunctions can then be expressed as:

$$f_n^{(o)} = A_n'\left[\sin \left(\alpha_n^{(o)} - 2\right)\theta - \frac{\sin\left(\alpha_n^{(o)} - 2\right)\beta}{\sin \alpha_n^{(o)}\beta} \sin \alpha_n^{(o)}\theta\right] \tag{2.25}$$

$$f_n^{(e)} = C_n'\left[\cos \left(\alpha_n^{(e)} - 2\right)\theta - \frac{\cos\left(\alpha_n^{(e)} - 2\right)\beta}{\cos \alpha_n^{(e)}\beta} \cos \alpha_n^{(e)}\theta\right] \quad . \tag{2.26}$$

Thus, for the odd problem, the stress expressions become:

$$\sigma_{rr} = \sum_n A_n \gamma^{\alpha_n^{(o)}-2}\left[\alpha_n^{(o)} f_n^{(o)} + f_n^{(o)''}\right] - 2B\gamma^{-1} \sin\theta \tag{2.27}$$

$$\sigma_{\theta\theta} = \sum_n A_n \gamma^{\alpha_n^{(o)}-2}\left[\alpha_n^{(o)}\left(\alpha_n^{(o)} - 1\right) f_n^{(o)}\right] \tag{2.28}$$

$$\tau_{r\theta} = -\sum_n A_n \gamma^{\alpha_n^{(o)}-2}\left[\left(\alpha_n^{(o)} - 1\right) f_n^{(o)'}\right] \tag{2.29}$$

where $\gamma = r/r_0$ and $f_n^{(o)}$ is the bracketed portion of Eq. (2.25).

For the even problem, the stress expressions become:

$$\sigma_{rr} = \sum_n C_n \gamma^{\alpha_n^{(e)}-2} \left[\alpha_n^{(e)} f_n^{(e)} + f_n^{(e)''}\right] + 2D\gamma^{-1} \cos\theta \tag{2.30}$$

$$\sigma_{\theta\theta} = \sum_n C_n \gamma^{\alpha_n^{(e)}-2} \left[\alpha_n^{(e)} \left(\alpha_n^{(e)} - 1\right) f_n^{(e)}\right] \tag{2.31}$$

$$\tau_{r\theta} = -\sum_n C_n \gamma^{\alpha_n^{(e)}-2} \left[\left(\alpha_n^{(e)} - 1\right) f_n^{(e)'}\right] \tag{2.32}$$

where, again $\gamma = r/r_0$ and $f_n^{(e)}$ is the bracketed portion of Eq. (2.26).

The transcendental equations, Eqs. (2.23) and (2.24), were solved on the CDC 3600 digital computer using a Newton-Raphson iteration technique in the complex plane since the eigenvalues will be complex numbers. However, the Newton-Raphson technique is quite sensitive to the initial guess for the root of the equation. Care must be taken so that the initial guess is in the neighborhood of the particular root being sought or roots may be skipped; i.e., the method may converge to a root other than the one being sought. To avoid this skipping of roots, asymptotic expressions for the roots were developed and these asymptotic values were used as the initial guess for each eigenvalue. For a detailed discussion of the development of these asymptotic values, see Appendix A.

For this specific wedge problem, only those eigenvalues which have negative real parts will be used to insure the solution goes to zero for large r.

For a number b = c + id which is a solution to the transcendental equation, note that its complex conjugate $\bar{b}$ = c - id is also a solution. This is the condition which must exist if the stresses, which are, of course, real quantities, are to be expressed in terms of complex eigenfunction expansions.

Having now satisfied the boundary conditions at infinity and at the wedge angle, β, the remaining boundary conditions at $r = r_0$ will be satisfied.

Using a generalized approach to orthogonality as outlined by P. F. Papkovich [17], orthogonality conditions were established for the eigenfunctions. For details of the method as applied to this particular problem, see Appendix B. The orthogonality condition for the eigenfunctions is:

$$\int_{-\beta}^{\beta}\left[\alpha_m(\alpha_m - 2)\alpha_n(\alpha_n - 2)f_m f_n + 4f_m' f_n' - f_m'' f_n''\right]d\theta = 0 \quad .$$
$$(m \neq n) \quad . \tag{2.33}$$

However, it was not possible to interpret this condition physically in terms of stress, displacement, or mixed boundary conditions on the end of the wedge.

Therefore, the last boundary conditions, those at $r = r_0$, were satisfied numerically by truncating the series expressions and determining the constants in a least squares sense.

On the end boundary, for the even problem, consider the expansion of the real functions σ^0_{rr} and $\tau^0_{r\theta}$ in terms of the complex eigenfunctions. Take

$$\sigma^0_{rr} = \sum_n A_n \Phi_n + \sum_n A'_n \overline{\Phi}_n + 2D \cos \theta \tag{2.34}$$

$$\tau^0_{r\theta} = \sum_n A_n \Psi_n + \sum_n A'_n \overline{\Psi}_n \tag{2.35}$$

where

$$\Phi_n = \alpha_n f_n + f''_n$$

$$\Psi_n = -(\alpha_n - 1) f'_n$$

and where α_n refers to $\alpha_n^{(e)}$ and f_n refers to $f_n^{(e)}$. The two conditions, Eqs. (2.34) and (2.35), are then sufficient to imply $A'_n = \overline{A}_n$. Thus, the boundary stresses can be written as

$$\sigma^0_{rr} = \sum_n A_n \Phi_n + \sum_n \overline{A}_n \overline{\Phi}_n + 2D \cos \theta \tag{2.36}$$

$$\tau^0_{r\theta} = \sum_n A_n \Psi_n + \sum_n \overline{A}_n \overline{\Psi}_n \tag{2.37}$$

Satisfying both the specified stresses in a least squares sense by using:

$$\sigma^0_{rr} + i\tau^0_{r\theta} = \sum_n^N A_n (\Phi_n + i\Psi_n) + \sum_n^N \overline{A}_n (\overline{\Phi}_n + i\overline{\Psi}_n) + 2D \cos \theta \tag{2.38}$$

leads to minimizing the integral:

$$\int_\Gamma \left[g^0 - \sum_n^N A_n(\Phi_n + i\Psi_n) - \sum_n^N \overline{A}_n(\overline{\Phi}_n + i\overline{\Psi}_n) - 2D\cos\theta\right]\left[\overline{g}^0 - \sum_n^N \overline{A}_n(\overline{\Phi}_n - i\overline{\Psi}_n) - \sum_n^N A_n(\Phi_n - i\Psi_n) - 2D\cos\theta\right]d\Gamma = \text{minimum} \tag{2.39}$$

where $g^0 = \sigma_{rr}^0 + i\tau_{r\theta}^0$. Note that the integrand is the modulus squared of the error in the series approximations.

After minimizing the integral with respect to the j^{th} constant, the following equations were developed:

$$\int_{-\beta}^{\beta}\left[\sum_n^N A_n(\overline{\Phi}_j\Phi_n + \overline{\Psi}_j\Psi_n) + \sum_n^N \overline{A}_n(\overline{\Phi}_j\overline{\Phi}_n + \overline{\Psi}_j\overline{\Psi}_n) + 2D\overline{\Phi}_j\cos\theta\right]d\theta = \frac{1}{2}\int_{-\beta}^{\beta}\left[(g^0 + \overline{g}^0)\overline{\Phi}_j - i(g^0 - \overline{g}^0)\overline{\Psi}_j\right]d\theta$$

$$(j = 1,2,\ldots,N) \tag{2.40}$$

$$\int_{-\beta}^{\beta}\left[\sum_n^N A_n(\Phi_j\Phi_n + \Psi_j\Psi_n) + \sum_n^N \overline{A}_n(\Phi_j\overline{\Phi}_n + \Psi_j\overline{\Psi}_n) + 2D\Phi_j\cos\theta\right]d\theta = \frac{1}{2}\int_{-\beta}^{\beta}\left[(g^0 + \overline{g}^0)\Phi_j - i(g^0 - \overline{g}^0)\Psi_j\right]d\theta$$

$$(j = 1,2,\ldots,N) \tag{2.41}$$

$$\int_{-\beta}^{\beta}\left[2\cos\theta\sum_n^N A_n\Phi_n + 2\cos\theta\sum_n^N \overline{A}_n\overline{\Phi}_n + 4D\cos^2\theta\right]d\theta = \int_{-\beta}^{\beta}(g^0 + \overline{g}^0)\cos\theta\, d\theta \quad . \tag{2.42}$$

The last equation is determined by minimizing Eq. (2.39) with respect to D.

In matrix form these equations generate a Hermitian matrix; i.e., $A_{ji} = \bar{A}_{ij}$. The maximum number of terms, N, was chosen so that the series representations for the specified boundary stresses converged within some ε error term. For all loading cases which were used, $N = 15$ yielded $\varepsilon < 0.5\%$. For all loading cases, the integrals were evaluated analytically.

The same method is applicable to the odd problem.

2.2. Results and Conclusions (Wedge Problem)

The roots of the transcendental equations, Eqs. (2.23) and (2.24), were determined by the Newton-Raphson method as discussed briefly in the previous section. The results are shown for several wedge angles in Tables (2.1)-(2.3).

The system of equations generated by Eqs. (2.40)-(2.42) was solved for the following loading cases:

	σ^0_{rr}	$\tau^0_{r\theta}$	Principal Decay for Stresses
(a)	$A + B\theta^2$	0	$r^{\alpha_1 - 2}$
(b)	0	$A \sin \frac{\pi\theta}{\beta} + B(\theta^3 - \beta^2\theta)$	$r^{\alpha_1 - 2}$
(c)	A	$B(\theta^3 - \beta^2\theta)$	r^{-1}
(d)	1.0	0	r^{-1}

where α_1 refers to $\alpha_1^{(e)}$ since these are even problems. Loading cases (a) and (c) are shown in Figs. (2.2) and (2.3), respectively. The constants A and B were chosen so the loading system itself would be in static equilibrium where the required conditions for static equilibrium are

$$\sum \overline{F} = 0 \tag{2.43}$$

$$\sum (\overline{r} \times \overline{F}) = 0 \quad ; \tag{2.44}$$

i.e., the resultant force and the resultant moment must be zero.

In 1945, von Mises [7] introduced a stronger condition of equilibrium denoted as astatic equilibrium and defined by the expressions

$$\sum \overline{F} = 0 \tag{2.45}$$

$$\sum \overline{r}\overline{F} = 0 \quad . \tag{2.46}$$

Note that astatic equilibrium implies static equilibrium but that the converse is not true.

Extending the above definition to include the distributed forces on the end boundary used for this particular wedge problem, the conditions for astatic equilibrium become:

$$\sum \overline{F} = 0: \quad \int_{-\beta}^{\beta} \left[\sigma_{rr}^{0} \cos\theta - \tau_{r\theta}^{0} \sin\theta\right] d\theta = 0 \tag{2.47}$$

$$\int_{-\beta}^{\beta} \left[\sigma_{rr}^{0} \sin\theta + \tau_{r\theta}^{0} \cos\theta\right] d\theta = 0 \tag{2.48}$$

$$\sum \overline{r}\,\overline{F} = 0: \int_{-\beta}^{\beta} \sigma_{rr}^{0}\, d\theta = 0 \tag{2.49}$$

$$\int_{-\beta}^{\beta} \tau_{r\theta}^{0}\, d\theta = 0 \tag{2.50}$$

$$\int_{-\beta}^{\beta}\left[\sigma_{rr}^{0}\cos\theta - \tau_{r\theta}^{0}\sin\theta\right]\cos\theta\, d\theta = 0 \qquad (2.51)$$

$$\int_{-\beta}^{\beta}\left[\sigma_{rr}^{0}\cos\theta - \tau_{r\theta}^{0}\sin\theta\right]\sin\theta\, d\theta = 0 \quad . \qquad (2.52)$$

von Mises shows that for the half plane, loadings in astatic equilibrium may generate faster decay rates for stresses than those loadings which are in simple static equilibrium.

However, the conditions of astatic equilibrium cannot be used to justify the faster decay rates in cases (a) and (b) since neither satisfies the required conditions. As was expected, case (a) did yield a faster decay rate than case (d) since (a) is in static equilibrium. The interaction of σ_{rr}^{0} and $\tau_{r\theta}^{0}$ for case (c) results in a slower decay rate than for cases (a) or (b).

