SOME ANALOGS OF THE PICONE IDENTITY APPLIED TO FOURTH ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

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In this paper we obtain Sturmian-type theorems and lower bounds for eigenvalues of some fourth order problems. Our results are attained by using some generalizations of the classical Picone identity, which are themselves derived from a generalized technique of Picard.

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To Mom and Dad, who gave me
life and the love of it.

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INTRODUCTION

Sturm-type comparison theorems and lower bounds for eigenvalues will be considered for various fourth order linear ordinary differential equations. The results are derived from certain extensions of Picone's [26] classical identity. In each case our studies are motivated by well known results for similar problems in connection with second order linear ordinary differential equations.

0.1 Sturmian Theorems

The classical Sturm-Picone comparison theorem (see [15], pp. 225-226) asserts that if u is a nontrivial solution to the boundary value problem

$$(a(x)u')' + c(x)u = 0 \quad \text{in} \quad (0,\ell)$$

$$u(0) = u(\ell) = 0$$

and if v is a solution of the equation

$$(A(x)v')' + C(x)v = 0 \quad \text{in} \quad (0,\ell)$$

where the coefficients satisfy the relationship

$$a \geq A > 0, \; C \geq c \quad \text{in} \quad (0,\ell)$$

then v must have a zero in $(0,\ell)$ unless u and v are linearly dependent. This fact follows readily from the Picone identity ([15], p. 226)

$$(0.1)\qquad (auu' - A\frac{u^2v'}{v})\Big|_0^\ell = \int_0^\ell [u(au')' - u^2\frac{(Av')'}{v} + A(u' - \frac{uv'}{v})^2 + (a - A)(u')^2]dx.$$

For the fourth order equations

$$(0.2)\qquad (a(x)u'')'' - c(x)u = 0$$

$$(0.3)\qquad (A(x)v'')'' - C(x)v = 0$$

the situation is somewhat more difficult as is evidenced by the following theorem of Leighton and Nehari ([22], p. 327).

<u>Theorem</u>. If v is a solution of

$$(0.3)\qquad (A(x)v'')'' - C(x)v = 0$$

with $A > 0$ and $C > 0$, and if the values of v, v', v'', and $(Av'')'$ are nonnegative but not all zero at $x = 0$, then the functions $v(x)$, $v'(x)$, $v''(x)$, and $(A(x)v''(x))'$ are all positive for $x > 0$.

Hence every equation of the type (0.3) has at least one solution without zeros. Therefore, if u is a nontrivial solution of

$$(au'')'' - cu = 0 \quad \text{in} \quad (0,\ell)$$

$$u(0) = u'(0) = u(\ell) = u'(\ell),$$

then it cannot be asserted that every solution v of (0.3) has a zero in $[0,\ell]$.

As a consequence, fourth order equations are studied either in the context of oscillatory behavior, as was done by Leighton and Nehari [22], or by considering classes of solutions which exclude those of the type occurring in the theorem, as was done by Diaz and Dunninger [5, 6], Dunninger [10], Kreith [17, 18, 20], Swanson [31], and Wong [35]. We shall follow the latter course.

In Chapter II we obtain Sturmian-type results for the fourth order equations (0.2) and (0.3) under the assumption that solutions v of (0.3) are positive at a point and satisfy $v'' \leq 0$ in the interval $(0,\ell)$. In fact, our results will be established for certain systems of second order equations which contain (0.2) and (0.3) as special cases.

Some oscillation results are also presented which compare the oscillatory behavior of (0.2) and a related second order equation.

Aside from resolving the above complication, (namely, the formulation of a fourth order Sturmian result) there has been difficulty in obtaining a natural analog of the second order Picone identity (0.1) which can be used to treat fourth order equations. One such identity was presented by Cimmino in 1930 [3], and later by Leighton [21] and Kreith [20], another by Kreith in 1969 [18], and a third by Dunninger in 1971 [10]. It is of interest that each of these identities as well as some new identities, can be derived from a more general identity which results from a technique due to Picard [25]. We shall carry this out in Chapter I.

0.2 Lower Bounds for Eigenvalues

If u is an eigenfunction corresponding to the lowest eigenvalue λ_1 of the problem

$$Lu \equiv (a(x)u')' + c(x)u = -\lambda\rho(x)u \qquad a > 0,\ \rho > 0 \quad \text{in} \quad (0,\ell)$$

$$u(0) = u(\ell) = 0,$$

then it is an immediate consequence of Picone's identity (0.1) that

$$\lambda_1 \geq \inf_{(0,\ell)} \frac{Lv}{\rho v}$$

where the infimum is taken over functions $v > 0$ in $[0,\ell]$. Lower bounds, of this nature, for eigenvalues were first established by Barta [2] and later considered by Duffin [8], Dunninger [10], Hersch [12, 13], Hersch and Payne [14], Ogawa and Protter [23], Protter [27], and Protter and Weinberger [30].

In Chapter III we establish analogous results for some fourth order eigenvalue problems.

Further lower bounds are established by comparing the lowest eigenvalue of fourth order problems with the lowest eigenvalues of related second order problems. These results are motivated by some recent works of Protter [28] and Hersch [13].

CHAPTER I

MAXIMUM PRINCIPLES AND BASIC IDENTITIES

I.1 Maximum Principles

In this paper we shall make use of some well known maximum principles (see e.g. [29], pp. 6-9) which for our purposes may be stated as follows.

Let u be a classical solution of the differential inequality

$$u'' + c(x)u \geq 0 \tag{1.1}$$

in a bounded interval $(0,\ell)$. Let $c(x) \in C[0,\ell]$.

<u>Principle I</u>. Assume $c(x) \leq 0$ in $(0,\ell)$. If u attains a nonnegative maximum value M at an interior point of $(0,\ell)$, then $u \equiv M$.

<u>Principle II</u>. Assume $c(x) \leq 0$ in $(0,\ell)$. If u (non-constant) has one-sided derivatives at $x = 0$ and $x = \ell$ and if $u \leq M$ in $(0,\ell)$ where $M \geq 0$, then $u'(0) < 0$ if $u(0) = M$, whereas $u'(\ell) > 0$ if $u(\ell) = M$.

<u>Principle III</u>. Assume there exists a classical solution $\zeta(x)$ of the differential inequality

$$\zeta'' + c(x)\zeta \leq 0$$

in $(0,\ell)$ such that $\zeta(x) > 0$ in $[0,\ell]$. Then the function u/ζ satisfies Principles I and II.

Remark. Analogous results hold for solutions of

$$u'' + c(x)u \leq 0$$

in $(0,\ell)$, yielding an associated minimum principle. These principles are obtained by applying the above results to the function $(-u)$.

I.2 Identities

Various integral identities have been employed by several authors ([3], [10], [18], [20], [21]) to obtain Sturmian-type comparison theorems for fourth order differential equations. We shall now indicate how these identities, as well as a new identity, may be derived from a single integral identity.

The method we use is a generalization of a technique used by Picard ([25], p. 151) in connection with certain problems for second order equations. For completeness we show that Picard's method readily yields Picone's identity

$$(1.2) \qquad \left(uu' - u^2 \frac{v'}{v}\right)\Big|_0^{\ell} = \int_0^{\ell}\left[\left(uu'' - u^2 \frac{v''}{v}\right) + \left(u' - u \frac{v'}{v}\right)^2\right]dx,$$

where for simplicity we have let $a \equiv A \equiv 1$ in (0.1).

To Green's identity

$$uu'\Big|_0^{\ell} = \int_0^{\ell}[uu'' + (u')^2]dx$$

we add the following identity for an arbitrary sufficiently smooth function P

$$Pu^2\Big|_0^{\ell} = \int_0^{\ell}(2Puu' + P'u^2)dx$$

and complete squares to obtain

$$(uu' + Pu^2)\Big|_0^{\ell} = \int_0^{\ell}[uu'' + (u' + Pu)^2 + (P' - P^2)u^2]dx \ .$$

Upon setting

$$P = -\frac{v'}{v} \ ,$$

which incidentally is a solution to the Riccati-type equation

$$P' - P^2 = -\frac{v''}{v},$$

we obtain the Picone identity (1.2).

In order to derive the fourth order identities, we begin with the Green's identity

$$(1.3) \qquad (uu''' - u'u'')\Big|_0^{\ell} = \int_0^{\ell}[uu^{(4)} - (u'')^2]dx$$

and add the obvious identity

$$(1.4) \qquad (P_1u^2 + P_2(u')^2 + 2P_3uu')\Big|_0^{\ell} = \int_0^{\ell}[(P_1u^2)' + (P_2(u')^2)' + (2P_3uu')']dx$$

for arbitrary sufficiently smooth functions P_1, P_2, and P_3. We thus obtain

$$(1.5) \qquad \begin{aligned}&[uu''' - u'u'' + P_1u^2 + P_2(u')^2 + 2P_3uu']\Big|_0^{\ell} \\ &= \int_0^{\ell}[uu^{(4)} - (u'')^2 + P_1'u^2 + 2P_1uu' + P_2'(u')^2 + 2P_2u'u'' \\ &\qquad + 2P_3'uu' + 2P_3(u')^2 + 2P_3uu'']dx .\end{aligned}$$

Guided by the form of the Picone identity (1.2) for second order equations we attempt to choose the functions P_1, P_2, and P_3 so that the integrand in (1.5) will be analogous in form to that of (1.2).

