STABILITY OF INTERCONNECTED SYSTEMS

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

WILEY E. THOMPSON

1968

This is to certify that the

thesis entitled

STABILITY OF INTERCONNECTED SYSTEMS

presented by

Wiley E. Thompson

has been accepted towards fulfillment
of the requirements for

Ph. D. degree in Elect. Engr.

Major professor

Date February 19, 1968

O-169

ABSTRACT

STABILITY OF INTERCONNECTED SYSTEMS

by Wiley E. Thompson

Virtually all engineering systems and certain classes of socio-economic systems are realized as interconnections of components or subsystems. As such, these interconnected systems can be defined in terms of their two fundamental structural features--that is, (1) the mathematical models of the unconstrained components or subsystems, and (2) the constraints imposed by the interconnections between the components.

Even though stability studies for dynamic systems have reached a remarkably high level of mathematical sophistication, most stability criteria still have at least two serious limitations: (1) The actual application to higher-order systems (particularly nonlinear systems) is impractical or virtually impossible; and (2) the results fail to adequately relate stability to the structural features of the system.

This thesis considers the stability of several classes of interconnected systems, consisting of multiterminal, nonlinear, time-varying components. In addition to dynamic components, algebraic

and algebraic-dynamic components are allowed. Scalar and vector Liapunov functions are constructed for the interconnected systems in terms of Liapunov functions of the individual components, the system structure, and a parameter vector determined in a prescribed optimal manner.

Sufficient conditions for various types of stability are obtained for the identified classes of interconnected systems. The conditions are given in forms which are particularly well suited for digital computation and design. A ninth-order, nonlinear system is considered for illustration.

STABILITY OF INTERCONNECTED SYSTEMS

By

Wiley E. Thompson

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1968

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to his major professor, Dr. H. E. Koenig, whose guidance and constant inspiration were invaluable. Thanks are also due to Dr. Y. Tokad for his efforts as the author's academic advisor.

The author also wishes to thank the Division of Engineering Research and the National Science Foundation, under whose sponsorship this research was performed.

Finally, the author wishes to thank his wife, Judy, for her patience, understanding, and encouragement throughout his entire graduate program.

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I INTRODUCTION

Most of the literature on stability analysis of dynamic systems assumes that a mathematical model of the system is given in state model form and that the analysis is to be based on that known model. For the stability studies of nonlinear systems, the most versatile and widely used method is Liapunov's Direct Method [HA-1, KA-1, KR-1, LI-1]. Although several procedures have been given for constructing Liapunov functions for systems of arbitrary order [IN-1, KR-2, LU-2, PE-1, SC-1, SZ-1, WA-1], the complexities involved in actually applying these procedures to nonlinear systems make their application impractical or virtually impossible for high-order systems.

Virtually all engineering and certain classes of socio-economic systems are realized as interconnected systems (subassemblies, or components) [KO-1, KO-3]. Generally speaking, the component models are simpler in form and of lower order than the model of the interconnected system, and thus lend more readily to analysis at the component level.

This thesis considers the stability of an interconnected system in terms of the two fundamental structural features--namely,

- the mathematical models of the components or subassemblies and
- their interconnection pattern.

This approach to the stability problem, in many cases, not only helps to circumvent some of the difficulties encountered in high order systems, but it gives stability in the form most useful for design--in terms of system structure. Using this approach [WI-1, WI-2], various classes of linear and nonlinear systems of multiterminal components can be identified for which the stability is relatively independent of the topology of the system. For given classes of components, conditions on the topology of the system which are necessary and/or sufficient for stability have been obtained. Moreover, for certain classes of components and a given system topology, system stability has been examined in terms of component characteristics.

Except for cases where the system topology is extremely simple, as in the problems of Aizerman [AI-1] and Lur'e [LU-1] where a linear, differential system and single, scalar, nonlinear algebraic component are connected in a feedback configuration, essentially nothing had been done in this direction until recently.

Williams [WI-1, WI-2] examined several classes of systems containing multiterminal, nonlinear and linear components with models of the forms

$$\frac{d}{dt}\Psi = F(Z_i, t)$$

$$Z_o = \Psi \qquad (1.0.1)$$

or

$$Z_o = G(Z_i)$$

and

$$\frac{d}{dt}\Psi = P(t)Z_i + F(t)$$

$$Z_o = \Psi \qquad (1.0.2)$$

or

$$Z_o = C(t)Z_i + G(t)$$

where Ψ is a state vector for the component and Z_o and Z_i are the vectors of complementary terminal variables for the component.

The particular classes of systems considered by Williams are defined in terms of certain restrictions on both the system interconnection pattern and the component equations. Some of these classes of linear and nonlinear systems are shown to be stable--a few requiring only minor restrictions on the interconnections. It is also shown that some of the restrictions on the component equations defining the classes are necessary for stability.

Williams obtained these results by constructing Liapunov functions from the structural features of the system. Linear graph theory and matroid theory are utilized in establishing the relevant properties of the functions so constructed.

Bailey [BA-1] obtained a sufficient condition for the equilibrium of a system to be asymptotically stable in the large when the component equations are of the form

$$\frac{d}{dt}\Psi = F(\Psi, t) + DU$$

$$Y = H\Psi \tag{1.0.3}$$

and are exponentially stable in the large. Vectors U and Y are the input and output vectors, respectively, and Ψ is the state vector. The condition applies to a restricted topology with no loading. The form (1.0.3) is further restrictive in that purely algebraic components are not allowed.

In Bailey's procedure Liapunov functions satisfying certain estimates are found for each of the components. Krasovskii [KR-1] showed that under certain conditions, exponential stability assures the existence of these Liapunov functions. A vector Liapunov function is formed from the component Liapunov functions, and its time derivative is evaluated along the solutions of the interconnected system. The estimates on the individual Liapunov functions are used to obtain a linear comparison system of differential inequalities with the vector Liapunov function as the dependent variable.

The idea of considering the Liapunov function as the dependent variable in a differential inequality, in the scalar case, was presented by Corduneanu [COR-1]. Under certain conditions on the system of differential inequalities, asymptotic stability of that linear system obtained by replacing the inequality by equality implies asymptotic stability of the original interconnected system.

Aggarwal and Bybee [AG-1] considered the stability of higher order systems obtained by coupling second order systems having the form

$$\ddot{\psi} + g(\dot{\psi}) + h(\psi) = 0, \quad (\psi \text{ a scalar}) \tag{1.0.4}$$

and gave sufficient conditions for the stability of the interconnected system. The approach is essentially that of constructing a Liapunov function for the interconnected system as a sum of the individual Liapunov functions for the components and then examining the time derivative of this function along the solution to the interconnected system. The Liapunov function used has a well known form for components of this type.

A body of results known as "frequency domain stability criterion" has recently been developed out of the work of V. M. Popov [PO-1]. These results as given in [BRO-1, KA-2, KU-1, NA-1, PO-1] apply to systems constructed by connecting a scalar, algebraic, nonlinear feedback element to a linear time-invariant system. In [AN-1, IB-1] multiple autonomous nonlinearities are allowed, while in [DE-1, KU-2, SA-1] the system is allowed to be forced with specific inputs, but a single nonlinearity is required. Jury and Lee [JU-1] allowed multiple nonlinearities and inputs.

The classes of problems investigated via the frequency domain approach have simple component interconnected patterns. For example, Anderson [AN-1] in generalizing the Popov criterion to the case of a system containing an arbitrary number of memoryless nonlinearities, limits the topology to a single feedback loop with alternately connected linear, time-invariant differential systems and nonlinear algebraic components.

The remainder of this chapter is devoted to some of the notation, definitions, and fundamental concepts necessary for an efficient development in this thesis.

1.1 Notation and Definitions

Denote by E^n the n-dimensional Euclidean space of real n-vectors. Capital letters in general are used to denote vectors and matrices, with lower case letters designating scalars--any exceptions are pointed out unless obvious. The transpose and conjugate transpose of a matrix are denoted by A^T and A^*, respectively. The norm $|X|$ of a n-vector is taken to be the Euclidean norm

$$|X| = (X^*X)^{\frac{1}{2}}$$

or
$$|X| = (X^TX)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}, \text{ if } X \in E^n \tag{1.1.1}$$

The Euclidean norm of a real m x n matrix A is then

$$|A| = \min\{\alpha : \alpha|X| \geq |AX|, \forall X \in E^n\} \tag{1.1.2}$$

It can be shown [HAL-1] that

$$|A| = \sqrt{\Lambda} \tag{1.1.3}$$

where Λ is the largest eigenvalue of the matrix A^*A.

Following Hahn [HA-1], let H_h denote the spherical neighborhood of the origin

$$H_h = \{X \in E^n : |X| \leq h\} \tag{1.1.4}$$

and let H_{h,t_o} denote the half-cylindrical neighborhood

$$H_{h,t_o} = \{(X,t) \in E^{n+1} : |X| \leq h \text{ and } t \geq t_o\} \tag{1.1.5}$$

of the t-axis in the motion space, where h is some finite constant.

<u>Definition 1.1.1</u> An n-dimensional vector function $F(\Psi, t)$ of the n-dimensional vector Ψ and t is said to belong to <u>class $\mathcal{G}$</u> in H_{h,t_o} if for all $(\Psi, t) \in H_{h,t_o}$ the following conditions hold:

- $F(\Psi, t)$ is continuous in Ψ and t, and
- has continuous first partial derivatives with respect to the components ψ_i of Ψ.

It can be shown [LE-1] that these conditions imply the Lipschitz condition

$$|F(\Psi,t) - F(\Psi',t)| \leq L|\Psi - \Psi'| \tag{1.1.6}$$

with respect to Ψ in H_{h,t_o}.

<u>Definition 1.1.2</u> A scalar function $\phi(r)$ of the real variable r is said to belong to <u>class $\mathcal{K}$</u> if

- $\phi(r)$ is continuous and real valued on $0 \leq r \leq h$,
- $\phi(0) = 0$, and
- $\phi(r)$ is strictly monotonically increasing with r.

<u>Definition 1.1.3</u> A scalar function $v(\Psi, t)$ is said to be <u>positive</u> (negative) <u>definite</u> if for some $\phi \in \mathcal{K}$

$$v(\Psi, t) \geq \phi(|\Psi|) \qquad (\leq -\phi(|\Psi|)) \tag{1.1.7}$$

is satisfied in H_{h, t_o}. (If h can be taken arbitrarily large, and $\phi(r)$ increases unboundedly with r, then $v(\Psi, t)$ is <u>radially unbounded</u>.)

<u>Definition 1.1.4</u> A scalar function $v(\Psi, t)$ is said to be <u>decrescent</u> if for some $\phi \in \mathcal{K}$

$$|v(\Psi, t)| \leq \phi(|\Psi|) \tag{1.1.8}$$

in H_{h, t_o}.

1.2 Systems Concepts

A <u>component</u> is an entity such as a piece of physical hardware or a part of a socio-economic or biological system for which there can be defined a finite number of interfaces or <u>terminals</u> (Fig. 1) at which the component significantly interacts with its environment. A component can be defined by a set of <u>component equations</u> and a <u>terminal linear graph</u>.

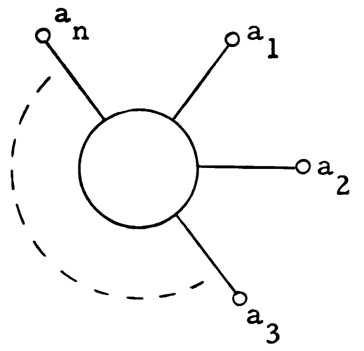


Fig. 1 Representation of an n-terminal component

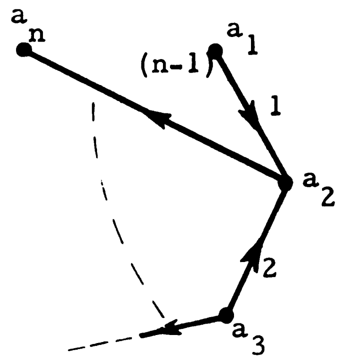


Fig. 2 Terminal graph for an n-terminal component

The essential features of a terminal graph (Fig. 2) for an n terminal component are that it has n vertices corresponding to the n terminals of the component and (n-1) directed edges (or lines) which form no circuits. Such a graph is called a tree. Associated with each edge of the terminal graph is a pair of complementary terminal variables--through and across variables--whose values are associated with the terminal measurements of the component.

The component equations as considered here are algebraic and/or differential constraint equations interrelating the component variables. This set of component variables is the minimal set of variables which is necessary to completely characterize the terminal behavior of the component. The set of component variables then consists of a set of terminal variables and a set of parametric or state variables. The set of state variables, which may or may not be disjoint from the set of terminal variables, are those variables which are necessary to completely describe the dynamical behavior of the component.

An interconnected system is then defined in terms of its structure which consists of a collection of component equations and a set of linear constraint equations derived from the system graph. The system graph is constructed operationally and uniquely from the component terminal graphs by coalescing vertices of these graphs that correspond to the same terminal of the system.

1.3 Stability Concepts

A solution $\hat{\Psi}(t) = \Psi(t; \hat{\Psi}_o, t_o)$ of the system

$$\dot{\Psi} = F(\Psi, t) \tag{1.3.1}$$

is said to be stable (in the sense of Liapunov) if for every $\epsilon > 0$, there exists a $\delta(\epsilon, t_o) > 0$, such that $|\Psi_o - \hat{\Psi}_o| < \delta$ implies $|\Psi(t; \Psi_o, t_o) - \Psi(t; \hat{\Psi}_o, t_o)| < \epsilon$ for all solutions $\Psi(t; \Psi_o, t_o)$ and all $t \geq t_o$.

Remarks 1. Geometrically, a stable solution is one which is not greatly disturbed by a small change in initial position. In fact, stability of a system is uniform (with respect to t) continuity of its solution in Ψ_o.

2. Stability with respect to parameters other than Ψ_o may be defined.

The change of variable $\breve{\Psi} = \Psi - \hat{\Psi}(t)$, $\hat{\Psi}(t)$ a known solution of (1.3.1), formally transforms (1.3.1) into

$$\begin{aligned}\dot{\breve{\Psi}} &= F(\Psi, t) - F(\hat{\Psi}(t), t) \\ &= F(\tilde{\Psi} + \hat{\Psi}(t),\ t) - F(\hat{\Psi}(t), t) \\ &= \tilde{F}(\tilde{\Psi}, t) \qquad (1.3.2)\end{aligned}$$

Thus there is a one-to-one correspondence between solutions of equation (1.3.1) and (1.3.2) with $\hat{\Psi}(t)$ corresponding to $\tilde{\Psi} \equiv 0$.

If $F(\Psi, t)$ is linear in Ψ so that

$$\dot{\Psi} = A\Psi + E(t) \qquad (1.3.3)$$

then

$$\dot{\tilde{\Psi}} = \tilde{F}(\tilde{\Psi}, t) = F(\tilde{\Psi}, t) = A\tilde{\Psi}$$

so that stability of the null solution, $\tilde{\Psi} = 0$, of the unforced system is equivalent to the stability of any solution of (1.3.3).

Following common practice, it is postulated that $F(0, t) \equiv 0$, and all further discussion of stability will refer to the stability of the null solution, $\Psi \equiv 0$.

