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ABSTRACT

AN APPLICATION OF OPTIMAL CONTROL THEORY TO POLICIES
FOR INTERNAL AND EXTERNAL BALANCE

By

Nguyen Thi Bich Thuy

This thesis examines the dynamic problem of quantitative economic policy for internal and external balance within the framework of optimal control theory.

Three cases are considered: the one-country, the two-country with common external balance, and the two-country with linear dependent external balance. When applying Chow's control approach, it is found that the optimal solution for achieving the joint balance can always be uniquely determined whenever Tinbergen's principle on the equality between the number of independent targets is met; however, there is no guarantee that the solution is feasible for a given economy.

In fact, when the optimal fiscal and monetary policies for achieving internal and external balance of the U.S. and Canada are analyzed for the period of 1961-1970, it is found that they are inconsistent and a trade-off between the attainment of joint balance and the feasibility or consistency of policy-instruments must occur when limits are imposed upon the instrument-magnitudes.

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It is also found that if more complicated assumptions, such as active responses from the second country, or conflicting balance of payments and growth targets are added, the optimal control framework is no longer appropriate for the analysis of internal and external balance.

To encourage future research, a two-player multistage non zero sum game with linear quadratic system and perfect information is presented. It is found that the non-cooperative (or so-called Nash) equilibrium strategies are always inferior in the case of two controllers which leads to the search for a non-inferior solution, in particular the Pareto-Optimum strategy.

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AN APPLICATION OF OPTIMAL CONTROL THEORY TO POLICIES FOR INTERNAL AND EXTERNAL BALANCE

By

Nguyen Thi Bich Thuy

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Economics

1974

"Con cố chí ần cần học tập,
Mẹ ra công gắng sức lo toan,
Cha con đả mất từ lâu,
Mọi đường có Mẹ chu tòan cho con."

(T. T. Ba, 1966)

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ACKNOWLEDGMENTS

I wish to express my deep gratitude and sincere appreciation to Professor Anthony Y. C. Koo, Professor Robert H. Rasche, and Professor Kwang Yun Lee. I would like to especially thank Professor Koo for suggesting the direction of the thesis and for his encouragement and support. Professor Rasche was of invaluable assistance throughout the preparation of this study. The time and attention he devoted to the work is greatly appreciated. In addition, I would like to acknowledge the assistance of Professor Lee. Special thanks are due to Dr. Thai Van Can of the International Monetary Fund for his suggestions in regard to an appropriate econometric model of this study.

J'adresse des remerciements sincères au professeur Alain Haurie et à M. Bernard Jacquet de l'Ecole des Hautes Etudes Commerciales pour l'aide inestimable qu'ils m'ont apportée en des étapes critiques de ma recherche. Je m'en voudrais de ne pas mentionner le soutien que m'ont toujours procurée Lise et Yves Lacroix. Je leur dis sincèrement merci.

Schließlich empfinde ich in nicht geringem Maße Dankbarkeit und große Wertschätzung meinem Lehrer gegenüber, der ebenfalls mein bester Freund Während meiner positivsten sowie auch negativsten Erfahrungen geworden ist. Herr Doktor V. Leroy Name ist es, dem ich für seinen überaus hilfreichen Einsatz für mich, seine Unterstutzung und Ermutigung danke, die mich dazu anspornten den wesetnlichen Wert und die Schöenheit

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der Wirtschaftswissenschaften und des Lebens während meiner Auslandsstudien zu entdecken. Erinnerung an und Verpflichtung für die schönen gemeinsam verbrachten Tage, wo immer ich auch sein werde, haben einen bleibenden Einfluß auf mein ganzes Leben.

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INTRODUCTION

The problem of internal-external balance was first raised by Tinbergen and Meade under the title of "Theory of Quantitative Economic Policy" (Meade:1951, Tinbergen:1952). Along this direction Mundell (1962) introduced the most controversial issue called "The Appropriate Use of Monetary and Fiscal Policies for Internal and External Stability." However, studies related to these problems are presented in a static context with one exception of Votey's work (1969).

Parallel to the Meade-Tinbergen-Mundell formulation of the static problem of internal-external balance, significant advances both by engineers in modern control theory and by econometricians in macroeconomic modeling led to interest in the problem of the dynamic economic policy-making. However, in spite of numerous studies on the application of optimal control theory to economics, a survey of which is found in Kendrick's unpublished paper (1972) as well as in Park's dissertation (1973), no studies have considered the problem of internal-external balance.

The purpose of this study is twofold: first, to show that optimal control theory is also applicable to internal-external models; secondly, to introduce a new framework, called differential game theory, for solving the problem of internal-external balance under more sophisticated assumptions than those of Meade-Tinbergen-Mundell models, e.g., active responses from other countries, conflicting targets, and decentralized decision-making.

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Votey's macrodynamic model will be borrowed to illustrate applications of optimal control tools. To derive optimal policies for internal-external balance Chow's deterministic optimal control approach (Chow:1970a, 1970b, 1972a, 1972b, 1972c) is preferred to that of Pindyck (1972, 1973), because the former approach involves a more straightforward application, given the structural form of Votey's model, and requires only basic matrix manipulation to compute the optimal solution.

Chapter I will present a survey of the literature on the theory of economic policy for internal-external balance and on the application of optimal control theory to problems of economic stabilization. The main concern of the chapter will be to show a gap between these two developments. Chapter II will deal with the application of optimal control techniques to the one-country model. The two-country optimal control problem will be treated in Chapter III for the case of common external balance, and in Chapter IV for the case of linear dependent external balance. Chapter V will introduce a two-player deterministic multistage game with a linear and quadratic system for further research on its application to problems of internal-external balance, or more generally to problems of conflicts in economics. Chapter VI will present the main conclusions of the study and suggest some recommendations.

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CHAPTER I

SURVEY OF LITERATURE

1.1. On the Theory of Economic Policy for Internal-External Balance

Forty years ago, John Maynard Keynes with his "General Theory" dealt a fatal blow to the classical notion that "internal balance", or full-employment equilibrium in the domestic economy, could be attained through an automatic adjustment mechanism with little government intervention. But the notion that the "automatic" price and income effects tend to restore "external balance" or balance-of-payments equilibrium persisted until the middle 1950's. Under the combined pressure of modern economic theory, and observation of the chronic international financial difficulties that plague the real world under fixed exchange rates, economists have come to regard "external balance" as one of the specific economic objectives of deliberate governmental action, rather than as something that will take care of itself (von Neumann Whitman:1970).

The pioneers in developing a formal body of analysis incorporating these views are Meade (1951) and Tinbergen (1952). They introduced a new approach to the problem of simultaneously achieving internal and external balance via quantitative economic policy, that is, of finding the values of policy variables given some desired levels of real nce and balance

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Table 1.1. Cases

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income and balance of payments. This approach is just the inverse of the traditional multiplier analysis.

In addition to the criteria on the effectiveness of policy instruments, Tinbergen in his work also formulated a rule on determining the existence of a solution to attain the joint balance, called the principle of equality between the number of independent targets and the number of independent instruments. Once Tinbergen's rule is satisfied, all the policy variables can be set at the necessary levels and all targets achieved simultaneously.

Defining external balance in terms of the current account, and assuming that fiscal and monetary policies are equivalent methods of controlling the aggregate demand, Meade (1951) developed a theory of economic policy to cure internal and external disequilibrium. Two of Meade's four cases have potential conflicts (Table 1.1). This analysis has been widely accepted for a considerable period.

Table 1.1. Cases of internal and external disequilibrium

	Disequilibrium				
	Internal		External		
Case	Deflation	Inflation	Deficit	Surplus	Cures
1	x			x	Inflationist policy
2		x	x		Deflationist policy
3 (conflict)	x		x		Devaluation
4 (conflict)		x		x	Appreciation of exchange rate

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In the 1960's, the problem of using monetary and fiscal policy to achieve internal and external stability received considerable attention. The analysis assumes that a country chooses to keep its exchange rate fixed and chooses to avoid the use of direct controls over trade and capital movements. Mundell, in his classic 1962 article, proposed a new theory based on the Principle of Effective Market Classification (PEMC) which states that "each policy instrument should be directed towards that target on which it has relatively the greatest impact." Unlike the old theory of Meade, Mundell defined external balance in terms of the current account and short term capital flows. The latter assumed to be responsive to changes in interest rate differentials among countries. Mundell also divided financial policy into two separate policies: monetary and fiscal. Under these assumptions Mundell concluded that in a disequilibrium situation "monetary policy ought to be aimed at external objectives and fiscal policy at internal objectives", on the grounds that to do the opposite would worsen the disequilibrium situation.

The main criticisms of Mundell's theory are (Yeager:1966, Votey:1969, Patrick:1968, Cooper:1969):

(1) Mundell's proposal and proof are set forth as a short-run solution.

(2) The analysis deals with one, very small country.

(3) The suggested policy mix may thus temporarily palliate a fundamental external imbalance.

(4) The proper assignment assumes full freedom in assigning instruments to variables. Therefore, so long as nations

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remain independent in their actions, some targets could not be reached.

(5) The "comparative advantage" of instruments may vary with the environment, the level of targets, and the distance from the targets of the variables to be controlled.

(6) The Mundellian model prescribes a set of policy responses that converges on equilibrium even when the policy makers have limited knowledge of the economic system. However, uncertainty about the analytical relationship linking instruments and targets can also lead to inefficient use of instruments.

(7) The analysis is mainly static. No account is taken of the speed of adjustment of the system, the effect of a proper policy assignment on growth, or cyclicity.

Attempts to remedy the shortcomings of Mundell's analysis under fixed exchange rate have been undertaken. Cooper (1969), Patrick (1968) and Votey (1969) extended Mundell's theory to a two-country case with emphasis on the interdependence of the economies.

Cooper's work (1969) is mainly concerned with the gains from coordinating the instruments of economic policy both within and between nations. He found that as the economic interdependence increases, the effectiveness of decentralized policy making a la Mundell will decrease, and the case for coordination of policy making, for directing all the policy instruments at all the targets, becomes more compelling. The analytical framework used in his study is ilar to that in

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similar to that introduced by Tinbergen (1952), involving targets and instruments of economic policy.

Patrick (1968) studied the stability of Mundell's assignment proposal for a two-country model under various assumptions: common balance-of-payments, passive responses from Country II, active responses from Country II with international cooperation as well as conflicting targets. He concluded that the explicit inclusion of the rest of the world does not invalidate Mundell's theory; however, the other complications introduced by the assumptions of active responses from Country II raise the possibility that Mundell's conclusions are inappropriate.

Recently, Patrick (1973) reexamined the convergence of assignment for a decentralized system using McFadden's criteria for stability (McFadden:1968, 1969). He discovered that for the two targets and two instruments case the minimum information necessary to establish convergent policy in a centralized system is virtually identical to that necessary to establish a convergent decentralized system.

Votey (1969) extended Mundell's analysis in two senses: first, Votey's model allows reactions from the rest of the world; and secondly, it is dynamic in that it adds a production function and allows for accumulation of the capital stock over time. Then, he studied the effects on stability, cyclic response, and growth from applying Mundellian policy, for the cases of a simple, open-economy, one-country model and of a two-country model. The main conclusions of Votey's study are:

(1) A prolonged solution based on Mundellian policy requires a higher degree of sensitivity of capital flows to interest rate differentials than Stein's (1973) results

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(2) If there exists international cooperation, with higher rather than lower growth rates as the goal of all parties, the adoption of Mundellian policy leads to a choice of action which may stimulate growth.

(3) International cooperation with respect to the establishment of interest rates can lead both to a more favorable balance of payments and more favorable rates of growth.

(4) Within the confines of Votey's model it appears that cyclicity is not a problem associated with the adoption of Mundell's solution.

In spite of the above results, Votey is not an advocate for the adoption of Mundellian policy. He recognized that monetary policy is a most important tool for achieving domestic goals and its abandonment to the external problem is too large a price to pay. Furthermore, the lags which must be accepted in the effectiveness of fiscal policy both in initiating action and in achieving results makes it unacceptable as the sole tool of domestic policy. To these two objections to the adoption of Mundellian policy, we would add a third one: It is not likely that the fluctuations in interest rates necessitated by foreign balance can be exactly offset in the domestic economy by government expenditure to the extent that the sectoral imbalances do not occur, at least without some very selective countermeasures.

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Another extension of Mundell is the recent work by Krainer (1973). He incorporated the production, investment, and financing decisions of multinational firms into a macroeconomic model. This provides a better understanding of how factor endowments influence the structure and interrelationship of trade and investment and how this might influence the impact of monetary and fiscal policies on the goals of full employment and equilibrium in the balance of payments in a fixed-exchange-rate world. He concluded that monetary and fiscal policies have different effects in resource rich and resource poor creditor countries. However, his model is fundamentally static as are all the previous ones with the exception of Votey's.

1.2. On the Application of Optimal Control Theory to Macroeconomic Stabilization Policy

In the 1960's optimal control theory found substantial applications in economics because of increasing interest in dynamic decision-making. This was facilitated by the development of quantitative econometrics and of computers. Numerous applications of control theory have been found in macroeconomics as well as in microeconomic fields, such as growth models, planning models for sectoral allocation of resources, short-run economic stabilization models, consumer choice problems, dynamic models of investment and pricing by firms, portfolio analysis models, and pollution control problems (Kendrick:1971, Park:1973).

This section surveys studies on optimal planning for economic stabilization via optimal control techniques.

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In the early fifties Tustin (1953) noted that the problem of determining macroeconomic policies is a feedback control problem. Work along this direction was also done by Phillips (1954, 1957). He proposed a number of stabilization policies and considered the stability properties of the system when these policies were implemented. In particular, he showed that the application of certain types of stabilization policies to multiplier-accelerator macroeconomic models could result in undesired oscillations and even instability in economic activity. These results are also found in other studies (Baumol:1961, Chow:1968, Allen:1967). However, Phillips' analysis was purely descriptive in the sense that while the alternative policies he considered were plausible, they were not derived from any optimizing behavior. Since that time attention has shifted to more normative questions and to the study of optimal stabilization policies.

Van Eijk and Sandee (1959), Holt (1962, 1965), and Theil (1964) applied a more modern analysis to macroeconomic systems without correlating the analysis with the control system aspects of the problem. Their works have generally related to the derivation of linear decision rules of the type first derived by Simon (1956). These decision rules minimize the distance between actual and desired levels of the target variables, i.e., the social welfare function is quadratic in form. But neither Holt (1962) nor Theil (1964) used modern control theory to derive their decision rules.

Recent advances in control theory have led to the development of new, more convenient techniques than the calculus of variations: Pontryagin's maximum principle (Pontryagin *et al.*:1962) and Bellman's amic programmin

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dynamic programming (Bellman:1957). As a result of these developments it is now known that with the assumption of quadratic utility functional and a linear model it is possible to obtain the optimal policy as a linear feedback law. This is a particularly convenient form in which to obtain the solution.

Deterministic control theory has been applied to macroeconomic models by Buchanan (1968), Sakakibara (1969, 1970), Sengupta (1970), Pindyck (1972, 1973), and Turnovsky (1973). They all used a quadratic welfare function and linear deterministic model.

Buchanan (1968) showed that modern control theory is applicable to the problem of economic policy determination for domestic stabilization of macroeconomic systems. Sakakibara (1969, 1970) integrated demand and foreign sectors to an ordinary growth model and applied dynamic optimization to evaluate actual economic policies (unemployment rates and investment - GNP ratio) for the United States and Japan. He found that growth policies have been too conservative while the movement of stabilization tools has been too erratic for 1952-1967. For Japan the growth policy in the 1960's was quite successful while that of the 1950's was too conservative. Both Sengupta (1970) and Turnovsky (1973) applied optimal control techniques to the stabilization of the deterministic Phillips' multiplier-accelerator model. Pindyck (1972, 1973) applied the deterministic control theory to study the optimal time path for the policy variables, using a linear econometric model of the U.S. economy. His analysis provides empirical measures of the trade-off between unemployment and inflation.

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However, the above studies are restrictive in three senses: the optimal control problem has been completely deterministic, the cost function is quadratic, and the econometric models used are linear. Recently, some attempts have been made to remedy these shortcomings.

Our knowledge of the economy is incomplete; the coefficients of an econometric model are themselves random variables, and each equation in the structural form of the model has an implicit error term associated with it. To cope with uncertainty, stochastic optimal control theory has been introduced. Chow has shown that there are two gains from the optimal stochastic control policy: the gain of the optimal stochastic control over the optimal deterministic control and the gain of the optimal deterministic control over the deterministic control rule of a constant growth rate for each policy variable (Chow:1972b). At the extreme, if the error terms are additive, uncorrelated normal random variables, the cost functional is quadratic and the system is linear, the principle of "certainty equivalence" (known as the "Separation Theorem" in the control literature) allows the stochastic control problem to be reduced to one that is essentially deterministic (Theil:1957, Wonham:1968, 1969, Sorenson:1968). The optimal control becomes a function of the expected value of the state vector, and if there is no measurement noise, the solution for the optimal control is the same as for the deterministic problem (Chow:1972b, 1972c).

Paryani (1972) has applied the tools of stochastic control to derive an optimal control policy for the U.S. national economy. He found that the optimal control variables differ from the actual values of these variables during the period 1954-1963, suggesting the use of ore flexible

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more flexible control policies by the decision-makers to reach the internal balance.

Instead of additive stochastic disturbances, Turnovsky and Henderson (1972, 1973) introduce stochastic parameters in a somewhat specialized dynamic context.

Benjamin Friedman (1972) extends Theil's stochastic optimal control approach to economic policy to the case where the welfare function may not be quadratic but is approximated by several quadratic segments. It is an attempt to solve dynamic optimization problems with more general cost functions.

Stein and Infante (1973) have sought optimal stabilization policies which drive the quadratic cost of deviation monotonically to zero instead of minimizing the cost.

Finally, the most serious restriction is that of a linear model. Most econometric models are at least quasi-linear in structure, but sometimes the more interesting aspects of their dynamic behavior arise from non-linearities. Recently, several studies on the application of optimal control theory to non-linear macroeconomic models have been conducted (Livesey:1971, Holbrook:1972, Norman:1972, Shupp:1972, Haurie and van Petersen:1973).

In spite of these efforts, further research needs to be done on the development of stochastic, non-linear and non-quadratic optimal control theory and its application to macroeconomic stabilization problems.

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(1) The problem of quantitative economic policy for internal and external balance has been basically static with the one exception of Votey's study.

(2) Optimal control theory has been mainly applied to the problem of domestic stabilization. No account is taken of the regulation of balance-of-payments targets, of the interdependence between countries and of the repercussion effects when more than one country is considered.

Therefore, the next three chapters will attempt to bridge this gap (Thai Van Can:1972) by applying optimal control tools to Votey's one- and two-country macrodynamic models to derive optimal policies for internal and external balance.

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CHAPTER II

ONE-COUNTRY MODEL

This chapter deals with a two-target and two-instrument case. Both fiscal and monetary authorities of Country I act to attain full employment and balance-of-payments equilibrium while Country II is assumed to be passive with respect to its targets of internal and external balance. To find the optimal policies for internal-external balance in Country I. First, Votey's one-country econometric model will be presented (Votey:1969). Secondly, the one-country optimal control problem will be formulated in terms of a linear-quadratic (L.Q.) system to find the optimal control solution which minimizes the quadratic cost function subject to the constraints of a linear dynamic system. Pindyck's and Chow's approaches (Pindyck:1973, Chow:1972a) will be used to put the reduced form of Votey's econometric model into the "state-space" system. Next, Chow's formulation will be used to derive the optimal solution both analytically and numerically with reference to the United States. Finally, the optimal solution will be appraised and amended.

2.1. Presentation of Votey's Model

The main assumptions of the one-country model are:

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(1) The export demand is given. It is assumed to grow at the same rate as it has in the past and is treated as an exogenous variable.

(2) The foreign interest rate is given. This assumes the passive cooperation of Country II such that Country I may adjust its own rate to achieve the external balance without any foreign interference or countermeasures.

(3) The short-term capital flows are sensitive to the interest rate differentials between the two countries.

The model has five equations and five unknowns which lead to a determinant system. The variables are classified in three classes:

(1) Endogenous variables:

B_{1t}: Balance of payments

C_{1t}: Consumption expenditures

I^G_{1t}: Gross investment

I^n_{1t}: Net investment

K_{1t}: Capital stock

M_{1t}: Imports

O_{1t}: Net capital outflow

Y_{1t}: National income

(2) Controlled exogenous variables:

G_{1t}: Government expenditures

r_{1t}: Rate of interest

(3) Non-co

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$P_{X_{1t}}$:

r_{2t}:

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The equations are

(1) $Y_{1t}=C_{1t}+$

(2) $B_{1t}=X_{1t}$ -

(3) $K_{1t}=I^{n}_{1t}$ -

(4) $I^{G}_{1t}= I^{n}_{1t}$

(5) $Y_{1t}=\beta_{10}+$

(6) $C_{1t}=\alpha_{10}+$

(7) $M_{1t}=\beta_{10}+$

$+ \beta_{12}$

(8) $I^{n}_{1t}=\gamma_{10}+$

$+\gamma_{13}K_{1}$

(3) Non-controlled exogenous variables:

L_{1t}: Labor force

$P_{M_{1t}}$: Import price

$P_{x_{1t}}$: Export price

r_{2t}: Foreign interest rate

T_{1t}: Tax receipts

X_{1t}: Exports

$TT_{1t} = \left(\frac{P_M}{P_x}\right)_{1t}$: Terms of Trade

The equations are as follows:

(I-1) $Y_{1t}=C_{1t}+I^G_{1t}+G_{1t}+X_{1t}-M_{1t}$ National income identity

(I-2) $B_{1t}=X_{1t} - M_{1t} - O_{1t}$ Balance of payments identity

(I-3) $K_{1t}=I^n_{1t} + K_{1t-1}$ Capital stock identity

(I-4) $I^G_{1t} = I^n_{1t} + \delta_* K_{1t}$ Gross investment identity

(E-1) $Y_{1t}=\delta_{10}+\delta_{11}K_{1t}+\delta_{12}L_{1t}$ Production function

(E-2) $C_{1t}=\alpha_{10}+\alpha_{11}(Y_{1t}-T_{1t}-\delta_* K_{1t})$ Consumption function

(E-3) $M_{1t}=\beta_{10}+\beta_{11}(Y_{1t}-T_{1t}-\delta_* K_{1t})$
$+ \beta_{12} TT_{1t}$ Import function

(E-4) $I^n_{1t}=\gamma_{10}+\gamma_{11}Y_{1t-1}-\gamma_{12}[q(\delta_*+r)]_{1t-1}$
$+\gamma_{13}K_{1t-1}$ Investment function

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(E-5) $O_{1t}=\eta_{10}+\eta_{11}(r_{2t}-r_{1t})$ Net outflows of short term capital

All the equations are linear functions, and none are greater than the first order of difference.

Equation (E-1) - The output is a linear function of both capital and labor. It embodies the assumption of perfect substitutability between the two factors of production.

Equation (E-2) - Consumption expenditures are made a function of disposable income, using Klein's approach in the econometric model of the United Kingdom (Klein:1961). Disposable income is GNP, less capital depreciation ($\delta_* K_{1t}$ where δ_* is the rate of replacement) and also less taxes (T_{1t}) which are determined by a linear function of the form: $T_{1t} = \lambda_{10} + \lambda_{11} Y_{1t}$.

Equation (E-3) - Imports are simply a function of disposable income and the ratio of foreign to domestic prices.

Equation (E-4) - Net investment expenditures, that is, net addition to the capital stock, are assumed to depend on money output, the capital stock, and the user cost of capital which prevail at the time the investment decision is made. In Votey's formulation, the user cost of capital has two principal components: the opportunity cost of funds tied in the capital, plus the cost of the actual capital consumed. This can be written in the form: $q(\delta_* + r)$, where q is the price of capital goods, δ_* is the rate of replacement of capital stock, and r is the rate of interest, which represents the opportunity cost of funds. To simplify the model, it is assumed that relative prices within the country do not change, in which case q = 1 over time.

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Equation (E-5) - The capital transfers are assumed to be sensitive to interest rate differentials. Two results are available: Stein's results (or minimum results) dealing only with short term capital movements, and maximum results dealing with all capital transfers as a direct function of the existing differentials. Stein's results will be chosen for the analysis of short-run stabilization.

2.2. Optimal Control Problem Without Constraints on Instrument-Variable Magnitudes

It is assumed that the Country I has two instruments: government expenditures G_{1t} and the short-run interest rate r_{1t}. It is also assumed that its goal is to attain simultaneously internal and external balance, that is, a situation of full employment without inflation combined with balance-of-payments equilibrium. The exchange rate throughout the study is assumed to be fixed; therefore, the decision-makers of Country I try to steer their GNP and balance of payments close to the targets by choosing the appropriate combination of monetary and budgetary measures (r^*_{1t}, G^*_{1t}).

The external balance is represented by the balance-of-payments equilibrium, that is, $\overline{B}_{1t} = 0$ while the internal balance is determined by the production function: $Y_{1t} = \delta_{10} + \delta_{11} K_{1t} + \delta_{12} L_{1t}$ (2.1) which gives the potential output. It is noted that the capital stock K_{1t} which is an endogenous variable in the reduced form of Votey's econometric model also figures in the production function for the determination of the potential output. To overcome this problem of "double entry" of K_{1t}, the following transformation in the variables of equation 2.1 is needed:

$\tilde{Y}_{1t} - \delta_{11} K_{1t} =$

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$\tilde{Y}_{1t} = \delta_{10} + \delta_{12}$

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$y_t = \bar{A} y_{t-1} + \bar{B}$

where $\stackrel{d}{=}$ means "d

$y_t \stackrel{d}{=} \left[\tilde{Y}_{1t}, B_{1t},\right.$

$x_t \stackrel{d}{=} \left[G_{1t}, r_{1t}\right]'$

$z_t \stackrel{d}{=} \left[1, \Pi_{1t}, I\right.$

$$\bar{A} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & 0 & A_{23} \\ \gamma_{31} & 0 & a_{34} \end{bmatrix};$$

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which lead to the

$$Y_{1t} - \delta_{11} K_{1t} = \delta_{10} + \delta_{12} L_{1t} \tag{2.2}$$

Let

$$\tilde{Y}_{1t} = Y_{1t} - \delta_{11} K_{1t} \tag{2.3}$$

be the output net of capital accumulation. Then equation 2.2 becomes:

$$\tilde{Y}_{1t} = \delta_{10} + \delta_{12} L_{1t} \tag{2.4}$$

which determines the internal balance. To apply the optimal control theory, the one-country econometric model has been put into the reduced form (see Appendix A-1):

$$y_t = \bar{A}\, y_{t-1} + \bar{B}\, u_t + \bar{C}\, u_{t-1} + \bar{D}\, z_t \tag{2.5}$$

where $\underset{=}{d}$ means "defined as" and

$$y_t \overset{d}{=} \left[\tilde{Y}_{1t},\ B_{1t},\ K_{1t}\right]'$$

$$u_t \overset{d}{=} \left[G_{1t},\ r_{1t}\right]'$$

$$Z_t \overset{d}{=} \left[1,\ TT_{1t},\ T_{1t},\ X_{1t},\ r_{2t}\right]'$$

$$\bar{A} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & 0 & A_{23} \\ \gamma_{11} & 0 & \alpha_{34} \end{bmatrix};\ \bar{B} = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & \eta_{11} \\ 0 & 0 \end{bmatrix};\ \bar{C} = \begin{bmatrix} 0 & A_{15} \\ 0 & A_{25} \\ 0 & \gamma_{12} \end{bmatrix};\ \bar{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & Q_{11} & 0 \\ D_{21} & D_{22} & D_{23} & 1-Q_{21} & \eta_{11} \\ \alpha_{36} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Given the structural form (2-5) the one-country optimal control problem can be formulated in either Pindyck's or Chow's terminology. Both lead to the same result. However, Chow's method will be applied derive t

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to derive the optimal solution first analytically then numerically, using the U.S. data for the period 1961-1970.

2.2.1. Formulation of the Problem

Given the reduced form of Votey's one-country model to use Pindyck's approach the following transformation to equation 2.5. must be made:

$$y_t - \overline{B}\, u_t - \overline{D}\, z_t = \overline{A}(y_{t-1} - \overline{B}\, u_{t-1} - \overline{D}\, z_{t-1}) + (\overline{AB} + \overline{C})u_{t-1} + \overline{A}\,\overline{D}\, z_{t-1} \tag{2.6}$$

to get Pindyck's "state-space" system (see Appendix A-2, equation A-2.1a). In other words, the "state-space" system for the one-country optimal control problem is given by:

$$x_t^P = A^P x_{t-1}^P + B^P u_{t-1}^P + C^P z_{t-1} \tag{2.7}$$

where:

$x_t^P \overset{d}{=} y_t - \overline{B}u_t - \overline{D}z_t \overset{d}{=} \left[\tilde{Y}Y_{1t}, BB_{1t}, KK_{1t}\right]'$ is the (3x1) state vector;

$u_{t-1}^P \overset{d}{=} u_{t-1} \overset{d}{=} \left[G_{1t-1}, r_{1t-1}\right]'$ is the (2x1) control vector;

$z_{t-1}^P \overset{d}{=} z_{t-1} \overset{d}{=} \left[1, TT_{1t-1}, T_{1t-1}, T_{1t-1}, X_{1t-1}, r_{2t-1}\right]'$ is the (5x1) exogenous variable vector;

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$$A^P \stackrel{d}{=} \bar{A} \stackrel{d}{=} \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & 0 & A_{23} \\ \gamma_{11} & 0 & \alpha_{34} \end{bmatrix} ; \; \bar{B} \stackrel{d}{=} \overline{AB} + \bar{C} \stackrel{d}{=} \begin{bmatrix} A_{11}Q_{11} & A_{15} \\ A_{21}Q_{11} & A_{25} \\ \gamma_{11}Q_{11} & -\gamma_{12} \end{bmatrix} ;$$

$$\bar{C} \stackrel{d}{=} \overline{AD} = \begin{bmatrix} A_{11}D_{11}+A_{13}\ a_{36} & A_{11}D_{12} & A_{11}D_{13} & A_{11}Q_{11} & 0 \\ A_{21}D_{11}+A_{23}\ a_{36} & A_{21}D_{12} & A_{21}D_{13} & A_{21}Q_{11} & 0 \\ \gamma_{11}D_{11}+a_{34}\ a_{36} & \gamma_{11}D_{12} & \gamma_{11}D_{13} & \gamma_{11}Q_{11} & 0 \end{bmatrix}$$

are known matrices. Since $R^P=0$ under the assumption of no constraints on instrument-variable magnitudes, the cost functional to be minimized is:

$$J^P = \frac{1}{2} \sum_{t=1}^{N} \left[(x^P_{t-1} - \bar{x}^P_{t-1})' Q^P (x^P_{t-1} - \bar{x}^P_{t-1}) \right] \tag{2.8}$$

where

$\bar{x}^P_{t-1} \stackrel{d}{=} y_{t-1} - \bar{B}u_{t-1} - \bar{D}z_{t-1} = \left[\tilde{Y}Y_{1t-1},\ BB_{1t-1},\ KK_{1t-1} \right]'$ is the nominal or ideal state variable vector;

$Q^P = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with q_{11} and q_{22} are weights attached to the respective quadratic deviations $(\tilde{Y}Y_{1t-1} - \bar{\tilde{Y}Y}_{1t-1})^2$ and $(BB_{1t-1} - \overline{BB}_{1t-1})^2$.

Unlike Pindyck's approach, Chow's state-space system (see equation A-2.17) does not require any transformation to the reduced form (2.5). In fact, in Chow's terminology the dynamic system of the one-country optimal control problem is given by:

$x_t^c = A^c x_{t-1}^c + B$

where $x_t^c \overset{d}{=} \Big[y_t$

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$$A^c = \begin{bmatrix} \bar{A} & \bar{C} \\ 0 & 0 \end{bmatrix}$$

$$B^c = \begin{bmatrix} \bar{D} \\ 0 \end{bmatrix} \overset{d}{=} \begin{bmatrix} D \\ D \\ a \\ (\\ (\end{bmatrix}$$

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Appendix A-2)

$$x_t^c = A^c x_{t-1}^c + B^c u_t^c + C^c z_t^c \tag{2.9}$$

where $x_t^c \overset{d}{=} \left[y_t \,\vdots\, u_t\right]' \overset{d}{=} \left[Y_{1t}, B_{1t}, K_{1t}\ G_{1t}, r_{1t}\right]'$ is the (5x1) vector of current endogenous and controlled variables; $u_t^c \overset{d}{=} u_t = \left[G_{1t}, r_{1t}\right]'$ is the (2x1) vector of controlled variables; $z_t^c \overset{d}{=} z_t \overset{d}{=} \left[1, TT_{1t}, T_{1t}, X_{1t}, r_{2t}\right]'$ is the (5x1) vector of exogenous variables;

$$A^c = \left[\begin{array}{c:c} \bar{A} & \bar{C} \\ \hdashline 0 & 0 \end{array}\right] \overset{d}{=} \begin{bmatrix} A_{11} & 0 & A_{13} & 0 & A_{15} \\ A_{21} & 0 & A_{23} & 0 & A_{25} \\ \gamma_{11} & 0 & a_{34} & 0 & -\gamma_{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad B^c = \left[\begin{array}{c} \bar{B} \\ \hdashline I \end{array}\right] = \begin{bmatrix} Q_{11} & 0 \\ -Q_{21} & \eta_{11} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\bar{C} = \left[\begin{array}{c} \bar{D} \\ \hdashline 0 \end{array}\right] \overset{d}{=} \begin{bmatrix} D_{11} & D_{12} & D_{13} & Q_{11} & 0 \\ D_{21} & D_{22} & D_{23} & 1-Q_{21} & -\eta_{11} \\ a_{36} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{are known matrices.}$$

Since there are no constraints on the policy-variable magnitudes, the weight matrix in the welfare cost function (see equation A-2.18 in Appendix A-2) has the form:

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$$Q^c = \left[\begin{array}{c|c} Q^P & 0 \\ \hline 0 & 0 \end{array}\right] \underline{\underline{d}} \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

instead of

$$Q^c = \left[\begin{array}{c|c} Q^P & 0 \\ \hline 0 & R^P \end{array}\right] \underline{\underline{d}} \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{11} & 0 \\ 0 & 0 & 0 & 0 & r_{22} \end{bmatrix}$$

where q_{11} and q_{22} are weights attached to the respective quadratic deviations from the internal and external balance, i.e., weights attached to $(\tilde{Y}_{1t}-\bar{\tilde{Y}}_{1t})^2$ and $(B_{1t}-\bar{B}_{1t})^2$. The weights r_{11} and r_{22} are attached to the respective quadratic deviations from the limits set on fiscal and monetary variable magnitudes, i.e., weights attached to $(G_{1t}-\bar{G}_{1t})^2$ and $(r_{1t}-\bar{r}_{1t})^2$.

Appendix A-3 shows the equivalence of these two approaches. However, this study will use Chow's approach to derive the optimal policy mix for internal and external balance for two reasons. First, Chow's state-space system formulation is more straightforward and simpler to apply, given the particular structural form of Votey's model, than Pindyck's formulation, which requires a transformation of variables before applying his result (see equation A-2.16). Secondly, Chow's method just requires basic matrix manipulations even when constraints x policy-vari

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on policy-variable magnitudes are introduced, while Pindyck's method requires the determination of Riccati and tracking equations (see Appendix A-2) in the case of $R^P \neq 0$.

Whether formulated in Pindyck's or Chow's frameworks, the one-country optimal control problem is typically a linear quadratic tracking problem. However, it is restrictive in two senses. First, the quadratic form of the welfare cost function is subject to criticism because it assumes that the deviations from either side of the targets--negative as well as positive deviations--are of equal cost. However, in the real world there is no such symmetry in the sharing of the burden for the adjustment of the balance of payments equilibrium; for example, along this critic Friedman (1972, 1973) has made a contribution to this problem by introducing a "piece-wise quadratic criterion function" which divides the range of possible values for each endogenous variable and each policy variable into three regions: values within the middle region are assigned zero cost, but values within the two extreme regions are penalized quadratically but asymmetrically. Still, the criterion function remains quadratic, and further research needs to be done in this field.

The second criticism is of the linear form of the econometric model, which is a very crude approximation of the real world. However, the disadvantage of a linear quadratic framework is compensated for by the computational advantages (Pindyck:1973).

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2.2.2. Optimal Solution for Internal and External Balance

Chow's optimal solution is given by A-2.20 in Appendix A-2.

After substitutions and computations (see Appendix A-4), it becomes:

$$\begin{bmatrix} C^f_t \\ I^f_t \end{bmatrix} = \begin{bmatrix} a_{11} A_{11} & 0 & a_{11} A_{13} & 0 & a_{11} A_{15} \\ \frac{1}{n_{11}} (\beta_{11} A_{11} + A_{21}) & 0 & \frac{1}{n_{11}} (\beta_{11} A_{13} + A_{23}) & 0 & \frac{1}{n_{11}} (\beta_{11} A_{15} + A_{25}) \end{bmatrix} \begin{bmatrix} \bar{Y}^f_{t-1} \\ B^f_{t-1} \\ K^f_{t-1} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ \frac{\beta_{11}}{n_{11}} & \frac{1}{n_{11}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{Y}_{1t} \\ B_{1t} \\ K_{1t} \\ \bar{G}_{1t} \end{bmatrix}$$

$$\begin{bmatrix} G^*_{1t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} a_{11} A_{11} & 0 & a_{11} A_{13} & 0 & a_{11} A_{15} \\ \frac{1}{\eta_{11}} (\beta_{11} A_{11} + A_{21}) & 0 & \frac{1}{\eta_{11}} (\beta_{11} A_{13} + A_{23}) & 0 & \frac{1}{\eta_{11}} (\beta_{11} A_{15} + A_{25}) \end{bmatrix} \begin{bmatrix} \tilde{Y}^*_{1t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ G^*_{1t-1} \\ r^*_{1t-1} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ \frac{\beta_{11}}{\eta_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{Y}_{1t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{G}_{1t} \\ \underline{r}_{1t} \end{bmatrix}$$

$$- \begin{bmatrix} a_{11} D_{11} & a_{11} D_{12} & a_{11} D_{13} & 1 & 0 \\ \frac{1}{\eta_{11}} (\beta_{11} D_{11} + D_{21}) & \frac{1}{\eta_{11}} (\beta_{11} D_{12} + D_{22}) & \frac{1}{\eta_{11}} (\beta_{11} D_{13} + D_{23}) & \frac{1}{\eta_{11}} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \Pi_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix} \qquad (2.10)$$

or after substituting $Y_{1t} = \delta_{10} + \delta_{12} L_{1t}$ and $\bar{B}_{1t} = 0$ into equation 2.10:

$$\begin{bmatrix} G^*_{1t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} -a_{11} A_{11} & 0 & -a_{11} A_{13} & 0 & -\gamma_{11} A_{15} \\ \frac{-1}{\eta_{11}} (\beta_{11} A_{11} + A_{21}) & 0 & \frac{-1}{\eta_{11}} (\beta_{11} A_{13} + A_{23}) & 0 & \frac{-1}{\eta_{11}} (\beta_{11} A_{15} + A_{25}) \end{bmatrix} \begin{bmatrix} \tilde{Y}^x_{1t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ G^*_{1t-1} \\ r^*_{1t-1} \end{bmatrix} + \begin{bmatrix} -a_{11} (D_{11} - \delta_{10}) \\ \frac{-1}{\eta_{11}} (\beta_{11} D_{11} - \beta_{11} \delta_{10} + D_{21}) \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \delta_{12} & -a_{11} D_{12} & -a_{11} D_{13} & -1 & 0 \\ \frac{\beta_{11} \delta_{12}}{\eta_{11}} & \frac{-1}{\eta_{11}} (\beta_{11} D_{12} + D_{22}) & \frac{-1}{\eta_{11}} (\beta_{11} D_{12} + D_{22}) & \frac{-1}{\eta_{11}} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ L_{1t} \\ \Pi_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix} \qquad (2.11)$$

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It is noted that the optimal control solution in equations 2.10 or 2.11 is independent of the weights attached to the deviations from the targets. This means that internal as well as external balance affects the optimal policy mix equally, and no trade-off between these two targets exists. The optimal path is obtained by substituting equation 2.11 into the dynamic system (equation 2.9):

$$
\begin{bmatrix} \tilde{Y}^*_{1t} \\ B^*_{1t} \\ K^*_{1t} \\ G^*_{1t} \\ r^*_{1t} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\gamma_{11} & 0 & a_{34} & 0 & -\gamma_{12} \\
-a_{11}A_{11} & 0 & -a_{11}A_{13} & 0 & -a_{11}A_{15} \\
\frac{-1}{\eta_{11}}(\beta_{11}\ A_{11} + A_{21}) & 0 & \frac{-1}{\eta_{11}}(\beta_{11}\ A_{13} + A_{23}) & 0 & \frac{-1}{\eta_{11}}(\beta_{11}\ A_{15} + A_{25})
\end{bmatrix}
\begin{bmatrix} Y^*_{1t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ G^*_{1t-1} \\ r^*_{1t-1} \end{bmatrix}
\tag{2.12}
$$

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+
\begin{bmatrix}
\delta_{10} & \delta_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-a_{11}(D_{11} - \delta_{10}) & a_{11}\delta_{12} & -a_{11}D_{12} & -1 & 0 \\
\frac{-1}{\eta_{11}}(\beta_{11}D_{11} - \beta_{11}\ 10 + D_{21}) & \frac{\beta_{11}\delta_{12}}{\eta_{11}} & \frac{-1}{\eta_{11}}(\beta_{11}D_{13} + D_{23}) & \frac{-1}{\eta_{11}} & 1
\end{bmatrix}
\begin{bmatrix} \frac{1}{L_{1t}} \\ TT_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix}
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Next, the decision-makers of Country I may wish to evaluate the performance of the economic system with respect to the targets. This can be done by measuring the welfare cost function (see equation A-2.18):

$$\hat{J}^c = \frac{1}{2} \sum_{t=1}^{N} \left[(x_t^{c*} - \bar{x}_t^c)' Q^c (x_t^{c*} - \bar{x}_t^c) \right]$$

After substitutions and computations (see Appendix A-5) it is found that $\hat{J}^c = 0$.

Several conclusions can be derived from this analysis.

First, Tinbergen's problem of quantitative economic policy (Tinbergen:1954) in the theory of economic policy is nothing but a linear tracking problem in the theory of optimal control when time intervenes. In other words, given (a) the structure of an economy, (b) the target variables and their numerical values, and (c) the nature of the instrument variables, the problem consists in finding the numerical values of the instruments as a function of the targets and certain structural data such that the optimum policy is obtained. The optimal control tools provide a systematic procedure to solve the dynamic problem of quantitative economic policy.

Second, unlike Mundell's "division labour" proposal (Mundell:1962) the interdependency of economic policies prevails under the auspices of the optimal control approach; that is, the values of the instrument variables are dependent, generally speaking, on all target sets and cannot be considered in isolation. However, this study, because of the particular structure of Votey's one-country model, notes that the optimal fiscal policy G^*_{1t} is governed only by the full-employment target $\bar{\bar{Y}}_{1t}$ and, in turn, the latter target can only be obtained by the correct fi

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the correct fiscal policy, while the optimal monetary policy r^*_{1t} has to obey the two targets together. The second target $\overline{B}_{1t}$, or the balance-of-payment equilibrium, depends on both instruments, but once G^*_{1t} has been fixed it can only be taken care of by r^*_{1t}. It is in this sense that we can conclude along with Mundell that the interest rate is assigned to the maintenance of balance-of-payments equilibrium, and government expenditure is assigned to full employment (Mundell:1962, Patrick:1968). This proper assignment leads to a stable system (Votey:1969).

Third, "Tinbergen's Rule" is verified within the framework of optimal control theory. If the number of independent variables to be controlled--the number of non-zero diagonal elements in the weight-matrix Q^c--is equal to the number of independent instruments, the variables will be on target exactly with $\hat{J}^c=0$ (Chow:1972c) and all the unknowns of the policy problem are solved. The optimal solution is unique and independent of the weights attached to the quadratic deviations from the targets. However, the optimal solution may not be feasible since a set of targets together with a choice of instrument variables in a given economy may be called inconsistent (Tinbergen:1954) if it requires values of the instrument variables which are declared inadmissible for practical consideration by certain constraints on the policy-variable magnitudes. Section 3 will deal with the appraisal and amendment of the optimal solution, if necessary, for a particular economy such as the United States.

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2.3. U.S. Optimal Policies for Internal and External Balance: Appraisal and Amendment of the Optimal Solution

Given the analytical result for the one-country optimal solution to the problem of internal-external balance (Appendix A-4), Votey's numerical values for the structural coefficients and historical data will be used to derive the U.S. optimal policy mix for the period 1961-1970. However, the optimal solution is not feasible for technical and practical reasons, such as negative values of interest rates. Therefore, limits on policy-variable magnitudes have to be set (Tinbergen:1954), and the amendment of optimal solution is required.

First, Votey's numerical values are substituted for the estimated structural constants (Votey:1969) into equation A-4.6 and the dynamic system equation A-2.18. The computation results are:

$$\begin{bmatrix} z^*_{1t} \\ r^*_{1t} \end{bmatrix} =$$

$$
\begin{bmatrix} G^*_{1t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} .0025 & 0 & .0014 & 0 & 434.2236 \\ 0 & 0 & .0005 & 0 & -0.2037 \end{bmatrix} \begin{bmatrix} \bar{\tilde{Y}}^*_{1t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ G^*_{1t-1} \\ r^*_{1t-1} \end{bmatrix} +
$$

$$
\begin{bmatrix} .3810 & 0 & 0 & 0 & 0 \\ .0028 & .0216 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\tilde{Y}}_{1t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{G}_{1t} \\ \bar{r}_{1t} \end{bmatrix} + \qquad (2.13)
$$

$$
\begin{bmatrix} -46.2265 & -28.3500 & 0.6190 & -1 & 0 \\ 0.3043 & 0.6126 & -0.0028 & -.0216 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ TT_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix}
$$

$$\begin{bmatrix} \tilde{Y}^*_{1t} \\ B^*_{1t} \\ K^*_{1t} \\ G^*_{1t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} -.0067 & 0 & -.00364 & 0 & -1139.7064 \\ 0.0009 & 0 & -.0212 & 0 & 158.7130 \\ -.0027 & 0 & 1.0618 & 0 & -461.7000 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{Y}^*_{1t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ G^*_{1t-1} \\ r^*_{1t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} 2.6247 & 0 \\ -0.3438 & 46.279 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G^*_{1t} \\ r^*_{1t} \end{bmatrix} \qquad (2.14)$$

$$+ \begin{bmatrix} 121.4358 & 74.4102 & -1.6246 & 2.6247 & 0 \\ -29.9914 & 18.6032 & 0.3438 & 0.6562 & -46.2790 \\ 9.1299 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ TT_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix}$$

Given these two equations, the optimal stabilization policies (G^*_{1t}, r^*_{1t}) for the United States over 10 periods from 1961 to 1970 can be derived with the initial conditions given by:

$\tilde{Y}_1(0) = 383.37$

$B_1(0) = 4.17$

$K_1(0) = 508.69$

$G_1(0) = 94.9$

$r_1(0) = 0.0215$

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Table 2.1. N

Period (Year)
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where (0) refers to 1960. All these values are historical. The nominal values of the state variables are given by Table 2.1.

Table 2.1. Nominal values for state variables - U.S.: 1961-1970

Period (Year)	$\tilde{\bar{Y}}_{1t}$	$\bar{B}_{1t}$	$\bar{K}_{1t}$	$\bar{G}_{1t}$	$\bar{r}_{1t}$
1(1961)	283.2452	0	523.9507	98.7	0.0215
2(1962)	283.8683	0	539.6692	102.65	0.0215
3(1963)	288.7687	0	555.8592	106.76	0.0215
4(1964)	293.8258	0	572.5349	111.03	0.0215
5(1965)	299.3091	0	589.7109	115.47	0.0215
6(1966)	304.5954	0	607.4022	120.09	0.0215
7(1967)	310.9349	0	625.6242	124.89	0.0215
8(1968)	316.5227	0	644.3929	129.89	0.0215
9(1969)	324.5467	0	663.7246	135.09	0.0215
10(1970)	332.5143	0	683.6363	140.49	0.0215

The potential output net of capital accumulation ($\tilde{\bar{Y}}_{1t}$) is determined by the labor force. The nominal balance of payments is assumed to be the balance-of-payments equilibrium that is $\bar{B}_{1t}=0$. Also, it is assumed that the nominal capital stock and the nominal government expenditures grow, respectively, at 3 percent and 4 percent per annum from their initial 1960 values, while the nominal interest rate $\bar{r}_{1t}$ is fixed at its initial 1960 value.

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Table 2.2. Historical values for exogenous variables - U.S.: 1961-1970

Period (Year)	1	TT_{1t}	T_{1t}	x_{1t}	r_{2t}
1(1961)	1	1.0000	144.63	20.99	0.0299
2(1962)	1	1.0101	157.03	21.68	0.0391
3(1963)	1	1.0000	168.76	23.28	0.0378
4(1964)	1	0.9902	174.07	26.63	0.082
5(1965)	1	1.0000	190.06	27.53	0.0454
6(1966)	1	1.0094	213.33	30.45	0.0496
7(1967)	1	1.0280	228.93	31.63	0.0595
8(1968)	1	1.0183	263.31	34.66	0.0624
9(1969)	1	1.0268	296.70	37.99	0.0781
10(1970)	1	1.0083	302.00	43.23	0.0444

The U.S. optimal stabilization policies for 10 periods (1961-1970) are obtained by the following steps:

(1) Compute G_1^* (1) and r_1^* (1) from equation 2.13 using the initial conditions: $\left[\bar{\bar{Y}}_1 (0), B_1 (0), K_1 (0), G_1 (0), r_1 (0)\right]'$ and the exogenous variables of period 1 given by Table 2.2.

(2) Compute $\left[\bar{\bar{Y}}_1^* (1), B_1^* (1), K_1^* (1), G_1^* (1), r_1^* (1)\right]'$ from equation 2.14. Now $\left[\bar{\bar{Y}}^*_1 (1), B_1^* (1), K_1^* (1), G_1^* (1), r_1^* (1)\right]'$ can be used in equation 2.13 to compute $\left[G_1^* (2), r_1^* (2)\right]'$, which can be used in equation 2.14 to compute

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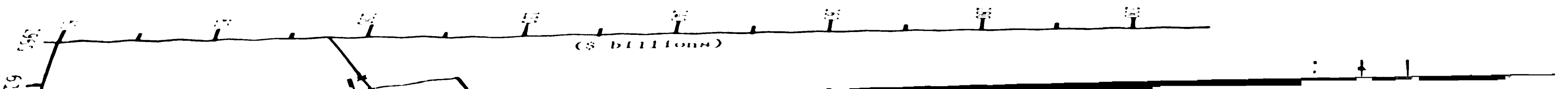
$\left[\bar{\tilde{Y}}_1^* (2),\ B_1^* (2),\ K_1^* (2),\ G_1^* (2),\ r_1^* (2)\right]'$, and so on. Continue the process until all of the control vectors $\left[G_1^*(t),\ r_1^*(t)\right]'$, t=1,...,10 and all of the state variable vectors $\left[\bar{\tilde{Y}}_1^*(t),\ B_1^*(t),\ K_1^*(t),\ G_1^*(t),\right.$ $\left.r_1^*(t)\right]'$, t=1,...,9, have been computed.

These two steps for obtaining computational optimal control solution require only basic matrix manipulations--additions, subtractions, multiplications and small inversions.

If the cost functional J^c (equation A-2.18) does not penalize for policy-instrument deviations from the nominal, the penalties q_{11} and q_{22} for deviations of the two endogenous variable $\tilde{Y}_{1t}$ and B_{1t} from their nominal paths do not appear in the analytical optimal control solution (equation A-4.6). Therefore, the optimal policy mix for attaining the joint balance is unique and the trade-off between the targets does not influence the optimal policy formulation. This comes to confirm Tinbergen's proposal on the uniqueness of the solution whenever the number of independent instruments is equal to the number of independent targets. The results are presented in graphical forms (Figures 2.1 to 2.4) with time on the horizontal axis so as to easily observe the general form and characteristics of the optimal solution.

Next, the effectiveness and performance of the U.S. optimal policy mix will be appraised. In optimal control theory, the values of the instrument variables become the unknowns, dependent upon the predetermined desired values of the target variables. Therefore, the criterion of "effectiveness" of a particular policy instrument with respect to a particular target variable is different than the one used in the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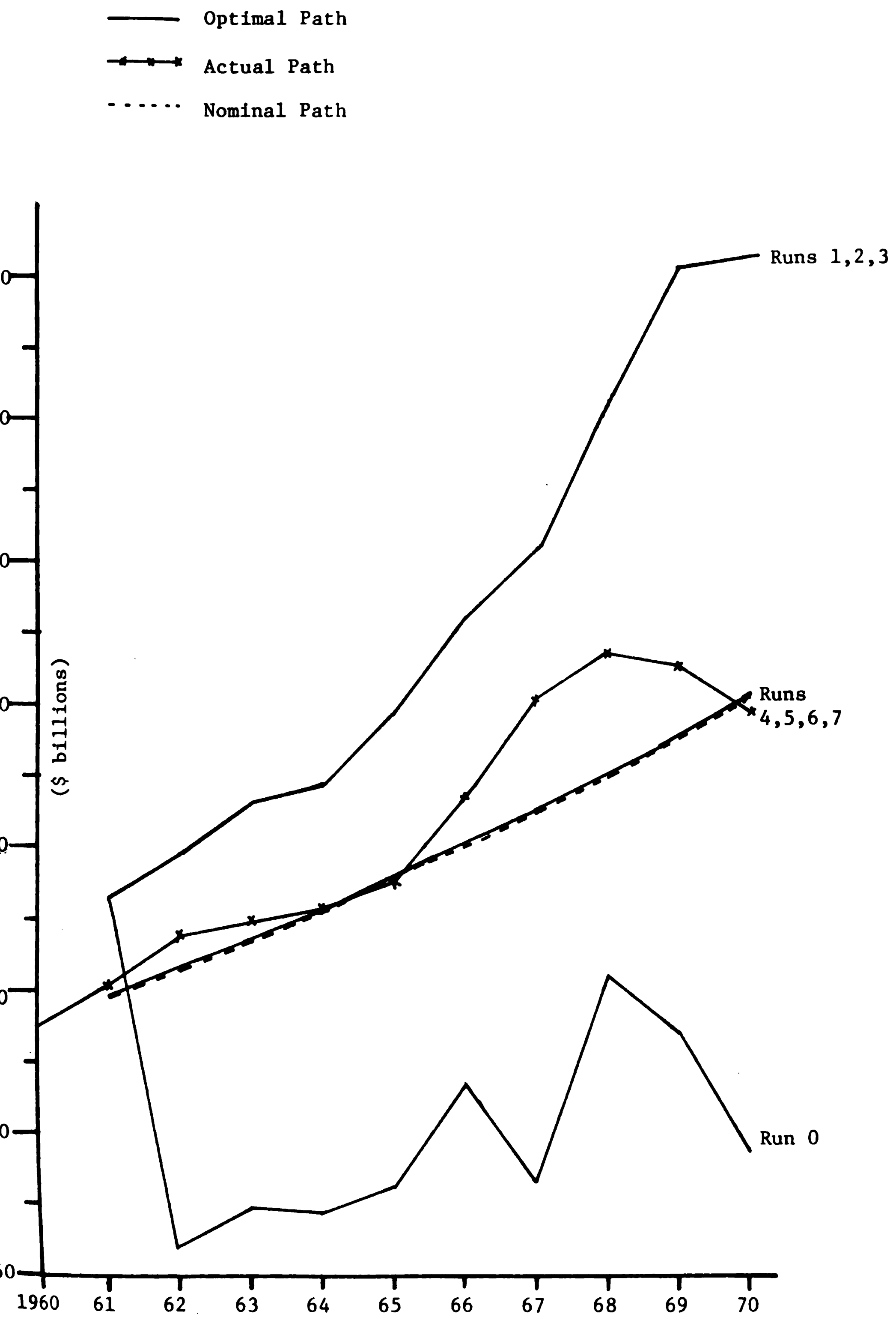

Figure 2.1. U.S. Government expenditures: optimal paths compared with actual and nominal paths (one-country model).