Case (c) presents another particularly interesting result. Within the range $r_0 < r < 1.4r_0$, the σ_{rr} component of stress at $\theta = \beta$ increases to a value approximately 500% its corresponding value on the boundary before it begins to decay. This can be seen in Fig. (2.5). Physically, this is fairly easy to justify since the large shear stress on the boundary near $\theta = \beta$ results in large radial stresses in the vicinity of the corner at $\theta = \beta$. For all the loading functions used, β was taken as 30°.

Tables (2.4) and (2.5) show how well the truncated series represent the specified loading functions for different values of N where N represents the number of pairs of eigenvalues used in the truncation.

Decay properties of the stresses are shown in Figures (2.4) and (2.5).

The solutions of Equations (2.23) and (2.24) which lie in the right half plane can be obtained from the left half solutions by the relations:

$$\alpha_n^{(\text{even-right})} = -\alpha_n^{(\text{even-left})} + 2$$

$$\alpha_n^{(\text{odd-right})} = -\alpha_n^{(\text{odd-left})} + 2$$

For the stresses to be bounded at infinity, the real part of $\alpha_n < 2$. Therefore, no eigenvalues in the right half plane enter into the series summations.

Table 2.1--Roots of transcendental Eqs. (2.23) and (2.24) (roots in left half plane).

n	$\alpha_n^{(e)}$		$\alpha_n^{(o)}$	
$\beta = 10°$				
1	- 11.0795	-i 6.3844	- 20.4864	-i 7.8711
2	29.6943	8.8302	38.8245	9.5441
3	47.9149	10.1141	56.9819	10.5889
4	66.0338	10.9959	75.0753	11.3522
5	84.1094	11.6690	93.1380	11.9542
6	102.1623	12.2135	111.1832	12.4514
7	120.2015	12.6710	129.2176	12.8750
8	138.2319	13.0654	147.2447	13.2439
9	156.2562	13.4120	165.2667	13.5708
10	174.2762	13.7212	183.2849	13.8642
$\beta = 20°$				
1	- 5.0578	-i 3.0954	- 9.7541	-i 3.8431
2	14.3550	4.3241	18.9184	4.6817
3	23.4625	4.9670	27.9952	5.2047
4	32.5206	5.4083	37.0410	5.5865
5	41.5577	5.7450	46.0717	5.8877
6	50.5836	6.0174	55.0939	6.1364
7	59.6028	6.2462	64.1107	6.3482
8	68.6178	6.4434	73.1241	6.5327
9	77.6297	6.6168	82.1349	6.6962
10	86.6395	6.7714	91.1438	6.8429

Asymptotic expressions are given by Eqs. (A.18), (A.19), (A.22), (A.23).

Table 2.2--Roots of transcendental Eqs. (2.23) and (2.24) (roots in left half plane).

n	$\alpha_n^{(e)}$		$\alpha_n^{(o)}$	
$\beta = 30°$				
1	- 3.0593	-i 1.9520	- 6.1820	-i 2.4557
2	9.2457	2.7780	12.2860	3.0171
3	15.3142	3.2078	18.3351	3.3665
4	21.3514	3.5024	24.3644	3.6213
5	27.3752	3.7271	30.3842	3.8222
6	33.3919	3.9088	36.3985	3.9881
7	39.4043	4.0614	42.4094	4.1294
8	45.4139	4.1929	48.4180	4.2525
9	51.4217	4.3085	54.4250	4.3615
10	57.4280	4.4116	60.4308	4.4593
$\beta = 45°$				
1	- 1.7396	-i 1.1190	- 3.8083	-i 1.4639
2	5.8451	1.6816	7.8688	1.8424
3	9.8856	1.9702	11.8981	2.0764
4	13.9079	2.1673	15.9158	2.2468
5	17.9223	2.3175	19.9278	2.3810
6	21.9325	2.4388	23.9365	2.4918
7	25.9401	2.5407	27.9432	2.5861
8	29.9460	2.6284	31.9485	2.6682
9	33.9508	2.7056	35.9528	2.7409
10	37.9547	2.7743	39.9564	2.8061

Asymptotic expressions are given by Eqs. (A.18), (A.19), (A.22), (A.23).

Table 2.3--Roots of transcendental Eqs. (2.23) and (2.24) (roots in left half plane).

n	$^{(e)}_{n}$		$^{(o)}_{n}$	
β = 60°				
1	- 1.0941	-i 0.6046	- 2.6307	-i 0.8812
2	4.1517	1.0493	5.6657	1.1720
3	7.1758	1.2690	8.6834	1.3493
4	10.1895	1.4179	11.6944	1.4779
5	13.1985	1.5311	14.7020	1.5789
6	16.2050	1.6223	17.7076	1.6622
7	19.2099	1.6989	20.7119	1.7330
8	22.2137	1.7649	23.7154	1.7947
9	25.2168	1.8228	26.7182	1.8493
10	28.2194	1.8744	29.7205	1.8983
β = 75°				
1	- 0.9130	-i 0.0000	- 1.9367	-i 0.3637
2	3.1455	0.5232	4.3518	0.6299
3	5.5567	0.7117	6.7605	0.7783
4	7.9636	0.8347	9.1662	0.8836
5	10.3684	0.9268	11.5703	0.9656
6	12.7720	1.0008	13.9735	1.0329
7	15.1748	1.0626	16.3759	1.0901
8	17.5770	1.1157	18.7779	1.1397
9	19.9788	1.1623	21.1796	1.1836
10	22.3803	1.2037	23.5809	1.2229

Asymptotic expressions are given by Eqs. (A.18), (A.19), (A.22), (A.23).

Table 2.4--Convergence of eigenfunction expansions (loading case (a)).

$\beta = 30°$

θ	Specified Function $\sigma^0_{rr} = A + B\theta^2$	No. of Paired Eigenvalues N = 6	N = 10	N = 15
0°	1.0000	0.9994	1.0006	1.0000
5°	0.9134	0.9143	0.9136	0.9133
10°	0.6537	0.6522	0.6534	0.6538
15°	0.2208	0.2230	0.2209	0.2209
20°	-0.3853	-0.3876	-0.3846	-0.3854
25°	-1.1646	-1.1646	-1.1652	-1.1647
30°	-2.1170	-2.1341	-2.1215	-2.1170

θ	Specified Function $\tau^0_{r\theta} = 0$	No. of Paired Eigenvalues N = 6	N = 10	N = 15
0°	0.0000	0	0	0
5°		-2×10^{-4}	2×10^{-4}	1×10^{-4}
10°		6×10^{-5}	-3×10^{-4}	-4×10^{-5}
15°		7×10^{-4}	-8×10^{-4}	5×10^{-5}
20°		-3×10^{-3}	-7×10^{-5}	-3×10^{-4}
25°		5×10^{-3}	6×10^{-5}	-2×10^{-4}
30°		0	0	0

$$B = -\frac{A \sin\beta}{2\beta\cos\beta + (\beta^2 - 2)\sin\beta},$$

$A = 1.0$

Table 2.5--Convergence of eigenfunction expansions (loading case (c)).

$\beta = 30°$

θ	Specified Function $\sigma^0_{rr} = A$	No. of Paired Eigenvalues N = 6	N = 10	N = 15
0°	1.0000	0.9845	0.9938	0.9982
5°		1.0107	0.9993	1.0009
10°		1.0035	1.0072	0.9995
15°		0.9732	1.0033	0.9986
20°		1.0549	0.9925	1.0043
25°		0.9483	1.0076	1.0017
30°		1.0646	0.9733	0.9582

θ	Specified Function $\tau^0_{r\theta} = B(\theta^3 - \beta^2\theta)$	No. of Paired Eigenvalues N = 6	N = 10	N = 15
0°	0	0	0	0
5°	2.2603	2.2745	2.2551	2.2588
10°	4.1332	4.1093	4.1336	4.1355
15°	5.2310	5.2552	5.2408	5.2308
20°	5.1665	5.1685	5.1690	5.1693
25°	3.5519	3.4850	3.5407	3.5529
30°	0	0	0	0

$$B = \frac{A \sin \beta}{2(\beta^2 - 3)\sin \beta + 6\beta \cos \beta},$$

$A = 1.0$

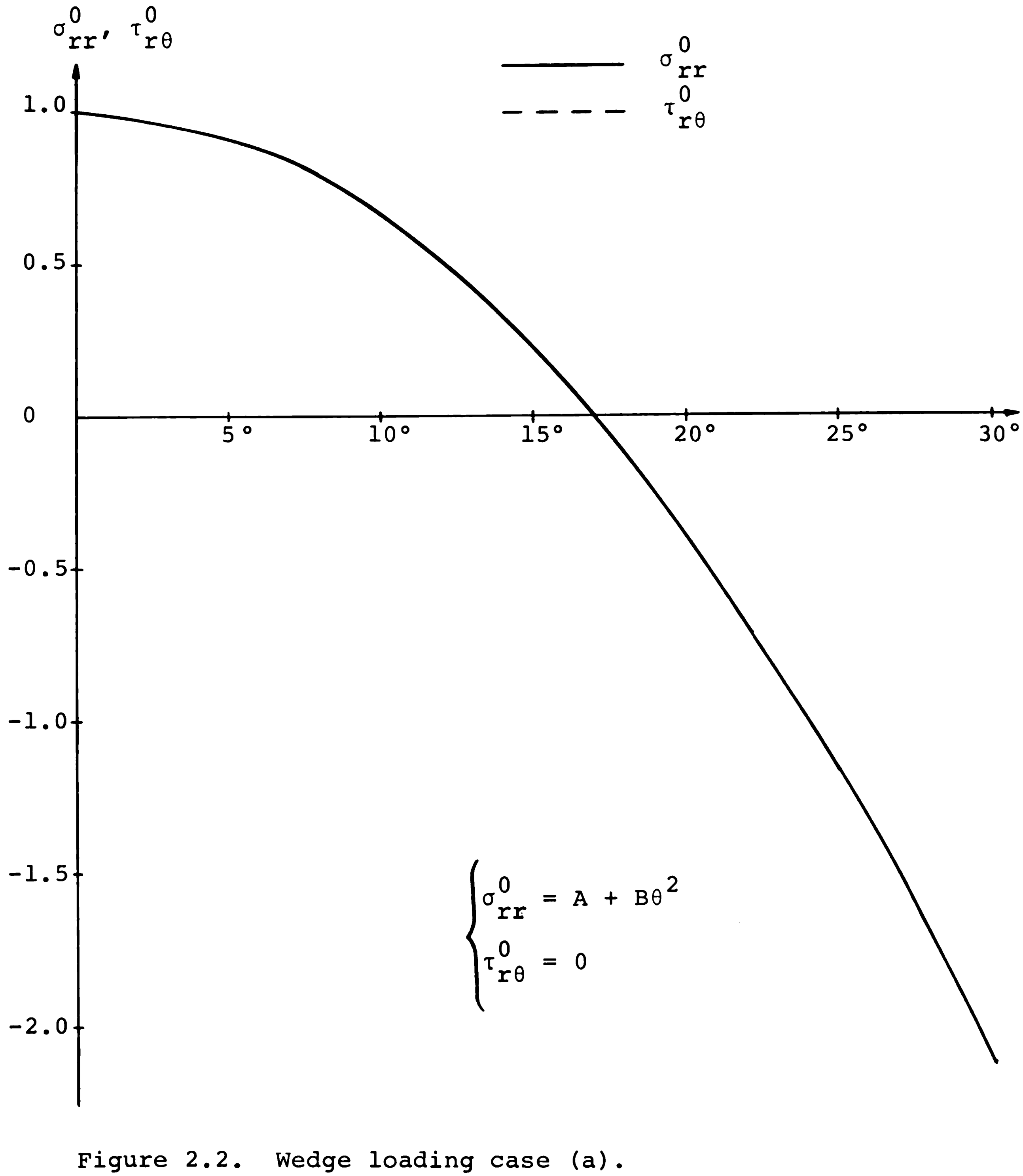

Figure 2.2. Wedge loading case (a).