Many types of stability have been defined [ANT-1, HA-1]. Those definitions important to this development are given in the following.

Consider the differential system

$$\dot{\Psi} = F(\Psi, t)\,, \qquad F(0, t) \equiv 0 \qquad (1.3.4)$$

for $t_o \leq t \leq +\infty$, and let $\Psi(t; \Psi_o, t_o)$ denote a solution of (1.3.4) such that $\Psi(t_o; \Psi_o, t_o) = \Psi_o$; let $\Psi(t)$ also denote a solution to (1.3.4). The scalar components of the n-vector Ψ are denoted by $\psi_1, \psi_2, \ldots, \psi_n$.

Definition 1.3.1 The solution $\Psi = 0$ of (1.3.4) is stable if for every $\epsilon > 0$, there exists a $\delta(\epsilon; t_o) > 0$, such that $|\Psi_o| < \delta$ implies $|\Psi(t; \Psi_o, t_o)| < \epsilon$ for $t \geq t_o$.

Definition 1.3.2 The solution $\Psi = 0$ of (1.3.4) is uniformly stable if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $|\Psi_o| < \delta$ implies $|\Psi(t; \Psi_o, t_o)| < \epsilon$ for $t \geq t_o$.

Definition 1.3.3 The solution $\Psi = 0$ of (1.3.4) is asymptotically stable if it is stable and, in addition, there exists a $\delta_o(t_o) > 0$ with the property that if $|\Psi_o| < \delta_o$ then

$$\lim_{t \to \infty} \Psi(t; \Psi_o, t_o) = 0$$

Definition 1.3.4 The solution $\Psi = 0$ of (1.3.4) is uniformly asymptotically stable if there exists a $\delta_o > 0$ and functions $\delta(\epsilon)$ and $T(\epsilon)$ with the property that $|\Psi_o| < \delta(\epsilon)$ implies $|\Psi(t; \Psi_o, t_o)| < \epsilon$ for $t \geq t_o$, and $|\Psi_o| < \delta_o$, $t \geq t_o + T(\epsilon)$ implies $|\Psi(t; \Psi_o, t_o)| < \epsilon$.

Remark: Halanay [HAL-1] has shown that the functions $\delta(\epsilon)$ and $T(\epsilon)$ in the definition can be chosen continuous and monotone. It is further shown that uniform asymptotic stability is equivalent to the existence of two scalar functions a(r) and b(t), the first monotone-increasing and the second monotone-decreasing, such that

$$|\Psi(t;\Psi_o,t_o)| \leq a(|\Psi_o|)b(t-t_o) \tag{1.3.5}$$

In particular, if $a(|\Psi_o|)$ is linear and $b(t-t_o)$ is exponential, the stability is exponential as defined next.

Definition 1.3.5 The solution $\Psi = 0$ of (1.3.4) is exponentially stable if there exist two constants $\alpha > 0$ and $\beta > 0$ which are independent of the initial values, such that for sufficiently small initial values the following inequality holds

$$|\Psi(t;\Psi_o,t_o)| \leq \beta|\Psi_o|\exp(-\alpha(t-t_o)) \tag{1.3.6}$$

If (1.3.6) is satisfied for all initial values in some region, then (1.3.4) will be said to be exponentially stable in this region. If (1.3.6) is satisfied for all initial conditions from which solutions originate, then (1.3.4) is said to be exponentially stable in the large.

If the initial conditions Ψ_o satisfy the relations

$$|\Psi_o| \leq h_o < h/\beta, \quad t_o \geq 0 \tag{1.3.7}$$

where β is defined in Definition 1.3.5, then denote $\Psi_o \in H_{h_o}$.

Let $v(\Psi,t)$ be a differentiable function of Ψ and t. Then the total time derivative of $v(\Psi,t)$ with respect to the system (1.3.1) is given by

$$\begin{aligned}\dot{v} \equiv \frac{d}{dt}v(\Psi,t) &= \frac{\partial v}{\partial t} + (\nabla v)^T\dot{\Psi} \\ &= v_t + (\nabla v)^T F(\Psi,t)\end{aligned} \tag{1.3.8}$$

where

$$v_t = \frac{\partial v}{\partial t}$$

and

$$\nabla v = \begin{bmatrix} \frac{\partial v}{\partial \psi_1} & \frac{\partial v}{\partial \psi_2} & \cdots & \frac{\partial v}{\partial \psi_n} \end{bmatrix}^T$$

A positive definite function $v(\Psi, t)$ satisfying any of Liapunov's stability or instability theorems, or any generalization thereof, for a system is called a Liapunov function for the system.

II EXPONENTIAL STABILITY OF A CLASS OF INTERCONNECTED SYSTEMS

Sufficient conditions are given for the exponential stability of the equilibrium of a system of interconnected components taken from a class of nonlinear, time-varying, multiterminal, differential, algebraic, and mixed components. A Liapunov function is constructed for the interconnected system from Liapunov functions of the individual components and the interconnection pattern. A parameter vector appearing in the function is to be selected in a prescribed optimal manner. The Liapunov function so constructed for the system can be used to establish a lower bound on the rate of decay of the system.

2.1 Interconnected Systems

Consider a system S for which the direct sum of the N component models forms a set of equations of the form

$$\begin{aligned} \dot{\Psi} &= F(\Psi, t) + DZ_1 \\ Z_2 &= G(\Psi, t) + CZ_1 \end{aligned} \qquad (2.1.1)$$

Let the component equations for the i-th component be denoted by

$$\begin{aligned} \dot{\Psi}^i &= F^i(\Psi^i, t) + D^i Z_1^i \\ Z_2^i &= G^i(\Psi^i, t) + C^i Z_1^i \end{aligned} \qquad (2.1.2)$$

where Ψ^i is an n_i-dimensional state vector, and Z_1^i and Z_2^i are e_i-dimensional vectors of terminal variables representing the through and across variables associated with the e_i edges of the component terminal graph $\mathcal{G}_i$. Each of these $2e_i$ terminal variables must appear once in either of the vectors Z_1^i or Z_2^i; however, a through and across variable associated with a particular edge of $\mathcal{G}_i$ may both appear in the same terminal vector, say Z_1^i. In such case, the terminal variables associated with some other edge of $\mathcal{G}_i$ must both appear in Z_2^i.

If $F^i(\Psi^i, t)$, D^i, and $G^i(\Psi^i, t)$ are zero--hence Ψ^i is null--then the i-th component is <u>purely algebraic</u>. If C^i is zero, the i-th component is <u>purely dynamic</u>. Otherwise, the component is said to be a <u>mixed component</u>. Thus the set of components having dynamic parts consists of the dynamic and mixed components.

Let m be the number of components in (2.1.1) with dynamic parts and let $n = \sum_{i=1}^{N} n_i$, $e = \sum_{i=1}^{N} e_i$. Then $F(\Psi, t)$ and $G(\Psi, t)$ are respectively n- and e-dimensional nonlinear, time-varying vector functions of the n vector Ψ. The constant matrices D and C have dimensions n x e and e x e respectively.

The class of interconnections considered is characterized by the system graph $\mathcal{G}$ having a tree T_1 and a tree T_2 such that the combined set of fundamental circuit and cut-set equations [KO-1] may be written in the form

$$Z_1 = \Phi Z_2 \qquad (2.1.3)$$

When these conditions are satisfied for a system S, the interconnection is said to satisfy <u>graphic constraint GC</u>. The trees T_1 and T_2 may or may not be identical. If $T_1 = T_2$ then it can be shown that $\Phi^T = -\Phi$.

What graphic constraint GC actually implies, in terms of restrictions on $\mathcal{G}$, is that the through variables appearing as components in Z_1 must be included in some tree T_1 of $\mathcal{G}$ and the across variables appearing as components in Z_1 must be included in a co-tree of some tree T_2 of $\mathcal{G}$. If graphic constraint GC is satisfied, then following Frame and Koenig [FR-1], the components of Z_1 and Z_2 are called the secondary and primary variables, respectively, associated with the trees T_1 and T_2 for $\mathcal{G}$--the vectors Z_1 and Z_2 are called secondary and primary terminal vectors and denoted by Z_s and Z_p.

If graphic constraint GC is satisfied, then applying the interconnection constraint equations (2.1.3) to the direct sum of the component equations (2.1.1) gives

$$\begin{aligned} \dot{\Psi} &= F(\Psi, t) + D\Phi Z_p \\ Z_p &= G(\Psi, t) + C\Phi Z_p \end{aligned} \tag{2.1.4}$$

for the system S.

If none of the eigenvalues of the matrix $C\Phi$ are equal to +1, then the matrix $(U - C\Phi)$ is nonsingular, and the interconnected system S is

$$\begin{aligned} \dot{\Psi} &= F(\Psi, t) + AG(\Psi, t) \\ Z_p &= BG(\Psi, t) \end{aligned} \tag{2.1.5}$$

where $$A = D\Phi(U - C\Phi)^{-1}$$

and $$B = (U - C\Phi)^{-1}$$

When this condition is satisfied, <u>graphic-component constraint GCC</u> is said to be satisfied. This condition is automatically satisfied for a system of purely dynamic components.

2.2 Exponential Stability

In general, even though a nonlinear system is asymptotically stable or uniformly asymptotically stable, nothing can be said about the rate of decay of the solutions or about the asymptotic behavior of the associated Liapunov functions. Thus in certain engineering design situations, where a certain rate of decay of the solutions is required, the assurance of asymptotic stability is inadequate. However if the system is exponentially stable, then under certain conditions it is possible to define a Liapunov function which satisfies certain estimates, yet to be defined. Further, a system having a Liapunov function satisfying these estimates is exponentially stable, and these estimates can be used to establish a lower bound on the rate of decay of the system solution. These concepts are stated more precisely in the following theorem which, along with parts of its proof, are suggested by the work of Krasovskii [KR-1].

<u>Theorem 2.2.1</u> Consider the n-dimensional system of differential equations

$$\dot{\Psi} = F(\Psi, t), \qquad F(0, t) = 0 \tag{2.2.1}$$

(i) If there exists a positive definite scalar function $v(\Psi, t)$ and a set of positive constants c_1, c_2, c_3, and c_4 satisfying (a) and (b) of the relations

$$\begin{aligned} &\text{(a)} \qquad c_1|\Psi|^2 \leq v(\Psi, t) \leq c_2|\Psi|^2 \\ &\text{(b)} \qquad \dot{v}(\Psi, t) \leq -c_3|\Psi|^2 \\ &\text{(c)} \qquad |\nabla v| \leq c_4|\Psi| \end{aligned} \tag{2.2.2}$$

for all $(\Psi, t) \in H_{h,\tau}$, $\tau \geq 0$, then the system (2.2.1) is exponentially stable for all $\Psi_o \in H_{h_o}$. Further, the inequality (1.3.6) is satisfied with $\alpha = c_3/2c_2$ and $\beta = \sqrt{c_2/c_1}$.

(ii) If $F(\Psi, t) \in \mathcal{B}$ in $H_{h,\tau}$, $\tau \geq 0$, and if (2.2.1) is exponentially stable for all $\Psi_o \in H_{h_o}$, then there exists a positive definite function $v(\Psi, t)$ and associated positive constants satisfying (2.2.2) for all $(\Psi, t) \in H_{h,\tau}$.

<u>Proof:</u>

(i) Suppose that there exists a positive definite scalar function $v(\Psi, t)$ satisfying (2.2.2). Then from (2.2.2a) and (2.2.2b)

$$\dot{v} \leq -c_3|\Psi|^2 \leq -\frac{c_3}{c_2} v \tag{2.2.3}$$

Integrating (2.2.3) gives

$$v(\Psi(t;\Psi_o,t_o),t) \le v(\Psi_o,t_o)\exp[-\frac{c_3}{c_2}(t-t_o)] \qquad (2.2.4)$$

Using (2.2.2a) to eliminate v from (2.2.4), one obtains the estimate

$$|\Psi(t;\Psi_o,t_o)|^2 \le \frac{c_2}{c_1}|\Psi_o|^2 \exp[-\frac{c_3}{c_2}(t-t_o)] \qquad (2.2.5)$$

Taking the square root of both sides of the inequality (2.2.5) gives

$$|\Psi(t;\Psi_o,t_o)| \le \sqrt{\frac{c_2}{c_1}}\,|\Psi_o|\,\exp[-\frac{c_3}{2c_2}(t-t_o)]$$

or

$$|\Psi(t;\Psi_o,t_o)| \le \beta|\Psi_o|\exp[-\alpha(t-t_o)]$$

where

$$\beta = \sqrt{\frac{c_2}{c_1}} > 0 \text{ and } \alpha = \frac{c_3}{2c_2} > 0.$$

This proves the assertion that (2.2.1) is exponentially stable.

(ii) It is now shown that under the hypothesis of the theorem exponential stability of (2.2.1) is sufficient for the existence of a function $v(\Psi,t)$ satisfying the estimates (2.2.2). Consider the function

$$v(\Psi_o,t_o) = \int_{t_o}^{t_o+T} |\Psi(s;\Psi_o,t_o)|^2\,ds \qquad (2.2.6)$$

where

$$T = \max\{\frac{\ln(\sqrt{2}\beta)}{\alpha}, \frac{\ln 2}{2L}\} \qquad (2.2.7)$$

with L the Lipschitz constant given in (1.1.6) and α, β the positive constants given in (1.3.6) in the definition of exponential stability. The point (Ψ_o, t_o) is an arbitrary point in $H_{h,\tau}$ subject to the constraint that $\Psi_o \in H_{h_o}$.

(a) Applying the result of Lemma A.1 (Appendix) to (2.2.6)

$$v(\Psi_o, t_o) \leq |\Psi_o|^2 \int_{t_o}^{t_o+T} e^{2L(s-t_o)} ds = \frac{|\Psi_o|^2}{2L} \left[e^{2L(s-t_o)} \right]_{t_o}^{t_o+T}$$

Then from (2.2.7) it follows that

$$v(\Psi_o, t_o) \leq c_2 |\Psi_o|^2$$

where $c_2 \geq \frac{1}{2L} > 0$. Since (Ψ_o, t_o) is arbitrary subject to $\Psi_o \in H_{h_o}$, and since (2.2.1) is exponentially stable, it follows that $\Psi(t)$ does not leave $H_{h,\tau}$. Thus

$$v(\Psi, t) \leq c_2 |\Psi|^2$$

In a similar manner it follows that

$$v(\Psi, t) \geq c_1 |\Psi|^2$$

where $c_1 \geq \frac{1}{4L} > 0$. Thus relation (2.2.2a) is satisfied by the $v(\Psi, t)$ given by (2.2.6) and (2.2.7).