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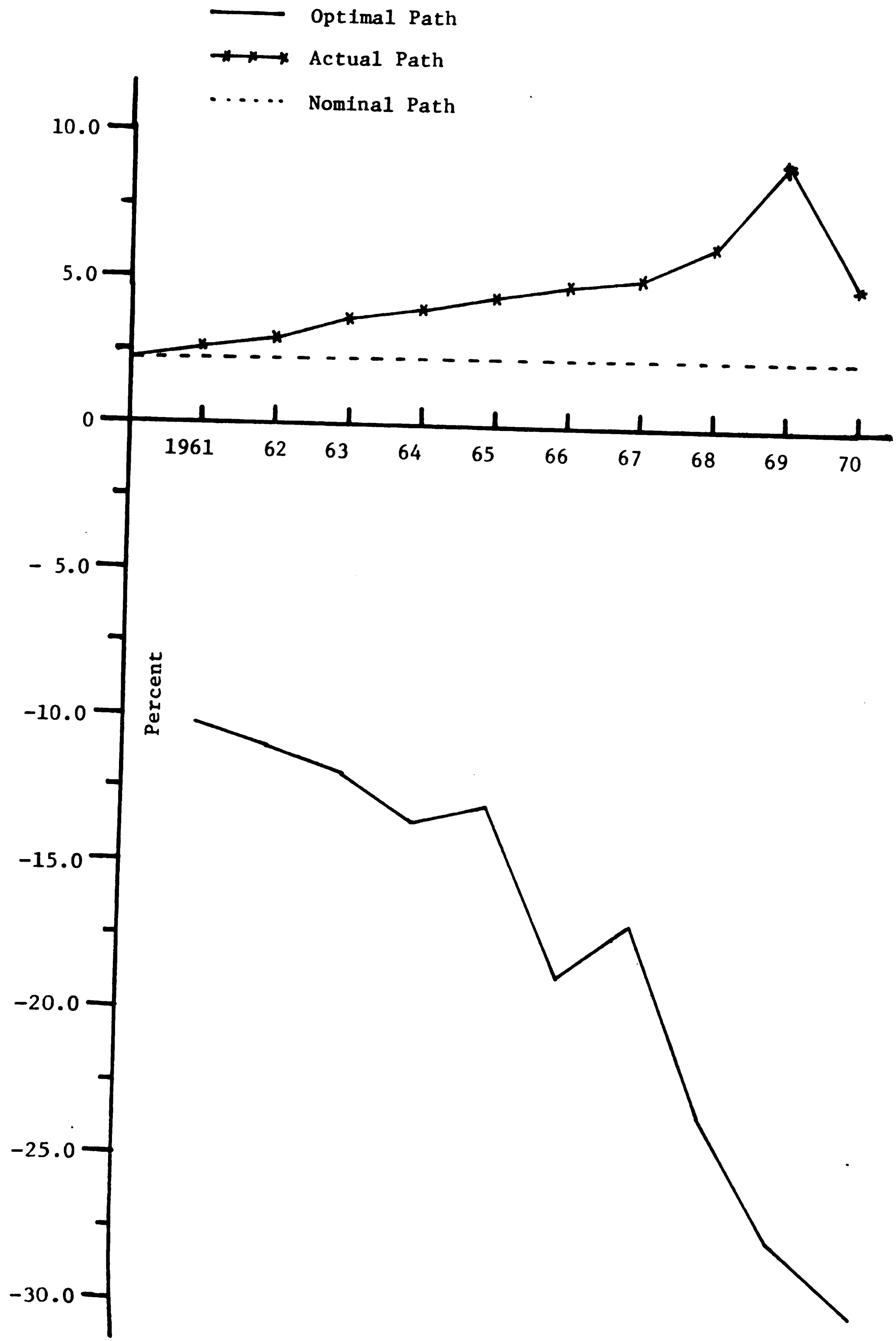
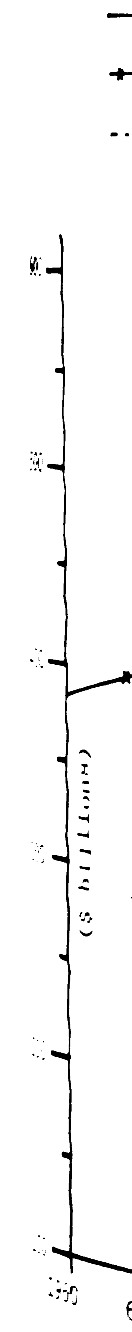


Figure 2.2. U.S. interest rate: optimal path compared with actual and nominal paths (one-country model).



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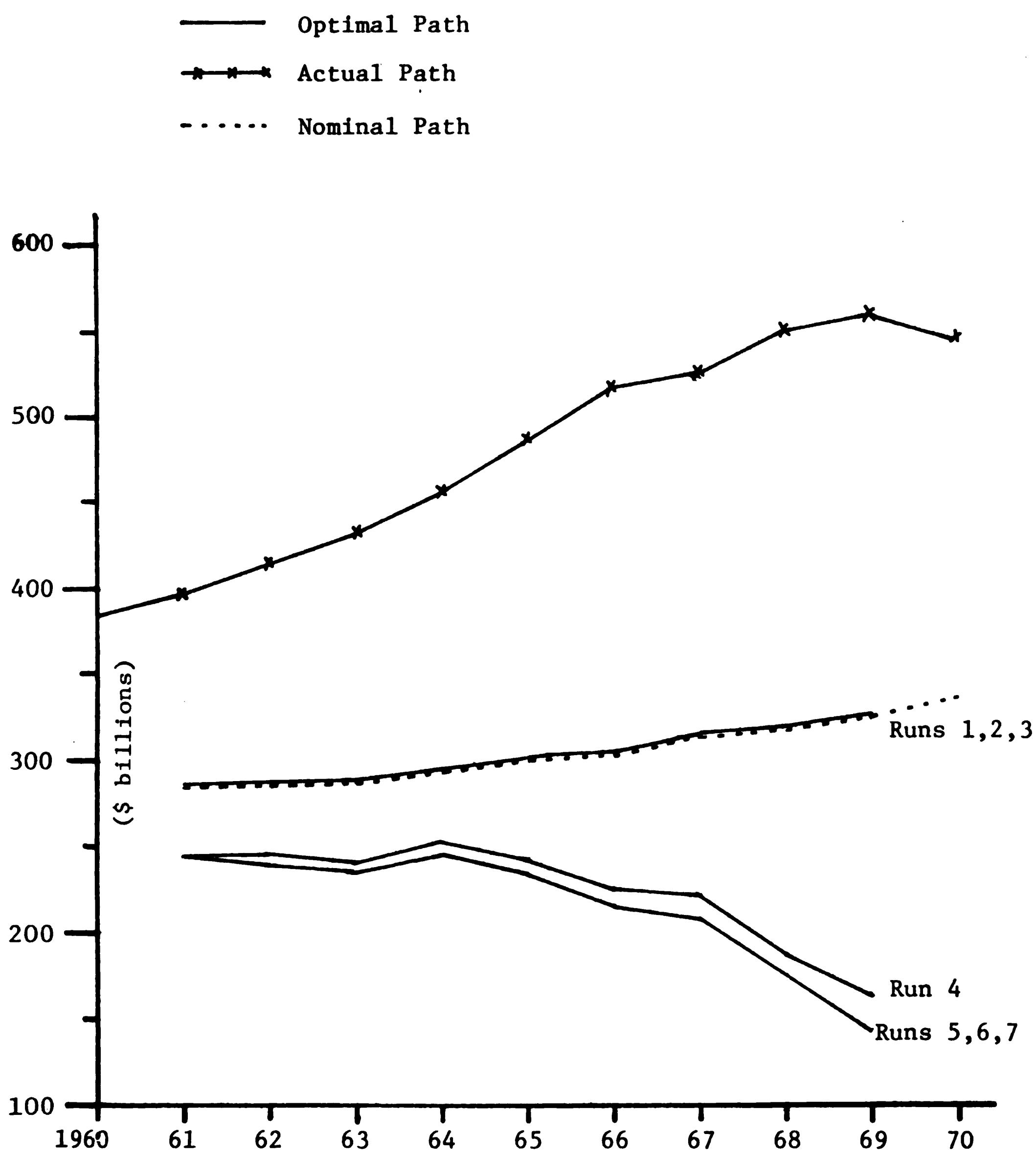
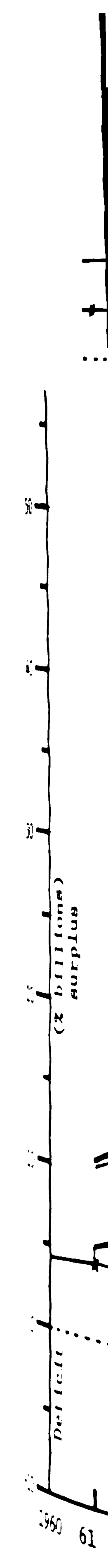


Figure 2.3. U.S. optimal GNP net of capital stock compared with the actual and potential GNP (one-country model).



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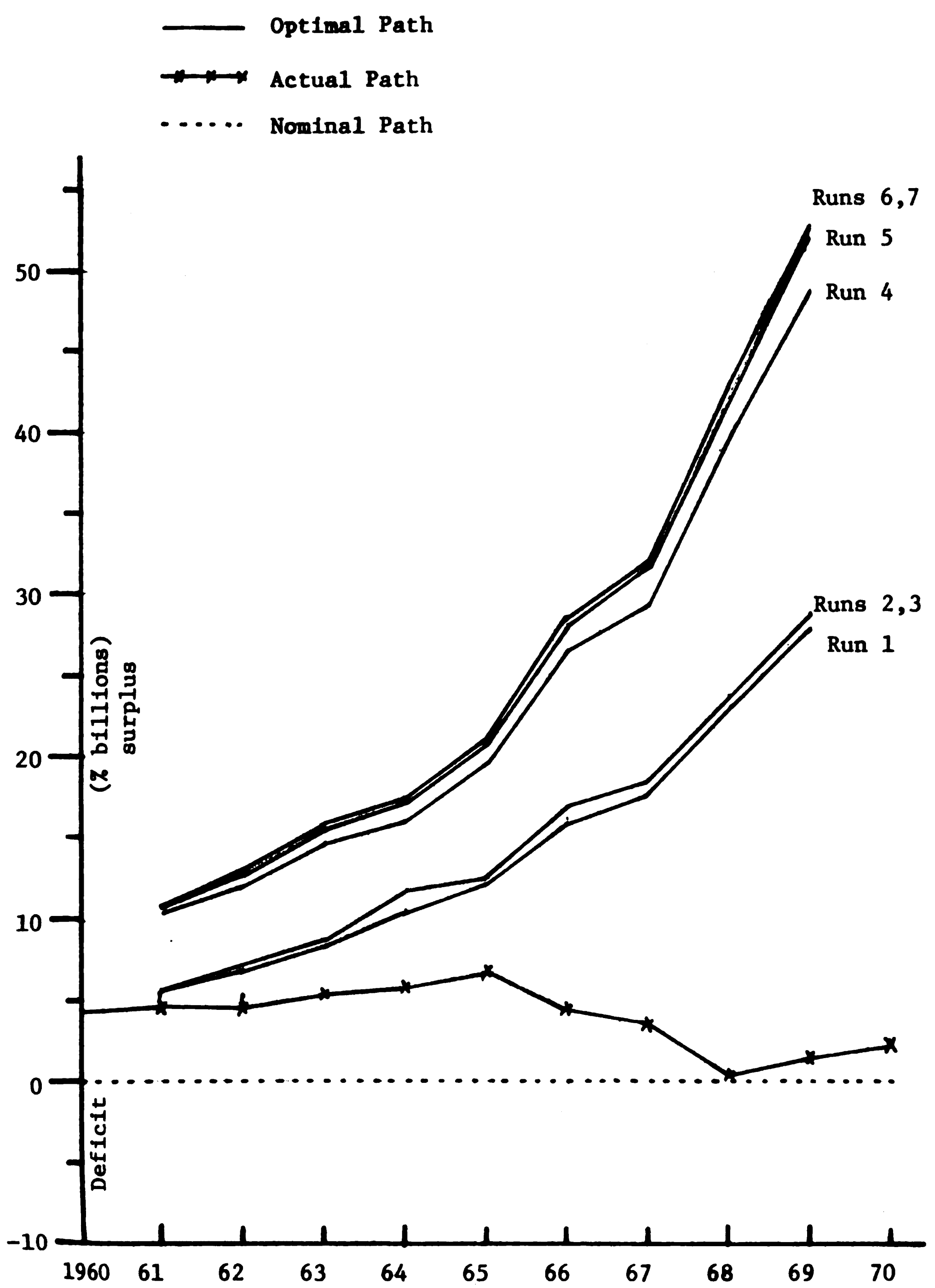
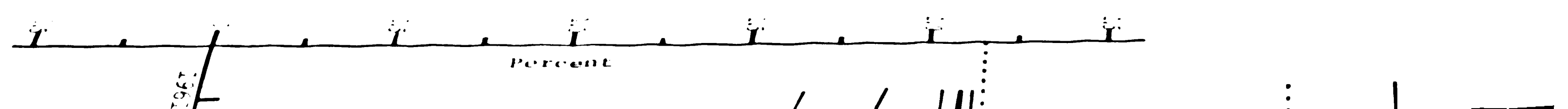


Figure 2.4. U.S. balance of payments: optimal paths compared with the actual path (one-country model).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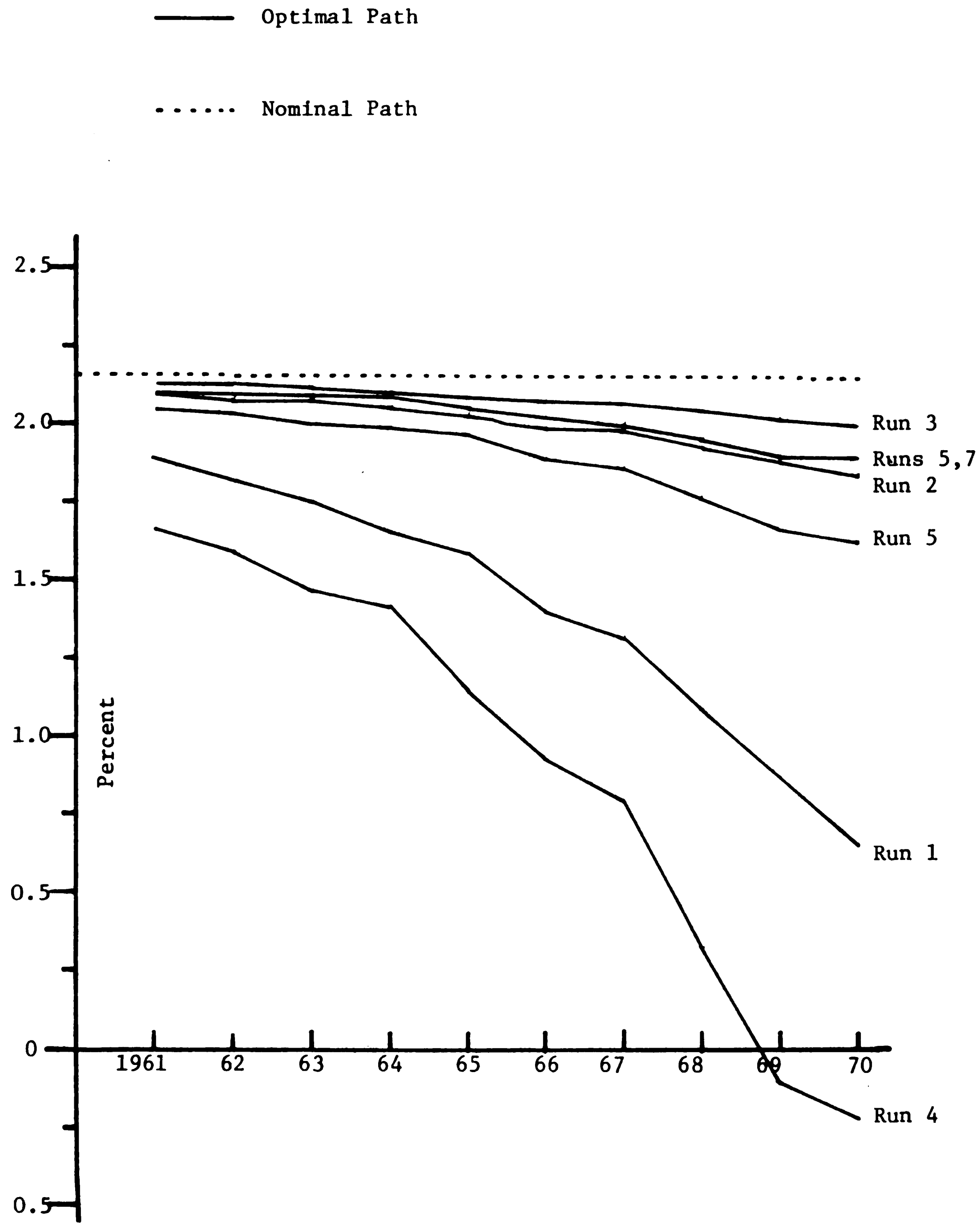

Figure 2.5. U.S. interest rate: optimal path compared with the nominal rate (one-country model).

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"traditional" or multiplier approach (Tinbergen:1954). In other words, a policy instrument generally is more effective the smaller the change in the policy variable required to bring about a given change in the target variable when all other targets are held constant, e.g., the smaller $\frac{1}{\delta G^*_{1t} / \delta \bar{\tilde{Y}}_{1t}}$, the more effective G^*_{1t}. Within the confines of Votey's model, the matrix

Row	$\bar{\tilde{Y}}_{1t}$	$\bar{B}_{1t}$
1	0.3810	0
2	0.0028	0.216

of equation 2.13 shows that the fiscal policy or government expenditure is the most effective instrument to regulate employment since it is governed only by the target $\bar{\tilde{Y}}_{1t}$ which, in turn, is determined by the labor force. The optimal path for the fiscal instrument is derived for the period 1961-1970. However, as Figure 2.1 shows, the optimal path is far below the actual or historical path. This means that during the period in question, fiscal policy was overused by the U.S. policymakers and, as a result, the target of internal balance was overshot. Figure 2.3 shows that the actual or historical path of the GNP fluctuates above the nominal path $\bar{\tilde{Y}}_{1t}$, which represents the situation of full employment without inflation. Therefore, it is not surprising that during the last decade a troublesome inflationary spiral has threatened the U.S. economy. This is because of an excess demand created by a boom in government expenditures.

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Unlike the fiscal policy G^*_{1t}, the optimal monetary policy r^*_{1t} has to obey both internal and external balance. But when using the "effectiveness" criterion described previously, the monetary policy is more effective in maintaining balance-of-payments equilibrium than in regulating full employment. Therefore, once the fiscal policy has been set at the optimal level, a level which brings the economy close to a situation of full employment without inflation, the maintenance of external balance can be taken care of only by the monetary policy. Equation 2.13 permits the computation of optimal path for the monetary instrument. It is found that the simultaneous achievement of internal and external balance would require high negative values in the short-term interest rate (Figure 2.2). Therefore, a limit or a so-called boundary condition (Tinbergen:1967) has to be set on this policy-variable magnitude to indicate that negative interest rates resulting from the application of optimal control theory to the U.S. economy are technically impossible.

Boundary conditions and their violation by the optimal policy mix, the so-called inconsistency of the optimal policy, require the problem of quantitative economic policy to be reformulated (Tinbergen:1967). Within the framework of optimal control theory it can be done by penalizing the deviations of the control variables from their nominal values, which represents the limits on acceptable instrument variable magnitudes.

For the one-country problem, the newly defined matrix Q^c of the cost function, including the boundary conditions, is:

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$$Q^c = \left[\begin{array}{c:c} Q^P & 0 \\ \hdashline 0 & R^P \end{array}\right] = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{11} & 0 \\ 0 & 0 & 0 & 0 & r_{22} \end{bmatrix}$$

The nominal values for the monetary and fiscal instruments to be tracked are given in Table 2.1.

A sensitivity analysis of the weighting factors on components of only the control vector is performed. Since the optimal control solution does not depend on the weights q_{11} and q_{22}, it is assumed that through the performance of seven experiments, both are equal to 1, for reasons of simplicity. The cost function for all the experiments is given in Table 2.3, and the results are shown in Figures 2-1 to 2.5.

Table 2.3. One-country model: penalty weights attached to the deviations of state and control variables from their nominal values

Run	q_{11}	q_{22}	q_{33}	r_{11}	r_{22}
0	1	1	0	0	0
1	1	1	0	0	10^5
2	1	1	0	0	$5x10^5$
3	1	1	0	0	10^6
4	1	1	0	10^5	10^5
5	1	1	0	$5x10^5$	$5x10^5$
6	1	1	0	10^5	10^6
7	1	1	0	$5x10^5$	10^6

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Results

Since only the optimal monetary policy is not feasible because of the negative value of interest rates, the first three experiments are assigned increasing penalty costs for deviations of the monetary instrument from the nominal value, fixed at a rate of 2.15 percent, while government spending is free to fluctuate to achieve the internal balance. Figure 2.5 shows that the higher the penalty cost, the closer the simulated optimal path for r_{1t} is to its nominal path. But the price paid is that the balance-of-payments equilibrium is no longer attained, and within the context of Votey's model, the more the interest rate is constrained to the limit of 2.15 percent, the higher the surplus in the balance of payments is (Figure 2.4). In summary, the joint internal and external balance cannot be achieved for the U.S. economy during the period 1961-1970 once the deviations of the monetary policy from the nominal path are penalized. Since the interest rate is the most effective instrument with respect to the external balance target, the latter has to be dropped.

Unlike the optimal monetary policy, which in the first three experiments gets closer and closer to its nominal value, the optimal path for the fiscal policy diverges strongly from the nominal path (Figure 2.1). This means that a feasibility trade-off exists between the two components of the control vector. However, even if the degree of freedom given to the fiscal tool allows internal balance, federal budget constraints as well as political constraints could prevent a large change in government spending which might be required for feasibility and stability in the monetary instrument.

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In the last four experiments, limits on both policy-variable magnitudes are introduced. Figures 2.1 and 2.5 show the feasibility trade-off between the two instruments. The tracking of government expenditure to the nominal $\overline{G}_{1t}$ is sensitive only to the increasing value of the penalty costs on deviations of monetary policy from 2.15 percent, while the tracking of the nominal interest rate depends on the weighting factors for both instruments. It is noted that for the same penalty costs r_{22}, the effectiveness of the interest rate in tracking $\overline{r}_{1t}$ diminishes with introduction of limits on government expenditure magnitudes. Furthermore, neither the internal nor the external balance target is achieved (Figures 2.3 and 2.4). In fact, within the confines of Votey's model, the U.S. economy must be in a situation of deflation combined with a balance of payments surplus for the monetary and fiscal policies to be feasible. Does this mean that the initial conditions for the one-country optimal control problem are far away from the "ideal" conditions or the arbitrary set of nominal values for monetary and fiscal policies to be tracked need to be more inflationist, or simply the policy for internal and external balance is not feasible within the optimal control framework due to political, social and ethical constraints established in any given economy? No definitive answer can be given.

Conclusion

Under the assumption of constant price, fixed exchange rate and passiveness on the part of the second country, i.e., fixed foreign interest rate, it is found for the case of one-country that:

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(1) The solution for simultaneously attaining internal and external balance can always be determined via the optimal control techniques if Tinbergen's principle on equality between the number of independent instruments and the number of independent targets is met, and if there are no constraints on instrument-variable magnitudes. There is no guarantee that this optimal policy mix will be feasible in a given economy.

(2) Policy inconsistency in Tinbergen's sense (Tinbergen:1967) arises in the U.S. economy; that is, the optimal monetary policies to achieve the overall balance violate the boundary conditions or limits on interest rate variable magnitudes that, for practical or political considerations, have to be set.

(3) One of the targets has to be dropped, and within the confines of Votey's dynamic model the choice of that target rests on Mundell's "division of labor" principle. For the U.S. case, the tracking for external balance has to be dropped at the expense of the tracking for the nominal interest rate fixed at 2.15 percent.

(4) If constraints on all policy-variable magnitudes are included in the optimal control problem, neither the internal nor the external balance is achieved

and a trade-off exists between attainment of the overall balance and feasibility of monetary and fiscal policies to be carried out toward the given targets.

CHAPTER III

TWO-COUNTRY MODEL

CASE A: PASSIVE RESPONSES AND COMMON EXTERNAL BALANCE

This chapter deals with a three-target and three-instrument problem: the fiscal authority of Country II acts to attain full employment in II while the fiscal and monetary authorities of Country I strive for both internal full employment and common external balance. To find the optimal policies for internal and external balance in both countries the procedure used in the one-country model case will be repeated. After the presentation of Votey's two-country model (Votey:1969), the two-country optimal control problem will be formulated into Pindyck's framework as well as that of Chow. Then for reasons similar to the previous case, Chow's approach will be chosen to derive the optimal solution both analytically and numerically with reference to the United States and Canada. Finally, the optimal solution will be appraised and amended.

3.1. Presentation of Votey's Model

The two-country model is obtained by adding to the one-country model a second set of behavioral equations for consumption, investment, imports, production function and identities defining capital stock and national income with Country II subscripts. Votey's two-count

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two-country model is considered under the following assumptions:

(1) The foreign rate of interest is given. This assumes Country II's passive cooperation such that Country I may adjust its own rate to achieve external balance without foreign interference or countermeasures.

(2) Stein's results dealing with short term interest rates will be considered to represent the degree of sensitivity of capital flows to interest rate differentials.

(3) Unlike the one-country model, the exports of Country I will be composed primarily of imports of Country II and will therefore be a dependent variable in the system.

(4) The value of imports of Country I from Country II is assumed to be fixed or exogenous to the system.

The variables are: $i = 1,2$

B_t: Common balance of payments

C_{it}: Consumption expenditures in Country i

G_{it}: Government expenditures of Country i

I^G_{it}: Gross investment expenditures in Country i

I^n_{it}: Net investment expenditures in Country i

IE_{1t}: Investment earnings of Country I from abroad

K_{it}: Capital stock of Country i

L_{it}: Labor force in Country i

M_{it}: Total imports demand of Country i

M_{3t}: Imports of Country III from I and II

M^{III}_{1t}: Imports of Country I from Country III

O_{1t}: Net short-term capital outflows of Country I

$_{it}$: Rate

$P_{M_{it}}$: Impo

$P_{X_{it}}$: Expo

T_{it}: Tax

Π_{it}: Capi

I_{it}: Tota

Y_{it}: Nati

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(I-1) $Y_{it} =$

(I-2) $K_{1t} =$

(I-3) $I^G_{1t} =$

(I-4) $B_{1t} =$

(I-5) $Y_{2t} =$

(I-6) $K_{2t} =$

(I-7) $I^G_{2t} =$

(II-1) $Y_{1t} =$

(II-2) $C_{1t} =$

(II-3) $M_{1t} =$

(II-4) $I^n_{1t} =$

r_{it}: Rate of interest in Country i

$P_{M_{it}}$: Import price of Country i

$P_{x_{it}}$: Export price of Country i

T_{it}: Tax receipts of Country i

TR_{2t}: Capital transfer of Country II

X_{it}: Total exports of Country i

Y_{it}: National income of Country i

The equations are as follows:

(I-1) $Y_{it} = C_{it} + I^G_{1t} + G_{it} + (M_{2t}+M_{3t}+IE_{1t})-M_{1t}$

(I-2) $K_{1t} = I^n_{1t} + K_{1t-1}$

(I-3) $I^G_{1t} = I^n_{1t} + \delta_* K_{1t}$

(I-4) $B_{1t} = (M_{2t}+M_{3t}+IE_{1t}) - M_{1t}-O_{1t}$

(I-5) $Y_{2t} = C_{2t}+I^G_{2t} + G_{2t}+(M_{1t}-M^{III}_{1t}) - M_{2t}$

(I-6) $K_{2t} = I^n_{2t} + K_{2t-1}$

(I-7) $I^G_{2t} = I^n_{2t} + \delta_* K_{2t}$

(E-1) $Y_{1t} = \delta_{10} + \delta_{11} K_{1t} + \delta_{12} L_{1t}$

(E-2) $C_{1t} = \alpha_{10} + \alpha_{11}(Y_{1t}-T_{1t} - \delta_* K_{1t})$

(E-3) $M_{1t} = \beta_{10} + \beta_{11}(Y_{1t}-T_{1t} - \delta_* K_{1t}) + \beta_{12}\left(\frac{P_M}{P_x}\right)_{1t}$

(E-4) $I^n_{1t} = \gamma_{10} + \gamma_{11} Y_{1t-1}-\gamma_{12} q(\delta_*+r)_{1t-1}+\gamma_{13} K_{1t-1}$

(3-5) $O_{1t} = \eta$

(3-6) $Y_{2t} = \delta$

(3-7) $C_{2t} = \alpha$

(3-8) $M_{2t} = \beta$

(3-9) $I^{n}_{2t} = \gamma$

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(E-5) $O_{1t} = \eta_{10} + \eta_{11} (r_{2t} - r_{1t})$

(E-6) $Y_{2t} = \delta_{20} + \delta_{21} K_{2t} + \delta_{22} L_{2t}$

(E-7) $C_{2t} = \alpha_{20} + \alpha_{21} (Y_{2t} - T_{2t} - TR_{2t} - \delta_* K_{2t})$

(E-8) $M_{2t} = \beta_{20} + \beta_{21} (Y_{2t} - T_{2t} - TR_{2t} - \delta_* K_{2t}) + \beta_{22} \left(\frac{P_M}{P_x}\right)_{2t}$

(E-9) $I^n_{2t} = \gamma_{20} + \gamma_{21} Y_{2t-1} - \gamma_{22} q(\delta_* + r)_{2t-1} + \gamma_{23} K_{2t-1}$

The comments on (E-1) to (E-5) have been presented in the preceding chapter.

Equation E-6 - The output of Country II is a linear function of both its factors of production: K_{2t}, L_{2t}.

Equation E-7 - Consumption expenditures of II are a function of its disposable income. Here the disposable income definition is different from that of Country I and it is represented by GNP less capital depreciation ($\delta_* K_{2t}$ where δ_* is the rate of replacement and is assumed to be the same for both countries), less taxes and less capital transfers of Country II (TR_{2t} is assumed to be exogenous).

Equation E-8 - Net investment expenditures in II are assumed to depend on money supply, the capital stock and the user cost of capital in II. The user cost of capital in II has the same definition as that in Country I, with the assumption q = 1 over time.

Identity I-4 - It represents the common external balance between the two countries.

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3.2. Optimal Control Problem without Constraints on Policy Variable Magnitudes

Under the assumption of common external balance, Country II is passive with respect to its balance of payments, allowing Country I to seek its balance-of-payments target unimpeded. Therefore, there are three independent targets to achieve simultaneously: full employment in both countries and maintenance of Country I's balance-of-payments equilibrium. To attain these targets, three independent instruments are considered: government expenditures in Countries I and II coupled with a short-term interest rate in Country I, while Country II's interest rate is regarded as fixed or given.

The external balance, as defined previously, is $\bar{B}_{1t} = 0$ while the two internal balances for Country I and II are determined by their respective production functions: $Y_{it} = \delta_{i0}+\delta_{i1} K_{it}+\delta_{i2} L_{it}$ (3.1) where the subscript i stands for country i (i = 1,2). As before, to overcome the problem of "double entry" of K_{it} in the reduced form, equation 3.1 is transformed to:

$$Y_{it}-\delta_{i1} K_{it} = \delta_{i0}+\delta_{i2} L_{it} \; ; \; i = 1,2 \tag{3.2}$$

Let $\tilde{Y}_{it} = Y_{it}-\delta_{i1} K_{it}$. Then equation 3.2 becomes:

$$\bar{\tilde{Y}}_{it} = \delta_{i0}+\delta_{i2} L_{it} \; ; \; i = 1,2 \tag{3.3}$$

which determines the internal balance for the respective country.

To apply the optimal control theory, Votey's two-country model has been formulated into the following reduced form (Appendix A-6):

$y_t = \bar{A}y_{t-1} + \bar{B}u_t +$

where

$y_t \stackrel{d}{=} \left[\tilde{Y}_{1t}, \tilde{Y}_{2t}, B\right.$

$u_t \stackrel{d}{=} \left[G_{1t}, G_{2t}, r\right.$

$z_t \stackrel{d}{=} \left[1, IE_{1t}, M^{I}_{1}\right.$

$$\bar{A} = \left[\begin{array}{ll} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \\ \gamma_{11} & 0 \\ 0 & \gamma_{21} \end{array}\right.$$

$$\bar{D} = \left[\begin{array}{ll} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ a_{46} & 0 \\ a_{56} & 0 \end{array}\right.$$

Then, given the

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the assumption

$$y_t = \bar{A}y_{t-1}+\bar{B}u_t+\bar{C}u_{t-1}+\bar{D}z_t \tag{3.4}$$

where

$$y_t \stackrel{d}{=} \left[\tilde{Y}_{1t},\tilde{Y}_{2t},B_{1t},K_{1t},K_{2t},G_{1t},G_{2t},r_{1t}\right]'$$

$$u_t \stackrel{d}{=} \left[G_{\ t},G_{2t},r_{1t}\right]'$$

$$z_t \stackrel{d}{=} \left[1, IE_{1t},M_{1t}^{III}, TT_{1t}, T_{1t}, r_{2t}, r_{2t-1}, TT_{2t}, T_{2t}, TR_{2t}, M_{3t}\right]'$$

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & A_{15} \\ A_{21} & A_{22} & 0 & A_{24} & A_{25} \\ A_{31} & A_{32} & 0 & A_{34} & A_{35} \\ \gamma_{11} & 0 & 0 & a_{45} & 0 \\ 0 & \gamma_{21} & 0 & 0 & a_{55} \end{bmatrix} ;\bar{B} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & \eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ;\bar{C} = \begin{bmatrix} 0 & 0 & A_{16} \\ 0 & 0 & A_{26} \\ 0 & 0 & A_{36} \\ 0 & 0 & -\gamma_{12} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & 0 & A_{17} & D_{16} & D_{17} & D_{17} & D_{18} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & 0 & A_{27} & D_{26} & D_{27} & D_{27} & D_{28} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & -\eta_{11} & A_{37} & D_{36} & D_{37} & D_{37} & D_{38} \\ a_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{56} & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, given the structural form of equation 3.4, the two-country optimal control problem under the assumption of common external balance will be formulated into Pindyck's as well as Chow's framework. Since the two approaches are equivalent (Appendix A-3), Chow's method will be considered for the derivation of the two-country optimal solution under the assumption of no constraints on policy-variable magnitudes.

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$x_t^P = A^P x_{t-1}^P +$

where $x_t^P \stackrel{d}{=} y_t -$

state vector;

and $z_t^P \stackrel{d}{=} \Big[1, IE$

(1x1) current

matrices:

$$A^P = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \gamma_{11} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_{11}Q_{11} \\ A_{21}Q_{11} \\ A_{31}Q_{11} \\ \gamma_{11}Q \\ \gamma_{21}Q \end{bmatrix}$$

3.2.1. Formulation of the Problem

For the two-country optimal control problem formulated into Pindyck's framework (Appendix A-2.1), the "state-space" system is:

$$x_t^P = A^P x_{t-1}^P + B^P u_{t-1}^P + C^P z_{t-1}^P \qquad (3.5)$$

where $x_t^P \stackrel{d}{=} y_t - \bar{B}u_t - \bar{D}z_t \stackrel{d}{=} \left[\tilde{Y}Y_{1t}, \tilde{Y}Y_{2t}, BB_{1t}, KK_{1t}, KK_{2t}\right]'$ is the (5x1) state vector; $u_t^P \stackrel{d}{=} u_t \stackrel{d}{=} \left[G_{1t}, G_{2t}, r_{1t}\right]'$ is the (3x1) control vector; and $z_t^P \stackrel{d}{=} \left[1, IE_{1t}, M_{1t}^{III}, TT_{1t}, T_{1t}, r_{2t}, TT_{2t}, T_{2t}, TR_{2t}, M_{3t}\right]'$ is the (11x1) current exogenous variable vector. A^P, B^P and C^P are known matrices:

$$A^P = \bar{A} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & A_{15} \\ A_{21} & A_{22} & 0 & A_{24} & A_{25} \\ A_{31} & A_{32} & 0 & A_{34} & A_{35} \\ \gamma_{11} & 0 & 0 & a_{45} & 0 \\ 0 & \gamma_{21} & 0 & 0 & a_{55} \end{bmatrix}$$

$$;B^P = \overline{AB} + \bar{C} = \begin{bmatrix} A_{11}Q_{11}+A_{12}Q_{21} & A_{21}Q_{12}+A_{12}Q_{22} & A_{16} \\ A_{21}Q_{11}+A_{22}Q_{21} & A_{21}Q_{12}+A_{22}Q_{22} & A_{26} \\ A_{31}Q_{11}+A_{32}Q_{21} & A_{31}Q_{12}+A_{32}Q_{22} & A_{36} \\ \gamma_{11}Q_{11} & \gamma_{11}Q_{12} & \\ \gamma_{21}Q_{21} & \gamma_{21}Q_{22} & \end{bmatrix}$$

$$\bar{C}=\overline{AD}=\left[\begin{array}{lllllll}
A_{11}D_{11}+A_{12}D_{21}+A_{14}a_{46}+A_{15}a_{56} & A_{11}D_{12}+A_{12}D_{22} & A_{11}D_{13}+A_{12}D_{23} & A_{11}D_{14}+A_{12}D_{24} & A_{11}D_{15}+A_{12}D_{25} & 0 & A_{11}A_{17}+A_{12}A_{27}-A_{15}\gamma_{22} \\
A_{21}D_{11}+A_{22}D_{21}+A_{24}a_{46}+A_{25}a_{56} & A_{21}D_{12}+A_{22}D_{22} & A_{21}D_{13}+A_{22}D_{23} & A_{21}D_{14}+A_{22}D_{24} & A_{21}D_{15}+A_{22}D_{25} & 0 & A_{21}A_{17}+A_{22}A_{27}-A_{25}\gamma_{22} \\
A_{31}D_{11}+A_{32}D_{21}+A_{34}a_{46}+A_{35}a_{56} & A_{31}D_{12}+A_{32}D_{22} & A_{21}D_{13}+A_{22}D_{23} & A_{21}D_{14}+A_{22}D_{24} & A_{21}D_{15}+A_{22}D_{25} & 0 & A_{31}A_{17}+A_{32}A_{27}-A_{35}\gamma_{22} \\
\gamma_{11}D_{11}+a_{45}a_{46} & \gamma_{11}D_{12} & \gamma_{11}D_{13} & \gamma_{11}D_{14} & \gamma_{11}A_{15} & 0 & \gamma_{11}D_{17} \\
\gamma_{21}D_{21}+a_{55}a_{56} & \gamma_{21}D_{22} & \gamma_{21}D_{23} & \gamma_{21}D_{24} & \gamma_{21}D_{25} & 0 & \gamma_{21}A_{27}-a_{55}\gamma_{12}
\end{array}\right.$$

$$\left.\begin{array}{llll}
A_{11}D_{16}+A_{12}D_{26} & A_{11}D_{17}+A_{12}D_{27} & A_{11}D_{17}+A_{12}D_{27} & A_{11}D_{18}+A_{12}D_{28} \\
A_{21}D_{16}+A_{22}D_{26} & A_{21}D_{17}+A_{22}D_{27} & A_{21}D_{17}+A_{22}D_{27} & A_{21}D_{18}+A_{22}D_{28} \\
A_{31}D_{16}+A_{32}D_{26} & A_{31}D_{17}+A_{32}D_{37} & A_{31}D_{17}+A_{32}D_{27} & A_{31}D_{18}+A_{32}D_{28} \\
\gamma_{11}A_{16} & \gamma_{11}D_{17} & \gamma_{11}D_{17} & \gamma_{11}D_{18} \\
\gamma_{21}D_{26} & \gamma_{21}D_{27} & \gamma_{21}D_{27} & \gamma_{21}D_{28}
\end{array}\right] ;$$

and the cost functional under the assumption of $R^P = 0$ is:

$$J^P = \frac{1}{2}\sum_{t=1}^{N} (x^P_{t-1}-\bar{x}^P_{t-1})' \, Q^P (x^P_{t-1}-\bar{x}^P_{t-1})$$

where $\bar{x}^P_{t-1} \overset{d}{=} y_{t-1}-\bar{B}u_{t-1}-\bar{D}z_{t-1} \overset{d}{=} \left[\bar{\bar{Y}}Y_{1t-1}, \bar{\bar{Y}}Y_{2t-1}, \overline{BB}_{1t-1}, \overline{KK}_{2t-1}\right]'$ is the (5x1) nominal state variable vector to be tracked;

$$Q^p = \begin{bmatrix} q_{11} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$x_t^c = A^c x^c_{t-1} +$

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$\overset{d}{=} \Big[G_{1t}, G_{2t},$

$\overset{d}{=} \Big[1, IE_{1t}, M$

vector of exoge

known matrices:

$$Q^P = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the weight matrix attached to the respective quadratic deviations from the nominal state variables $\bar{\tilde{YY}}_{1t-1}$, $\bar{\tilde{YY}}_{2t-1}$, and $\overline{BB}_{1t-1}$. Under Chow's framework no transformation of variables for equation 3.4 is needed, and for the two-country problem the dynamic system is the following:

$$x_t^c = A^c x_{t-1}^c + B^c u_t^c + C^c z_t^c \qquad (3.6)$$

where $x_t^c \overset{d}{=} \left[y_t \,\vdots\, u_t \right]' \overset{d}{=} \left[\tilde{Y}_{1t}, \tilde{Y}_{2t}, B_{1t}, K_{1t}, K_{2t} \quad G_{1t}, G_{2t}, r_{1t} \right]'$

is the (8x1) vector of current endogenous and controlled variables;

$u_t^c \overset{d}{=} \left[G_{1t}, G_{2t}, r_{1t} \right]'$ is the (3x1) vector of controlled variables;

$z_t^c \overset{d}{=} \left[1, IE_{1t}, M_{1t}^{III}, TT_{1t}, T_{1t}, r_{2t}, r_{2t-1}, TT_{2t}, T_{2t}, TR_{2t}, M_{3t} \right]'$ is the (11x1) vector of exogenous non-controlled variables; and A^c, B^c and C^c are known matrices:

$$A^c = \left[\begin{array}{c:c} \overline{A} & \overline{C} \\ \hdashline O & 0 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & A_{15} & 0 & 0 & A_{16} \\ A_{21} & A_{22} & 0 & A_{24} & A_{25} & 0 & 0 & A_{26} \\ A_{31} & A_{32} & 0 & A_{34} & A_{35} & 0 & 0 & A_{36} \\ \gamma_{11} & 0 & 0 & a_{45} & 0 & 0 & 0 & -\gamma_{12} \\ 0 & \gamma_{21} & 0 & 0 & a_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ;$$

$$B^c = \left[\begin{array}{c} \overline{B} \\ \hdashline I \end{array}\right] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & \eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ;$$

$$C^c = \left[\begin{array}{c} \overline{D} \\ \hdashline O \end{array}\right] = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & 0 & A_{17} & D_{16} & D_{17} & D_{17} & D_{18} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & 0 & A_{27} & D_{26} & D_{27} & D_{27} & D_{28} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & \eta_{11} & A_{37} & D_{36} & D_{37} & D_{37} & D_{38} \\ a_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{56} & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Appendix A-2,

$Q^C = $ Q^P 0

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$Q^C = $ Q^P 0

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and under the assumption of no limits on instrument variable magnitudes the weight matrix Q^c in the welfare cost function equation A-2.18, Appendix A-2, is:

$$Q^c = \left[\begin{array}{c|c} Q^P & 0 \\ \hline O & 0 \end{array}\right] = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

instead of

$$Q^c = \left[\begin{array}{c|c} Q^P & 0 \\ \hline O & R^P \end{array}\right] = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{33} \end{bmatrix}$$

where q_{11}, q_{22} and q_{33} are weights attached to the quadratic deviations from the situation of full employment in both countries and from the common external balance, i.e., weights attached to $(\tilde{Y}_{1t}-\bar{\tilde{Y}}_{1t})^2$,

$(\dot{Y}_{2t}-\bar{\dot{Y}}_{2t})^2$, an

weights attach

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(Appendix A-2)

(Appendix A-8)

$(\tilde{Y}_{2t}-\bar{\tilde{Y}}_{2t})^2$, and $(B_{1t}-\bar{B}_{1t})^2$, respectively, while r_{11}, r_{22} and r_{33} are weights attached to the quadratic deviations from the limits set on both countries' fiscal policies and Country I's monetary policy, i.e., weights attached to $(G_{1t}-\bar{G}_{1t})^2$, $(G_{2t}-\bar{G}_{2t})^2$, and $(r_{1t}-\bar{r}_{1t})^2$.

3.2.2. Optimal Solution for Internal and External Balance

Using Chow's result (equation A-2.20) for the optimal control (Appendix A-2), the optimal solution for the two-country problem is (Appendix A-8):

$$\begin{bmatrix} G^{*}_{1t} \\ G^{*}_{2t} \\ r^{*}_{1t} \end{bmatrix} = \begin{bmatrix} -\theta_{11} & -\theta_{12} & 0 & -\theta_{14} & -\theta_{15} & 0 & 0 & -\theta_{16} \\ -\theta_{21} & -\theta_{22} & 0 & -\theta_{24} & -\theta_{25} & 0 & 0 & -\theta_{26} \\ -\theta_{31} & -\theta_{32} & 0 & -\theta_{34} & -\theta_{35} & 0 & 0 & -\theta_{36} \end{bmatrix} \begin{bmatrix} \tilde{Y}^{*}_{1t-1} \\ \tilde{Y}^{*}_{2t-1} \\ B^{*}_{1t-1} \\ K^{*}_{1t-1} \\ K^{*}_{2t-1} \\ G^{*}_{1t-1} \\ G^{*}_{2t-1} \\ r^{*}_{1t-1} \end{bmatrix} + \begin{bmatrix} Q_{22}/Q & -Q_{12}/Q & 0 & 0 & 0 & 0 & 0 & 0 \\ -Q_{21}/Q & Q_{11}/Q & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\eta_{11}}\frac{(Q_{21}Q_{32}-Q_{22}Q_{31})}{Q} & \frac{-1}{\eta_{11}}\frac{(Q_{11}Q_{32}-Q_{12}Q_{31})}{Q} & \frac{1}{\eta_{11}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{\bar{Y}}_{1t} \\ \bar{\bar{Y}}_{2t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \end{bmatrix} + \begin{bmatrix} -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & -\Delta_{15} & 0 & -\Delta_{19} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} \\ -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & -\Delta_{25} & 0 & -\Delta_{29} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} \\ -\Delta_{31} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & -\Delta_{35} & 0 & -\Delta_{39} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} \end{bmatrix} \begin{bmatrix} I \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix} \qquad (3.7)$$

After substituting $\bar{\bar{Y}}_{it} = \delta_{i0}+\delta_{i2}\, L_{it}$ $(i=1,2)$ and $\bar{B}_{1t} = 0$ into equation 3.7:

$$\begin{bmatrix} G^{*}_{1t} \\ G^{*}_{2t} \\ r^{*}_{1t} \end{bmatrix} = \begin{bmatrix} -\theta_{11} & -\theta_{12} & 0 & -\theta_{14} & -\theta_{15} & 0 & 0 & -\theta_{16} \\ -\theta_{21} & -\theta_{22} & 0 & -\theta_{24} & -\theta_{25} & 0 & 0 & -\theta_{26} \\ -\theta_{31} & -\theta_{32} & 0 & -\theta_{34} & -\theta_{35} & 0 & 0 & -\theta_{36} \end{bmatrix} \begin{bmatrix} Y^{*}_{1t-1} \\ Y^{*}_{2t-1} \\ B^{*}_{1t-1} \\ K^{*}_{1t-1} \\ K^{*}_{2t-1} \\ G^{*}_{1t-1} \\ G^{*}_{2t-1} \\ r^{*}_{1t-1} \end{bmatrix} + \begin{bmatrix} -\Delta_{11} + \frac{Q_{22}\delta_{10}-Q_{12}\delta_{20}}{Q} & \frac{Q_{22}\delta_{12}}{Q} \\ -\Delta_{21} + \frac{Q_{21}\delta_{10}+Q_{11}\delta_{20}}{Q} & \frac{-Q_{21}\delta_{12}}{Q} \\ -\Delta_{31}+\frac{1}{\eta_{11}Q}\left[\delta_{10}(Q_{21}Q_{32}-Q_{22}Q_{31})-\delta_{20}(Q_{11}Q_{32}-Q_{12}Q_{31})\right] & \frac{\delta_{12}}{\eta_{11}Q}(Q_{12}Q_{32}-Q_{22}Q_{31}) \end{bmatrix} \qquad (3.8)$$

$$\begin{bmatrix} \frac{-Q_{12}\delta_{22}}{Q} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & -\Delta_{15} & 0 & -\Delta_{19} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} \\ \frac{Q_{11}\delta_{22}}{Q} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & -\Delta_{25} & 0 & -\Delta_{29} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} \\ \frac{-\delta_{22}}{\eta_{11}Q}(Q_{11}Q_{32}Q_{12}Q_{31}) & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & -\Delta_{35} & 1 & -\Delta_{39} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} \end{bmatrix} \begin{bmatrix} 1 \\ L_{1t} \\ L_{2t} \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix}$$

The optimal path is obtained by substituting equation 3.8 into the dynamic system (equation 3.6):

$$\begin{bmatrix} Y^*_{1t} \\ \vdots \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots \end{bmatrix} \begin{bmatrix} Y^*_{1t-1} \\ \vdots \end{bmatrix} \quad \begin{bmatrix} \delta_{10} \\ \vdots \end{bmatrix} \qquad \delta_{12}$$

The optimal path is obtained by substituting equation 3.8 into the dynamic system (equation 3.6):

$$
\begin{bmatrix} \tilde{Y}^*_{1t} \\ \tilde{Y}^*_{2t} \\ B^*_{1t} \\ K^*_{1t} \\ K^*_{2t} \\ G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_{11} & 0 & 0 & a_{45} & 0 & 0 & 0 & -\gamma_{12} \\
0 & \gamma_{21} & 0 & 0 & a_{55} & 0 & 0 & 0 \\
-\theta_{11} & -\theta_{12} & 0 & -\theta_{14} & -\theta_{15} & 0 & 0 & -\theta_{16} \\
-\theta_{21} & -\theta_{22} & 0 & -\theta_{24} & -\theta_{25} & 0 & 0 & -\theta_{26} \\
-\theta_{31} & -\theta_{32} & 0 & -\theta_{34} & -\theta_{35} & 0 & 0 & -\theta_{36}
\end{bmatrix}
\begin{bmatrix} \tilde{Y}^*_{1t-1} \\ \tilde{Y}^*_{2t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
+ \tag{3.9}
$$

$$
\begin{bmatrix}
\delta_{10} & \delta_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta_{20} & 0 & \delta_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\Delta_{11} + \frac{1}{Q}(Q_{22}\delta_{10}-Q_{12}\delta_{20}) & (Q_{22}\delta_{12})/Q & -(Q_{12}\delta_{22})/Q & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & -\Delta_{15} & 0 & -\Delta_{19} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} \\
-\Delta_{21} + \frac{1}{Q}(-Q_{21}\delta_{10}+Q_{11}\delta_{20}) & -(Q_{21}\delta_{12})/Q & (Q_{11}\delta_{22})/Q & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & -\Delta_{25} & 0 & -\Delta_{29} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} \\
-\Delta_{31} + \frac{1}{\eta_{11}Q}\ \delta_{10}(Q_{21}Q_{32}-Q_{22}Q_{31})-\delta_{20}(Q_{11}Q_{32}-Q_{12}Q_{31}) & \frac{\delta_{12}}{\eta_{11}Q}(Q_{12}Q_{32}-Q_{22}Q_{31}) & -\frac{\delta_{22}}{\eta_{11}Q}(Q_{11}Q_{32}-\theta_{12}Q_{31}) & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & -\Delta_{35} & 1 & -\Delta_{39} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38}
\end{bmatrix}
\begin{bmatrix} 1 \\ L_{1t} \\ L_{2t} \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix}
$$

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To evaluate the performance of the economic system with respect to the target set, compute the optimal welfare cost:

$$\hat{J}^c = \frac{1}{2} \sum_{t=1}^{N} (x_t^{*c} - \overline{x}_t^c)' Q^c (x^{*c} - \overline{x}_t^c)$$

It is found that $\hat{J}^c = 0$ (Appendix A-9).

The results of the two-country optimal control problem under common external balance assumption (Case A) are not fundamentally different from those obtained in the one-country problem of Chapter II. The concluding remarks which can be drawn from these results are:

(1) The economic interdependence of the two countries implies the interdependence of optimal policies. However, within the confines of Votey's model, the internal balance in both countries are reached by the fiscal policies of both countries while the overall interdependent balance--full employment without inflation--is attained by monetary policy in Country I. The relative impact of these three instruments on the three targets depends on the coefficient-magnitudes of the following matrix:

Row	$\overline{\tilde{Y}}_{1t}$	$\overline{\tilde{Y}}_{2t}$	$\overline{B}_{1t}$
1	$\frac{Q_{22}}{Q}$	$\frac{-Q_{12}}{Q}$	0
2	$\frac{-Q_{21}}{Q}$	$\frac{Q_{11}}{Q}$	0
3	$\frac{1}{\eta_{11}} \frac{(Q_{21}Q_{32}-Q_{22}Q_{31})}{Q}$	$\frac{-1}{\eta_{11}} \frac{(Q_{11}Q_{32}-Q_{12}Q_{31})}{Q}$	η_{11}

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This will be computed in the next section by substituting Votey's numerical values for the estimated coefficients into the two-country model.

(2) The variables $\tilde{Y}^*_{1t}$, $\tilde{Y}^*_{2t}$ and B^*_{1t} are on target exactly (equation 3.9) and the optimal cost is zero since the number of independent variables to be controlled (the number of non-zero diagonal elements in Q^c) is equal to the number of independent instruments (Chow:1972b), i.e., Tinbergen's rule is met.

(3) Given the analytical solutions (equation 3.9), the simultaneous achievement of internal and external balance for the two-country model is possible within the framework of optimal control theory, and the optimal control vector is independent of the penalty costs assigned to deviations of $\tilde{Y}_{it}$ $(i=1,2)$ and B_{1t} from their targets, and therefore is unique. However, nothing guarantees the feasibility of this optimal policy mix in a given economy. Therefore, a test of feasibility will be done in the next section for the United States and Canada during the period 1961-1970.

3.3. U.S. and Canada Optimal Policies for Internal and External Balance: Appraisal and Amendment of the Optimal Solution

Similar to the one-country model of Chapter II, Votey's numerical values for the structural coefficients and the historical data for the United St

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the United States as well as Canada will be used to derive the U.S. and Canadian optimal policy mix for the period 1961-1970, during which Canada was under the regime of fixed exchange rate. It is found that the optimal solution for both countries is not feasible for technical, practical and political reasons; therefore, limits on policy-variable magnitudes have to be set (Tinbergen:1954) and the optimal solution amended.

First, using Votey's numerical values for the estimated structural constants (Votey:1969), the matrices of coefficients for equation 3.7 and equation 3.6 are computed:

$$\begin{bmatrix} G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} 0.0025 & -0.0051 & 0 & 0.0013 & -0.0039 & 0 & 0 & 434.2240 \\ 0.0001 & -0.1477 & 0 & -0.0218 & 0.0225 & 0 & 0 & 9.4831 \\ 0 & -0.0001 & 0 & 0.0012 & -0.0007 & 0 & 0 & -0.5356 \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t-1} \\ \tilde{Y}_{2t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix} + \begin{bmatrix} 0.3810 & -0.0977 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1315 & 0.3632 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0028 & -0.0021 & 0.0216 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{Y}_{1t} \\ \bar{Y}_{2t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix} + \begin{bmatrix} -33.5392 & -1.0000 & 0 & -28.3566 & 0.6185 & 0 & 10.7499 & 0 & 0.0977 & 0.0977 & -1 \\ -41.9366 & 0.0000 & -1 & 28.3566 & 0.1315 & 0 & 309.8858 & 0 & 0.6368 & 0.6368 & 0 \\ 0.5924 & -0.0216 & 0 & -0.6127 & -0.0028 & 1 & 0.2323 & 0 & 0.0021 & 0.0021 & -0.0216 \end{bmatrix} \begin{bmatrix} 1 \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix} \tag{3.10}$$

$$
\begin{bmatrix} \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \\ B^*_{1t} \\ K^*_{1t} \\ K^*_{2t} \\ G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
=
\begin{bmatrix}
-0.0073 & 0.1298 & 0 & 0.0131 & 0.0806 & 0 & 0 & -1263.7107 \\
-0.0028 & 0.4538 & 0 & 0.0648 & -0.0329 & 0 & 0 & -483.6579 \\
0.0008 & 0.0324 & 0 & -0.0524 & 0.0201 & 0 & 0 & 143.7013 \\
-0.0027 & 0 & 0 & 1.0618 & 0 & 0 & 0 & -461.7 \\
0 & 0.1668 & 0 & 0 & 1.1035 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \tilde{Y}_{1t-1} \\ \tilde{Y}^*_{2t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
+
\begin{bmatrix}
2.8933 & 0.7782 & 0 \\
1.0476 & 3.0351 & 1 \\
-0.2781 & 0.1942 & 46.279 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
+
$$

$$
\begin{bmatrix}
129.6743 & 2.8933 & 0.7782 & 59.9756 & -1.8918 & 0 & -272.2641 & 0 & -0.7782 & -0.7782 & 2.8933 \\
162.4181 & 1.0476 & 3.0351 & -56.3613 & -1.0470 & 0 & -951.8105 & 0 & -2.0351 & -2.0351 & 1.0476 \\
-28.6009 & 0.7219 & 0.1942 & 14.9641 & 0.2779 & -46.279 & -67.9307 & 0 & -0.1942 & -0.1942 & 0.7219 \\
9.1298 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-18.6889 & 0 & 0 & 0 & 0 & 0 & -349.8519 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} 1 \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix}
\qquad (3.11)
$$

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Given these two equations, the optimal stabilization policy mix (G^*_{1t}, G^*_{2t}, r^*_{1t}) is derived for the United States and Canada over 10 periods running from 1961 to 1970, with the following initial conditions:

$$\tilde{Y}_1(0) = 383.37$$
$$\tilde{Y}_2(0) = 20.06$$
$$B_1(0) = 4.17$$
$$K_1(0) = 508.69$$
$$K_2(0) = 49.33$$
$$G_1(0) = 94.90$$
$$G_2(0) = 6.97$$
$$r_1(0) = 0.0215$$

where (0) refers to 1960. All these values are historical. The nominal values for state variables are given in Table 3.1. The nominal values for U.S. state variables ($\bar{\tilde{Y}}_{1t}$, $\bar{B}_{1t}$, $\bar{K}_{1t}$, $\bar{G}_{1t}$) are the same as those in Table 2.1. Similarly, for Canada the potential output net of capital accumulation ($\bar{\tilde{Y}}_{2t}$) is determined by the Canadian labor force. The nominal capital stock $\bar{K}_{2t}$ and government expenditures $\bar{G}_{2t}$ are assumed to grow at the same rate as those in the United States, that is, respectively, at 3 percent and 4 percent per annum from their initial 1960 values. The historical values for exogenous variables are given in Table 3.2. The U.S. and Canadian optimal stabilization policies for 10 periods (1961-1970) are obtained by the following steps which require only basic matrix manipulations--additions, subtractions, multiplications and small inversions:

Table 3.1.