A. Cimmino's identity.

Recalling that the Wronskian $W(\alpha_1,\alpha_2,\ldots,\alpha_n)$ of the n functions $\alpha_1,\alpha_2,\ldots,\alpha_n$ is defined as

$$W(\alpha_1,\alpha_2,\ldots,\alpha_n) = \begin{vmatrix} \alpha_1 & \alpha_2 & & \alpha_n \\ \alpha_1' & \alpha_2' & & \alpha_n' \\ \vdots & & & \vdots \\ \alpha_1^{(n-1)} & \alpha_2^{(n-1)} & \cdots & \alpha_n^{(n-1)} \end{vmatrix} , \quad W(\alpha_1) = \alpha_1 ,$$

we observe that the integral in identity (1.2) can be written in the form

$$\int_0^{\ell} \{(uu'' - \frac{u^2}{v} v'') + [\frac{W(v,u)}{W(v)}]^2\}dx .$$

It is then natural to seek a fourth order identity with an integral of the form

(1.6) $$\int_0^{\ell} \{(uu^{(4)} - \frac{u^2}{v} v^{(4)}) - [\frac{W(u,y,v)}{W(y,v)}]^2\}dx$$

where u is a nontrivial solution of

(1.7) $$u^{(4)} - c(x)u = 0 \quad \text{in} \quad (0,\ell) ,$$

and y and v are linearly independent solutions of the boundary value problem

(1.8) $$v^{(4)} - C(x)v = 0 \quad \text{in} \quad (0,\ell)$$

(1.9) $$v(0) = v(\ell) = v'(0) = v'(\ell) = 0 .$$

Observing that

$$W(u,y,v) = \begin{vmatrix} u & y & v \\ u' & y' & v' \\ u'' & y'' & v'' \end{vmatrix} = u \begin{vmatrix} y' & v' \\ y'' & v'' \end{vmatrix} - u' \begin{vmatrix} y & v \\ y'' & v'' \end{vmatrix} + u'' \begin{vmatrix} y & v \\ y' & v' \end{vmatrix}$$

we find, upon defining

$$\sigma = \begin{vmatrix} y & v \\ y' & v' \end{vmatrix} \quad , \quad \omega = \begin{vmatrix} y' & v' \\ y'' & v'' \end{vmatrix}$$

that

$$W(u,y,v) = u\omega - u'\sigma' + u''\sigma$$

and

$$W(y,v) = \sigma \ .$$

The integral (1.6) can thus be written

$$(1.10) \qquad \int_0^{\ell} [(uu^{(4)} - u^2 \frac{v^{(4)}}{v}) - (u'' - \frac{\sigma'}{\sigma} u' + \frac{\omega}{\sigma} u)^2] dx \ .$$

By rearranging terms, the right hand side of (1.5) can be written as

$$(1.11) \qquad \begin{aligned} &\int_0^{\ell} [uu^{(4)} - (u'' - P_2 u' - P_3 u)^2 + (u')^2 (2P_3 + P_2' + P_2^2) \\ &\quad + 2uu'(P_2 P_3 + P_1 + P_3') + u^2 (P_1' + P_3^2)] dx, \end{aligned}$$

and by comparing (1.11) with (1.10), we are led to choose

$$P_2 = \frac{\sigma'}{\sigma}$$

and

$$P_3 = -\frac{\omega}{\sigma} \ .$$

It follows (see e.g. [21]) from (1.7), (1.8), (1.9), and the definitions of σ and ω that

$$(1.12) \qquad \sigma'' - 2\omega = 0 \quad \text{in} \quad (0,\ell)$$

$$\omega^2 + C\sigma^2 + \sigma\omega'' - \omega'\sigma' = 0 \quad \text{in} \quad (0,\ell). \tag{1.13}$$

In view of (1.12) we find that

$$2P_3 + P_2' + P_2^2 = 0 \ .$$

Furthermore, the choice

$$P_1 = \frac{\omega'}{\sigma}$$

implies that

$$P_2P_3 + P_1 + P_3' = 0 \ ,$$

and then (1.13) implies

$$P_1' + P_3^2 = \frac{\sigma\omega'' - \omega'\sigma' + \omega^2}{\sigma^2} = -C = -\frac{v^{(4)}}{v} \ .$$

With the above choices of P_1, P_2, and P_3 identity (1.5) reduces to

$$\begin{aligned}[t] &\left[\frac{u}{\sigma}(\sigma u''' - \omega u') - \frac{u'}{\sigma}(\sigma u'' - \sigma' u') - \frac{u}{\sigma}(\omega u' - u\omega')\right]\Big|_0^\ell \\ &= \int_0^\ell \left[\left(uu^{(4)} - u^2\frac{v^{(4)}}{v}\right) - \left(u'' - \frac{\sigma'}{\sigma}u' + \frac{\omega}{\sigma}u\right)^2\right]dx \end{aligned} \tag{1.14}$$

which is the desired identity.

This identity was first introduced by Cimmino in 1930 [3], and was recently found by Leighton [21] and Kreith [20].

B. Kreith's identity.

Upon setting $P_3 = 0$, the right hand side of (1.5) reduces to

(1.15) $$\int_0^\ell [uu^{(4)} - (u'' - P_2u')^2 + (u')^2(P_2' + P_2^2) + 2P_1uu' + P_1'u^2]dx.$$

Choosing

$$P_2 = \frac{v''}{v'}$$

the expression (1.15) becomes

(1.16) $$\int_0^\ell [uu^{(4)} - (u'' - u' \frac{v''}{v'})^2 + \frac{v'''}{v'} (u' + \frac{P_1v'}{v'''} u)^2 + (P_1' - P_1^2 \frac{v'}{v'''})u^2]dx.$$

Hence, by the choice

$$P_1 = - \frac{v'''}{v}$$

(1.16) becomes

$$\int_0^\ell [uu^{(4)} - (u'' - \frac{u'v''}{v'})^2 + \frac{v'''}{v'} (u' - \frac{uv'}{v})^2 - \frac{u^2v^{(4)}}{v}]dx .$$

With the above choices of P_1 and P_2 the identity (1.5) now yields Kreith's identity [18]

(1.17) $$[\frac{u}{v} (vu''' - uv''') - \frac{u'}{v'} (v'u'' - u'v'')]\Big|_0^\ell = \int_0^\ell [(uu^{(4)} - u^2 \frac{v^{(4)}}{v}) - (u'' - u' \frac{v''}{v'})^2 + \frac{v'''}{v'} (u' - \frac{uv'}{v})^2]dx.$$

C. Dunninger's identity.

Upon setting $P_2 = 0$, the right hand side of (1.5) reduces to

(1.18) $$\int_0^\ell [uu^{(4)} - (u'' - P_3u)^2 + 2P_3(u')^2 + 2(P_1 + P_3')uu' + (P_1' + P_3^2)u^2]dx.$$

Choosing

$$P_3 = \frac{v''}{v}$$

the expression (1.18) becomes

$$\int_0^{\ell}\{uu^{(4)} - (u'' - \frac{uv''}{v})^2 + 2\frac{v''}{v}[u' + \frac{v}{2v''}(P_1 + \frac{v'''}{v} - \frac{v'v''}{v^2})u]^2 \tag{1.19}$$
$$+ u^2[P_1' + (\frac{v''}{v})^2 - \frac{v}{2v''}(P_1 + \frac{v'''}{v} - \frac{v'v''}{v^2})^2]\}dx\ .$$

Hence, by the choice

$$P_1 = -\frac{(v'v'' + vv''')}{v^2}$$

(1.19) becomes

$$\int_0^{\ell}[uu^{(4)} - (u'' - \frac{uv''}{v})^2 + 2\frac{v''}{v}(u' - \frac{uv'}{v})^2 - u^2\frac{v^{(4)}}{v}]dx.$$

With the above choices of P_1 and P_3, the identity (1.5) now yields Dunninger's identity [10]

$$[\frac{u}{v}(vu''' - uv''') + \frac{v''}{v}[\frac{u}{v}(vu' - uv')] - \frac{u'}{v}(vu'' - uv'')]\Big|_0^{\ell}$$
$$(1.20)\qquad = \int_0^{\ell}[(uu^{(4)} - \frac{u^2v^{(4)}}{v}) - (u'' - \frac{uv''}{v})^2 + 2\frac{v''}{v}(u' - u\frac{v'}{v})^2]dx.$$

D. A new identity.

If we now set $P_1 = P_3 = 0$ the identity (1.5) reduces to

$$[uu''' - u'u'' + P_2(u')^2]\Big|_0^{\ell} = \int_0^{\ell}[uu^{(4)} - (u'' - P_2u')^2 \tag{1.21}$$
$$+ (P_2^2 + P_2')(u')^2]dx\ .$$

Upon choosing

$$P_2 = \frac{v'}{v}$$

we arrive at the following new identity

$$(1.22) \quad [uu''' - u'u'' + \frac{v'(u')^2}{v}]\Big|_0^{\ell} = \int_0^{\ell}[uu^{(4)} + \frac{v''}{v}(u')^2 - (u'' - \frac{u'v'}{v})^2]dx .$$

Remark: Although the identities, as they have been presented, can only be applied to equations of the form

$$u^{(4)} - c(x)u = 0$$

$$v^{(4)} - C(x)v = 0,$$

they may easily be extended to cover the more general equations

$$(1.23) \qquad (a(x)u'')'' + 2(b(x)u')' - c(x)u = 0$$

$$(1.24) \qquad (A(x)v'')'' + 2(B(x)v')' - C(x)v = 0.$$

We shall illustrate this point later on.

In Chapter II we will show how Dunninger's identity (1.20) can be extended to an identity which is valid for systems of second order equations which contain the equations (1.23) and (1.24) as a special case. Moreover, we shall obtain Sturmian-type comparison theorems for such systems.

The identity (1.22) (actually a slight generalization of it) will be used subsequently in connection with various oscillation problems (Chapter II) and eigenvalue problems (Chapter III).

CHAPTER II

COMPARISON AND OSCILLATION

II.1 Sturm-type Comparison Theorems

In this section and the succeeding sections all solutions are assumed to be classical, and any coefficients appearing are assumed sufficiently differentiable. No further mention of these facts will be made.