(b) Using Liebnitz's Rule [OL-1] for differentiating $v(\Psi, t)$ with respect to t along a solution $\Psi(t; \Psi_o, t_o)$ of (2.2.1) gives

$$\frac{d}{dt}\, v(\Psi(t;\Psi_o, t_o), t) = \frac{d}{dt}\left[\int_t^{t+T} |\Psi(s;\Psi(t;\Psi_o, t_o), t)|^2 ds\right]$$

$$= -\,|\Psi(t;\Psi(t;\Psi_o, t_o), t)|^2 + |\Psi(t+T;\Psi(t;\Psi_o, t_o), t)|^2$$

$$+ \int_t^{t+T} \frac{d}{dt}\, |\Psi(s;\Psi(t;\Psi_o,t_o), t)|^2 ds \qquad (2.2.8)$$

By definition,

$$\Psi(s;\Psi(t+\Delta t;\Psi_o, t_o), t+\Delta t) \equiv \Psi(s;\Psi(t;\Psi_o, t_o), t)$$

so that

$$\int_t^{t+T} \frac{d}{dt}\, |\Psi(s;\Psi(t;\Psi_o, t_o), t)|^2 ds = 0$$

and hence

$$\frac{d}{dt} v(\Psi, t) = -\,|\Psi(t;\Psi_o, t_o)|^2 + |\Psi(t+T;\Psi_o, t_o)|^2 \qquad (2.2.9)$$

However, since (2.2.1) is exponentially stable it follows that

$$|\Psi(t+T;\Psi_o, t_o)| \leq \beta\, e^{-\alpha T}\, |\Psi(t;\Psi_o, t_o)|$$

so that (2.2.9) becomes

$$\frac{d}{dt} v(\Psi, t) \leq -\,|\Psi(t;\Psi_o, t_o)|^2 + \beta^2 e^{-2\alpha T}\, |\Psi(t;\Psi_o, t_o)|^2 \qquad (2.2.10)$$

Using (2.2.7) in (2.2.10) gives

$$\frac{d}{dt}\, v(\Psi, t) \leq -c_3\, |\Psi(t;\Psi_o, t_o)|^2$$

where $c_3 \geq \frac{1}{2}$. Thus relation (2.2.2b) is established.

(c) Denote by ψ_{o_j} the j-th component of the vector Ψ_o. Since $F(\Psi, t) \in \mathcal{C}$ in $H_{h,\tau}$, the partial derivatives $\dfrac{\partial \psi_i(t;\Psi_o, t_o)}{\partial \psi_{o_j}}$ exist and are continuous [CO-1, SAN-1], so that $\nabla v(\Psi_o, t_o)$ exists and can be computed by differentiation under the integral [FU-1]

$$\nabla v(\Psi_o, t_o) = \int_{t_o}^{t_o+T} \nabla |\Psi(t;\Psi_o, t_o)|^2 dt$$

Carrying out this operation,

$$\nabla v(\Psi_o, t_o) = 2\int_{t_o}^{t_o+T} J(\Psi(t;\Psi_o, t_o))\,\Psi(t;\Psi_o, t_o)dt \tag{2.2.11}$$

where $J(\Psi(t;\Psi_o, t_o))$ is the Jacobian matrix of $\Psi(t;\Psi_o, t_o)$ with respect to Ψ_o. Taking norms in (2.2.11)

$$|\nabla v(\Psi_o, t_o)| = 2\left|\int_{t_o}^{t_o+T} J(\Psi(t;\Psi_o, t_o))\,\Psi(t;\Psi_o, t_o)dt\right| \quad .$$

$$\leq 2\int_{t_o}^{t_o+T} |J(\Psi(t;\Psi_o, t_o))|\,|\Psi(t;\Psi_o, t_o)|dt \tag{2.2.12}$$

But from Lemma A.2,

$$\left|\frac{\partial \Psi(t;\Psi_o, t_o)}{\partial \psi_{o_i}}\right| \leq e^{L(t-t_o)}, \quad i = 1, 2, \ldots, n$$

so that

$$|J(\Psi(t;\Psi_o, t_o))| \leq e^{L(t-t_o)} \tag{2.2.13}$$

Substituting (2.2.13) in (2.2.12) and using the hypothesis that (2.2.1) is exponentially stable gives

$$|\nabla v(\Psi_o, t_o)| \leq 2\beta|\Psi_o| \int_{t_o}^{t_o+T} e^{L(t-t_o)} e^{-\alpha(t-t_o)} dt$$

$$= c_4 |\Psi_o| \qquad (2.2.14)$$

where $\qquad c_4 = 2\beta \int_{t_o}^{t_o+T} \exp[(L-\alpha)(t-t_o)]\, dt > 0.$

But following the same reasoning as in (a), (2.2.14) implies

$$|\nabla v| \leq c_4|\Psi|, \quad c_4 > 0$$

and (2.2.2c) is proved.

<u>Remark:</u> Note that in part (iia) of the proof of the theorem, the only use made of the fact that (2.2.1) is exponentially stable is to insure that $\Psi(t)$ remains in the region $H_{h,\tau}$ where $F(\Psi, t) \in \mathscr{B}$. Thus if $F(\Psi, t) \in \mathscr{B}$ for all $\Psi \in E^n$, then (2.2.2a) is satisfied even if (2.2.1) is not exponentially stable.

2.3 Exponential Stability of a Class of Interconnected Exponentially Stable Components

Consider now the exponential stability of a class of interconnected systems with component models of the form (2.1.2) and interconnection constraints (2.1.3). Let the interconnections satisfy graphic constraint GC and let graphic-component constraint GCC be satisfied so that the interconnected system model is given by (2.1.5).

A component or system

$$\dot{\Psi} = F(\Psi, t, Z_s) \quad , \quad F(0, t, 0) \equiv 0$$
$$Z_p = G(\Psi, t, Z_s) \tag{2.3.1}$$

is said to be stable if the null solution of

$$\dot{\Psi} = F(\Psi, t, 0) \tag{2.3.2}$$

is stable. If (2.3.2) is exponentially stable then it is said to belong to class $\mathcal{E}$, denoted $F(\Psi, t, 0) \in \mathcal{E}$.

To provide a better understanding of exponential stability and how it relates to other types of stability, and to give an indication of the importance of the class $\mathcal{E}$, some properties of this class are:

- The linear, constant coefficient system $\dot{\Psi} = F\Psi + DE(t)$, is asymptotically stable, uniformly asymptotically stable, and exponentially stable, if and only if, the eigenvalues of F have negative real parts [SAN-1].

- For the continuous, time-varying system $\dot{\Psi} = F(t)\Psi$, uniform asymptotic stability of the solution $\Psi = 0$ is equivalent to both uniform asymptotic stability in the large and exponential stability in the large of every solution [SAN-1].

- If the solution $\Psi(t)$ of the system $\dot{\Psi} = F(t)\Psi + E(t)$, with F(t) bounded, is bounded for all continuous, bounded inputs E(t), then the system is exponentially stable [HAL-1].

- The solution $\Psi = 0$ of the system $\dot{\Psi} = F(\Psi, t)$, $F(0, t) \equiv 0$, is exponentially stable for all $\Psi_o \in H_{h_o}$ if there exists some constant $c_3 > 0$ such that $\Psi^T F(\Psi, t) \leq -c_3 |\Psi|^2$ for all $(\Psi, t) \in H_{h,\tau}$. If h can be taken sufficiently large so that all $\Psi_o \in H_{h_o}$, then the solution $\Psi = 0$ is exponentially stable in the large. (Proof follows from Theorem 2.2.1 with $v(\Psi, t) = \Psi^T \Psi$.)

- Let L(r) be a continuous, scalar function on some interval, $0 \leq r \leq s$, with $L(0) = 0$, $L(r) > 0$ for $0 < r \leq s$, and

$$\int_{0+}^{s} \frac{dr}{L(r)} = +\infty$$

If a(r) in (1.3.5) is linear and if $|F(\Psi, t)| \leq L(r)|\Psi|$ for $|\Psi| < r$, then the system $\dot{\Psi} = F(\Psi, t)$ is exponentially stable [HAL-1].

- If $F(\Psi, t)$ is homogeneous in Ψ of first degree, then uniform asymptotic stability implies exponential stability for $\dot{\Psi} = F(\Psi, t)$ [HAL-1].

Of particular interest here are systems for which $AG(\Psi, t)$ in (2.1.5) has one of the three following properties or forms

$$AG(\Psi, t) = M(t)\Psi \tag{2.3.3}$$

$$|AG(\Psi, t)| < \gamma_1 |\Psi| \tag{2.3.4}$$

$$AG(\Psi, t) = M(t)\Psi + \tilde{G}(\Psi, t) \tag{2.3.5}$$

with

$$|\tilde{G}(\Psi, t)| < \gamma_2 |\Psi|$$

Clearly the three forms are not disjoint; however, each is important enough to warrant individual attention.

The new stability results which follow begin with a theorem giving sufficient conditions on the non-dynamic parts of the component equations and the interconnection pattern for exponential stability of a system of interconnected components from Class $\mathcal{E}$ in terms of (2.3.4).

Theorem 2.3.1 Consider an interconnected system S given by (2.1.5) with component models of the form (2.1.2) and interconnection constraints (2.1.3). For each of the m components S_i having dynamic parts, let $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E})$ and let $v_i(\Psi^i, t)$, c_1^i, c_2^i, c_3^i, c_4^i be Liapunov functions and associated positive constants satisfying Theorem 2.2.1. Then all S for which

$$\gamma_1 < \min_i \frac{c_3^i}{c_4^i} \tag{2.3.6}$$

are exponentially stable, where γ_1 is related to S by (2.3.4).

Proof: Let

$$v(\Psi, t) = \sum_{i=1}^{m} v_i(\Psi^i, t) \tag{2.3.7}$$

Denote by $\dot{v}(\Psi, t)$ the total derivative of (2.3.7) with respect to the system

$$\dot{\Psi} = F(\Psi, t) \tag{2.3.8}$$

and by $\dot{v}_S(\Psi, t)$ the total derivative of (2.3.7) with respect to the system (2.1.5). Then from (2.2.2a)

$$v(\Psi, t) \geq \sum_{i=1}^{m} c_1^i \, |\Psi^i|^2$$

$$\geq \sum_{i=1}^{m} c_1 |\Psi^i|^2 = c_1 |\Psi|^2$$

where $$c_1 = \min_i c_1^i > 0 \tag{2.3.9}$$

Similarly,

$$v(\Psi, t) \leq c_2 |\Psi|^2$$

where $$c_2 = \max_i \; c_2^i > 0 \tag{2.3.10}$$

From (2.2.2c) one obtains

$$|\nabla v(\Psi, t)| = \left(\sum_{i=1}^{m} |\nabla v_i(\Psi^i, t)|^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^{m} (c_4^i \, |\Psi^i|)^2 \right)^{\frac{1}{2}}$$

$$\leq c_4 |\Psi|$$

where $$c_4 = \max_i c_4^i > 0 \tag{2.3.11}$$

$$\dot{v}_S = \dot{v} + (\nabla v)^T AG(\Psi, t)$$

$$\leq \dot{v} + |\nabla v| \, |AG(\Psi, t)| \tag{2.3.12}$$

Applying inequalities (2.2.2) to (2.3.12),

$$\dot{v}_S \leq \sum_{i=1}^{m} -c_3^i |\Psi^i|^2 + \sum_{i=1}^{m} c_4^i |\Psi^i| |AG(\Psi, t)|$$

Using (2.3.4),

$$\dot{v}_S \leq - \sum_{i=1}^{m} (c_3^i - c_4^i \gamma_1) |\Psi^i|^2$$

$$\leq -c_3 |\Psi|^2 \tag{2.3.13}$$

where $c_3 = \min_i (c_3^i - c_4^i \gamma_1)$. Now if

$$c_3 = \min_i (c_3^i - c_4^i \gamma_1) > 0 \tag{2.3.14}$$

then S is exponentially stable by Theorem 2.2.1. But (2.3.14) is satisfied if (2.3.6) is satisfied.

The next theorem relates the stability in the more general situation, where $AG(\Psi, t)$ has the form (2.3.5), to that of the reduced system

$$\dot{\Psi} = F(\Psi, t) + M(t)\Psi \tag{2.3.15}$$

that is, where $\tilde{G}(\Psi, t) = 0$.

<u>Theorem 2.3.2</u> Let $(F(\Psi, t) + M(t)\Psi) \in (\mathcal{B} \cap \mathcal{E})$ and let $v(\Psi, t)$, c_1, c_2, c_3, c_4 be a Liapunov function and positive constants satisfying (2.2.2) of Theorem 2.2.1. Then the system (2.1.5) is exponentially stable if

$$\gamma_2 < \frac{c_3}{c_4} \tag{2.3.16}$$

where γ_2 is given in (2.3.5).

<u>Proof</u>: Let $\dot{v}_S$ be the total derivative of $v(\Psi, t)$ with respect to the system (2.1.5) and $\dot{v}$ for (2.3.15). Then

$$\begin{aligned} \dot{v}_S &= \dot{v} + (\nabla v)^T \tilde{G}(\Psi, t) \\ &\leq \dot{v} + |\nabla v| \, |\tilde{G}(\Psi, t)| \end{aligned}$$

or using (2.2.2) and (2.3.5)

$$\begin{aligned} \dot{v}_S &\leq -c_3 |\Psi|^2 + c_4 |\Psi| \, |\tilde{G}(\Psi, t)| \\ &\leq -(c_3 - c_4 \gamma_2) \, |\Psi|^2 \end{aligned}$$

Let $a_1 = c_1$, $a_2 = c_2$, $a_3 = (c_3 - c_4 \gamma_2)$, and $a_4 = c_4$. Then

$$a_1 |\Psi|^2 \leq v(\Psi, t) \leq a_2 |\Psi|^2$$

$$\dot{v}_S \leq -a_3 |\Psi|^2$$

$$|\nabla v(\Psi, t)| \leq a_4 |\Psi|^2$$

and by Theorem 2.2.1, the system (2.1.4) is exponentially stable if $a_3 > 0$ or

$$\gamma_2 < \frac{c_3}{c_4}$$

The next theorem gives sufficient conditions for interconnected systems of a class to be exponentially stable, in terms of component Liapunov functions, non-dynamic and interconnection properties, and an optimally selected parameter vector K.