Period (Year)	
1(1961)	28
2(1962)	2
3(1963)	2
4(1964)	2
5(1965)	2
6(1966)	3
7(1967)	1
8(1968)	1
9(1969)	
10(1970)	

Table 3.1. Nominal values for the state variables - U.S. and Canada: 1961-1970

Period (Year)	$\bar{\tilde{Y}}_{1t}$	$\bar{\tilde{Y}}_{2t}$	$\bar{B}_{1t}$	$\bar{K}_{1t}$	$\bar{K}_{2t}$	$\bar{G}_{1t}$	$\bar{G}_{2t}$	$\bar{r}_{1t}$
1(1961)	283.2462	2.3165	0	523.9507	50.8099	98.7	7.2488	0.0215
2(1962)	283.8683	2.3499	0	539.6692	52.3342	102.65	7.5388	0.0215
3(1963)	288.7687	2.3972	0	555.8592	53.9042	106.76	7.8404	0.0215
4(1964)	293.8258	2.4629	0	572.5349	55.5213	111.03	8.1540	0.0215
5(1965)	299.3091	2.5368	0	589.7109	57.1869	115.47	8.4802	0.0215
6(1966)	304.5954	2.6359	0	607.4022	58.9025	120.09	8.8194	0.0215
7(1967)	310.9349	2.7332	0	625.6242	60.6696	124.89	9.1722	0.0215
8(1968)	316.5227	2.8132	0	644.3929	62.4897	124.89	9.5391	0.0215
9(1969)	324.5467	2.8995	0	663.7246	64.3644	135.09	9.9207	0.0215
10(1970)	332.5143	2.9748	0	683.6363	66.2953	140.49	10.3175	0.0215

Period (Year)	IE_{1t}	M_{1t}^{III}	TT_{1t}	T_{1t}	r_{2t}	r_{2t-1}	TT_{2t}	T_{2t}	TR_{2t}	M_{3t}

Table 3.2. *Historical values for the exogenous variables - U.S. and Canada: 1961-1970*

Period (Year)	IE_{1t}	M_{1t}^{III}	TT_{1t}	T_{1t}	r_{2t}	r_{2t-1}	TT_{2t}	T_{2t}	TR_{2t}	M_{3t}
1(1961)	3.55	2.22	1.0000	144.63	0.0299	0.0325	1.0408	9.57	3.44	4.5431
2(1962)	4.05	2.45	1.0101	151.03	0.0391	0.0299	1.0309	10.53	3.72	4.9299
3(1963)	4.15	2.52	1.0000	168.76	0.0378	0.0391	1.0000	11.18	3.85	5.4886
4(1964)	4.87	2.83	0.9902	174.07	0.0382	0.0378	1.0000	12.73	4.13	5.9344
5(1965)	5.29	3.32	1.0000	190.06	0.0454	0.0382	1.0198	14.29	4.57	6.279
6(1966)	5.37	4.12	1.0094	213.33	0.0496	0.0454	1.0490	16.59	5.05	6.6656
7(1967)	5.89	4.46	1.0280	228.93	0.0595	0.0496	1.0583	18.51	6.22	6.5062
8(1968)	6.22	5.89	1.0183	263.31	0.0624	0.0595	1.0566	20.99	7.19	7.1595
9(1969)	5.97	5.80	1.0268	296.70	0.0781	0.0624	1.0648	24.44	6.06	8.1595
10(1970)	6.27	6.61	1.0083	302.00	0.0444	0.0781	1.0614	26.27	6.80	10.3021

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Some remarks

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Chapter II (s

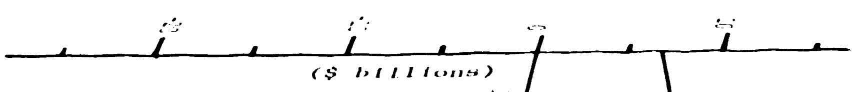
(1) Compute $[G_1^*(1),\ G_2^*(1),\ r_1^*(1)]'$ from equation 3.10 using the initial conditions given by the historical data for $[\bar{\tilde{Y}}_1(0),\ \bar{\tilde{Y}}_2(0),\ B_1(0),\ K_1(0),\ G_1(0),\ G_2(0),\ r_1(0)]$ and the exogenous variables of period 1 given in Table 3.2.

(2) Compute $[\tilde{Y}^*(1),\ \tilde{Y}_2(1),\ B_1^*(1),\ K_1^*(1),\ K_2^*(1),\ G_1^*(1),\ G_2^*(1),\ r_1^*(1)]'$ from equation 3.11. Now $[\tilde{Y}^*(1),\ \tilde{Y}_2^*(1),\ B_1^*(1),\ K_1^*(1),\ K_2^*(1),\ G_1^*(1),\ G_2^*(1),\ r_1^*(1)]'$ can be used in equation 3.10 to compute $[G_1^*(2),\ G_2^*(2),\ r_1^*(2)]'$, which can be used in equation 3.11 to compute $[\tilde{Y}_1^*(2),\ \tilde{Y}_2^*(2),\ B_1^*(2),\ K_1^*(2),\ K_2^*(2),\ G_1^*(2),\ G_2^*(2),\ r_1^*(2)]'$, and so on. Continue the process until all of the control vectors $[G_1^*(t),\ G_2^*(t),\ r_1^*(t)]'$, $t = 1,\ldots, 10$, and all the state variable vectors $[\tilde{Y}_1^*(t),\ \tilde{Y}_2^*(t),\ B_1^*(t),\ K_1^*(t),\ K_2^*(t),\ G_1^*(t),\ G_2^*(t),\ r_1^*(t)]'$, $t = 1,\ldots, 9$, have been computed.

The results are presented in graphical form (Figures 3.1 to 3.6) with time on the horizontal axis. First, based on the following matrix (equation 3.7)

Row	$\bar{\tilde{Y}}_{1t}$	$\bar{\tilde{Y}}_{2t}$	$\bar{B}_{1t}$
1	0.3810	-0.0977	0
2	-0.1315	0.3632	0
3	0.0028	-0.0021	0.0216

some remarks will be made on the effectiveness of policy-instrument variables. Using the same concept of effectiveness as described in Chapter II (section 2.3), it is noted that the fiscal policy of each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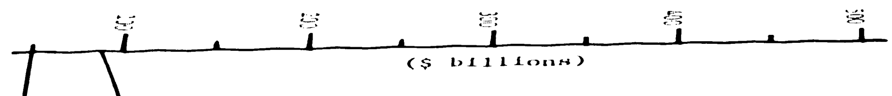
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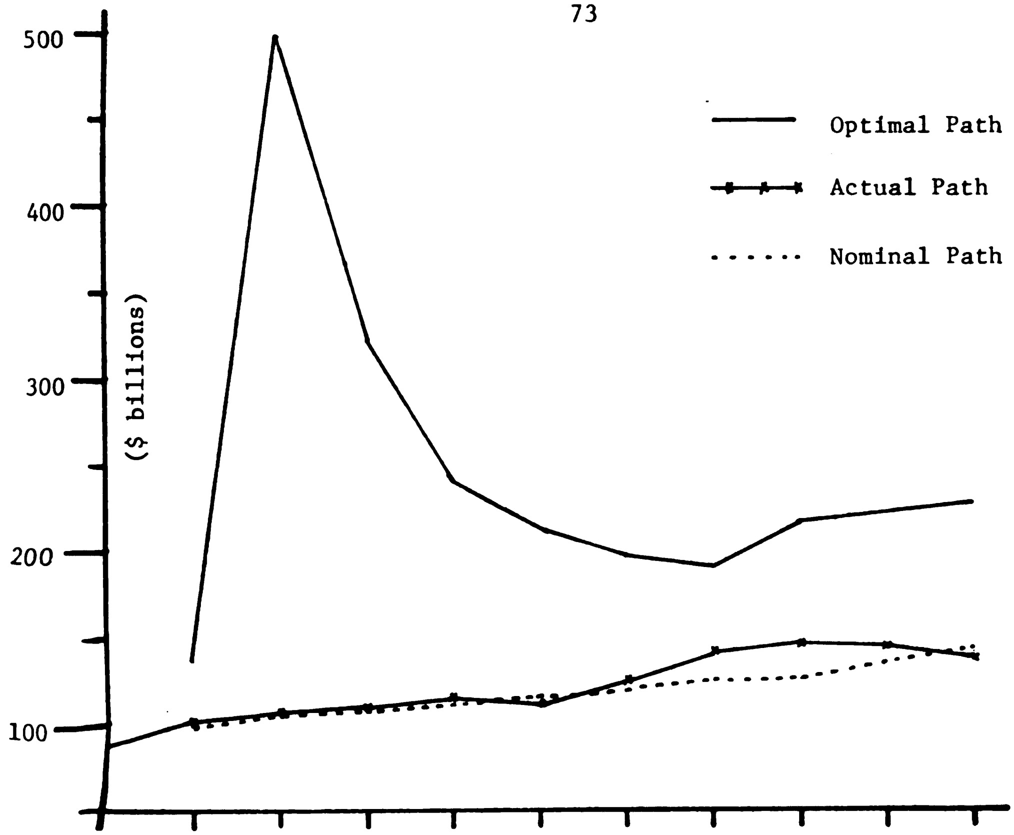


Figure 3.1. Two-country model (Case A) - U.S. Government expenditures: the optimal path compared with actual and nominal paths.

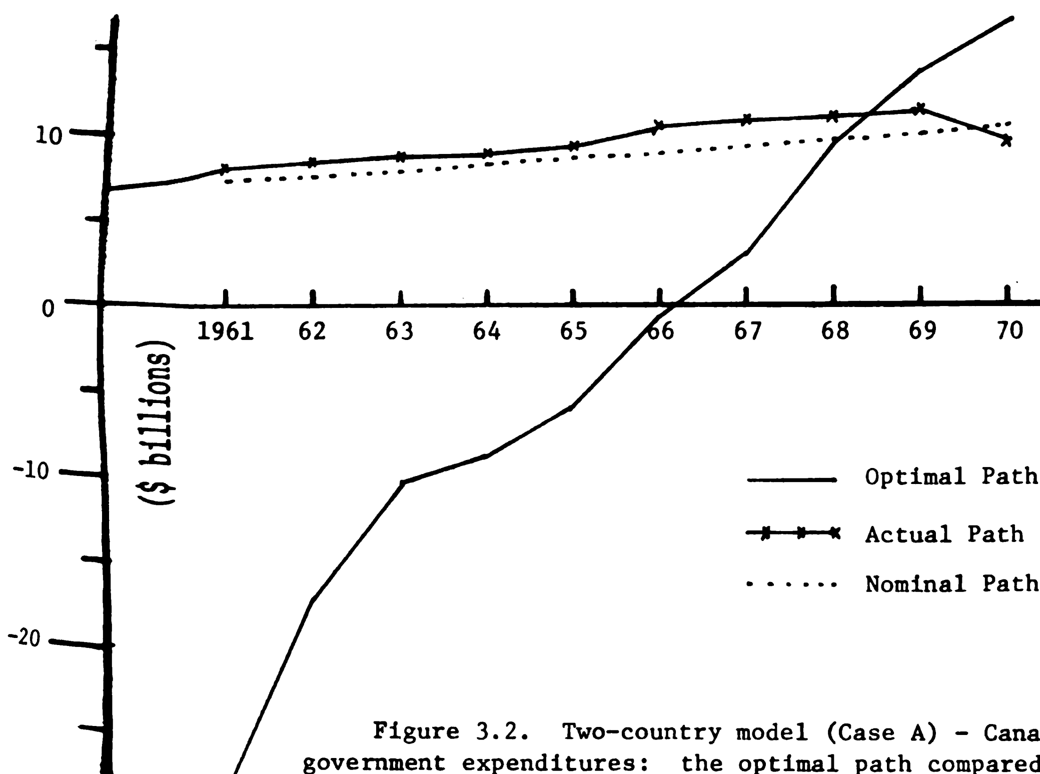
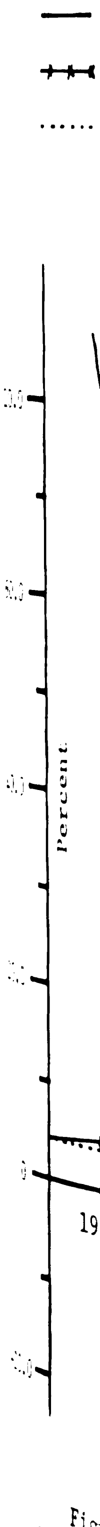


Figure 3.2. Two-country model (Case A) - Canadian government expenditures: the optimal path compared with actual and nominal paths.



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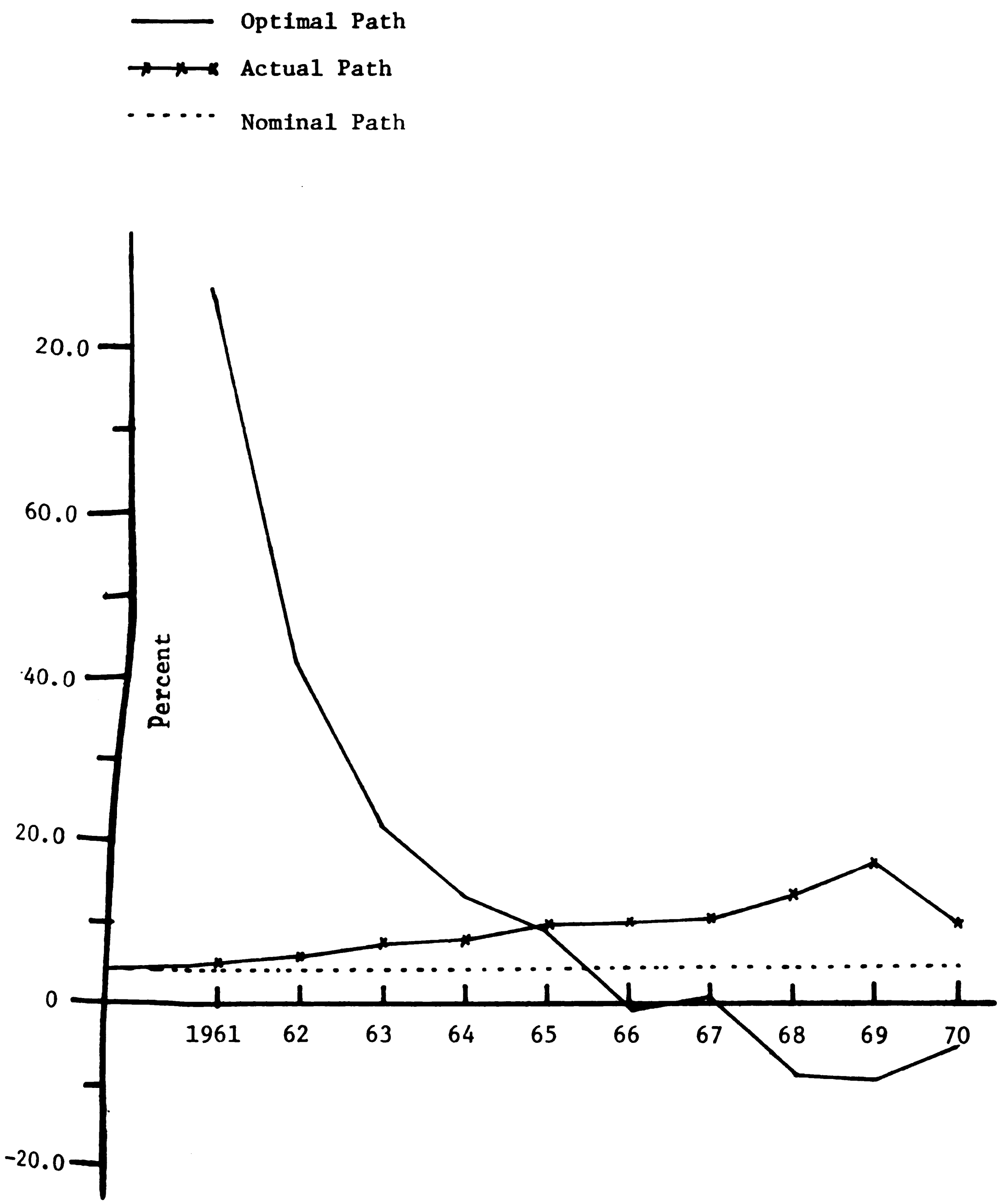
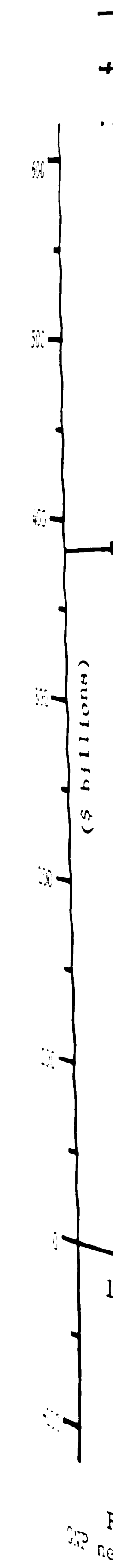


Figure 3.3. Two-country model (Case A) - U.S. interest rate: the optimal path compared with actual and nominal paths.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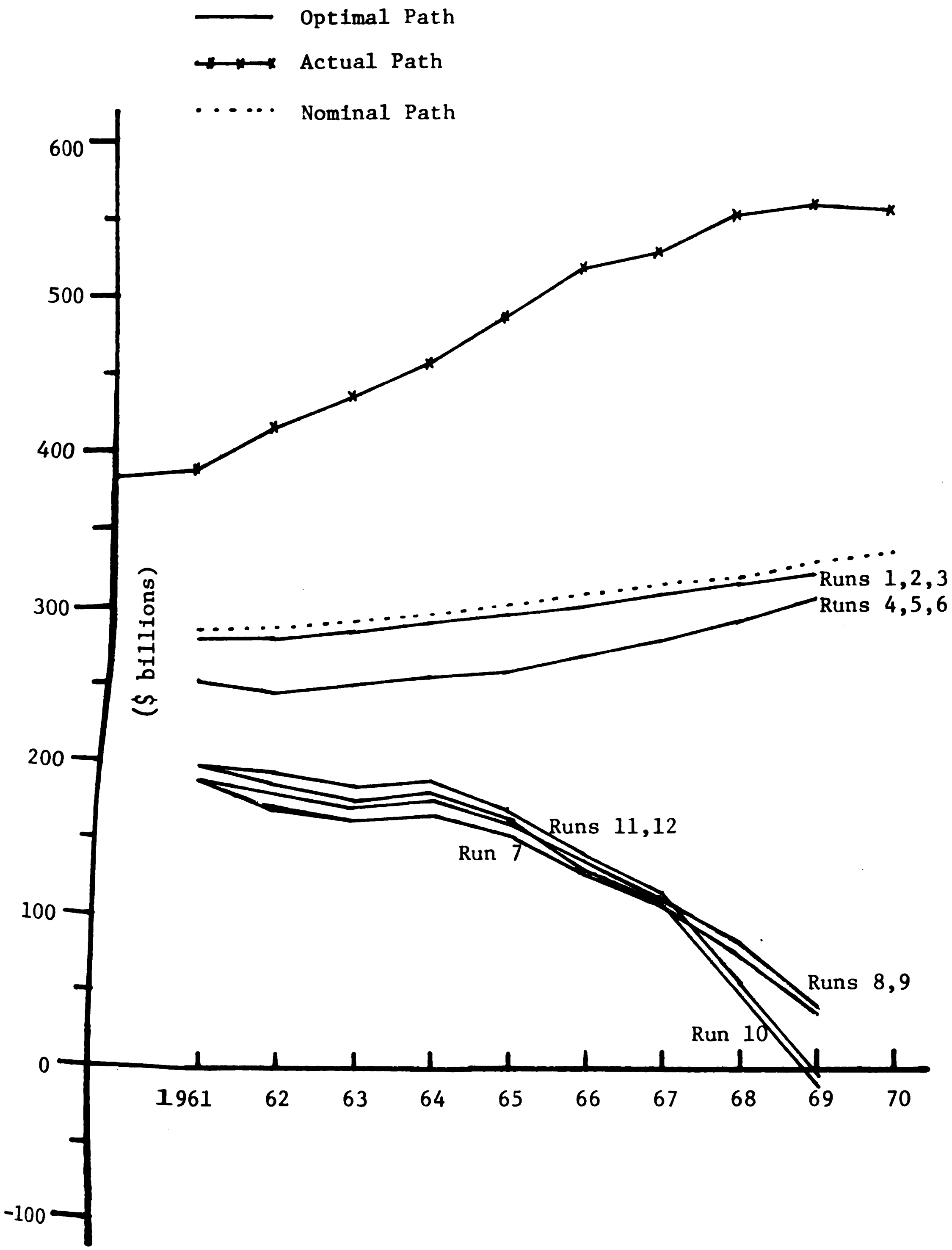
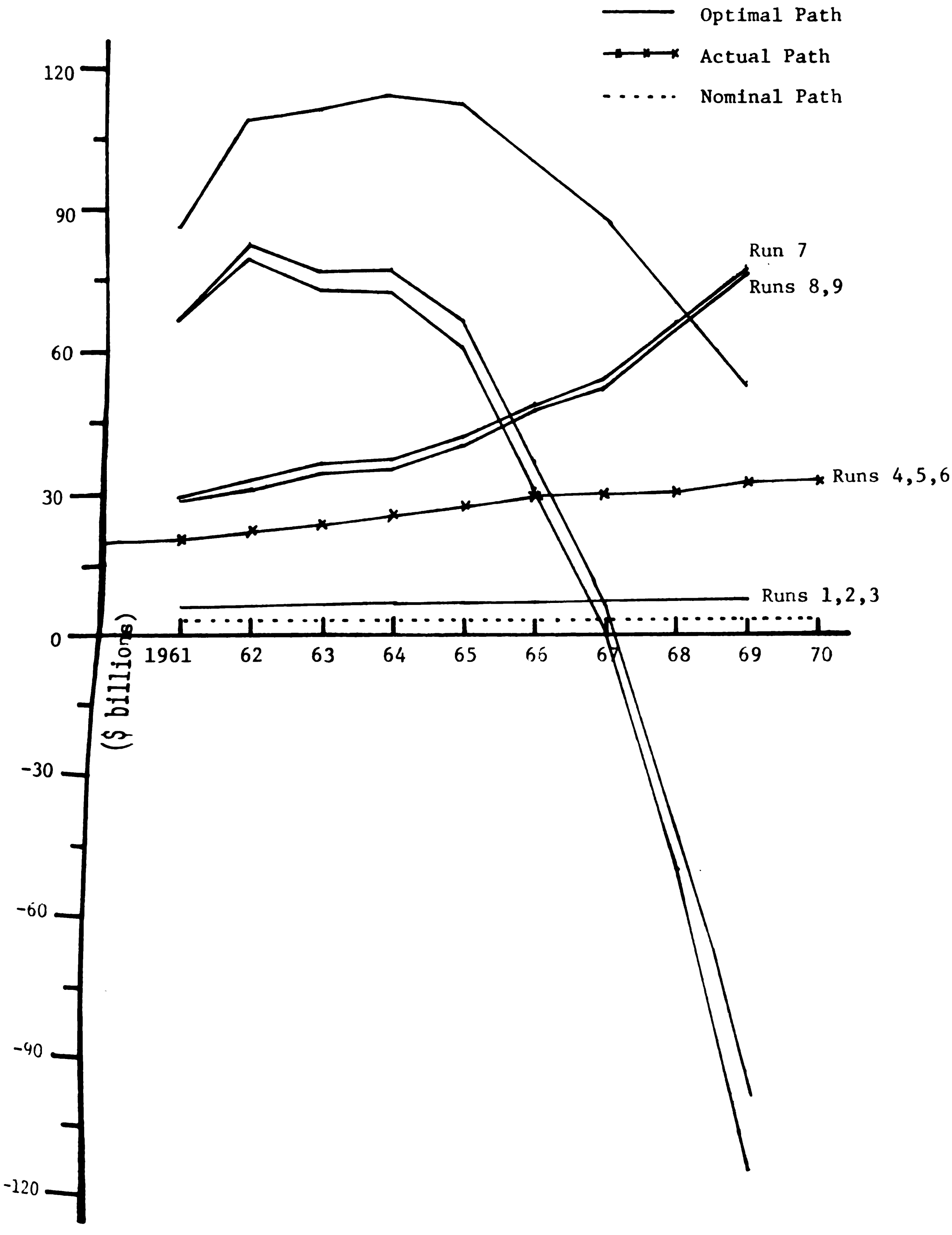
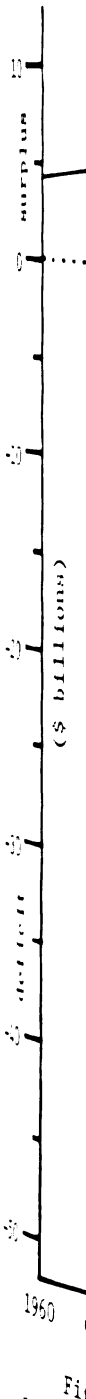


GNP Figure 3.4. Two-country model (Case A) - United States: optimal net of capital stock compared with the actual and potential GNP.



GNP Figure 3.5. Two-country model (Case A) - Canada: optimal net of capital stock compared with the actual and potential GNP.



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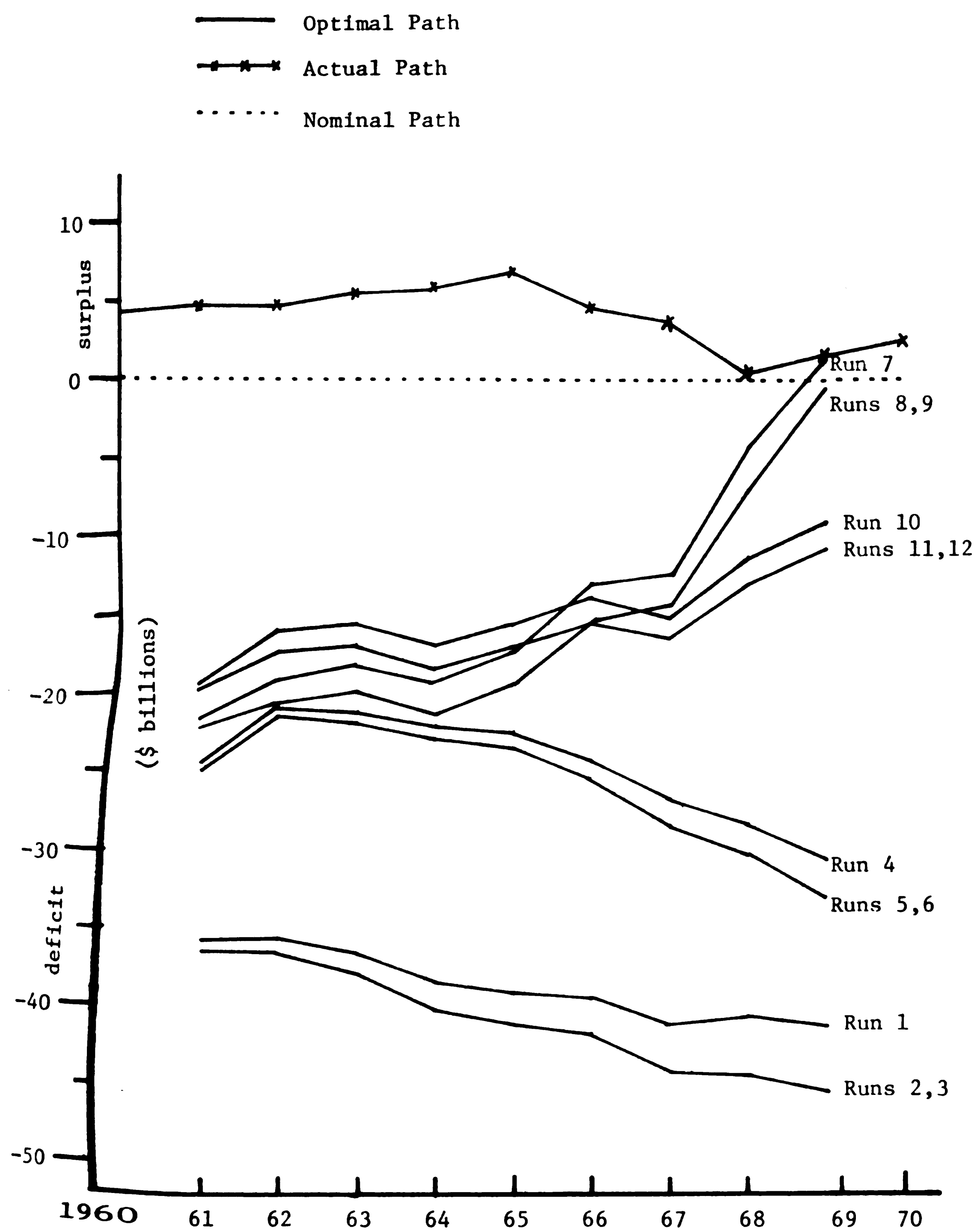

Figure 3.6. Two-country model (Case A) - U.S. balance of payments: optimal paths compared with the actual and nominal paths.

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country is not only more effective in achieving its own internal balance than that of the other country, but also that a negative change in its own fiscal policy is required to bring about a given change in the other country's internal target variable. This means that the economic interdependence of the two countries lowers the effectiveness of each country's fiscal instrument in reaching its own internal balance. Country I's monetary policy depends on all three targets, but with varying degree. It is more effective in achieving the common external balance than the internal balances which have already been taken care of by the fiscal policies. Furthermore, its overall effectiveness is reduced due to the existence of Country II's internal target. In short, the effectiveness of each country's policy making will decline as economic interdependence increases.

Next, an appraisal of the results obtained by the techniques of optimal control will be conducted to see if they are feasible in the U.S. and Canadian economies. Figure 3.1 shows that the optimal fiscal policy in the United States diverges from both its actual and nominal path, and that the economic interdependence prevailing in the two-country model requires much higher U.S. government expenditures to reach the situation of full employment than in the one-country case. Furthermore, the GNP resulting from using the optimal fiscal policy is exactly on the internal balance path (Figure 3.5). Again it is noted that the actual path for the GNP diverges upward from the full employment situation. In other words, for the last decade U.S. economy dealt with inflation created by an excess-demand.

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For Canada, which is a small country compared with the United States, the attainment of its own internal balance, when impeded by the U.S. pursuit of domestic stabilization, required a tightening of fiscal policy with a high accumulation of government savings during the first half of the planning period, then a steady increase of government expenditures starting at 1967 (Figure 3.2). The optimal path also diverges from the nominal and actual paths, the growth of which is more stable. Figure 3.5 shows excess demand of an amount equal to the deviation of the actual GNP from the situation of internal balance; therefore, inflation occurs in Canada too.

Once the U.S. government expenditures are set at the level for attaining the internal balance, the common external balance is carried out by the U.S. monetary policy, defined as the U.S. short-term interest rate. Unlike the one-country case in which the value is negative, the optimal interest rate starts with a very high and even unrealistic level around 82 percent, then decreases over time with negative values for the three last periods.

In short, the actual situation in the United States and Canada, compared with the given targets, is as follows: both countries are facing inflation combined with a surplus in the U.S. balance of payments, which is termed as a situation of potential conflict in policy goals (Johnson:1966). To change this conflict situation to an overall balance by using traditional policies rather than appreciation of the exchange rate would require, within the context of optimal control theory, high government expenditures and a negative interest rate in the United States, while government savings would be forced in Canada.

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All three of these optimal policies are declared to be inadmissible for either social, political, or practical reasons. Therefore, the boundary conditions or limits on policy-variable magnitudes which have been violated are to become active. Within the framework of control theory this can be done by introducing into the cost function the penalty costs for deviations of the policy variables from their limits, that is, by changing the diagonal elements of the matrix Q^c as follows:

$$Q^c = \left[\begin{array}{c|c} Q^P & 0 \\ \hline 0 & R^P \end{array}\right] = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{33} \end{bmatrix}$$

Similar to the one-country case, only a sensitivity analysis of the weighting factors on the components of the control vector is performed, i.e., when the weights for the endogenous variables remain unchanged. The cost function for all the experiments is given in Table 3.3 and the results are shown in Figures 3.7, 3.8 and 3.9.

Table 3.3.

Runs	
Test 1	1
	2
	3
Test 2	4
	5
	6
	7
	8
	[illegible]
Test 3	1
	1
	1

Table 3.3. Two-country model (Case A): penalty weights attached to the deviations of state and control variables from their nominal values

	Runs	q_{11}	q_{22}	q_{33}	r_{11}	r_{22}	r_{33}
Test 1	1	1	1	1	0	0	10^5
	2	1	1	1	0	0	$5x10^5$
	3	1	1	1	0	0	10^6
Test 2	4	1	1	1	0	10^5	10^5
		1	1	1	0	$5x10^5$	10^5
	5	1	1	1	0	$5x10^5$	$5x10^5$
		1	1	1	0	10^5	$5x10^5$
	6	1	1	1	0	$5x10^5$	10^6
		1	1	1	0	10^5	10^6
		1	1	1	0	10^6	10^6
	7	1	1	1	10^5	0	10^5
		1	1	1	$5x10^5$	0	10^5
		1	1	1	10^6	0	10^5
	8	1	1	1	10^5	0	$5x10^5$
		1	1	1	$5x10^5$	0	$5x10^5$
		1	1	1	10^6	0	$5x10^5$
	9	1	1	1	10^5	0	10^6
		1	1	1	$5x10^5$	0	10^6
		1	1	1	10^6	0	10^6
Test 3	10	1	1	1	10^5	10^5	10^5
		1	1	1	$5x10^5$	$5x10^5$	10^5
		1	1	1	10^5	$5x10^5$	10^5
		1	1	1	$5x10^5$	10^5	10^5
	11	1	1	1	10^5	10^5	$5x10^5$
		1	1	1	$5x10^5$	10^5	$5x10^5$
		1	1	1	10^5	$5x10^5$	$5x10^5$
		1	1	1	$5x10^5$	$5x10^5$	$5x10^5$
	12	1	1	1	10^5	10^5	10^6
		1	1	1	$5x10^5$	10^5	10^6
		1	1	1	10^5	$5x10^5$	10^6
		1	1	1	$5x10^5$	$5x10^5$	10^6
		1	1	1	10^6	10^6	10^6

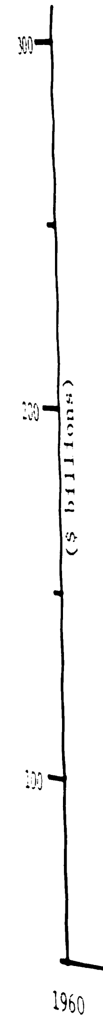


Figure 3.7. Two-country model (Case A) - U.S. Government expenditures: optimal paths compared with nominal path.

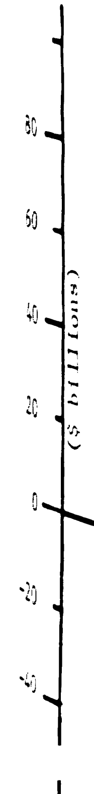


Figure 3.8. Two-country model (Case A) - Canadian government expenditures: optimal paths compared with nominal path.

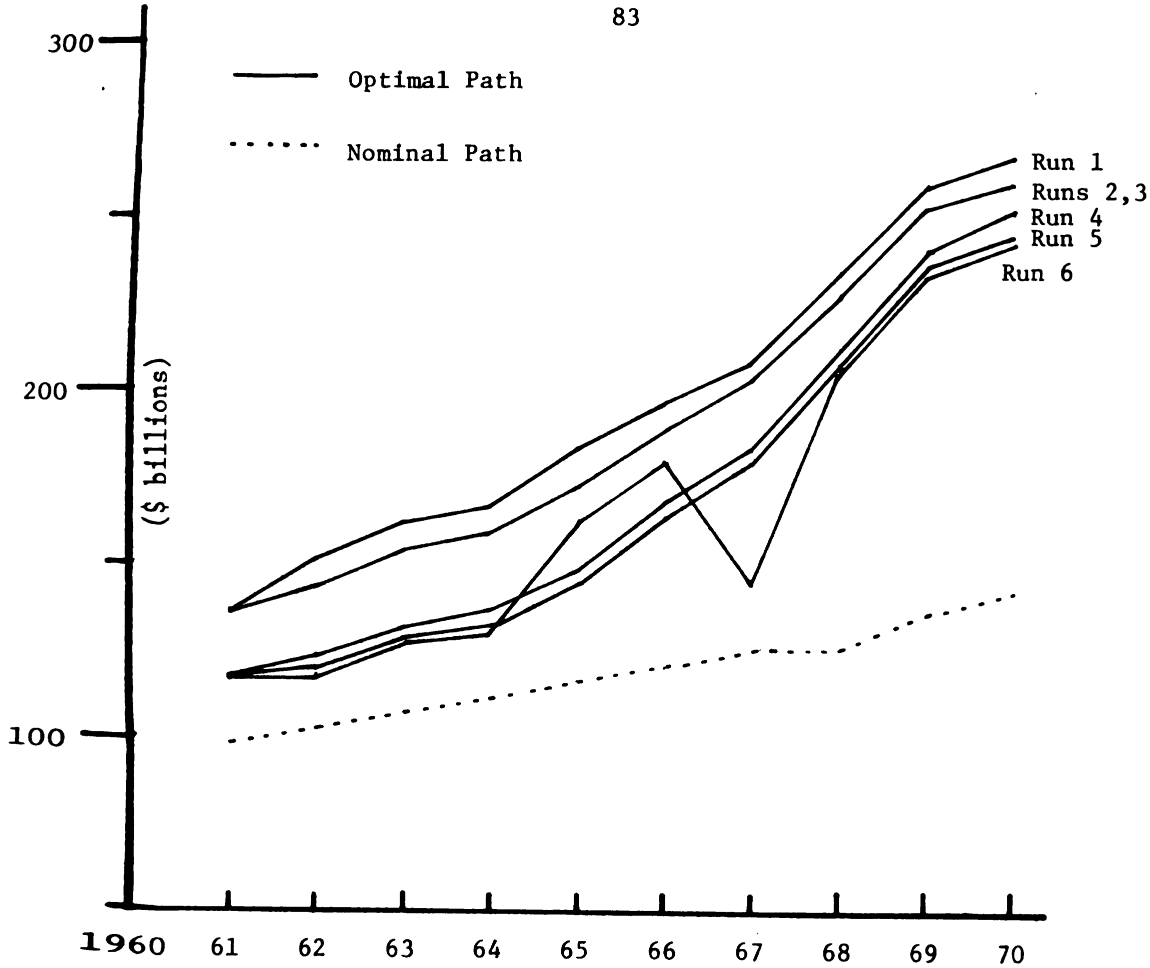


Figure 3.7

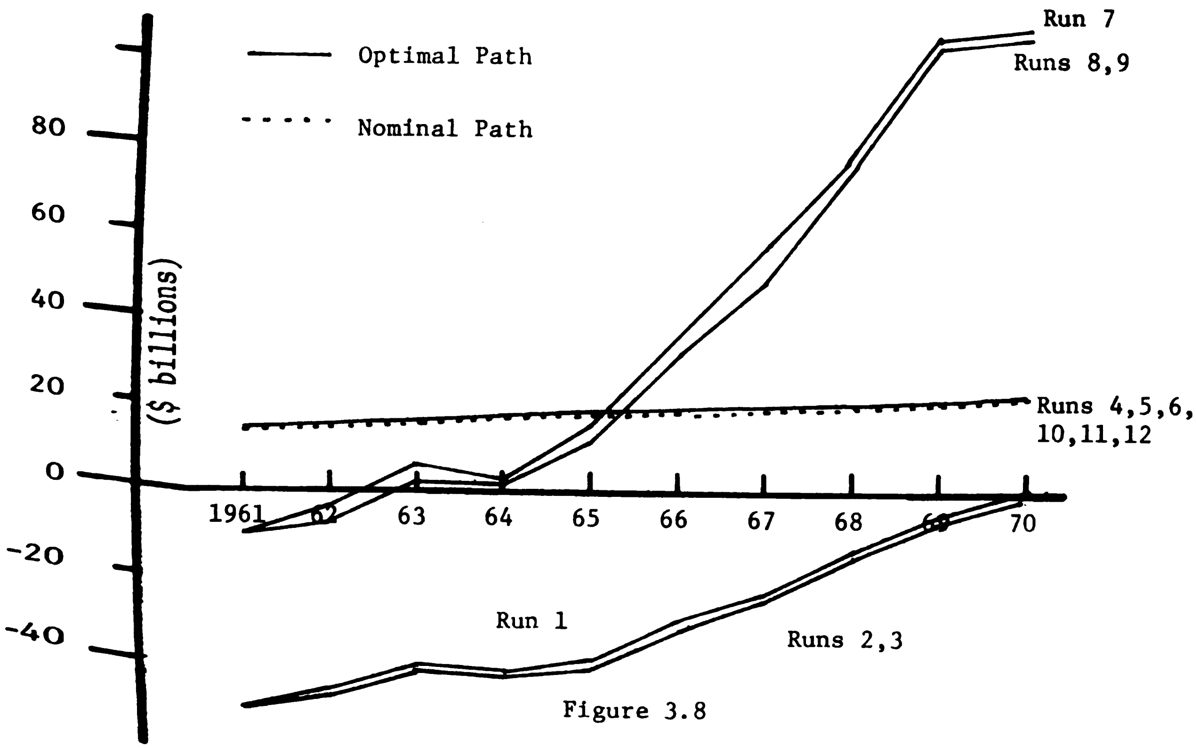
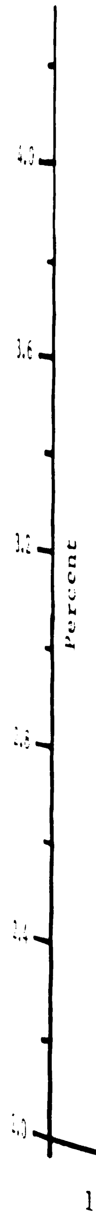


Figure 3.8



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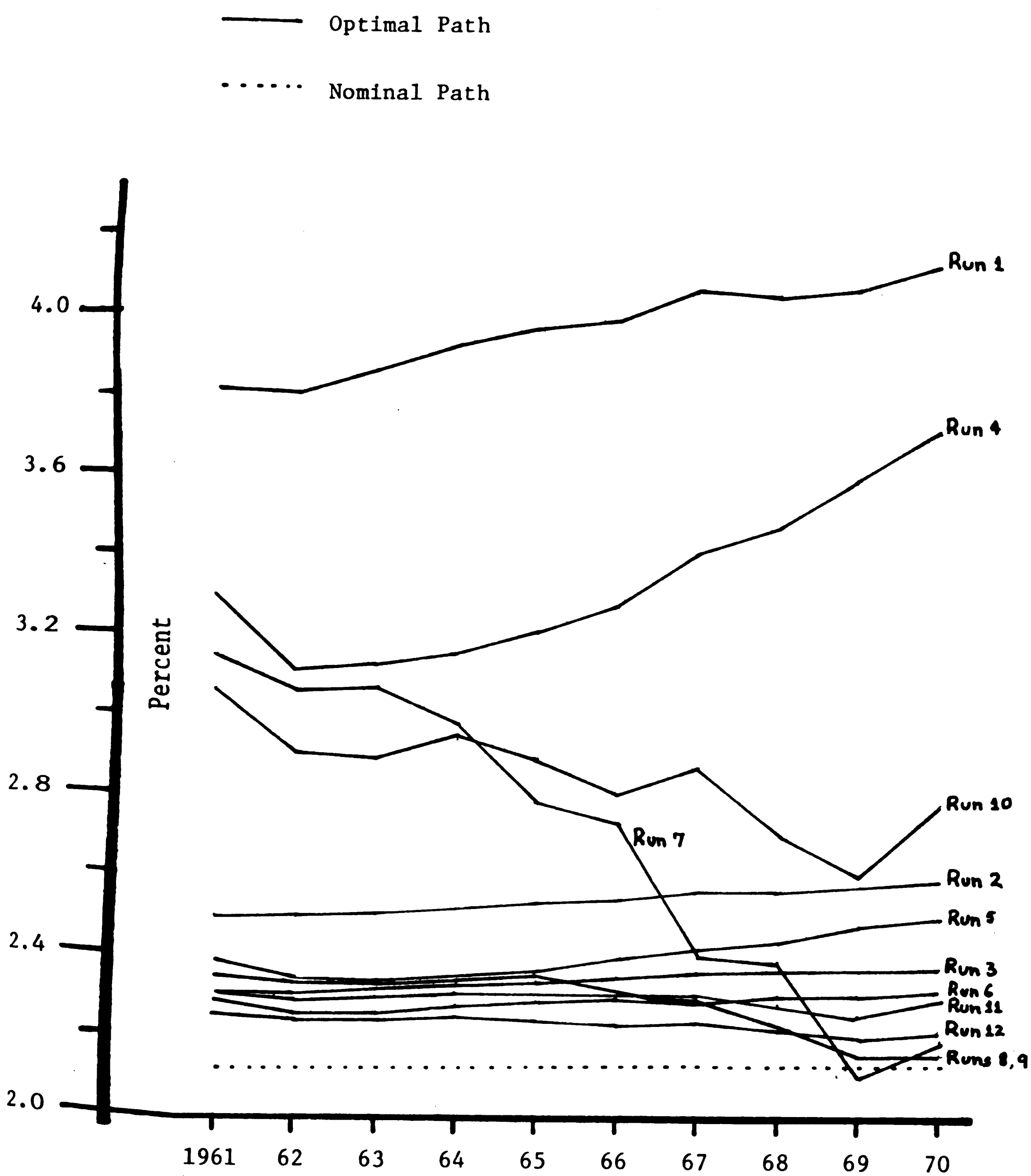

Figure 3.9. Two-country model (Case A) - U.S. interest rate: optimal paths compared with nominal path.

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Three tests of sensitivity are performed with penalty weights assigned first to one instrument, then to two instruments, and finally to all the three instruments.

For the first test (runs 1,2,3) in which only Country I's monetary instrument is constrained to the limit fixed at 2.15 percent, it is observed that the more weight given to the interest rate r_{1t} the closer it gets to the nominal path, while the two other unconstrained instruments G_{1t} and G_{2t} diverge strongly from their boundary conditions upward and downward, respectively. Therefore, they are declared inadmissible for practical and political reasons. Furthermore, there is a tracking trade-off between endogenous variables and instruments. In fact, because a division of labor in achieving the joint balance prevails in Votey's model when r_{1t} is constrained within limits, a large deficit occurs in the U.S. balance of payments, while the GNP in both the United States and Canada keeps tracking the full-employment situation at the expense of non-feasibility of fiscal policies G_{1t} and G_{2t}.

The second test concerns penalty weights given to two of the instruments--Country I's interest rate and government spending of either country. It is found that:

(1) If both Country I's instruments are constrained by the boundary conditions, then its interest rate will track the nominal path more closely than if limits are set on either Country I's interest rate alone or on the pair of Country II's fiscal instrument and Country I's interest rate. In short, limits set on

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G_{1t} reinforce the feasibility of Country I's monetary policy, while limits on G_{2t} do the opposite.

(2) Since two boundary conditions are active, it is expected that at least two endogenous variables would not be on target exactly because a tracking trade-off between targets and instruments occurs. Because of the particular structural form of Votey's model, the choice of targets to be dropped is based on Mundell's "division of labor" principle. That is, the target on which the constraint policy instrument is the most effective has to be changed numerically or given up, at the expense of instruments tracking for limits.

Finally, the third test in which weights are assigned to all the three instruments is performed. It is found that:

(1) Each of the three instruments is tracking its limits. However, giving increasing weight to r_{11} leads to a closer tracking of Country I's monetary policy on its limit, while the tracking for nominal government expenditures in both countries remains the same, whatever the weight number assigned to them.

(2) Compared to the second test (runs 7,8,9), the tracking for the fixed interest rate is less

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effective. This is due to the additional boundary condition set on Country II's government expenditures.

(3) Similar to the first two tests, the tracking of all endogenous variables $\tilde{Y}_{1t}$, $\tilde{Y}_{2t}$ and B_{1t} is lost. The deviations from their respective targets are larger than those obtained in tests 1 and 2.

It can be concluded that:

(1) As economic interdependence increases, the effectiveness and impact of each country's policy on its own target will decline.

(2) If Tinbergen's principle on equality between the number of targets and the number of instruments is met, it is always possible to steer the endogenous variables to be controlled exactly to targets by using the quadratic welfare function (equation A-2.16) and assigning positive weights only to the deviations of the variables selected.

(3) However, nothing guarantees the feasibility of optimal policies to achieve the internal and external balance. When the boundary conditions become active, a tracking trade-off between endogenous variables and instruments occurs.

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CHAPTER IV

TWO-COUNTRY MODEL

CASE B: PASSIVE RESPONSES AND LINEAR DEPENDENT EXTERNAL BALANCE

Unlike the preceding chapter, this one deals with a four-target and three-instrument case: the fiscal authority in each country will act to attain its own internal balance while the monetary authority in Country I strives not only for its own external balance but also for that of Country II under the assumptions of passive responses from Country II and non-common but non-conflicting balance-of-payments targets. Since the number of targets is greater than the number of instruments, Tinbergen's rule is no longer satisfied. Therefore, it is expected that the optimal results for internal and external balance will differ from those obtained in the two preceding cases. To show this, the same analysis procedure will be repeated. First, Votey's two-country model with modifications on the foreign sector will be presented. Second, the optimal control problem will be formulated to derive the optimal solution analytically and numerically with reference to the United States and Canada. Finally, the economic evaluation of the optimal policy mix will be made as well as its amendment when boundary conditions become active.

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4.1. Presentation of Votey's Model with Modifications on the Foreign Sector

Votey's two-country model presented in Chapter III will be modified by redefining all the variables in the foreign sector and by adding an identity for the determination of Country II's balance of payments. This is done because Votey ignores the trade of Countries I and II with the rest of the world by assuming the total exports of I (II) equal the total imports of II (I) and by using a single equation--Country I's balance of payments--as the common external balance of both countries. Now consider the "modified" two-country model under the following assumptions:

(1) Country II's interest rate is given, that is, Country II is assumed to be in the position of passive cooperation with Country I for dealing with the external balance.

(2) Stein's results for short-term interest rates represent the degree of sensitivity of capital flows to interest rate differentials.

(3) Unlike Case A, the total exports and imports of Countries I and II will include the trade with the third country which represents the rest of the world.

The variables of Votey's econometric model with modifications on the foreign sector are:

for $i = 1,2$

$B_{it} =$ Balance of payments in Country i

$C_{it} =$ Consumption expenditures of Country i

I_{it}^{G} = Gross investment expenditures of Country i

I_{it}^{n} = Net investment expenditures of Country i

IE_{1t} = Investment earnings of Country I from abroad

K_{it} = Capital stock of Country i

L_{it} = Labor force of Country i

M_{it} = Total imports demand of Country i

M_{it}^{III} = Imports of Country i from Country III

O_{it} = Net short term capital outflows of Country i

r_{it} = Rate of interest in Country i

Px_{it} = Export price in Country i

PM_{it} = Import price in Country i

T_{it} = Tax receipts of Country i

TR_{2t} = Transfer payments of Country II

TT_{it} = $\left(\frac{P_M}{P_x}\right)_{it}$ = Terms of trade of Country i

X_{it} = Total exports of Country i

X_{it}^{III} = Exports of Country i to Country III

Y_{it} = National income of Country i

The equations are as follows:

(I-1) $$Y_{1t} = C_{1t} + I_{1t}^{G} + G_{1t} + X_{1t} - M_{1t}$$

(I-2) $$B_{1t} = X_{1t} - M_{1t} - O_{1t}$$

(I-3) $$K_{1t} = I_{1t}^{n} + K_{1t-1}$$

(I-4) $$I_{1t}^{G} = I_{1t}^{n} + \delta_{*} K_{1t}$$

(I-5) $$X_{1t} = X_{1t}^{III} + (M_{2t} - M_{2t}^{III}) + IE_{1t}$$

(I-6) $Y_{2t} = C_{2t} + I^G_{2t} + G_{2t} + X_{2t} - M_{2t}$

(I-7) $B_{2t} = X_{2t} - M_{2t} + 0_{1t}$

(I-8) $K_{2t} = I^n_{2t} + K_{2t-1}$

(I-9) $I^G_{2t} = I^n_{2t} + \delta_* K_{2t}$

(I-10) $X_{2t} = X^{III}_{2t} + (M_{1t} - M^{III}_{1t})$

(E-1) $Y_{1t} = \delta_{10} + \delta_{11} K_{1t} + \delta_{12} L_{1t}$

(E-2) $C_{1t} = \alpha_{10} + \alpha_{11} (Y_{1t} - T_{1t} - \delta_* K_{1t})$

(E-3) $M_{1t} = \beta_{10} + \beta_{11} (Y_{1t} - T_{1t} - \delta_* K_{1t}) + \beta_{12} TT_{1t}$

(E-4) $I^n_{1t} = \gamma_{10} + \gamma_{11} Y_{1t-1} - \gamma_{12} \; q(\delta_* + r)_{1t-1} + \gamma_{13} K_{1t-1}$

(E-5) $0_{1t} = \eta_{10} + \eta_{11} (r_{2t} - r_{1t})$

(E-6) $Y_{2t} = \delta_{20} + \delta_{21} K_{2t} + \delta_{22} L_{2t}$

(E-7) $C_{2t} = \alpha_{20} + \alpha_{21} (Y_{2t} - T_{2t} - TR_{2t} - \delta_* K_{2t})$

(E-8) $M_{2t} = \beta_{20} + \beta_{21} (Y_{2t} - T_{2t} - TR_{2t} - \delta_* K_{2t}) + \beta_{22} TT_{2t}$

(E-9) $I^n_{2t} = \gamma_{20} + \gamma_{21} Y_{2t-1} - \gamma_{22} \; (\delta_* + r)_{2t-1} + \gamma_{23} K_{2t-1}$

The comments on equations (E-1) to (E-5) have been presented in Chapter II.

Equation E-6 - The output of Country II is a linear function of both its factors of production: capital stock (K_{2t}) and labor force (L_{2t}).

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Equation E-7 - Consumption expenditures are made a function of disposable income. But the disposable income definition for Country II is slightly different from that of Country I (Votey:1969), and it is represented by GNP less capital depreciation ($\delta_* K_{2t}$ where δ_* is the rate of replacement and is assumed to be the same for both countries) less taxes and also less transfer payments of Country II (TR_{2t} assumed to be exogenous).

Equation E-8 - Import demand of II is simply a function of disposable income and ratio of foreign to domestic prices in Country II (TT_{2t}).

Equation E-9 - Net investment expenditures in Country II are assumed to depend on money output, the capital stock and the user cost of capital in Country II. The user cost of capital in Country II has the same definition as that of Country I with assumption $q = 1$ over time.