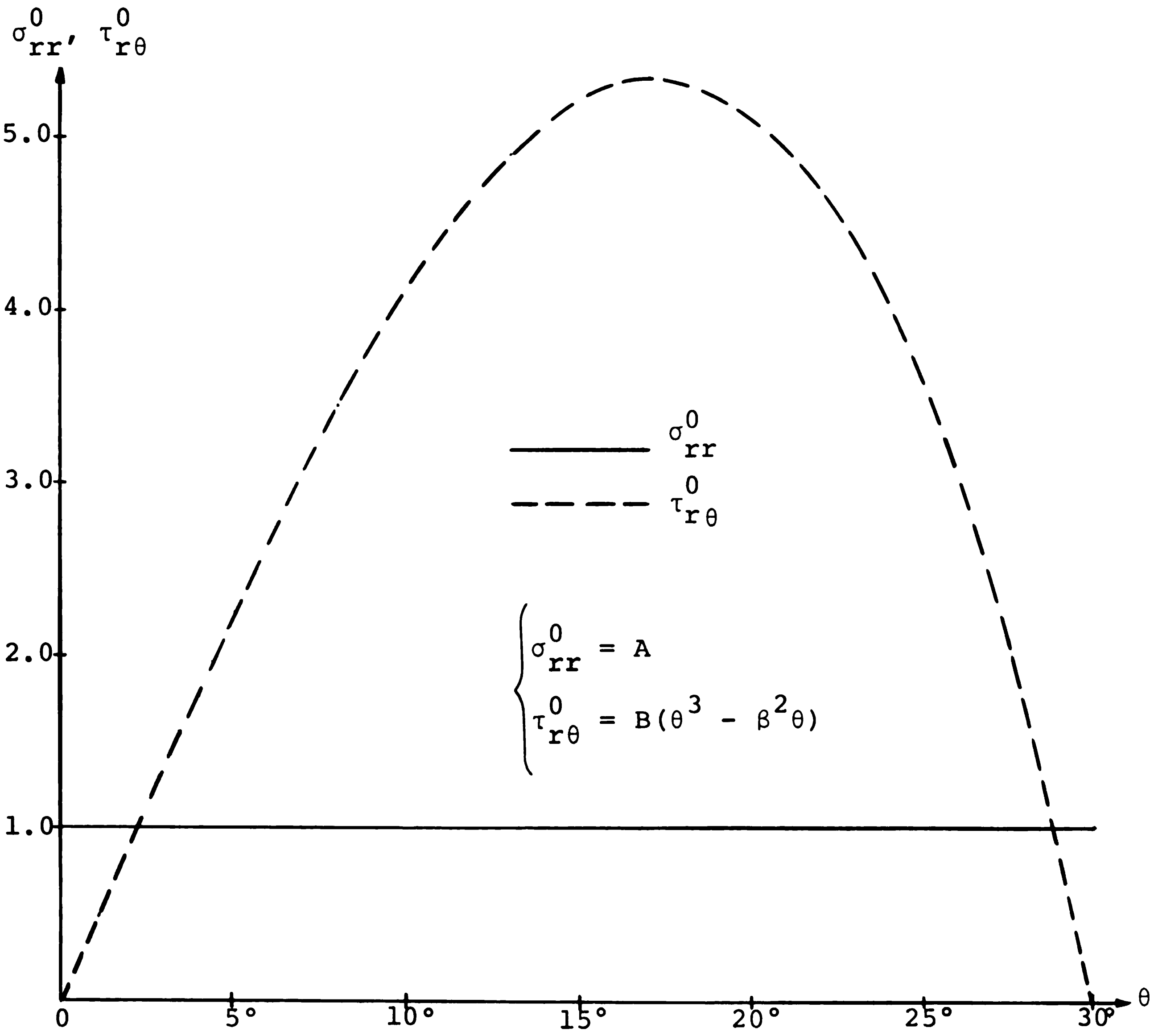

Fig. 2.3. Wedge loading case (c).

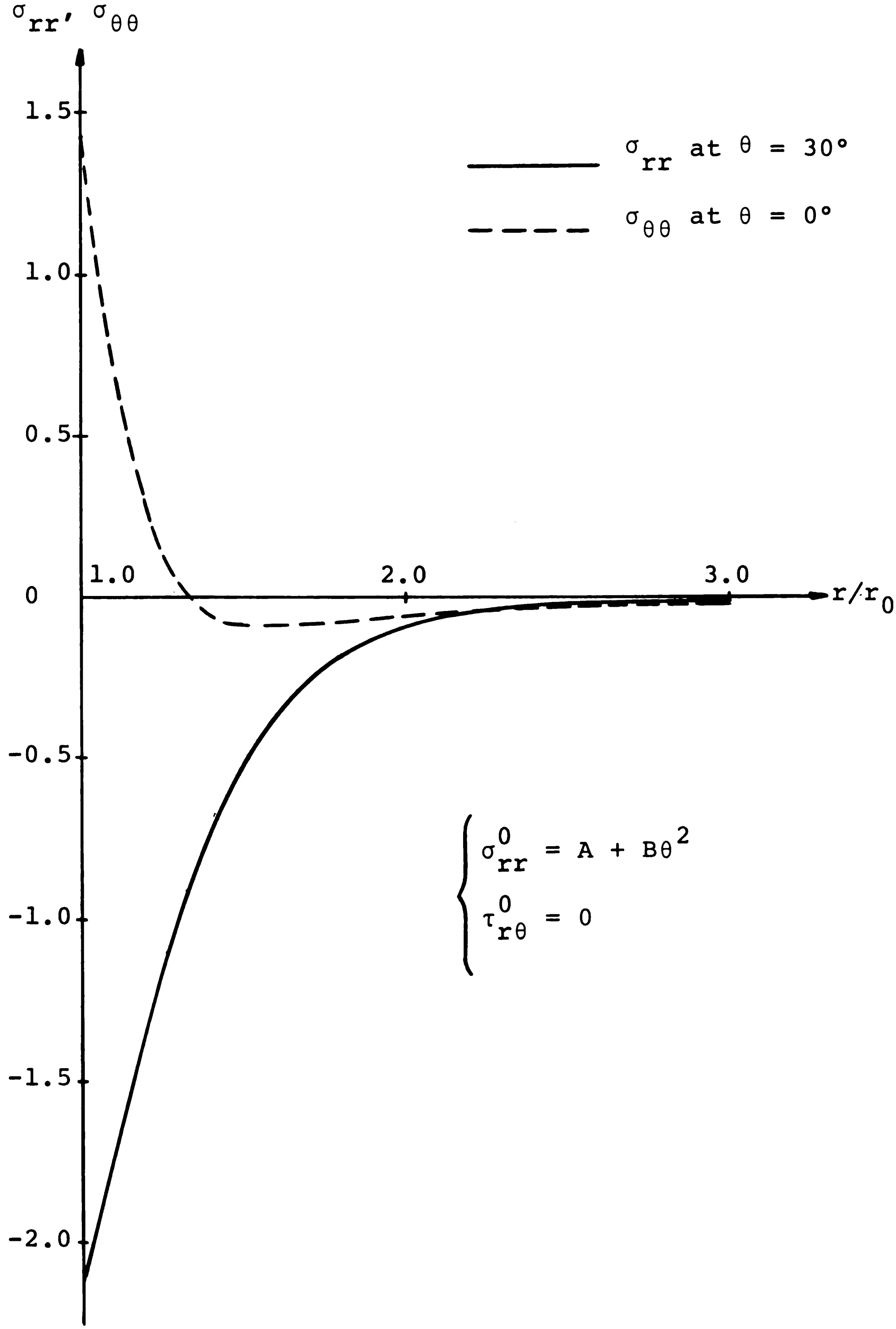

Fig. 2.4. Decay properties of σ_{rr} at $\theta = 30°$ and $\sigma_{\theta\theta}$ at $\theta = 0°$ for loading case (a).

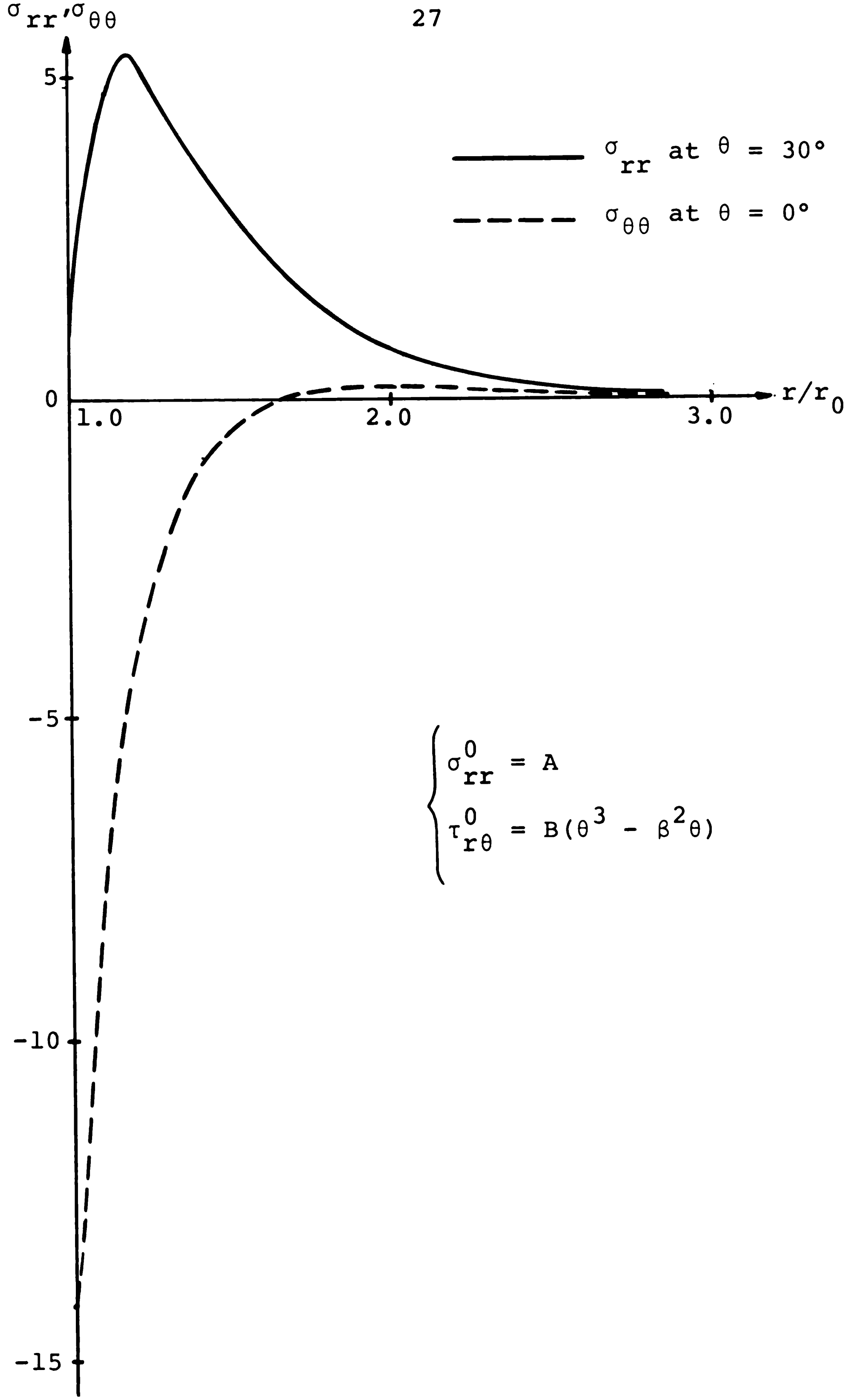

Fig. 2.5. Decay properties of σ_{rr} at $\theta = 30°$ and $\sigma_{\theta\theta}$ at $\theta = 0°$ for loading case (c).

III. THE CONE PROBLEM

3.1. Formulation of the Cone Problem

Consider the cone shown in Fig. 3.1. The boundary conditions to be specified are:

$$\text{Solution} \to 0 \text{ as } R \to \infty \tag{3.1}$$

$$\sigma_{\phi\phi}(R,\beta) = 0 \tag{3.2}$$

$$\tau_{R\phi}(R,\beta) = 0 \tag{3.3}$$

$$\sigma_{RR}(R_0,\phi) = \sigma^0_{RR}(\phi) \tag{3.4}$$

$$\tau_{R\phi}(R_0,\phi) = \tau^0_{R\phi}(\phi) \tag{3.5}$$

where $\sigma^0_{RR}(\phi)$ and $\tau^0_{R\phi}(\phi)$ are the specified loading functions.