<u>Theorem 2.3.3</u> Consider an interconnected system (2.1.5) with component models of the form (2.1.2) and interconnection constraints (2.1.3). For each of the m components S_i having dynamic parts, let $F^i(\Psi^i, t) \in (\mathcal{C} \cap \mathcal{E})$ and let $v_i(\Psi^i, t)$, c_1^i, c_2^i, c_3^i, c_4^i be Liapunov functions and positive constants satisfying (2.2.2) of Theorem 2.2.1. Suppose that there exists a parameter vector $K = (k_1, k_2, \ldots, k_m)^T$ with positive components such that $c_3 > 0$, with

$$\begin{aligned} c_3 = \max \Big\{ & \big(\min_i k_i c_3^i - |M(t)| \max_j k_j c_4^j\big), \\ & \big(\min_i [k_i c_3^i - (1 + (k_i c_4^i)^2)] \frac{|M(t)|}{2}\big), \\ & \big(\min_i [k_i c_3^i - \frac{1}{2} \sum_{j=1}^{m} (k_i c_4^i |M_{ij}(t)| + k_j c_4^j |M_{ji}(t)|)]\big) \Big\} \end{aligned} \qquad (2.3.17)$$

where M(t) is defined by (2.3.3) or (2.3.5) and $M_{ij}(t)$ is the submatrix of M(t) whose rows correspond to $\dot{\Psi}^i$ in (2.1.5) and whose columns correspond to Ψ^j. Then the interconnected system (2.1.5) is exponentially stable if

$$\gamma_2 < \frac{c_3}{\max_j k_j c_4^j} \qquad (2.3.18)$$

<u>Proof:</u> Consider the function

$$v(\Psi, t) = \sum_{i=1}^{m} k_i v_i(\Psi^i, t) \qquad (2.3.19)$$

for the reduced system (2.3.15). Then from (2.2.2a),

$$v(\Psi, t) \geq \sum_{i=1}^{m} k_i c_1^i \, |\Psi^i|^2$$

$$\geq \sum_{i=1}^{m} c_1 |\Psi^i|^2 = c_1 |\Psi|^2$$

where $$c_1 = \min_i k_i c_1^i \tag{2.3.20}$$

In a similar manner, one obtains

$$v(\Psi, t) \leq c_2 |\Psi|^2$$

where $$c_2 = \max_i k_i c_2^i \tag{2.3.21}$$

For the gradient vector ∇v of (2.3.19), one has

$$|\nabla v(\Psi, t)| = \Big(\sum_{i=1}^{m} |k_i \nabla v_i(\Psi^i, t)|^2 \Big)^{\frac{1}{2}}$$

Then applying (2.2.2b) to the right-hand side gives

$$|\nabla v(\Psi, t)| \leq \Big(\sum_{i=1}^{m} (k_i c_4^i)^2 |\Psi^i|^2 \Big)^{\frac{1}{2}}$$

$$\leq c_4 |\Psi|$$

where $$c_4 = \max_i k_i c_4^i \tag{2.3.22}$$

The total derivative of (2.3.19) with respect to the system (2.3.15) is

$$\dot{v}(\Psi, t) = \sum_{i=1}^{m} k_i \dot{v}_i(\Psi^i, t) + (\nabla v(\Psi, t))^T M(t)\Psi$$

$$\leq \sum_{i=1}^{m} k_i \dot{v}_i(\Psi^i, t) + |\nabla v(\Psi, t)|\,|M(t)|\,|\Psi| \qquad (2.3.23)$$

Using (2.2.2c) to eliminate $|\nabla v|$ and (2.2.2b) to eliminate $\dot{v}_i(\Psi^i, t)$ from (2.3.23) gives

$$\dot{v}(\Psi, t) \leq \sum_{i=1}^{m} -k_i c_3^i |\Psi^i|^2 + c_4 |M(t)|\,|\Psi|^2$$

$$\leq -c_3 |\Psi|^2$$

where
$$c_3 = \min_i k_i c_3^i - c_4 |M(t)|$$

$$= \min_i k_i c_3^i - |M(t)| \max_j k_j c_4^j \qquad (2.3.24)$$

An alternate expression for c_3 may be obtained by returning to (2.3.23) and noting that by Lemma A.4

$$|\nabla v(\Psi, t)|\,|\Psi| \leq \frac{|\nabla v(\Psi, t)|^2 + |\Psi|^2}{2}$$

It follows then from (2.2.2) that

$$\dot{v}(\Psi, t) \leq \sum_{i=1}^{m} -k_i c_3^i |\Psi^i|^2 + \left[\sum_{i=1}^{m} |k_i \nabla v_i(\Psi^i, t)|^2 + \sum_{i=1}^{m} |\Psi^i|^2\right] \frac{|M(t)|}{2}$$

$$\leq -\sum_{i=1}^{m} \left(k_i c_3^i - [1 + (k_i c_4^i)^2] \frac{|M(t)|}{2}\right) |\Psi^i|^2$$

$$\leq -c_3 |\Psi|^2$$

where

$$c_3 = \min_i \left(k_i c_3^i - [1 + (k_i c_4^i)^2] \frac{|M(t)|}{2}\right) \tag{2.3.25}$$

A third expression for c_3 is obtained as follows

$$\dot{v}(\Psi, t) = \sum_{i=1}^{m} k_i \dot{v}_i(\Psi^i, t) + \sum_{i=1}^{m} \sum_{j=1}^{m} k_i (\nabla v_i(\Psi^i, t))^T M_{ij}(t) \Psi^j \tag{2.3.26}$$

Taking norms in (2.3.26) and using (2.2.2) yields

$$\dot{v}(\Psi, t) \leq \sum_{i=1}^{m} -k_i c_3^i |\Psi^i|^2 + \sum_{i=1}^{m} \sum_{j=1}^{m} k_i c_4^i |\Psi^i| \, |M_{ij}(t)| \, |\Psi^j| \tag{2.3.27}$$

Using the inequality

$$|\Psi^i| \, |\Psi^j| \leq \frac{|\Psi^i|^2 + |\Psi^j|^2}{2}$$

in (2.3.27) gives

$$\dot{v}(\Psi, t) \leq \sum_{i=1}^{m} -k_i c_3^i |\Psi^i|^2 + \sum_{i=1}^{m} \sum_{j=1}^{m} k_i c_4^i |M_{ij}(t)| \left[\frac{|\Psi^i|^2 + |\Psi^j|^2}{2} \right]$$

$$= \sum_{i=1}^{m} -k_i c_3^i |\Psi^i|^2 + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} (k_i c_4^i |M_{ij}(t)| + k_j c_4^j |M_{ji}(t)|) |\Psi^i|^2$$

$$= \sum_{i=1}^{m} \left[-k_i c_3^i + \frac{1}{2} \sum_{j=1}^{m} (k_i c_4^i |M_{ij}(t)| + k_j c_4^j |M_{ji}(t)|) \right] |\Psi^i|^2$$

$$\leq -c_3 |\Psi|^2$$

where

$$c_3 = \min_i \left[k_i c_3^i - \frac{1}{2} \sum_{j=1}^{m} (k_i c_4^i |M_{ij}(t)| + k_j c_4^i |M_{ji}(t)|) \right] \tag{2.3.28}$$

Thus, if any of the values for c_3 given by (2.3.24), (2.3.25), or (2.3.28) is positive, the reduced system (2.3.15) is exponentially stable. Further, if (2.3.18) is satisfied, then by Theorem 2.3.2 the system (2.1.5) is exponentially stable, and the theorem is proved.

Remark 1 If $AG(\Psi, t)$ has the form (2.3.3), then any positive parameter vector K which gives a $c_3 > 0$ is optimal, in the sense that this is sufficient for exponential stability.

Remark 2 If $AG(\Psi, t)$ has the form (2.3.5) then any K for which (2.3.18) is satisfied is optimal. However, if (2.3.18) cannot be satisfied, but a $c_3 > 0$ can be found, then an optimal K is one which gives $c_3 > 0$ and also maximizes the ratio c_3/c_4. Under these conditions, (2.1.5) is exponentially stable in some smaller region H_{h', t_o} for which

$$|G(\Psi, t)| < \gamma'_2, \qquad (\Psi, t) \in H_{h', t_o}$$

and

$$\gamma'_2 < \frac{c_3}{c_4}$$

Remark 3 The expressions obtained for c_3 are by no means unique, but they are sufficient to illustrate the general concepts involved. The answer to the question of which expression for c_3 is "best" is clearly a function of the particular system involved.

Remark 4 If no K can be found such that $c_3 > 0$, then the results are inconclusive.

The problem of determining the parameter vector K can thus be stated in terms of a static parameter optimization problem, where the objective is to maximize the ratio c_3/c_4, with c_3 and c_4 given by (2.3.17) and (2.3.22) respectively. This problem may appear to be an extremely difficult one to solve in terms of analytical approaches; however, it is a very practical problem for numerical computation and can be solved on a digital computer by a number of different techniques.

For a more restricted class of components, a more specific set of sufficient conditions can be obtained. Consider in particular that class of component models for which there exists Liapunov functions satisfying (2.2.2) which have a quadratic form. Denote this class by $\mathcal{E}_Q$. The class $\mathcal{E}_Q$ not only allows a large class of important components, but the restriction is intuitively reasonable, considering the conditions (2.2.2). The next theorem gives sufficient conditions for an interconnection of components from class $\mathcal{E}_Q$ to be exponentially stable.

Theorem 2.3.4 Consider an interconnected system (2.1.5) with component models of the form (2.1.2) and interconnection constraints (2.1.3). For each of the m components S_i having dynamic parts, let $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E}_Q)$ and let

$$v_i(\Psi^i, t) = \Psi^{i^T} P^i(t)\Psi^i \tag{2.3.29}$$

$c_1^i, c_2^i, c_3^i, c_4^i$ be Liapunov functions and associated positive constants satisfying relations (2.2.2). If there exists a positive parameter vector $K = (k_1, k_2, \ldots, k_m)^T$ such that

$$c_3 = \min_i k_i c_3^i - \sigma > 0 \tag{2.3.30}$$

where σ is the maximum value assumed by any of the eigenvalues of the symmetric part of the matrix

$$(P(K,t) + P(K,t)^T)\, M(t) \tag{2.3.31}$$

where

$$P(K,t) = \begin{bmatrix} k_1 P^1(t) & & & & \\ & k_2 P^2(t) & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & k_m P^m(t) \end{bmatrix} \tag{2.3.32}$$

then the reduced system (2.3.15) is exponentially stable. Further, if

$$\gamma_2 < \frac{\min_i k_i c_3^i - \sigma}{\max_i k_i c_4^i} \tag{2.3.33}$$

with γ_2 defined by (2.3.5) then the system (2.1.5) is exponentially stable.

Proof: For the reduced system (2.3.15), choose the positive definite function

$$v(\Psi, t) = \Psi^T P(K, t)\Psi \tag{2.3.34}$$

where P(K,t) is defined in (2.3.32). Then if

$$\begin{aligned} c_1 &= \min_i k_i c_1^i \\ c_2 &= \max_i k_i c_2^i \\ c_4 &= \max_i k_i c_4^i \end{aligned} \tag{2.3.35}$$

it follows that

$$\begin{aligned} c_1 |\Psi|^2 \leq v(\Psi, t) \leq c_2 |\Psi|^2 \\ |\nabla v(\Psi, t)| \leq c_4 |\Psi| \end{aligned} \tag{2.3.36}$$

Now the total derivative of (2.3.34) with respect to (2.3.15) is

$$\begin{aligned} \dot{v}(\Psi, t) &= \sum_{i=1}^{m} k_i \dot{v}_i(\Psi^i, t) + (\nabla v(\Psi, t))^T M(t)\Psi \\ &= \sum_{i=1}^{m} k_i \dot{v}_i(\Psi^i, t) + \Psi^T(P(K, t) + P(K, t)^T)M(t)\Psi \end{aligned} \tag{2.3.37}$$

But from Lemma A.3

$$\Psi^T(P(K, t) + P(K, t)^T) M(t)\Psi \leq \sigma \Psi^T\Psi = \sigma |\Psi|^2 \tag{2.3.38}$$

where σ is the maximum value assumed by any of the eigenvalues of the symmetrix matrix.

$$\frac{1}{2}([P(K,t) + P(K,t)^T]M(t) + M(t)^T[P^T(K,t) + P(K,t)]) \quad (2.3.39)$$

over the interval $t_o \leq t < \infty$ so that

$$\dot{v}(\Psi, t) \leq -c_3|\Psi|^2 \quad (2.3.40)$$

where c_3 is given by (2.3.30). Consequently if $c_3 > 0$, (2.3.15) is exponentially stable by Theorem 2.2.1, and if (2.3.33) is satisfied, then the system (2.1.5) is exponentially stable by Theorem 2.3.2.

Remarks: The discussion following Theorem 2.3.3 regarding its application also applies to Theorem 2.3.4. It may also be noted that in the case where the matrix (2.3.39) is constant, σ is its maximum eigenvalue.

A special form of Theorem 2.3.4 results when the Liapunov functions of the system components satisfying (2.2.2) can be taken as

$$v_i(\Psi^i, t) = \Psi^{i^T}\Psi^i \quad (2.3.41)$$

Corollary 2.3.4 If in Theorem 2.3.4, the component Liapunov functions satisfying (2.2.2) are given by (2.3.41), then the conclusions of Theorem 2.4.4 hold with σ taken as the maximum value assumed by the eigenvalues of

$$P(K)M(t) + M(t)^T P(K) \quad (2.3.42)$$

where P(K) is the diagonal matrix

$$P(K) = \begin{bmatrix} k_1 U & & & & & \\ & k_2 U & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & k_m U \end{bmatrix} \tag{2.3.43}$$

and the submatrices $k_i U$ are of order n_i.

Remark 1 The assumption (2.3.41) about the component Liapunov functions is valid, in particular, when the component equations are such that for some $c_3^i > 0$, one has

$$\Psi^{i^T} F^i(\Psi^i, t) \leq -c_3^i |\Psi^i|^2 \tag{2.3.44}$$

Remark 2 It follows that under the hypothesis of Corollary 2.3.4, that a negative semidefinite M(t) is sufficient for $c_3 > 0$.

2.4 Exponential Stability of a Class of Interconnected Systems with Unstable Components

In the preceding sections, only interconnections of exponentially stable components have been considered. No explicit consideration has been given to the concept that even though a component may be unstable when isolated, it may become stable--even exponentially stable--when interconnected into a system of other components.

This problem is considered in the following theorem.

<u>Theorem 2.4.1</u> Consider an interconnected system (2.1.5) with component models of the form (2.1.2) and interconnection constraints (2.1.3). Suppose that:

(i) $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E}_Q)$, $i \neq q$, for (m-1) of the m components S_i having dynamic parts.

(ii) The Liapunov functions for the components S_i, $i \neq q$ satisfying (2.2.2) have the form

$$v_i(\Psi^i, t) = \Psi^{i^T} P^i(t)\Psi^i \tag{2.3.45}$$

with associated constants c_1^i, c_2^i, c_3^i and c_4^i.

(iii) The q-th dynamic component S_q is unstable.

(iv)
$$c_3 = \min_i k_i c_3^i - \sigma \tag{2.3.46}$$

where σ is the maximum value assumed by any eigenvalue of the symmetric part of the matrix

$$(P(K,t) + P(K,t)^T)M(t) \tag{2.3.47}$$

with

$$P(K,t) = \begin{bmatrix} k_1 P^1(t) & & & & \\ & \ddots & & & \\ & & k_q P^q(t) & & \\ & & & \ddots & \\ & & & & k_m P^m(t) \end{bmatrix} \tag{2.3.48}$$

If there exists a positive definite $n_q \times n_q$ matrix $P^q(t)$ such that for

$$v_q(\Psi^q; t) = \Psi^{q^T} P^q(t)\, \Psi^q \tag{2.3.49}$$

one has

$$\dot{v}_q(\Psi^q; t) \leq -c_3^q\, |\Psi|^2, \quad c_3^q \text{ a constant} \tag{2.3.50}$$

and a positive parameter vector K such that $c_3 > 0$, then the reduced system (2.3.15) is exponentially stable. Further, if (2.3.33) is satisfied then the system (2.1.5) is exponentially stable.

Proof: The proof is similar to that of Theorem 2.3.4.

Remark 1 In this case, a necessary condition for Theorem 2.4.1 to be satisfied is that σ be negative, and hence, that the matrix (2.3.47) be negative definite.