Identity I-2 - The identity for the determination of the balance of payments in Country I is different from that of Votey. The same definition as that of the one-country model is used, but X_{1t} and M_{1t} redefined to take into account the trade between Countries I and II and the rest of the world (called Country III).

Identity I-5 - The total exports of Country I are equal to the exports of Country I to Country III, plus exports of Country I to Country II ($X_{1t}^{II} = M_{2t} - M_{2t}^{III}$) plus investment earnings of Country I from abroad.

Similarly, total imports of Country I are equal to imports of Country I from Country III (M_{1t}^{III}) plus imports of Country I from Country II ($M_{1t}^{II} = X_{2t} - X_{2t}^{III}$).

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Identity I-7 - Added to Votey's model is the Country II balance-of-payments identity. It is equal to total exports of Country II (X_{2t}) less total imports of Country II (M_{2t}) less net short-term capital outflows of Country II ($0_{2t} = -0_{1t}$), which is equivalent to net short-term capital inflow from Country I.

Identity I-10 - Total exports of Country II are equal to exports of Country II to Country III (X_{2t}^{III}) plus exports of Country II to Country I ($X_{2t}^{I} = M_{1t} - M_{1t}^{III}$). Similarly, total imports of Country II are equal to imports of Country II from Country III (M_{2t}^{III}) plus imports of Country II from Country I ($M_{2t}^{I} = X_{1t} - X_{1t}^{III}$).

Then substituting X_{2t} and M_{2t} into $B_{2t} = X_{2t} - M_{2t} + 0_{1t}$ results in:

$$B_{2t} = X_{2t}^{III} + M_{1t} - M_{1t}^{III} - X_{2t}^{III} - X_{1t} + X_{1t}^{III} + 0_{1t}$$

$$= (X_{1t}^{III} + X_{2t}^{III}) - (M_{1t}^{III} + M_{2t}^{III}) - (X_{1t} - M_{1t} - 0_{1t})$$

Or defining X_{3t}: total exports of Country III $= M_{1t}^{III} + M_{2t}^{III}$

M_{3t}: total imports of Country III $= X_{1t}^{III} + X_{2t}^{III}$

B_{1t}: $X_{1t} - M_{1t} - 0_{1t}$

results in:

$$B_{2t} = -X_{3t} + M_{3t} - B_{1t}$$

Therefore, B_{2t} is linearly dependent on B_{1t}.

4.2. Optimal Control Problem Without Constraints on Policy-Variable Magnitudes

Unlike Case A in Chapter III, the assumption of common balance of payments is no longer held when the third country block is introduced to rep

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to represent the rest of the world. Therefore, there are four targets to achieve simultaneously, but only three are independent. Internal balance in both countries and Country I's balance-of-payments equilibrium are independent, while Country II's balance of payments is a linear function of that of Country I instead of being equal to Country I's balance of payments with the opposite sign. As for the instruments, they are in the number of three as in Case A, i.e., government expenditures in both countries and Country I's short-term interest rate are used to achieve the joint balance, while the short-term interest rate of Country II is considered as fixed or given under the assumption of passive responses from Country II.

The definition of external and internal balance is similar to that used in Case A (section 3.2) and the reduced form of Votey's modified two-country model given by Appendix A-10:

$$y_t = \bar{A}\, y_{t-1} + \bar{B}u_t + \bar{C}u_{t-1} + \bar{D}z_t \tag{4.1}$$

where

$$y_t \stackrel{d}{\equiv} \left[Y_{1t},\ Y_{2t},\ B_{1t},\ B_{2t},\ K_{1t},\ K_{2t},\ G_{1t},\ G_{2t},\ r_{1t}\right]'$$

$$u_t \stackrel{d}{\equiv} \left[G_{1t},\ G_{2t},\ r_{1t}\right]'$$

$$z_t \stackrel{d}{\equiv} \Big[1,\ IE_{1t},\ X_{1t}^{III},\ M_{1t}^{III},\ TT_{1t},\ T_{1t},\ r_{2t},\ r_{2t-1},\ TT_{2t},\ T_{2t}$$

$$TR_{2t},\ X_{2t}^{III},\ M_{2t}^{III}\Big]'$$

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{13} & A_{14} \\ A_{21} & A_{22} & 0 & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & 0 & A_{33} & A_{34} \\ -A_{31} & A_{32} & 0 & 0 & -A_{33} & A_{34} \\ \gamma_{11} & 0 & 0 & 0 & a_{55} & 0 \\ 0 & {}_{21} & 0 & 0 & 0 & a_{66} \end{bmatrix} ; \bar{B} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & -Q_{32} & \eta_{11} \\ -Q_{31} & Q_{32} & -\eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \bar{C} = \begin{bmatrix} 0 & 0 & A_{15} \\ 0 & 0 & A_{25} \\ 0 & 0 & -A_{35} \\ 0 & 0 & A_{35} \\ 0 & 0 & -\gamma_{12} \\ 0 & 0 & 0 \end{bmatrix}$$

$$;\bar{D} = \begin{bmatrix} D_{11} & Q_{11} & Q_{11} & -Q_{12} & D_{12} & D_{13} & 0 & A_{16} & D_{14} & D_{15} & D_{15} & Q_{12} & -Q_{11} \\ D_{21} & Q_{21} & Q_{21} & -Q_{22} & D_{22} & D_{23} & 0 & A_{26} & D_{24} & D_{25} & D_{25} & Q_{22} & -Q_{21} \\ D_{31} & 1+Q_{31} & 1+Q_{31} & Q_{32} & -D_{32} & D_{33} & -\eta_{11} & A_{36} & -D_{34} & -D_{35} & -D_{35} & -Q_{32} & -(1+Q_{31}) \\ D_{31} & -(1+Q_{31}) & -Q_{31} & -(1+Q_{32}) & D_{32} & -D_{33} & \eta_{11} & -A_{36} & D_{34} & D_{35} & D_{35} & 1+Q_{32} & Q_{31} \\ a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Next, the optimal control problem for the "modified" two-country model will be formulated into Pindyck's and Chow's framework but only the latter will be used to derive the optimal solution.

4.2.1. Formulation of the Problem

Expressed in Pindyck's terminology, the "state-space" for the "modified" two-country optimal control problem is:

$$x_t^P = A^P x_{t-1}^P + B^P u_{t-1}^P + C^P z_{t-1}^{P} \qquad (4.2)$$

where $x_t^P \stackrel{d}{=} y_t - \bar{B}u_t - \bar{D}z_t \stackrel{d}{=} \left[\tilde{YY}_{1t}, \tilde{YY}_{2t}, BB_{1t}, BB_{2t}, KK_{1t}, KK_{2t}\right]'$ is the (6x1) state vector; $u_{t-1}^P \stackrel{d}{=} u_{t-1} \stackrel{d}{=} \left[G_{1t-1}, G_{2t-1}, r_{1t-1}\right]'$ is the (3x1) control vector; $z_t^P \stackrel{d}{=} \left[1, IE_{1t}, X_{1t}^{III}, M_{1t}^{III}, TT_{1t}, T_{1t}, r_{2t}, TT_{2t}, T_{2t}, TR_{2t}, X_{2t}^{III}, M_{2t}^{III}\right]'$ is the (12x1) vector of current exogenous variable; z_{t-1}^P is the (12x1) vector of lagged exogenous variable; and A^P, B^P and C^P are known matrices.

Matrix $A^P = \bar{A}$

Row	$\tilde{YY}_{1t}$	$\tilde{YY}_{2t}$	BB_{1t}	BB_{2t}	KK_{1t}	KK_{2t}
1	A_{11}	A_{12}	0	0	A_{13}	A_{14}
2	A_{21}	A_{22}	0	0	A_{23}	A_{24}
3	A_{31}	$-A_{32}$	0	0	A_{33}	$-A_{34}$
4	$-A_{31}$	A_{32}	0	0	$-A_{33}$	A_{34}
5	γ_{11}	0	0	0	a_{55}	0
6	0	γ_{21}	0	0	0	a_{66}

Matrix $B^P = \overline{AB} + \overline{C}$

Row	G_{1t-1}	G_{2t-1}	r_{1t-1}
1	$A_{11}Q_{11}+A_{12}Q_{21}$	$A_{11}Q_{12}+A_{12}Q_{22}$	A_{15}
2	$A_{21}Q_{11}+A_{22}Q_{21}$	$A_{21}Q_{12}+A_{22}Q_{22}$	A_{25}
3	$A_{31}Q_{11}-A_{32}Q_{21}$	$A_{31}Q_{12}-A_{32}Q_{22}$	$-A_{35}$
4	$-A_{31}Q_{11}+A_{32}Q_{21}$	$-A_{31}Q_{12}+A_{32}Q_{22}$	A_{35}
5	$\gamma_{11}Q_{11}$	$\gamma_{11}Q_{12}$	$-\gamma_{12}$
6	$\gamma_{21}Q_{21}$	$\gamma_{21}Q_{22}$	0

Matrix $C^P = \bar{A}\,\bar{D}$

Row	1	IE_{1t-1}	X^{III}_{1t-1}	M^{III}_{1t-1}	TT_{1t-1}	T_{1t-1}
1	$A_{11}D_{11}+A_{12}D_{21}+A_{13}a_{57}+A_{14}a_{67}$	$A_{11}Q_{11}+A_{12}Q_{21}$	$A_{11}Q_{11}+A_{12}Q_{21}$	$-(A_{11}Q_{12}+A_{12}Q_{22})$	$A_{11}D_{12}+A_{12}D_{22}$	$A_{11}D_{13}+A_{12}D_{23}$
2	$A_{21}D_{11}+A_{22}D_{21}+A_{23}a_{57}+A_{24}a_{67}$	$A_{21}Q_{11}+A_{22}Q_{21}$	$A_{21}Q_{11}+A_{22}Q_{21}$	$-(A_{21}Q_{12}+A_{22}Q_{22})$	$A_{21}D_{12}+A_{22}D_{22}$	$A_{21}D_{13}+A_{22}D_{23}$
3	$A_{31}D_{11}-A_{32}D_{21}+A_{33}a_{57}-A_{34}a_{67}$	$A_{31}Q_{11}-A_{32}Q_{21}$	$A_{31}Q_{11}-A_{32}Q_{21}$	$-(A_{31}Q_{12}+A_{32}Q_{22})$	$A_{31}D_{12}-A_{32}D_{22}$	$A_{31}D_{13}-A_{32}D_{23}$
4	$-A_{31}D_{11}+A_{32}D_{21}-A_{33}a_{57}+A_{34}a_{67}$	$-A_{31}Q_{11}+A_{32}Q_{21}$	$-A_{31}Q_{11}+A_{32}Q_{21}$	$-(-A_{31}Q_{12}+A_{32}Q_{22})$	$-A_{31}D_{12}+A_{32}D_{22}$	$-A_{31}D_{13}+A_{32}D_{23}$
5	$\gamma_{11}D_{11}+a_{55}a_{57}$	$\gamma_{11}Q_{11}$	$\gamma_{11}Q_{11}$	$-\gamma_{11}Q_{12}$	$\gamma_{11}D_{12}$	$\gamma_{11}D_{13}$
6	$\gamma_{21}D_{21}+a_{66}a_{67}$	$\gamma_{21}Q_{21}$	$\gamma_{21}Q_{21}$	$-\gamma_{21}Q_{22}$	$\gamma_{21}D_{22}$	$\gamma_{21}D_{23}$

Row	r_{2t-1}	TT_{2t-1}	T_{2t-1}	TR_{2t-1}	X^{III}_{2t-1}	M^{III}_{2t-1}
1	$A_{11}A_{16}+A_{12}A_{26}-A_{14}\gamma_{22}$	$A_{11}D_{14}+A_{12}D_{24}$	$A_{11}D_{15}+A_{12}D_{25}$	$A_{11}D_{15}+A_{12}D_{25}$	$A_{11}Q_{12}+A_{12}Q_{22}$	$-(A_{11}Q_{11}+A_{12}Q_{21})$
2	$A_{21}A_{16}+A_{22}A_{26}-A_{24}\gamma_{22}$	$A_{21}D_{14}+A_{22}D_{24}$	$A_{21}D_{15}+A_{22}D_{25}$	$A_{21}D_{15}+A_{22}D_{25}$	$A_{21}Q_{12}+A_{22}Q_{22}$	$-(A_{21}Q_{11}+A_{22}Q_{21})$
3	$A_{31}A_{16}-a_{32}A_{26}+A_{34}\gamma_{22}$	$A_{31}D_{14}-A_{32}D_{24}$	$A_{31}D_{15}-A_{32}D_{25}$	$A_{31}D_{15}-A_{32}D_{25}$	$A_{31}Q_{12}-A_{32}Q_{22}$	$-(A_{31}Q_{11}-A_{32}Q_{21})$
4	$-A_{31}A_{16}+A_{32}A_{26}-A_{34}\gamma_{22}$	$-A_{31}D_{14}+A_{32}D_{24}$	$-A_{31}D_{15}+A_{32}D_{25}$	$-A_{31}D_{15}+A_{32}D_{25}$	$-A_{31}Q_{12}+A_{32}Q_{22}$	$-(A_{31}Q_{11}+A_{32}Q_{21})$
5	$\gamma_{11}A_{16}$	$\gamma_{11}D_{14}$	$\gamma_{11}D_{15}$	$\gamma_{11}D_{15}$	$\gamma_{11}Q_{12}$	$-\gamma_{11}Q_{11}$
6	$\gamma_{21}A_{26}-a_{66}\gamma_{22}$	$\gamma_{21}D_{24}$	$\gamma_{21}D_{25}$	$\gamma_{21}D_{25}$	$\gamma_{21}Q_{22}$	$-\gamma_{21}Q_{21}$

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and the cost function under the assumption of $R^P = 0$ is:

$$J^P = \frac{1}{2} \sum_{t=1}^{N} (x^P_{t-1} - \bar{x}^P_{t-1})' \, Q^P (x^P_{t-1} - \bar{x}^P_{t-1})$$

where

$$\bar{x}^P_{t-1} \overset{d}{=} \bar{y}_{t-1} - \bar{B}u_{t-1} - \bar{D}z_{t-1} \overset{d}{=} \left[\tilde{YY}_{1t-1}, \tilde{YY}_{2t-1}, \bar{BB}_{1t-1}, \bar{BB}_{2t-1}, \bar{KK}_{1t-1}, \bar{KK}_{2t-1}\right]'$$

is the (6x1) nominal state variable vector to be tracked; and

$$Q^P = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is the weight matrix

attached to the respective quadratic deviations from the nominal state variables $\tilde{YY}_{1t-1}$, $\tilde{YY}_{2t-1}$, $\bar{BB}_{1t-1}$, $\bar{BB}_{2t-1}$, $\bar{KK}_{1t-1}$, and $\bar{KK}_{2t-1}$.

Unlike that of Pindyck, Chow's framework is very simple to apply to the one-country reduced form (equation 4.1). There is no transformation of variables to be done and the dynamic system is of the following:

$$x^c_t \overset{d}{=} A^c x^c_{t-1} + B^c u^c_t + C^c z^c_t \tag{4.3}$$

where

$$x^c_t \overset{d}{=} \left[y_t \vdots u_t\right]' \overset{d}{=} \left[\tilde{Y}_{1t}, \tilde{Y}_{2t}, B_{1t}, B_{2t}, K_{1t}, K_{2t}, G_{1t}, G_{2t}, r_{1t}\right]'$$

is the (9x1) vector of current endogenous and controlled variables;

$u^c_t \overset{d}{=} u_t \overset{d}{=} \left[G_{1t}, G_{2t}, r_{1t}\right]'$ is the (3x1) vector of controlled variables;

$z_t^c \stackrel{d}{=} z_t \stackrel{d}{=} \left[1, IE_{1t}, X_{1t}^{III}, M_{1t}^{III}, TT_{1t}, T_{1t}, r_{2t}, r_{2t-1}, TT_{2t}, T_{2t}, TR_{2t},\right.$

$\left. X_{2t}^{III}, M_{2t}^{III}\right]'$ is the (13x1) vector of exogenous noncontrolled variables;

and A^c, B^c and C^c are known matrices:

$$\left[\begin{array}{c} \vdots \end{array}\right] \left[\begin{array}{ccccccccc} \Lambda_{11} & \Lambda_{12} & 0 & 0 & \Lambda_{13} & \Lambda_{14} & 0 & 0 & \Lambda_{15} \\ \Lambda_{21} & \Lambda_{22} & 0 & 0 & \Lambda_{23} & \Lambda_{24} & 0 & 0 & \Lambda_{2[illegible]} \end{array}\right] \qquad \left[\quad\right] \quad \left[\begin{array}{ccc} Q_{11} & Q_{12} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$A^c = \left[\begin{array}{c:c} \bar{A} & \bar{C} \\ \hdashline 0 & 0 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{13} & A_{14} & 0 & 0 & A_{15} \\ A_{21} & A_{22} & 0 & 0 & A_{23} & A_{24} & 0 & 0 & A_{25} \\ A_{31} & -A_{32} & 0 & 0 & A_{33} & -A_{34} & 0 & 0 & -A_{35} \\ -A_{31} & A_{32} & 0 & 0 & -A_{33} & A_{34} & 0 & 0 & A_{35} \\ \gamma_{11} & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & -\gamma_{12} \\ 0 & \gamma_{21} & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ; \quad B^c = \left[\begin{array}{c} \bar{B} \\ \hdashline I \end{array}\right] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & -Q_{32} & \eta_{11} \\ -Q_{31} & Q_{32} & -\eta_{11} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C^c = \left[\begin{array}{c} \bar{D} \\ \hdashline 0 \end{array}\right] = \begin{bmatrix} D_{11} & Q_{11} & Q_{11} & -Q_{12} & D_{12} & D_{13} & 0 & A_{16} & D_{14} & D_{15} & D_{15} & Q_{12} & -Q_{11} \\ D_{21} & Q_{21} & Q_{21} & -Q_{22} & D_{22} & D_{23} & 0 & A_{26} & D_{24} & D_{25} & D_{25} & Q_{22} & -Q_{21} \\ D_{31} & 1+Q_{31} & 1+Q_{31} & Q_{32} & -D_{32} & D_{33} & -\eta_{11} & A_{36} & D_{34} & -D_{35} & -D_{35} & -Q_{32} & -(1+Q_{31}) \\ -D_{31} & -(1+Q_{31}) & -Q_{31} & -(1+Q_{32}) & D_{32} & -D_{33} & \eta_{11} & -A_{36} & -D_{34} & D_{35} & D_{35} & 1+Q_{32} & Q_{31} \\ a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & -\,{}_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Under the assumption of no limits on instrument-variable magnitudes, the weight matrix Q^c in the welfare cost function (equation A-2.18, Appendix A-2) is:

$$Q^c = \left[\begin{array}{c|c} Q^P & 0 \\ \hline 0 & 0 \end{array}\right] = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

instead of:

$$Q^c = \left[\begin{array}{c|c} Q^P & 0 \\ \hline 0 & R^P \end{array}\right] = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{33} \end{bmatrix}$$

where q_{11}, q_{22}, q_{33} and q_{44} are weights attached to the quadratic deviations from the internal and external balance in both countries, i.e., weights attached to $(\tilde{Y}_{1t}-\bar{\tilde{Y}}_{1t})^2$, $(\tilde{Y}_{2t}-\bar{\tilde{Y}}_{2t})^2$, $(B_{1t}-\bar{B}_{1t})^2$, and $(B_{2t}-\bar{B}_{2t})$, resp

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respectively, while r_{11}, r_{22} and r_{33} are weights attached to the quadratic deviations from the limits set on both fiscal policies and Country I's monetary policy, i.e., weights attached to $(G_{1t}-\bar{G}_{1t})^2$, $(G_{2t}-\bar{G}_{2t})^2$ and $(r_{1t}-\bar{r}_{1t})^2$.

4.2.2. Optimal Solution for Internal and External Balance

Using Chow's result (equation A-2.20) for the optimal control (Appendix A-2), the optimal solution for the "modified" two-country problem is computed (Appendix A-11):

$$\begin{bmatrix} G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} -\theta_{11} & -\theta_{12} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15} \\ -\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25} \\ -\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t-1} \\ \tilde{Y}_{2t-1} \\ B^*_{1t-1} \\ B^*_{2t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{-Q_{22}}{Q} & \frac{+Q_{12}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{+Q_{21}}{Q} & \frac{-Q_{11}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \frac{-(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{-q_{44}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{Y}_{1t} \\ \bar{Y}_{2t} \\ \bar{B}_{1t} \\ \bar{B}_{2t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{F}_{1t} \end{bmatrix} \tag{4.4}$$

$$+ \begin{bmatrix} -\Delta_{10} & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\ -\Delta_{20} & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\ -\Delta_{30} & -\Delta_{31} & -\Delta_{31} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & 1 & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39} \end{bmatrix} \begin{bmatrix} 1 \\ IE_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}$$

Substituting $\bar{\bar{Y}}_{it} = \delta_{10} + \delta_{12}\, L_{it}$ $(i = 1,2)$ B and $\bar{B}_{it} = 0$ $(i = 1,2)$ into equation 4.4 results in:

$$
\begin{bmatrix} G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
=
\begin{bmatrix}
-\theta_{11} & -\theta_{12} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15} \\
-\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25} \\
-\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35}
\end{bmatrix}
\begin{bmatrix} \tilde{Y}^*_{1t-1} \\ \tilde{Y}^*_{2t-1} \\ B^*_{1t-1} \\ B^*_{2t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
+
\begin{bmatrix}
-\Delta_{10} + \frac{1}{Q}\,(Q_{22}\delta_{10} - Q_{12}\delta_{20}) \\
-\Delta_{20} + \frac{1}{Q}\,(-Q_{21}\delta_{10} + Q_{11}\delta_{20}) \\
-\Delta_{30} + \frac{1}{\eta_{11}Q}\left[(Q_{21}Q_{32} + Q_{22}Q_{31})\delta_{10} - \delta_{20}(Q_{11}Q_{32} + Q_{12}Q_{31})\right]
\end{bmatrix}
\qquad (4.5)
$$

$$
\begin{bmatrix}
\frac{+Q_{22}\delta_{12}}{Q} & \frac{Q_{12}\delta_{22}}{Q} & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\
\frac{-Q_{21}\delta_{12}}{Q} & \frac{+Q_{11}\delta_{22}}{Q} & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\
\frac{\delta_{12}}{\eta_{11}Q}\,(Q_{21}Q_{32} + Q_{22}Q_{31}) & \frac{-\delta_{22}}{\eta_{11}Q}\,(Q_{11}Q_{32} + Q_{12}Q_{31}) & -\Delta_{31} & -\Delta_{31} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & 1 & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39}
\end{bmatrix}
\begin{bmatrix} 1 \\ L_{1t} \\ L_{2t} \\ IE_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

The optimal path is obtained by substituting equation 4.4 into the dynamic system (equation 4.3):

$$
\begin{bmatrix} \bar{Y}^*_{1t} \\ \bar{Y}^*_{2t} \\ B^*_{1t} \\ B^*_{2t} \\ K^*_{1t} \\ K^*_{2t} \\ G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_{11} & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & -\gamma_{12} \\
0 & \gamma_{21} & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 \\
-\theta_{11} & -\theta_{22} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15} \\
-\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25} \\
-\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35}
\end{bmatrix}
\begin{bmatrix} \bar{Y}^*_{1t-1} \\ \bar{Y}^*_{2t-1} \\ B^*_{1t-1} \\ B^*_{2t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
$$

$$
+
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-q_{33}}{q_{33}+q_{44}} & \frac{q_{44}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-Q_{22}}{Q} & \frac{Q_{12}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-Q_{21}}{Q} & \frac{-Q_{11}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \frac{-(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{-q_{44}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \bar{\bar{Y}}_{1t} \\ \bar{\bar{Y}}_{2t} \\ \bar{B}_{1t} \\ \bar{B}_{2t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix}
\qquad (4.6)
$$

$$
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q_{44}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{44}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} \\
0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{33}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{33}}{q_{33}+q_{44}} \\
a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \\
-\Delta_{10} & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\
-\Delta_{20} & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\
-\Delta_{30} & -\Delta_{31} & -\Delta_{31} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & 0 & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39}
\end{bmatrix}
\begin{bmatrix} 1 \\ IE_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

Substituting $\bar{\bar{Y}}_{1t} = \delta_{10} + \delta_{12} L_{1t}$ and $\bar{B}_{1t} = 0$ into equation 4.6:

$$
\begin{bmatrix} Y^*_{1t} \\ Y^*_{2t} \\ B^*_{1t} \\ B^*_{2t} \\ K^*_{1t} \\ K^*_{2t} \\ G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_{11} & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & -\gamma_{12} \\
0 & \gamma_{21} & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 \\
-\theta_{11} & -\theta_{12} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15} \\
-\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25} \\
-\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35}
\end{bmatrix}
\begin{bmatrix} \tilde{Y}^*_{1t-1} \\ \tilde{Y}^*_{2t-1} \\ B^*_{1t-1} \\ B^*_{2t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
\tag{4.7}
$$

$$
+ \left[\begin{array}{ccccccccccccccc}
\delta_{10} & \delta_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta_{20} & 0 & \delta_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{q_{44}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{44}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} \\
0 & 0 & 0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{33}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{33}}{q_{33}+q_{44}} \\
a_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{61} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \\
-\Delta_{10}+\frac{1}{Q}(-Q_{22}\delta_{10}+Q_{12}\delta_{20}) & \frac{-Q_{22}\delta_{12}}{Q} & \frac{-Q_{12}\delta_{22}}{Q} & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\
-\Delta_{20}+\frac{1}{Q}(Q_{21}\delta_{10}-Q_{11}\delta_{20}) & \frac{Q_{21}\delta_{12}}{Q} & \frac{-Q_{11}\delta_{22}}{Q} & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\
-\Delta_{30}+\frac{1}{\eta_{11}Q}\left[\delta_{10}(Q_{21}Q_{32}+Q_{22}Q_{31})-\delta_{20}(Q_{11}Q_{32}-Q_{12}Q_{31}\right] & \frac{\delta_{12}}{\eta_{11}Q}(Q_{21}Q_{32}+Q_{22}Q_{31}) & \frac{\delta_{22}}{\eta_{11}Q}(Q_{11}Q_{32}+Q_{12}Q_{31}) & -\Delta_{31} & -\Delta_{41} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & 1 & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39}
\end{array}\right]
\begin{bmatrix} 1 \\ L_{1t} \\ L_{2t} \\ IK_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

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Next, the optimal welfare cost is computed:

$$\hat{J}^c = \frac{1}{2}\sum_{t=1}^{N}(x_t^{c*}-\bar{x}_t^{c*})'Q^c(x_t^{c*}-\bar{x}_t^{c})$$

And it is found that (Appendix A-12):

$$\hat{J}^c(u*) = \frac{1}{2}\frac{(q_{33}q_{44})}{q_{33}+q_{44}}\sum_{t=1}^{N}(M_{3t}-X_{3t})^2 \qquad (4.8)$$

The results for the two-country optimal control problem under the assumption of linear dependent balance of payments differ from those obtained in Case A of Chapter III. From equation 4.4 it is noted that only the variables $\tilde{Y}*_{it}$ $(i = 1,2)$ are on the targets exactly, while the variables B^*_{it} $(i = 1,2)$ deviate from their equilibrium. How much Country I's balance of payments B^*_{1t} deviates from its equilibrium depends on the welfare weight q_{33} assigned to it, and Country III's balance of trade. Similarly, the deviation of Country II's balance of payments depends on the weight q_{44} assigned to it and on Country III's balance of trade. Unlike Case A, the optimal welfare cost is no longer equal to zero, but it depends on the penalty costs that each country assigns to the deviation of its balance of payments from the equilibrium as well as Country III's trade balance.

In summary, the concluding remarks for Case B are:

(1) Similar to Case A, the economic interdependence between countries is reflected in the policy interdependence; that is, no policy can be considered in isolation. Again within the confines of Votey's model, there is one peculiarity to be noted. Both fiscal policies G^*_{1t} and G^*_{2t} affect

the full-employment situation in both countries, while Country I's monetary policy is directed at all four targets (internal and external balance in both countries) and the relative effectiveness of these three instruments is measured by the matrix:

$$\begin{array}{ccccc}
\text{Row} & \bar{\tilde{Y}}_{1t} & \bar{\tilde{Y}}_{2t} & \bar{B}_{1t} & \bar{B}_{2t} \\
1 & \frac{-Q_{22}}{Q} & \frac{Q_{12}}{Q} & 0 & 0 \\
2 & \frac{Q_{21}}{Q} & \frac{-Q_{11}}{Q} & 0 & 0 \\
3 & \frac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \frac{-(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{-q_{44}}{\eta_{11}(q_{33}+q_{44})}
\end{array}$$

which will be evaluated in the next section by substituting Votey's numerical values for the estimated coefficients into the two-country model.

(2) Along with Chow (1972c), it is concluded here that if the number of variables to be controlled is larger than the number of instruments, the variables will not reach the targets exactly, and their deviations from the targets will depend on the welfare weights in Q^c assigned to them. However, due to this study's formulation of the optimal control problem and the particular structural form of Votey's model which has been constructed to investigate the effects of Mundellian policy assignment on the

stability of the system, the conclusion holds only for the external situation where there is only one instrument, i.e., Country I's interest rate to deal with two external targets or balance-of-payments equilibrium in both countries. As for the internal situation of both countries where the number of variables to be controlled (the GNP of both countries $\bar{\tilde{Y}}_{it}$ (i = 1,2) is equal to the number of instruments (the fiscal policies of both countries G_{it} [i = 1,2]), Tinbergen's rule is met and the former variables are exactly on target.

(3) Unlike Case A in Chapter III, the optimal policies are no longer unique in attaining the joint balance in both countries. In Case B there is a family of optimal policies, depending on the welfare weights assigned to the deviations of both balance of payments from the equilibrium, because there are not enough instruments to attain the targets fixed by policy-makers.

(4) Similar to Case A, nothing guarantees the feasibility of the optimal policy mix in a given economy. Therefore, in addition to a trade-off between the two external targets due to an insufficient orchestration of instruments to secure the simultaneous attainment of external balance in both countries, there may exist another trade-off

between attainment of targets and consistency of policies.

4.3. U.S. and Canada Optimal Policies for Internal and External Balance: Appraisal and Amendment of the Optimal Solution

Similar to Case A in Chapter III, Votey's numerical values for the structural coefficients and the historical data for the United States, Canada and the European Economic Community (EEC), which represents the third country in our "modified" two-country model, will be used to derive the U.S. and Canadian optimal policy mix for the 1960's. Since there is a trade-off between the U.S. and Canadian balance-of-payments targets, a sensitivity analysis of the weighting factors on the components of the target vector will be performed. Then the result of each run of the test will be appraised to determine if the policies obtained by the techniques of optimal control are feasible. In other words, if the optimal policy is in accordance with the boundary condition that the instrument variable cannot surpass certain numerical values for practical or political reasons, the boundary condition does not interfere. However, if the values found for the unknown instruments violate the boundary condition, the optimal solution has to be perfected and the boundary condition becomes active by redefining the weight matrix Q^c to include the penalty cost assigned to deviations of instruments from their limits.

First, using Votey's numerical values for the estimated structural constants (Votey:1969), the matrices of coefficients for equation 4.4 and the dynamic system of equation 4.3 are computed:

$$
\begin{bmatrix} G^e_{1t} \\ G^e_{2t} \\ r^e_{1t} \end{bmatrix}
=
\begin{bmatrix}
0.0025 & -0.0051 & 0 & 0 & -0.0013 & -0.0339 & 0 & 0 & 434.2278 \\
0.0001 & -0.1477 & 0 & 0 & -0.0218 & 0.0225 & 0 & 0 & 9.4818 \\
0 & -0.0001 & 0 & 0 & 0.0005 & -0.0007 & 0 & 0 & -0.2051
\end{bmatrix}
\begin{bmatrix} \tilde{Y}^e_{1t-1} \\ \tilde{Y}^e_{2t-1} \\ B^e_{1t-1} \\ B^e_{2t-1} \\ K^e_{1t-1} \\ K^e_{2t-1} \\ G^e_{1t-1} \\ G^e_{2t-1} \\ r^e_{1t-1} \end{bmatrix}
$$

$$
+
\begin{bmatrix}
0.3810 & -0.0977 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.1315 & 0.3632 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0028 & -0.0021 & 0.0216\left(\frac{q_{33}}{q_{33}+q_{44}}\right) & -0.0216\left(\frac{q_{44}}{q_{33}+q_{44}}\right) & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \bar{\bar{Y}}_{1t} \\ \bar{\bar{Y}}_{2t} \\ \bar{B}_{1t} \\ \bar{B}_{2t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ Y_{1t} \end{bmatrix}
\qquad (4.9)
$$

$$
+
\begin{bmatrix}
-33.5374 & -1.0000 & -1.0000 & 0.0000 & -28.3566 & 0.6185 & 0 & 10.7500 & 0 & 0.0977 & 0.0977 & 0.0000 & 1.0000 \\
-41.9370 & 0.0000 & 0.0000 & 1.0000 & 28.3564 & 0.1315 & 0 & 309.8857 & 0 & 0.6368 & 0.6368 & -1.0000 & 0.0000 \\
0.5859 & -0.0216 & -0.0216\left(\frac{q_{33}}{q_{33}+q_{44}}\right) & -0.0216\left(\frac{q_{44}}{q_{33}+q_{44}}\right) & -0.6127 & -0.0028 & 1 & 0.2324 & 0 & 0.0021 & 0.0021 & 0.0216\left(\frac{q_{44}}{q_{33}+q_{44}}\right) & 0.0216\left(\frac{q_{33}}{q_{33}+q_{44}}\right)
\end{bmatrix}
$$

$$
\begin{bmatrix} 1 \\ IR_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

$$
\begin{bmatrix} \tilde{Y}^*_{1t} \\ \tilde{Y}^*_{2t} \\ B^*_{1t} \\ B^*_{2t} \\ K^*_{1t} \\ K^*_{2t} \\ G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
=
\begin{bmatrix}
-0.0073 & 0.1298 & 0 & 0 & 0.0131 & 0.0806 & 0 & 0 & -1263.7107 \\
-0.0028 & 0.4538 & 0 & 0 & 0.0648 & -0.0329 & 0 & 0 & -483.6539 \\
0.0075 & 0.0324 & 0 & 0 & -0.0172 & 0.0201 & 0 & 0 & 128.5835 \\
-0.0075 & -0.0324 & 0 & 0 & 0.0172 & -0.0201 & 0 & 0 & -128.5835 \\
-0.0027 & 0 & 0 & 0 & 1.0618 & 0 & 0 & 0 & -461.7 \\
0 & 0.1668 & 0 & 0 & 0 & 1.1033 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \tilde{Y}^*_{1t-1} \\ \tilde{Y}^*_{2t-1} \\ B^*_{1t-1} \\ B^*_{2t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
+
\begin{bmatrix}
2.8932 & 0.7782 & 0 \\
1.04755 & 3.0351 & 0 \\
-0.2785 & 0.1942 & 46.279 \\
0.2785 & -0.1942 & -46.279 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
\qquad (4.10)
$$

$$
+
\begin{bmatrix}
129.6688 & 2.8932 & 2.8932 & -0.7783 & 59.9751 & -1.8918 & 0 & -272.2641 & 0 & -0.7782 & -0.7782 & 0.7783 & -2.8932 \\
162.4172 & 1.0475 & 1.0475 & -3.0351 & -56.3610 & -1.0470 & 0 & -951.8105 & 0 & -2.0351 & -2.0351 & 3.0351 & -1.0475 \\
-28.3120 & 0.7215 & 0.7215 & -0.1942 & 14.9535 & 0.2782 & -46.279 & -67.9307 & 0 & -0.1942 & -0.1942 & 0.1942 & -0.7215 \\
28.3120 & -0.7215 & 0.2785 & -0.8055 & -14.9535 & -0.2782 & 46.279 & 67.9307 & 0 & 0.1942 & 0.1942 & 0.8058 & -0.2785 \\
9.1299 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18.6889 & 0 & 0 & 0 & 0 & 0 & 0 & -349.8519 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} 1 \\ IE_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

Given these two equations, the optimal policy mixes (G^*_{1t}, G^*_{2t}, r^*_{1t}) for the United States and Canada over 10 periods running from 1961 to 1970 can be derived with the following initial conditions:

$\tilde{Y}_1(0) = 383.37$

$\tilde{Y}_2(0) = 20.06$

$B_1(0) = 4.17$

$B_2(0) = 1.24$

$K_1(0) = 508.69$

$K_2(0) = 49.33$

$G_1(0) = 94.90$

$G_2(0) = 6.97$

$r_1(0) = 0.0215$

where (0) refers to 1960. All these values are historical.

The nominal values for state variables are given in Table 4.1. The nominal value for state variables is similar to that in Table 3.1, to which is added the value zero for Country II's balance-of-payments equilibrium. The historical values for exogenous variables are given in Table 4.2.

The U.S.-Canadian optimal policies for internal and external balance are obtained by the following steps, which require only some basic matrix manipulations:

(1) Compute $\left[G^*_1(1),\ G^*_2(1),\ r_1(1)\right]'$ from equation 4.9 using the initial conditions: $\left[\tilde{Y}_1(0),\ \tilde{Y}_2(0),\ B_1(0),\ B_2(0),\ K_1(0),\ K_2(0),\ G_1(0),\ G_2(0),\ r_1(0)\right]'$ and the exogenous variables of period 1 given by Table 4.2.

Table 4.1. Nominal values for the state variables - United States and Canada: 1961-1970

Period (Year)	$\bar{\tilde{Y}}_{1t}$	$\bar{\tilde{Y}}_{2t}$	$\bar{B}_{1t}$	$\bar{B}_{2t}$	$\bar{K}_{1t}$	$\bar{K}_{2t}$	$\bar{G}_{1t}$	$\bar{G}_{2t}$	$\bar{r}_{1t}$
1(1961)	283.2462	2.3165	0	0	523.9507	50.8099	98.7	7.2488	0.0215
2(1962)	283.8683	2.3499	0	0	539.6692	52.3342	102.65	7.5388	0.0215
3(1963)	288.7687	2.3972	0	0	555.8592	53.9042	106.76	7.8404	0.0215
4(1964)	293.8258	2.4629	0	0	572.5349	55.5213	111.03	8.1540	0.0215
5(1965)	299.3091	2.5368	0	0	589.7109	57.1869	115.47	8.4802	0.0215
6(1966)	304.5954	2.6359	0	0	607.4022	58.9025	120.09	8.8194	0.0215
7(1967)	310.9349	2.7332	0	0	625.6242	60.6696	124.89	9.1722	0.0215
8(1968)	316.5227	2.8132	0	0	644.3929	62.4897	129.89	9.5391	0.0215
9(1969)	324.5467	2.8995	0	0	663.7246	64.3644	135.09	9.9207	0.0215
10(1970)	332.5143	2.9748	0	0	683.6363	66.2953	140.49	10.3175	0.0215

Table 4.2. Historical values for the exogenous variables - U.S.

Table 4.2. Historical values for the exogenous variables - U.S., Canada and EEC: 1961-1970

Period (Year)	1	IE_{1t}	X^{III}_{1t}	M^{III}_{1t}	TT_{1t}	T_{1t}	r_{2t}	r_{2t-1}	TT_{2t}	T_{2t}	TR_{2t}	X^{III}_{2t}	M^{III}_{2t}
1(1961)	1	3.55	3.57	2.22	1.0000	144.63	0.0299	0.0325	1.0408	9.57	3.44	0.49	0.32
2(1962)	1	4.05	4.56	2.45	1.0101	157.03	0.0391	0.0299	1.0309	10.53	3.72	0.43	0.31
3(1963)	1	4.15	4.92	2.52	1.0000	168.76	0.0378	0.0391	1.0000	11.18	3.85	0.45	0.32
4(1964)	1	4.87	5.29	2.83	0.9903	174.07	0.0382	0.0378	1.0000	12.73	4.13	0.52	0.37
5(1965)	1	5.29	5.26	3.32	1.0000	190.06	0.0454	0.0382	1.0198	14.29	5.57	0.59	0.47
6(1966)	1	5.37	5.54	4.12	1.0094	213.33	0.0496	0.0454	1.0490	16.59	5.05	0.60	0.51
7(1967)	1	5.89	5.67	4.46	1.0280	228.93	0.0595	0.0496	1.0583	18.51	6.22	0.64	0.58
8(1968)	1	6.22	6.14	5.89	1.0183	263.31	0.0624	0.0595	1.0566	20.99	7.19	0.71	0.61
9(1969)	1	5.97	6.98	5.80	1.0268	296.70	0.0781	0.0624	1.0648	24.44	6.06	0.79	0.73
10(1970)	1	6.27	8.42	6.61	1.0083	302.00	0.0444	0.0781	1.0614	26.27	6.80	1.15	0.77

(2) Compute $\left[\tilde{Y}_1^*(1), \tilde{Y}_2^*(1), K_1^*(1), K_2^*(1), G_1^*(1), G_2^*(1), r_1^*(1)\right]'$ from equation 4.10. Now $\left[\tilde{Y}_1^*(1), \tilde{Y}_2^*(1), B_1^*(1), B_2^*(1), K_1^*(1), K_2^*(1), G_1^*(1), G_2^*(1), r_1^*(1)\right]'$ can be used in equation 4.9 to compute $\left[G_1^*(2), G_2^*(2), r_1^*(2)\right]'$ which can be used to compute $\left[\tilde{Y}_1^*(2), \tilde{Y}_2^*(2), B_1^*(2), B_2^*(2), K_1^*(2), K_2^*(2), G_1^*(2), G_2^*(2), r_1^*(2)\right]'$ and so on. Continue the process until all the control vectors $\left[G_1^*(t), G_2^*(t), r_1^*(t)\right]'$, $t = 1,\ldots, 10$ and all the state variable vectors $\left[Y_1^*(t), Y_2^*(t), B_1^*(t), B_2^*(t), K_1^*(t), K_2^*(t), G_1^*(t), G_2^*(t), r_1^*(t)\right]'$, $t = 1,\ldots, 9$, have been computed.

Unlike the first two cases, the optimal control solution for achieving the overall balance is no longer unique, but is a function of the weights q_{33} and q_{44} attached to the deviations of the U.S. and Canadian balances-of-payments from their equilibriums. To study the effects of the trade-off between the two external targets on the optimal policy mix, three experiments will be performed.

Table 4.3. Two-country model (Case B): penalty weights attached to the deviations of U.S. and Canadian balance of payments from the equilibrium

Trade-Off Experiment	q_{33}	q_{44}
A	10^{-4}	10^{-5}
B	10^{5}	10^{5}
C	10^{5}	10^{-4}

First, observations will be made on the relative effectiveness of the impact of three instruments on the internal and external targets. Based on the following matrix (equation 4.9)

Row	$\bar{\tilde{Y}}_{1t}$	$\bar{\tilde{Y}}_{2t}$	$\bar{B}_{1t}$	$\bar{B}_{2t}$
1	0.3810	-0.0977	0	0
2	-0.1315	0.3632	0	0
3	0.0028	-0.0021	$0.0216(\frac{q_{33}}{q_{33}+q_{44}})$	$-0.0216(\frac{q_{44}}{q_{33}+q_{44}})$

it is noted that the policy interdependency prevails, but in a strict sense within the confines of Votey's model. That is, both fiscal policies affect the internal balance in both countries, but with a positive impact on its own target and a negative impact on the other country's target, while the U.S. monetary policy affects the overall balance positively with respect to the U.S. internal and external targets and negatively with respect to Canada's joint balance. Furthermore, the impact of the U.S. monetary policy is greater on the U.S. external target than on the U.S. internal target. Therefore, due to the particular structural form of Votey's model, Mundellian policy assignment to internal and external balance is inherent in the two-country model. It is also noted that the effectiveness of the impact of U.S. monetary policy on the U.S. and Canadian balance of payments depends on the welfare weights q_{33} and q_{44}: the greater the weights, the greater the impact.

Next is an appraisal of optimal policies for internal and external balance in the United States and Canada. The results are presented in graphical form (Figures 4.1 to 4.5) with time on the horizontal axis.

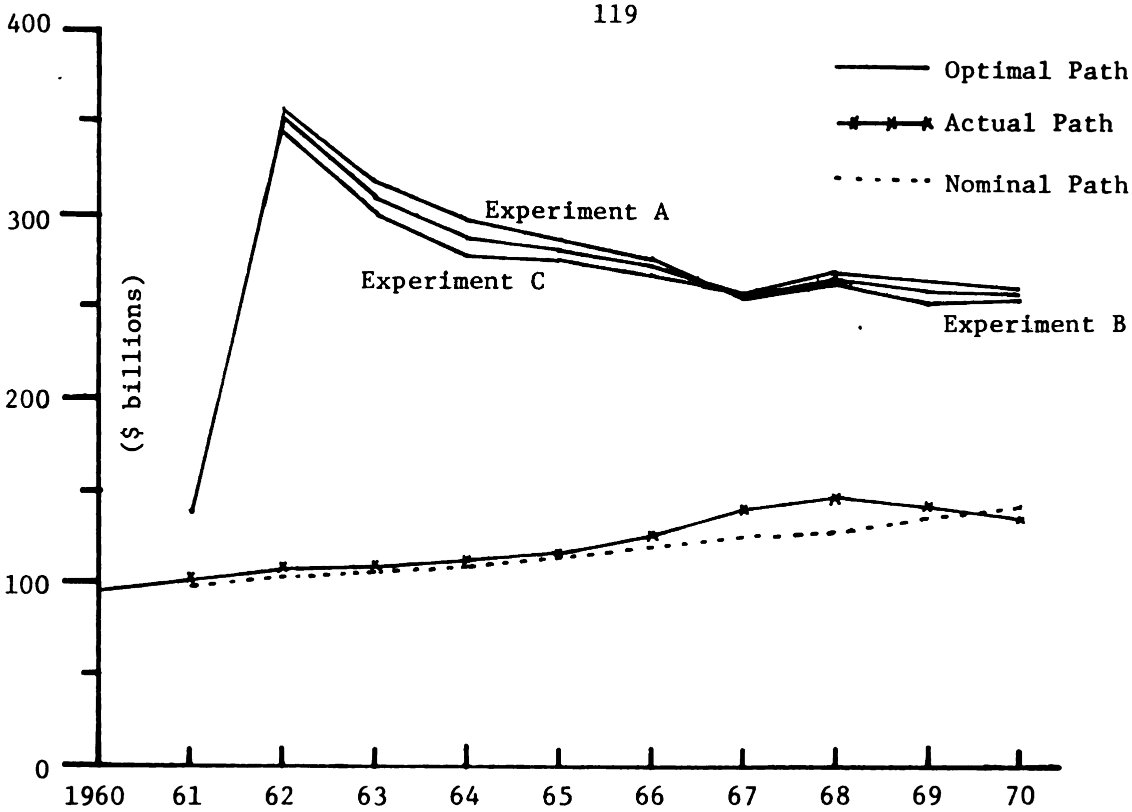

Figure 4.1. Two-country model (Case B) - Effects of trade-off between the two external targets on the U.S. government expenditures: optimal paths compared with actual and nominal paths.

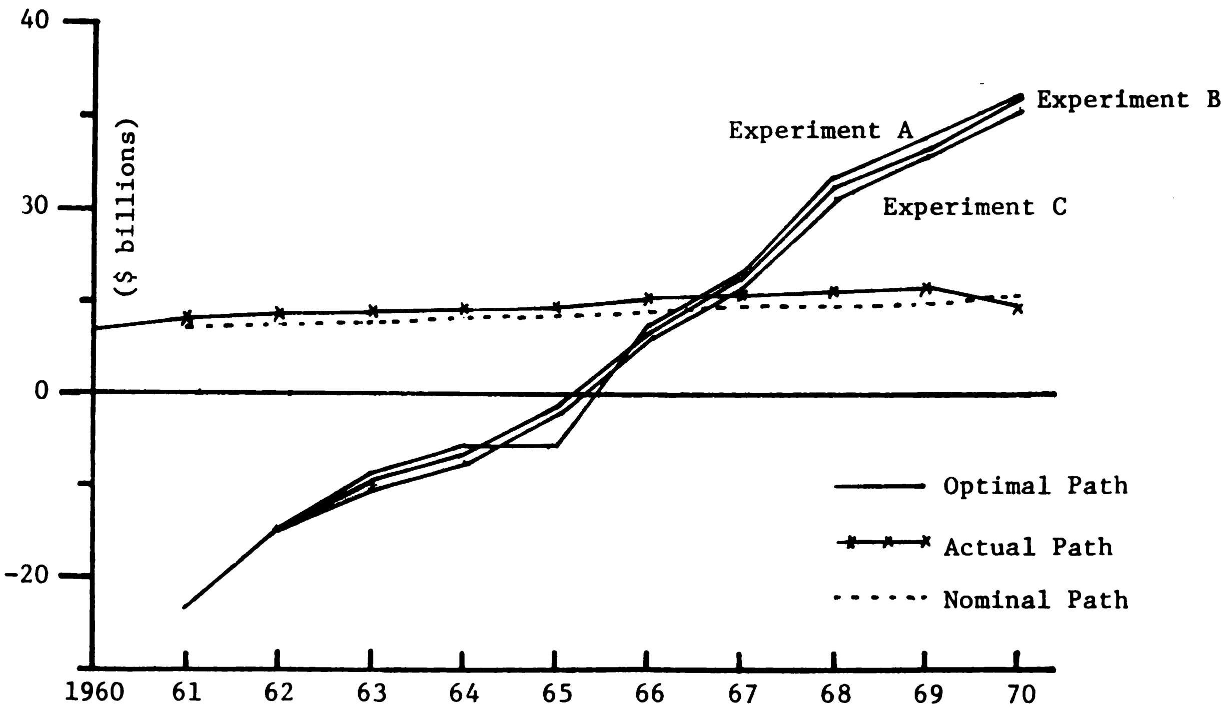

Figure 4.2. Two-country model (Case B) - Effects of trade-off between the two external targets on the Canadian government expenditures: optimal paths compared with actual and nominal paths.

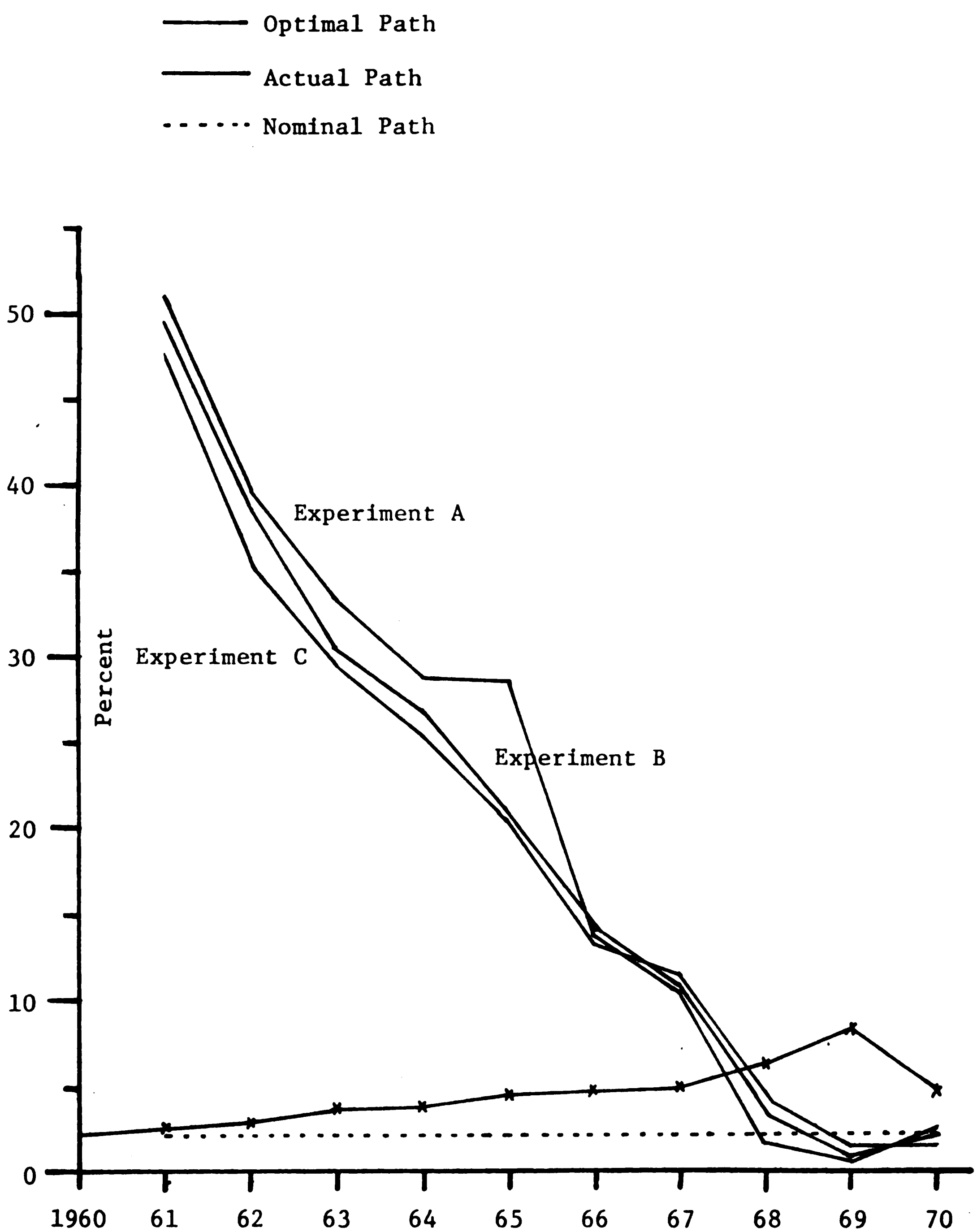

Figure 4.3. Two-country model (Case B) - Effects of trade-off between the two external targets on the U.S. interest rate: optimal paths compared with actual and nominal paths.

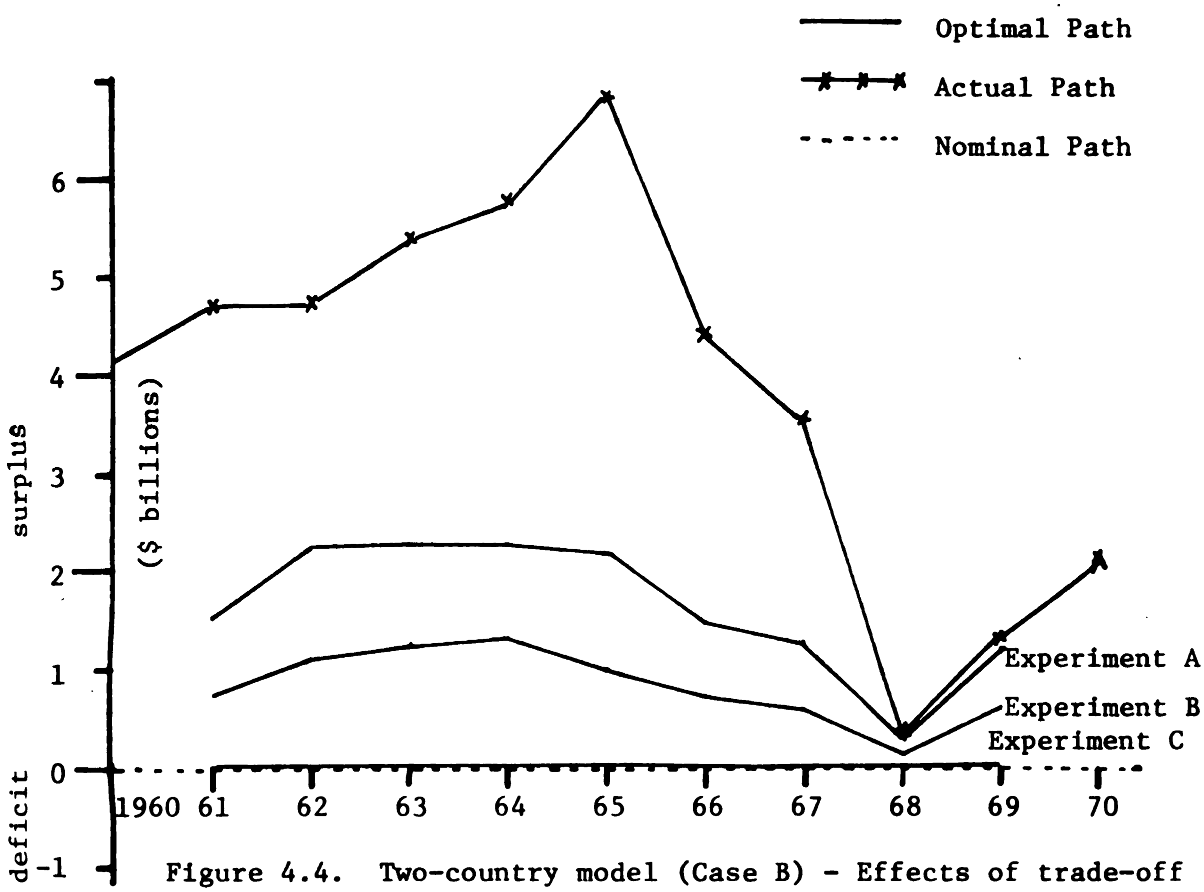

Figure 4.4. Two-country model (Case B) - Effects of trade-off between the two external targets on the U.S. balance of payments: optimal paths compared with actual and nominal paths.