Consider the fourth order equations

$$(2.1)\qquad (a(x)u'')'' + 2(\hat{b}(x)u')' - \hat{c}(x)u = 0 \quad \text{in} \quad (0,\ell)$$

$$(2.2)\qquad (A(x)v'')'' + 2(B(x)v')' - C(x)v = 0 \quad \text{in} \quad (0,\ell).$$

Upon setting

$$e = \frac{1}{a} \qquad\qquad f = \frac{1}{A}$$

$$b = \frac{\hat{b}}{a} \qquad\qquad d = \frac{B}{A}$$

$$c = -(\hat{c} + \frac{\hat{b}^2}{a} + \hat{b}'') \qquad\qquad h = -(C + \frac{B^2}{A} + B''),$$

Whyburn showed in 1930 [33] that (2.1) and (2.2) were equivalent to the following systems

$$(2.3)\qquad \begin{aligned} u'' + bu &= ew \\ w'' + bw &= -cu \end{aligned} \qquad \text{in} \quad (0,\ell)$$

$$v'' + dv = fz$$

(2.4) $$\text{in } (0,\ell),$$

$$z'' + dz = -hv$$

respectively, when a and A, and hence e and f, never vanish in $(0,\ell)$.

In what follows we shall consider the systems (2.3) and (2.4) under somewhat less restrictive hypotheses. Namely, we do not require e and f to have fixed signs. Consequently, the systems (2.3) and (2.4) will be more general than the equations (2.1) and (2.2)

To compare solutions of (2.3) and (2.4) we derive an identity which is closely related to Dunninger's identity (1.20). We could again use Picard's technique. However, proceeding more directly, and being motivated by the form of the left hand side of (1.20) we consider the following expression

$$\left\{\frac{u}{v}(vw' - uz') + \frac{z}{v}\left[\frac{u}{v}(vu' - uv')\right] - \frac{u'}{v}(vw - uz)\right\}\Big|_0^\ell$$

(2.5)
$$= \int_0^\ell \left\{\frac{u}{v}(vw' - uz') + \frac{z}{v}\left[\frac{u}{v}(vu' - uv')\right] - \frac{u'}{v}(vw - uz)\right\}'dx.$$

Upon performing the differentiation under the integral, rearranging terms, and making use of (2.3) and (2.4) the right hand side of (2.5) becomes

$$\int_0^\ell \left[uw'' - u''w + \frac{2z}{v}\left(u' - u\frac{v'}{v}\right)^2 - u^2\frac{z''}{v} + 2uu''\frac{z}{v} - u^2\frac{v''z}{v^2}\right]dx$$

$$= \int_0^\ell \left[(h-c)u^2 + \frac{2z}{v}\left(u' - u\frac{v'}{v}\right)^2 + 2(d-b)u^2\frac{z}{v} - ew^2 + 2eu\frac{wz}{v}\right.$$
$$\left. - f\frac{u^2z^2}{v^2}\right]dx.$$

Then completing squares on the last three terms and substituting back into (2.5) we obtain the following formal identity for solution pairs (u,w) and (v,z) of (2.3) and (2.4), respectively

$$\left[\frac{u}{v}(vw' - uz') + \frac{z}{v}\left[\frac{u}{v}(vu' - uv')\right] - \frac{u'}{v}(vw - uz)\right]\Big|_0^{\ell}$$

(2.6)

$$= \int_0^{\ell}\{(h-c)u^2 - e(w - \frac{uz}{v})^2 + \frac{2z}{v}(u' - \frac{uv'}{v})^2 + 2(d-b)u^2\frac{z}{v}$$

$$+ (e-f)(\frac{uz}{v})^2\}dx.$$

Theorem 2.1. Suppose there exists a function w such that u is a nontrivial solution of (2.3), and suppose there exists a function z such that v satisfies system (2.4) and $v > 0$ at some point in $(0,\ell)$. If

(1) $f \geq e \geq 0$, $d \geq b$ in $(0,\ell)$,

and either

(2) $\int_0^{\ell}(c-h)u^2dx \geq 0$

(3) $z < 0$ in $(0,\ell)$

(4) $u \not\equiv kv$, k a constant

or

(5) $\int_0^{\ell}(c-h)u^2dx > 0$

(6) $z \leq 0$ in $(0,\ell)$

then, under any boundary conditions on u, v, w, and z such that the left hand side of identity (2.6) is nonnegative, v must have a zero at some point in $[0,\ell]$.

Proof. Suppose to the contrary that v has no zero in $[0,\ell]$. Since $v > 0$ at some point in $(0,\ell)$, we have $v > 0$ in $[0,\ell]$, and the identity (2.6) is valid. In view of the hypotheses (1), (2), and (3) and the fact that the left side of (2.6) is non-negative, we readily find that

$$0 \leq \int_0^\ell \frac{z}{v}\left(u' - \frac{uv'}{v}\right)^2 dx \ . \tag{2.7}$$

On the other hand hypothesis (3) and the fact that $v > 0$ in $[0,\ell]$ imply

$$\int_0^\ell \frac{z}{v}\left(u' - \frac{uv'}{v}\right)^2 dx \leq 0 \ . \tag{2.8}$$

Inequalities (2.7) and (2.8) together imply that

$$\left(\frac{u}{v}\right)' = \frac{1}{v}\left(u' - \frac{uv'}{v}\right) \equiv 0$$

which contradicts hypothesis (4).

In a similar manner hypotheses (5) and (6) lead to the contradiction

$$0 \leq \int_0^\ell (h-c)u^2 dx < 0 \ .$$

Hence, in either case, v must vanish at some point in $[0,\ell]$.

Remark. Theorem 2.1 holds for a variety of general boundary conditions. Of importance, from a physical point of view, we cite the following boundary conditions

(I) $u(0) = u'(0) = u(\ell) = u'(\ell) = 0$

(II) $u(0) = w(0) = u(\ell) = w(\ell) = 0$

which correspond to clamped ends and supported ends respectively in the boundary value problem for the vibrating rod (see [4], pp. 295-296).

Remark. Under boundary conditions (I) or (II) the conclusion of Theorem 2.1 is valid without hypothesis (4). Indeed, if v has no zero in $[0,\ell]$, then $v > 0$ in $[0,\ell]$. But $u(0) = u(\ell) = 0$, and hence u and v must be linearly independent.

Our next results show that under some additional hypotheses, namely that the coefficient d is nonpositive, when the boundary conditions (I) or (II) hold, the conclusion of Theorem 2.1 can be sharpened to assert that v must have a zero in the open interval $(0,\ell)$.

Theorem 2.2. Suppose there exists a function w such that u is a nontrivial solution of (2.3), and suppose there exists a function z such that v satisfies system (2.4) and $v > 0$ at some point in $(0,\ell)$. If

(1) $f \geq e \geq 0 \ , \ 0 \geq d \geq b$ in $(0,\ell)$

and either

$$\begin{cases} (2) & \int_0^\ell (c-h)u^2 dx \geq 0 \\ (3) & z < 0 \text{ in } (0,\ell) \end{cases}$$

or

$$\begin{cases} (4) & \int_0^\ell (c-h)u^2 dx > 0 \\ (5) & z \leq 0 \text{ in } (0,\ell), \end{cases}$$

and if u satisfies boundary condition

(I) $u(0) = u'(0) = u(\ell) = u'(\ell) = 0,$

then v must vanish in $(0,\ell)$.

Proof. Suppose v does not vanish in $(0,\ell)$. Then $v > 0$ in $(0,\ell)$, and by continuity $v \geq 0$ in $[0,\ell]$.

Consequently the identity (2.6) is valid on the interval (α,β) where $0 < \alpha < \beta < \ell$.

$$\left[\frac{u}{v}(vw' - uz') + \frac{z}{v}\left[\frac{u}{v}(vu' - uv')\right] - \frac{u'}{v}(vw - uz)\right]\Big|_{\alpha}^{\beta}$$

(2.9)
$$= \int_{\alpha}^{\beta}\{(h-c)u^2 - e(w - u\frac{z}{v})^2 + 2\frac{z}{v}(u' - \frac{uv'}{v})^2 + 2(d-b)u^2\frac{z}{v} + (e-f)(\frac{uz}{v})^2\}dx.$$

To establish that (2.6) is valid in $[0,\ell]$ we consider the various possible cases depending on the behavior of v at the boundary points.

Case I. $v(0) \neq 0$ and $v(\ell) \neq 0$.

In this case identity (2.6) is obviously valid in $[0,\ell]$, and moreover the boundary term is zero. Hence the conclusion follows from Theorem 2.1.

Case II. $v(0) = 0$ or $v(\ell) = 0$.

Suppose $v(0) = 0$ and $v(\ell) \neq 0$. From the first equation of system (2.4) and the hypotheses (1) and (3), or (1) and (5), it follows that

$$v'' + dv = fz \leq 0 .$$

Since $d \le 0$, Principle II implies $v'(0) > 0$. Hence an application of L'Hôpital's rule yields

(2.10)
$$\lim_{x\to 0^+} \frac{u}{v} = \lim_{x\to 0^+} \frac{u'}{v'} = 0$$

and consequently

$$\lim_{x\to 0^+} \frac{u}{v}(vw' - uz') = 0$$

$$\lim_{x\to 0^+} \frac{z}{v}\left[\frac{u}{v}(vu' - uv')\right] = \lim_{x\to 0^+} \frac{u}{v} \cdot \lim_{x\to 0^+} \left(zu' - \frac{u}{v} zv'\right) = 0$$

$$\lim_{x\to 0^+} \frac{u'}{v}(vw - uz) = \lim_{x\to 0^+} \left(u'w - \frac{u}{v} u'z\right) = 0 \ .$$

Another application of L'Hôpital's rule (if necessary) readily verifies that the integrand in (2.9) may be extended to a continuous function in the interval $[0,\ell]$. Therefore, it now follows that (2.9) is valid in $[0,\ell]$, and moreover the boundary term is zero. Since $u(\ell) = 0$ but $v(\ell) \neq 0$, u and v are linearly independent. Proceeding as in Theorem 2.1 we infer that v must have a zero in $(0,\ell)$. A similar argument is valid in the case $v(0) \neq 0$ and $v(\ell) = 0$.