Remark 2 In the actual application of Theorem 2.4.1 it is operationally expedient to assume a diagonal form for $P^q(t)$ when possible.

III VECTOR LIAPUNOV FUNCTIONS FOR A CLASS OF INTERCONNECTED SYSTEMS

In this chapter, sufficient conditions are given for various types of stability for interconnected systems from a subclass of those systems considered in Chapter II. The approach given here differs from that of Chapter II in that a vector Liapunov function is constructed from the Liapunov functions for the individual components. A system of linear differential inequalities, satisfied by the vector Liapunov function, is constructed from the component Liapunov functions. This system of inequalities depends upon the component equations and Liapunov functions, the interconnection pattern, and also upon a parameter vector whose optimal selection increases the domain of systems for which this stability criterion becomes valuable. The stability properties of the interconnected system are then examined in terms of those of the linear comparison system obtained from the system of linear differential inequalities by replacing the inequality by an equality.

Under several additional restrictions on the system topology and component characteristics, and equating all the components of the parameter vector to unity, one of the results, Corollary 3.3.1.1, reduces to the main result of Bailey [BA-1].

3.1 Interconnected Systems

The class of systems considered here can be defined in terms of a restriction on the class considered in Chapter II. Specifically, it is assumed that $AG(\Psi, t)$ in (2.1.5) is linear in Ψ, that is,

$$AG(\Psi, t) = M(t)\Psi \tag{3.1.1}$$

Condition (3.1.1) can be satisfied in either of two ways. If $G(\Psi, t)$ is linear in Ψ, then (3.1.1) will be satisfied; or if $G(\Psi, t)$ is not linear in Ψ, then A may be such that (3.1.1) is still satisfied.

3.2 Vector Liapunov Functions

Corduneanu [COR-1] gave a generalization of Liapunov stability criteria, viewing the Liapunov function as the dependent variable in a first order differential inequality. The basic concepts involved had been used earlier by Conti [CON-1] in connection with existence and uniqueness studies for ordinary differential equations. Stability results of this type are also given in [BR-1, SAN-1].

Bellman [BEL-1], Matrosov [MA-1], and Lakshmikantham [LA-1] have given stability results in terms of vector Liapunov functions and have shown that their use is indeed advantageous for certain problems.

In this section, results of Wazewski [WAZ-1] on differential inequalities are used to extend the stability results of Corduneanu [COR-1] to vector Liapunov functions.

Consider the m-dimensional differential system

$$\dot{R} = W(R, t) \tag{3.2.1}$$

where the components w_i of W are defined and continuous in some open region Ω of the (m+1) dimensional space, and W is such that there exists a unique solution passing each point $(R_o, t_o) \in \Omega$. Let each function w_i be non-decreasing with respect to $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_m$ in Ω, i.e., for arbitrary points

$$(R, t) = (r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_m, t) \in \Omega$$

and

$$(R', t) = (r'_1, \ldots, r'_{i-1}, r_i, r'_{i+1}, \ldots, r'_m, t) \in \Omega$$

satisfying

$$R \geq R'$$

it follows that

$$W(R, t) \geq W(R', t)$$

Further, let the m-dimensional vector function V(t) be continuously differentiable in the interval $[t_o, \theta)$, and such that $V(t_o) = R_o$, $(V(t), t) \in \Omega$ when $t \in [t_o, \theta)$. Then it follows from the results of Wazewski [WAZ-1], that if

$$\dot{V}(t) \leq W(V, t) \quad , \quad t \in [t_o, \theta) \tag{3.2.2}$$

then

$$V(t) \leq R(t;\, R_o, t_o) \quad , \; t \in [t_o, \theta) \tag{3.2.3}$$

The point θ may be taken as $\theta = \infty$, or θ may be chosen so that $t \to \theta$ as the solution of (3.2.1) approaches the boundary of Ω.

<u>Theorem 3.2.1</u> Let $F(\Psi, t) \in \mathscr{E}$ in H_{h, t_o} for the n-dimensional system

$$\dot{\Psi} = F(\Psi, t) \; , \qquad F(0, t) \equiv 0 \tag{3.2.4}$$

and consider the m-dimensional system

$$\dot{R} = W(R, t) \quad , \qquad W(0, t) \equiv 0 \tag{3.2.5}$$

where the components w_i of W are defined, real, continuous, non-decreasing with respect to $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_m$ in

$$\Omega = \{ (R, t) : \; |R| < \hat{R} \leq \infty \, , \; t \geq 0 \}$$

and are such that there exists a unique solution passing through each point $(R_o, t_o) \in \Omega$. Let $V(\Psi, t) \geq 0$ be a differentiable m-dimensional vector function, and let the total time derivative of V with respect to (3.2.4) satisfy

$$\dot{V}(\Psi, t) \leq W(V(\Psi, t), t) \tag{3.2.6}$$

Then:

(1) If the solution $R = 0$ of (3.2.5) is stable and $|V(\Psi, t)|$ is positive definite, then the solution $\Psi = 0$ of (3.2.4) is stable. Further,

if $|V(\Psi, t)|$ is decrescent, then the solution $\Psi = 0$ is uniformly stable.

(2) If the solution $R = 0$ of (3.2.5) is asymptotically stable and $|V(\Psi, t)|$ is positive definite, then the solution $\Psi = 0$ of (3.2.4) is asymptotically stable. Further, if $|V(\Psi, t)|$ is decrescent, then the solution $\Psi = 0$ is uniformly asymptotically stable.

Proof: (1) Since $|V(\Psi, t)|$ is positive definite, there exists a function $a \in \mathcal{K}$ such that

$$|V(\Psi, t)| \geq a(|\Psi|) \tag{3.2.7}$$

Let (3.2.5) be stable, and let $\epsilon > 0, t_o \geq 0$ be given. Then from the stability of (3.2.5) there exists an $\eta(\epsilon, t_o) > 0$ such that $0 < |R_o| < \eta$ implies

$$|R(t;\ R_o, t_o)| < a(\epsilon) \tag{3.2.8}$$

From the continuity of V, it follows that there exists a $\delta(\epsilon, t_o) > 0$ such that $|\Psi_o| < \delta$ implies $|V(\Psi_o, t_o| < \eta$. Since from (3.2.6) one has

$$\dot{V}(\Psi(t\ ;\ \Psi_o, t_o), t) \leq W(V(\Psi(t; \Psi_o, t_o), t), t) \tag{3.2.9}$$

it follows from Wazewski's results (3.2.3) that

$$V(\Psi(t; \Psi_o, t_o), t) \leq R(t; V(\Psi_o, t_o), t_o) \tag{3.2.10}$$

Combining (3.2.7), (3.2.10), and (3.2.8) gives

$$a(|\Psi(t; \Psi_o, t_o)|) \leq |V(\Psi(t; \Psi_o, t_o), t)| \leq |R(t; V(\Psi_o, t_o), t_o)| < a(\epsilon)$$

so that

$$a(|\Psi(t;\Psi_o,t_o)|) \;<\; a(\epsilon)$$

and since $a \in \mathcal{K}$

$$|\Psi(t;\Psi_o,t_o)| \;<\; \epsilon$$

for $t \geq t_o$, if $|\Psi_o| < \delta(\epsilon, t_o)$. Thus (3.2.4) is stable.

If (3.2.5) is uniformly stable, then η in the preceding may be chosen independent of t_o. Further, if $|V(\Psi, t)|$ is decrescent, then there exists a function $b \in \mathcal{K}$ such that

$$|V(\Psi, t)| \;\leq\; b(|\Psi|) \tag{3.2.11}$$

and δ may also be chosen independent of t_o. In fact, choose $\delta < \xi$, where $b(\xi) = \eta$. Then proceeding as above, uniform stability of (3.2.4) is obtained.

(2) If (3.2.5) is asymptotically stable, then

$$\lim_{t \to \infty} R(t; V(\Psi_o, t_c), t_o) = 0$$

This condition and (3.2.10) imply

$$\lim_{t \to \infty} V(\Psi(t;\Psi_o,t_o),) = 0 \tag{3.2.12}$$

Thus from (3.2.7) and (3.2.12)

$$\lim_{t \to \infty} a(|\Psi(t;\Psi_o,t_o)|) \;=\; 0$$

which implies that

$$\lim_{t \to \infty} \Psi(t; \Psi_o, t_o) = 0$$

and (3.2.4) is asymptotically stable.

If (3.2.5) is uniformly asymptotically stable, then there exists an $\eta_o > 0$, and for $\epsilon > 0$, a quantity $T(\epsilon) > 0$ such that $|R_o| < \eta_o$ implies

$$|R(t; R_o, t_o)| < a(\epsilon), \quad \text{for } t > t_o + T(\epsilon) \tag{3.2.13}$$

If δ_o is chosen such that $b(\delta_o) < \eta_o$, then $|\Psi_o| < \delta_o$ implies that

$$|V(\Psi_o, t_o)| < b(|\Psi_o|) < b(\delta_o) < \eta_o \tag{3.2.14}$$

Thus from (3.2.13)

$$|R(t; V(\Psi_o, t_o), t_o)| < a(\epsilon), \text{ for } t \geq t_o + T(\epsilon) \tag{3.2.15}$$

From (3.2.15), (3.2.10), and (3.2.7) it follows that

$$a(|\Psi(t; \Psi_o, t_o)|) < a(\epsilon), \quad \text{for } t \geq t_o + T(\epsilon) \tag{3.2.16}$$

Finally from (3.2.16) and $a \in \mathcal{K}$

$$|\Psi(t; \Psi_o, t_o)| < \epsilon, \text{ for } t \geq t_o + T(\epsilon), \quad |\Psi_o| < \delta_o$$

and (3.2.4) is uniformly asymptotically stable, thus completing this proof.

Remark This theorem is somewhat related to a theorem on vector Liapunov functions given in [MA-1] ; however, in that theorem the requirements on the vector $V(\Psi, t)$ are too stringent for the theorem to be applicable here. In Matrosov's theorem it is required that

$$\sum_{i=1}^{k} v_i(\Psi, t) \quad , \quad 1 \le k \le m$$

be positive definite. But if the partial sums $v_1(\Psi, t)$, $v_1(\Psi, t) + v_2(\Psi, t), \ldots,$ are not all functions of all the components of Ψ, then clearly they cannot be positive definite functions of Ψ.

3.3 Vector Liapunov Functions and Stability of the Interconnected System

The theory developed in the preceding can now be applied to the central problem of investigating the stability properties of the interconnected system. As before, it is assumed that the interconnection of components (2.1.2) given by (2.1.3) satisfies graphic constraint GC and that graphic-component constraint GCC is satisfied so that the system model is given by (2.1.5). The following theorem gives one such result.

Theorem 3.3.1 Consider an interconnected system S having the form (2.1.5), satisfying (3.1.1), with component models of the form (2.1.2) and interconnection constraints (2.1.3). For each of the m components S_i having dynamic parts, let $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E})$ and let $v^i(\Psi^i, t)$, c_1^i, c_2^i, c_3^i, c_4^i be Liapunov functions and positive constants satisfying (2.2.2).

Define the m-dimensional linear system

$$\dot{R} = W(K, t)R \tag{3.3.1}$$

where the matrix $W(K, t)$ is given by

$$w_{ij} = \frac{(k_i c_4^i)^2 \tilde{m}_i |M_{ij}(t)|^2}{4k_{m+i} k_j c_1^j} \, , \quad i \neq j \tag{3.3.2}$$

$$w_{ii} = \begin{cases} \dfrac{q_i}{k_i c_2^i} \, , & \text{if } q_i < 0 \\[2ex] 0 \, , & \text{if } q_i = 0 \\[2ex] \dfrac{q_i}{k_i c_1^i} \, , & \text{if } q_i > 0 \end{cases} \tag{3.3.3}$$

with

$$q_i = k_{m+i} + k_i (c_4^i |M_{ii}(t)| - c_3^i) \tag{3.3.4}$$

The submatrix $M_{ij}(t)$ of $M(t)$ is as defined in Theorem 2.3.3; $\tilde{m}_i$ is the number of nonvanishing norms $|M_{ij}(t)| \neq 0$ for $j \neq i$; and K is a 2m-dimensional parameter vector.

If there exist positive values for the components of K such that the linear comparison system (3.3.1) is stable (uniformly stable, asymptotically stable, or uniformly asymptotically stable), then the interconnected system S is stable (uniformly stable, asymptotically stable, or uniformly asymptotically stable).

<u>Proof</u>: Since $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E}_o)$, the existence of the component Liapunov functions satisfying (2.2.2) is assured by Theorem 2.2.1. Consider the m-dimensional vector V whose i-th component is

given by

$$v_i = k_i v^i \tag{3.3.5}$$

where v^i is the Liapunov function corresponding to the i-th dynamic component and $k_i > 0$ is a constant. The total derivative of v_i with respect to the system S is

$$\dot{v}_i = k_i \frac{\partial v^i}{\partial t} + k_i(\nabla v^i)^T \dot{\Psi}^i \tag{3.3.6}$$

and from (2.1.5) and (3.1.1)

$$\dot{v}_i = k_i \dot{v}^i + k_i(\nabla v^i)^T M_i(t)\Psi \tag{3.3.7}$$

where $M_i(t)$ is the row submatrix of M(t) formed by the rows of M(t) corresponding to $\dot{\Psi}^i$ in the system model S. Thus (3.3.7) can be written as

$$\dot{v}_i = k_i \dot{v}^i + k_i(\nabla v^i)^T \sum_{j=1}^{m} M_{ij}(t)\Psi^j \tag{3.3.8}$$

Applying the Schwartz inequality and inequalities (2.2.2) to the right side of (3.3.8) gives

$$\dot{v}_i \leq k_i(c_4^i \; |M_{ii}(t)| - c_3^i)\,|\Psi^i|^2 + k_i c_4^i |\Psi^i| \sum_{\substack{j=1 \\ j\neq i}}^{m} |M_{ij}(t)|\,|\Psi^j| \tag{3.3.9}$$

Using Lemma A.7 to eliminate the product terms $|\Psi^i|\,|\Psi^j|$ from (3.3.9), one obtains

$$\dot{v}_i \leq [k_{m+i} + k_i(c_4^i|M_{ii}(t)| - c_3^i)]\,|\Psi^i|^2$$

$$+ \frac{1}{4k_{m+i}}\,(k_i c_4^i \sum_{\substack{j=1\\ j\neq i}}^{m} |M_{ij}(t)|\,|\Psi^j|)^2 \qquad (3.3.10)$$

Applying Lemma A.6 to the last term in (3.3.10) gives

$$\dot{v}_i \leq [k_{m+i} + k_i(c_4^i|M_{ii}(t)| - c_3^i)]|\Psi^i|^2$$

$$+ \frac{(k_i c_4^i)^2}{4k_{m+i}}\,\tilde{m}_i \sum_{\substack{j=1\\ j\neq i}}^{m} |M_{ij}(t)|^2|\Psi^j|^2 \qquad (3.3.11)$$

Using (2.2.2) to eliminate $|\Psi^i|^2$ and $|\Psi^j|^2$ from (3.3.11)

$$\dot{v}_i \leq \begin{cases} \dfrac{q^i}{c_2^i}\, v^i + \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{m} w_{ij}k_j v^j, & \text{if } q_i < 0 \\[2ex] \dfrac{q^i}{c_1^i}\, v^i + \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{m} w_{ij}k_j v^j, & \text{if } q_i > 0 \end{cases} \qquad (3.3.12)$$

Then from (3.3.5) the system

$$\dot{V} \leq W(K,t)V$$

results with the entries w_{ij} of the matrix $W(K,t)$ as given in the theorem and $V = [v_1 v_2 \dots v_m]^T$. Since the $v^i(\Psi^i, t)$ and hence $v_i(\Psi^i, t)$ are positive definite functions of their arguments Ψ^i, it can be shown that $|V(\Psi, t)|$ is a positive definite function of its argument Ψ.