The problem will be solved using Papkovich-Neuber functions. The displacement field may be expressed in the form [15]

$$\bar{u} = \bar{B} + \bar{\nabla}\chi \tag{3.6}$$

where $\bar{B}$ is a vector function to be determined and χ is a scalar function to also be determined. Substituting Eq. (3.6) into Navier's equation, it can be shown that

$$\chi = -\frac{1}{4(1-\nu)}(\bar{R}\cdot\bar{B} + B_0) \tag{3.7}$$

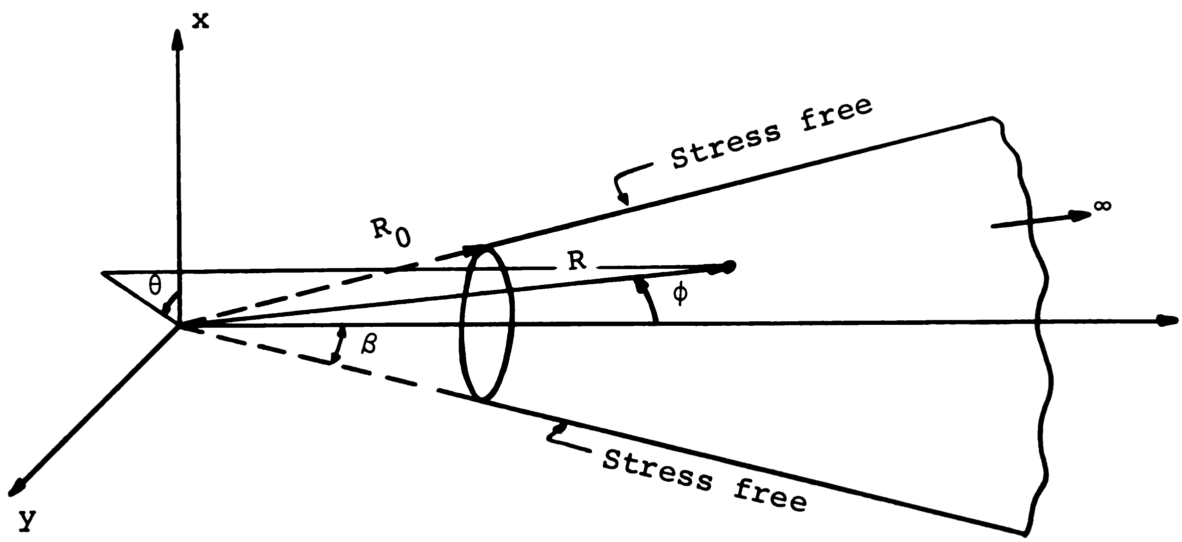

Fig. 3.1. Truncated semi-infinite three-dimensional cone.

where $\overline{B}$ and B_0 are harmonic functions. $\overline{R}$ is the position vector.

Thus, the displacement equation becomes

$$\overline{u} = \overline{B} - \frac{1}{4(1 - \nu)} \overline{\nabla}(\overline{R}\cdot\overline{B} + B_0) \tag{3.8}$$

where

$$\nabla^2\overline{B} = 0 \quad \text{and} \quad \nabla^2 B_0 = 0 \quad .$$

Formulating the problem for the cone in terms of the spherical coordinates R, θ, ϕ it is necessary to determine the functions $\overline{B}$ and B_0 so that

$$\nabla^2\overline{B} = 0 \tag{3.9}$$

and

$$\nabla^2 B_0 = 0 \tag{3.10}$$

where, for an axisymmetric problem; i.e., no θ dependence,

$$\nabla^2 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\right) + \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial}{\partial\phi}\right) \tag{3.11}$$

$$\overline{B} = \overline{e}_R B_R(R,\phi) + \overline{e}_\phi B_\phi(R,\phi) \tag{3.12}$$

$$B_0 = B_0(R,\phi) \tag{3.13}$$

$$\overline{R} = R\overline{e}_R \quad . \tag{3.14}$$

The non-zero displacement components for the cone become

$$u_R = B_R - \frac{1}{4(1 - \nu)} \frac{\partial}{\partial R} (B_0 + RB_R) \tag{3.15}$$

$$u_\phi = B_\phi - \frac{1}{4(1-\nu)} \frac{1}{R} \frac{\partial}{\partial\phi} (B_0 + RB_R) \quad . \tag{3.16}$$

It should be noted that neither B_R nor B_ϕ is a harmonic function. However, it is possible [15] to find the components of $\bar{B}$ such that

$$\nabla^2(B_z) = 0 \tag{3.17}$$

and

$$\nabla^2\left(B_\rho e^{i\theta}\right) = 0 \tag{3.18}$$

where B_ρ and B_z are the components of $\bar{B}$ in cylindrical coordinates. See Appendix C for the solution of these equations. Then the components of $\bar{B}$ in spherical coordinates can be determined by using the transformation equations:

$$B_R = B_\rho \sin\phi + B_z \cos\phi \tag{3.19}$$

$$B_\phi = B_\rho \cos\phi - B_z \sin\phi \quad . \tag{3.20}$$

For a solution which approaches zero as $R \to \infty$ Eqs. (3.17) and (3.18) yield

$$B_\rho = \sum_n A_n R^{-\alpha_n - 1} \frac{dP_{\alpha_n}(\mu)}{d\phi} \tag{3.21}$$

$$B_z = \sum_n A_n' R^{-\alpha_n - 1} P_{\alpha_n}(\mu) \tag{3.22}$$

where

$$\mu = \cos\phi \tag{3.23}$$

$$P_{\alpha_n}(\mu) = \sum_{k=0}^{\infty} \frac{(-\alpha_n)_k (\alpha_n + 1)_k}{(k!)^2} \left(\frac{1-\mu}{2}\right)^k \qquad (3.24)$$

$$(\gamma)_k = \gamma(\gamma + 1)(\gamma + 2)\ldots(\gamma + k - 1) \quad , \quad k \geq 1 \qquad (3.25)$$

$$(\gamma)_0 = 1 \quad . \qquad (3.26)$$

Equation (3.24) is one of the possible hypergeometric series representations for Legendre functions [16] and is convergent for

$$|1 - \mu| < 2 \quad . \qquad (3.27)$$

For the cone problem, the range of μ will be $0 < \mu \leq 1$.

Equations (3.19) and (3.20) give the components of $\bar{B}$ in spherical coordinates as:

$$B_R = \sum_n R^{-\alpha_n - 1}\left[A_n(\mu^2 - 1)P'_{\alpha_n} + A'_n \mu P_{\alpha_n}\right] \qquad (3.28)$$

$$B_\phi = -\sum_n R^{-\alpha_n - 1} \sin\phi \left[A_n \mu P'_{\alpha_n} + A'_n P_{\alpha_n}\right] \qquad (3.29)$$

where

$$P' = \frac{dP_{\alpha_n}(\mu)}{d\mu} \quad .$$

It is now desired to find B_0 so that

$$\nabla^2 B_0 = 0 \qquad (3.30)$$

and so that $\bar{B}$ and B_0 both contribute the same power of R to the displacements. This will then allow the boundary conditions at $\phi = \beta$ to be satisfied in a tractable manner.

If B_0 is of the form

$$B_0 = \sum_n B_n R^{-\alpha_n} P_{\alpha_n - 1}(\mu) \quad , \tag{3.31}$$

these conditions are satisfied.

Thus, the displacement expressions for the cone become

$$u_R = \sum_n R^{-\alpha_n - 1} \Big\{ \left(A_n\alpha_n + A_n'\right)(1 + k\alpha_n)\mu P_{\alpha_n} - \alpha_n \left[(1 + k\alpha_n)A_n - kB_n\right] P_{\alpha_n - 1} \Big\} \tag{3.32}$$

$$u_\phi = \sum_n R^{-\alpha_n - 1} \Big\{ \left[(1 + k\alpha_n)A_n - kB_n - k\left(A_n\alpha_n + A_n'\right)\right] \frac{dP_{\alpha_n - 1}}{d\phi} - \left(A_n\alpha_n + A_n'\right)\left[1 - k(\alpha_n + 1)\right] P_{\alpha_n} \sin\phi \Big\} \tag{3.33}$$

where $k = \frac{1}{4(1 - \nu)}$. However, without any loss of generality, the two equations above can be expressed in terms of the two arbitrary complex constants C_n' and D_n' as

$$u_R = \sum_n R^{-\alpha_n - 1} \Big\{ C_n'\left(1 + k_{\alpha_n}\right)\mu P_{\alpha_n} - \alpha_n D_n' P_{\alpha_n - 1} \Big\} \tag{3.34}$$

$$u_\phi = \sum_n R^{-\alpha_n - 1} \Big\{ \left[D_n' - kC_n'\right] \frac{dP_{\alpha_n - 1}}{d\phi} - C_n'\left[1 - k(\alpha_n + 1)\right] P_{\alpha_n} \sin\phi \Big\} \tag{3.35}$$

where

$$C_n' = A_n \alpha_n + A_n'$$

$$D_n' = (1 + k\alpha_n)A_n - kB_n \quad .$$

As will be shown later, this is not the complete solution for the displacements.

Applying the strain-displacement relations and the constitutive equations, the non-zero stresses can be shown to be:

$$\sigma_{RR} = \sum_n \gamma^{-\alpha_n - 2}\left\{\left[-\left[\alpha_n^2 + 5\alpha_n + 2(2-\nu)\right]\mu P_{\alpha_n} + 2\nu\alpha_n P_{\alpha_n - 1}\right]kC_n + \alpha_n(\alpha_n + 1)P_{\alpha_n - 1}D_n\right\} + C_0\gamma^{-2}\left[1 + \cos\beta - \frac{2(2-\nu)}{(1-2\nu)}\cos\phi\right] \tag{3.36}$$

$$\sigma_{\theta\theta} = \sum_n \gamma^{-\alpha_n - 2}\left\{\left[(1-2\nu)(1+2\alpha_n)\mu P_{\alpha_n} + 2\nu\alpha_n P_{\alpha_n - 1} + \mu P'_{\alpha_n - 1}\right]kC_n - \left[\alpha_n P_{\alpha_n - 1} + \mu P'_{\alpha_n - 1}\right]D_n\right\} + C_0\gamma^{-2}\left[\frac{\cos\phi - \cos\beta}{1 + \cos\phi} + \cos\phi - 1\right] \tag{3.37}$$

$$\sigma_{\phi\phi} = \sum_n \gamma^{-\alpha_n - 2}\left\{\left[\left[\alpha_n^2 - \alpha_n + (1-2\nu)\right]\mu P_{\alpha_n} + 2(1-\nu)\alpha_n P_{\alpha_n - 1} - \mu P'_{\alpha_n - 1}\right]kC_n - \left[\alpha_n^2 P_{\alpha_n - 1} - \mu P'_{\alpha_n - 1}\right]D_n\right\} + C_0\gamma^{-2}\left[\frac{\cos\phi - \cos\beta}{1 + \cos\phi}\right] \tag{3.38}$$

$$\tau_{R\phi} = \sum_n \gamma^{-\alpha_n - 2} \sin\phi \left\{ -\left[\left[(\alpha_n + 1)^2 - 2(1 - \nu) \right] P_{\alpha_n} + \left[(\alpha_n + 1) + 2(1 - \nu) \right] P'_{\alpha_n - 1} \right] kC_n + (\alpha_n + 1) P'_{\alpha_n - 1} D_n \right\} + C_0 \gamma^{-2} \sin\phi \left[\frac{\cos\phi - \cos\beta}{1 + \cos\phi} \right] \tag{3.39}$$

where $\gamma = R/R_0$. The C_0 term is included in order to incorporate that portion of the solution corresponding to $\alpha_n = 0$. That portion of the solution was determined independently using the Navier equations. The necessity of this term will be apparent after the next few steps.