<u>Case III</u>. $v(0) = v(\ell) = 0$.

By Principle II, $v'(0) \neq 0$ and $v'(\ell) \neq 0$. Since $u'(0) = u'(\ell) = 0$, u and v are linearly independent. The rest of the proof follows as in Case II.

<u>Corollary</u>. Suppose there exists a function w such that u is a nontrivial solution of system (2.3), and suppose there exists a function z such that v satisfies system (2.4) and $v > 0$ at some point in $(0,\ell)$. If

(1) $f \geq e \geq 0,\ 0 \geq d \geq b$ in $(0,\ell)$

(2) $\int_0^\ell (c-h)u^2 dx \geq 0$

(3') $z < 0$ at some point in $(0,\ell)$,

and if u satisfies boundary condition

(I) $u(0) = u'(0) = u(\ell) = u'(\ell) = 0$,

then either v or z must vanish at some point in $(0,\ell)$.

Proof. If z does not vanish in $(0,\ell)$, then by (3') $z < 0$ in $(0,\ell)$, and the result now follows from Theorem 2.2.

Remark. In Theorem 2.2 the hypothesis $d \leq 0$ was only used, in conjunction with Principle II, to obtain that v' is nonzero at an endpoint where v vanishes. However, this condition on the coefficient d can be removed if we assume the existence of a positive function ζ satisfying the hypotheses of Principle III. That is, we assume there exists a function $\zeta > 0$ in $[0,\ell]$ such that

$$\zeta'' + d\zeta \leq 0 \quad \text{in} \quad (0,\ell).$$

The following analysis shows that under this new hypothesis we can again conclude that v' is nonzero at endpoints where v is zero.

Since $z \leq 0$, we have from the first equation in system (2.4) that

$$v'' + dv = fz \leq 0.$$

Then if, for example, $v(0) = 0$, by Principle III

$$(\frac{v}{\zeta})'|_0 > 0.$$

Hence

$$0 < \frac{\zeta(0)v'(0) - v(0)\zeta'(0)}{\zeta^2(0)} = \frac{v'(0)}{\zeta(0)} \quad .$$

Consequently $v'(0) > 0$, and we can proceed to prove Theorem 2.2 as before.

Theorem 2.3. Suppose there exists a function w such that u is a nontrivial solution of (2.3), and suppose there exists a function z such that v satisfies system (2.4) and $v > 0$ at some point in $(0,\ell)$. If

(1) $f \geq e \geq 0,\ 0 \geq d \geq b$ in $(0,\ell)$

(2) $z(0) = z(\ell) = 0$

and either

$$\begin{cases} (3) & h \leq 0,\ h \not\equiv 0 \text{ in } (0,\ell) \\ (4) & \int_0^\ell (c-h)u^2 dx \geq 0 \end{cases}$$

or

$$\begin{cases} (5) & h \leq 0 \text{ in } (0,\ell) \\ (6) & \int_0^\ell (c-h)u^2 dx > 0, \end{cases}$$

and if u and w satisfy the boundary condition

(II) $u(0) = w(0) = u(\ell) = w(\ell) = 0$

then v must have a zero in $(0,\ell)$.

Proof. Suppose v does not vanish in $(0,\ell)$. Then $v > 0$ in $(0,\ell)$, and $v \geq 0$ in $[0,\ell]$. From the second equation in system (2.4) together with hypothesis (3)

$$z'' + dz = -hv \geq 0, \not\equiv 0 \quad \text{in} \quad (0,\ell).$$

Then hypothesis (2) and Principle I imply $z < 0$ in $(0,\ell)$. Similarly under hypothesis (5)

$$z'' + dz = -hv \geq 0 \quad \text{in} \quad (0,\ell),$$

and hypothesis (2) and Principle I imply $z \leq 0$ in $(0,\ell)$. The remainder of the proof consists of showing that (2.9) is valid in $[0,\ell]$ from which the conclusion follows. The details are similar to those shown in (2.10), and hence are omitted. Note that here we only know that

$$\lim_{x \to 0^+} \frac{u}{v} = \lim_{x \to 0^+} \frac{u'}{v'}$$

exists and is finite. Hypothesis (2) is used to show that the boundary term vanishes.

Remark. In the special case when the coefficients c,b, and e in (2.3) are identically equal to h,d, and f, respectively, in (2.4), the above theorems are separation theorems for the system (2.3). That is, under the given hypotheses, the zeros of linearly independent solutions of (2.3) separate each other.

Remark. In the case $f,e > 0$ the systems (2.3) and (2.4) can be transformed back to the fourth order equations (2.1) and (2.2) by the substitutions indicated on page 15. Consequently, our results contain some recent results of Dunninger [10].

Remark. Theorem 2.1 is also valid for the systems of differential inequalities

$$\begin{aligned} u'' + bu &= ew \\ uw'' + buw &\le -cu^2 \end{aligned} \qquad \text{in} \quad (0,\ell)$$

and

$$\begin{aligned} zv'' + dvz &\ge fz^2 \\ vz'' + dvz &\ge -hv^2 \ . \end{aligned} \qquad \text{in} \quad (0,\ell)$$

Furthermore, if $z < 0$ in Theorem 2.2, and if $h \not\equiv 0$ in Theorem 2.3, these theorems also hold for the above systems of inequalities.

Remark. We can treat in the same manner nonlinear differential inequalities by allowing coefficients $c(x,u,w)$, $b(x,u,w)$, $e(x,u,w)$, $d(x,v,z)$, $f(x,v,z)$, and $h(x,v,z)$ as long as the coefficients are continuous in x for all values of the other variables.

II.2 Oscillation Results

Under consideration now is the boundary value problem

$$(a(x)u'')'' - c(x)u = 0 \quad \text{in} \quad (0,\ell) \tag{2.11}$$

$$u(0) = u'(0) = u(\ell) - u'(\ell) = 0 \tag{2.12}$$

and the second order equation

$$(A(x)v')' + C(x)v = 0 \quad \text{in} \quad (0,\ell). \tag{2.13}$$

The identity (1.22)

$$[uu''' - u'u'' + \frac{v'}{v}(u')^2]\Big|_0^\ell = \int_0^\ell [uu^{(4)} + \frac{v''}{v}(u')^2 - (u'' - u'\frac{v'}{v})^2]dx$$

is readily generalized to the following identity.

$$\begin{aligned}&[u(au'')' - au'u'' + \frac{Av'}{v}(u')^2]\Big|_0^\ell \\ &= \int_0^\ell [u(au'')'' + \frac{(Av')'}{v}(u')^2 - A(u'' - u'\frac{v'}{v})^2 + (A-a)(u'')^2]dx.\end{aligned} \tag{2.14}$$

In fact, in Picard's technique, identity (2.14) is derived merely by adding the Green's identity

$$[u(au'')' - au'u'']\Big|_0^\ell = \int_0^\ell [u(au'')'' - a(u'')^2]dx$$

to the identity

$$P_2(u')^2\Big|_0^\ell = \int_0^\ell [P_2(u')^2]'dx = \int_0^\ell [P_2'(u')^2 + 2P_2u'u'']dx$$

to obtain

$$[u(au'')' - au'u'' + P_2(u')^2]\Big|_0^{\ell}$$

(2.15)
$$= \int_0^{\ell}[u(au'')'' - A(u'' - \frac{P_2}{A}u')^2 + (A-a)(u'')^2 + (P_2' + \frac{P_2^2}{A})(u')^2]dx \ .$$

Identity (2.14) follows upon setting

$$P_2 = \frac{Av'}{v} \ .$$

Provided identity (2.14) is valid, the following observation can be made.

<u>Remark</u>. If u is a solution of (2.11), (2.12) and v is a solution of (2.13) with $A > 0$, and if

$$\int_0^{\ell}(a-A)(u'')^2 dx \geq 0,$$

then identity (2.14) reduces to the inequality

(2.16)
$$0 \leq \int_0^{\ell}[cu^2 - C(u')^2]dx \ .$$

Moreover, equality in (2.16) implies, from (2.14), that

$$0 = \int_0^{\ell} A(u'' - \frac{u'v'}{v})^2 dx \ .$$

Then since $A > 0$, we conclude that

$$u'' - \frac{u'v'}{v} = v(\frac{u'}{v}) \equiv 0,$$

and hence $u' \equiv kv$ for some constant k.

We shall use inequality (2.16) to relate conjugate points of solutions to the second order equation (2.13) and solutions to the fourth order equation (2.11).

Definition. (cf. [34], p. 17). For the second order equation (2.13), the first conjugate point $\eta_1(x_0)$ of $x = x_0$ is the smallest $x_1 > x_0$ such that there exists a nontrivial solution v of (2.13) with

$$v(x_0) = v(x_1) = 0 \ .$$

Definition. (cf. [34], p. 82). For the fourth order equation (2.11), the first conjugate point $\hat{\eta}_1(x_0)$ of $x = x_0$ is the smallest $\hat{x}_1 > x_0$ such that there exists a nontrivial solution u of (2.11) with

$$u(x_0) = u'(x_0) = u(\hat{x}_1) = u'(\hat{x}_1) = 0.$$

Theorem 2.5. Let u be a nontrivial solution of (2.11) such that $\hat{\eta}_1(0)$ exists. Let v be a nontrivial solution of (2.13) such that $\eta_1(0)$ exists. If

(1) $A > 0$ in $(0, \hat{\eta}_1(0))$

(2) $\int_0^{\hat{\eta}_1(0)} (a-A)(u'')^2 dx \geq 0$

(3) $\int_0^{\hat{\eta}_1(0)} cu^2 dx \leq \int_0^{\hat{\eta}_1(0)} C(u')^2 dx$

then

$$\hat{\eta}_1(0) \geq \eta_1(0).$$

Proof. Suppose $\hat{\eta}_1(0) < \eta_0$. Then v does not vanish in $(0, \hat{\eta}_1(0)]$. Although $v(0) = 0$, $v'(0) \neq 0$ since nontrivial solutions of second order linear equations have simple zeros. Hence, as before, an application of L'Hôpital's rule as $x \to 0^+$ establishes the validity of identity (2.14).