Further, from (2.2.2a) the $v^i(\Psi^i, t)$ and hence $v_i(\Psi^i, t)$ are decrescent, so that $|V(\Psi, t)|$ is decrescent. Thus the proof follows from Theorem 3.2.1.

At various steps throughout the derivation of the matrix $W(K, t)$ in the preceding theorem, there is a certain amount of choice as to exactly what inequalities are used and how they are used. Clearly, the answer to which is the better choice depends upon the particular system structure. The following corollaries give some additional relations for $W(K, t)$, which are closely related to those in the theorem.

Corollary 3.3.1.1 Theorem 3.3.1 holds for $W(K, t)$ with

$$w_{ij} = \frac{(k_i c_4^i)^2 \sum\limits_{\substack{j=1 \\ j \neq i}}^{m} |M_{ij}(t)|^2}{4k_{m+i} k_j c_1^j}, \quad i \neq j \tag{3.3.13}$$

and w_{ii} given by (3.3.3) and (3.3.4).

Proof: The result follows from the proof of the theorem by applying Lemma A.5 instead of Lemma A.6 to the last term in (3.3.10).

An application of Corollary 3.1.1.1 gives the result of Bailey [BA-1] as a special case. In the notation of this thesis, Bailey considers component models of the form

$$\dot{\Psi}^i = F(\Psi^i, t) + D^i Z_1^i \tag{B.1}$$

$$Z_2^i = G^i \Psi^i$$

In addition to the hypothesis of Theorem 3.2.1, he assumes that no loading occurs and that the system topology is restricted so that in Theorem 3.2.1

$$M_{ii} = 0, \qquad i = 1, 2, \ldots, m \tag{B.2}$$

In his main theorem, Bailey gives the comparison system (after corrections)

$$w_{ii} = - \frac{c_3^i}{2c_2^i}, \quad i = j \tag{B.3}$$

$$w_{ij} = \frac{(c_4^i)^2 \sum_{j=1}^{m} |M_{ij}|^2}{2c_3^i c_1^j}, \quad i \neq j \tag{B.4}$$

If in Corollary 3.3.1.1, one takes $M_{ii} = 0$; $C = 0$ and $G(\Psi, t)$ has the form $G\Psi$ in (2.1.1); $k_i = 1$ and $k_{m+i} = c_3^i/2$, $i = 1, 2, \ldots, m$; then (3.3.13), (3.3.3), and (3.3.4) reduce to (B.3) and (B.4).

<u>Corollary 3.3.1.2</u> Theorem 3.3.1 holds for $W(K, t)$ with $\tilde{m}_i$ in (3.3.2) replaced by $\hat{m}_i$, where $\hat{m}_i$ is the number of nonvanishing norms $|M_{ij}(t)| \neq 0$, all j; and w_{ii} is given by (3.3.3) with

$$q_i = k_{m+i} - k_i c_3^i + \frac{(k_i c_4^i)^2 \hat{m}_i |M_{ii}(t)|^2}{4k_{m+i}} \tag{3.3.14}$$

Proof: The proof of this result follows from that of the theorem by considering inequality (3.3.9) in the form

$$\dot{v}_i \leq -k_i c_3^i |\Psi^i|^2 + k_i c_4^i |\Psi^i| \sum_{j=1}^{m} |M_{ij}(t)| \, |\Psi^j| \tag{3.3.15}$$

Corollary 3.3.1.3 Theorem 3.3.1 holds with W(K, t) given by

$$w_{ij} = \frac{(k_i c_4^i)^2 \sum_{j=1}^{m} |M_{ij}(t)|^2}{4k_{m+i} k_j c_1^j}, \quad i \neq j \tag{3.3.16}$$

and w_{ii} by (3.3.3) with

$$q_i = k_{m+i} - k_i c_3^i + \frac{(k_i c_4^i)^2}{4k_{m+i}} \left(\sum_{j=1}^{m} |M_{ij}(t)|^2 \right) \tag{3.3.17}$$

Proof: The proof follows from the proof of the theorem with both of the modifications given in the proofs of Corollary 3.3.1.1 and Corollary 3.3.1.2.

The next theorem and corollaries give stability criterion in terms of a comparison system that does not depend on a parameter vector. A parameter vector is not involved, because in this particular derivation, the parameters as appearing in (3.3.5) would have no effect on the stability of the comparison system. Further, the inequality which introduces the second m components of K, in the preceding, is not used.

<u>Theorem 3.3.2</u> Consider an interconnected system S having the form (2.1.5), satisfying (3.1.1), with component models of the form (2.1.2) and interconnection constraints (2.1.3). For each of the m components S_i having dynamic parts, let $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E})$ and let $v_i(\Psi^i, t)$, c_1^i, c_2^i, c_3^i, c_4^i be Liapunov functions and positive constants satisfying (2.2.2). If the m-dimensional, linear comparison system

$$\dot{R} = W(t)R \tag{3.3.18}$$

with

$$w_{ij} = \frac{c_4^i}{2c_1^j}, \qquad i \neq j \tag{3.3.19}$$

$$q_i = -c_3^i + \frac{c_4^i}{2}\left(1 + \sum_{j=1}^{m} |M_{ij}(t)|^2\right) \tag{3.3.20}$$

and

$$w_{ii} = \begin{cases} \dfrac{q_i}{c_2^i}, & \text{for } q_i < 0 \\[2ex] 0, & \text{for } q_i = 0 \\[2ex] \dfrac{q_i}{c_1^i}, & \text{for } q_i > 0 \end{cases} \tag{3.3.21}$$

is stable (uniformly stable, asymptotically stable, or uniformly asymptotically stable), then the interconnected system S is stable (uniformly stable, asymptotically stable, or uniformly asymptotically stable).

Proof: Since $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E})$, the component Liapunov functions $v_i(\Psi^i, t)$ satisfying (2.2.2) exist by Theorem 2.2.1. Consider the m-dimensional vector V with i-th component $v_i(\Psi^i, t)$. The total derivative of v_i with respect to the system S is

$$\dot{v}_i = \frac{\partial v_i}{\partial t} + (\nabla v_i)^T \dot{\Psi}^i \qquad (3.3.22)$$

or

$$\dot{v}_i(\Psi, t) = \dot{v}_i(\Psi^i, t) + (\nabla v_i(\Psi^i, t))^T M_i(t)\Psi \qquad (3.3.23)$$

where $\dot{v}_i(\Psi^i, t)$ is the total derivative of v_i with respect to the component model, $\nabla v_i(\Psi^i, t)$ is the gradient vector of v_i with respect to its argument Ψ^i, and $M_i(t)$ is the row submatrix of M(t) defined in the proof of Theorem 3.3.1. Rewriting (3.3.23) as

$$\dot{v}_i(\Psi, t) = \dot{v}_i(\Psi^i, t) + (\nabla v_i(\Psi^i, t))^T \sum_{j=1}^{m} M_{ij}(t)\Psi^j \qquad (3.3.24)$$

and applying the Schwartz inequality and inequalities (2.2.2) gives

$$\dot{v}_i \leq -c_3^i \, |\Psi^i|^2 + c_4^i \sum_{j=1}^{m} |M_{ij}(t)|\,|\Psi^i|\,|\Psi^j| \qquad (3.3.25)$$

From Lemma A.4 with $a = |M_{ij}(t)|\,|\Psi^i|$ and $b = |\Psi^j|$, one has

$$\dot{v}_i \leq -c_3^i \, |\Psi^i|^2 + \frac{c_4^i}{2} \sum_{j=1}^{m} (|M_{ij}(t)|^2 |\Psi^i|^2 + |\Psi^j|^2) \qquad (3.3.26)$$

Rearranging terms in (3.3.26) and applying inequalities (2.2.2) gives the result. By the same reasoning as in Theorem 3.3.1, $|V(\Psi, t)|$ is positive definite and decrescent, thus the theorem follows from Theorem 3.2.1.

<u>Corollary 3.3.2.1</u> Theorem 3.3.2 holds for W(t) given by

$$w_{ij} = \frac{c_4^i \, |M_{ij}(t)|^2}{2c_1^j} , \quad i \neq j \tag{3.3.27}$$

and

$$q_i = -c_3^i + \frac{(m+|M_{ii}(t)|^2)c_4^i}{2} \tag{3.3.28}$$

<u>Proof:</u> The proof follows from the proof of the theorem by taking $a = |\Psi^i|$ and $b = |M_{ij}(t)|\,|\Psi^j|$ when applying Lemma A.4.

<u>Corollary 3.3.2.2</u> Theorem 3.3.2 holds for W(t) with w_{ij} given by (3.3.19) and

$$q_i = -c_3^i + c_4^i \left(|M_{ii}(t)| + \frac{1}{2} \sum_{\substack{j=1 \\ j\neq i}}^{m} |M_{ij}(t)|^2\right) \tag{3.3.29}$$

<u>Proof:</u> The proof follows from the proof of the theorem with (3.3.25) replaced by

$$\begin{aligned} \dot{v}_i \leq & -c_3^i \, |\Psi^i|^2 + c_4^i |M_{ii}(t)|\,|\Psi^i|^2 \\ & + c_4^i \left(\sum_{\substack{j=1 \\ j\neq i}}^{m} |M_{ij}(t)|\,|\Psi^i|\,|\Psi^j|\right) \end{aligned} \tag{3.3.30}$$

<u>Corollary 3.3.2.3</u> Theorem 3.3.2 holds for W(t) with

$$w_{ij} = \frac{c_4^i \, |M_{ij}(t)|^2}{2c_1^j} , \quad i \neq j \tag{3.3.31}$$

and

$$q_i = -c_3^i + c_4^i \left(|M_{ii}(t)| + \frac{m-1}{2} \right) \tag{3.3.32}$$

Proof: The proof follows from the proof of the theorem with both of the modifications given in the proofs of Corollary 3.3.2.1 and Corollary 3.3.2.2.

Corollary 3.3.2.4 Theorem 3.3.2 holds for W(t) with

$$w_{ij} = \frac{c_4^i |M_{ij}(t)|}{2c_1^j}, \quad i \neq j \tag{3.3.33}$$

and

$$q_i = -c_3^i + \frac{c_4^i}{2} \left(|M_{ii}(t)| + \sum_{j=1}^{m} |M_{ij}(t)| \right) \tag{3.3.34}$$

Proof: The proof follows from the proof of the theorem by taking $a = |\Psi^i|$ and $b = |\Psi^j|$ when applying Lemma A.4.

If it is assumed that the component models belong to class $\mathcal{E}_Q$, more specific sets of sufficient conditions can be obtained corresponding to Theorem 3.3.1 and Theorem 3.3.2 and their corollaries. One such result given for class $\mathcal{E}_Q$ in the following theorem corresponds to Corollary 3.3.2.4 for class $\mathcal{E}$.

Theorem 3.3.3 Consider an interconnected system S having the form (2.1.5), satisfying (3.1.1), with component models of the form (2.1.2) and interconnection constraints (2.1.3). For each of the m components S_i having dynamic parts, let $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E}_Q)$ and let

$$v_i(\Psi^i, t) = \Psi^{i^T} P^i(t)\Psi^i \tag{3.3.35}$$

c_1^i, c_2^i, c_3^i, c_4^i be Liapunov functions and positive constants satisfying (2.2.2). Define the m-dimensional linear comparison

$$\dot{R} = W(t)R \tag{3.3.36}$$

with the matrix W(t) given by

$$w_{ij} = \frac{|(P^i(t) + P^i(t)^T)M_{ij}(t)|}{2c_1^j}, \; i \neq j \tag{3.3.37}$$

and

$$w_{ii} = \begin{cases} \dfrac{q^i}{c_2^i}, & \text{if } q_i < 0 \\ 0 \;, & \text{if } q_i = 0 \\ \dfrac{q^i}{c_1^i}, & \text{if } q_i > 0 \end{cases} \tag{3.3.38}$$

where

$$q_i = -c_3^i + \sigma_i + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^{m} |(P^i(t) + P^i(t)^T)M_{ij}(t)| \tag{3.3.39}$$

and σ_i the maximum value assumed by an eigenvalue of the symmetric part of the matrix

$$(P^i(t) + P^i(t)^T)M_{ii}(t) \tag{3.3.40}$$

If the linear system (3.3.36) is stable (uniformly stable, asymptotically stable, or uniformly asymptotically stable), then the interconnected system S is stable (uniformly stable, asymptotically stable, or uniformly asymptotically stable).

Proof: Since $F^i(\Psi^i, t) \in (\mathcal{B} \cap \mathcal{E}_{\rho Q})$, the Liapunov functions (3.3.35) satisfying (2.2.2) exist. The total derivative of v_i with respect to the system S is

$$\dot{v}_i(\Psi, t) = \dot{v}_i(\Psi^i, t) + (\nabla v_i(\Psi^i, t))^T M_i(t)\Psi \tag{3.3.41}$$

or from (3.3.35)

$$\dot{v}_i(\Psi, t) = \dot{v}_i(\Psi^i, t) + \Psi^{i^T} \sum_{j=1}^{m} (P^i(t) + P^i(t)^T)M_{ij}(t)\Psi^j \tag{3.3.42}$$

From (2.2.2) and Lemma A.3, (3.3.42) gives

$$\dot{v}_i \le (-c_3^i + \sigma_i)|\Psi^i|^2 + \Psi^{i^T} \sum_{\substack{j=1 \\ j\ne i}}^{m} (P^i(t) + P^i(t)^T)M_{ij}(t)\Psi^j \tag{3.3.43}$$

Then using the Schwartz inequality and applying Lemma A.4 with $a = |\Psi^i|$, $b = |\Psi^j|$, one obtains

$$\dot{v}_i \le (-c_3^i + \sigma_i + \frac{1}{2} \sum_{\substack{j=1 \\ j\ne i}}^{m} |(P^i(t) + P^i(t)^T)M_{ij}(t)|)|\Psi^i|^2$$

$$+ \frac{1}{2} \sum_{\substack{j=1 \\ j\ne i}}^{m} |(P^i(t) + P^i(t)^T)M_{ij}(t)|\,|\Psi^j|^2 \tag{3.3.44}$$

Inequality (2.2.2a) applied to (3.3.44) gives the desired result for $W(t)$, and the theorem then follows from Theorem 3.2.1.