The boundary conditions at $\phi = \beta$ are:

$$\sigma_{\phi\phi}\Big|_{\phi = \beta} = 0$$

$$\tau_{R\phi}\Big|_{\phi = \beta} = 0$$

for all $R \geqslant R_0$; i.e., for $\gamma \geqslant 1$. Applying these conditions to Eqs. (3.38) and (3.39) results in the transcendental equation for the determination of allowable α_n. After much simplification, the transcendental equation can be written as:

$$\alpha_n^2\Bigg\{\mu_0\left[2k\left(\mu_0^2 - 1\right)\alpha_n^2 + 2k\left(\mu_0^2 - 1\right)\alpha_n + \mu_0^2\right]P_{\alpha_n}^2(\mu_0)$$
$$+ \mu_0\left[2k\left(\mu_0^2 - 1\right)\alpha_n^2 + 2k\left(\mu_0^2 - 1\right)\alpha_n + 1\right]P_{\alpha_n-1}^2(\mu_0)$$
$$- 2k\Bigg[2\mu_0^2\left(\mu_0^2 - 1\right)\alpha_n^2 + \left(\mu_0^2 - 1\right)\left(3\mu_0^2 - 1\right)\alpha_n$$
$$+ \left[\mu_0^4 + 2(1 - 2\nu)\mu_0^2 + 1\right]\Bigg]P_{\alpha_n}(\mu_0)P_{\alpha_n-1}(\mu_0)\Bigg\} = 0 \qquad (3.40)$$

where $\mu_0 = \cos\beta$. Thus, it is obvious that $\alpha_n = 0$ is a solution to this equation and must be included for a complete solution. For $\alpha_n \neq 0$, the transcendental equation is

$$\mu_0\left[2k\left(\mu_0^2 - 1\right)\alpha_n^2 + 2k\left(\mu_0^2 - 1\right)\alpha_n + \mu_0^2\right]P_{\alpha_n}^2(\mu_0)$$
$$+ \mu_0\left[2k\left(\mu_0^2 - 1\right)\alpha_n^2 + 2k\left(\mu_0^2 - 1\right)\alpha_n + 1\right]P_{\alpha_n-1}^2(\mu_0)$$
$$- 2k\Bigg[2\mu_0^2\left(\mu_0^2 - 1\right)\alpha_n^2 + \left(\mu_0^2 - 1\right)\left(3\mu_0^2 - 1\right)\alpha_n$$
$$+ \left[\mu_0^4 + 2(1 - 2\nu)\mu_0^2 + 1\right]\Bigg]P_{\alpha_n}(\mu_0)P_{\alpha_n-1}(\mu_0) = 0 \quad . \qquad (3.41)$$

The above equation was solved on the CDC 3600 digital computer by calculating the value of the equation at points of a grid system in the complex plane and plotting curves along which either the real or imaginary part of the equation was zero. The intersection points of the curves then correspond to the roots of the equation. Using this intersection point of the k^{th} set of curves as an initial guess for the k^{th} eigenvalue, the modulus of the equation was determined at that point and at the four neighboring points

$$\alpha_n \pm h$$

$$\alpha_n \pm ih$$

where h is the spacing of a new grid. The smallest modulus of the five calculated was then taken as the improved solution and the process repeated until the center point represented the smallest modulus. Then h was reduced by a factor of ten and the process repeated until the modulus was less than ε_1. In all the numerical work, $\varepsilon_1 = 10^{-5}$.

As in the wedge problem, for a number $b = c + id$ which is a solution to the transcendental equation, its complex conjugate $\bar{b} = c - id$ is also a solution.

Having satisfied the boundary conditions at $\phi = \beta$, the non-zero stresses can be expressed as:

$$\sigma_{RR} = \sum_n C_n \gamma^{-\alpha_n - 2} \Big\{ - \left[\alpha_n^2 + 5\alpha_n + 2(2 - \nu)\right] k\mu P_{\alpha_n} + 2k\nu\alpha_n P_{\alpha_n - 1} + \omega_n \alpha_n (\alpha_n + 1) P_{\alpha_n - 1} \Big\} + C_0 \gamma^{-2} \left[1 + \cos\beta - \frac{2(2 - \nu)}{(1 - 2\nu)} \cos\phi\right] \tag{3.42}$$

$$\sigma_{\theta\theta} = \sum_n C_n \gamma^{-\alpha_n - 2} \Big\{ (1 - 2\nu)(1 + 2\alpha_n) k\mu P_{\alpha_n} + 2k\nu\alpha_n P_{\alpha_n - 1} + k\mu P'_{\alpha_n - 1} - \omega_n \left(\alpha_n P_{\alpha_n - 1} + \mu P'_{\alpha_n - 1}\right) \Big\} + C_0 \gamma^{-2} \left[\frac{\cos\phi - \cos\beta}{1 + \cos\phi} + \cos\phi - 1\right] \tag{3.43}$$

$$\sigma_{\phi\phi} = \sum_n C_n \gamma^{-\alpha_n - 2} \left\{ \left[\alpha_n^2 - \alpha_n + (1 - 2\nu)\right] k\mu P_{\alpha_n} + 2k(1-\nu)\alpha_n P_{\alpha_n - 1} - k\mu P'_{\alpha_n - 1} - \omega_n \left(\alpha_n^2 P_{\alpha_n - 1} - \mu P'_{\alpha_n - 1}\right) \right\} + C_0 \gamma^{-2} \left[\frac{\cos\phi - \cos\beta}{1 + \cos\phi}\right] \tag{3.44}$$

$$\tau_{R\phi} = \sum_n C_n \gamma^{-\alpha_n - 2} \sin\phi \left\{ - \left[(\alpha_n + 1)^2 - 2(1 - \nu)\right] k P_{\alpha_n} - \left[(\alpha_n + 1) + 2(1 - \nu)\right] k P'_{\alpha_n - 1} + \omega_n (\alpha_n + 1) P'_{\alpha_n - 1} \right\} + C_0 \gamma^{-2} \sin\phi \left[\frac{\cos\phi - \cos\beta}{1 + \cos\phi}\right] \tag{3.45}$$

where

$$\omega_n = \frac{\left[(\alpha_n + 1)^2 - 2(1 - \nu)\right] k P_{\alpha_n}(\mu_0) + \left[(\alpha_n + 1) + 2(1 - \nu)\right] k P'_{\alpha_n - 1}(\mu_0)}{(\alpha_n + 1) P'_{\alpha_n - 1}(\mu_0)}$$

and α_k is the k^{th} root of Eq. (3.41). The C_0 terms represent the portion of the stress field contributed by $\alpha_n = 0$.

The displacement components corresponding to $\alpha_n = 0$ which must be added to Eqs. (3.34) and (3.35) for the complete displacement field are:

$$u_R = \frac{C_0}{R}\left[\frac{(1 - 2\nu)}{(3 - 4\nu)} (1 + \cos\beta) - \frac{4(1 - \nu)}{(3 - 4\nu)} \cos\phi\right] \tag{3.46}$$

$$u_\phi = \frac{C_0}{R}\left[- \frac{(1 - 2\nu)}{(3 - 4\nu)} (1 + \cos\beta) \frac{\sin\phi}{1 + \cos\phi} + \sin\phi\right] \tag{3.47}$$

The last boundary conditions, those at $R = R_0$, will now be satisfied. These conditions had to be satisfied numerically as was the case for the wedge problem. The method of least squares was again used to determine the constants for the truncated series.

Using a similar argument as in the wedge problem, the stresses on the end of the cone can be expressed as

$$\sigma_{RR}^{0} = \sum_{n} C_n \Phi_n + \sum_{n} \overline{C}_n \overline{\Phi}_n + C_0 \eta \tag{3.48}$$

$$\tau_{R\phi}^{0} = \sum_{n} C_n \Psi_n + \sum_{n} \overline{C}_n \overline{\Psi}_n + C_0 \lambda \tag{3.49}$$

where

$$\Phi_n = -\left[\alpha_n^2 + 5\alpha_n + 2(2 - \nu)\right] k\mu P_{\alpha_n} + 2k\nu\alpha_n P_{\alpha_n - 1}$$

$$+ \omega_n \alpha_n (\alpha_n + 1) P_{\alpha_n - 1}$$

$$\Psi_n = -\left[(\alpha_n + 1)^2 - 2(1 - \nu)\right] kP_{\alpha_n} \sin \phi$$

$$- \left[(\alpha_n + 1) + 2(1 - \nu)\right] kP'_{\alpha_n - 1} \sin \phi$$

$$+ \omega_n (\alpha_n + 1) P'_{\alpha_n - 1} \sin \phi$$

$$\eta = 1 + \cos \beta - \frac{2(2 - \nu)}{(1 - 2\nu)} \cos \phi$$

$$\lambda = \frac{\sin \phi (\cos \phi - \cos \beta)}{(1 + \cos \phi)} .$$

Following the same procedure as outlined in Section 2.1 for the wedge problem, the following equations were developed for the determination of the constants:

$$\int_0^\beta \left[\sum_n^N c_n(\overline{\Phi}_j \Phi_n + \overline{\Psi}_j \Psi_n) + \sum_n^N \overline{c}_n(\overline{\Phi}_j \overline{\Phi}_n + \overline{\Psi}_j \overline{\Psi}_n) + c_0(\overline{\Phi}_j \eta \right.$$

$$\left. + \overline{\Psi}_j \lambda) \right] \sin\phi \, d\phi = \frac{1}{2} \int_0^\beta \left[(g^0 + \overline{g}^0)\overline{\Phi}_j - i(g^0 - \overline{g}^0)\overline{\Psi}_j \right] \sin\phi \, d\phi$$

$$(j = 1,2,\ldots,N) \qquad (3.50)$$

$$\int_0^\beta \left[\sum_n^N c_n(\Phi_j \Phi_n + \Psi_j \Psi_n) + \sum_n^N \overline{c}_n \, (\Phi_j \, \overline{\Phi}_n + \Psi_j \overline{\Psi}_n) + c_0(\Phi_j \eta \right.$$

$$\left. + \Psi_j \lambda) \right] \sin\phi \, d\phi = \frac{1}{2} \int_0^\beta \left[(g^0 + \overline{g}^0)\Phi_j - i(g^0 - \overline{g}^0)\Psi_j \right] \sin\phi \, d\phi$$

$$(j = 1,2,\ldots,N) \qquad (3.51)$$

$$\int_0^\beta \left[\sum_n^N c_n(\eta\Phi_n + \lambda\Psi_n) + \sum_n^N \overline{c}_n(\eta\overline{\Phi}_n + \lambda\overline{\Psi}_n) + c_0(\eta^2 + \lambda^2) \right] \sin\phi \, d\phi$$

$$= \frac{1}{2}\int_0^\beta \left[(g^0 + \overline{g}^0)\eta - i(g^0 - \overline{g}^0)\lambda \right] \sin\phi \, d\phi \qquad (3.52)$$

where

$$g^0 = \sigma_{RR}^0 + i\tau_{R\phi}^0 \quad .$$

In matrix form, these equations also generate a Hermitian matrix. Again, as for the wedge, N was chosen so that the series representations for the specified loading functions converged within some ε_2 error term. For the loading cases which were used, N = 5 yielded $\varepsilon_2 \leq 16\%$ where the maximum error occurred only in the neighborhood of $\phi = \beta$. The integration required for each matrix element was performed numerically using the Newton-Cotes method.

Because of the lengthy computer time required for $N > 5$, the solution was not determined for larger values of N. Therefore, the convergence near $\phi = \beta$ for the cone as shown in Tables (3.2) and (3.3) is not quite as accurate as one may desire. However, over 70% of the boundary, the agreement was within 5%.

3.2 Results and Conclusions (Cone Problem)

The roots of the transcendental equation, Eq. (3.41), were determined by the method as outlined in the previous section. The results for several cone angles are shown in Table 3.1.

The system of equations generated by Eqs. (3.50)-(3.52) was solved for the following loading cases:

	σ^0_{RR}	$\tau^0_{R\phi}$	Principal Decay for Stresses
(a)	$1.0 + A\phi^2 + B\phi^3$	0	$R^{-\alpha_1-2}$
(b)	A	$B(\phi^3 - \beta^2\phi)$	$R^{-\alpha_1-2}$
(c)	1.0	0	R^{-2}

Loading cases (a) and (b) are shown in Figs. (3.2) and (3.3), respectively. The constants A and B were chosen so the loading system would be in static equilibrium.

For this axisymmetric cone, the conditions for astatic equilibrium become

$$\sum \overline{F} = 0 : \quad \int_0^{\beta} \left[\sigma_{RR}^0 \cos \phi - \tau_{R\phi}^0 \sin \phi \right] \sin \phi \, d\phi = 0 \quad (3.53)$$

$$\sum \overline{r}\overline{F} = 0 : \quad \int_0^{\beta} \sigma_{RR}^0 \sin \phi \, d\phi = 0$$

$$\int_0^{\beta} \left[\sigma_{RR}^0 \cos \phi - \tau_{R\phi}^0 \sin \phi \right] \sin \phi \cos \phi \, d\phi = 0 \quad . \quad (3.54)$$

Similar to the results of the wedge problem, loading case (a) did yield a faster decay rate than case (c) since (a) is in static equilibrium. However, for the cone, case (b) yields a faster decay than for the similar loading function on the wedge. None of the loading cases is in astatic equilibrium.