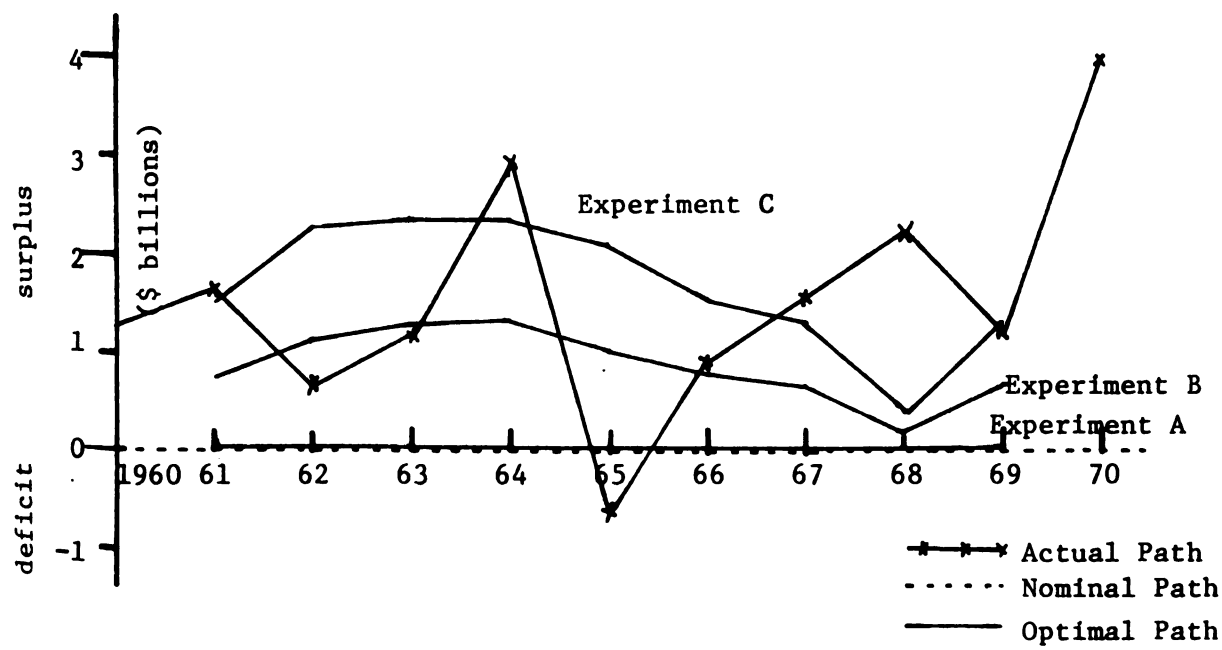

Figure 4.5. Two-country model (Case B) - Effects of trade-off between the two external targets on the Canadian balance of payments: optimal paths compared with actual and nominal paths.

The following remarks on the performance of optimal policies can be made:

(1) All three optimal policies are sensitive to the trade-off between the two balance-of-payments targets. This trade-off is shown by Figures 4.4 and 4.5. The more the deviation of the balance of payments from its equilibrium is penalized, the closer the external balance. In fact, in experiment A where the U.S. balance of payments diverges strongly from equilibrium, the Canadian balance of payments is exactly on the target, and vice versa in experiment C. However, if equal penalty costs are assigned to both deviations (experiment B), both balance of payments diverge equally from the equilibrium. Therefore, the external balances in both countries can never be attained simultaneously. A trade-off exists between them.

(2) The actual paths of fiscal and monetary policies deviate from their optimal ones. This implies that the joint balance is neither attained in the United States nor in Canada during the 1960's. In fact, the United States was faced with inflation and balance-of-payments surplus as was Canada, with the exception of 1965 when Canada had a deficit.

(3) When using the optimal fiscal policies, the U.S. and Canadian economies are found to be on the

targets exactly no matter what the trade-off between the balance of payment targets. On the contrary, the U.S. optimal monetary policy by itself cannot bring the U.S. as well as the Canadian balance of payments to the equilibrium. This requires additional constraints: heavy penalty costs have to be assigned to deviations from the external balance.

(4) It is found that the optimal policies for internal and external balance are inconsistent, i.e., inadmissible for political, social, and technical reasons. In other words, they violate the boundary conditions which have been set up to limit the instrument-magnitudes. Therefore, these additional constraints become active by reformulating the cost functional J_c (equation A-2.18) in the two-country optimal control problem (Case B) with the assumption $R^P \neq 0$.

Next, the amendment of optimal solution generated by the introduction of boundary conditions will be examined. The newly defined weight matrix Q_c under the assumption $R^P \neq 0$ is:

Row	$\bar{\tilde{Y}}_{1t}$	$\bar{\tilde{Y}}_{2t}$	$\bar{B}_{1t}$	$\bar{B}_{2t}$	$\bar{K}_{1t}$	$\bar{K}_{2t}$	$\bar{G}_{1t}$	$\bar{G}_{2t}$	$\bar{r}_{1t}$
1	q_{11}	0	0	0	0	0	0	0	0
2	0	q_{22}	0	0	0	0	0	0	0
3	0	0	q_{33}	0	0	0	0	0	0
4	0	0	0	q_{44}	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	r_{11}	0	0
8	0	0	0	0	0	0	0	r_{22}	0
9	0	0	0	0	0	0	0	0	r_{33}

A sensitivity analysis of the weighting factor on components of the control vector is performed. Unlike Case A in Chapter III because of a trade-off between the two countries' external targets, the sensitivity analysis will be done for each trade-off experiment with the cost functional given in Table 4.4.

Three sensitivity tests will be performed for each trade-off experiment: in test 1, only the deviations of U.S. monetary policy from its constant limit set at 2.15 percent are penalized; in test 2, the deviations of two out of three instruments policies--the U.S. monetary policy coupled with the fiscal policy in either country--from their nominal values are penalized; and in test 3, all three instruments are tracked to their limits.

Table 4.4. Two-country model (Case B): penalty weights attached to the deviations of control variables from their nominal values

	Runs	r_{11}	r_{22}	r_{33}
Test 1	1	0	0	1
	2	0	0	10^5
	3	0	0	$5x10^5$
	4	0	0	10^6
Test 2	5	0	1	10^6
	6	0	10^5	10^6
		0	$5x10^5$	10^6
		0	10^6	10^6
	7	1	0	10^6
	8	10^5	0	10^6
		$5x10^5$	0	10^6
		10^6	0	10^6
Test 3	9	1	1	1
	10	10^5	10^5	10^5
	11	$5x10^5$	$5x10^5$	$5x10^5$
	12	10^6	10^6	10^6
	13	1	1	10^5
	14	1	1	10^6
	15	1	10^5	10^5
	16	10^6	1	10^5
	17	$5x10^5$	10^5	10^5
		10^5	$5x10^5$	10^5
	18	1	10^6	10^6
	19	10^6	1	10^6
	20	10^5	10^6	10^6
		10^6	10^5	10^6

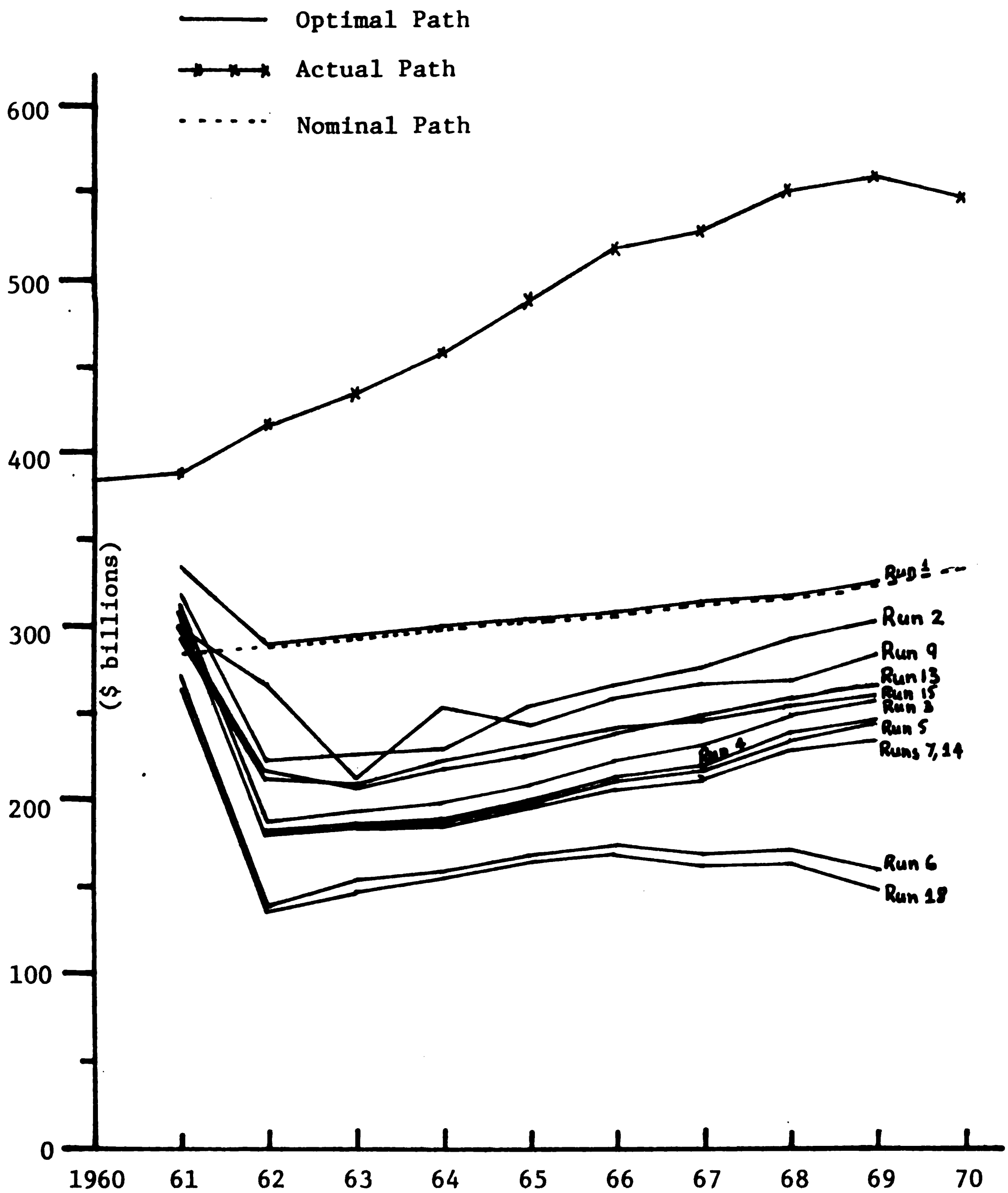

Figure 4.6(A). Two-country model (Case B) - United States: optimal GNP trajectories compared with the desired and actual GNP (trade-off experiment A where $q_{33} = 10^{-4}$, $q_{44} = 10^{5}$).

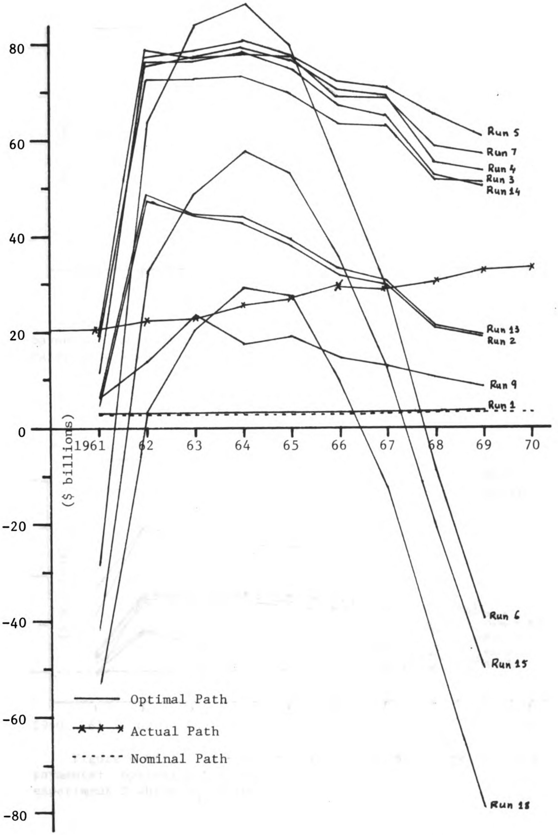

Figure 4.7(A). Two-country model (Case B) - Canada: optimal GNP trajectories compared with the desired and actual GNP (trade-off experiment A where $q_{33} = 10^{-4}$, $q_{44} = 10^{5}$).

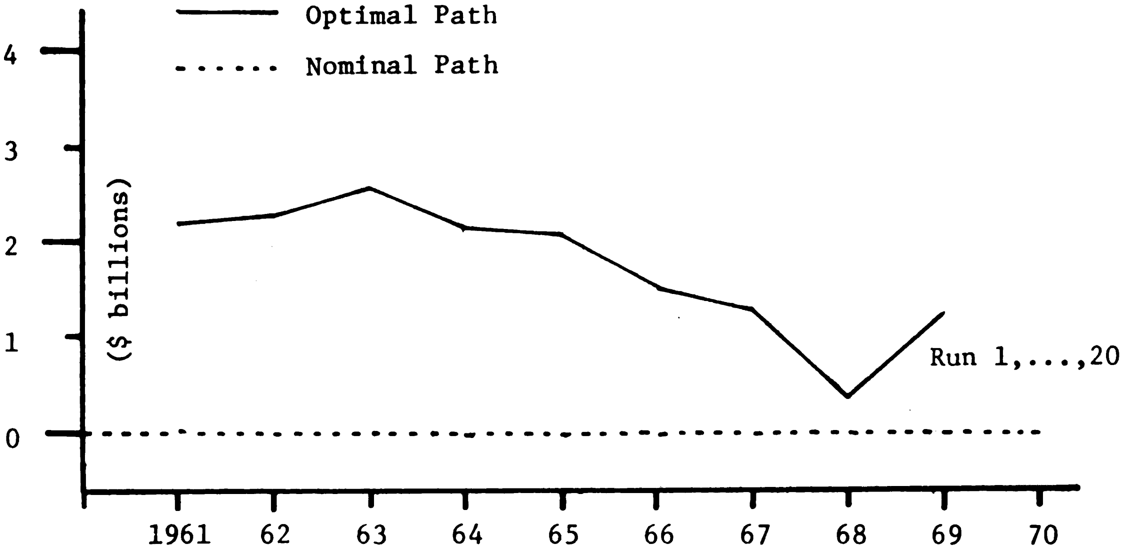

Figure 4.8(A). Two-country model (Case B) - U.S. balance of payments: optimal paths compared with the equilibrium (trade-off experiment A where $q_{33} = 10^{-4}$, $q_{44} = 10^{5}$).

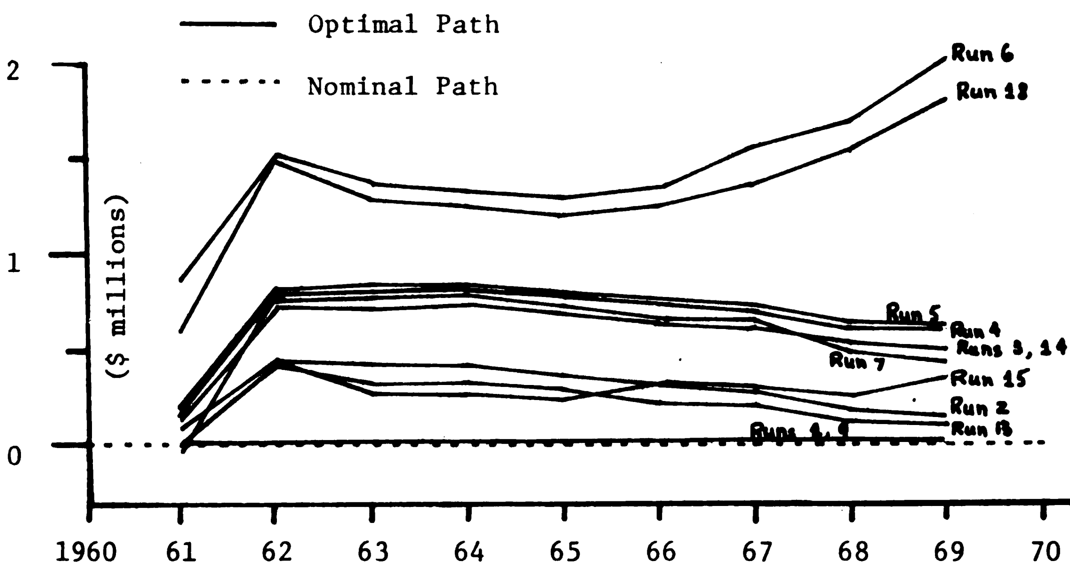

Figure 4.9(A). Two-country model (Case B) - Canadian balance of payments: optimal paths compared with the equilibrium (trade-off experiment A where $q_{33} = 10^{-4}$, $q_{44} = 10^{5}$).

Figure 4.10(A). Two-country model (Case B) - U.S. government expenditures: optimal paths compared with nominal path (trade-off experiment A where $q_{33} = 10^{-4}$, $q_{44} = 10^{5}$).

Figure 4.11(A). Two-country model (Case B) - Canadian government expenditures: optimal paths compared with nominal path (trade-off experiment A where $q_{33} = 10^{-4}$, $q_{44} = 10^{5}$).

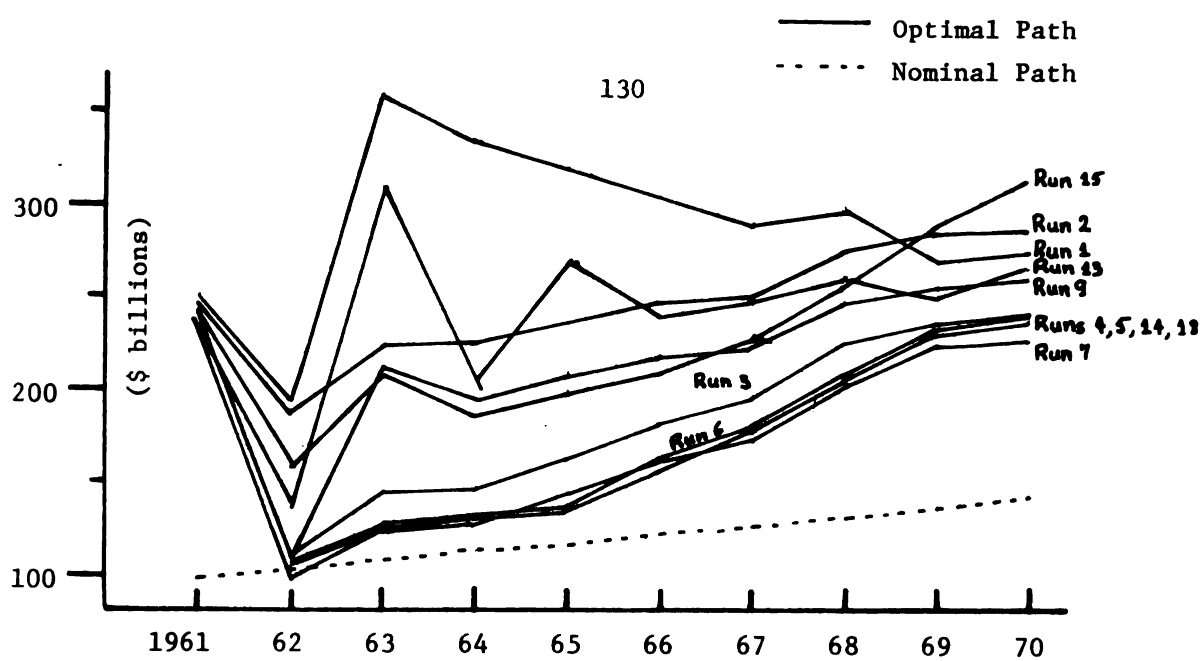

Figure 4.10(A)

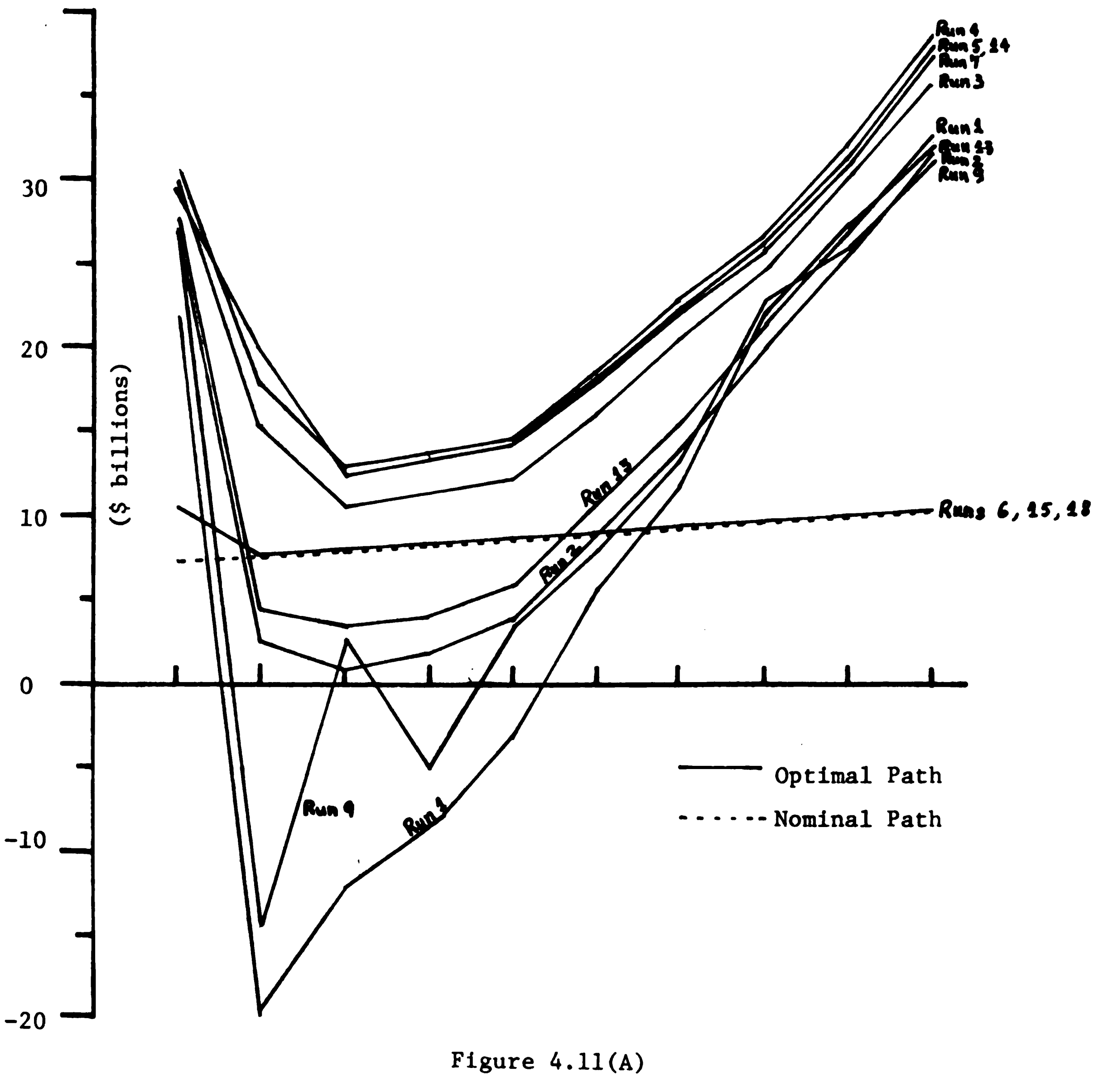

Figure 4.11(A)

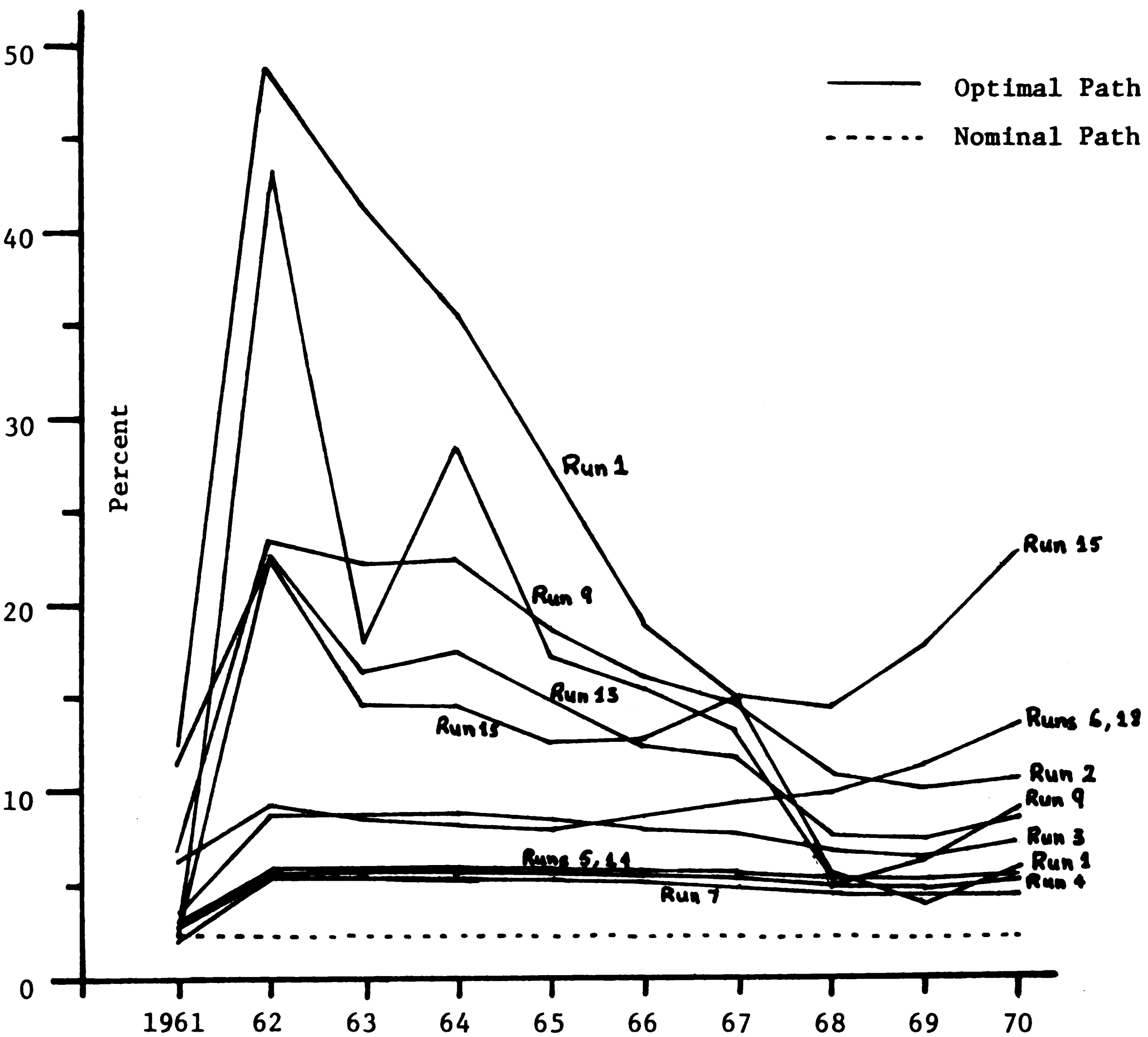

Figure 4.12(A). Two-country model (Case B) - U.S. interest rate: optimal paths compared with nominal path (trade-off experiment A where $q_{33} = 10^{-4}$, $q_{44} = 10^5$).

(i) Trade-off experiment A [Figures 4.6(A) to 4.12(A)]

In test 1 (runs 1 to 4) it is observed that the tracking for the U.S. nominal interest rate gets closer the greater the value given to the welfare weights r_{33}. Because of the policy interdependency (equation 4.8) the U.S. and Canadian fiscal instruments are also tracking their nominal paths. However, the tracking for U.S. nominal government expenditures is positively correlated with that of the U.S. nominal interest rate, while there is a negative correlation between the latter tracking and that of the Canadian nominal government spending. Furthermore, it is found that the optimal path of U.S. balance of payments on which the U.S. monetary policy has the greatest impact remains insensitive to the tracking for the U.S. nominal interest rate; and only for the three other endogenous variables, $\tilde{Y}_{1t}$, $\tilde{Y}_{2t}$, and B_{2t} is there a trade-off between attainment of targets and feasibility of the U.S. monetary policy.

In test 2 (runs 5 to 8) it is found that the additional boundary condition reinforces the tracking for both the nominal U.S. interest rate and Canadian government expenditures, but only if a very low weight is attached to deviations of U.S. government spending from their nominal path (r_{22}) compared to that of U.S. monetary policy (r_{44}). Otherwise, there would be deviations from both nominal paths $\bar{r}_{1t}$, $\bar{G}_{2t}$ and the amended optimal control solution could be explosive. Similar to test 1, it generates a trade-off between attainment of three other targets, U.S. internal balance and Canada's overall balance, and the tracking for the nominal values of two instruments--U.S. fiscal and monetary policies or on Canada's fiscal policy and U.S. monetary policy.

The results of test 2 are also found in test 3 in which the limits are set upon all three instruments.

(ii) Trade-off experiment C [Figures 4.6(C) to 4.12(C)]

This is the opposite case of trade-off experiment A, in which a high penalty cost is assigned to deviations of U.S. balance of payments from its equilibrium. The same results are obtained, with one exception. Since the external balance trade-off between the United States and Canada has been reversed (the United States attach more importance in welfare weights to the performance of its external balance than Canada does), it is found that the tracking for limits on policy-magnitude leaves the optimal path of Canada's balance of payments unchanged [Figure 4.9(C)]. However, the tracking affects the attainment of external balance in the United States by using the inconsistent or nonfeasible optimal policies. Furthermore, trade-off experiment A assigned heavy penalty costs to deviations of Canada's balance of payments from its external balance, resulting in a surplus of Canada's balance of payments at the expense of tracking for limits on policy-magnitudes. Unlike experiment A, it is found in the trade-off experiment C that the tracking for the nominal policies brings a deficit in the U.S. balance of payments.

(iii) Trade-off experiment B [Figures 4.6(B) to 4.12(B)]

When both countries give equal importance to the achievement of their external balance, a trade-off occurs between attainment of the four targets--internal and external balance in the United States and Canada--and the feasibility of all three policies. In other words, to

Figure 4.6(C). Two-country model (Case B) - United States optimal GNP trajectories compared with the desired GNP (trade-off experiment C where $q_{33} = 10^5$, $q_{44} = 10^{-4}$).

Figure 4.7(C). Two-country model (Case B) - Canada: optimal GNP trajectories compared with the desired GNP (trade-off experiment C where $q_{33} = 10^5$, $q_{44} = 10^{-4}$).

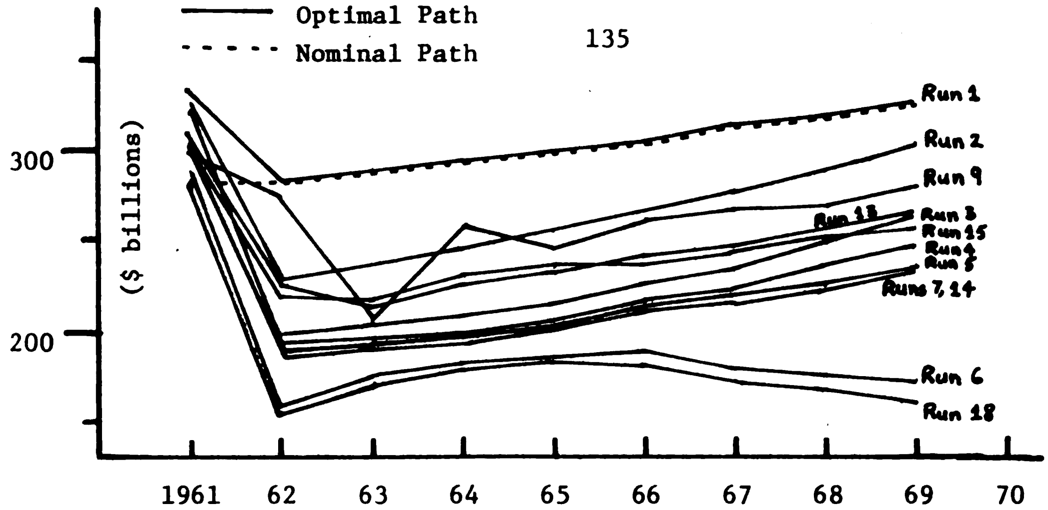

Figure 4.6(C)

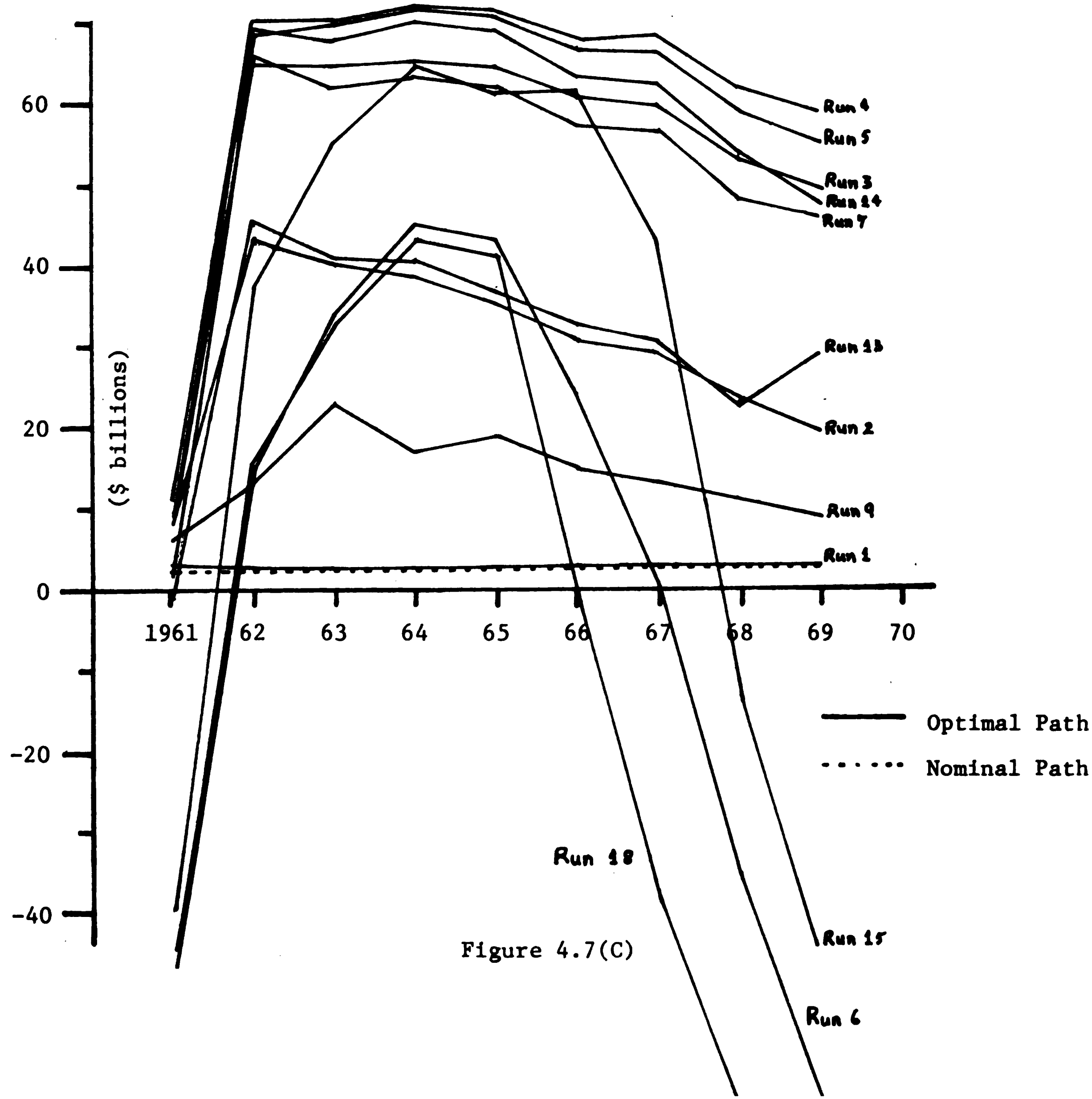

Figure 4.7(C)

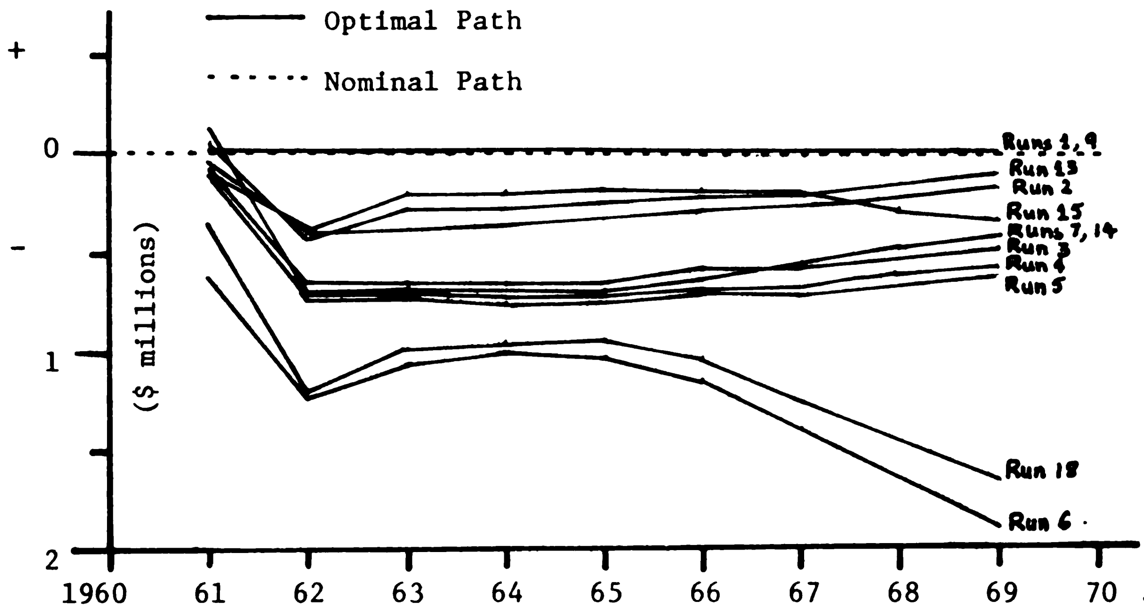

Figure 4.8(C). Two-country model (Case B) - U.S. balance of payments: optimal paths compared with the equilibrium (trade-off experiment C where $q_{33} = 10^5$, $q_{44} = 10^{-4}$).

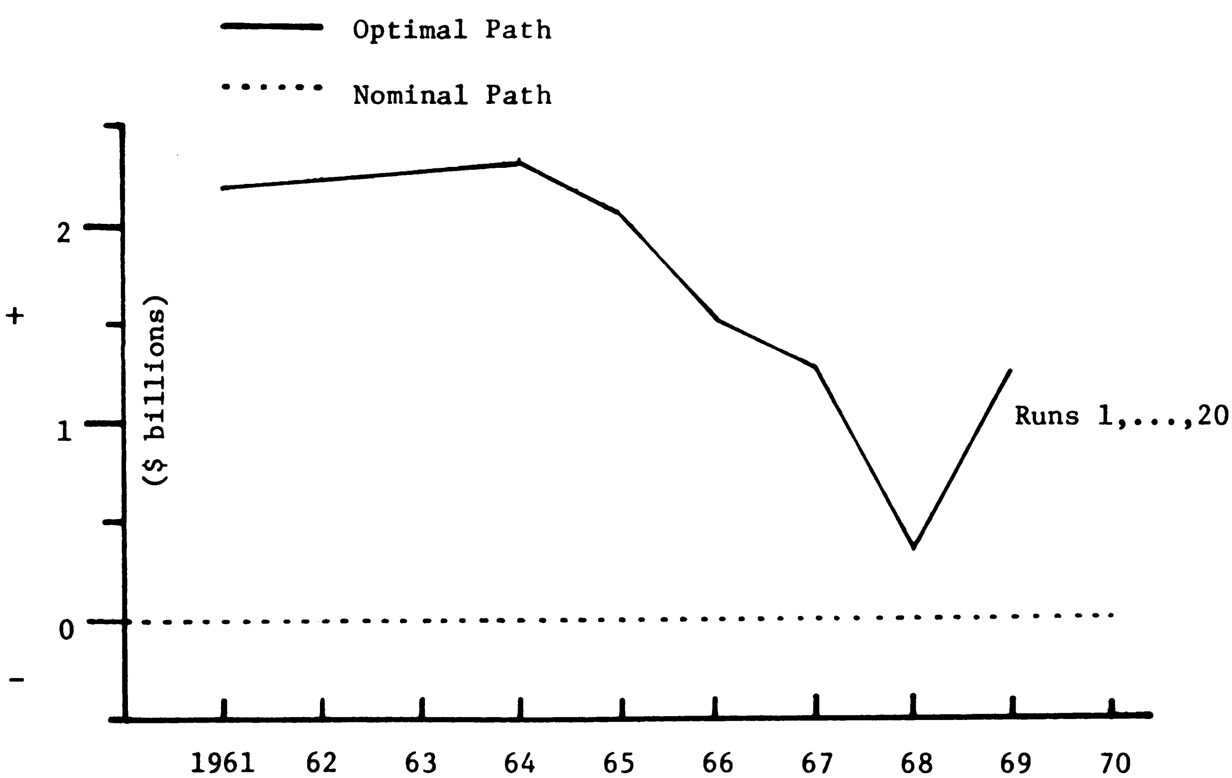

Figure 4.9(C). Two-country model (Case B) - Canadian balance of payments: optimal path compared with the equilibrium (trade-off experiment C where $q_{33} = 10^5$, $q_{44} = 10^{-4}$).

Figure 4.10(C). Two-country model (Case B) - U.S. government expenditures: optimal paths compared with nominal path (trade-off experiment C where $q_{33} = 10^5$, $q_{44} = 10^{-4}$).

Figure 4.11(C). Two-country model (Case B) - Canadian government expenditures: optimal paths compared with nominal path (trade-off experiment C where $q_{33} = 10^5$, $q_{44} = 10^{-4}$).

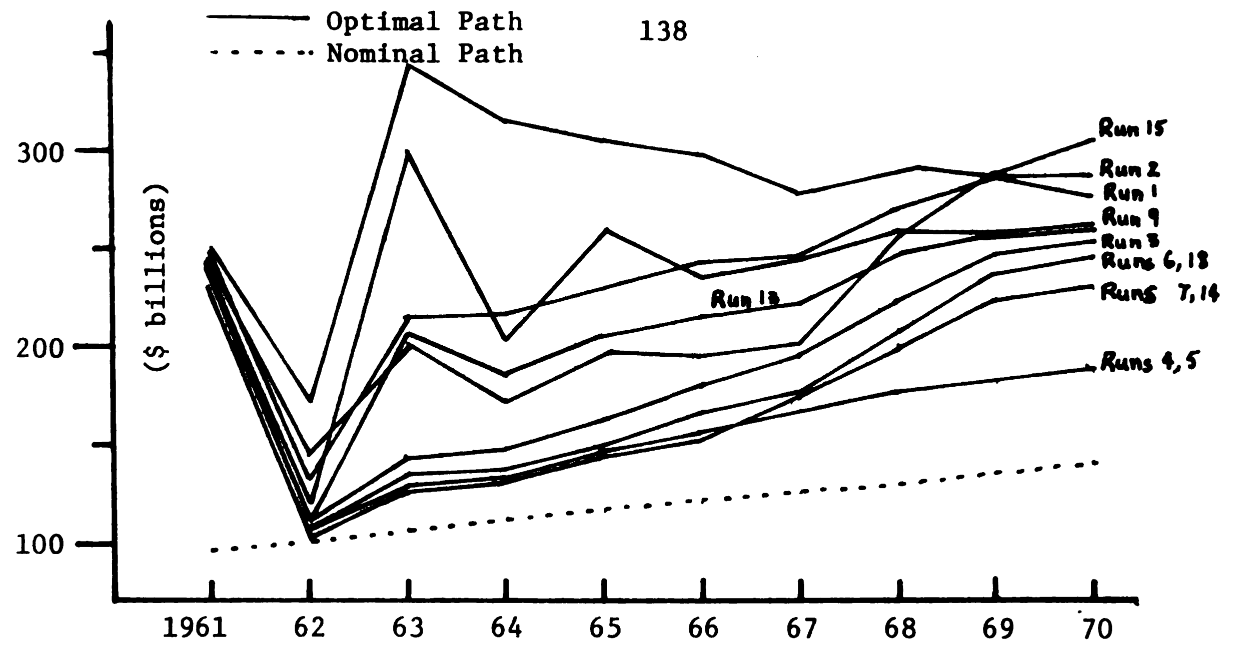

Figure 4.10(C)

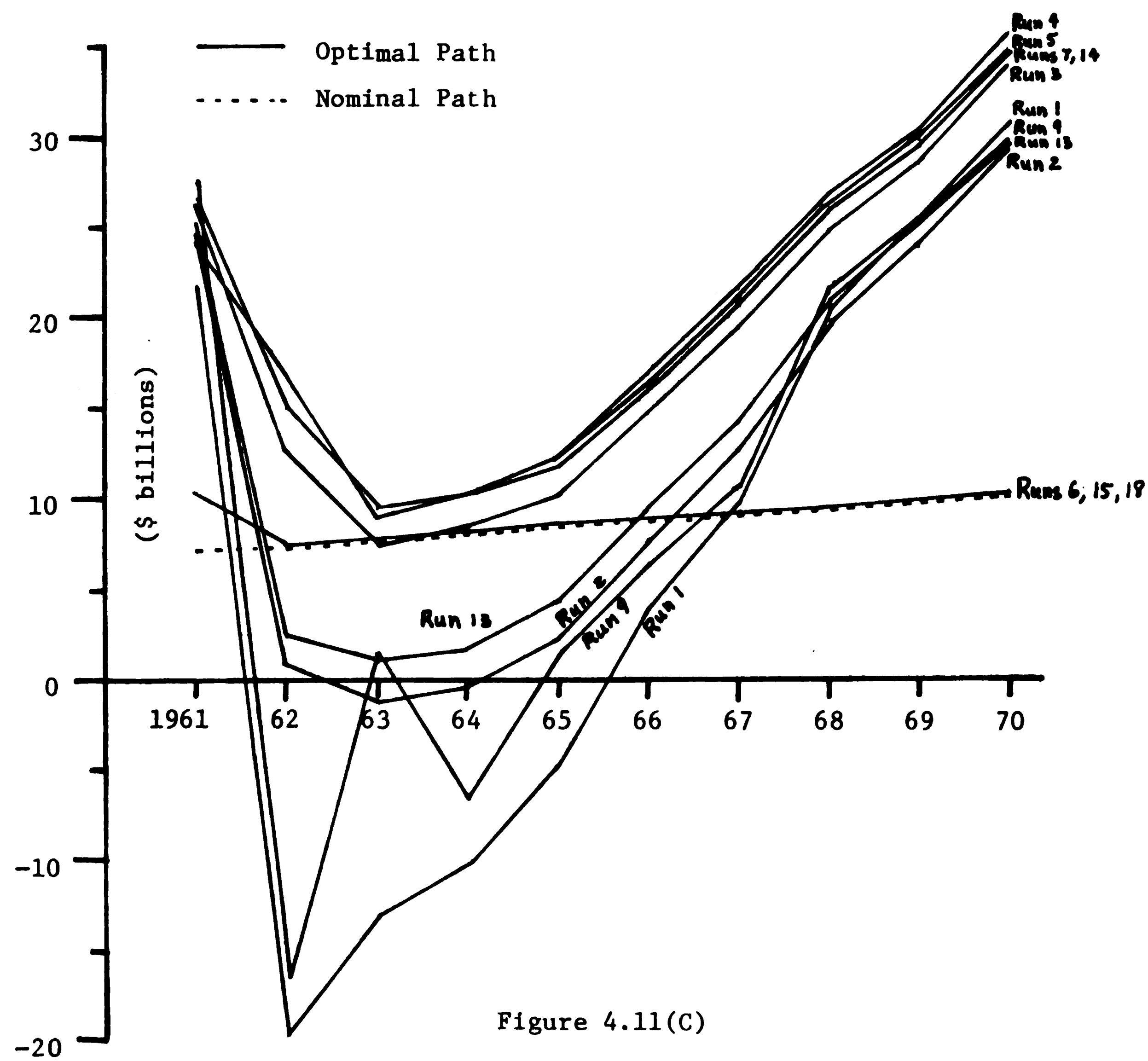

Figure 4.11(C)

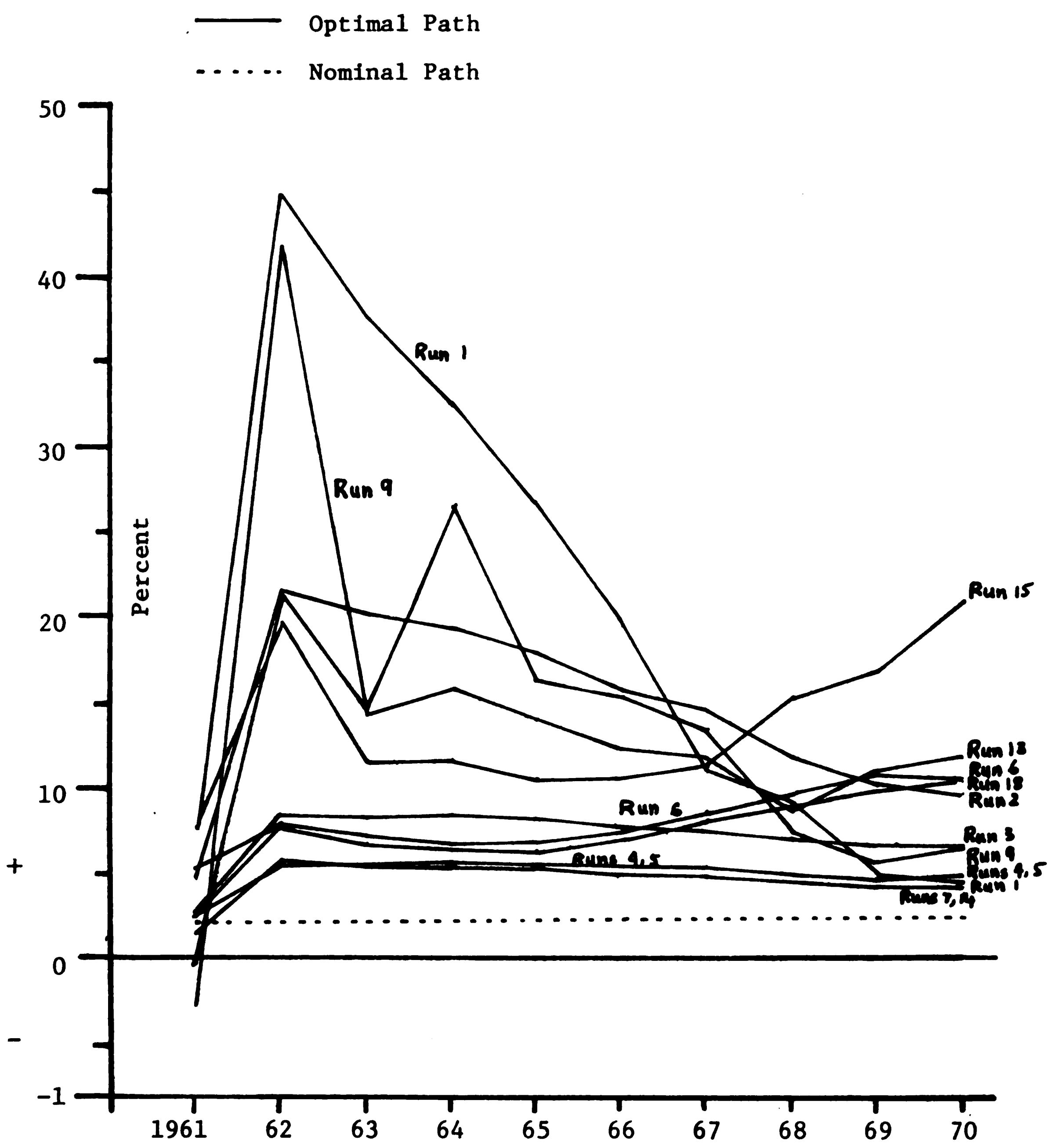

Figure 4.12(C). Two-country model (Case B) - U.S. interest rate: optimal paths compared with nominal path (trade-off experiment C where $q_{33} = 10^5$, $q_{44} = 10^{-4}$).

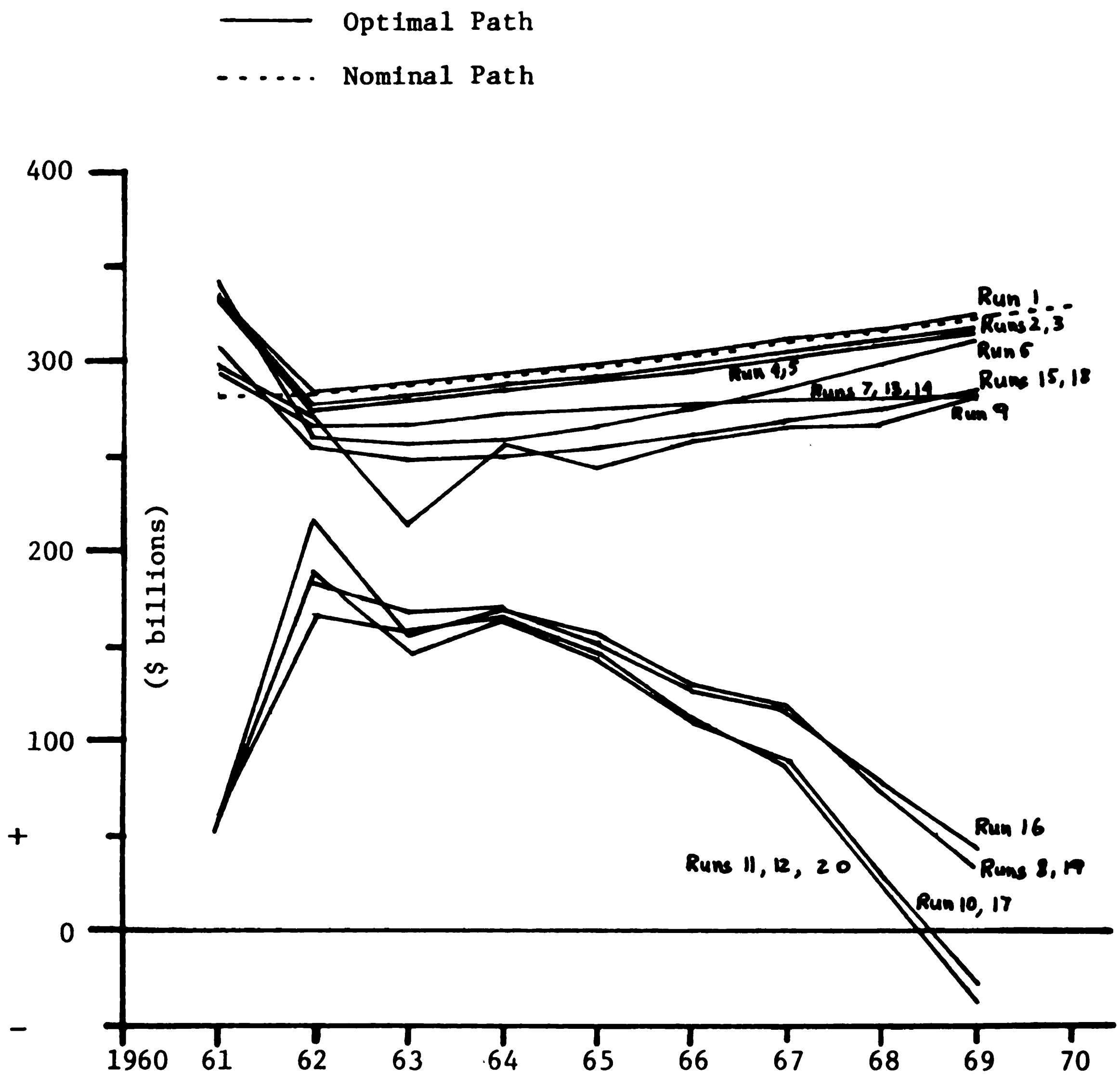

Figure 4.6(B). Two-country model (Case B) - United States: optimal GNP trajectories compared with the desired GNP (trade-off experiment B where $q_{33} = 10^5$, $q_{44} = 10^5$).