Corollary 3.3.3 Suppose that the component Liapunov functions $v_i(\Psi^i, t)$ in Theorem 3.3.3 can be chosen as

$$v_i(\Psi^i, t) = |\Psi^i|^2 \tag{3.3.45}$$

Then the theorem holds with

$$w_{ij} = \frac{|M_{ij}(t)|}{c_1^j}, \quad i \neq j \tag{3.3.46}$$

$$q_i = -c_3^i + \sigma_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} |M_{ij}(t)| \tag{3.3.47}$$

and σ_i the maximum value assumed by any of the eigenvalues of the matrix

$$M_{ii}(t) + M_{ii}(t)^T \tag{3.3.48}$$

IV APPLICATIONS

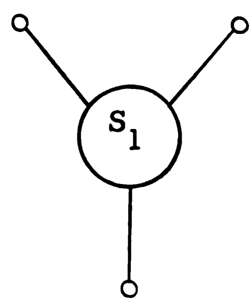
The purpose of this chapter is to illustrate, by means of example, some of the stability theory developed in this thesis. In so doing, an attempt is made to indicate the importance of these concepts to design work.

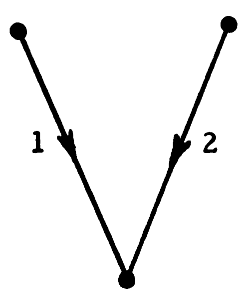
Consider a hypothetical interconnected system S composed of the following subsystems

S_1: a second order, three terminal, nonlinear, mixed component

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = \begin{bmatrix} -\psi_1^3 & -\psi_1 \\ -\psi_2 & -\frac{1}{2}\sin 2\psi_2 \end{bmatrix} + \begin{bmatrix} -.549 & 1.943 \\ .086 & -.239 \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.000 & .500 \\ 0 & .200 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} .300 & 0 \\ .400 & .100 \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \tag{4.1}$$



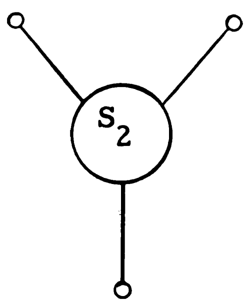


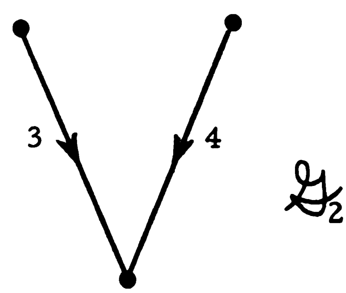
$\mathcal{G}_1$

S_2: a third order, three terminal, linear, dynamic component

$$\begin{bmatrix} \dot{\psi}_3 \\ \dot{\psi}_4 \\ \dot{\psi}_5 \end{bmatrix} = \begin{bmatrix} -.100 & .700 & -.300 \\ -.700 & -.200 & .500 \\ .400 & -.500 & -.300 \end{bmatrix} \begin{bmatrix} \psi_3 \\ \psi_4 \\ \psi_5 \end{bmatrix} + \begin{bmatrix} .799 & .626 \\ -.157 & .213 \\ 1.000 & -.652 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.000 & .500 & 0 \\ 0 & -.500 & 2.000 \end{bmatrix} \begin{bmatrix} \psi_3 \\ \psi_4 \\ \psi_5 \end{bmatrix} \tag{4.2}$$

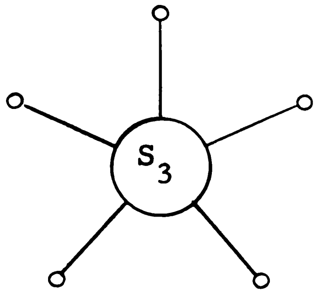


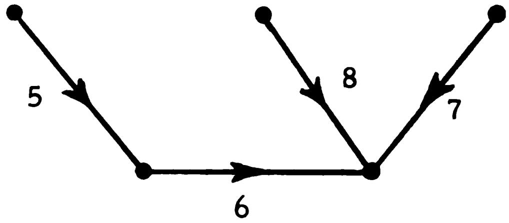


S_3: a second order, five terminal, linear, mixed component

$$\begin{bmatrix} \dot{\psi}_6 \\ \dot{\psi}_7 \end{bmatrix} = \begin{bmatrix} -.300 & .200 \\ -.200 & -.100 \end{bmatrix} \begin{bmatrix} \psi_6 \\ \psi_7 \end{bmatrix} + \begin{bmatrix} -.223 & -.315 & -.156 & .071 \\ 1.000 & .461 & -.978 & .603 \end{bmatrix} \begin{bmatrix} x_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} \tag{4.3}$$

$$\begin{bmatrix} y_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 1.000 & .600 \\ -.300 & 4.000 \\ 0 & 1.000 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_6 \\ \psi_7 \end{bmatrix} + \begin{bmatrix} -2.661 & -2.910 & 1.166 & -.899 \\ .368 & .273 & -1.375 & 1.465 \\ 0 & 1.000 & 0 & 0 \\ .199 & -.200 & 0 & -1.000 \end{bmatrix} \begin{bmatrix} x_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix}$$



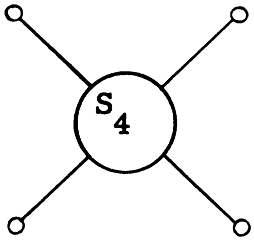


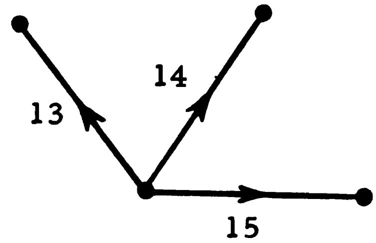
$\mathcal{G}_3$

S_4: a second order, four terminal, nonlinear dynamic component

$$\begin{bmatrix} \dot{\psi}_8 \\ \dot{\psi}_9 \end{bmatrix} = \begin{bmatrix} -3\psi_8 - \psi_9 \\ \psi_8 - 2\psi_9 - \psi_9^{3} \end{bmatrix} + \begin{bmatrix} .500 & -.001 & .298 \\ -.019 & .197 & .397 \end{bmatrix} \begin{bmatrix} x_{13} \\ y_{14} \\ x_{15} \end{bmatrix}$$

$$\begin{bmatrix} y_{13} \\ x_{14} \\ y_{15} \end{bmatrix} = \begin{bmatrix} .039 & 1.994 \\ g & -.378 \\ -.071 & -.395 \end{bmatrix} \begin{bmatrix} \psi_8 \\ \psi_9 \end{bmatrix} \tag{4.4}$$

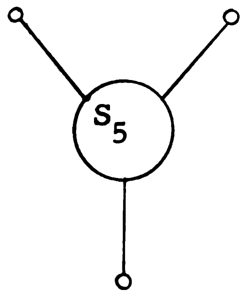


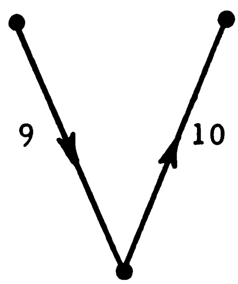


$\mathcal{G}_4$

S_5: a three terminal, linear, algebraic component

$$\begin{bmatrix} y_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} 0 & -1.000 \\ 1.000 & .400 \end{bmatrix} \begin{bmatrix} x_9 \\ y_{10} \end{bmatrix} \tag{4.5}$$

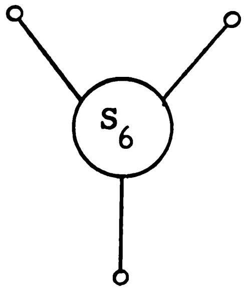


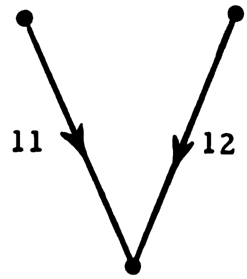


$\mathcal{G}_5$

S_6: a three terminal, linear, algebraic component

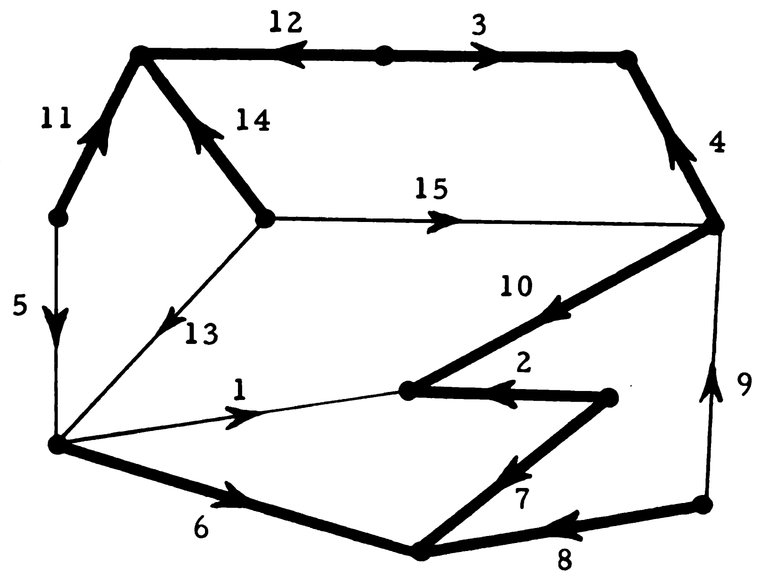
$$\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} -1.000 & .300 \\ 0 & -.400 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} \tag{4.6}$$





$\mathcal{G}_6$

The system graph $\mathcal{G}$ for the interconnected system S is



$\mathcal{G}$

and the linear interconnection constraint equations derived from $\mathcal{G}$ are

$$
\begin{bmatrix} x_1 \\ y_2 \\ y_3 \\ y_4 \\ x_5 \\ y_6 \\ y_7 \\ y_8 \\ x_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ x_{13} \\ y_{14} \\ x_{15} \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & -1 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & -1 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix} y_1 \\ x_2 \\ x_3 \\ x_4 \\ y_5 \\ x_6 \\ x_7 \\ x_8 \\ y_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ y_{13} \\ x_{14} \\ y_{15} \end{bmatrix}
\tag{4.7}
$$

Problem 1 Determine a value for the parameter g in the component equations for S_4 so that the interconnected system S is exponentially stable. The value of g must satisfy the design constraint

$$-1.485 \leq g \leq .915 \tag{4.8}$$

Solution: Consider first the matrices $F^{(2)}$ and $F^{(3)}$ corresponding to the dynamic parts of the linear components S_2 and S_3 respectively.

The eigenvalues of $F^{(2)}$ are determined to be

$$\lambda_1 = -.1871$$

$$\lambda_2 = -.2065 + j0.9220$$

$$\lambda_3 = -.2065 - j0.9220$$

and those of $F^{(3)}$ are

$$\lambda_1 = -.2000 + j0.1732$$

$$\lambda_2 = -.2000 - j0.1732$$

Since the eigenvalues of S_2 and S_3 all have negative real parts, these components are exponentially stable and have Liapunov functions satisfying (2.2.2). To determine the component Liapunov functions v_2 and v_3 for S_2 and S_3, respectively, assume they have the quadratic forms

$$v_2 = \Psi^{(2)^T} P^{(2)} \Psi^{(2)} \tag{4.9}$$

$$v_3 = \Psi^{(3)^T} P^{(3)} \Psi^{(3)} \tag{4.10}$$

Assume further, that the total time derivatives of v_2 and v_3 with respect to the component equations (4.2) and (4.3), respectively, are

$$\dot{v}_2 = -\Psi^{(2)^T} \Psi^{(2)} = -(\psi_3^2 + \psi_4^2 + \psi_5^2) \tag{4.11}$$

and

$$\dot{v}_3 = -\Psi^{(3)^T} \Psi^{(3)} = -(\psi_6^2 + \psi_7^2) \tag{4.12}$$

Frame [FR-2] has given a very elegant method for calculating the positive definite matrix of a quadratic Liapunov function, for time-invariant linear systems, from the negative definite matrix in the derivative of the Liapunov function. Application of this procedure to (4.11) and (4.12) give the following positive definite matrices for $P^{(2)}$ and $P^{(3)}$.

$$P^{(2)} = \begin{bmatrix} 2.723 & .326 & .002 \\ .326 & 2.779 & .345 \\ .002 & .345 & 2.240 \end{bmatrix} \tag{4.13}$$

$$P^{(3)} = \begin{bmatrix} 2.143 & -.714 \\ -.714 & 3.571 \end{bmatrix} \tag{4.14}$$

Consider now the nonlinear component S_1. If a Liapunov function v_1 is assumed as

$$v_1 = \frac{1}{2} \Psi^{(1)^T} \Psi^{(1)} = \frac{1}{2} (\psi_1^2 + \psi_2^2) \tag{4.15}$$

then

$$\nabla v_1 = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{4.16}$$

The total time derivative of v_1 with respect to the component equations (4.1) becomes

$$\dot{v}_1 = 2\Psi^{(2)^T} F^{(2)}$$

$$= -\psi_2^2 - \frac{1}{2}\psi_2 \sin 2\psi_2 - \psi_1^4 - \psi_1^2 \tag{4.17}$$

Since $\frac{\sin\theta}{\theta} > -.218$,

$$a\psi_2^2 < (\psi_2^2 + \frac{1}{2}\psi_2 \sin 2\psi_2) \tag{4.18}$$

for all $a \leq .782$, so that

$$\dot{v}_1 \leq -(.782\psi_2^2 + \psi_1^2)$$

$$\leq -.78(\psi_1^2 + \psi_2^2) = -.78|\Psi^{(2)}|^2 \tag{4.19}$$

Thus S_1 is exponentially stable by Theorem 2.2.1, with v_1 satisfying (2.2.2).

For the nonlinear system S_4, assume a Liapunov function v_4 as

$$v_4 = \Psi^{(4)^T} \Psi^{(4)} = |\Psi^{(4)}|^2 \tag{4.20}$$

Then

$$\nabla v_4 = 2\Psi^{(4)} \tag{4.21}$$

and the total time derivative of v_4 with respect to (4.4) is given by

$$\dot{v}_4 = -(3\psi_8^2 + 2\psi_9^2 + 4\psi_9^4)$$

$$\leq -2(\psi_8^2 + \psi_9^2) = -2|\Psi^{(4)}|^2 \tag{4.22}$$

The function v_4 satisfies (2.2.2) and thus S_4 is exponentially stable.

Having determined component Liapunov functions, the positive constants for which the functions satisfy (2.2.2) of Theorem 2.2.1 are

S_1:	$c_1^1 = .500$	$c_2^1 = .500$	$c_3^1 = .780$	$c_4^1 = 1.000$
S_2:	$c_1^2 = 2.038$	$c_2^2 = 3.156$	$c_3^2 = 1.000$	$c_4^2 = 6.312$
S_3:	$c_1^3 = 1.847$	$c_2^3 = 3.876$	$c_3^3 = 1.000$	$c_4^3 = 7.735$
S_4:	$c_1^4 = 1.000$	$c_2^4 = 1.000$	$c_3^4 = 2.000$	$c_4^4 = 2.000$

Since the component Liapunov functions have the quadratic form (2.3.29), Theorem 2.3.4 is applicable. For the parameter vector $K = [1 \;\; 1 \;\; 1 \;\; 1]^T$, the following values of c_3, as determined by (2.3.30) in Theorem 2.3.4, are computed for several values of g in the allowable interval

Value of g	Computed value of c_3
-1.485	.18
-1.085	.16
- .685	.09
- .385	- .01
- .285	- .06
.115	- .36
.515	- .82
.915	-1.36

By Theorem 2.3.4, S is exponentially stable for any g giving a positive value for c_3.