Similar to the decay in the wedge, loading case (b) for the cone results in an increase of the σ_{RR} component of stress within a small region before it begins to decay. Again, the interpretation of this result is quite similar to that of the wedge.

Tables (3.2) and (3.3) show how well the truncated series represent the specified loading functions.

Decay properties of the stresses are shown in Figs. (3.4) and (3.5).

Table 3.1--Roots of transcendental Eq. (3.41)(roots in right half plane).

n	α_n		n	α_n	
$\beta = 15°$:			$\beta = 30°$:		
1	9.9170 + i	5.0850	1	4.7409 + i	2.3589
2	22.6618	6.1604	2	11.1025	2.9327
3	34.9063	6.8925	3	17.2181	3.3041
4	47.0343	7.4239	4	23.2783	3.5723
			5	29.3163	3.7814
$\beta = 45°$:			$\beta = 60°$:		
1	3.0371 + i	1.3520	1	2.2189 + i	0.7425
2	7.2610	1.7771	2	5.3518	1.1135
3	11.3298	2.0309	3	8.3933	1.3111
4	15.3657	2.2118	4	11.4151	1.4493
5	19.3883	2.3525	5	14.4289	1.5560
$\beta = 75°$:					
1	1.7530 + i	0.0460			
2	4.2211	0.5707			
3	6.6413	0.7429			
4	9.0523	0.8582			
5	11.4593	0.9457			

Table 3.2--Convergence of eigenfunction expansions (loading case (a)).

$\beta = 60°$

ϕ	Specified Function $\sigma^0_{R\phi} = 1.0 + A\phi^2 + B\phi^3$	No. of Paired Eigenvalues N = 5
0°	1.0000	0.9033
10°	0.8418	0.7914
20°	0.4716	0.4597
30°	0.0458	0.0681
40°	-0.2791	-0.2654
50°	-0.3465	-0.3412
60°	0	-2×10^{-2}

ϕ	Specified Function $\tau^0_{R\phi} = 0$	No. of Paired Eigenvalues N = 5
0°	0	0
10°		4×10^{-2}
20°		5×10^{-2}
30°		2×10^{-2}
40°		-1×10^{-3}
50°		-1×10^{-2}
60°		0

$$A = \frac{4\beta^3 - 3(2\beta^2 - 1)\sin 2\beta - 6\beta\cos 2\beta}{2\beta^4\sin 2\beta + 4\beta^3\cos 2\beta - 3\beta^2\sin 2\beta + 2\beta^3} ,$$

$$B = \frac{2\beta^2 - 2\beta\sin 2\beta + 1 - \cos 2\beta}{-\beta^4\sin 2\beta - 2\beta^3\cos 2\beta + 1.5\beta^2\sin 2\beta - \beta^3} .$$

Table 3.3--Convergence of eigenfunction expansions (loading case (b))

$\beta = 60°$

ϕ	Specified Function $\sigma^0_{R\phi} = A$	No. of Paired Eigenvalues $N = 5$
0°	1.0000	1.1230
10°		1.1033
20°		1.0478
30°		0.9415
40°		0.9686
50°		0.9953
60°		1.1621

ϕ	Specified Function $\tau^0_{R\phi} = B(\phi^3 - \beta^2\phi)$	No. of Paired Eigenvalues $N = 5$
0°	0	0
10°	0.7627	0.7125
20°	1.3947	1.2788
30°	1.7652	1.6995
40°	1.7434	1.7588
50°	1.1986	1.2062
60°	0	0

$$B = \frac{-\,8A \sin^2\beta}{(2\beta^4 + 2\beta^2 + 3) - 6\beta \sin 2\beta + (4\beta^2 - 3)\cos 2\beta},$$

$A = 1.0$

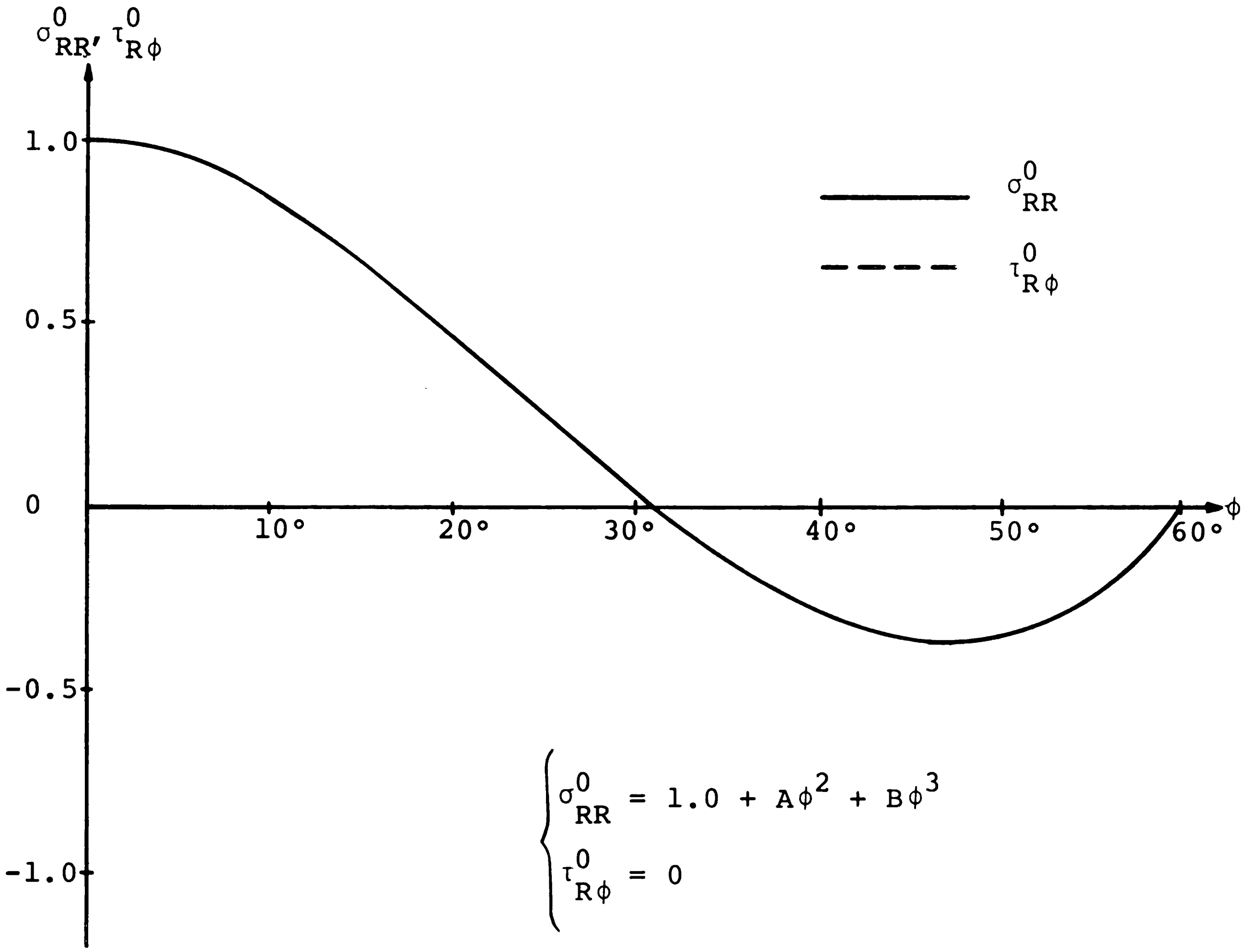

Fig. 3.2. Cone loading case (a).

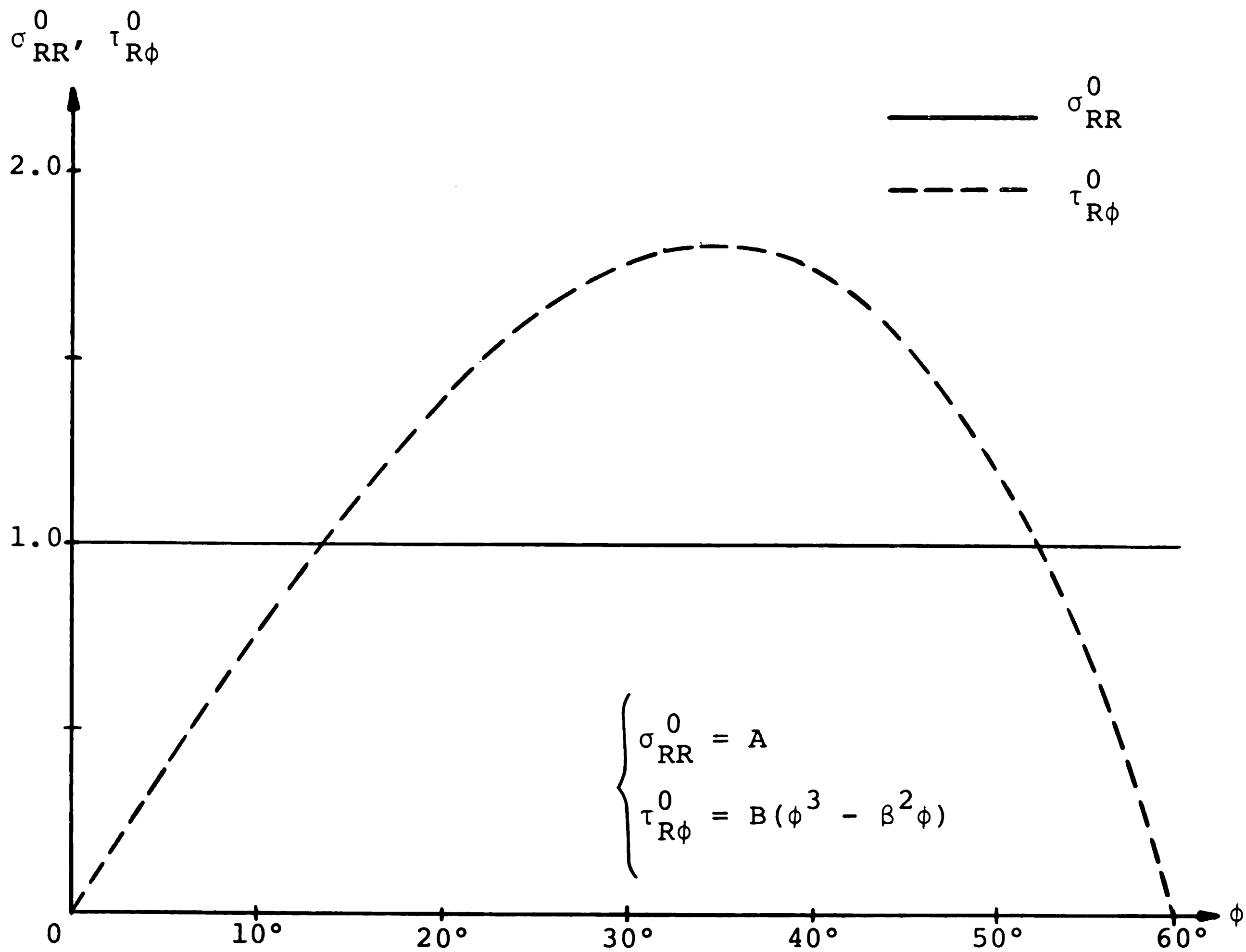

Fig. 3.3. Cone loading case (b).