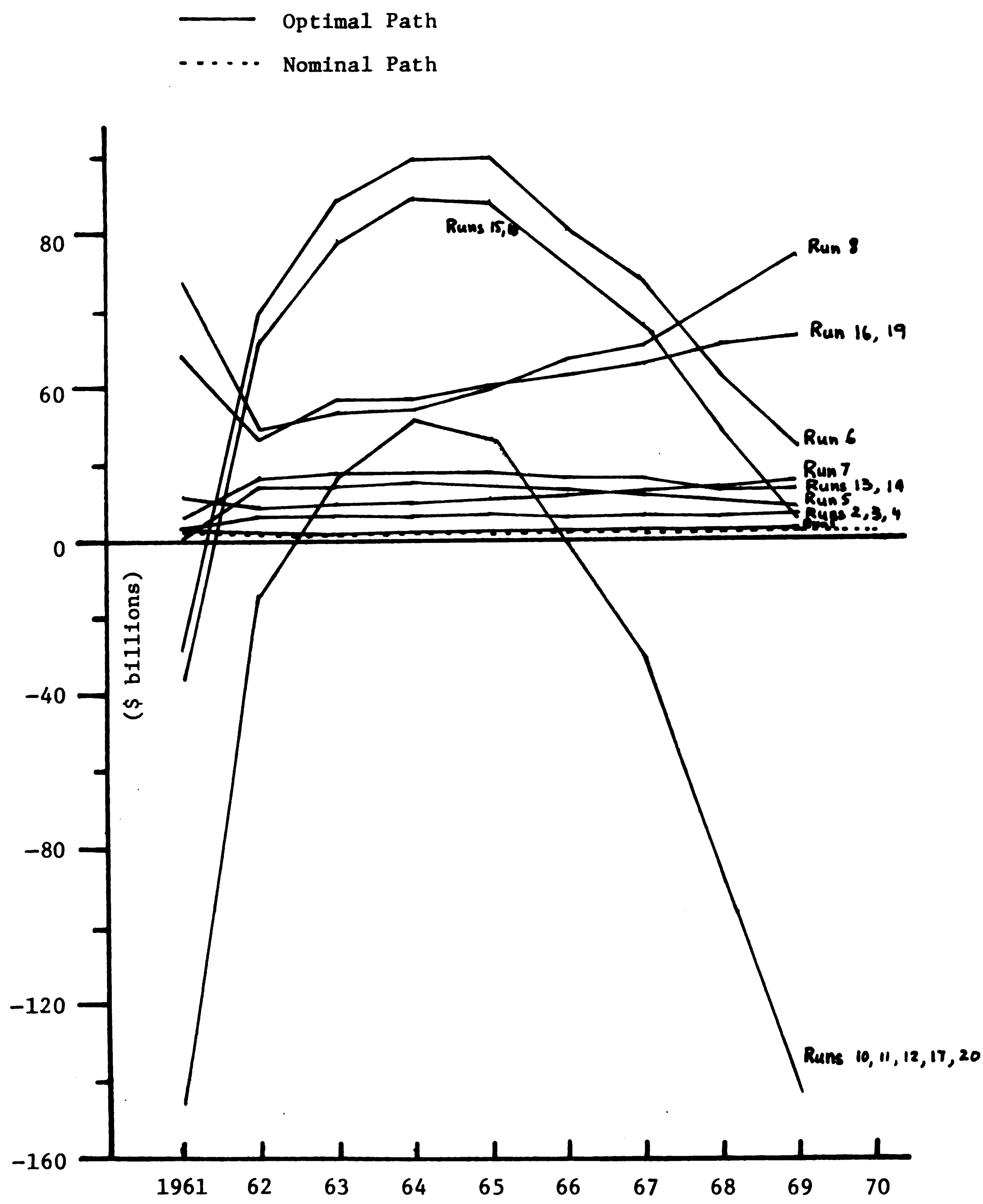

Figure 4.7(B). Two-country model (Case B) - Canada: optimal GNP trajectories compared with the desired GNP (trade-off experiment B where $q_{33} = 10^5$, $q_{44} = 10^5$).

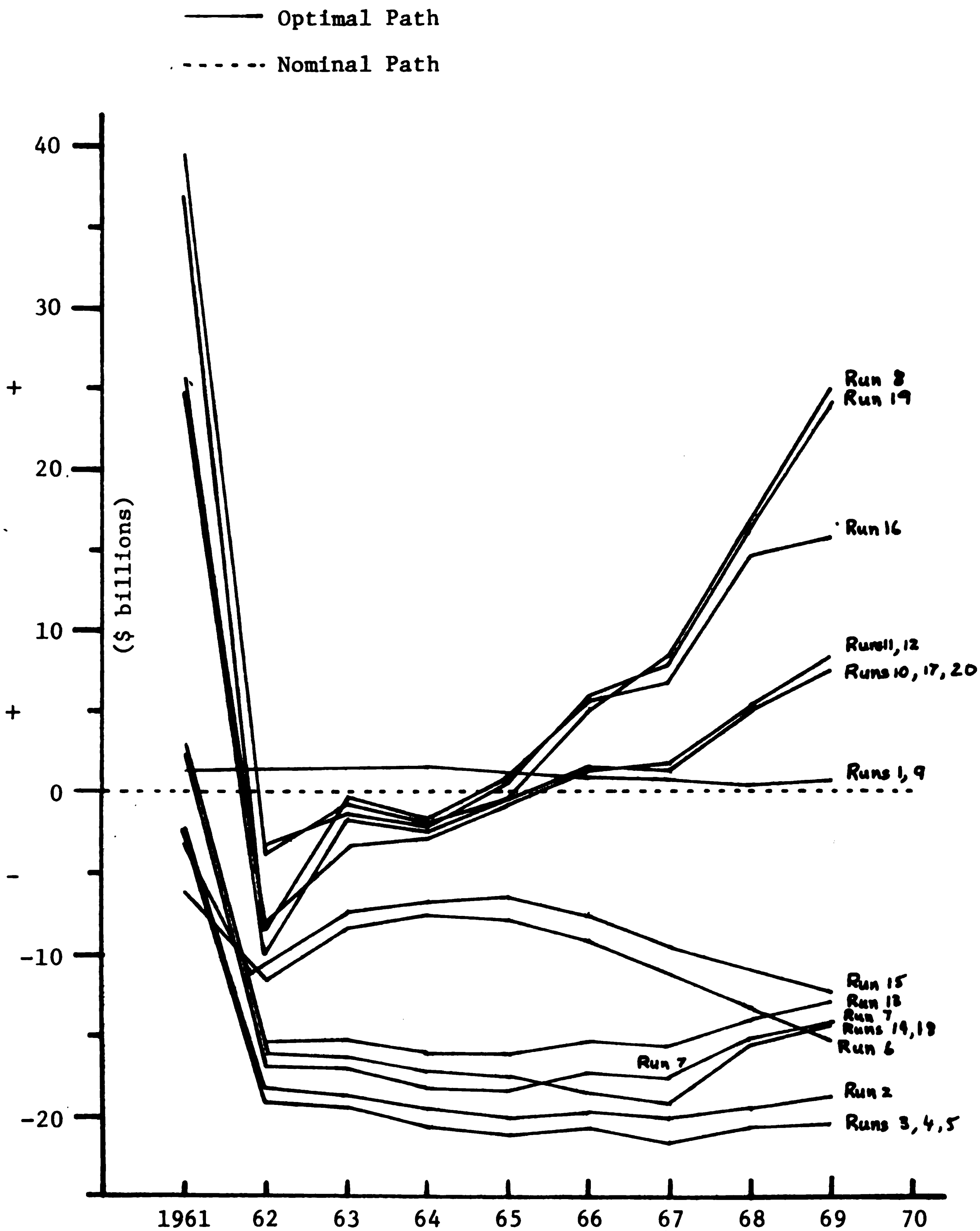

Figure 4.8(B). Two-country model (Case B) - U.S. balance of payments: optimal paths compared with the equilibrium (trade-off experiment B where $q_{33} = 10^5$, $q_{44} = 10^5$).

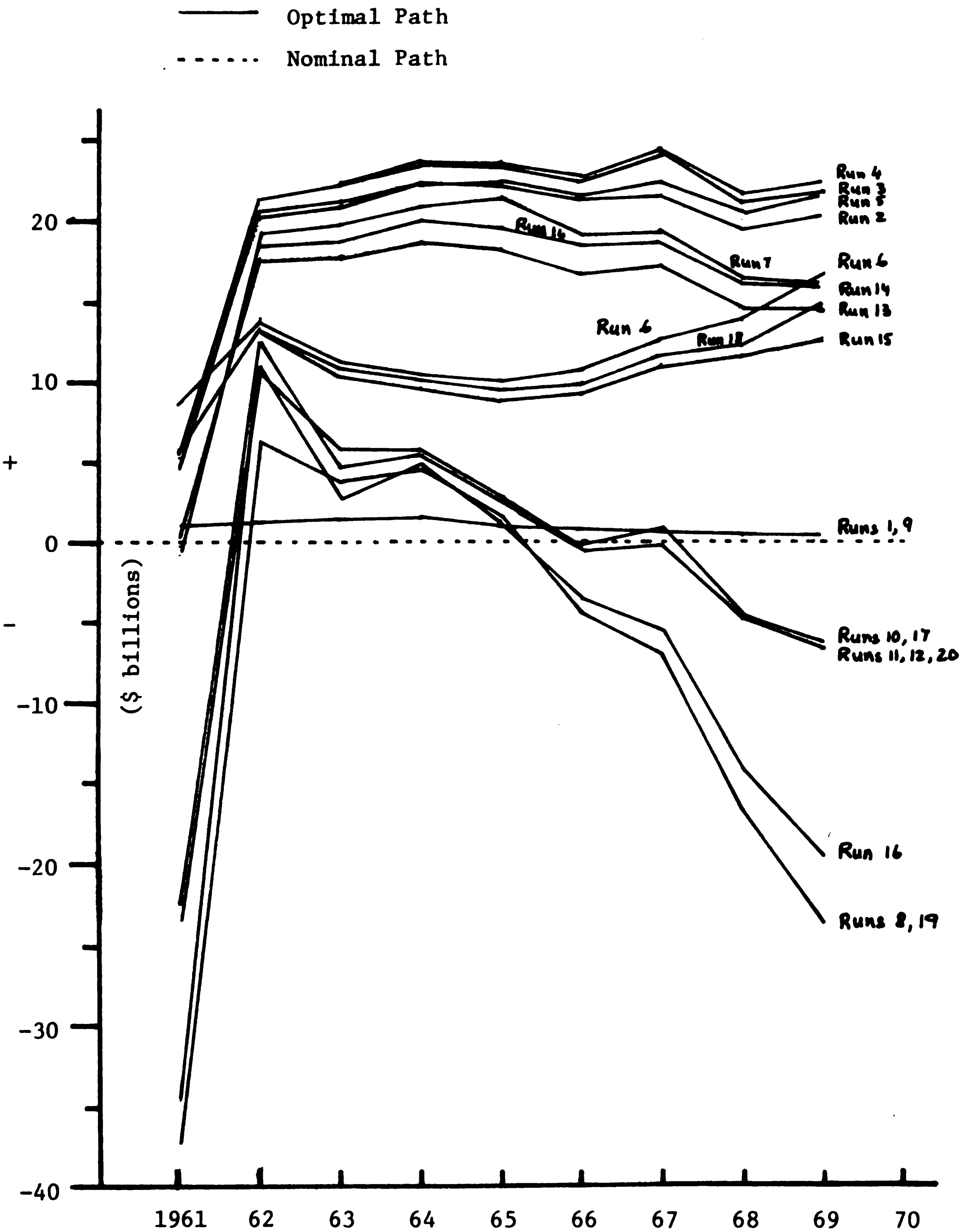

Figure 4.9(B). Two-country model (Case B) - Canadian balance of payments: optimal paths compared with the equilibrium (trade-off experiment B where $q_{33} = 10^5$, $q_{44} = 10^5$).

Figure 4.10(B). Two-country model (Case B) - U.S. government expenditures: optimal paths compared with nominal path (trade-off experiment B where $q_{33} = 10^5$, $q_{44} = 10^5$).

Figure 4.11(B). Two-country model (Case B) - Canadian government expenditures: optimal paths compared with nominal path (trade-off experiment B where $q_{33} = 10^5$, $q_{44} = 10^5$).

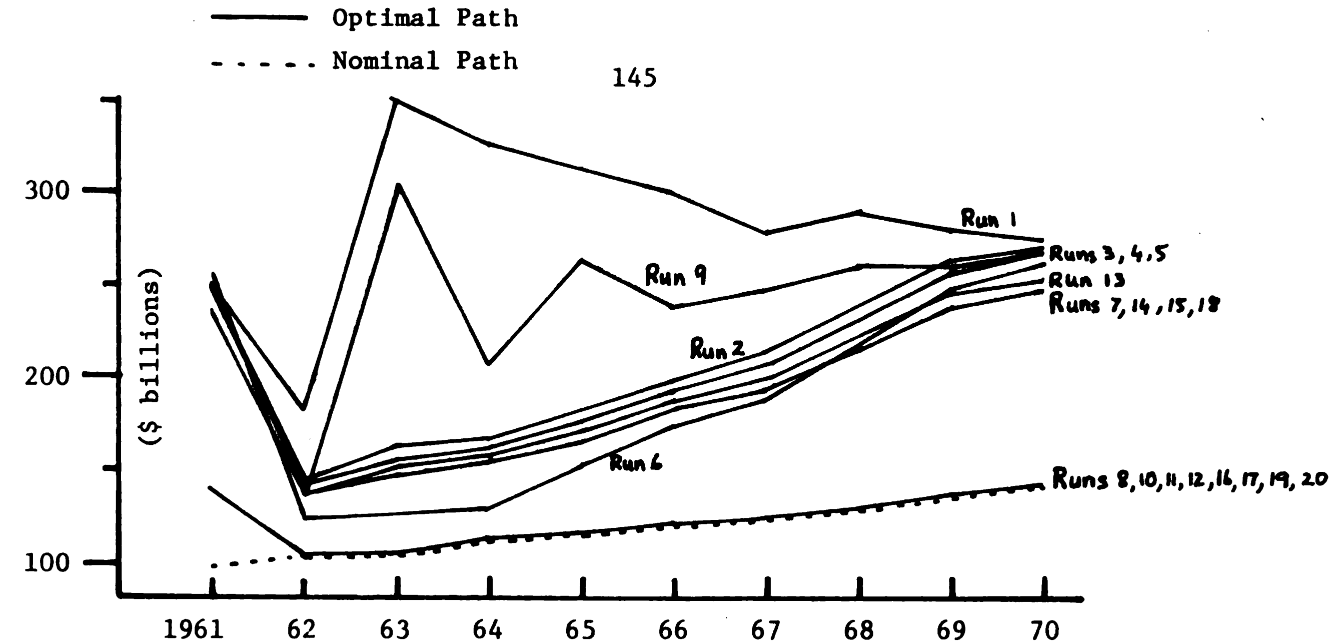

Figure 4.10(B)

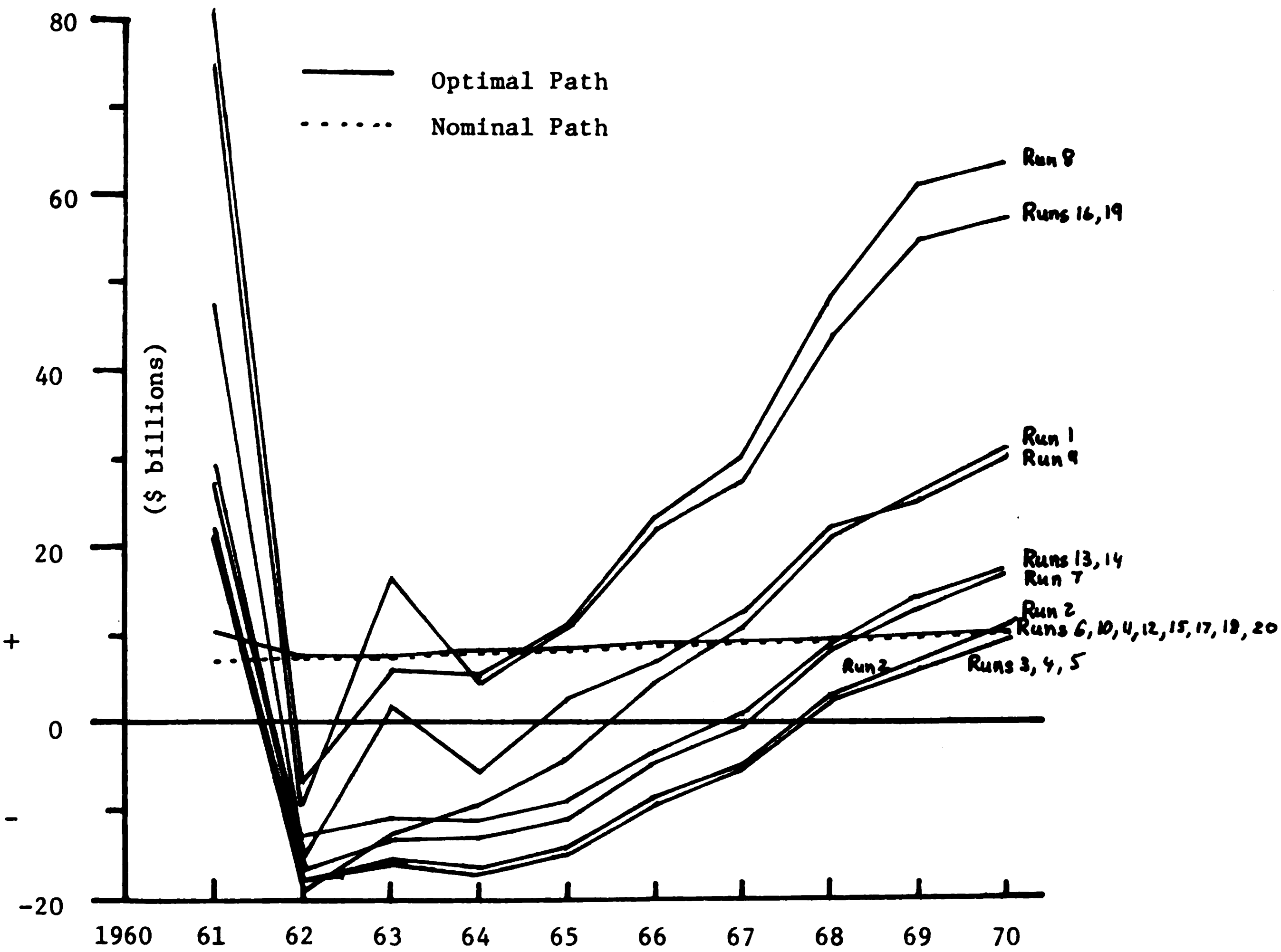

Figure 4.11(B)

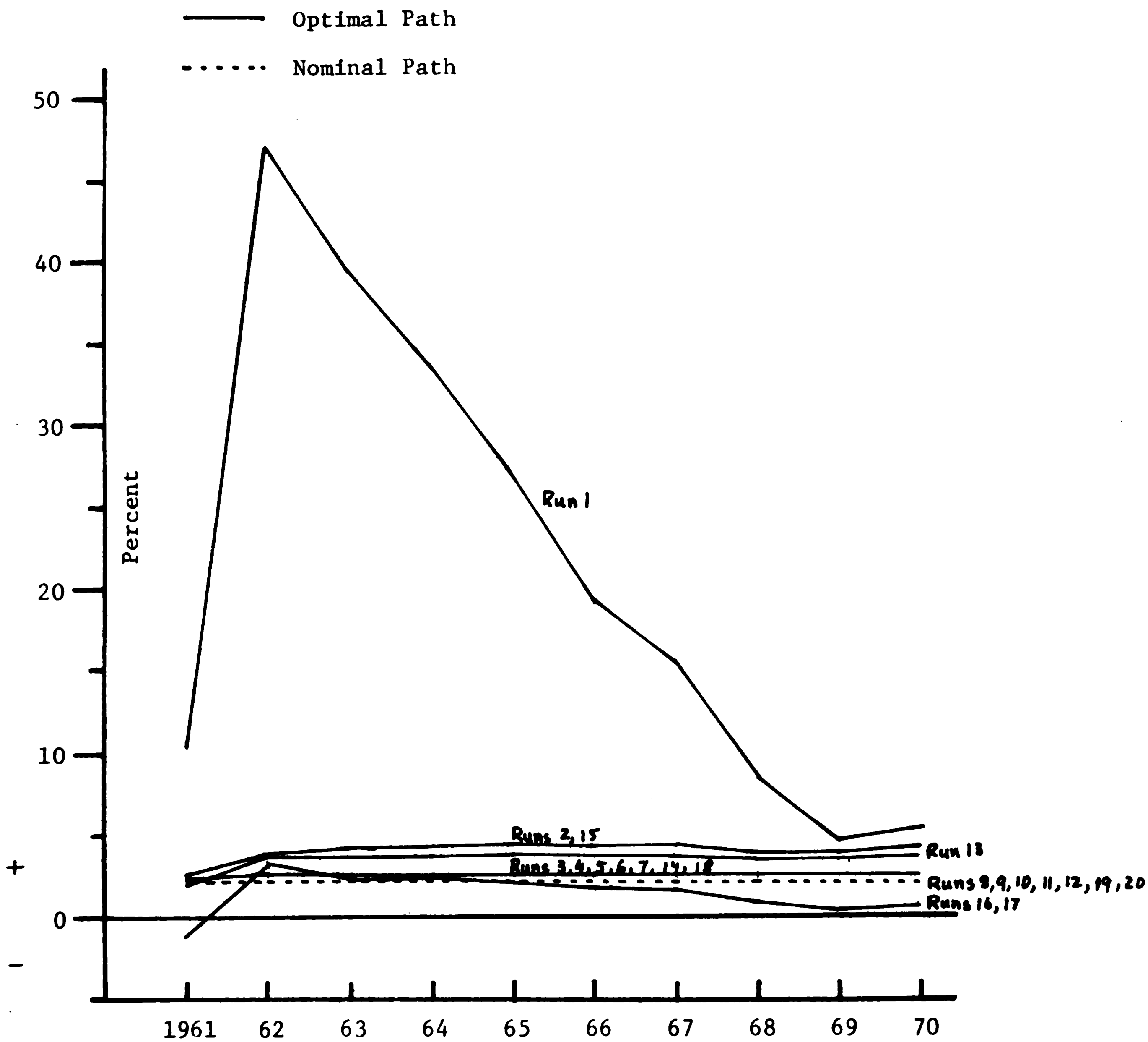

Figure 4.12(B). Two-country model (Case B) - U.S. interest rate: optimal path compared with nominal path (trade-off experiment C where $q_{33} = 10^5$, $q_{44} = 10^5$).

track the two nominal fiscal policies (4 percent growth per annum) and the nominal U.S. interest rate (2.15 percent), both the United States and Canada must give up their targets of internal and external balance--deflation coupled with balance-of-payments deficit in the United States and inflation paired with balance-of-payments surplus in Canada.

Under the assumptions of fixed exchange rate, passive responses from the second country and linear dependent balance of payments, it is concluded:

(1) Within the optimal control framework, if the number of variables to be controlled is larger than the number of instruments, the variables will not reach the targets exactly, and their deviations from the targets will depend on the welfare weights in Q^c assigned to them (Chow:1972c). As a result, the solution for simultaneously achieving internal and external balance in both countries is no longer uniquely determined. Within the confines of Votey's model where an inherent Mundellian policy assignment prevails for internal and external balance, only the variables of balance of payments (B_{1t}, B_{2t}) are not on target exactly since there is only one instrument, Country I's interest rate, to hit both external balances. Therefore, the solution obtained by the optimal control techniques is found to be a function of trade-off between the

two countries' balance of payments or, more specifically, a function of welfare weights q_{33} and q_{44} assigned to their deviations from equilibrium. Furthermore, unlike the two preceding cases in which the optimal cost $\hat{J}_c$ is zero, the trade balance of the third country with Countries I and II as well as the welfare weights q_{33} and q_{44} determine the optimal cost $\hat{J}_c$ (equation 4.8).

(2) The larger the welfare weight q_{33} (q_{44}), the closer the variables B_{1t} (B_{2t}) reach the equilibrium targets. A trade-off exists between the two external targets. For one country to hit its external balance the other one must consent to attach only a small importance to its own external balance. Otherwise, if both attach equal importance to the attainment of their own external balances, neither will reach the targets and both will face a balance-of-payments disequilibrium.

(3) When boundary conditions become active, a second trade-off occurs between attainment of targets and feasibility or consistency of policy-instruments, in addition to the trade-off between the external targets.

CHAPTER V

TWO-COUNTRY MODEL

CASE C: CONFLICT OF INTERESTS AND GAME THEORETICAL APPROACH

This chapter introduces a new framework for analyzing the problem of internal and external balance under the following assumptions:

(1) Economic interdependence between two countries,

(2) Active responses from the second country,

(3) Conflicting internal as well as external targets pursued by both countries but with the possibility of international cooperation.

The case of conflict of interests can be resolved by presenting a systematic and concise introduction to a two-player or controller multistage non-zero-sum game with linear quadratic system and perfect information will be presented. Its application to policy mix for internal and external balance will be left to future research, because more realistic assumptions will imply a more complicated model than that of Votey.

According to Y. C. Ho (Ho:1970), Differential Games (DG) is nothing but an extension of optimal control theory within the context of "Generalized Control Theory" (GCT) which incorporates all optimization problems possessing the three main ingredients: the criterion function, the controller (or player), and the information set available to the controller.

As in the optimal control problem, the differential game problems consist of finding strategies which minimize the performance criterion subject to the constraints described by the state system, with the difference that there is more than one controller.

Furthermore, in the non-zero-sum (NZS) game the players' interests are not in direct conflict. Therefore, contrary to the zero sum (ZS) game it is no longer possible to have a unique definition of optimality (Starr and Ho:1963a, 1963b). In this chapter, the noncooperative (or so-called Nash) equilibrium strategies will be determined first, then the inferiority of Nash equilibrium strategies will be proved, which leads to the search for a noninferior solution in particular the so-called Pareto Optimum Strategy.

5.1. Formulation of the Problem

The two-player NZS game is defined by:

(1) A linear, time-invariant, dynamic system, so-called "kinematic" equation (Isaacs:1965):

$$x_{t+1} = Ax_t + B_1u_{1t} + B_2u_{2t} + Cz_t \qquad (5.1a)$$

$$x_{t+1} - x_t = Fx_t + B_1u_{1t} + B_2u_{2t} + Cz_t \qquad (5.1b)$$

where $x \epsilon E^n$ is the state variable, $u_1 \epsilon E^{r_1}$ is player (Country) I's control variable, $u_2 \epsilon E^{r_2}$ is player (Country) II's control variable, $z \epsilon E^s$ is the exogenous noncontrolled variable,

$A = I+F$ is a nxn matrix, B_1 is a nxr_1 matrix, B_2 is a nxr_2 matrix, C is a nxs matrix, and all the matrices are assumed to be time-invariant and known.

(2) Quadratic performance criteria:

$$J_1 = \frac{1}{2} \sum_{t=0}^{N} \{[x_t - \overline{x}_t]^T Q_1 [x_t - \overline{x}_t] + u_{1t}{}^T R_{11} u_{1t} + u_{2t}{}^T R_{12} u_{2t}\} \tag{5.2a}$$

$$J_2 = \frac{1}{2} \sum_{t=0}^{N} \{[x_t - \overline{x}_t]^T Q_2 [x_t - \overline{x}_t] + u_{1t}{}^T R_{21} u_{1t} + u_{2t}{}^T R_{22} u_{2t}\} \tag{5.2b}$$

where J_1 and J_2 are players (Countries) I and II's performance criteria to be minimized; Q_1 Q_2, R_{11}, R_{12}, R_{21} and R_{22} are assumed to be symmetric; R_{11} and R_{22} are positive definite matrices; and Q_1 and Q_2 are non-singular matrices.

Therefore, the problem consists of finding a pair of strategies $(\tilde{u}_1, \tilde{u}_2)$, called Nash equilibrium or non-cooperative equilibrium strategies such that:

$$J_1(\tilde{u}_1, \tilde{u}_2) \leqslant J_1(\tilde{u}_1, \tilde{u}_2)$$

$$J_2(\tilde{u}_1, \tilde{u}_2) \leqslant J_2(\tilde{u}_1, \tilde{u}_2)$$

5.2. Determination of a Nash Equilibrium Set $(\tilde{u}_1, \tilde{u}_2)$

5.2.1. Definition

Following Starr and Ho (1963a), the strategy set $\tilde{u} = \{\tilde{u}_1, \ldots, \tilde{u}_k\}$ is called a Nash equilibrium strategy set if for $i = 1,2,\ldots k$:

$$J_i(\tilde{u}_1,\ldots,\tilde{u}_1,\ldots,\tilde{u}_k) \leq J_i(\tilde{u}_1,\ldots,\tilde{u}_i,\ldots,\tilde{u}_k)$$

for all admissible $\tilde{u}_i$, that is, for controls which belong to a Euclidian space.

5.2.2. Necessary Conditions for a Nash Equilibrium Solution

Starr and Ho (1963a, 1963b) derive the necessary conditions for a Nash equilibrium in the continuous case while Haurie's work (1970, 1971) is in connection with the discrete case.

Applying Haurie's results, a Nash equilibrium strategy pair for the linear quadratic NZS game problem is obtained by solving the following equations:

$$\left.\frac{\delta\mathcal{H}_{1t+1}}{\delta u_{1t}}\right|_{\substack{\tilde{x}_t \\ \tilde{u}_{1t}}} = 0 \tag{5.3a}$$

$$\left.\frac{\delta\mathcal{H}_{2t+1}}{\delta u_{2t}}\right|_{\substack{\tilde{x}_t \\ \tilde{u}_{2t}}} = 0 \tag{5.3b}$$

$$\tilde{x}_{t+1} - \tilde{x}_t = \frac{\delta\mathcal{H}_{1t+1}}{\delta_{1t+1}} = \frac{\delta\mathcal{H}_{2t+1}}{\delta_{2t+1}} = F\tilde{x}_t + B_1\tilde{u}_{1t} + B_2\tilde{u}_{2t} \tag{5.4}$$

$$\tilde{\lambda}_{1t+1} - \tilde{\lambda}_{1t} = -\left[\frac{\delta\mathcal{H}_{1t+1}}{\delta_{x_t}} + \frac{\delta\mathcal{H}_{1t+1}}{\delta u_{2t}}\,\frac{\delta\tilde{u}_{2t}}{\delta_{x_t}}\right]^T \tag{5.5a}$$

$$\tilde{\lambda}_{2t+1} - \tilde{\lambda}_{2t} = -\left[\frac{\delta \mathcal{H}_{2t+1}}{x_t} + \frac{\delta \mathcal{H}_{2t+1}}{\delta u_{1t}} \frac{\delta \tilde{u}_{1t}}{\delta x_t}\right]^T \tag{5.5b}$$

with the boundary conditions:

$$\tilde{x}(0) = \tilde{x}_0 \tag{5.6}$$

$$\tilde{\lambda}_1(N) = Q_1\left[\tilde{x}(N) - \tilde{x}(N)\right] \tag{5.7a}$$

$$\tilde{\lambda}_2(N) = Q_2\left[\tilde{x}(N) - \tilde{x}(N)\right] \tag{5.7b}$$

where $\mathcal{H}_{1t+1}$ and $\mathcal{H}_{2t+1}$ are the respective hamiltonians for the two players.

5.2.3. Determination of the Solution

Along the optimal path, the Hamiltonians for the two players are defined as follows (Starr and Ho:1963a, Haurie:1970, 1971):

$$\begin{aligned}\mathcal{H}_{1t+1} = \frac{1}{2} & \Big[(\tilde{x}_t - \bar{x}_t)^T Q_1 (\tilde{x}_t - \bar{x}_t) + \tilde{u}_{1t}{}^T R_{11} \tilde{u}_{1t} + \tilde{u}_{2t} \\ & R_{12} \tilde{u}_{2t}\Big] + \tilde{\lambda}^T_{1t+1}\Big[F\tilde{x}_t + B_1\tilde{u}_{1t} + B_2\tilde{u}_{2t} + Cz_t\Big]\end{aligned} \tag{5.8a}$$

$$\begin{aligned}\mathcal{H}_{2t+1} = \frac{1}{2} & \Big[(\tilde{x}_t - \bar{x}_t)^T Q_2 (\tilde{x}_t - \bar{x}_t) + \tilde{u}_{1t}{}^T R_{21} \tilde{u}_{1t} + \tilde{u}_{2t} \\ & R_{22} \tilde{u}_{2t}\Big] + \tilde{\lambda}^T_{2t+1}\Big[F\tilde{x}_t + B_1\tilde{u}_{1t} + B_2\tilde{u}_{2t} + C\,{}_t\Big]\end{aligned} \tag{5.8b}$$

where $\tilde{\lambda}_{1t+1}$, $\tilde{\lambda}_{2t+1}$ are co-state variables.

Then, solving equations 5.4a and 5.4b yields

$$\tilde{u}_{1t} = -R_{11}^{-1} B_1^T \tilde{\lambda}_{1t+1} \tag{5.9a}$$

$$\tilde{u}_{2t} = -R_{22}^{-1} B_2^T \tilde{\lambda}_{2t+1} \tag{5.9b}$$

Therefore $\mathcal{H}_{1t+1}$ and $\mathcal{H}_{2t+1}$ are at their minimum with respect to u_1 and u_2 since R_{11} and R_{22} are positive definite. Since the performance criteria are assumed to be quadratic, the closed-loop Nash equilibrium strategies of the form $\tilde{u}_{jt} = \tilde{S}_j(\tilde{x}_t) + s_j$ can be derived (Starr and Ho: 1963a). Similar to the linear quadratic optimal control problem, it is known that (Lee *et al.*:1972, Athans and Falb:1966):

$$\tilde{\lambda}_{1t+1} = K_{1t+1} \tilde{x}_{t+1} + k_{1t+1} \tag{5.10a}$$

$$\tilde{\lambda}_{2t+1} = K_{2t+1} \tilde{x}_{t+1} + k_{2t+1} \tag{5.10b}$$

Substituting equations 5.10a and 5.10b into equations 5.9a and 5.9b:

$$\begin{aligned}\tilde{u}_{1t} &= -R_{11}^{-1} B_1^T K_{1t+1} \tilde{x}_{t+1} - R_{11}^{-1} B_1^T k_{1t+1} \\ &\stackrel{d}{=} D_{1t+1} \tilde{x}_{t+1} + d_{1t+1}\end{aligned} \tag{5.11a}$$

$$\begin{aligned}\tilde{u}_{2t} &= -R_{22}^{-1} B_2^T K_{2t+1} \tilde{x}_{t+1} - R_{22}^{-1} B_2^T k_{2t+1} \\ &\stackrel{d}{=} D_{2t+1} \tilde{x}_{t+1} + d_{2t+1}\end{aligned} \tag{5.11b}$$

Again, substituting equations 5.11a and 5.11b into the "kinematic" equation 5.1b:

$$\tilde{x}_{t+1}-x_t = F\tilde{x}_t + B_1D_{1t+1}\,\tilde{x}_{t+1} + B_2D_{2t+1}\,\tilde{x}_{t+1} + B_1d_{1t+1} + B_2d_{2t+1} + Cz_t \tag{5.12}$$

Since $\left[G_{t+1}\right]^{-1} \stackrel{d}{=} \left[I-B_1D_{1t+1}-B_2D_{2t+1}\right]^{-1}$ exists for all t, t {0,...,N-1}, then:

$$\begin{aligned}\tilde{x}_{t+1} &= \left(G_{t+1}\right)^{-1} A\tilde{x}_t + \left(G_{t+1}\right)^{-1} B_1d_{1t+1} + \left(G_{t+1}\right)^{-1} B_2d_{2t+1} + \left(G_{t+1}\right)^{-1} Cz_t \quad ; \quad A \stackrel{d}{=} I + F \\ &= \left(G_{t+1}\right)^{-1} A\tilde{x}_t + g_{t+1} + G_{t+1}{}^{-1}\, Cz_t\end{aligned} \tag{5.13}$$

Substituting equation 5.13 into equations 5.11a and 5.11b:

$$\tilde{u}_{1t} = H_{1t+1}\,\tilde{x}_t + h_{1t+1} + D_{1t+1}\left(G_{t+1}\right)^{-1} Cz_t \tag{5.14a}$$

$$\tilde{u}_{2t} = H_{2t+1}\,\tilde{x}_t + h_{2t+1} + D_{2t+1}\left(G_{t+1}\right)^{-1} Cz_t \tag{5.14b}$$

where

$$H_{1t+1} \stackrel{d}{=} D_{1t+1}\left(G_{t+1}\right)^{-1} A$$

$$h_{1t+1} \stackrel{d}{=} D_{1t+1}\, g_{t+1} + d_{1t+1}$$

$$H_{2t+1} \stackrel{d}{=} D_{2t+1}\left(G_{t+1}\right)^{-1} A$$

$$h_{2t+1} \stackrel{d}{=} D_{2t+1}\, g_{t+1} + d_{2t+1}$$

Solving the canonical equations 5.5a and 5.5b, and rearranging the terms:

$$\tilde{\lambda}_{1t} = \left[Q_1+H^T_{2t+1}\, R_{12}H_{2t+1}\right]\tilde{x}_t+A^T\left[G^{-1}_{t+1}\right]^T\tilde{\lambda}_{1t+1}+H^T_{2t+1}R_{12}h_{2t+1}$$

$$+\, H^T_{2t+1}R_{12}D_{2t+1}G^{-1}_{t+1}Cz_t-Q_1\bar{x}_t \tag{5.15a}$$

$$\tilde{\lambda}_{2t} = \left[Q_2+H^T_{1t+1}R_{21}H_{1t+1}\right]\tilde{x}_t+A^T\left[G^{-1}_{t+1}\right]^T\tilde{\lambda}_{2t+1}+H^T_{1t+1}+H^T_{1t+1}R_{21}D_{1t+1}$$

$$G^{-1}_{t+1}Cz_t-Q_2\bar{x}_t \tag{5.15b}$$

Substituting equations 5.10a and 5.10b into equations 5.15a and 5.15b:

$$K_{1t}\tilde{x}_t+k_{1t} = \left[Q_1+H^T_{2t+1}R_{12}H_{2t+1}+A^T(G^{-1}_{t-1})^TK_{1t-1}G^{-1}_{t+1}A\right]$$

$$\tilde{x}_t+A^T(G^{-1}_{t+1})^T\; K_{1t+1}g_{t+1}+k_{1t+1} \;+\, H^T_{2t+1}R_{12} \tag{5.16a}$$

$$h_{2t+1}+\left[H^T_{2t+1}\; R_{12}D_{2t+1}+A^T(G^{-1}_{t+1})^TK_{1t+1}\right]$$

$$G^{-1}_{t+1}\; Cz_t \,-\, Q_1\, \bar{x}_t$$

$$K_{2t}\tilde{x}_t+k_{2t} = \left[Q_2+H^T_{1t+1}R_{21}H_{1t+1}+A^T(G^{-1}_{t-1})^TK_{2t+1}G^{-1}_{t-1}\; A\right]$$

$$\tilde{x}_t+A^T(G^{-1}_{t-1})^T\; K_{2t+1}g_{t+1}+k_{2t+1} \;+\, H^T_{1t+1} \tag{5.16b}$$

$$R_{21}h_{1t+1}+\left[H^T_{1t+1}\, R_{21}D_{t+1}+A^T(G^{-1}_{t+1})^TK_{2t+1}\right]G^{-1}_{t+1}\; Cz_t-Q_2\; \bar{x}_t$$

Then equating the coefficient yields:

$$K_{1t} = Q_1+H^T_{2t+1}\; R_{12}H_{2t+1}+A^T(G^{-1}_{t+1})^TK_{1t+1}G^{-1}_{t+1}\; A \tag{5.17a}$$

$$K_{2t} = Q_2+H^T_{1t+1}\; R_{21}H_{1t+1}+A^T(G^{-1}_{t+1})^TK_{2t+1}G^{-1}_{t+1}\; A \tag{5.17b}$$

$$k_{1t} = A^T(G_{t+1}^{-1})^T\left[K_{1t+1}g_{t+1}+k_{1t+1}\right] + H_{2t+1}^T\ R_{12}\ h_{2t+1}$$
$$+\left[H_{2t+1}^T\ R_{12}D_{2t+1}+A^T(G_{t+1}^{-1})^TK_{1t+1}\right]G_{t+1}^{-1}\ Cz_t - Q_1\ \overline{x}_t \tag{5.18a}$$

$$k_{2t} = A^T(G_{t+1}^{-1})^T\left[K_{2t+1}g_{t+1}+k_{2t+1}\right] + H_{2t+1}^T\ R_{21}\ h_{1t+1}$$
$$+\left[H_{1t+1}^T\ R_{21}\ D_{1t+1}+A^TK_{2t+1}\right]G_{t+1}^{-1}\ Cz_t - Q_2\ \overline{x}_t \tag{5.18b}$$

And

$$K_{1N} = Q_1 \tag{5.19a}$$

$$K_{2N} = Q_2 \tag{5.19b}$$

$$k_{1N} = Q_1\ \overline{x}_N \tag{5.20a}$$

$$k_{2N} = Q_2\ \overline{x}_N \tag{5.20b}$$

Summary of the solution

The Nash equilibrium strategy pair $(\tilde{u}_1,\ \tilde{u}_2)$ is found by the following steps:

(1) Solve the Riccati equations 5.17a and 5.17b with their boundary conditions 5.19a and 5.19b backwards in time to get values for K_t, $t = 1,\ldots,N$.

(2) Solve the tracking equations 5.18a and 5.18b with their boundary conditions 5.20a and 5.20b backwards in time to get values for k_t, $t = 1,\ldots,N$.

(3) Compute the Nash equilibrium strategy set $u(0) = \{\tilde{u}_1(0),\ \tilde{u}_2(0)\}$ from equations 5.14a, 5.14b:

$$\tilde{u}_{1t} = H_{1t+1}\,\tilde{x}_t + \left[h_{1t+1} + D_{1t+1}G_{t+1}^{-1}Cz_t\right] \tag{5.14a}$$

$$\tilde{u}_{2t} = H_{2t+1}\,\tilde{x}_t + \left[h_{2t+1} + D_{2t+1}G_{t+1}^{-1}Cz_t\right] \tag{5.14b}$$

where:

$$H_{1t+1} \overset{d}{=} D_{1t+1}\left(G_{t+1}\right)^{-1} A$$

$$H_{2t+1} \overset{d}{=} D_{2t+1}\left(G_{t+1}\right)^{-1} A$$

$$h_{1t+1} \overset{d}{=} D_{1t+1}\, g_{t+1} + d_{1t+1}$$

$$h_{2t+1} \overset{d}{=} D_{2t+1}\, g_{t+1} + d_{2t+1}$$

$$D_{1t+1} \overset{d}{=} -R_{11}^{-1}\, B_1^T\, K_{1t+1}$$

$$D_{2t+1} \overset{d}{=} -R_{22}^{-1}\, B_2^T\, K_{2t+1}$$

$$d_{1t+1} \overset{d}{=} -R_{11}^{-1}\, B_1^T\, k_{1t+1}$$

$$d_{2t+1} \overset{d}{=} -R_{22}^{-1}\, B_2^T\, k_{2t+1}$$

$$\left(G_{t+1}\right)^{-1} \overset{d}{=} \left[I - B_1D_{1t+1} - B_2D_{2t+1}\right]^{-1}$$

$$g_{t+1} \overset{d}{=} \left(G_{t+1}\right)^{-1}\left[B_1d_{1t+1} + B_2d_{2t+1}\right]$$

using $\tilde{x}(0) = \tilde{x}_0$. Then compute $x(1)$ from equation 5.4. Now $\tilde{x}(1)$ can be used in equation 5.14a and 5.14b to compute $u(2) \overset{d}{=} \{\tilde{u}_1(2), \tilde{u}_2(2)\}$ which can be used in equation 5.4 to compute $\tilde{x}(2)$, etc. Continue this process until all of the $u_t \overset{d}{=} \{\tilde{u}_{1t}, \tilde{u}_{2t}\}$, $t = 0, 1, \ldots, N-1$ and all of the $\tilde{x}_t$, $t = 1, \ldots, N$, have been computed.

(4) The costs $\tilde{J}_1$, $\tilde{J}_2$ can be computed from equations 5.2a and 5.2b.

5.2.4. Principle of Optimality in the NZS Game

The strategy set u is optimal in the Nash sense; that is, the equilibrium solution is secured against any attempt by one player unilaterally to alter his strategy. If every player is using his Nash control, and if a given player plays non-Nash optimally, he will do no better.

The conditions of Section 5.2.2 are necessary for the Nash equilibrium strategy set to exist, but these solutions are not protected against cheating and thus are unstable in a noncooperative sense. In other words, it is possible for the "prisoner's dilemma" to occur and the optimality is non-unique in the NZS differential game (Starr and Ho:1963a, 1963b). Therefore, there are solutions other than Nash's, such as noninferior solutions and minimax solutions.

This study is concerned with the noninferior solutions, and in particular with the Pareto optimal strategy set. The noninferior solution has the property that it is not dominated by any other solution point in its neighborhood. In other words, any deviation from the noninferior solution cannot result in simultaneous improvement of all J_i's ($i = 1,\ldots,k$).

First, the sufficient conditions for the Nash equilibrium to be inferior are derived to check if Nash strategies are inferior. If they are, the Pareto optimal solution, which is noninferior and therefore superior to the Nash solution, will be determined.

5.3. On the Inferiority of Nash Equilibrium for a Two Player Multistage Game

5.3.1. The Basic Sufficient Condition for Inferiority

Consider k players with respective controls $u_i \in E^{r_i}$, r_i given integer and respective criteria:

$$J_i : E^r \rightarrow R \qquad i = 1,\ldots,k$$

where $r = r_1 + r_2 + \ldots + r_k$.

Definition 5.3.1:

The control $u \in E^r$ is inferior if there exists $u \in E^r$ such that:

$$J_i(u) \leq J_i(\tilde{u}) \text{ for all } i \; \{1,\ldots,k\}$$

$$J_j(u) < J_j(\tilde{u}) \text{ for some } j \; \{1,\ldots,k\}$$

Lemma 5.3.1: (Rekasius & Schmitendorf:1971)

If each functional is C^1 in a neighborhood of $\hat{u}$, then a sufficient condition for u to be inferior is that there exists a vector h E^k such that:

$$h < 0 \tag{5.21}$$

and

$$r[\Lambda] = r[\Lambda \vdots h] \tag{5.22}$$

where Λ is the following kxr matrix:

$$\Lambda \overset{d}{=} \begin{bmatrix} \dfrac{\delta\mathcal{H}_1}{\delta_u} \\ \vdots \\ \dfrac{\delta\mathcal{H}_k}{\delta_u} \end{bmatrix}_{\tilde{u}}$$

and $\mathcal{H}_1,\ldots,\mathcal{H}_k$ are the Hamiltonians:

$$\mathcal{H}_i = \mathcal{H}_i(x,u_i, \lambda_i, t) \qquad i = 1,\ldots,k.$$

5.3.2. Application to Two-Player L.Q. Multistage Game

Consider the linear system:

$$x_{t+1} = Ax_t + B_1u_{1t} + B_2u_{2t} + Cz_t$$

and the quadratic criteria

$$J_1 = \frac{1}{2}\sum_{t=0}^{N} \left[x_t-\bar{x}_t\right]^T Q_1\left[x_t-\bar{x}_t\right] + u_{1t}^T R_{11}u_{1t}$$
$$+ u_{2t}^T R_{12}\, u_{2t}$$

$$J_2 = \frac{1}{2}\sum_{t=0}^{N} \left[x_t-\bar{x}_t\right]^T Q_2\left[x_t-\bar{x}_t\right] + u_{1t}^T R_{21}u_{1t} + u_{2t}^T R_{22}\, u_{2t} \quad .$$

Construct the matrix Λ:

$$\Lambda = \begin{bmatrix} \dfrac{\delta\mathcal{H}_1}{\delta u_1} & \dfrac{\delta\mathcal{H}_1}{\delta u_2} \\ \\ \dfrac{\delta\mathcal{H}_2}{\delta u_1} & \dfrac{\delta\mathcal{H}_2}{\delta u_2} \end{bmatrix}$$

where

$$\mathcal{H}_{1t+1} = \frac{1}{2}\left[\tilde{x}_t^T Q_1\tilde{x}_t+\tilde{u}_{1t}^T R_{11}\,\tilde{u}_{1t}+\tilde{u}_{2t}^T R_{12}\tilde{u}_{2t}\right]$$
$$+ \tilde{\lambda}_{1t+1}^T\left[F\tilde{x}_t + B_1\tilde{u}_{1t} + B_2\tilde{u}_{2t} + Cz_t\right]$$

$$\mathcal{H}_{2t+1} = \frac{1}{2}\left[x_t{}^T Q_2 x_t + u_{1t}{}^T R_{21}\, u_{1t} + u_{2t}{}^T R_{22} u_{1t}\right] + \lambda_{2t+1}^T$$

$$\left[F x_t + B_1 u_{1t} + B_2 u_{2t} + C z_t\right]$$

Since $\tilde{u} \stackrel{d}{=} (\tilde{u}_1, \tilde{u}_2)$ are a Nash equilibrium, the following conditions are necessarily satisfied:

$$\left.\frac{\delta\mathcal{H}_1}{\delta u_1}\right|_{\tilde{u}_1} = \left.\frac{\delta\mathcal{H}_2}{\delta u_2}\right|_{\tilde{u}_2} = 0$$

Thus the matrix Λ has the following form:

$$\Lambda = \begin{bmatrix} 0 & \dfrac{\delta\mathcal{H}_1}{u_2} \\ \dfrac{\delta\mathcal{H}_2}{u_1} & 0 \end{bmatrix}$$

where:

$$\frac{\delta\mathcal{H}_{1t+1}}{\delta u_{2t}} = \tilde{u}_{2t}^T\, R_{12} + B_2^T\, \tilde{\lambda}_{1t+1}$$

$$\frac{\delta\mathcal{H}_{2t+1}}{\delta u_{1t}} = \tilde{u}_{1t}^T\, R_{21} + B_1^T\, \tilde{\lambda}_{2t+1}$$

According to the previous results (Section 5.2.3), a Nash equilibrium is given by:

$$\tilde{u}_{1t} = -R_{11}^{-1}\, B_1^T\, \tilde{\lambda}_{1t+1}$$

$$\tilde{u}_{2t} = -R_{22}^{-1}\, B_2^T\, \tilde{\lambda}_{2t+1}$$

therefore, it is rather unlikely that

$$\frac{\delta \mathcal{H}_1}{\delta u_2} = \frac{\delta \mathcal{H}_2}{\delta u_1} = 0.$$

Thus Λ is non-singular and has a rank 2. And rank $\left[\Lambda \vdots h\right]$ = rank $\Lambda = 2$ for $h < 0$. By Lemma 5.3.1, $\tilde{u}$ is inferior; that is, the Nash solution of a two-player NZS game will usually be inferior.

5.4. Determination of the Pareto-Optimal Set

Nash equilibrium strategy set $(\tilde{u}_1, \tilde{u}_2)$ has been shown to be inferior and it is assumed that the two players agree to cooperate. Therefore, the two-player multistage NZS game is reduced to a vector-valued criterion optimal control problem in which the decision maker is a team of two players, and several optimality criteria are relevant to the players, although their relative importance is not obvious.

Different approaches to the vector-valued criterion optimization problem have been developed. Basile and Vincent (1970a); Vincent and Leitman (1970b); Leitman, Rocklin and Vincent (1972); and Stalford (1972) used a geometric approach based on the concepts of convex cost cones to derive the Pareto optimal set for both static and continuous games. Schmitendorf *et al.* (1971, 1972, 1973) also derived the necessary conditions for Pareto optimal solutions for static and continuous games. This approach does not require any local convexity assumptions like the method of Vincent *et al.*, but is based instead on the rank conditions of the cost matrix. Above all, the most popular technique is the scalarization technique, where the vector performance index problem is converted into a family of scalar index problems by forming an auxiliary scalar index as a function of the vector index and a vector of weighting parameters (Zadeh:1963, Klinger:1964, Da Cunha and Polak:1967, Starr and Ho:1963a, 1963b, Reid and Citron:1971, Haurie: 1970, 1971, 1973).

Haurie's approach and results (1970) will be applied to the two-player multistage NZS game with LQ system, perfect information and complete cooperation.