Consequently hypotheses (1) and (2) together with

$$u(0) = u'(0) = u(\hat{\eta}_1(0)) = u'(\hat{\eta}_1(0)) = 0$$

imply inequality (2.16) holds for $\ell = \hat{\eta}_1(0)$. In view of hypothesis (3) we have equality in (2.16), and hence $u' \equiv kv$. Therefore $v(0) = v(\hat{\eta}_1(0)) = 0$, which contradicts the assumption that $\hat{\eta}_1(0) < \eta_1(0)$. Hence $\hat{\eta}_1(0) \geq \eta_1(0)$.

Inequality (2.16) can also be used to obtain a disconjugary result.

Definition. (cf. [34], p. 17). The second order equation (2.13) is said to be disconjugate in $[0,\ell)$, $0 < \ell \leq \infty$, if $\eta_1(0)$ does not exist in $[0,\ell)$.

Definition. (cf. [1]). The fourth order equation (2.11) is said to be disconjugate in $[0,\ell)$, $0 < \ell \leq \infty$, if $\hat{\eta}_1(0)$ does not exist in $[0,\ell)$.

Theorem 2.6. Suppose that

(1) $A > 0$ in $(0,\ell)$

(2) $\int_0^{\xi} (a-A)(u'')^2 dx \geq 0$

(3) $\int_0^{\xi} cu^2 dx \leq \int_0^{\xi} C(u')^2 dx$

for all $\xi \in (0,\ell)$, where u is any solution of (2.11) in $(0,\ell)$. If (2.13) is disconjugate in $[0,\ell)$, then (2.11) is also disconjugate in $[0,\ell)$.

<u>Proof</u>. Suppose, to the contrary, that (2.11) is not disconjugate in $[0,\ell)$. Then there exists a solution u of (2.11) such that for some $\hat{\eta}_1(0) \in (0,\ell)$

$$u(0) = u'(0) = u(\hat{\eta}_1(0)) = u'(\hat{\eta}_1(0)) = 0. \tag{2.17}$$

Since any v which satisfies (2.13) is disconjugate in $[0,\ell)$, v does not vanish in $(0, \hat{\eta}_1(0)]$. Again, by an application of L'Hôpital's rule at $x = 0$, identity (2.14) is valid in $[0, \hat{\eta}_1(0)]$. The boundary condition (2.17) and hypotheses (1) and (2) imply inequality (2.16) holds in $(0, \hat{\eta}_1(0))$. Moreover, in view of hypothesis (3) for $\xi = \hat{\eta}_1(0)$ we have equality in (2.16), and hence $u' \equiv kv$. But then by (2.17)

$$v(0) = v(\hat{\eta}_1(0)) = 0$$

which contradicts the fact that v is disconjugate in $[0,\ell)$. Therefore (2.11) must be disconjugate in $[0,\ell)$.

CHAPTER III

EIGENVALUE PROBLEMS

III.1 Barta-type Lower Bounds

In this section we shall be concerned with obtaining Barta-type [2] lower bounds for the smallest eigenvalues of the following problems

$$(I)\quad \begin{cases} \mathcal{L}u - \Omega u \equiv (a(x)u'')'' - c(x)u - \Omega u = 0 \quad \text{in} \quad (0,\ell) \\ u(0) = u'(0) = u(\ell) = u'(\ell) = 0 \end{cases}$$

$$(II)\quad \begin{cases} \mathcal{L}u - \Lambda u \equiv (a(x)u'')'' - c(x)u - \Lambda u = 0 \quad \text{in} \quad (0,\ell) \\ u(0) = u''(0) = u(\ell) = u''(\ell) = 0 \end{cases}$$

$$(III)\quad \begin{cases} (a(x)u'')'' + \gamma u'' = 0 \quad \text{in} \quad (0,\ell) \\ u(0) = u'(0) = u(\ell) = u'(\ell) = 0 \,. \end{cases}$$

Problem (I):

Let u be an eigenfunction which corresponds to the lowest eigenvalue Ω_1 of problem (I). If in addition u is positive in $(0,\ell)$, then the following Barta-type inequality is valid [9]

$$(3.1)\qquad \Omega_1 \geq \inf_{(0,\ell)} \frac{\mathcal{L}v}{v} \,.$$

Here v is any positive function which satisfies the boundary condition in (I).

The proof of (3.1) is quite simple. Indeed, from (I)

$$\int_0^{\ell} v(\mathcal{L}u - \Omega_1 u)\,dx = 0,$$

and hence an integration by parts yields

$$\int_0^{\ell} u(\mathcal{L}v - \Omega_1 v)\,dx = 0.$$

But $u > 0$ in $(0,\ell)$, and so (3.1) follows.

It is of interest to note that in general it is not known whether the eigenfunction corresponding to Ω_1 has a fixed sign in $(0,\ell)$ (see [9]). Hence, using a suitable modification of Dunninger's identity (1.20), we will construct similar bounds without this assumption.

If the operator L is defined as

$$Lv \equiv (A(x)v'')'' - C(x)v,$$

then we obtain the following formal identity which is a generalization of (1.20) (see [10]).

$$\begin{aligned} &\{\frac{u}{v}[v(au'')' - u(Av'')'] + \frac{Av''}{v}[\frac{u}{v}(vu' - uv')] \\ (3.2)\quad &+ \frac{u'}{v}(Auv'' - avu'')\}\Big|_0^{\ell} = \int_0^{\ell}[(A-a)(u'')^2 + (c-C)u^2]dx \\ &+ \int_0^{\ell}[2\frac{Av''}{v}(u' - \frac{uv'}{v})^2 - A(u'' - \frac{uv''}{v})^2]dx + \int_0^{\ell}\frac{u}{v}(v\mathcal{L}u - uLv)dx\ . \end{aligned}$$

<u>Theorem 3.1</u>. Suppose there exists a function v satisfying

(1) $v > 0$ in $(0,\ell)$

(2) $v'' \leq 0$ in $(0,\ell)$.

If u is an eigenfunction corresponding to the lowest eigenvalue Ω_1 of (I), and if $A \geq 0$ in $(0,\ell)$ and

$$\text{(3)} \quad V[u] \equiv \int_0^\ell [(a-A)(u'')^2 + (C-c)u^2]dx \geq 0,$$

then

$$\Omega_1 \geq \inf_{(0,\ell)} \frac{Lv}{v}. \tag{3.3}$$

<u>Proof</u>. We first note that from hypotheses (1) and (2) and Principle II, it follows that if $v(0) = 0$, then $v'(0) \neq 0$. Similarly if $v(\ell) = 0$, then $v'(\ell) \neq 0$. Consequently the validity of identity (3.2) can be established following the procedure in Theorem 2.2. In view of the above hypotheses (3.2) reduces to

$$\int_0^\ell (\Omega_1 - \frac{Lv}{v})u^2 dx = V[u] - \int_0^\ell [2\frac{Av''}{v}(u' - u\frac{v'}{v})^2 - A(u'' - u\frac{v''}{v})^2]dx \geq 0.$$

Hence,

$$\sup_{(0,\ell)} (\Omega_1 - \frac{Lv}{v}) \geq 0$$

from which inequality (3.3) follows.

<u>Remark</u>. The above theorem is a slight improvement of a recent result of Dunninger [10]. Namely, we have eliminated the hypothesis that v is positive at the boundary points.

<u>Remark</u>. In comparing inequalities (3.3) and (3.1) we note that we have not only succeeded in removing the fixed sign condition, but we have also removed the condition that v must satisfy

the boundary conditions in (I). However, we must pay for this by adding the requirement that $v'' \le 0$, and consequently, it is easily seen that the eigenfunction u is not an admissible function in the inequality (3.3).

<u>Remark.</u> Although we are not able to obtain a complementary upper bound, it should be pointed out that upper bounds are usually much more readily found. For example, the Rayleigh quotient characterization (see [11], p. 393) of the eigenvalue

$$\Omega_1 = \min \frac{\int_0^{\ell}[a(\varphi'')^2 - c\varphi^2]dx}{\int_0^{\ell} \varphi^2 dx} ,$$

where the minimum is taken over all functions φ satisfying the boundary conditions in (I), readily yields good upper bounds for Ω_1.

Problem (II):

Proceeding exactly as in Theorem 3.1 we can readily establish an analogous result for the eigenvalue problem (II).

<u>Theorem 3.2</u>. Suppose there exists a function v satisfying

(1) $v > 0$ in $(0,\ell)$

(2) $v'' \le 0$ in $(0,\ell)$,

and suppose $A \ge 0$ in $(0,\ell)$. If u is an eigenfunction corresponding to the lowest eigenvalue Λ_1 of problem (II), and if

(3) $V[u] \ge 0$,

then

$$\Lambda_1 \geq \inf_{(0,\ell)} \frac{Lv}{v} . \tag{3.4}$$

In the special case that $c \equiv 0$ we can obtain the following complementary upper bound.

Theorem 3.3. Suppose there exists a function v satisfying

(1) $v > 0$ in $(0,\ell)$

(2) $v(0) = v''(0) = v(\ell) = v''(\ell) = 0$

(3) $V[v] \leq 0$.