From (2.3.35), one also has

$c_1 = .500 \qquad c_2 = 3.867 \qquad c_4 = 7.735$

for the Liapunov function constructed for the interconnected system S.

Referring back to the proof of Theorem 2.2.1, one can see that there is more information here than just the assurance that S is exponentially stable for say, $g = -1.485$. Following equation (2.2.5) one has

$$|\Psi(t;\, \Psi_o, t_o)| \leq \beta |\Psi_o| \exp[-\alpha(t-t_o)]$$

for S, where

$$\beta = \sqrt{\frac{c_2}{c_1}} = \sqrt{\frac{3.867}{.500}} = 2.8$$

and

$$\alpha = \frac{c_3}{2c_2} = \frac{.18}{7.734} = .023$$

Clearly this upper bound on the solutions of S is not a least upper bound, because of various approximations made and because of the non-uniqueness of the Liapunov functions involved. However, this does give a lower bound on the rate of decay of the solutions of S. Further, if $F(\Psi, t) \in \mathcal{B}$ only for Ψ in some $H_{h,\tau}$ with h fixed, then since a value is known for β, the value h_o, defining H_{h_o}, can be determined from (1.3.7). Then for all $\Psi_o \in H_{h_o}$, one is assured of exponential stability.

Problem 2 In the preceding problem, a value of $g = -.285$ gave $c_3 = -.06$. Does this mean that S is not exponentially stable for this value of g ?

Solution: The first question one should ask is whether a positive c_3 might not be obtained for $g = -.285$ by using values for the parameter vector K other than $K = [1 \quad 1 \quad 1 \quad 1]^T$, as used in the preceding.

A gradient directed search in the K space, using Theorem 2.3.4 and beginning at $K = [1 \quad 1 \quad 1 \quad 1]^T$, yields the results shown below.

k_1	k_2	k_3	k_4	Computed c_3
1.0000	1.0000	1.0000	1.0000	-.06
1.0139	1.0000	.9999	.9854	-.04
1.0333	1.0000	.9998	.9649	-.01
1.0604	1.0000	.9997	.9361	.03
1.0984	1.0000	.9995	.8960	.08
1.1516	1.0000	.9992	.8397	.14
1.2262	1.0000	.9988	.7609	.17
1.2527	1.0923	1.0062	.7328	.18
1.2527	1.1427	1.0501	.7751	.19
1.3480	1.2309	1.1221	.8060	.20
1.3581	1.2481	1.1404	.8412	.21

It can be seen from these results, since an admissable K can be found which results in a $c_3 > 0$, that the system S is indeed exponentially stable for $g = -.285$. This clearly shows the value of the dependence of the system Liapunov function upon K.

V SUMMARY AND CONCLUSIONS

This thesis considers the stability of several classes of interconnected systems, consisting of multiterminal, nonlinear, time-varying components. The central objective is to obtain original stability criteria which relate to the two fundamental structural features of the system and which are practical to apply.

Exponential stability of several classes of interconnected systems is considered in Chapter II and a number of original results are obtained. Specifically, Theorem 2.3.1. gives sufficient conditions for exponential stability of a class of interconnected, exponentially stable components. This condition is given in terms of a bound on the norm of a vector in the interconnected system model. The bound is a function of the stability properties of the dynamic parts of the component models. Theorem 2.3.2 provides a basis for studying exponential stability of certain classes of systems in terms of the exponential stability of a less complicated or reduced system. Theorem 2.3.3 gives a sufficient condition for exponential stability of a class of interconnected exponentially stable components. The condition is in terms of an algebraic relation which depends upon properties of component Liapunov functions, structural features of the system, and a parameter vector. The condition on the algebraic expression is obtained by constructing a Liapunov function for the interconnected system in terms of the individual component Liapunov functions. Theorem 2.3.4 and Corollary 2.3.4 give more specific conditions for exponential stability of systems constructed from a more restricted classes of components. Theorem 2.4.1 extends the results of Theorem 2.3.4 to allow for systems with unstable components.

In Chapter III, the study of the stability of interconnected systems is based upon the construction of vector Liapunov functions for the systems. Theorem 3.2.1 extends some results of Corduneanu [COR-1], on scalar Liapunov functions, to vector Liapunov functions and establishes a basis for examining the stability of interconnected systems in terms of vector Liapunov functions. Theorem 3.3.1 and its corollaries give sufficient conditions for stability, (uniform stability, asymptotic stability, or uniform asymptotic stability) of a class of interconnected systems in terms of that of a linear comparison system, derived from a vector Liapunov function for the system. The comparison system is a function of the properties of the component Liapunov functions, the system structure, and a parameter vector. Theorem 3.3.2 and its corollaries give similar results, except that the comparison system, derived in a different manner, is not a function of a parameter vector. Theorem 3.3.3 and Corollary 3.3.3 give more specific comparison systems for more restricted classes of components.

In Chapter IV a ninth-order, nonlinear system is considered for illustration. The example indicates an important design application and shows the value of the parameter vector as an argument of the system Liapunov function.

The stability criteria obtained are particularly well suited for digital computation and computer-aided design. Considering stability in terms of the system structure, in many cases, helps to circumvent some of the mathematical complexities involved with higher-order nonlinear systems.

The exponential stability results in Chapter II are particularly valuable in engineering design applications, since a bound on the decay or response of the interconnected system is obtained as a by-product of the stability analysis.

As with most sufficiency conditions, there is always the problem that the conditions may be overly sufficient. This problem can be alleviated, to a certain extent, by the selection of proper values in the parameter vector in the Liapunov function. This is clearly shown in the example considered in Chapter IV.

The results of Chapter III on vector Liapunov functions for interconnected systems are the most general results of this type published in that they encompass a larger class of components and a larger class of interconnections. Further, the parameter dependence increases the usefulness of the criteria.

A number of areas for future research related to the work here seem rather promising. Some specific problems are:

- The results here apply mainly to system of components from Class $\mathcal{E}$. Perhaps similar results could be obtained for other classes of components.
- The problem of obtaining necessary conditions for stability or instability results could be investigated.
- The hypothesis of part (ii) of Theorem 2.2.1 might be weakened so that instead of a differentiability condition on the system, one has a less restrictive condition, such as a Lipschitz condition.
- The component equation forms and interconnections allowed cover a very broad class of components. Perhaps some useful results could be obtained by applying the concepts here to systems with very special assumptions on the component models and the interconnections.
- An investigation to determine explicit information as to the manner in which the parameter vector affects the stability of the comparison systems in Chapter III could prove quite valuable. Some initial efforts have already been made, by the author, on this problem.

- Concepts developed here could be valuable in investigating conditional stability for interconnected systems.

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APPENDIX A

In the following several essential results used in this thesis are proved or stated for reference.

<u>Lemma A.1</u> For the n-dimensional system of ordinary differential equations

$$\dot{\Psi} = F(\Psi, t) , \qquad \Psi(t_o; \Psi_o, t_o) = \Psi_o \tag{A.1}$$

$$F(0, t) \equiv 0$$

let $F(\Psi, t) \in \mathcal{C}$ in $H_{h,\tau}$, $\tau \geq 0$, and suppose that (A.1) is exponentially stable for all $\Psi_o \in H_{h_o}$, $h_o < h/\beta$. Then for all $\Psi_o \in H_{h_o}$

$$|\Psi_o| \, e^{-L(t-t_o)} \leq |\Psi(t; \Psi_o, t_o)| \leq |\Psi_o| e^{L(t-t_o)} \tag{A.2}$$

where L is the Lipschitz constant for $F(\Psi, t)$ in $H_{h,\tau}$.

<u>Proof</u>: Since $F(\Psi, t) \in \mathcal{C}$ in $H_{h,\tau}$, a Lipschitz condition is satisfied in this region, and since (A.1) is exponentially stable, for $\Psi_o \in H_{h_o}$, $\Psi(t; \Psi_o, t_o)$ will remain in $H_{h,\tau}$. Thus with $F(0, t) \equiv 0$, one has

$$|F(\Psi, t)| \leq L|\Psi| \tag{A.3}$$

so that

$$\frac{d}{dt}(\Psi^T\Psi) = 2\,\Psi^T F(\Psi, t)$$
$$\leq 2\,|\Psi|\,|F(\Psi, t)|$$
$$\leq 2L|\Psi|^2 \tag{A.4}$$

But (A. 4) implies that

$$|\Psi(t; \Psi_o, t_o)|^2 \leq |\Psi_o|^2\, e^{2L(t-t_o)}, \qquad \Psi(t_o; \Psi_o, t_o) = \Psi_o \tag{A.5}$$

and hence

$$|\Psi(t; \Psi_o, t_o)| \leq |\Psi_o|\, e^{L(t-t_o)}$$

From (A. 4) it follows that

$$\frac{d}{dt}(|\Psi|^2) \geq -2L|\Psi|^2$$

which similarly gives

$$|\Psi(t; \Psi_o, t_o)| \geq |\Psi_o|\, e^{-L(t-t_o)}$$

<u>Lemma A. 2</u> Let $F(\Psi, t) \in \mathcal{L}$ in $H_{h,\tau}$, $\tau \geq 0$ for the system (A. 1), and denote by $\dfrac{\partial\Psi(t; \Psi_o, t_o)}{\partial\psi_{o_i}}$, $t_o \geq \tau$ the vector of partial derivatives of the solution $\Psi(t; \Psi_o, t_o)$ of (A. 1) with respect to the i-th component of the vector Ψ_o at time t. Then

$$\left|\frac{\partial\Psi(t; \Psi_o, t_o)}{\partial\psi_{o_i}}\right| \leq e^{L(t-t_o)}, \quad i = 1, 2, \ldots, n \tag{A.6}$$

in $H_{h,\tau}$ where L is the Lipschitz constant for $F(\Psi, t)$ in $H_{h,\tau}$.

Proof: Since $F(\Psi, t) \in \mathscr{B}$, the partial derivatives exist and are continuous [CO-1]. Let Ψ'_o be a point in H_h such that

$$|\Psi_o - \Psi'_o| < \delta \tag{A.7}$$

Let $\Psi(t)$ and $\Psi'(t)$ be solutions of (A.1) passing through Ψ_o and Ψ'_o respectively at $t=t_o$. Then they may be written as

$$\Psi(t) = \Psi(t_o) + \int_{t_o}^{t} F(\Psi(s), s)ds$$

and

$$\Psi'(t) = \Psi'(t_o) + \int_{t_o}^{t} F(\Psi'(s), s)ds$$

so that

$$\Psi(t) - \Psi'(t) = \Psi(t_o) - \Psi'(t_o) + \int_{t_o}^{t} [F(\Psi(s), s) - F(\Psi'(s), s)]ds$$

or

$$|\Psi(t) - \Psi'(t)| \leq |\Psi(t_o) - \Psi'(t_o)| + \int_{t_o}^{t} |F(\Psi(s), s) - F(\Psi'(s), s)|ds$$

Let

$$r(t) = |\Psi(t) - \Psi'(t)|$$

so that

$$r(t) \leq r(t_o) + L \int_{t_o}^{t} r(s)ds \tag{A.8}$$

since $F(\Psi, t) \in \mathcal{C}$.

Defining

$$R(t) = \int_{t_o}^{t} r(s)ds$$

and using (A. 7), the relation (A. 8) becomes

$$\dot{R}(t) - LR(t) \leq \delta \qquad (A.9)$$

Multiplying both sides of (A. 9) by $e^{-L(t-t_o)}$ and integrating the resulting expression from t_o to t, gives

$$e^{-L(t-t_o)} R(t) \leq \frac{\delta}{L} (1 - e^{-L(t-t_o)})$$

and hence

$$R(t) \leq \frac{\delta}{L} (e^{L(t-t_o)} - 1) \qquad (A.10)$$

Substituting (A. 10) into (A. 8) gives

$$r(t) \leq \delta e^{L(t-t_o)}$$

Finally

$$\left| \frac{\partial \Psi (t; \Psi_o, t_o)}{\partial \psi_{o_i}} \right| \leq \lim_{\delta \to 0} \left| \frac{\Psi(t) - \Psi'(t)}{\delta} \right| \leq e^{L(t-t_o)}$$

<u>Lemma A. 3</u> Let A(t) be an arbitrary real, continuous, bounded n x n matrix. Then

$$\sigma_1 X^T X \leq X^T A(t) X \leq \sigma_n X^T X \qquad (A.11)$$

where σ_1, σ_n are the minimum and maximum values assumed by any of the eigenvalues of the symmetric part of the matrix A(t).

Proof: Let $A_s(t)$ denote the symmetric part of A(t), i.e.,

$$A_s(t) = \frac{1}{2}(A(t) + A(t)^T)$$

then

$$X^T AX = X^T A_s(t)X$$

For each $t \in [t_o, \infty)$ one has defined a constant matrix A_{s_t} for which it is known that [BEL-2]

$$\lambda_{1_t} = \min_X \frac{X^T A_{s_t} X}{X^T X}$$

$$\lambda_{n_t} = \max_X \frac{X^T A_{s_t} X}{X^T X}$$

where λ_{1_t} and λ_{n_t} are the minimum and maximum eigenvalues of A_{s_t}, respectively. Thus if

$$\sigma_1 = \min_t \lambda_{1_t}$$

and

$$\sigma_n = \max_t \lambda_{n_t}$$

then one has

$$\sigma_1 X^T X \leq X^T A(t)X \leq \sigma_n X^T X$$

Lemma A.4 For any two real numbers a and b

$$ab \leq \frac{1}{2}(a^2 + b^2) \tag{A.12}$$

Proof: (A.12) follows from $(a-b)^2 \geq 0$

Lemma A.5 (Cauchy-Schwartz Inequality) For any two real vectors X and Y one has

$$(X^T Y)^2 \leq (X^T X)(Y^T Y) \tag{A.13}$$

Lemma A.6 For any real vector X with components x_i, $i = 1, 2, \ldots, n$, one has

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n \sum x_i^2 \tag{A.14}$$

Proof: Inequality (A.14) follows from Lemma A.5 by taking $y_i = 1$, $i = 1, 2, \ldots, n$.

Lemma A.7 For all real constants a, b and all $\Psi \in E^n$,

$$a|\Psi|^2 + b|\Psi| \leq (k+a)|\Psi|^2 + \frac{b^2}{4k} \tag{A.15}$$

if $k > 0$.

Proof: If $b = 0$, then clearly (A.15) is satisfied. Assume $b \neq 0$ and consider the quadratic inequality

$$a|\Psi|^2 + b|\Psi| \leq d|\Psi|^2 + e$$

or

$$(d-a)|\Psi|^2 - b|\Psi| + e \geq 0 \tag{A.16}$$

But (A.16) is satisfied for all Ψ, if and only if, $(d-a) > 0$ and

$$b^2 - 4(d-a)e \leq 0 \tag{A.17}$$

Let $d = k+a$ for any $k > 0$. Then with

$$e = \frac{b^2}{4(d-a)} \tag{A.18}$$

(A.16) is satisfied and the result (A.15) follows.