5.4.1. Definition

A set of control functions $u \stackrel{d}{=} (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_k)$ is said to be Pareto optimal if for each set of control functions $u \stackrel{d}{=} (u_1, u_2, \ldots, u_k)$ there exists:

either $J_i(u) = J_i(\hat{u})$ for all i $\{1,\ldots,k\}$

or at least for one i $\{1,\ldots,k\}$: $J_i(u) > J_i(u)$

5.4.2. Scalarization Procedure

The Pareto optimal set $\hat{u}$ could be obtained by minimizing the following scalar performance criterion

$$J = \mu_1 J_1 + \mu_2 J_2$$

$$\mu_1 + \mu_2 = 1$$

$$\mu_1 > 0, \ \mu_2 > 0$$

provided that the set of cost vectors (J_1, J_2) generated by all the admissible controls is convex (Starr and Ho:1963a). Therefore, the problem consists of finding a set of controls $\hat{u}(\mu_j, x, t) \stackrel{d}{=} \hat{u} = (\hat{u}_1, \hat{u}_2)$ such that:

$$\mu_1 J_1 (\hat{u}) + \mu_2 J_2 (\hat{u}) < \mu_1 J_1 (u) + \mu_2 J_2 (u).$$

5.4.3. Determination of the Solution

The problem of finding the Pareto optimal set $\hat{u}$ is formulated as follows: minimize the performance criterion

$$J \overset{d}{=} \mu_1 J_1 + \mu_2 J_2 = \frac{1}{2} \sum_{t=0}^{N} \{ [x_t - \bar{x}_t]^T Q [x_t - \bar{x}_t] + u_t^T R u_t \} \tag{5.23}$$

given the dynamic system

$$x_{t+1} - x_t = F x_t + B u_t + C z_t \tag{5.24}$$

where

$$Q = \begin{bmatrix} \mu_1 Q_1 & 0 \\ 0 & \mu_2 Q_2 \end{bmatrix}$$

$$R = \begin{bmatrix} \mu_1 R_{11} + \mu_2 R_{12} & 0 \\ 0 & \mu_1 R_{21} + \mu_2 R_{22} \end{bmatrix}$$

$$B \overset{d}{=} [B_1, B_2]$$

$$u \overset{d}{=} [u_1, u_2]$$

To get the necessary conditions for the Pareto optimal solution, the Hamiltonian is constructed:

$$\mathcal{H}_{t+1} = \frac{1}{2} \{ [x_t - \bar{x}_t]^T Q [x_t - \bar{x}_t] + u_t^T R u_t \} + \hat{\lambda}^T_{t+1} \left[Fx_t + Bu_t + Cz_t \right] \quad (5.25)$$

Similar to the linear quadratic optimal control problem, it is known that (Lee *et al.*:1972, Athans and Falb:1966):

$$\hat{\lambda}_{t+1} \overset{d}{=} K_{t+1} x_{t+1} + k_{t+1} \quad (5.26)$$

Then the minimization of the Hamiltonian is written as follows:

$$\frac{\delta \mathcal{H}_{t+1}}{\delta u_t} = R\hat{u}_t + B^T \hat{\lambda}_{t+1} = 0 \quad (5.27)$$

and the canonical equations are:

$$x_{t+1} - x_t = \frac{\delta \mathcal{H}_{t+1}}{\delta \hat{\lambda}_{t+1}} = Fx_t + Bu_t + Cz_t \quad (5.28)$$

$$\hat{\lambda}_{t+1} - \hat{\lambda}_t = \frac{\delta \mathcal{H}_{t+1}}{\delta x_t} = -Q \ \hat{x}_t - \bar{x}_t \ - F^T \hat{\lambda}_{t+1} \quad (5.29)$$

with the split boundary conditions:

$$x(0) = x_0 \quad (5.30)$$

$$\hat{\lambda}_N = Q(\hat{x}_N - \bar{x}_N) \quad (5.31)$$

From equation 5.27:

$$\hat{u}_t = -R^{-1} B^T \hat{\lambda}_{t+1} \quad (5.32)$$

Substituting equation 5.26 into equation 5.32:

$$\hat{x}_t = -R^{-1} B^T K_{t+1} \hat{x}_{t+1} - R^{-1} B^T k_{t+1} \quad (5.33)$$

Substituting equation 5.33 into equation 5.28:

$$\hat{x}_{t+1} - \hat{x}_t = F\hat{x}_t - BR^{-1}B^T K_{t+1} \hat{x}_{t+1} - BR^{-1}B^T k_{t+1} + Cz_t \quad (5.34)$$

Rearranging the terms, equation 5.34 yields:

$$\hat{x}_{t+1} = W^{-1}A\hat{x}_t - W^{-1}BR^{-1}B^Tk_{t+1} + W^{-1}\,Cz_t \tag{5.35}$$

where: $A \stackrel{d}{=} I + F$

$W \stackrel{d}{=} I + BR^{-1}\,B^TK_{t+1}$

$W^{-1} \stackrel{d}{=} \; I + BR^{-1}\,B^TK_{t+1}\;^{-1}$

Substituting equation 5.35 into equation 5.33:

$$\hat{u}_t = -R^{-1}B^TK_{t+1}\left[W^{-1}A\hat{x}_t - W^{-1}BR^{-1}B^Tk_{t+1} + W^{-1}Cz_t\right] - R^{-1}B^Tk_{t+1} \tag{5.36}$$

Now re-examine W^{-1}. Using the matrix identity (Pindyck:1973), W^{-1} becomes:

$$W^{-1} = I - B\left[R+B^TK_{t+1}B\right]^{-1}\,B^TK_{t+1} \tag{5.37}$$

Substituting equation 5.37 into equation 5.36:

$$\begin{aligned}\hat{u}_t = {} & -R^{-1}B^TK_{t+1}\left[I-B(R+B^TK_{t+1}B)^{-1}B^TK_{t+1}\right]A\hat{x}_t + R^{-1}\,B^TK_{t+1} \\ & \left[I-B(R+B^TK_{t+1}B)^{-1}\,B^TK_{t+1}\right]BR^{-1}B^Tk_{t+1} - R^{-1}B^TK_{t+1}\left[I-B(R+B^TK_{t+1}B)^{-1}\right. \\ & \left. B^TK_{t+1}\right]Cz_t - R^{-1}B^Tk_{t+1}\end{aligned} \tag{5.38}$$

And after rearranging:

$$\begin{aligned}\hat{u}_t = {} & -R^{-1}\{I-B^TK_{t+1}B\left[R+B^TK_{t+1}B\right]^{-1}\}\,B^TK_{t+1}A\hat{x}_t + R^{-1}\{I-B^TK_{t+1}B \\ & \left[R+B^TK_{t+1}B\right]^{-1}\}\,B^TK_{t+1}BR^{-1}\,B^Tk_{t+1} - R^{-1}B^Tk_{t+1} - R^{-1}\{\,I-B^TK_{t+1}B \\ & \left[R+B^TK_{t+1}B\right]^{-1}\}\,B^TK_{t+1}\,Cz_t.\end{aligned} \tag{5.39}$$

Now, using the identity:

$$I-X\ (Y+X)^{-1} = Y(Y+X)^{-1} \tag{5.40}$$

Letting $X = B^TK_{t+1}B$ and $Y = R$, simplify equation 5.39 to:

$$\begin{aligned}\hat{u}_t = &-\left[R+B^TK_{t+1}B\right]^{-1} B^TK_{t+1}A\hat{x}_t + \left[R+B^TK_{t+1}B\right]^{-1} B^TK_{t+1}\ BR^{-1}B^Tk_{t+1} \\ &- R^{-1}B^Tk_{t+1} - \left[R+B^TK_{t+1}B\right]^{-1} B^TK_{t+1}\ Cz_t\end{aligned} \tag{5.41}$$

Next, solve the second canonical equation 5.29:

$$\hat{\lambda}_t = Q[\hat{x}_t-\overline{x}_t] + (I+F)^T\ \hat{\lambda}_{t+1} = Q[\hat{x}_t-\overline{x}_t] + A^T\hat{\lambda}_{t+1} \tag{5.42}$$

Substituting equation 5.26 into equation 5.42:

$$K_t\hat{x}_t+k_t = Q[\hat{x}_t-\overline{x}_t] + A^TK_{t+1}\hat{x}_{t+1}+A^Tk_{t+1} \tag{5.43}$$

After rearranging the terms:

$$A^TK_{t+1}\hat{x}_{t+1} + Q\hat{x}_t-K_t\hat{x}_t = -A^Tk_{t+1} + k_t + Q\overline{x}_t. \tag{5.44}$$

Substituting equation 5.35 into equation 5.44:

$$\begin{aligned}A^TK_{t+1}\ \{&W^{-1}A\hat{x}_t-W^{-1}BR^{-1}B^Tk_{t+1} + W^{-1}Cz_t\} + Q\hat{x}_t - K_t\hat{x}_t = -A^Tk_{t+1} \\ &+ k_t + Q\overline{x}_t.\end{aligned} \tag{5.45}$$

Rearranging the terms:

$$\begin{aligned}&\left[Q + A^TK_{t+1}W^{-1}A\right]\hat{x}_t - A^TK_{t+1}W^{-1}BR^{-1}B^Tk_{t+1} + A^Tk_{t+1} - Q\overline{x}_t + \\ &A^TK_{t+1}W^{-1}Cz_t = K_t\hat{x}_t + k_t\end{aligned} \tag{5.46}$$

Equating the coefficients of the left-hand side and the right-hand side yields:

$$K_t = Q + A^T K_{t+1} W^{-1} A \tag{5.47}$$

$$k_t = -A^T K_{t+1} W^{-1} B R^{-1} B^T k_{t+1} + A^T k_{t+1} - Q\overline{x}_t + A^T K_{t+1} W^{-1} C z_t \tag{5.48}$$

with the boundary conditions:

$$K_N = Q \tag{5.49}$$

$$k_N = -Q\overline{x}_N \tag{5.50}$$

In conclusion, equation 5.41 determines the family of Pareto optimal strategies in terms of the present optimal state $\hat{x}_t$ and the solutions to the Riccati equation 5.47 and the "tracking" equation 5.48. It is noted that the determination of the Pareto optimal strategies is similar to that of the optimal control except that instead of having a unique optimal solution, there is a family of optimal solutions, functions of the parameters μ_1 and μ_2. In fact,

$$Q = \begin{bmatrix} \mu_1 Q_1 & 0 \\ 0 & \mu_2 Q_2 \end{bmatrix} \text{ and } R = \begin{bmatrix} \mu_1 R_{11} + \mu_2 R_{12} & 0 \\ 0 & \mu_1 R_{21} + \mu_2 R_{22} \end{bmatrix}$$

CHAPTER VI

CONCLUSIONS AND RESEARCH RECOMMENDATIONS

6.1. Summary of Conclusions

This study has applied Chow's deterministic optimal control approach to Votey's macrodynamic one- and two-country models. Summarizing the main conclusions:

(1) Optimal policies for internal and external balance can be determined by optimal control techniques.

(2) Chow's propositions concerning optimal control are verified in this study, i.e., if the number of variables to be controlled (the number of non-zero diagonal elements in Q^c) is equal to the number of instruments, that is, if Tinbergen's rule is satisfied, the variables will be on target exactly and the optimal cost is minimal. If the number of variables to be controlled is larger than the number of instruments or if Tinbergen's rule is violated, the variables will not reach their targets exactly. Their deviations from the targets will depend on the welfare weights in Q^c assigned to them and, therefore, the optimal cost is no longer minimal (Chow:1972b).

(3) Optimal policies for internal-external balance may be viewed as inadmissible for a given economy because they violate the limits set upon policy variable-magnitudes for political, social or technical reasons. Whenever these boundary conditions become active, a trade-off occurs between attainment of the joint balance and feasibility or consistency of policy-instruments.

(4) If Tinbergen's rule is not satisfied, an additional trade-off occurs among the targets and optimal policies for internal-external balance are no longer uniquely determined, but a function of the trade-off. Analysis of a range of penalty functions reveals changes in optimal control policies with welfare weights q_{33} and q_{44}.

(5) Optimal control theory provides the appropriate framework for structuring and investigating the problem of internal-external balance under the assumption of the existence of only one policy-maker or of a centralization of powers. However, this assumption certainly does not hold as the interdependence between countries with different goals increases. In this study, the optimal control tools have failed to take the case of decentralized economic policy, the conflicts in targets and the reactions from other countries to our own optimal

policies into account. Therefore, the suggested picture of the mechanism of economic policy derived from recognition of the existence of more than one controller is one of a game between various controllers. Only the theoretical presentation of a two-player deterministic multistage game has been introduced. Its applications are left to further researches.

6.2. Further Research Recommendations

(1) When the system under consideration is a macroeconomic model the assumptions of a linear model and a quadratic cost function are made for their convenient analytical properties rather than for their relationship to economic reality. For example, the potential output of the economy, $\bar{\tilde{Y}}_{it}$, it was taken to be a linear function of the labor force and capital stock. This was an approximation to the nonlinear Cobb-Douglas production function which is generally held by economists to be the proper form. This approximation was made to more easily fit the problem into control system format. Thus, the effects of nonlinearities on the performance of a control system should be done in this case.

(2) In this study, along with Pindyck (1972, 1973), deterministic control theory has been applied to the one- and two-country models since for the class of

linear and quadratic optimal control problems use can be made of the "Separation Theorem" or certainty-equivalence principle to treat stochastic addition disturbances in a neat way. The stochastic optimal solution is the same as obtained under the deterministic control problem. However, recently Chow has shown and measured the welfare gains of stochastic optimal control over deterministic optimal control (Chow:1972c). It would be very useful to build a "noise" model and to recommend further research on the nongaussian, nonlinear and nonquadratic aspects of optimal control theory and differential games.

(3) The sensitivity analysis of the weighting factors on the components of target as well as of control vector reveals significant changes in "amended" optimal control policies. However, in addition to its quadratic property, the penalty function is restrictive in two senses. First, all of the off-diagonal elements are zero, meaning that no interaction exists between targets and between instruments. Secondly, the welfare weights are chosen arbitrarily to perform the sensitivity analysis. Therefore, quantitative research by economists is recommended to establish the magnitude and shape of a more general penalty function.

(4) To be able to apply differential games theory, more complex and detailed models than that of Votey's using

quarterly data are required to incorporate the assumptions of decentralization in decision-making, of active responses from Country II, of conflicting targets and of policy interdependence.

(5) Many other areas of research related to differential games areas and its application to economics are interesting and appear to be fruitful. A selected bibliography on this particular topic is presented in Appendix A-13.

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APPENDICES

APPENDIX A-1

THE REDUCED FORM OF ONE-COUNTRY MODEL

The Votey's model is composed of:

5 functional equations:

(E-1) $Y_{1t} = \delta_{10}+\delta_{11} K_{1t}+\delta_{12} L_{1t}$

(E-2) $C_{1t} = \alpha_{10}+\alpha_{11} Y_{1t} - \alpha_{11} T_{1t} - \alpha_{11} \delta^* K_{1t}$

(E-3) $M_{1t} = \beta_{10}+\beta_{11} Y_{1t} - \beta_{11} T_{1t} - \beta_{11} \delta^* K_{1t}+\beta_{12} TT_{1t}; \; TT_{1t} = \left(\frac{Px}{PM}\right)_{1t}$

(E-4) $I^n_{1t} = (\gamma_{10} - \gamma_{12} \delta^*) + \gamma_{11} Y_{1t-1} + \gamma_{13} K_{1t-1} - \gamma_{12} r_{1t-1}$

(E-5) $O_{1t} = \eta_{10} + \eta_{11} r_{2t} - \eta_{11} r_{1t}$

4 identities:

(I-1) $Y_{1t} = C_{1t} + I^G_{1t} + G_{1t} + X_{1t} - M_{1t}$

(I-2) $B_{1t} = X_{1t} - M_{1t} - O_{1t}$

(I-3) $K_{1t} = I^n_{1t} + K_{1t-1}$

(I-4) $K^G_{1t} = I^n_{1t} + \delta^* K_{1t}$

Substituting (I-4), (E-2), (E-3), (E-4) into (I-1); (E-3), (E-5) into (I-2); and (E-4) into (I-3):

(I-1a) $(1-\alpha_{11}+\beta_{11})\ Y_{1t} + \delta*(\alpha_{11}-\beta_{11}-1)K_{1t} = \gamma_{11}Y_{1t-1}+\gamma_{13}\ K_{1t-1}-\gamma_{12}r_{1t-1}+G_{1t}$

$+ (\alpha_{10}+\gamma_{10}-\gamma_{12}\ \delta*-\beta_{10})-(\alpha_{11}-\beta_{11})T_{1t}+X_{1t} - \beta_{12}\ TT_{1t}$

(I-2a) $\beta_{11}Y_{1t}+\beta_{1t}-\beta_{11}\ \delta*\ K_{1t} = \eta_{11}r_{1t}-(\beta_{10}+\eta_{10}) + \beta_{11}T_{1t} + X_{1t}-\beta_{12}\ TT_{1t}$

$- \eta_{11}\ r_{2t}$

(I-3a) $K_{1t} = \gamma_{11}\ Y_{1t-1} + (1 + \gamma_{13})\ K_{1t-1}-\gamma_{12}\ r_{1t-1} + (\gamma_{10}-\gamma_{12}\ \delta*)$

Let us make the following transformation:

(I-5a) $Y_{1t} = \tilde{Y}_{1t} + \delta_{11}\ K_{1t}$

(I-6a) $Y_{1t-1} = \tilde{Y}_{1t-1} + \delta_{11}\ K_{1t-1}$

where $\tilde{Y}_{1t}$ is the potential output net of capital depreciation.

Substituting (I-5a) and (I-6a) into (I-1a), (I-2a) and (I-3a):

(I-1b) $a_{11}\ \tilde{Y}_{1t} + a_{12}\ K_{1t} = \gamma_{11}\ \tilde{Y}_{1t-1} + a_{14}\ K_{1t-1} - \gamma_{12}\ r_{1t-1} + G_{1t}$

$+ a_{16} - a_{17}\ T_{1t} + X_{1t} - \beta_{12}\ TT_{1t}$

(I-2b) $\beta_{11}\ \tilde{Y}_{1t} + B_{1t} + a_{22}\ K_{1t} = \eta_{11}\ r_{1t} - a_{26} + \beta_{11}\ T_{1t} + X_{1t}$

$- \beta_{12}\ TT_{1t} - \eta_{11}\ r_{2t}$

(I-3b) $K_{1t} = \gamma_{11}\ \tilde{Y}_{1t-1} + a_{34}\ K_{1t-1} - \gamma_{12}\ r_{1t-1} + a_{36}$

where:

$a_{11} = 1-\alpha_{11} + \beta_{11}$

$a_{12} = a_{11}\ (\delta_{11} - \delta*)$

$a_{22} = \beta_{11}\ (\delta_{11} - \delta*)$

$a_{26} = \beta_{10} + \eta_{10}$

$a_{14} = \gamma_{13} + \gamma_{11}\,\delta_{11}$

$a_{16} = \alpha_{10} + \gamma_{10} - \gamma_{12}\,\delta^* - \beta_{10}$ $\qquad$ $a_{34} = 1 + \gamma_{13} + \gamma_{11}\,\delta_{11}$

$a_{17} = \alpha_{11} - \beta_{11}$ $\qquad$ $a_{36} = \gamma_{10} - \gamma_{12}\,\delta^*$.

In matrix notation, the above set of difference equations becomes:

$$\begin{bmatrix} a_{11} & 0 & a_{12} \\ \beta_{11} & 1 & a_{22} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t} \\ \beta_{1t} \\ K_{1t} \end{bmatrix} = \begin{bmatrix} 11 & 0 & a_{14} \\ 0 & 0 & 0 \\ \gamma_{11} & 0 & a_{34} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t-1} \\ \beta_{1t-1} \\ K_{1t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \eta_{11} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{1t} \\ r_{1t} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\gamma_{12} \\ 0 & 0 \\ 0 & -\gamma_{12} \end{bmatrix} \begin{bmatrix} G_{1t-1} \\ r_{1t-1} \end{bmatrix} + \begin{bmatrix} a_{16} & -\beta_{12} & -a_{17} & 1 & 0 \\ a_{26} & -\beta_{12} & \beta_{11} & 1 & -\eta_{11} \\ a_{36} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ TT_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix}$$

Multiplying both sides of the matrix-equation by:

$$\begin{bmatrix} Q_{11} & 0 & -Q_{13} \\ -Q_{21} & 1 & Q_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & a_{12} \\ 11 & 1 & a_{22} \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

we get the reduced form for the one-country econometric model:

$$\begin{bmatrix} Y_{1t} \\ B_{1t} \\ K_{1t} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & 0 & A_{23} \\ 11 & 0 & a_{34} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t-1} \\ B_{1t-1} \\ K_{1t-1} \end{bmatrix} + \begin{bmatrix} Q_{11} & 0 \\ -Q_{21} & \eta_{11} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{1t} \\ r_{1t} \end{bmatrix} + \begin{bmatrix} 0 & A_{15} \\ 0 & A_{25} \\ 0 & -\gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} G_{1t-1} \\ r_{1t-1} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{13} & Q_{11} & 0 \\ D_{21} & D_{22} & D_{23} & 1-Q_{21} & -\eta_{11} \\ a_{36} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ TT_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix}$$

where:

$Q_{11} = 1/a_{11}$

$Q_{13} = a_{12}/a_{11}$

$Q_{21} = \beta_{11}/a_{11}$

$Q_{23} = \dfrac{\beta_{11}\ a_{12}}{a_{11}} - a_{22}$

$D_{11} = Q_{11}\ a_{16} - Q_{13}\ a_{36}$

$D_{12} = -Q_{11}\ \beta_{12}$

$D_{13} = -Q_{11}\ a_{17}$

$D_{21} = Q_{23}\ a_{36} - Q_{21}\ a_{16} - a_{26}$

$A_{11} = (Q_{11} - Q_{13})\ \gamma_{11}$

$A_{13} = Q_{11}\ a_{14} - Q_{13}\ a_{34}$

$A_{15} = -\gamma_{12}\ (Q_{11} - Q_{13})$

$D_{22} = \beta_{12}\ (Q_{21} - 1)$

$D_{23} = Q_{21}\ a_{17} + \beta_{11}$

$A_{21} = (-Q_{21} + Q_{23})\ \gamma_{11}$

$A_{23} = -Q_{21}\ a_{14} + Q_{23}\ a_{34}$

$A_{25} = -(-Q_{21} + Q_{23})\ \gamma_{12}$

In matrix notation, the structural form of the one-country econometric model is:

$$\text{(A-1.1)} \quad y_t = \bar{A}\ y_{t-1} + \bar{B}\ u_t + \bar{C}\ u_{t-1} + \bar{D}\ z_t$$

And using Votey's estimated structural constants:

$\alpha_{10} = 52.27$	$\beta_{10} = 14.59$	$\gamma_{10} = 31.707$	$\delta_{10} = 0$	$\eta_{10} = -0.6915$
$\alpha_{11} = 0.75$	$\beta_{11} = 0.131$	$\gamma_{11} = -0.0027$	$\delta_{11} = .2051$	$\eta_{11} = 46.279$
$\alpha_{12} = $ N.S.	$\beta_{12} = -28.35$	$\gamma_{12} = 461.7$	$\delta_{12} = 4.02$	
		$\gamma_{13} = 0.0624$	$\delta^* = .0489$	

We compute the numerical values for the matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$:

$$\bar{A} = \begin{bmatrix} -.0067 & 0 & -.00364 \\ .0092 & 0 & -.02129 \\ -.0027 & 0 & 1.0618 \end{bmatrix} ; \quad \bar{B} = \begin{bmatrix} 2.6247 & 0 \\ -.3438 & 46.279 \\ 0 & 0 \end{bmatrix} ; \quad \bar{C} = \begin{bmatrix} 0 & -1139.7064 \\ 0 & 158.7518 \\ 0 & -461.7 \end{bmatrix}$$

$$; \bar{D} = \begin{bmatrix} 121.4358 & 74.4102 & -1.6246 & 2.6247 & 0 \\ -29.9921 & 18.6032 & 0.3438 & 0.6562 & -46.279 \\ 9.1299 & 0 & 0 & 0 & 0 \end{bmatrix}$$

APPENDIX A-2

THE DETERMINISTIC LINEAR QUADRATIC TRACKING PROBLEM: A SUMMARY OF CHOW'S AND PINDYCK'S APPROACHES

A-2.1. Pindyck's Approach

R. S. Pindyck in his recent book (Pindyck:1973) studies the application of optimal control theory to the problem of short-term economic stabilization policy. He formulates the above problem as a linear quadratic tracking problem; that is, given the standard "state-space" system:

(A-2.1a) $x^P_{t+1} = A^P x^P_t + B^P u^P_t + C^P z^P_t.$

or

(A-2.1b) $x^P_{t+1} - x^P_t = F^P x^P_t + B^P u^P_t + C^P z^P_t$ with $A^P \overset{d}{=} I + F^P$ and the

initial condition:

(A-2.2) $x(0) = x_0$

The problem consists of minimizing the social welfare cost function

(A-2.3) $J^P = \frac{1}{2} \sum_{t=1}^{N} \{[x^P_t - \overline{x}^P_t]' \, Q^P \, [x^P_t - \overline{x}^P_t] + [u^P_t - \overline{u}^P_t]' \, R^P$

$[u^P_t - \overline{u}^P_t]\}$

where:

$x^P \epsilon E^n$ is the state variable vector, $u^P \epsilon E^r$ is the control variable fector, $\overline{x}^P \epsilon E^n$ and $\overline{u}^P \epsilon E^r$ are the nominal state and control vectors to be tracked, $z^P \epsilon E^s$ is the exogenous noncontrolled variable vector, $A^P \stackrel{d}{=} I + F^P$ is an n x n matrix, B^P is an n x r matrix, C^P is an n x s matrix; the elements of A^P, B^P and C^P are assumed to be known and time invariant; Q^P is an n x n positive semi-definite matrix, R^P is an r x r positive semi-definite matrix.

Using the minimum principle method, Pindyck solves the discrete linear quadratic tracking problem. In other words, for u_t^P to be minimal the following necessary conditions must be satisfied:

(A-2.4) $$\left.\frac{\delta \mathcal{H}_t}{\delta u_t^P}\right|_{x_t^{P*}} = 0$$

with the canonical equations:

(A-2.5) $$x_{t+1}^{P*} - x_t^{P*} = \left.\frac{\delta \mathcal{H}_t}{\delta \lambda_{t+1}^P}\right|_{x_t^{P*}} = F^P x_t^{P*} + B^P u_t^{P*} + C^P z_t^P$$

(A-2.6) $$\lambda_{t+1}^{P*} - \lambda_t^{P*} = \left.\frac{\delta \mathcal{H}_t}{\delta x_t^P}\right|_{x_t^{P*}} = -Q^P \left[x_t^{P*} - \overline{x}_t^{P*} \right] - F^{P'} x_{t+1}^{P*}$$

and the boundary conditions:

(A-2.2) $$x(0) = x_0$$

(A-2.7) $$x_N^{P*} = \frac{1}{2} \frac{\delta}{\delta x} \{[x_N^{P*} - \overline{x}_N^P]' \, Q^P \, [x_N^{P*} - \overline{x}_N^P]\} = Q^P \, [x_N^{P*} - \overline{x}_N^P]$$

where $\mathcal{H}$ is the Hamiltonian and is given by:

$$\text{(A-2.8)} \quad \mathcal{H}[x_t^P, x_{t+1}^P, u_t^P] = \frac{1}{2}[x_t^P - \bar{x}_t^P]' \; Q^P \; [x_t^P - \bar{x}_t^P] + \frac{1}{2}[u_t^P - \bar{u}_t^P]'$$

$$R^P [u_t^P - \bar{u}_t^P] + x_{t+1}^{P'} [F^P x_t^P + B^P u_t^P + C^P z_t^P]$$

The procedure to solve equation A-2.5 through equation A-2.8 has been exposed in details in Pindyck's book. Here, a summary of the different steps to obtain the optimal control u_t^{P*} is presented:

(i) Solve the Riccati equation given by:

$$\text{(A-2.9)} \quad K_t^P = Q^P + A^{P'} \{K_{t+1}^P - K_{t+1}^P \; B^P [R^P + B^{P'} K_{t+1}^P \; B^P] \; A^P\}$$

with the boundary condition:

$$\text{(A-2.10)} \quad K_N^P = Q^P$$

backwards in time to get values for K_t; $t = 1,\ldots,N$.

Store the resulting matrices.

(ii) Solve the tracking equation given by:

$$\text{(A-2.11)} \quad k_t^P = -A^{P'} \{K_{t+1}^P - K_{t+1}^P \; B^P [R^P + B^{P'} K_{t+1}^P \; B^P]^{-1} \; B_{t+1}^{P'} \; K_{t+1}^P\}$$

$$B^P (R^P)^{-1} B^{P'} k_{t+1}^P + A^{P'} k_{t+1}^P + A^{P'} \{K_{t+1}^P - K_{t+1}^P \; B^P [R^P +$$

$$B^{P'} K_{t+1}^P \; B^P]^{-1} \; B^{P'} K_{t+1}^P\} [B^P \bar{u}_t^P + C^P z_t^P] - Q^P \; \bar{x}_t^P$$

with the boundary condition:

$$\text{(A-2.12)} \quad k_N^P = -Q^P \bar{x}_N$$

backwards in time to get values for k_t; $t = 1,\ldots,N$. Store the resulting vectors.

(iii) Compute the optimal control $u^*(0)$ from the following equation:

$$\text{(A-2.13)}\quad u^*_t = -\left[R^P + B^{P'}K^P_{t+1}B^P\right]^{-1} B^{P'}K^P_{t+1}A^P x^{P*}_t + \left[R^P + B^{P'}K^P_{t+1}B^P\right]^{-1} B^{P'}K^P_{t+1}B^P (R^P)^{-1}B^{P'}k^P_{t+1} - (R^P)^{-1}B^{P'}k^P_{t+1} - \left[R^P + B^{P'}K^P_{t+1}B^P\right]^{-1}B^{P'}K^P_{t+1}\left[B^P\bar{u}^P_t + C^P z^P_t\right] + \bar{u}^P_t$$

using $x(0) = x_0$. Then compute $x^*(1)$ from the "state-space" equation (A-2.1a). Now $x^*(1)$ can be used in equation A-2.13 to compute $u^*(2)$, which can be used in equation A-2.1a to compute $x^*(2)$ and so on. Continue the process until all of the u^*_t, $t = 1,\ldots,N-1$ and all of the x^*_t, $t = 1,\ldots,N$ have been computed.

(iv) The optimal cost $\hat{J}^P$ can be computed from equation A-2-3. It is noted that when there is no limit on the instrument variable magnitudes that is when $R^P = 0$, the Ricatti and tracking equations are reduced to (Park:1973):

$$\text{(A-2.14)}\quad -K_t = Q^P \qquad \forall t \quad \{0,\ldots,N\}$$

$$\text{(A-2.15)}\quad k_t = -Q^P\bar{x}^P_t \qquad \forall t \quad \{0,\ldots,N\}$$

and Pindyck's result or equation A-2.13 becomes:

$$\text{(A-2.16)}\quad u^{P*}_t = -(B^{P'}Q^PB^P)^{-1} B^{P'} Q^PA^Px^{P*}_t + (B^{P'} Q^PB^P)^{-1} B^{P'}Q^P\bar{x}^P_{t+1} - (B^{P'} Q^PB^P)^{-1} B^{P'} Q^PC^Pz^P_t.$$

A-2.2. Chow's Approach

G. Chow, in a series of papers (Chow:1970a, 1970b, 1972a, 1972b, 1972c), has formulated the linear quadratic (L.Q.) optimal tracking problem in a different way from the standard one, as follows: minimize the welfare cost function

$$\text{(A-2.17)} \quad J^c = \frac{1}{2} \sum_{t=1}^{N} \{[x_t^c - \bar{x}_t^c]^T Q_t^c [x_t^c - \bar{x}_t^c]\}$$

given the dynamic system

$$\text{(A-2.18)} \quad x_t^c = A_t^c x_{t-1}^c + B_t^c u_t^c + C_t^c z_t^c$$

where:

$x_t^c \stackrel{d}{=}$ a vector of current of lagged dependent variables as well as current and lagged control variables.

$\bar{x}_t^c \stackrel{d}{=}$ a target vector which has the same dimension as x_t^c.

$u_t^c \stackrel{d}{=}$ a vector of controlled variables.

$z_t^c \stackrel{d}{=}$ a vector including the exogenous and noncontrolled variables and the constant term. It is assumed to be given constant.

$Q_t^c \stackrel{d}{=}$ a known symmetric, diagonal and positive semi-definite matrix.

A_t^c, B_t^c, $C_t^c \stackrel{d}{=}$ given constant matrices.

Then using either Lagrange multiplier method (Chow:1972b) or Bellman's dynamic programming (Chow:1972c) he arrives at the following results:

(i) for the last period $t = N$

$$u_N^{c*} = K_N^c\, x_N^{c*} + k_N^c$$

$$x_N^{c*} = \left[A_N^c + B_N^c K_N^c\right] x_{N-1}^{c*} + C_N^c\, z_N^c + B_N^c\, k_N^c$$

where:

$$K_N^c = -\left[B_N^{cT}\, H_B^c\, B_N^c\right]^{-1} B_N^{cT}\, H_N^c\, A_N^c$$

$$k_N^c = -\left[B_N^{cT}\, H_N^c\, B_N^c\right]^{-1} B_N^{cT}\left[H_N^c\, C_N^c\, z_N^c - h_N^c\right]$$

$$H_N^c = Q_N^c$$

$$h_N^c = Q_N^c\, \bar{x}_N^c$$

(ii) for the period $t = N-1$

$$u_{N-1}^{c*} = K_{N-1}^c\, x_{N-1}^{c*} + k_{N-1}^c$$

$$x_{N-1}^{c*} = \left[A_{N-1}^c + B_{N-1}^c K_{N-1}^c\right] x_{N-2}^{c*} + C_{N-1}^c\, z_{N-1}^c + B_{N-1}^c\, k_{N-1}^c$$

where:

$$K_{N-1}^c = -\left[B_{N-1}^{c\ T}\, H_{N-1}^c\, B_{N-1}^c\right]^{-1} B_{N-1}^{c\ T}\, H_{N-1}^c\, A_{N-1}^c$$

$$k_{N-1}^c = -\left[B_{N-1}^{c\ T}\, H_{N-1}^c\, B_{N-1}^c\right]^{-1} B_{N-1}^{c\ T}\left[H_{N-1}^c\, C_{N-1}^c\, z_{N-1}^c - h_{N-1}^c\right]$$

$$H_{N-1}^c = Q_{N-1}^c + A_N^{cT}\, H_N^c\left[A_N^c + C_N^c\, K_N^c\right]$$

$$h_{N-1}^c = Q_{N-1}^c\, \bar{x}_{N-1}^c - A_{N-1}^{c\ T}\, H_N^c\left[C_N^c\, z_N^c + B_N^c\, k_N^c\right] + A_N^{cT}\, h_N^c$$

(iii) and so on until the period $t = t_1$.

It is interesting to note that for the time-invariant optimal control problem, that is, a problem in which $A_t^c = A^c$, $B_t^c = B^c$, $C_t^c = C^c$, and $Q_t^c = Q$, Chow's results are reduced to:

$$\text{(A-2.19)} \quad u_t^{c*} = K^c x_{t-1}^{c*} + k_t^c$$

where:

$$K^c = -(B^{c'} H^c B^c)^{-1} B^{c'} H^c A^c$$

$$k_t^c = -(B^{c'} H^c B^c)^{-1} \left[H^c C^c z_t^c - h_t^c \right]$$

$$H^c = Q^c + (A^c + B^c K^c)^T Q (A^c + B^c K^c) + \left[(A^c + B^c K^c)^T \right]^2 Q^c (A^c + B^c Q^c)^2 + \ldots$$

$$h_t^c = Q^c \bar{x}_t^c + (A^c + B^c K^c)^T \left[Q^c \bar{x}_{t+1}^c - H^c C^c z_{t+1}^c \right].$$

Substituting $K^c = -(B^{c'} H^c B^c)^{-1} B^{c'} H^c A^c$, the expressions H^c and h_t^c are reduced to:

$$H^c = Q^c$$

$$h_t^c = Q^c \bar{x}_t$$

Therefore, equation A-2.19 becomes:

$$\text{(A-2.20)} \quad u_t^{c*} = -(B^{c'} Q^c B^c)^{-1} B^{c'} Q^c A^c x_{t-1}^{c*} + (B^{c'} Q^c B^c)^{-1} B^{c'} Q^c \bar{x}_t^c$$

$$- (B^{c'} Q^c B^c)^{-1} B^{c'} Q^c C^c z_t^c.$$

APPENDIX A-3

ON THE EQUIVALENCE OF PINDYCK'S AND CHOW'S OPTIMAL CONTROL SOLUTION WHEN THERE IS NO CONSTRAINT ON INSTRUMENT-VARIABLE MAGNITUDES

Generally speaking, the optimal control problem consists of minimizing a social welfare cost function subject to a given dynamic "state-space" system. The main difference between Pindyck's and Chow's optimal control frameworks rests upon the definition of the "state-space" system. It will be shown that, when there is no constraint on instrument-variable magnitudes, that is, when the weights attached to the deviations from the instrument limits are zero, i.e., $R^P = 0$, both formulations lead to the same optimal result in terms of endogenous and exogenous variables of the econometric model.

Let the reduced form of a linear econometric model be of the following form:

$$\text{(A-3.1)} \quad y_t = \bar{A}\, y_{t-1} + \bar{B}\, u_t + \bar{C}\, u_{t-1} + \bar{D}\, z_t$$

where $y \in E^n$ is the endogenous variable, $u \in E^r$ is the exogenous controlled variable, $\bar{A}$ is a n x n matrix, $\bar{B}$ and $\bar{C}$ are n x r matrices, $\bar{D}$ is a n x s matrix.

First, the optimal control problem is formulated into Pindyck's terminology and then into Chow's terminology. According to Pindyck, the above macroeconometric model must be arranged into the following state-space system:

$$\text{(A-3.2)} \quad x_t^P = A^P x_{t-1}^P + B^P u_{t-1}^P + C^P z_{t-1}^P$$

where $x_t^P \overset{d}{=} \left[y_t - \bar{B}\, u_t - \bar{D}\, z_t\right] \in E^n$ is the state variable vector after transformation, $u_{t-1}^P \overset{d}{=} u_{t-1} \in E^n$ is the control vector, $z_{t-1}^P \overset{d}{=} z_{t-1} \in E^s$ is the exogenous variable vector, $A^P \overset{d}{=} \bar{A}$ is n x n matrix, $B^P \overset{d}{=} \bar{A}\,\bar{B} + \bar{C}$ is n x r matrix, $C^P \overset{d}{=} \bar{A}\,\bar{D}$ is n x s matrix. Given this "state-space" system (A-3.2) the optimal control problem consists of finding u_t^{P*} which minimizes the following cost functional:

$$\text{(A-3.3)} \quad J^P = \frac{1}{2} \sum_{t=1}^{N} \Big[(x_{t-1}^P - \bar{x}_{t-1}^P)'\, Q^P\, (x_{t-1}^P - \bar{x}_{t-1}^P)$$

$$+ (u_{t-1}^P - \bar{u}_{t-1}^P)'\, R^P\, (u_{t-1}^P - \bar{u}_{t-1}^P) \Big]$$

where $\bar{x}_{t-1}^P \overset{d}{=} \left[\bar{Y}_{t-1} - \bar{B}\, u_{t-1} - \bar{D}\, z_{t-1}\right] \in E^n$ is the nominal state vector, $\bar{u}_{t-1}^P \overset{d}{=} \bar{u}_{t-1} \in E^r$ is the nominal control vector, Q^P is (n x n) symmetric, positive definite matrix, R^P is (r x r) symmetric, positive definite matrix. Under the assumption of $R^P = 0$, Pindyck's optimal solution is (Park:1973):

$$\text{(A-3.4)} \quad u_{t-1}^{P*} = -(B^{P'} Q^P B^P)^{-1} B^{P'} Q^P A^P x_{t-1}^{P*} + (B^{P'} Q^P B^P)^{-1} B^{P'} Q^P \bar{x}_t^P$$

$$- (B^{P'} Q^P B^P)^{-1} B^{P'} Q^P C^P z_{t-1}^P$$

Now in Chow's terminology [Chow:1972(a,b)] the deterministic optimal control problem is to find u_t^{c*} which minimizes the following welfare cost function

$$\text{(A-3.5)} \quad J^c = \frac{1}{z} \sum_{t=1}^{N} (x^c - \overline{x}_t^c)' \, Q^c \, (x^c - \overline{x}_t^c)$$

given the dynamic system

$$\text{(A-3.6)} \quad x_t^c = A^c x_{t-1}^c + B^c u_t^c + C^c z_t^c$$

where $x_t^c \overset{d}{=} \left[y_t \,\vdots\, u_t \right]' \in E^{n+r}$ is the vector of current endogenous and controlled variable, $u_t^c \overset{d}{=} u_t \in E^r$ is the control vector, $z_t^c \overset{d}{=} z_t \in E^s$ is the vector of exogenous noncontrolled variables, $\overline{x}_t^c \overset{d}{=} \left[y_t \,\vdots\, \overline{u}_t \right]' \in E^{n+r}$ is the vector of nominal endogenous and controlled variables;

where

$$A^c \overset{d}{=} \left[\begin{array}{c:c} \overline{A} & \overline{C} \\ nxn & nxr \\ \hdashline 0 & 0 \\ rxn & mxr \end{array} \right] \quad \text{is } (n+r) \text{ x } (n+r) \text{ matrix}$$

$$B^c \overset{d}{=} \left[\begin{array}{c} \overline{B} \\ nxr \\ \hdashline I \\ rxr \end{array} \right] \quad \text{is } (n+r) \text{ x r matrix}$$

$$C^c \stackrel{d}{\equiv} \begin{bmatrix} \overline{D} \\ n\times s \\ \hdashline 0 \\ r\times s \end{bmatrix} \quad \text{is } (n+r) \times s \text{ matrix}$$

$$Q^c \stackrel{d}{\equiv} \left[\begin{array}{c:c} Q^P & 0 \\ n\times r & n\times r \\ \hdashline 0 & 0 \\ r\times n & r\times r \end{array}\right] \quad \text{is } (n+r) \times (n+r) \text{ symmetric and positive definite matrix}$$

Under the assumption of time invariant coefficient matrices, Chow's optimal solution is:

$$\text{(A-3.7)} \quad u_t^{c*} = -(B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^cA^c\overline{x}_{t-1}^{c*} + (B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^c\overline{x}_t^c - (B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^cC^cz_t^c$$

And now it is shown that under the assumption of no constraints on instrument-variable magnitudes, i.e., $R^P = 0$, Pindyck's approach leads to the same optimal solution as Chow's approach.

First expressing Chow's optimal control solution in terms of y_t, u_t, z_t, $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ and Q^P, equation A-3.7 becomes:

$$u_t^* = - \left[(\overline{B}' \;\vdots\; I) \begin{pmatrix} Q^P & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \begin{pmatrix} \overline{B} \\ I \end{pmatrix} \right]^{-1} (\overline{B}' \;\vdots\; I) \begin{bmatrix} Q^P & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} \overline{A} & \vdots & \overline{C} \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} Y_{t-1}^* \\ \cdots \\ u_{t-1}^* \end{bmatrix} + \left[(\overline{B}' \;\vdots\; I) \begin{pmatrix} Q^P & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \begin{pmatrix} \overline{B} \\ \cdots \\ I \end{pmatrix} \right]^{-1}$$

$$(\overline{B}' \;\vdots\; I) \begin{bmatrix} Q^P & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} \overline{y}_t \\ \cdots \\ \overline{u}_t \end{bmatrix} - (\overline{B}' \;\vdots\; I) \begin{pmatrix} Q^P & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \begin{pmatrix} \overline{B} \\ \cdots \\ I \end{pmatrix} \Bigg]^{-1}$$

$$\left[(\overline{B}' \;\vdots\; I) \begin{pmatrix} Q^P & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \begin{pmatrix} \overline{D} \\ \cdots \\ 0 \end{pmatrix} \right] z_t$$

(A-3.9)
$$u_t^* = - \left[(\overline{B}'Q^P\overline{B})^{-1}\, \overline{B}'Q^P\overline{A} \;\vdots\; (\overline{B}'Q^P\overline{B})^{-1}\, \overline{B}'Q^P\overline{C} \right] \begin{bmatrix} y_{t-1}^* \\ \cdots \\ u_{t-1}^* \end{bmatrix}$$

$$+ \left[(\overline{B}'Q^P\overline{B})^{-1}\, \overline{B}'Q^P \;\vdots\; 0 \right] \begin{bmatrix} \overline{y}_t \\ \cdots \\ \overline{u}_t \end{bmatrix} - (\overline{B}'Q^P B)^{-1}\, \overline{B}'Q^P\overline{D}\, z_t$$

or

$$\text{(A-3.10)} \quad u_t^* = -(\bar{B}'Q^P\bar{B})^{-1}\,\bar{B}'Q^P\bar{A}\,y_{t-1}^* + (\bar{B}'Q^P\bar{B})^{-1}\,\bar{B}'Q^P\,\bar{y}_t$$

$$- (\bar{B}'Q^P\bar{B})^{-1}\,\bar{B}'Q^P\bar{C}\,u_{t-1}^* - (\bar{B}Q^P\bar{B})^{-1}\,\bar{B}'Q^P\bar{D}\,z_t$$

Then substituting $x_{t-1}^{P*} = y_{t-1}^* - \bar{B}\,u_{t-1}^* - \bar{D}\,z_{t-1}$ and $\bar{x}_t^P = \bar{y}_t - \bar{B}\,u_t^* - \bar{D}\,z_t$

into equation A-3.10 , it becomes:

$$\text{(A-3.11)} \quad u_{t-1}^* = -(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{A}\left[y_{t-1}^* - \bar{B}\,u_{t-1}^* - \bar{D}\,z_{t-1}\right]$$

$$+ (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\left[\bar{y}_t - \bar{B}\,u_t^* - \bar{D}\,z_t\right] - (B^{P\prime}Q^PB^P)^{-1}$$

$$B^{P\prime}Q^PC^Pz_{t-1}$$

Substituting $C^P = \bar{A}\,\bar{D}$ into equation A-3.11 and rearranging the terms:

$$\text{(A-3.12)} \quad u_{t-1}^* = -(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{A}\,y_{t-1}^* + (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{A}\,\bar{B}\,u_{t-1}^*$$

$$+ (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}\,Q^P\bar{y}_t - (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{B}\,u_t^*$$

$$- (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{D}\,z_t$$

Then u_t^* is given by:

$$\text{(A-3.13)} \quad \left[(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{B}\right]u_t^* = -(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{A}\,y_{t-1}^*$$

$$+ \left[(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{A}\,\bar{B} - I\right]u_{t-1}^* + (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{y}_t$$

$$- (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{D}\,z_t.$$

$$(\text{A-3.14})\quad u_t^* = \left[(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{B}\right]^{-1}\{-(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{A}\;y_{t-1}^*$$

$$+\left[(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{A}\;\bar{B}-I\right]u_{t-1}^* + (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{y}_t$$

$$-(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{D}\;z_t\}\;.$$

Let us examine the following expression:

$$(\text{A-3.15})\quad (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{B} \equiv (B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P(Q^P)^{-1}\;(\bar{B}')^{-1}\;(\bar{B}'\;Q^P\bar{B})$$

And the inverse of expression A-3.15 is:

$$(\text{A-3.16})\quad \left[(B^{P\prime}Q^PB^P)^{-1}\,B^{P\prime}Q^P\bar{B}\right]^{-1} = (\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P(Q^P)^{-1}(B^{P\prime})^{-1}\;(B^{P\prime}Q^PB^P)$$

Substituting A-3.16 into equation A-3.14:

$$(\text{A-3.17})\quad u_t^* = -(\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P(Q^P)^{-1}(B^P)^{-1}\;(B^{P\prime}Q^PB^P)\;(B^{P\prime}Q^PB^P)^{-1}$$

$$B^{P\prime}Q^P\bar{A}\;y_{t-1}^* + (\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P(Q^P)^{-1}\;(B^{P\prime})^{-1}\;(B^{P\prime}Q^PB^P)$$

$$(B^{P\prime}Q^PB^P)^{-1}\;B^{P\prime}Q^P\bar{A}\;\bar{B}\;u_{t-1}^* - (\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P(Q^P)^{-1}$$

$$(B^{P\prime})^{-1}\;(B^{P\prime}Q^PB^P)\;u_{t-1}^* + (\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P(Q^P)^{-1}\;(B^{P\prime})^{-1}$$

$$(B^{P\prime}Q^PB^P)\;(B^{P\prime}Q^PB^P)^{-1}\;B^{P\prime}Q^P\;\bar{y}_t - (\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P$$

$$(Q^P)^{-1}\;(B^{P\prime})^{-1}\;(B^{P\prime}Q^PB^P)\;(B^{P\prime}Q^PB^P)^{-1}\;B^{P\prime}Q^P\bar{D}\;z_t$$

Rearranging the terms, equation A-3.17 becomes:

$$(\text{A-3.18})\quad u_t^* = -(\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P\bar{A}\;y_{t-1}^* + (\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P(\bar{A}\;\bar{B}-B^P)\;u_{t-1}^*$$

$$+(\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P\;\bar{y}_t - (\bar{B}'Q^P\bar{B})^{-1}\;\bar{B}'Q^P\bar{D}\;z_t$$

Substituting $B^P = \bar{A}\,\bar{B} + \bar{C}$ into equation A-3.18:

$$(\text{A-3.19}) \quad u^*_t = -(\bar{B}'Q^P\bar{B})^{-1}\,\bar{B}'Q^P\bar{A}\;y^*_{t-1} - (\bar{B}'Q^P\bar{B})^{-1}\,\bar{B}'Q^P\bar{C}\;u^*_{t-1}$$

$$+ (\bar{B}'Q^P\bar{B})^{-1}\,\bar{B}'Q^P\;\bar{y}_t - (\bar{B}'Q^P\bar{B})\;\bar{B}'Q^P\bar{D}\;z_t \qquad \text{Q.E.D.}$$

Therefore, Pindyck's optimal solution expressed in terms of the endogenous, exogenous-controlled and noncontrolled-variables is equivalent to that of Chow (equation A-3.10) even if the formulation of the state-space system is different. This equivalence verifies the property of uniqueness of the optimal control; that is, different formulations or approaches always lead to the same unique solution.

APPENDIX A-4

COMPUTATIONS OF THE OPTIMAL SOLUTION FOR THE ONE-COUNTRY CONTROL PROBLEM

Chow's optimal solution is given by equation A-2.20 (Appendix A-2):

$$\text{(A-2.20)} \quad u_t^* = -(B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^cA^cx_{t-1}^{c*} + (B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^c\overline{x}_t^c$$
$$- \, (B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^cC^cz_t^c.$$

where

$$x_t^c \overset{d}{=} \left[\tilde{Y}_{1t}^*,\ B_{1t}^*,\ K_{1t}^*,\ G_{1t}^*,\ r_{1t}^*\right]'$$

$$u_t^{c*} \overset{d}{=} \left[G_{1t}^*,\ r_{1t}^*\right]'$$

$$z_t^c \overset{d}{=} \left[1,\ TT_{1t},\ T_{1t},\ X_{1t},\ r_{2t}\right]'$$

$$\overline{x}_t^c \overset{d}{=} \left[\overline{\tilde{Y}}_{1t},\ \overline{B}_{1t},\ \overline{K}_{1t},\ \overline{G}_{1t},\ \overline{r}_{1t}\right]' \overset{d}{=} \left[\delta_{10}+\delta_{12}L_{1t},\ 0,\ \overline{K}_{1t},\ \overline{G}_{1t},\ \overline{r}_{1t}\right]'$$

$$A^c = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 & A_{15} \\ A_{21} & 0 & A_{23} & 0 & A_{25} \\ \gamma_{11} & 0 & a_{34} & 0 & -\gamma_{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ;B^c = \begin{bmatrix} Q_{11} & 0 \\ -Q_{21} & \eta_{11} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$;C^c = \begin{bmatrix} D_{11} & D_{12} & D_{13} & Q_{11} & 0 \\ D_{21} & D_{22} & D_{23} & 1-Q_{21} & -\eta_{11} \\ a_{36} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ;Q^c = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Computing

$$B^{c\prime}Q^c = \begin{bmatrix} Q_{11} & -Q_{21} & 0 & 1 & 0 \\ 0 & \eta_{11} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} q_{11}Q_{11} & -q_{22}Q_{21} & 0 & 0 & 0 \\ 0 & q_{22}\ {}_{11} & 0 & 0 & 0 \end{bmatrix}$$

$$B^{c\prime}Q^cB^c = \begin{bmatrix} q_{11}Q_{11} & -q_{22}Q_{21} & 0 & 0 & 0 \\ 0 & q_{22}\ {}_{11} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{11} & 0 \\ -Q_{21} & \eta_{11} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} q_{11}\,Q_{11}^2 + q_{22}\,Q_{21}^2 & -q_{22}\,\eta_{11}\,Q_{21} \\ -q_{22}\eta_{11}\,Q_{21} & q_{22}\,\eta_{11}^2 \end{bmatrix}$$

$$(B^{c\prime}Q^cB^c)^{-1} = \begin{bmatrix} \dfrac{1}{q_{11}\,Q_{11}^2} & \dfrac{Q_{21}}{q_{11}\,\eta_{11}\,Q_{11}^2} \\ \dfrac{Q_{21}}{q_{11}\eta_{11}Q_{11}^2} & \dfrac{1}{q_{22}\eta_{11}^2} + \dfrac{Q_{21}}{q_{11}\eta_{11}^2Q_{11}^2} \end{bmatrix}$$

Now

(A-4.1) $$(B^{c\prime}Q^cB^c)^{-1}\,B^{c\prime}Q^c = \begin{bmatrix} \frac{1}{q_{11}Q_{11}^2} & \frac{Q_{21}}{q_{11}\eta_{11}Q_{11}^2} \\ \frac{Q_{21}}{q_{11}\eta_{11}Q_{11}^2} & \frac{1}{q_{22}\eta_{11}^2} + \frac{Q_{21}^2}{q_{11}\eta_{11}^2Q_{11}^2} \end{bmatrix} \begin{bmatrix} q_{11}Q_{11} & -q_{22}Q_{21} & 0 & 0 & 0 \\ 0 & q_{22}\eta_{11} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{Q_{11}} & 0 & 0 & 0 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix}$$

(A-4.2) $$(B^{c\prime}Q^cB^c)^{-1}\,B^{c\prime}Q^cA^c = \begin{bmatrix} \frac{1}{Q_{11}} & 0 & 0 & 0 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & 0 & A_{13} & 0 & A_{15} \\ A_{21} & 0 & A & 0 & A_{25} \\ \gamma_{11} & 0 & a_{34} & 0 & -\gamma_{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{A_{11}}{Q_{11}} & 0 & \frac{A_{13}}{Q_{11}} & 0 & \frac{A_{15}}{Q_{11}} \\ \frac{1}{\eta_{11}}\left(\frac{A_{11}}{Q_{11}}Q_{21}+A_{21}\right) & 0 & \frac{1}{\eta_{11}}\left(\frac{A_{13}}{Q_{11}}Q_{23}+A_{23}\right) & 0 & \frac{1}{\eta_{11}}\left(\frac{A_{15}}{Q_{11}}Q_{21}+A_{25}\right) \end{bmatrix}$$

(A-4.3) $$(B^{c\prime}Q^cB^c)^{-1}\,B^{c\prime}Q^cC^c = \begin{bmatrix} \frac{1}{Q_{11}} & 0 & 0 & 0 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} & Q_{11} & 0 \\ D_{21} & D_{22} & D_{23} & 1-Q_{21} & -\eta_{11} \\ a_{36} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{D_{11}}{Q_{11}} & \frac{D_{12}}{Q_{11}} & \frac{D_{13}}{Q_{11}} & 1 & 0 \\ \frac{1}{\eta_{11}}\left(\frac{D_{11}}{Q_{11}}Q_{21}+D_{21}\right) & \frac{1}{\eta_{11}}\left(\frac{D_{12}}{Q_{11}}Q_{21}+D_{22}\right) & \frac{1}{\eta_{11}}\left(\frac{D_{13}}{Q_{11}}Q_{21}+D_{23}\right) & \frac{1}{\eta_{11}} & -1 \end{bmatrix}$$

Substituting $Q_{11} = \frac{1}{a_{11}}$; $Q_{21} = \frac{\beta_{11}}{a_{11}}$ (Appendix A-1) into $(B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^c$, $(B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^cA^c$ and $(B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^cC^c$ we get:

$$\text{(A-4.4)} \quad (B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^c = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ \frac{\beta_{11}}{\eta_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(A-4.5)} \quad (B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^cA^c = \begin{bmatrix} a_{11}A_{11} & 0 & a_{11}A_{13} & 0 & a_{11}A_{15} \\ \frac{1}{\eta_{11}}(\beta_{11}A_{11}+A_{21}) & 0 & \frac{1}{\eta_{11}}(\beta_{11}A_{13}A_{23}) & 0 & \frac{1}{\eta_{11}}(\beta_{11}A_{15}+A_{25}) \end{bmatrix}$$

$$\text{(A-4.6)} \quad (B\ 'Q\ B\ )^{-1}B\ 'Q\ C \ = \begin{bmatrix} a_{11}D_{11} & a_{11}D_{12} & a_{11}D_{13} & 1 & 0 \\ \frac{1}{\eta_{11}}(\beta_{11}D_{11}+D_{21}) & \frac{1}{\eta_{11}}(\beta_{11}D_{12}+D_{22}) & \frac{1}{\eta_{11}}(\beta_{11}D_{13}+D_{23}) & \frac{1}{\eta_{11}} & -1 \end{bmatrix}$$

Then the optimal solution is:

$$(A\text{-}4.6)\quad \begin{bmatrix} G^*_{1t} \\ r^*_{1t} \end{bmatrix} = -\begin{bmatrix} a_{11}A_{11} & 0 & a_{11}A_{13} & 0 & a_{11}A_{15} \\ \frac{1}{\eta_{11}}(\beta_{11}A_{11}+A_{21}) & 0 & \frac{1}{\eta_{11}}(\beta_{11}A_{13}+A_{23}) & 0 & \frac{1}{\eta_{11}}(\beta_{11}A_{15}+A_{25}) \end{bmatrix} \begin{bmatrix} \bar{Y}^*_{1t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ G^*_{1t-1} \\ r^*_{1t-1} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ \frac{\beta_{11}}{\eta_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{Y}_{1t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{G}_{1t} \\ \bar{r}_{1t} \end{bmatrix}$$

$$- \begin{bmatrix} a_{11}D_{11} & a_{11}D_{12} & a_{11}D_{13} & 1 & 0 \\ \frac{1}{\eta_{11}}(\beta_{11}D_{11}+D_{21}) & \frac{1}{\eta_{11}}(\beta_{11}D_{12}+D_{22}) & \frac{1}{\eta_{11}}(\beta_{11}D_{13}D_{23}) & \frac{1}{\eta_{11}} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \Pi_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix}$$

Substituting $Y_{1t} = \delta_{10}+\delta_{12}L_{1t}$ and $\bar{B}_{1t} = 0$ into the optimal solution and rearranging the terms:

$$(A\text{-}4.7)\quad \begin{bmatrix} G^*_{1t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} -a_{11}A_{11} & 0 & -a_{11}A_{13} & 0 & -a_{11}A_{15} \\ \frac{-1}{\eta_{11}}(\beta_{11}A_{11}+A_{21}) & 0 & \frac{-1}{\eta_{11}}(\beta_{11}A_{13}+A_{23}) & 0 & \frac{-1}{\eta_{11}}(\beta_{11}A_{15}+A_{25}) \end{bmatrix} \begin{bmatrix} \bar{Y}^*_{1t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ G^*_{1t-1} \\ r^*_{1t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} -a_{11}(D_{11}-\delta_{10}) & a_{11}\delta_{12} & -a_{11}D_{12} & -a_{11}D_{13} & -1 & 0 \\ \frac{1}{\eta_{11}}(\beta_{11}D_{11}-\beta_{11}\delta_{10}+D_{21}) & \frac{\beta_{11}\delta_{12}}{\eta_{11}} & \frac{1}{\eta_{11}}(\beta_{11}D_{12}+D_{22}) & \frac{1}{\eta_{11}}(\beta_{11}D_{13}+D_{23}) & \frac{-1}{\eta_{11}} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ L_{1t} \\ \Pi_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix},$$

APPENDIX A-5

EVALUATION OF THE WELFARE COST FOR THE ONE-COUNTRY CONTROL PROBLEM

The welfare cost function is given by:

$$\text{(A-5.1)}\quad \hat{J}^c = \frac{1}{2}\sum_{t=1}^{N} (x_t^{c*} - \bar{x}_t^c)' Q^c (x_t^{c*} - \bar{x}_t^c)$$

First substituting the optimal solution

$$\text{(A-5.2)}\quad u_t^{c*} = -(B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^cA^c x_{t-1}^{c*} + (B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^c\bar{x}_t^c$$
$$- (B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^cC^cz_t^c$$

into the dynamic system

$$\text{(A-5.3)}\quad x_t^{c*} = A^c x_{t-1}^{c*} + B^c u_t^{c*} + C^cz_t^c$$

we get:

$$\text{(A-5.4)}\quad x_t^{c*} = A^cx_{t-1}^{c*} + B^c\Big[-(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^cA^cx_{t-1}^{c*} + (B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^c\bar{x}_t^c$$
$$- (B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^cC^cz_t^c\Big] + C^cz_t^c$$
$$= \Big[I-B\,(B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^c\Big] A^cx_{t-1}^{c*} + B^c(B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^c\, x_t^c$$
$$+ \Big[I-B^c(B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^c\Big] C^cz_t^c$$

And

$$(A\text{-}5.5)\quad x_t^{c*} - \overline{x}_t^c = \left[I - B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]A^c x_{t-1}^{c*} + \left[B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c - I^c\right]\overline{x}_t^c$$

$$+ \left[I - B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]C^c z_t^c$$

Let us compute

$$B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c = \begin{bmatrix} Q_{11} & 0 \\ -Q_{21} & \eta_{11} \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{Q_{11}} & 0 & 0 & 0 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{Q_{11}} & 0 & 0 & 0 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix}$$

$$I-B^{c}(B^{c\prime}Q^{c}B^{c})^{-1}B^{c\prime}Q^{c} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{Q_{11}} & 0 & 0 & 0 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{-1}{Q_{11}} & 0 & 0 & 1 & 0 \\ \frac{-Q_{21}}{\eta_{11}Q_{11}} & \frac{-1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix}$$

$$\left[I-B^{c}(B^{c\prime}Q^{c}B^{c})^{-1}B^{c\prime}Q^{c}\right]A^{c} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{-1}{Q_{11}} & 0 & 0 & 1 & 0 \\ \frac{-Q_{21}}{\eta_{11}Q_{11}} & \frac{-1}{\eta_{11}} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & 0 & A_{13} & 0 & A_{15} \\ A_{21} & 0 & A_{23} & 0 & A_{25} \\ \gamma_{11} & 0 & a_{34} & 0 & -\gamma_{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \gamma_{11} & 0 & a_{34} & 0 & -\gamma_{12} \\ -A_{11}/Q_{11} & 0 & \frac{-A_{13}}{Q_{11}} & 0 & \frac{-A_{15}}{Q_{11}} \\ \frac{1}{\eta_{11}}\left(\frac{-A_{11}}{Q_{11}}Q_{21}-A_{21}\right) & 0 & \frac{1}{\eta_{11}}\left(\frac{A_{13}}{Q_{11}}Q_{21}-A_{23}\right) & 0 & \frac{1}{\eta_{11}}\left(\frac{A_{15}}{Q_{11}}Q_{21}-A_{25}\right) \end{bmatrix}$$

$$\left[I - B\,(B\,'Q\,B\,)^{-1}B\,'Q\right] C^{c} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{Q_{11}} & 0 & 0 & 1 & 0 \\ \frac{-Q_{21}}{\eta_{11}Q_{11}} & \frac{-1}{\eta_{11}} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} & Q_{11} & 0 \\ D_{21} & D_{22} & D_{23} & 1-Q_{21} & -\gamma_{11} \\ a_{36} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{36} & 0 & 0 & 0 & 0 \\ \frac{-D_{11}}{Q_{11}} & \frac{-D_{12}}{Q_{11}} & \frac{-D_{13}}{Q_{11}} & -1 & 0 \\ \frac{1}{\eta_{11}}\left(\frac{D_{11}}{Q_{11}}Q_{21}-D_{21}\right) & \frac{1}{\eta_{11}}\left(\frac{D_{12}}{Q_{11}}Q_{21}-D_{21}\right) & \frac{1}{\eta_{11}}\left(\frac{D_{13}}{Q_{11}}Q_{21}-D_{23}\right) & \frac{-1}{\eta_{11}} & 1 \end{bmatrix}$$

$$B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c-I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{Q_{11}} & 0 & 0 & 0 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \frac{1}{Q_{11}} & 0 & 0 & -1 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & -1 \end{bmatrix}$$

Then equation A-6.5 becomes after substitution of Votey's variables:

$$\text{(A-5.6)}\quad \begin{bmatrix} \tilde{Y}^*_{1t} \\ B^*_{1t} \\ K^*_{1t} \\ G^*_{1t} \\ r^*_{1t} \end{bmatrix} - \begin{bmatrix} \tilde{Y}_{1t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{G}_{1t} \\ \bar{r}_{1t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \gamma_{11} & 0 & a_{34} & 0 & -\gamma_{12} \\ \frac{-A_{11}}{Q_{11}} & 0 & \frac{-A_{13}}{Q_{11}} & 0 & \frac{-A_{15}}{Q_{11}} \\ \frac{1}{\eta_{11}}\left(\frac{-A_{11}}{Q_{11}}Q_{21}-A_{21}\right) & 0 & \frac{1}{\eta_{11}}\left(\frac{-A_{13}}{Q_{11}}Q_{21}-A_{23}\right) & 0 & \frac{1}{\eta_{11}}\left(\frac{-A_{15}}{Q_{11}}Q_{21}-A_{25}\right) \end{bmatrix} \begin{bmatrix} \tilde{Y}^*_{1t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ G^*_{1t-1} \\ r^*_{1t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \frac{1}{Q_{11}} & 0 & 0 & -1 & 0 \\ \frac{Q_{21}}{\eta_{11}Q_{11}} & \frac{1}{\eta_{11}} & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{G}_{1t} \\ \bar{r}_{1t} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{36} & 0 & 0 & 0 & 0 \\ \frac{-D_{11}}{Q_{11}} & \frac{-D_{12}}{Q_{11}} & \frac{-D_{13}}{Q_{11}} & -1 & 0 \\ \frac{1}{\eta_{11}}\left(\frac{-D_{11}}{Q_{11}}(Q_{21}-D_{21}\right) & \frac{1}{\eta_{11}}\left(\frac{-D_{12}}{Q_{11}}Q_{21}-D_{22}\right) & \frac{1}{\eta_{11}}\left(\frac{-D_{13}}{Q_{11}}Q_{21}-D_{23}\right) & \frac{-1}{\eta_{11}} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ TT_{1t} \\ T_{1t} \\ X_{1t} \\ r_{2t} \end{bmatrix}$$

Since

$$Q^c = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

(A-5.7) $(x_t^{c*} - \overline{x}_t^c)' \, Q^c (x_t^{c*} - \overline{x}_t^c) = 0$

Therefore

$$\hat{J} = \frac{1}{2} \sum_{t=1}^{N} (x_t^{c*} - \overline{x}_t^c)' \, Q^c (x_t^{c*} - \overline{x}_t^c) = 0 \qquad \text{Q.E.D.}$$

APPENDIX A-6

THE REDUCED FORM OF THE TWO-COUNTRY MODEL (CASE A)

Under the assumption of common external balance, the variables of Votey's econometric model are classified as follows:

(1) Endogenous variables: B_{it}, C_{it}, I^G_{it}, I^n_{it}, K_{it}, M_{it}, O_{1t}, Y_{it}, $i = 1,2$.