If u is an eigenfunction corresponding to the lowest eigenvalue Λ_1 of problem (II) with $c \equiv 0$ and $a > 0$, then

$$\Lambda_1 \leq \sup_{(0,\ell)} \frac{Lv}{v} . \tag{3.5}$$

Proof. Interchanging the roles of u and v, a and A, and c and C in identity (3.2) we obtain the identity

$$\begin{aligned} &\{\frac{v}{u}[u(Av'')' - v(au'')'] + a\frac{u''}{u}[\frac{v}{u}(uv' - vu')] \\ &+ \frac{v'}{u}(avu'' - Auv'')\}\Big|_0^\ell = \int_0^\ell [(a-A)(v'')^2 + (C-c)v^2]dx \\ &+ \int_0^\ell [2a\frac{u''}{u}(v' - v\frac{u'}{u})^2 - a(v'' - v\frac{u''}{u})^2]dx \\ &+ \int_0^\ell \frac{v}{u}(uLv - v\ell u)dx. \end{aligned} \tag{3.6}$$

We first must establish that (3.6) is valid. To this end it suffices to show that $u > 0$ in $(0,\ell)$. For then it readily follows from

(II), Principle I, and the fact that Λ_1 is positive that $u'' \leq 0$ in $(0,\ell)$. Hence the validity of (3.6) is established in the same way that (3.2) was established.

To obtain the positivity of u we first note that problem (II) can be expressed as the composition of the two problems

$$(3.7)\qquad \begin{aligned} u'' &= w/a \\ u(0) &= u(\ell) = 0 \end{aligned}$$

$$(3.8)\qquad \begin{aligned} w'' &= \Lambda_1 u \\ w(0) &= w(\ell) = 0. \end{aligned}$$

Let $G(x,\xi)$ denote the Green's function associated with the boundary value problem

$$(3.9)\qquad \begin{aligned} y'' &= f(x) \\ y(0) &= y(\ell) = 0. \end{aligned}$$

Then from (3.7) we have

$$(3.10)\qquad u(x) = -\int_0^\ell G(x,\xi)\,\frac{w(\xi)}{a(\xi)}\,d\xi,$$

and from (3.8) we have

$$(3.11)\qquad w(x) = -\int_0^\ell \Lambda_1 G(x,\xi)u(\xi)\,d\xi\ .$$

Inserting (3.11) into (3.10) we obtain

$$u(x) = \int_0^\ell\Big[\int_0^\ell \frac{\Lambda_1}{a(\xi)}\,G(x,\xi)G(\xi,\eta)\,d\xi\Big]u(\eta)\,d\eta\ .$$

Since $G(x,\xi) > 0$, it follows from Jentzsch's Theorem [16] that $u \geq 0$ in $(0,\ell)$. By applying Principle I to (3.8) and then to (3.7), it is easily seen that $u > 0$ in $(0,\ell)$.

Returning to identity (3.6), in view of the hypotheses, we obtain

$$0 \leq \int_0^\ell \left(\frac{Lv}{v} - \Lambda_1\right)v^2 dx$$

from which (3.5) follows.

Remark. Since $u > 0$ in $(0,\ell)$, it follows that u is an admissible function in (3.5). Hence upon setting $A \equiv a > 0$, $C \equiv c = 0$ in (3.5), equality holds and therefore

$$\Lambda_1 = \inf \{ \sup_{(0,\ell)} \frac{\mathcal{L}v}{v} \}$$

where the infimum is taken over the class of functions v satisfying $v > 0$ in $(0,\ell)$ and the boundary conditions $v(0) = v''(0) = v(\ell) = v''(\ell) = 0$.

Moreover, since $u'' \leq 0$ in $(0,\ell)$, it follows that u is also an admissible function in (3.4). Hence upon setting $A \equiv a > 0$, $C \equiv c = 0$ in (3.4), equality holds, and therefore

$$\Lambda_1 = \sup \{ \inf_{(0,\ell)} \frac{\mathcal{L}v}{v} \}$$

where the supremum is taken over the class of functions v satisfying $v > 0$, $v'' \leq 0$ in $(0,\ell)$.

Problem (III):

For eigenvalue problem (III) we obtain the following result from the identity (2.14) which is repeated here

$$[u(au'')' - au'u'' + A\frac{v'(u')^2}{v}]\Big|_0^{\ell} = \int_0^{\ell}[u(au'')'' + \frac{(Av')'}{v}(u')^2 - A(u'' - \frac{u'v'}{v})^2 + (A-a)(u'')^2]dx. \tag{3.13}$$

Theorem 3.4. Suppose there exists a function v satisfying

(1) $v > 0$ in $(0,\ell)$

(2) v has at most simple zeros at 0 and ℓ,

and suppose $A > 0$ in $(0,\ell)$. If u is an eigenfunction corresponding to the lowest eigenvalue γ_1 of problem (III), and if

(3) $\int_0^{\ell}(a-A)(u'')^2dx \geq 0,$

then

$$\gamma_1 > \inf_{(0,\ell)} -\frac{(Av')'}{v} . \tag{3.14}$$

Proof. From identity (3.13), whose validity is established in the usual fashion, it follows that

$$0 \leq \int_0^{\ell}[u(au'')'' + \frac{(Av')'}{v}(u')^2]dx = \int_0^{\ell}[-\gamma_1 uu''dx + \frac{(Av')'}{v}(u')^2]dx.$$

Applying Green's first identity (integration by parts) to the first term on the right we have

$$0 \leq \int_0^{\ell}[\gamma_1 + \frac{(Av')'}{v}](u')^2dx.$$

Hence

$$\gamma_1 \geq \inf_{(0,\ell)} [-\frac{(Av')'}{v}]$$

with equality only if $u' \equiv kv$, as was shown in the remark on page 27. But since $v > 0$ in $(0,\ell)$, $u' \equiv kv$ implies u' has a fixed sign in $(0,\ell)$ which contradicts the boundary conditions $u(0) = u(\ell) = 0$. Consequently the inequality is strict and (3.14) is established.

III.2 Comparison Theorems for Eigenvalues

It is the purpose of this section to establish comparison theorems for the lowest eigenvalue Ω_1 of the problem

$$(3.15)\qquad \begin{aligned} &(a(x)u'')'' - \Omega\,\rho(x)u = 0,\ \rho(x) > 0 \quad \text{in} \quad (0,\ell) \\ &u(o) = u'(0) = u(\ell) = u'(\ell) = 0 \end{aligned}$$

and the lowest eigenvalues λ_1 and μ_1, which are known to be positive (see [4], pp. 292-295), of the second order problems

$$(3.16)\qquad \begin{aligned} &(A(x)v')' + \lambda v = 0,\ A > 0 \quad \text{in} \quad (0,\ell) \\ &v(0) = v(\ell) = 0 \end{aligned}$$

and

$$(3.17)\qquad \begin{aligned} &z'' + \mu z = 0 \quad \text{in} \quad (0,\ell) \\ &z(0) = z(\ell) = 0\ . \end{aligned}$$

In the case $a \equiv 1$ problem (3.15) governs the vibration of a non-homogeneous rod with linear density $\rho(x)$, and (3.17) governs the vibration of a homogeneous string.

<u>Theorem 3.5</u>. Let λ_1 be the lowest eigenvalue of problem (3.16), and let v be a corresponding eigenfunction. Let μ_1 be the lowest eigenvalue of the problem (3.17). If u is an eigenfunction corresponding to the lowest eigenvalue Ω_1 of problem (3.15) and

$$\int_0^\ell (a-A)(u'')^2 dx \geq 0,$$

then

$$\Omega_1 > \frac{\lambda_1 \mu_1}{\rho_0}$$

where $\rho_0 = \max\limits_{[0,\ell]} \rho$.

<u>Proof</u>. It is well known (see [4], p. 452) that $v > 0$ in $(0,\ell)$, and moreover (cf. Theorem 2.5) $v'(0) \neq 0$ and $v'(\ell) \neq 0$. Hence identity (3.13), whose validity is established in the usual fashion, yields

$$0 \leq \int_0^\ell \Omega_1 \rho \, u^2 dx - \int_0^\ell \lambda_1 (u')^2 dx \ . \tag{3.18}$$

From Rayleigh's principle (see [11], p. 393) it follows that

$$\mu_1 \leq \frac{\int_0^\ell (\varphi')^2 dx}{\int_0^\ell \varphi^2 dx} , \tag{3.19}$$

for sufficiently smooth functions φ vanishing on the boundary.