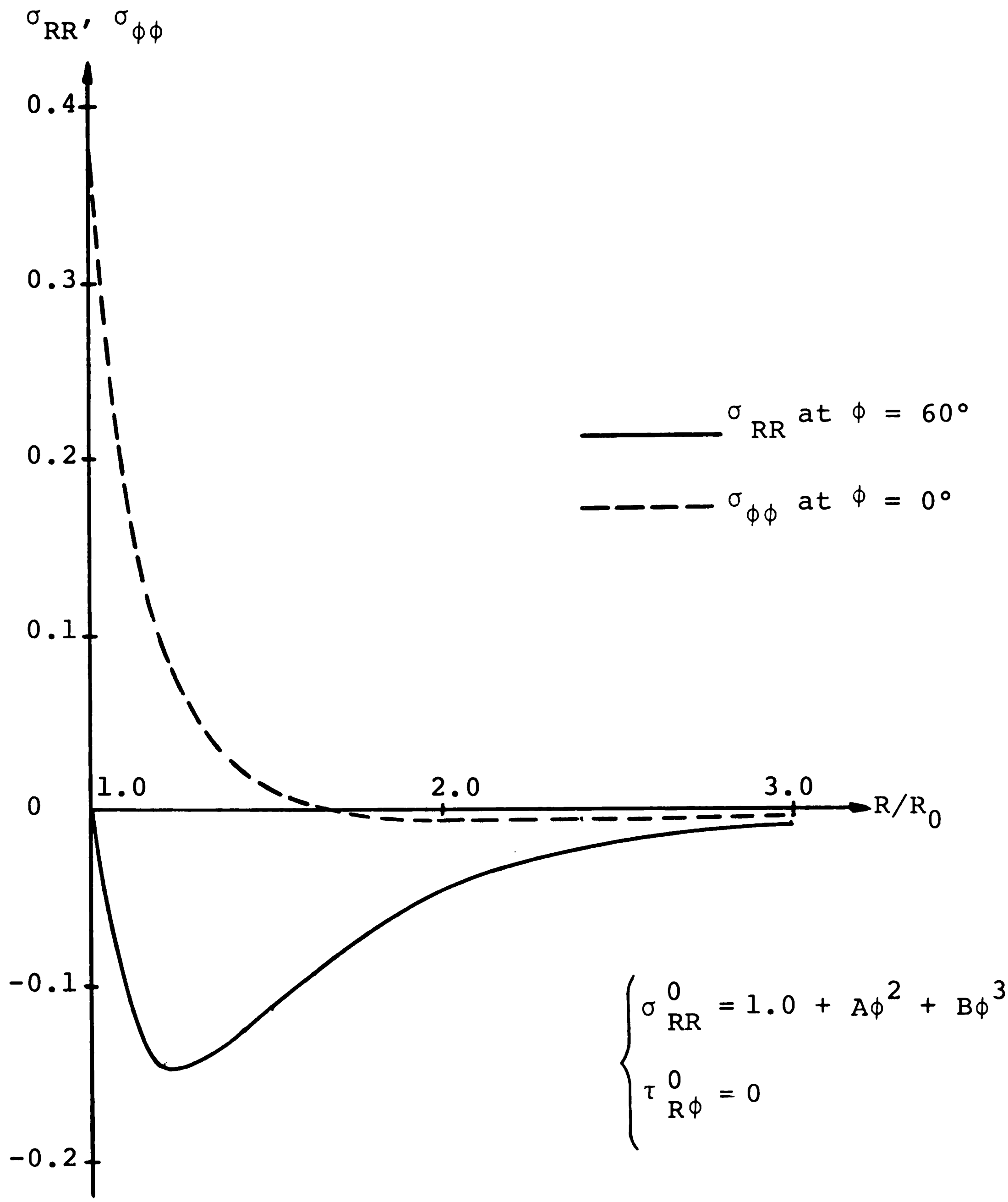

Fig. 3.4. Decay properties of σ_{RR} at $\phi = 60°$ and $\sigma_{\phi\phi}$ at $\phi = 0°$ for loading case (a).

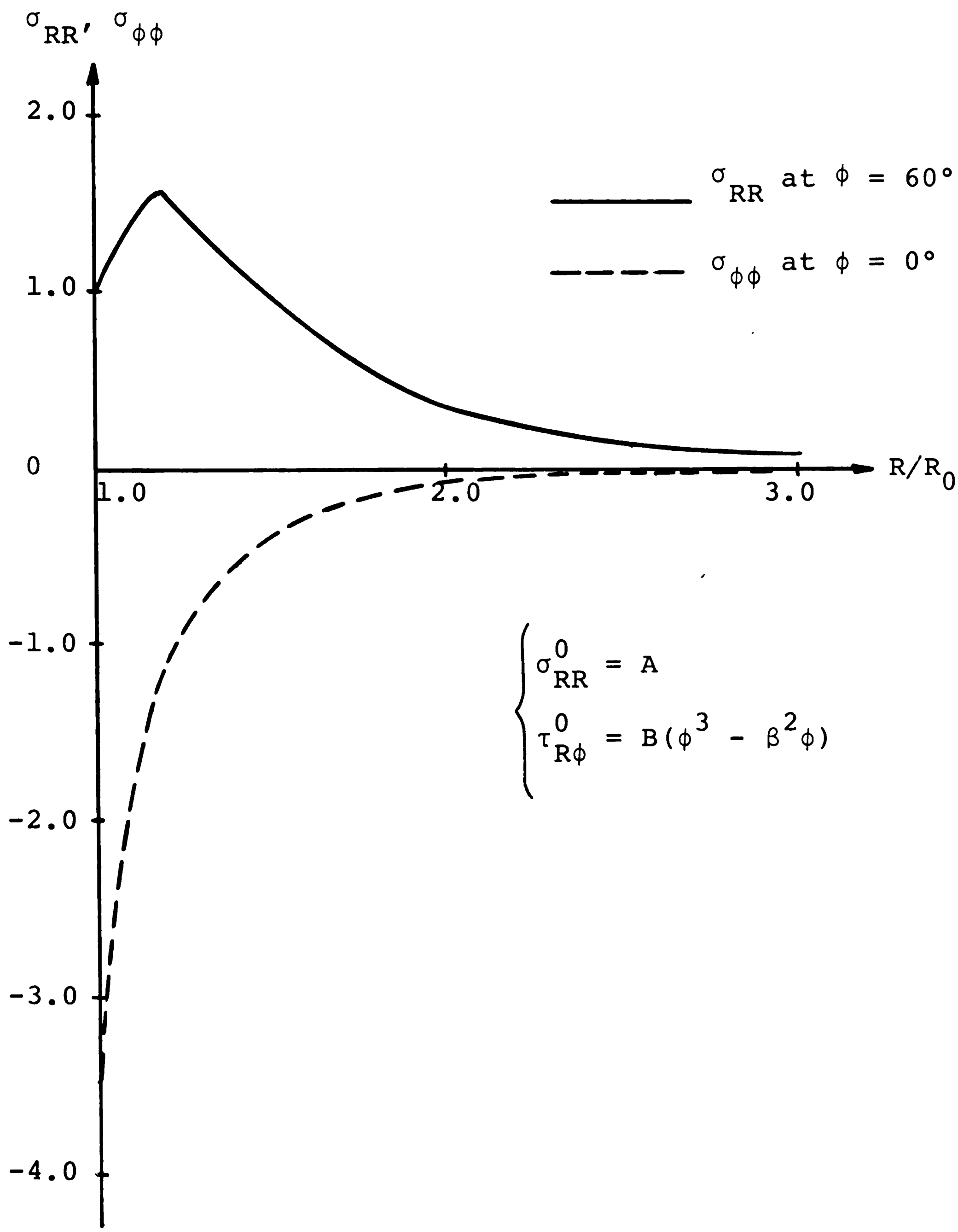

Fig. 3.5. Decay properties of σ_{RR} at $\phi = 60°$ and $\sigma_{\phi\phi}$ at $\phi = 0°$ for loading case (b).

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APPENDICES

APPENDIX A

DEVELOPMENT OF THE ASYMPTOTIC EIGENVALUES OF THE WEDGE PROBLEM

Consider the transcendental equation for the even eigenvalues, Eq. (2.24), in the form:

$$\alpha_n \beta \sin (2\beta) + 2\beta \cos (\alpha_n \beta) \sin \left[(\alpha_n - 2)\beta\right] = 0 \qquad \text{(A.1)}$$

For α_n complex, seek solutions in the first quadrant of the complex plane in the form:

$$\alpha_n = x_n + iy_n \qquad \text{(A.2)}$$

Substituting this expression for α_n into Eq. (A.1) yields:

$$(x_n + iy_n)\beta \sin (2\beta) + 2\beta \cos \left[(x_n + iy_n)\beta\right] \sin \left[\left((x_n - 2) + iy_n\right)\beta\right] = 0 \qquad \text{(A.3)}$$

Making use of the elementary trigonometric relations:

$$\sin (a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\sin (i\gamma) = i \sinh \gamma$$

$$\cos (i\gamma) = \cosh \gamma$$

yields the following coupled algebraic equations for the real and imaginary parts of α_n:

$$x_n\beta \sin(2\beta) + 2\beta\left\{\cos(x_n\beta)\sin\left[(x_n - 2)\beta\right]\cosh^2(y_n\beta)\right.$$

$$\left. + \sin(x_n\beta)\cos\left[(x_n - 2)\beta\right]\sinh^2(y_n\beta)\right\} = 0 \qquad \text{(A.4)}$$

and,

$$y_n\beta \sin(2\beta) + 2\beta\cos\left[2(x_n - 1)\beta\right]\sinh(y_n\beta)\cosh(y_n\beta) = 0 \quad . \qquad \text{(A.5)}$$

It is desired to develop asymptotic solutions for these equations: i.e., solutions which will be accurate for the higher eigenvalues. Substituting the exponential expressions for the hyperbolic functions and neglecting higher order terms, Eqs. (A.4) and (A.5) reduce to:

$$\phi_1 \sin 2\beta + \beta e^{\phi_2} \sin\phi_1 = 0 \qquad \text{(A.6)}$$

and

$$\phi_2 \sin 2\beta + \beta e^{\phi_2} \cos\phi_1 = 0 \qquad \text{(A.7)}$$

where

$$\phi_1 = 2(x_n - 1)\beta$$
$$\phi_2 = 2y_n\beta \quad .$$

These, then are the equations which hold for large ϕ_1 and ϕ_2.

Since solutions are being sought in the first quadrant, x_n and y_n are restricted to be positive numbers. However, for Eqs. (A.6) and (A.7) to be satisfied, $\sin\phi_1$ and $\cos\phi_1$ must be negative.

For Eq. (A.7) to be satisfied, $\cos\phi_1$ must be a small negative number since $e^{\phi_2} >> \phi_2$. Thus, take

$$\phi_1 = (4n - 1)\frac{\pi}{2} - \varepsilon_n \quad , \quad n = 1,2,3,... \qquad \text{(A.8)}$$

where ε_n is a small correction term to be determined shortly. From Eq. (A.6)

$$\phi_2 = \ell n \left[- \frac{\sin 2\beta}{\beta} \frac{\phi_1}{\sin \phi_1} \right] \quad . \qquad \text{(A.9)}$$

But from Eq. (A.8), $\sin \phi_1 \simeq -1$ and for an approximation of ϕ_2, use only the first term of ϕ_1 in Eq. (A.8); i.e., take

$$\phi_2 = \ell n \left[\frac{\sin 2\beta}{\beta} (4n - 1) \frac{\pi}{2} \right] \quad , \quad n = 1,2,3,\ldots \qquad \text{(A.10)}$$

Thus, ϕ_1 and ϕ_2 become:

$$\phi_1 = (4n - 1) \frac{\pi}{2} - \varepsilon_n \qquad \text{(A.11)}$$

$$\phi_2 = \ell n \left[\frac{\sin 2\beta}{\beta} (4n - 1) \frac{\pi}{2} \right] \qquad \text{(A.12)}$$

for $n = 1,2,3,\ldots$

Now substitute Eqs. (A.11) and (A.12) into Eqs. (A.6) and (A.7) to solve for ε_n. Equation (A.6) is satisfied approximately and Eq. (A.7) yields, for small ε_n:

$$\sin 2\beta \left\{ \ell n \left[\frac{\sin 2\beta}{\beta} (4n - 1) \frac{\pi}{2} \right] - (4n - 1) \frac{\pi}{2} \varepsilon_n \right\} = 0 \; . \qquad \text{(A.13)}$$

Thus, for $\sin 2\beta \neq 0$,

$$\varepsilon_n = \frac{\ell n \left[\frac{\sin 2\beta}{\beta} (4n - 1) \frac{\pi}{2} \right]}{(4n - 1)\frac{\pi}{2}} \quad , \quad n = 1,2,3,\ldots \qquad \text{(A.14)}$$

Therefore, the asymptotic eigenvalues in the right half plane for the even problem are:

$$x_n = \frac{1}{2\beta}\left\{(4n-1)\frac{\pi}{2} + 2\beta - \frac{\ln\left[\frac{\sin 2\beta}{\beta}(4n-1)\frac{\pi}{2}\right]}{(4n-1)\frac{\pi}{2}}\right\} \tag{A.15}$$

$$y_n = \frac{1}{2\beta}\ln\left[\frac{\sin 2\beta}{\beta}(4n-1)\frac{\pi}{2}\right] \tag{A.16}$$

where $\alpha_n^{(e)} = x_n + iy_n$ (even solution in right half plane).