In particular, since u is an admissible function in (3.19) we have

$$\int_0^\ell (u')^2 dx \geq \mu_1 \int_0^\ell u^2 dx$$

which combined with (3.18) yields

$$0 \leq \int_0^\ell u^2 (\Omega_1 \rho - \lambda_1 \mu_1) dx,$$

and thus

$$\Omega_1 \geq \frac{\lambda_1 \mu_1}{\rho_0} \ . \tag{3.20}$$

If equality holds in (3.20), then equality must hold in (3.18). Consequently, by the remark on page 27, $u' \equiv kv$. But then the positivity of v in $(0,\ell)$ again contradicts the boundary conditions $u(0) = u(\ell) = 0$. Therefore we have

$$\Omega_1 > \frac{\lambda_1\mu_1}{\rho_0} .$$

Remark. If $0 < \rho \le 1$, then (3.20) reduces to

$$\Omega_1 > \lambda_1\mu_1,$$

and if $A \equiv 1$ in (3.16), then $\lambda_1 = \mu_1$, and we have the well known result (see [24])

$$\Omega_1 > \mu_1^2 .$$

We wish to obtain similar lower bounds for Ω_1 for various classes of density functions ρ. To this end, we return to the derivation of identity (2.14) on page 26. Instead of making the choice

$$P_2 = \frac{Av'}{v}$$

in identity (2.15), we leave the function P_2 unspecified and consider the identity

$$\begin{aligned} &[u(au'')' - au'u'' + P_2(u')^2]\Big|_0^\ell \\ (3.21) \quad &= \int_0^\ell [u(au'')'' - A(u'' - \frac{P_2}{A}u')^2 + (\frac{P_2^2}{A} + P_2')(u')^2 + (A-a)(u'')^2]dx. \end{aligned}$$

If u is an eigenfunction corresponding to the lowest eigenvalue Ω_1 of problem (3.15), if

$$\int_0^{\ell}(a-A)(u'')^2 dx \geq 0,$$

and if identity (3.21) is valid and the boundary term vanishes, then we have

$$0 \leq \int_0^{\ell} \Omega_1 \rho \, u^2 dx - \int_0^{\ell} Q(x)(u')^2 dx \tag{3.22}$$

where

$$-Q = \frac{P_2^2}{A} + P_2' .$$

If $Q > 0$ and if α_1 is the lowest (positive) eigenvalue for the problem

$$\begin{aligned} &(Qz')' + \alpha Q z = 0 \quad \text{in} \quad (0,\ell) \\ &z(0) = z(\ell) = 0, \end{aligned} \tag{3.23}$$

then by Rayleigh's principle

$$\alpha_1 \leq \frac{\int_0^{\ell} Q(\varphi')^2 dx}{\int_0^{\ell} Q \, \varphi^2 dx}$$

where φ is any function vanishing on the boundary. In particular, since u is an admissible function, we have

$$\int_0^{\ell} - Q(u')^2 dx \leq -\alpha_1 \int_0^{\ell} Q \, u^2 dx ,$$

and inequality (3.22) becomes

$$0 \leq \int_0^{\ell} u^2(\Omega_1 \rho - \alpha_1 Q) dx .$$

Hence we obtain the formal inequality

$$\Omega_1 \geq \inf_{(0,\ell)} \frac{\alpha_1 Q}{\rho} . \tag{3.24}$$

Motivated by some recent works of Protter [28] and Hersch [13] in which a similar problem for second order equations was considered, we let

$$P_2 = \frac{\tau_0 A v'}{\tau v} + \frac{A\tau'}{2\tau} , \tag{3.25}$$

where v is an eigenfunction corresponding to the lowest eigenvalue λ_1 in problem (3.16), τ is an arbitrary smooth positive function in $(0,\ell)$, and

$$\tau_0 \equiv \min_{[0,\ell]} \tau .$$

A simple computation yields

$$-Q = \frac{P_2^2}{A} + P_2'$$

$$= - \frac{\tau_0 \lambda_1}{\tau} + \frac{A\tau_0}{\tau} \left(\frac{\tau_0}{\tau} - 1\right)\left(\frac{v'}{v}\right)^2 - \frac{A(\tau')^2}{4\tau^2} + \frac{(A\tau')'}{2\tau} \tag{3.26}$$

$$\leq - \frac{\tau_0 \lambda_1}{\tau} - \frac{A(\tau')^2}{4\tau^2} + \frac{(A\tau')'}{2\tau} ,$$

which implies

$$Q > 0$$

if we impose the further condition that

$$(A\tau')' \leq 0 \quad \text{in} \quad (0,\ell).$$

Therefore, by the choice for P_2 given in (3.25), and for $\tau > 0$, $(A\tau')' \leq 0$ in $(0,\ell)$ inequality (3.24) becomes the formal bound

$$\Omega_1 \geq \inf_{(0,\ell)} \frac{\alpha_1}{\rho}\left[\frac{\tau_0\lambda_1}{\tau} + \frac{A(\tau')^2}{4\tau^2} - \frac{(A\tau')'}{2\tau}\right] . \tag{3.27}$$

For choices of τ motivated by Protter [28], we obtain the following lower bounds for Ω_1.

Theorem 3.6. Let Ω_1 be the lowest eigenvalue and let u be the corresponding eigenfunction for problem (3.15), and let v be an eigenfunction corresponding to the lowest eigenvalue $\lambda_1 > 0$ of problem (3.16) with

$$\int_0^\ell (a-A)(u'')^2 dx \geq 0 .$$

If

$$\tau = \frac{1}{\rho} > 0$$

and if

$$[A(\frac{1}{\rho})']' \leq 0 \quad \text{in} \quad (0,\ell), \tag{3.28}$$

then we have the inequality

$$\Omega_1 > \frac{\alpha_1\lambda_1}{\rho_0} + \alpha_1 N_1 + \alpha_1 N_2 \tag{3.29}$$

where $\alpha_1 > 0$ is the lowest eigenvalue of problem (3.23)

$$\rho_0 \equiv \max_{[0,\ell]} \rho > 0$$

$$N_1 \equiv \frac{1}{4} \min_{[0,\ell]} A\rho[(\frac{1}{\rho})']^2 \geq 0$$

$$N_2 \equiv -\frac{1}{2} \max_{[0,\ell]} [A(\frac{1}{\rho})']' \geq 0 .$$

Proof. Since $v > 0$ in $(0,\ell)$, it follows (cf. Theorem 2.5) that $v'(0) \neq 0$ and $v'(\ell) \neq 0$. Therefore, with the choice of P_2 as given by (3.25), the validity of (3.21) follows as before, and moreover, the boundary term vanishes. Hence the formal bound (3.27) is now valid. For $\tau = \frac{1}{\rho}$ we note that $\tau_0 = \min_{[0,\ell]} \tau = \frac{1}{\rho_0}$ and (3.27) becomes

$$(3.30) \qquad \Omega_1 \geq \frac{\alpha_1\lambda_1}{\rho_0} + \alpha_1 N_1 + \alpha_2 N_2 \ .$$

Equality holds in (3.30) only if we have equality in (3.26) and in (3.22). Equality in (3.26) implies $\tau = \tau_0$, a constant, while equality in (3.22) taken together with (3.21) yields

$$(3.31) \qquad u'' - \frac{P_2}{A} u' \equiv 0 \quad \text{in} \quad (0,\ell).$$

Since $\tau \equiv \tau_0$, from (3.25) we have simply that $P_2 = \frac{Av'}{v}$, and hence (3.31) yields $u' \equiv kv$ for some constant k. But then the positivity of v in $(0,\ell)$ contradicts the boundary conditions on u. Therefore the inequality in (3.30) is strict.

Remark. For $\rho \equiv 1$, (3.26) shows that $Q = \lambda_1$, and thus problem (3.23) is equivalent to problem (3.17). As a result, $\alpha_1 = \mu_1$, and (3.29) simplifies to the previous bound of Theorem 3.5

$$\Omega_1 > \lambda_1\mu_1.$$

Remark. In the case $A \equiv 1$, condition (3.28) becomes $(\frac{1}{\rho})'' \leq 0$ which implies $\rho'' \geq 0$ in $(0,\ell)$. In other words, the vibrating rod with governing equation (3.15) is assumed less dense near the center, but may be of large density near the ends.

By choosing $\tau = \log(\frac{1}{\rho})$ we are led in a similar manner to the following result.

Theorem 3.6. Let Ω_1 be the lowest eigenvalue and let u be a corresponding eigenfunction for problem (3.15), and let v be an eigenfunction corresponding to the lowest eigenvalue $\lambda_1 > 0$ of problem (3.16) with

$$\int_0^{\ell} (a-A)(u'')^2 dx \geq 0 .$$

If

$$\tau = \log(\frac{1}{\rho}) > 0 \quad \text{where} \quad 0 < \rho < 1$$

and if

$$[A(\log \rho)'] \geq 0 \quad \text{in} \quad (0,\ell),$$

then

$$\Omega_1 > \frac{\alpha_1 \lambda_1 \tau_0}{N_3} + \frac{\alpha_1 N_4}{N_3^2} + \alpha_1 \frac{N_5}{N_3}$$

where $\alpha_1 > 0$ is the lowest eigenvalue of problem (3.23)

$$\tau_0 = \min_{[0,\ell]} (-\log \rho) > 0$$

$$N_3 \equiv \max_{[0,\ell]} (-\rho \log \rho) > 0$$

$$N_4 \equiv \frac{1}{4} \min_{[0,\ell]} \frac{A(\rho')^2}{\rho} \geq 0$$

$$N_5 \equiv \frac{1}{2} \min_{[0,\ell]} [A(\log \rho)']' \geq 0 .$$

In a similar manner the lowest eigenvalue Ω_1 of problem (3.15) can be compared to the lowest eigenvalue λ_1 of the more general second order problem

$$(3.32)\quad \begin{aligned} &(A(x)v')' + \lambda\sigma(x)v = 0, \ A > 0, \ \sigma > 0 \quad \text{in} \quad (0,\ell) \\ &v(0) = v(\ell) = 0 \ . \end{aligned}$$

Choosing P_2 as given by (3.25), and thus replacing λ_1 by $\lambda_1\sigma$, (3.26) becomes

$$-Q \le -\frac{\tau_0\lambda_1\sigma}{\tau} - \frac{A(\tau')^2}{4\tau^2} + \frac{(A\tau')'}{2\tau} ,$$

and the formal inequality (3.27) becomes

$$\Omega_1 \ge \inf_{(0,\ell)} \frac{\alpha_1}{\rho}\left[\frac{\tau_0\lambda_1\sigma}{\tau} + \frac{A(\tau')^2}{4\tau^2} - \frac{(A\tau')'}{2\tau}\right] .$$

The validity of identity (3.21) is established in the usual fashion, and we obtain the following bound.

<u>Theorem 3.7</u>. Let Ω_1 be the lowest eigenvalue and let u be a corresponding eigenfunction of problem (3.15), and let v be an eigenfunction corresponding to the lowest eigenvalue $\lambda_1 > 0$ of problem (3.32) with

$$\int_0^\ell (a-A)(u'')^2 dx \ge 0 \ .$$

If

$$\tau = \frac{\sigma}{\rho} > 0 \quad \text{in} \quad (0,\ell),$$

and

$$[A(\frac{\sigma}{\rho})']' \le 0 \quad \text{in} \quad (0,\ell),$$

then

$$\Omega_1 > \alpha_1 \tau_0 \lambda_1 + \alpha_1 N_6 + \alpha_1 N_7$$

where $\alpha_1 > 0$ is the lowest eigenvalue of (3.23)

$$\tau_0 \equiv \min_{[0,\ell]} \frac{\sigma}{\rho} > 0$$

$$N_6 \equiv \frac{1}{4} \min_{[0,\ell]} \frac{\rho A[(\frac{\sigma}{\rho})']^2}{\sigma^2} \geq 0$$

$$N_7 \equiv -\frac{1}{2} \max_{[0,\ell]} \frac{[A(\frac{\sigma}{\rho})']'}{\sigma} \geq 0 \ .$$

CHAPTER IV

REMARKS ON PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we shall comment on the extension of the preceeding results to the case of elliptic partial differential equations. We consider the following analogs of equations (2.1) through (2.4), namely,

$$(4.1) \quad \Delta(a(x)\Delta u) + 2\,\mathrm{div}(b(x)\ \mathrm{grad}\ u) - c(x)u = 0 \quad \text{in } G$$

$$\left(\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right)$$

$$(4.2) \quad \Delta(A(x)\Delta v) + 2\,\mathrm{div}(B(x)\ \mathrm{grad}\ v) - C(x)v = 0 \quad \text{in } G$$

and the corresponding systems

$$(4.3) \quad \begin{aligned} \Delta u + b(x)u &= e(x)w \\ \Delta w + b(x)w &= -c(x)u \end{aligned} \qquad \text{in } G$$

and

$$(4.4) \quad \begin{aligned} \Delta v + d(x)v &= f(x)z \\ \Delta z + d(x)z &= -h(x)v. \end{aligned} \qquad \text{in } G$$

We assume all solutions are classical and all coefficients are sufficiently differentiable in a bounded domain G with piecewise smooth boundary Γ in n-dimensional Euclidean space E^n.

Although the analogs of the identities used in Chapters II and III can be derived by Picard's technique, this method is rather cumbersome. In fact, in only two dimensions, Picard's method would involve the introduction of twelve arbitrary (sufficiently smooth) functions via the obvious identity

$$\int_{\Gamma}\{(P_{11}u^2 + P_{21}u_x^2 + P_{22}u_y^2 + P_{23}u_xu_y + 2P_{31}uu_x + 2P_{32}uu_y)\frac{\partial x}{\partial n}$$

$$+ (P_{12}u^2 + P_{24}u_x^2 + P_{25}u_y^2 + P_{26}u_xu_y + 2P_{33}uu_x + 2P_{34}uu_y)\frac{\partial y}{\partial n}\}dS$$

$$= \iint_G\{[P_{11}u^2 + P_{21}u_x^2 + P_{22}u_y^2 + P_{23}u_xu_y + 2(P_{31}uu_x + P_{32}uu_y)]_x$$

$$+ [P_{12}u^2 + P_{24}u_x^2 + P_{25}u_y^2 + P_{26}u_xu_y + 2(P_{33}uu_x + P_{34}uu_y)]_y\}dxdy\ .$$

A more natural and enlightening derivation of the identities for partial differential equations involves making use of the known one dimensional identities. For example, if we write equation (3.2) in differential form rather than in integral form and rearrange terms in the boundary, we obtain

$$[u(au'')' - au'u'' + 2Auu'\frac{v''}{v} - u^2(\frac{(Av'')'}{v} + \frac{Av'v''}{v^2})]'$$

$$= u(au'')'' - u^2\frac{(Av'')''}{v} + 2\frac{Av''}{v}(u' - \frac{uv'}{v})^2 - A(u'' - \frac{uv''}{v})^2 + (A-a)(u'')^2.$$

Letting $D_i = \frac{\partial}{\partial x_i}$ and $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, $(i,j = 1,\dots,n)$, we readily find that

$$D_j[uD_j(aD_{ii}u) - aD_juD_{ii}u + 2AuD_ju\frac{D_{ii}v}{v} - u^2(\frac{D_j(AD_{ii}v)}{v} + \frac{AD_jvD_{ii}v}{v^2})]$$

$$= uD_{jj}(aD_{ii}u) - \frac{u^2}{v}D_{jj}(AD_{ii}v) + 2A\frac{D_{ii}v}{v}(D_ju - u\frac{D_jv}{v})^2$$

$$- A(D_{ii}uD_{jj}u - 2uD_{jj}u\frac{D_{ii}v}{v} + u^2\frac{D_{ii}vD_{jj}v}{v^2}) + (A-a)D_{ii}uD_{jj}u.$$

Summing over i and j and using the divergence theorem, we easily obtain Dunninger's identity [10]

$$\int_{\Gamma}\{\frac{u}{v}[v\frac{\partial(a\Delta u)}{\partial n} - u\frac{\partial(A\Delta v)}{\partial n}] + A\frac{\Delta v}{v}[\frac{u}{v}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})]$$

$$+\frac{1}{v}\frac{\partial u}{\partial n}(Au\Delta v - av\Delta u)\}dS \tag{4.5}$$

$$= \int_G \frac{u}{v}[v\Delta(a\Delta u) - u\Delta(A\Delta v)]dx$$

$$+\int_G [2A\frac{\Delta v}{v}|\text{grad } u - u\frac{\text{grad } v}{v}|^2 - A(\Delta u - u\frac{\Delta v}{v})^2]dx,$$

where $\frac{\partial}{n}$ denotes the exterior normal derivative on the boundary Γ.

The same technique applied to the differential form of identity (2.6) yields the following identity for systems (4.3) and (4.4)

$$\int_{\Gamma}\{\frac{u}{v}(v\frac{\partial w}{\partial n} - u\frac{\partial z}{\partial n}) + \frac{z}{v}[\frac{u}{v}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})]$$

$$+\frac{1}{v}\frac{\partial u}{\partial n}(uz - vw)\}dS$$

$$= \int_G\{(h-c)u^2 - e(w - u\frac{z}{v})^2 + (e-f)\frac{u^2z^2}{v^2} + 2(d-b)\frac{u^2z}{v} \tag{4.6}$$

$$+ 2\frac{z}{v}|\text{grad } u - u\frac{\text{grad } v}{v}|^2\}dx .$$

Similarly, from the differential form of identity (1.22) we obtain the following analog of the new identity

$$\int_{\Gamma}[u\frac{\partial\Delta u}{\partial n} - \frac{1}{2}\frac{\partial u}{\partial n}\frac{\partial|\text{grad } u|^2}{\partial n} + \frac{|\text{grad } u|^2}{v}\frac{\partial v}{\partial n}]dS$$

$$= \int_G\{u\Delta^2 u + \frac{\Delta v}{v}|\text{grad } u|^2 - \sum_{i,j=1}^{n}(D_{ij}u - D_i u\frac{D_j v}{v})^2\}dx.$$

On the basis of (4.6) the following comparison theorem analogous to Theorem 2.1 is valid.

Theorem 4.1. Suppose there exists a function w such that u is a nontrivial solution of (4.3), and suppose there exists a function z such that v satisfies system (4.4) and $v > 0$ at some point in G. If

(1) $f \geq e \geq 0$, $d \geq b$ in G,

and either

$$\begin{cases} (2) & \int_G (c-h)u^2 dx \geq 0 \\ (3) & z < 0 \text{ in } G \\ (4) & u \not\equiv kv,\ k \text{ a constant} \end{cases}$$

or

$$\begin{cases} (5) & \int_G (c-h)u^2 dx > 0 \\ (6) & z \leq 0 \text{ in } G \end{cases}$$

then, under any boundary conditions on u,v,w, and z such that the left hand side of identity (4.6) is nonnegative, v must have a zero at some point in $\bar{G} = G \cup \Gamma$.

Proof. Suppose v has no zero in $\bar{G}$. Then $v > 0$ in $\bar{G}$, and identity (4.6) is valid. In view of hypotheses (1), (2), and (3) and the fact that the left hand side of (4.6) is nonnegative, we readily find that

$$(4.7) \qquad 0 \leq \int_G \frac{z}{v} \left|\text{grad } u - u \frac{\text{grad } v}{v}\right|^2 dx .$$

On the other hand hypothesis (3) and the fact that $v > 0$ in G imply

$$\int_G \frac{z}{v} \left|\text{grad } u - u \frac{\text{grad } v}{v}\right|^2 dx \le 0. \tag{4.8}$$

Inequalities (4.7) and (4.8) together imply that

$$\text{grad } \left(\frac{u}{v}\right) = \frac{1}{v} \left(\text{grad } u - u \frac{\text{grad } v}{v}\right) \equiv 0$$

which contradicts hypothesis (4).

In a similar manner hypotheses (5) and (6) lead to the contradiction

$$0 \le \int_G (h-c)u^2 dx < 0 .$$

Consequently, in either case v must vanish at some point in $\bar{G}$.

The extensions of the other results of Chapters II and III do not follow quite so easily. The primary difficulty stems from a need for an analog of L'Hôpital's rule for functions of several variables in order to establish the validity of the identities in the case $v = 0$ at a boundary point. Some recent results along this line have been obtained by Kreith [19], Swanson [32], Diaz and McLaughlin [7], and Dunninger [10]. However, it is not yet apparent that these techniques apply to the problems at hand, although the Diaz and McLaughlin result appears most promising. We hope to be able to discuss this problem at a future date.

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