A similar development for the decaying solution requires seeking solutions to Eq. (2.24) in the left half plane. Taking the solution for α_n in the form

$$\alpha_n = -u_n - iv_n \quad , \tag{A.17}$$

the asymptotic values in the left half plane for the even problem can be determined in a manner very consistent with the previous development as:

$$u_n = \frac{1}{2\beta}\left\{(4n-1)\frac{\pi}{2} - 2\beta - \frac{\ln\left[\frac{\sin 2\beta}{\beta}(4n-1)\frac{\pi}{2}\right]}{(4n-1)\frac{\pi}{2}}\right\} \tag{A.18}$$

$$v_n = \frac{1}{2\beta}\ln\left[\frac{\sin 2\beta}{\beta}(4n-1)\frac{\pi}{2}\right] \tag{A.19}$$

where $\alpha_n^{(e)} = -u_n - iv_n$. (even solution in left half plane)

For the odd problem, working with the odd transcendental equation, Eq. (2.23), the asymptotic values become:

$$p_n = \frac{1}{2\beta}\left\{(4n-3)\frac{\pi}{2} + 2\beta - \frac{\ln\left[\frac{\sin 2\beta}{\beta}(4n-3)\frac{\pi}{2}\right]}{(4n-3)\frac{\pi}{2}}\right\} \tag{A.20}$$

$$q_n = \frac{1}{2\beta}\ln\left[\frac{\sin 2\beta}{\beta}(4n-3)\frac{\pi}{2}\right] \tag{A.21}$$

where $\alpha_n^{(o)} = p_n + iq_n$ (odd solution in right half plane).

The decaying solution is:

$$r_n = \frac{1}{2\beta}\left\{(4n - 3)\frac{\pi}{2} - 2\beta - \frac{\ell n\left[\frac{\sin 2\beta}{\beta}(4n - 3)\frac{\pi}{2}\right]}{(4n - 3)\frac{\pi}{2}}\right\} \tag{A.22}$$

$$s_n = \frac{1}{2\beta}\,\ell n\left[\frac{\sin 2\beta}{\beta}(4n - 3)\frac{\pi}{2}\right] \tag{A.23}$$

where $\alpha_n^{(o)} = -r_n - is_n$ (odd solution in left half plane).

These values represent the approximate solutions to the appropriate transcendental equations for the higher eigenvalues; however, they are also quite accurate for the lower eigenvalues as may be observed using Tables (2.1)-(2.3).

APPENDIX B

ORTHOGONALITY CONDITIONS FOR THE EIGENFUNCTIONS OF THE WEDGE PROBLEM

Using the method of generalized orthogonality by P. F. Papkovich [17], the defining equation for the m^{th} eigenfunction of the wedge problem is the fourth order differential equation:

$$f_m^{IV}(\theta) + \left[\alpha_m^2 + (\alpha_m - 2)^2\right] f_m''(\theta) + \alpha_m^2(\alpha_m - 2)^2 f_m(\theta) = 0 \quad . \tag{B.1}$$

Similarly, the n^{th} eigenfunction is given by:

$$f_n^{IV}(\theta) + \left[\alpha_n^2 + (\alpha_n - 2)^2\right] f_n''(\theta) + \alpha_n^2(\alpha_n - 2)^2 f_n(\theta) = 0 \quad . \tag{B.2}$$

Multiply Eq. (B.1) by $\alpha_n(\alpha_n - 2)f_n$, Eq. (B.2) by $\alpha_m(\alpha_m - 2)f_m$, subtracting the equations, and integrating over the boundary yields:

$$\int_{-\beta}^{\beta} \left\{ \alpha_m(\alpha_m - 2) f_m f_n^{IV} - \alpha_n(\alpha_n - 2) f_n f_m^{IV} + \alpha_m(\alpha_m - 2)\left[\alpha_n^2 + (\alpha_n - 2)^2\right] f_m f_n'' - \alpha_n(\alpha_n - 2)\left[\alpha_m^2 + (\alpha_m - 2)^2\right] f_n f_m'' + \left[\alpha_m(\alpha_m - 2)\alpha_n^2(\alpha_n - 2)^2 - \alpha_n(\alpha_n - 2)\alpha_m^2(\alpha_m - 2)^2\right] f_m f_n \right\} d\theta = 0 \quad . \tag{B.3}$$

The boundary conditions for the particular wedge problem formulated in the earlier chapters require $f_m(\pm\beta)$, $f_m'(\pm\beta)$, $f_n(\pm\beta)$, $f_n'(\pm\beta)$ to vanish. Applying these conditions after integrating by parts twice for the first two terms and once for the next two, Eq. (B.3) reduces to

$$\int_{-\beta}^{\beta}\Bigg\{\Big[\alpha_m(\alpha_m - 2) - \alpha_n(\alpha_n - 2)\Big]f_m''f_n'' + \Big[-\alpha_m(\alpha_m - 2)\Big(\alpha_n^2 + (\alpha_n - 2)^2\Big) + \alpha_n(\alpha_n - 2)\Big(\alpha_m^2 + (\alpha_m - 2)^2\Big)\Big]f_m'f_n' - \alpha_m(\alpha_m - 2)\alpha_n(\alpha_n - 2)\Big[\alpha_m(\alpha_m - 2) - \alpha_n(\alpha_n - 2)\Big]f_mf_n\Bigg\}\,d\theta = 0 \quad . \tag{B.4}$$

Expanding and simplifying the coefficient of $f_m'f_n'$ and factoring out $\Big[\alpha_m(\alpha_m - 2) - \alpha_n(\alpha_n - 2)\Big]$ from each term yields:

$$\Big[\alpha_m(\alpha_m - 2) - \alpha_n(\alpha_n - 2)\Big]\int_{-\beta}^{\beta}\Big[\alpha_m(\alpha_m - 2)\alpha_n(\alpha_n - 2)f_mf_n + 4\,f_m'f_n' - f_m''f_n''\Big]d\theta = 0 \quad . \tag{B.5}$$

Thus, the orthogonality condition is:

$$\int_{-\beta}^{\beta}\Big[\alpha_m(\alpha_m - 2)\alpha_n(\alpha_n - 2)f_mf_n + 4\,f_m'f_n' - f_m''f_n''\Big]d\theta = 0, \quad (m \neq n) \quad . \tag{B.6}$$

APPENDIX C

DEVELOPMENT OF THE LEGENDRE FUNCTIONS USED IN THE CONE PROBLEM

Solving the cone problem by the method outlined in Chapter III requires solving the equations

$$\nabla^2 (B_z) = 0 \tag{C.1}$$

$$\nabla^2 \left(B_\rho e^{i\theta} \right) = 0 \tag{C.2}$$

where $B_z = B_z(R,\phi)$ and $B_\rho = B_\rho(R,\phi)$.

Assume a solution of the form

$$B_z = R^{-\alpha_n - 1} f_n(\phi) \tag{C.3}$$

and substitute into Eq. (C.1). Then

$$\nabla^2 \left[R^{-\alpha_n - 1} f_n(\phi) \right] = 0 \tag{C.4}$$

where ∇^2 is the Laplace operator in spherical coordinates. This leads to the following equation for f_n:

$$(1 - x^2) f_n''(x) - 2x\, f_n'(x) + \alpha_n(\alpha_n + 1) f_n(x) = 0 \tag{C.5}$$

where $x = \cos \phi$. This is Legendre's equation.

Change variables [16] by letting $t = \frac{1}{2}(1 - x)$. Eq. (C.5) becomes

$$t(1-t)f_n''(t) + (1-2t)f_n'(t) + \alpha_n(\alpha_n + 1)f_n(t) = 0 \tag{C.6}$$

which has a solution

$$f_n = (f_n)_1 = F\left(-\alpha_n,\ \alpha_n + 1; 1;\ \frac{1-x}{2}\right) \tag{C.7}$$

where $F(\alpha,\beta;\gamma;z)$ is the hypergeometric series defined by

$$F(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} z^k \tag{C.8}$$

$$(\lambda)_k = \lambda(\lambda + 1)(\lambda + 2)\ldots(\lambda + k - 1) \quad , \quad k \geqslant 1 \tag{C.9}$$

$$(\lambda)_0 = 1 \quad . \tag{C.10}$$

A second independent solution is generated [16] by letting $t = x^{-2}$ in Eq. (C.5). This leads to an equation whose solution is

$$f_n = (f_n)_2 = \frac{\sqrt{\pi}\ \Gamma(\alpha_n + 1)}{\Gamma\left(\alpha_n + \frac{3}{2}\right)(2x)^{\alpha_n+1}} F\left(\frac{\alpha_n}{2} + 1,\ \frac{\alpha_n}{2} + \frac{1}{2}; \alpha_n + \frac{3}{2}; \frac{1}{x^2}\right) . \tag{C.11}$$

The solutions to Eq. (C.5) are called the Legendre functions of degree α_n of the first and second kind, denoted by $P_{\alpha_n}(x)$ and $Q_{\alpha_n}(x)$, respectively; i.e.,

$$P_{\alpha_n}(x) = F\left(-\alpha_n, \alpha_n + 1; 1; \frac{1-x}{2}\right) \quad , \quad |x - 1| < 2 \tag{C.12}$$

$$Q_{\alpha_n}(x) = \frac{\sqrt{\pi}\ \Gamma(\alpha_n + 1)}{\Gamma\left(\alpha_n + \frac{3}{2}\right)(2x)^{\alpha_n+1}} F\left(\frac{\alpha_n}{2} + 1,\ \frac{\alpha_n}{2} + \frac{1}{2}; \alpha_n + \frac{3}{2}; \frac{1}{x^2}\right) ,$$

$$|x| > 1 \quad . \tag{C.13}$$

The general solution for Eq. (C.5) can then be written as

$$f_n = AP_{\alpha_n}(x) + BQ_{\alpha_n}(x) \quad . \tag{C.14}$$

The cone problem outlined in Chapter III requires B = 0 for a finite solution along the cone axis. The allowable solution for f_n becomes

$$f_n = AP_{\alpha_n}(x) \quad . \tag{C.15}$$

Thus, take

$$B_z = AR^{-\alpha_n - 1} P_{\alpha_n}(x) \quad . \tag{C.16}$$

Similarly, for the solution of B_ρ, assume

$$B_\rho = R^{-\alpha_n - 1} g_n(\phi) \tag{C.17}$$

and substitute into Eq. (C.2). This leads to the following equation for g_n:

$$(1 - x^2) g_n''(x) - 2x g_n'(x) + \left[\alpha_n(\alpha_n + 1) - \frac{1}{1 - x^2}\right] g_n(x) = 0$$

where $x = \cos\phi$. This is Legendre's associated equation which has as its solution

$$g_n(x) = C' P^1_{\alpha_n}(x) + D' Q^1_{\alpha_n}(x) \tag{C.19}$$

where

$$P^1_{\alpha_n}(x) = -\frac{\alpha_n(\alpha_n + 1)}{2}(1 - x^2)F\left(-\alpha_n + 1, \alpha_n + 2; 2; \frac{1 - x}{2}\right) ,$$

$$|1 - x| < 2 \qquad (C.20)$$

$$Q^1_{\alpha_n}(x) = \frac{-\sqrt{\pi}\,\Gamma(\alpha_n + 2)}{2^{\alpha_n+1}\,\Gamma\left(\alpha_n + \frac{3}{2}\right)x^{\alpha_n+2}}(x^2 - 1)^{\frac{1}{2}} F\left(\frac{\alpha_n}{2} + \frac{3}{2}, \frac{\alpha_n}{2}\right.$$

$$\left. + 1; \alpha_n + \frac{3}{2}; \frac{1}{x^2}\right) , \qquad |x| > 1 . \qquad (C.21)$$

Thus for the cone problem, $D' = 0$ and the allowable solution for g_n become

$$g_n = C' P^1_{\alpha_n}(x) \qquad (C.22)$$

which can be written as

$$g_n = C \frac{dP_{\alpha_n}}{d\phi} \qquad (C.23)$$

since

$$P^1_{\alpha_n} = (1 - x^2)^{\frac{1}{2}} \frac{dP_{\alpha_n}(x)}{dx} \qquad (C.24)$$

Thus, take

$$B_\rho = CR^{-\alpha_n - 1} \frac{dP_{\alpha_n}}{d\phi} . \qquad (C.25)$$

The components of $\overline{B}$ can now be determined in spherical coordinates using the transformation equations.