(2) Controlled Exogenous Variables: G_{1t}, G_{2t}, r_{1t}.

(3) Noncontrolled Exogenous Variables: IE_{1t}, L_{it}, P_{Mit}, P_{xit}, r_{2t}, T_{it}, TR_{2t}, X_{it}, M_{3t}, M^{III}_{it}, $i = 1,2$

And the structural model is constituted of: 9 equations:

(E-1) $Y_{1t} = \delta_{10} + \delta_{11} K_{1t} + \delta_{12} L_{1t}$

(E-2) $C_{1t} = \alpha_{10} + \alpha_{11} Y_{1t} - \alpha_{11} T_{1t} - \alpha_{11} \delta^* K_{1t} + TT_{1t}$

(E-3) $M_{1t} = \beta_{10} + \beta_{11} Y_{1t} - \beta_{11} T_{1t} - \beta_{11} \delta^* K_{1t} + \beta_{12} TT_{1t} +$;

$$TT_{1t} = \left(\frac{P_M}{P_x}\right)_{1t}$$

(E-4) $I^n_{1t} = \gamma_{10} + \gamma_{11} Y_{1t-1} - \gamma_{12} \delta^* - \gamma_{12} r_{1t-1} + \gamma_{13} K_{1t-1}$

(E-5) $O_{1t} = \eta_{10} + \eta_{11} r_{2t} - \eta_{11} r_{1t}$

(E-6) $Y_{2t} = \delta_{20} + \delta_{21} K_{2t} + \delta_{22} L_{2t}$

(E-7) $C_{2t} = \alpha_{20} + \alpha_{21} Y_{2t} + \alpha_{21} T_{2t} - \alpha_{21} TR_{2t} - \alpha_{21} \delta^* K_{2t}$

(E-8) $M_{2t} = \beta_{20} + \beta_{21} Y_{2t} - \beta_{21} T_{2t} - \beta_{21} TR_{2t} - \beta_{21} \delta^* K_{2t}$

$+ \beta_{22} TT_{2t}$; $TT_{2t} = \left(\frac{P_M}{P_X}\right) 2t$

(E-9) $I^n_{2t} = \gamma_{20} + \gamma_{21} Y_{2t-1} - \gamma_{22} \delta^* - \gamma_{22} r_{2t-1} + \gamma_{23} K_{2t-1}$

7 Identities:

(I-1) $Y_{1t} = C_{1t} + I^G_{1t} + G_{1t} + (M_{2t} + M_{3t} + IE_{1t}) - M_{1t}$

(I-2) $K_{1t} = I^n_{1t} + K_{1t-1}$

(I-3) $I^G_{1t} = I^n_{1t} + \delta^* K_{1t}$

(I-4) $B_{1t} = (M_{2t} + M_{3t} + IE_{1t}) - M_{1t} - O_{1t}$

(I-5) $Y_{2t} = C_{2t} + I^G_{2t} + G_{2t} + (M_{1t} - M^{III}_{1t}) - M_{2t}$

(I-6) $K_{2t} = I^n_{2t} + K_{2t-1}$

(I-7) $I^G_{2t} = I^n_{2t} + \delta^* K_{2t}$

Substituting {(I-3), (E-2), (E-3), (E-4)} into (I-1); {(I-7), (E-7), (E-3), (E-8), (E-9)} into (I-5); {E-8, E-3, E-5} into (I-4); (E-4) into (I-2); (E-9) into (I-6) and regrouping the terms, we get the following system of equations:

$$\text{(I-1a)}\quad (1-\alpha_{11} + \beta_{11})Y_{1t} - \beta_{21} Y_{2t} - \delta^*(1+\alpha_{11} + \beta_{11}) K_{1t} + \beta_{21} \delta^* K_{2t}$$

$$= \gamma_{11} Y_{1t-1} + \gamma_{13} K_{1t-1} + G_{1t} - \gamma_{12} r_{1t-1}$$

$$+ (\alpha_{10} - \beta_{10} + \beta_{20} + \gamma_{10} - \gamma_{12} \delta^*) + IE_{1t} - \beta_{12} TT_{1t}$$

$$+ (\beta_{11} - \alpha_{11}) T_{1t} + \beta_{22} TT_{2t} - \beta_{21} T_{2t} + \beta_{21} TR_{2t} + M_{3t}$$

$$\text{(I-5a)}\quad -\beta_{11} Y_{1t} + (1-\alpha_{21} + \beta_{21}) Y_{2t} + \beta_{11} \delta^* K_{1t} - \delta^* (1-\alpha_{21} + \beta_{21}) K_{2t}$$

$$= \gamma_{21} Y_{2t-1} + \gamma_{23} K_{2t-1} + G_{2t} - \gamma_{22} r_{2t-1}$$

$$+ (\alpha_{20} + \beta_{10} - \beta_{20} + \gamma_{20} - \gamma_{22} \delta^*) - M_{1t}^{III} + \beta_{12} TT_{1t}$$

$$- \beta_{11} T_{1t} - \beta_{22} TT_{2t} + (\beta_{21} - \alpha_{21}) T_{2t} + (\beta_{21} - \alpha_{21}) TR_{2t}$$

$$\text{(I-4a)}\quad \beta_{11} Y_{1t} - \beta_{21} Y_{2t} + B_{1t} - \beta_{11} \delta^* K_{1t} + \beta_{21} \delta^* K_{2t}$$

$$= \eta_{11} r_{1t} - \eta_{11} r_{2t} + (\beta_{20} - \beta_{10} - \eta_{10}) + IE_{1t} - \beta_{12} TT_{1t}$$

$$+ \beta_{11} T_{1t} + \beta_{22} TT_{2t} - \beta_{21} T_{2t} - \beta_{21} TR_{2t} + M_{3t}$$

$$\text{(I-2a)}\quad K_{1t} = \gamma_{11} Y_{1t-1} + (1 + \gamma_{13}) K_{1t-1} - \gamma_{12} r_{1t-1} + (\gamma_{10} - \gamma_{12} \delta^*)$$

$$\text{(I-6a)}\quad K_{2t} = \gamma_{21} Y_{2t-1} + (1 + \gamma_{23}) K_{2t-1} - \gamma_{22} r_{2t-1} + (\gamma_{20} - \gamma_{22} \delta^*)$$

Then doing the same transformation of variables as for the case of the one-country model:

$$Y_{it} = \delta_{i0} + \delta_{i1} K_{it} + \delta_{i2} L_{it} \qquad i = 1,2$$

or (I-7a) $Y_{it} - \delta_{i1} K_{it} = \delta_{i0} + \delta_{i2} L_{it}$

let (I-7b) $\tilde{Y}_{it} = Y_{it} - \delta_{i1} K_{it} = \delta_{i0} + \delta_{i2} L_{it}$

then (I-8) $Y_{it} = \tilde{Y}_{it} + \delta_{i1} K_{it}$

(I-9) $Y_{it-1} = \tilde{Y}_{it-1} + \delta_{i1} K_{it-1}$

substituting (I-8) and (I-9) into the above system of equations:

(I-1b) $(1-\alpha_{11} + \beta_{11}) \tilde{Y}_{1t} - \beta_{21} \tilde{Y}_{2t} - (1-\alpha_{11} + \beta_{11}) (\delta^* - \delta_{11}) K_{1t}$

$+ \beta_{21} (\delta^* - \delta_{21}) K_{2t} = \gamma_{11} \tilde{Y}_{1t-1} + (\gamma_{11} \delta_{11} + \gamma_{13}) K_{1t-1}$

$+ G_{1t} - \gamma_{12} r_{1t-1} + (\alpha_{10} - \beta_{10} + \beta_{20} + \gamma_{10} - \gamma_{12} \delta^*) + IE_{1t}$

$- \beta_{12} TT_{1t} + (\beta_{11} - \alpha_{11}) T_{1t} + \beta_{22} TT_{2t} - \beta_{21} T_{2t} - \beta_{21} TR_{2t} + M_{3t}$

(I-5b) $-\beta_{11} \tilde{Y}_{1t} + (1-\alpha_{21} + \beta_{21}) \tilde{Y}_{2t} + \beta_{11}(\delta^* - \delta_{11}) K_{1t}$

$- (1-\alpha_{21} + \beta_{21}) (\delta^* - \delta_{21}) K_{2t} = \gamma_{21} \tilde{Y}_{2t-1} + (\gamma_{21} + \delta_{21} + \gamma_{23}) K_{2t-1}$

$+ G_{2t} - \gamma_{22} r_{2t-1} + (\alpha_{20} + \beta_{10} - \beta_{20} + \gamma_{20} - \gamma_{22} \delta^*) - M_{1t}^{III}$

$+ \beta_{12} TT_{1t} - \beta_{11} T_{1t} - \beta_{22} TT_{2t} + (\beta_{21} - \alpha_{21}) T_{2t} + (\beta_{21} + \alpha_{21}) TR_{2t}$

(I-4b) $\beta_{11} \tilde{Y}_{1t} - \beta_{21} \tilde{Y}_{2t} + B_{1t} - \beta_{11} (\delta^* - \delta_{11}) K_{1t} + \beta_{21} (\delta^* - \delta_{21}) K_{2t}$

$= \eta_{11} r_{1t} - \eta_{11} r_{2t} + (\beta_{20} - \beta_{10} - \eta_{10}) + IE_{1t} - \beta_{12} TT_{1t}$

$+ \beta_{11} T_{1t} + \beta_{22} TT_{2t} - \beta_{21} T_{2t} - \beta_{21} TR_{2t} + M_{3t}$

$$\text{(I-2b)}\quad K_{1t} = \gamma_{11}\,\tilde{Y}_{1t-1} + (1 + \gamma_{13} + \gamma_{11}\,\delta_{11})\,K_{1t-1} - \gamma_{12}\,r_{1t-1} + (\gamma_{10} - \gamma_{12}\,\delta^{*})$$

$$\text{(I-6b)}\quad K_{2t} = \gamma_{21}\,\tilde{Y}_{2t-1} + (1 + \gamma_{23} + \gamma_{21}\,\delta_{21})\,K_{2t-1} - \gamma_{22}\,r_{2t-1} + (\gamma_{20} - \gamma_{22}\,\delta^{*})$$

And in matrix notation, the above system becomes:

$$
\text{(A-6.1)}\quad
\begin{bmatrix}
a_{11} & -\beta_{21} & 0 & a_{13} & a_{14} \\
-\beta_{11} & a_{22} & 0 & -a_{23} & -a_{24} \\
\beta_{11} & -\beta_{21} & 1 & -a_{33} & a_{34} \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{Y}_{1t} \\ \tilde{Y}_{2t} \\ B_{1t} \\ K_{1t} \\ K_{2t}
\end{bmatrix}
=
\begin{bmatrix}
\gamma_{11} & 0 & 0 & a_{15} & 0 \\
0 & \gamma_{21} & 0 & 0 & a_{25} \\
0 & 0 & 0 & 0 & 0 \\
\gamma_{11} & 0 & 0 & a_{45} & 0 \\
0 & \gamma_{21} & 0 & 0 & a_{55}
\end{bmatrix}
\begin{bmatrix}
\tilde{Y}_{1t-1} \\ \tilde{Y}_{2t-1} \\ B_{1t-1} \\ K_{1t-1} \\ K_{2t-1}
\end{bmatrix}
+
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \eta_{11} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
G_{1t} \\ G_{2t} \\ r_{1t}
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
a_{16} & 1 & 0 & -\beta_{12} & a_{17} & 0 & 0 & \beta_{22} & \beta_{21} & \gamma_{21} & 1 \\
a_{26} & 0 & 1 & \beta_{12} & -\beta_{11} & 0 & -\gamma_{22} & -\beta_{22} & a_{27} & a_{27} & 0 \\
a_{36} & 1 & 0 & -\beta_{12} & \beta_{11} & -\eta_{11} & 0 & \beta_{22} & -\beta_{21} & -\beta_{21} & 1 \\
a_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{56} & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t}
\end{bmatrix}
$$

where

$a_{11} = 1 - \alpha_{11} + \beta_{11}$

$a_{22} = 1 - \alpha_{21} + \beta_{21}$

$a_{13} = -(1 - \alpha_{11} + \beta_{11})(\delta_{11} - \delta^*)$

$a_{23} = \beta_{11}\ (\delta^* - \delta_{11})$

$a_{14} = \beta_{21}\ (\delta^* - \delta_{21})$

$a_{24} = (1 - \alpha_{21} + \beta_{21})(\delta^* - \delta_{21})$

$a_{15} = \gamma_{11}\ \delta_{11} + \gamma_{13}$

$a_{25} = \gamma_{21}\ \delta_{21} + \gamma_{23}$

$a_{16} = \alpha_{10} - \beta_{10} + \beta_{20} + \gamma_{10} - \gamma_{12}\delta^*$

$a_{26} = \alpha_{10} + \beta_{10} - \gamma_{20} + \gamma_{20} - {}_{22}\delta^*$

$a_{17} = \beta_{11} - \alpha_{11}$

$a_{27} = \beta_{21} - \alpha_{21}$

$a_{33} = \beta_{11}\ (\delta^* - \delta_{11})$

$a_{45} = 1 + \gamma_{13} + \gamma_{11}\ \delta_{11}$

$a_{34} = \beta_{21}\ (\delta^* - \delta_{21})$

$a_{46} = \gamma_{10} - \gamma_{12}\ \delta^*$

$a_{36} = \beta_{20} - \beta_{10} - \eta_{10}$

$a_{55} = 1 + \gamma_{23} + \gamma_{21}\ \delta_{11}$

$a_{56} = \gamma_{20} - \gamma_{22}\ \delta^*$

Then multiplying both sides of the matrix equation by:

$$\begin{bmatrix} a_{11} & -\beta_{21} & 0 & a_{13} & a_{14} \\ -\beta_{11} & a_{22} & 0 & a_{23} & -a_{24} \\ \beta_{11} & -\beta_{21} & 1 & -a_{33} & a_{34} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & Q_{14} & Q_{15} \\ Q_{21} & Q_{22} & 0 & Q_{24} & Q_{25} \\ Q_{31} & Q_{32} & 1 & Q_{34} & Q_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$|Q| = a_{11}a_{22} - \beta_{11}\beta_{21}$$

$$Q_{11} = \frac{a_{22}}{|Q|}$$

$$Q_{21} = \frac{\beta_{11}}{|Q|}$$

$$Q_{31} = \frac{\beta_{11}(\beta_{21} - a_{22})}{|Q|}$$

$$Q_{14} = \frac{-(a_{13}a_{22} + a_{23}\ \beta_{21})}{|Q|}$$

$$Q_{24} = \frac{-(a_{11}a_{23} + a_{13}\beta_{11})}{|Q|}$$

$$Q_{34} = \frac{1}{Q}\Big[-a_{11}(a_{23}\beta_{21}-a_{22}a_{33})$$

$$- \beta_{11}\beta_{21}(a_{33} - a_{23})-a_{13}\beta_{11}$$

$$(\beta_{21} - a_{22})\Big]$$

$$Q_{12} = \frac{\beta_{21}}{|Q|}$$

$$Q_{22} = \frac{a_{11}}{|Q|}$$

$$Q^{32} = \frac{\beta_{21}(a_{11}-\beta_{11})}{|Q|}$$

$$Q_{15} = \frac{a_{24}\beta_{21} - a_{14}a_{22}}{|Q|}$$

$$Q_{25} = \frac{a_{11}a_{24} - a_{14}\beta_{11}}{|Q|}$$

$$Q_{35} = \frac{1}{|Q|}\Big[-a_{11}(a_{22}a_{34}-a_{24}\beta_{21})$$

$$- \beta_{11}\beta_{21}(a_{24}-a_{34})-a_{14}\beta_{11}$$

$$(\beta_{21} - a_{22})\Big]$$

We get:

$$\begin{bmatrix} \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \\ B_{1t} \\ K_{1t} \\ K_{2t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & A_{15} \\ A_{21} & A_{22} & 0 & A_{24} & A_{25} \\ A_{31} & A_{32} & 0 & A_{34} & A_{35} \\ \gamma_{11} & 0 & 0 & a_{45} & 0 \\ 0 & \gamma_{21} & 0 & 0 & a_{55} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t-1} \\ \tilde{Y}_{2t-1} \\ B_{1t-1} \\ K_{1t-1} \\ K_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & \eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G_{1t} \\ G_{2t} \\ r_{1t} \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_{16} \\ 0 & 0 & A_{26} \\ 0 & 0 & A_{36} \\ 0 & 0 & -\gamma_{12} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G_{1t-1} \\ G_{2t-1} \\ r_{1t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & 0 & A_{17} & D_{16} & D_{17} & D_{17} & D_{18} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & 0 & A_{27} & D_{26} & D_{27} & D_{27} & D_{28} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & -\gamma_{11} & A_{37} & D_{36} & D_{37} & D_{37} & D_{38} \\ a_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{56} & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ IE_{1t} \\ M_{1t}^{III} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix}$$

where

$$A_{11} = \gamma_{11}(Q_{11} + Q_{14}) \; ; \; A_{14} = a_{15}Q_{11} + a_{45}Q_{14} \; ; \; A_{16} = -\gamma_{12}(Q_{11}+Q_{14})$$

$$A_{21} = \gamma_{11}(Q_{21} + Q_{24}) \; ; \; A_{24} = a_{15}Q_{21} + a_{45}Q_{24} \; ; \; A_{26} = -\gamma_{12}(Q_{21}+Q_{24})$$

$$A_{31} = \gamma_{11}(Q_{31} + Q_{34}) \; ; \; A_{34} = a_{15}Q_{31} + a_{45}Q_{34} \; ; \; A_{36} = -\gamma_{12}(Q_{31}+Q_{34})$$

$$A_{12} = \gamma_{21}(Q_{12} + Q_{15}) \; ; \; A_{15} = a_{25}Q_{12} + a_{55}Q_{15} \; ; \; A_{17} = -\gamma_{22}(Q_{12}+Q_{15})$$

$$A_{22} = \gamma_{21}(Q_{22} + Q_{25}) \; ; \; A_{25} = a_{25}Q_{22} + a_{55}Q_{25} \; ; \; A_{27} = -\gamma_{22}(Q_{22}+Q_{25})$$

$$A_{32} = \gamma_{31}(Q_{32} + Q_{35}) \; ; \; A_{35} = a_{25}Q_{32} + a_{55}Q_{35} \; ; \; A_{37} = -\gamma_{22}(Q_{32}+Q_{35})$$

$$D_{11} = a_{16}Q_{11} + a_{26}Q_{12} + a_{46}Q_{14} + a_{56}Q_{15} \; ; \; D_{12} = Q_{11} \; ; \; D_{13} = Q_{12}$$

$$D_{21} = a_{16}Q_{21} + a_{26}Q_{22} + a_{46}Q_{24} + a_{56}Q_{25} \; ; \; D_{22} = Q_{21} \; ; \; D_{23} = Q_{22}$$

$$D_{31} = a_{16}Q_{31} + a_{26}Q_{32} + a_{36} + a_{46}Q_{34} + a_{56}Q_{35} \; ; \; D_{32} = 1 + Q_{31}$$

$$D_{33} = Q_{32}$$

$$D_{14} = -\beta_{12}(Q_{11} - Q_{12}) \qquad D_{15} = a_{17}Q_{11} - \beta_{11}Q_{12}$$

$$D_{24} = -\beta_{12}(Q_{21} - Q_{22}) \qquad D_{25} = a_{17}Q_{21} - \beta_{11}Q_{22}$$

$$D_{34} = -\beta_{12}(Q_{31} - Q_{32}-1) \qquad D_{35} = a_{17}Q_{31} - \beta_{11}(Q_{32} - 1)$$

$$D_{16} = \beta_{22}(Q_{11} - Q_{12}) \qquad D_{17} = -\beta_{21}Q_{11} + a_{27}Q_{12}$$

$$D_{26} = \beta_{22}(Q_{21} - Q_{22}) \qquad D_{27} = -\beta_{21}Q_{21} + a_{27}Q_{22}$$

$$D_{36} = \beta_{22}(Q_{31} - Q_{32} + 1) \qquad D_{37} = -\beta_{21}(1 + Q_{31}) + a_{27}Q_{32}$$

$D_{18} = Q_{11}$

$D_{28} = Q_{21}$

$D_{38} = 1 + Q_{31}$

Therefore, the structural form of the two-country econometric model is:

$$\text{(A-6.3)} \quad y_{t-1} = \overline{A}y_{t-1} + \overline{B}u_t + \overline{C}u_{t-1} + \overline{D}z_t$$

And using Votey's estimated structural coefficients:

$\alpha_{10} = 52.57$	$\beta_{10} = 14.5131$	$\gamma_{10} = 31.707$	$\delta_{10} = 0$
$\alpha_{11} = 0.75$	$\beta_{11} = 0.131499$	$\gamma_{11} = -0.0027$	$\delta_{11} = 0.2051$
$\alpha_{12} =$ N.S.	$\beta_{12} = -28.3566$	$\gamma_{12} = 461.7$	$\delta_{12} = 4.02$
$\alpha_{20} = 31.26$	$\beta_{20} = -12.53$	$\gamma_{13} = 0.0624$	$\delta^* = 0.0489$
$\alpha_{21} = 0.7345$	$\beta_{21} = 0.09769$	$\gamma_{20} = 1.5812$	$\delta_{20} = 0$
	$\beta_{22} =$ N.S.	$\gamma_{21} = 0.1668$	$\delta_{21} = .36345$
	$\eta_{10} = -0.6915$	$\gamma_{22} = 349.8519$	$\delta_{22} = .35525$
	$\eta_{11} = 46.279$	$\gamma_{23} = 0.04291$	

We compute the numerical values for the matrices $\overline{A}$, $\overline{B}$ $\overline{C}$ and $\overline{D}$:

Matrix $\bar{A}$

Row	$\tilde{Y}_{1t-1}$	$\tilde{Y}_{2t-1}$	B_{1t-1}	K_{1t-1}	K_{2t-1}
1	-0.0073	0.1298	0	0.0131	0.0806
2	-0.0028	0.4538	0	0.0648	-0.0329
3	0.0008	0.0324	0	-0.0524	0.0201
4	-0.0027	0	0	1.0618	0
5	0	0.1668	0	0	1.1035

Matrix $\bar{B}$

Row	G_{1t}	G_{2t}	r_{1t}
1	2.8933	0.7782	0
2	1.0476	3.0351	0
3	-0.2781	0.1942	46.279
4	0	0	0
5	0	0	0

Matrix $\bar{C}$

Row	G_{1t-1}	G_{2t-1}	r_{1t-1}
1	0	0	-1263.7107
2	0	0	- 483.6579
3	0	0	143.7013
4	0	0	- 461.7
5	0	0	0

Matrix $\overline{D}$

Row	1	IE_{1t}	M_{1t}^{III}	TT_{1t}	T_{1t}	r_{2t}	r_{2t-1}	TT_{2t}	T_{2t}	TR_{2t}	M_{3t}
1	129.6743	2.8933	0.7782	59.9756	-1.8918	0	-272.2641	N.S.	-0.7782	-0.7782	2.8933
2	162.4181	1.0476	3.0351	-56.3613	-1.0470	0	-951.8105	N.S.	-2.0351	-2.0351	1.0476
3	-28.6009	0.7219	0.1942	14.9641	0.2779	-46.279	-67.9307	N.S.	-0.1942	-0.1942	0.7219
4	9.1298	0	0	0	0	0	0	0	0	0	0
5	-18.6889	0	0	0	0	-349.8519	0	0	0	0	0

APPENDIX A-7

COMPUTATION OF THE INVERSE OF A SYMMETRIC MATRIX (Ayres:1962)

Let us compute the inverse of the following matrix by partitioning:

$$(B^{c\prime}Q^cB^c) \stackrel{d}{=} \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{21} & b_{22} & b_{32} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

First, partition $(B^{c\prime}Q^cB^c)$ into:

$$(B^{c\prime}Q^cB^c) \stackrel{d}{=} \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{21} & b_{22} & b_{32} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \stackrel{d}{=} \left[\begin{array}{c:c} \Gamma_{11} & \Gamma_{12} \\ \hdashline \Gamma_{21} & \Gamma_{22} \end{array}\right]$$

where

$$\Gamma_{11} = \begin{bmatrix} b_{11} & b_{21} \\ b_{21} & b_{22} \end{bmatrix} ; \Gamma_{12} = \begin{bmatrix} b_{31} \\ b_{32} \end{bmatrix} ; \Gamma_{21} = \begin{bmatrix} b_{31}, & b_{32} \end{bmatrix} ; \Gamma_{22} = \begin{bmatrix} b_{33} \end{bmatrix}$$

Now

$$\Gamma_{11}^{-1} = \frac{1}{b_{11}b_{22}-b_{21}^2} \begin{bmatrix} b_{22} & -b_{21} \\ -b_{21} & b_{11} \end{bmatrix};$$

$$\Gamma_{11}^{-1}\Gamma_{12} = \frac{1}{b_{11}b_{22}-b_{21}^2} \begin{bmatrix} b_{22} & -b_{21} \\ -b_{21} & b_{11} \end{bmatrix} \begin{bmatrix} b_{31} \\ b_{32} \end{bmatrix} = \begin{bmatrix} \dfrac{-b_{22}\, b_{31} - b_{21}\, b_{32}}{b_{11}b_{22} - b_{21}^2} \\ \dfrac{-b_{21}\, b_{31} + b_{11}\, b_{32}}{b_{11}\, b_{22} - b_{21}^2} \end{bmatrix}$$

$$\xi = \Gamma_{22} - \Gamma_{21}\ \Gamma_{11}^{-1}\ \Gamma_{12}) = \left[b_{33} \right] - \left[b_{31}, b_{32} \right] \begin{bmatrix} \dfrac{b_{22}b_{31} - b_{21}b_{32}}{b_{11}b_{22} - b_{21}^2} \\ \dfrac{-b_{21}b_{31} + b_{11}b_{32}}{b_{11}b_{22} - b_{21}^2} \end{bmatrix}$$

$$= b_{33} - \frac{b_{31}\,(b_{22}b_{31} - b_{21}b_{32}) + b_{32}\,(-b_{21}b_{31} + b_{11}b_{32})}{b_{11}\,b_{22} - b_{21}^2}$$

$$= b_{33} - \frac{b_{22}b_{31}^2 - 2b_{21}b_{31}b_{32} + b_{11}b_{32}^2}{b_{11}b_{22} - b_{31}^2}$$

$$= \frac{b_{11}b_{22}b_{33} - b_{31}^2 b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}}{b_{11}b_{22} - b_{21}^2}$$

and $$\xi^{-1} = \frac{b_{11}b_{22} - b_{21}^2}{(b_{11}b_{22} - b_{21}^2)\, b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}}$$

then

$$\Lambda_{11} = \Gamma_{11}^{-1} + (\Gamma_{11}^{-1}\Gamma_{12})\xi^{-1}(\Gamma_{11}^{-1}\Gamma_{12})' = \frac{1}{b_{11}b_{22} - b_{21}^2}\begin{bmatrix} b_{22} & -b_{21} \\ -b_{21} & b_{11} \end{bmatrix} + \begin{bmatrix} \dfrac{b_{22}b_{31} - b_{21}b_{32}}{b_{11}b_{22} - b_{21}^2} \\ \dfrac{-b_{21}b_{31} + b_{11}b_{32}}{b_{11}b_{22} - b_{21}^2} \end{bmatrix} \frac{b_{11}b_{22} - b_{21}^2}{(b_{11}b_{22} - b_{21}^2)\,b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}} \begin{bmatrix} \dfrac{b_{22}b_{31} - b_{21}b_{32}}{b_{11}b_{22} - b_{21}^2} \, , & \dfrac{-b_{21}b_{31} + b_{11}b_{32}}{b_{11}b_{22} - b_{21}^2} \end{bmatrix}$$

$$\Lambda_{11} = \frac{1}{b_{11}b_{22} - b_{21}^2}\begin{bmatrix} b_{22} & -b_{21} \\ -b_{21} & b_{11} \end{bmatrix} + \frac{1}{(b_{11}b_{22} - b_{21}^2) \,/\, \left[(b_{11}b_{22} - b_{21}^2)\,b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}\right]}\begin{bmatrix} (b_{22}b_{31} - b_{21}b_{32})^2 & (b_{22}b_{31} - b_{21}b_{32})\,(b_{11}b_{32} - b_{21}b_{31}) \\ (b_{22}b_{31} - b_{21}b_{32})\,(b_{11}b_{32} - b_{21}b_{31}) & (b_{11}b_{32} - b_{21}b_{31})^2 \end{bmatrix}$$

$$= \frac{1}{(b_{11}b_{22} - b_{21}^2)\,b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}}\begin{bmatrix} b_{22}b_{33} - b_{32}^2 & b_{31}b_{32} - b_{21}b_{33} \\ b_{31}b_{32} - b_{21}b_{33} & b_{11}b_{33} - b_{31}^2 \end{bmatrix}$$

$$\Lambda_{12} = -(\Gamma_{11}^{-1}\ \Gamma_{12})\ \xi^{-1} = -\frac{1}{(b_{11}b_{22} - b_{21}^2)\ b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}} \begin{bmatrix} b_{22}b_{31} - b_{21}b_{32} \\ b_{11}b_{32} - b_{21}b_{31} \end{bmatrix}$$

$$\Lambda_{21} = \Lambda_{12}' = \frac{-1}{(b_{11}b_{22} - b_{21}^2)\ b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}} \begin{bmatrix} b_{22}b_{31} - b_{21}b_{32}\ ,\ b_{11}b_{32} - b_{21}b_{31} \end{bmatrix}$$

$$\Lambda_{22} = \xi^{-1} = \frac{1}{(b_{11}b_{22} - b_{21}^2)\ b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}} \begin{bmatrix} b_{11}b_{22} - b_{21}^2 \end{bmatrix}$$

And

$$(B^{c\prime}Q^cB^c)^{-1} = \left[\begin{array}{c|c} \Lambda_{11} & \Lambda_{12} \\ \hline \Lambda_{21} & \Lambda_{22} \end{array}\right] \stackrel{d}{=} \frac{1}{|\Lambda|} \left[\begin{array}{cc|c} b_{22}b_{33} - b_{32}^2 & b_{31}b_{32} - b_{21}b_{33} & b_{22}b_{31} - b_{21}b_{32} \\ b_{31}b_{32} - b_{21}b_{33} & b_{11}b_{33} - b_{31}^2 & b_{11}b_{32} - b_{21}b_{31} \\ \hline b_{22}b_{31} - b_{21}b_{32} & b_{11}b_{32} - b_{21}b_{31} & b_{11}b_{22} - b_{21}^2 \end{array}\right]$$

where

$$|\Lambda| \stackrel{d}{=} (b_{11}b_{22} - b_{21}^2)\ b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}$$

APPENDIX A-8

COMPUTATIONS OF OPTIMAL SOLUTION FOR THE TWO-COUNTRY CONTROL PROBLEM (CASE A)

The optimal solution is given by equation A-2.20 (Appendix A-2):

$$(\text{A-2.20})\quad u_t^{c*} = -(B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^cA^cx_{t-1}^{c*} + (B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^c\overline{x}_t^c$$
$$- (B^{c\prime}Q^cB^c)^{-1} B^{c\prime}Q^cC^cz_t^c$$

where

$$x_t^{c*} = \left[\tilde{Y}_{1t}^*,\ \tilde{Y}_{2t}^*,\ B_{1t}^*,\ K_{2t}^*,\ G_{1t}^*,\ G_{2t}^*,\ r_{1t}^*\right]'$$

$$\overline{x}_t^c = \left[\tilde{Y}_{1t},\ \tilde{Y}_{2t},\ \overline{B}_{1t},\ \overline{K}_{1t},\ \overline{G}_{1t},\ \overline{G}_{2t},\ \overline{r}_{1t}\right]' \overset{d}{=} \left[\delta_{10} + \delta_{12}\, L_{1t},\ \delta_{20}\right.$$
$$\left. + \delta_{22}\, L_{2t},\ 0,\ \overline{K}_{1t},\ \overline{G}_{1t},\ \overline{G}_{2t},\ \overline{r}_{1t}\right]'$$

$$u_t^{c*} = \left[G_{1t}^*,\ G_{2t}^*,\ \overline{r}_{1t}^*\right]'$$

$$z_t^c = \left[1,\ IE_{1t},\ M_{1t}^{III},\ TT_{1t},\ T_{1t},\ r_{2t},\ r_{2t-1},\ TT_{2t},\ T_{2t},\ TR_{2t},\ M_{3t}\right]'$$

$$A^c = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & A_{15} & 0 & 0 & A_{16} \\ A_{21} & A_{22} & 0 & A_{24} & A_{25} & 0 & 0 & A_{26} \\ A_{31} & A_{32} & 0 & A_{34} & A_{35} & 0 & 0 & A_{36} \\ \gamma_{11} & 0 & 0 & a_{45} & 0 & 0 & 0 & \gamma_{12} \\ 0 & \gamma_{21} & 0 & 0 & a_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ; B^c = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & \eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$; C^c = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & 0 & A_{17} & D_{16} & D_{17} & D_{17} & D_{18} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & 0 & A_{27} & D_{26} & D_{27} & D_{27} & D_{28} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & -\gamma_{11} & A_{37} & D_{36} & D_{37} & D_{37} & D_{38} \\ a_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{56} & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q^c = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Computing:

$$B^{c\prime}Q^{c} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} & 0 & 0 & 1 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{32} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & n_{11} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} q_{11}Q_{11} & q_{22}Q_{21} & q_{13}Q_{31} & 0 & 0 & 0 & 0 & 0 \\ q_{11}Q_{12} & q_{22}Q_{22} & q_{33}Q_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33}n_{11} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(B^{c\prime}Q^{c}B^{c}) = \begin{bmatrix} q_{11}Q_{11} & q_{22}Q_{21} & q_{33}Q_{31} & 0 & 0 & 0 & 0 & 0 \\ q_{11}Q_{12} & q_{22}Q_{22} & q_{33}Q_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33}n_{11} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & n_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} q_{11}Q_{11}^2 + q_{22}Q_{21}^2 + q_{33}Q_{31}^2 & q_{11}Q_{11}Q_{12} + q_{22}Q_{21}Q_{22} + q_{33}Q_{31}Q_{32} & q_{33}n_{11}Q_{31} \\ q_{11}Q_{11}Q_{12} + q_{22}Q_{21}Q_{22} + q_{33}Q_{31}Q_{32} & q_{11}Q_{12}^2 + q_{22}Q_{22}^2 + q_{33}Q_{32}^2 & q_{33}n_{11}Q_{32} \\ q_{33}n_{11}Q_{31} & q_{33}n_{11}Q_{32} & q_{33}n_{11}^2 \end{bmatrix}$$

We note that $(B^{c\prime}Q^cB^c)$ is a symmetric matrix and the inverse of $(B^{c\prime}Q^cB^c)$ is obtained by the method of partitioning (Ayres:1962) instead of computing the adjoint $(B^{c\prime}Q^cB^c)$. Appendix A-7 gives the steps of computing $(B^{c\prime}Q^cB^c)^{-1}$ which lead to the following result:

$$(B^{c\prime}Q^cB^c)^{-1} = \frac{1}{(b_{11}b_{22} - b_{21}^2)\, b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32}} \begin{bmatrix} b_{22}b_{33}-b_{32}^2 & b_{31}b_{32}-b_{21}b_{33} & b_{22}b_{31}-b_{21}b_{32} \\ b_{31}b_{32}-b_{21}b_{33} & b_{11}b_{33}-b_{31}^2 & b_{11}b_{32}-b_{21}b_{31} \\ b_{22}b_{31}-b_{21}b_{32} & b_{11}b_{32}-b_{21}b_{31} & b_{11}b_{22}-b_{21}^2 \end{bmatrix}$$

where

$b_{11} \stackrel{d}{=} q_{11}Q_{11}^2 + q_{22}Q_{21}^2 + q_{33}Q_{31}^2$ $\qquad$ $b_{21} \stackrel{d}{=} q_{11}Q_{11}Q_{12}+q_{22}Q_{21}Q_{22}+q_{33}Q_{31}Q_{32}$

$b_{22} \stackrel{d}{=} q_{11}Q_{12}^2 + q_{22}Q_{22}^2 + q_{33}Q_{32}^2$ $\qquad$ $b_{31} \stackrel{d}{=} q_{33}\eta_{11}Q_{31}$

$b_{33} \stackrel{d}{=} q_{33}\eta_{11}^2$ $\qquad$ $b_{32} \stackrel{d}{=} q_{33}\eta_{11}Q_{32}$

Substituting the bij's and bii's (i = 1,2,3; j = 1,2,3) by their corresponding values, we get:

$$(b_{11}b_{22}-b_{21}^2)\ b_{33}-b_{11}b_{32}^2-b_{22}b_{31}^2+2b_{21}b_{31}b_{32} \overset{d}{=} q_{11}q_{22}q_{33}\eta_{11}\ (Q_{11}Q_{22}-Q_{12}Q_{21})^2$$

$$b_{22}b_{33} - b_{32}^2 \overset{d}{=} q_{11}q_{33}\eta_{11}^2\ Q_{12}^2 + q_{22}q_{33}\eta_{11}^2\ Q_{22}^2$$

$$b_{11}b_{33} - b_{31}^2 \overset{d}{=} q_{11}q_{33}\eta_{11}^2 Q_{11}^2 + q_{22}q_{33}\eta_{11}^2 Q_{21}^2$$

$$b_{11}b_{22} - b_{21}^2 \overset{d}{=} q_{11}q_{22}(Q_{11}Q_{22}-Q_{12}Q_{21})^2+q_{11}q_{33}(Q_{11}Q_{32}-Q_{12}Q_{31})^2$$

$$+ q_{22}q_{33}(Q_{21}Q_{32}-Q_{22}Q_{31})^2$$

$$b_{31}b_{32} - b_{21}b_{33} \overset{d}{=} -q_{33}\eta_{11}^2(q_{11}Q_{11}Q_{12}+q_{22}Q_{21}Q_{22})$$

$$b_{22}b_{31}-b_{21}b_{32} \overset{d}{=} q_{11}q_{33}\eta_{11}Q_{12}(Q_{12}Q_{31}-Q_{11}Q_{32})+q_{22}q_{33}\eta_{11}Q_{22}(Q_{22}Q_{31}-Q_{21}Q_{32})$$

$$b_{11}b_{32}-b_{21}b_{31} \overset{d}{=} q_{11}q_{33}\eta_{11}Q_{11}(Q_{11}Q_{32}-Q_{12}Q_{31})+q_{22}q_{33}\eta_{11}Q_{21}(Q_{21}Q_{32}-Q_{22}Q_{31})$$

and

$$(B^{c\prime}Q^{c}B^{c})^{-1} = \begin{bmatrix} \dfrac{q_{11}\,Q_{12}^2 + q_{22}\,Q_{22}^2}{q_{11}q_{22}(Q_{11}Q_{22}-Q_{12}Q_{21})^2} & \dfrac{-(q_{11}Q_{11}Q_{12}+q_{22}Q_{21}Q_{22})}{q_{11}q_{22}(Q_{11}Q_{22}-Q_{12}Q_{21})^2} & \dfrac{q_{11}Q_{12}(Q_{12}Q_{31}-Q_{11}Q_{32})+q_{22}Q_{22}(Q_{22}Q_{31}-Q_{21}Q_{32})}{q_{11}q_{22}n_{11}(Q_{11}Q_{22}-Q_{12}Q_{21})^2} \\ \dfrac{-(q_{11}Q_{11}Q_{12}+q_{22}Q_{21}Q_{22})}{q_{11}q_{22}(Q_{21}Q_{22}-Q_{12}Q_{21})^2} & \dfrac{q_{11}Q_{11}^2 + q_{22}Q_{21}^2}{q_{11}q_{22}(Q_{11}Q_{22}-Q_{12}Q_{21})^2} & \dfrac{q_{11}Q_{11}(Q_{11}Q_{32}-Q_{12}Q_{31})+q_{22}Q_{21}(Q_{21}Q_{32}-Q_{21}Q_{32}}{q_{11}q_{22}n_{11}(Q_{11}Q_{22}-Q_{12}Q_{21})^2} \\ \dfrac{q_{11}Q_{12}(Q_{12}Q_{31}-Q_{11}Q_{32})+q_{22}Q_{22}(Q_{22}Q_{31}-Q_{21}Q_{32}}{q_{11}q_{22}n_{11}(Q_{11}Q_{22}-Q_{12}Q_{21})^2} & \dfrac{q_{11}Q_{11}(Q_{11}Q_{32}-Q_{12}Q_{31}+q_{22}Q_{21}(Q_{21}Q_{32}-Q_{22}Q_{31}}{q_{11}q_{22}n_{11}(Q_{11}Q_{22}-Q_{12}Q_{21})^2} & \dfrac{1}{q_{33}n_{11}^2} + \dfrac{(Q_{11}Q_{32}-Q_{12}Q_{31})^2}{q_{22}n_{11}^2(Q_{11}Q_{22}-Q_{12}Q_{21})^2} + \dfrac{(Q_{21}Q_{32}-Q_{22}Q_{31})^2}{q_{11}n_{11}^2(Q_{11}Q_{22}-Q_{12}Q_{21})^2} \end{bmatrix}$$

$$\text{(A-8.1)}\quad (B^{c\prime}Q^{c}B^{c})^{-1}B^{c\prime}Q^{c} = \begin{bmatrix} \dfrac{Q_{22}}{Q_{11}Q_{22} - Q_{12}Q_{21}} & \dfrac{-Q_{12}}{Q_{11}Q_{22} - Q_{12}Q_{21}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{-Q_{21}}{Q_{11}Q_{22} - Q_{12}Q_{21}} & \dfrac{Q_{11}}{Q_{11}Q_{22} - Q_{12}Q_{21}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{Q_{21}Q_{32} - Q_{22}Q_{31}}{n_{11}(Q_{11}Q_{22}-Q_{12}Q_{21})} & \dfrac{-(Q_{11}Q_{32} - Q_{12}Q_{31})}{n_{11}(Q_{11}Q_{22}-Q_{12}Q_{21}} & \dfrac{1}{n_{11}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{d}} \begin{bmatrix} \dfrac{Q_{22}}{Q} & \dfrac{-Q_{12}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{-Q_{21}}{Q} & \dfrac{Q_{11}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{Q_{21}Q_{32}-Q_{22}Q_{31}}{n_{11}Q} & \dfrac{-(Q_{11}Q_{32}-Q_{12}Q_{31})}{n_{11}Q} & \dfrac{1}{n_{11}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(A-8.2)} \quad (B^c Q^c B^c)^{-1} B^{c\prime} Q^c A^c = \begin{bmatrix} \theta_{11} & \theta_{12} & 0 & \theta_{14} & \theta_{15} & 0 & 0 & \theta_{16} \\ \theta_{21} & \theta_{22} & 0 & \theta_{24} & \theta_{25} & 0 & 0 & \theta_{26} \\ \theta_{31} & \theta_{32} & 0 & \theta_{34} & \theta_{35} & 0 & 0 & \theta_{36} \end{bmatrix}$$

where

$$Q \overset{d}{=} Q_{11}Q_{22} - Q_{12}Q_{21}$$

$$\theta_{11} \overset{d}{=} \frac{1}{Q}(A_{11}Q_{22} - A_{21}Q_{12})$$

$$\theta_{12} \overset{d}{=} \frac{1}{Q}(A_{12}Q_{22} - A_{22}Q_{12})$$

$$\theta_{14} \overset{d}{=} \frac{1}{Q}(A_{14}Q_{22} - A_{24}Q_{12})$$

$$\theta_{15} \overset{d}{=} \frac{1}{Q}(A_{15}Q_{22} - A_{25}Q_{12})$$

$$\theta_{16} \overset{d}{=} \frac{1}{Q}(A_{16}Q_{22} - A_{26}Q_{12})$$

$$\theta_{21} \overset{d}{=} \frac{1}{Q}(A_{21}Q_{11} - A_{11}Q_{21})$$

$$\theta_{21} \overset{d}{=} \frac{1}{Q}(A_{21}Q_{11} - A_{11}Q_{21})$$

$$\theta_{22} \overset{d}{=} \frac{1}{Q}(A_{22}Q_{11} - A_{12}Q_{21})$$

$$\theta_{24} \overset{d}{=} \frac{1}{Q}(A_{24}Q_{11} - A_{14}Q_{21})$$

$$\theta_{25} \overset{d}{=} \frac{1}{Q}(A_{25}Q_{11} - A_{15}Q_{21})$$

$$\theta_{26} \overset{d}{=} \frac{1}{Q}(A_{26}Q_{11} - A_{16}Q_{21})$$

$$\theta_{31} \overset{d}{=} \frac{1}{\eta_{11}}\left[A_{31} + \frac{A_{11}(Q_{21}Q_{32}-Q_{22}Q_{31})}{} \frac{-A_{21}(Q_{11}Q_{32} - Q_{12}Q_{31})}{Q}\right]$$

$$\theta_{32} \overset{d}{=} \frac{1}{\eta_{11}}\left[A_{32} + \frac{A_{12}(Q_{21}Q_{32}-Q_{22}Q_{31})}{} - \frac{A_{22}(Q_{11}Q_{32} - Q_{12}Q_{31})}{Q}\right]$$

$$\theta_{34} \overset{d}{=} \frac{1}{\eta_{11}}\left[A_{34} + \frac{A_{14}(Q_{21}Q_{32}-Q_{22}Q_{31})}{} - \frac{A_{24}(Q_{11}Q_{32} - Q_{12}Q_{31})}{Q}\right]$$

$$\theta_{35} \overset{d}{=} \frac{1}{\eta_{11}}\left[A_{35} + \frac{A_{15}(Q_{21}Q_{32}-Q_{22}Q_{31})}{} - \frac{A_{25}(Q_{11}Q_{32} - Q_{12}Q_{31})}{Q}\right]$$

$$\theta_{36} \stackrel{d}{=} \frac{1}{\eta_{11}}\left[A_{36} + \frac{A_{16}(Q_{21}Q_{32}-Q_{22}Q_{31})}{} - \frac{A_{26}(Q_{11}Q_{32} - Q_{12}Q_{31})}{Q}\right]$$

(A-8.3)

$$(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^cC^c = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} & \Delta_{15} & 0 & \Delta_{19} & \Delta_{16} & \Delta_{17} & \Delta_{17} & \Delta_{18} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \Delta_{24} & \Delta_{25} & 0 & \Delta_{29} & \Delta_{26} & \Delta_{27} & \Delta_{27} & \Delta_{28} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} & \Delta_{35} & -1 & \Delta_{39} & \Delta_{36} & \Delta_{37} & \Delta_{37} & \Delta_{38} \end{bmatrix}$$

where

$$\Delta_{11} \stackrel{d}{=} \frac{1}{Q}(D_{11}Q_{22} - D_{21}Q_{12}) \qquad \Delta_{21} \stackrel{d}{=} \frac{1}{Q}(-D_{11}Q_{21} + D_{21}Q_{11})$$

$$\Delta_{12} \stackrel{d}{=} \frac{1}{Q}(D_{12}Q_{22} - D_{22}Q_{12}) \qquad \Delta_{22} \stackrel{d}{=} \frac{1}{Q}(-D_{12}Q_{21} + D_{22}Q_{11})$$

$$\Delta_{13} \stackrel{d}{=} \frac{1}{Q}(D_{13}Q_{22} - D_{23}Q_{12}) \qquad \Delta_{23} \stackrel{d}{=} \frac{1}{Q}(-D_{13}Q_{21} + D_{23}Q_{11})$$

$$\Delta_{14} \stackrel{d}{=} \frac{1}{Q}(D_{14}Q_{22} - D_{24}Q_{12}) \qquad \Delta_{24} \stackrel{d}{=} \frac{1}{Q}(-D_{14}Q_{21} + D_{24}Q_{11})$$

$$\Delta_{15} \stackrel{d}{=} \frac{1}{Q}(D_{15}Q_{22} - D_{25}Q_{12}) \qquad \Delta_{25} \stackrel{d}{=} \frac{1}{Q}(-D_{15}Q_{21} + D_{25}Q_{11})$$

$$\Delta_{16} \stackrel{d}{=} \frac{1}{Q}(D_{16}Q_{22} - D_{26}Q_{12}) \qquad \Delta_{26} \stackrel{d}{=} \frac{1}{Q}(-D_{16}Q_{21} + D_{26}Q_{11})$$

$$\Delta_{17} \stackrel{d}{=} \frac{1}{Q}(D_{17}Q_{22} - D_{27}Q_{12}) \qquad \Delta_{27} \stackrel{d}{=} \frac{1}{Q}(-D_{17}Q_{21} + D_{27}Q_{11})$$

$$\Delta_{18} \stackrel{d}{=} \frac{1}{Q}(D_{18}Q_{22} - D_{28}Q_{12}) \qquad \Delta_{28} \stackrel{d}{=} \frac{1}{Q}(-D_{18}Q_{21} + D_{28}Q_{11})$$

$$\Delta_{19} \stackrel{d}{=} \frac{1}{Q}(A_{17}Q_{22} - A_{27}Q_{12}) \qquad \Delta_{29} \stackrel{d}{=} \frac{1}{Q}(-A_{17}Q_{21} + A_{27}Q_{11})$$

$$\Delta_{31} \stackrel{d}{=} \frac{1}{\eta_{11}Q}\left[D_{11}(Q_{21}Q_{31} - Q_{22}Q_{31}) - D_{21}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{D_{31}}{\eta_{11}}$$

$$\Delta_{32} \stackrel{d}{=} \frac{1}{\eta_{11}Q}\left[D_{12}(Q_{21}Q_{31} - Q_{22}Q_{31}) - D_{22}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{D_{32}}{\eta_{11}}$$

$$\Delta_{33} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{13}(Q_{21}Q_{31} - Q_{22}Q_{31}) - D_{23}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{D_{33}}{\eta_{11}}$$

$$\Delta_{34} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{14}(Q_{21}Q_{31} - Q_{22}Q_{31}) - D_{24}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{D_{34}}{\eta_{11}}$$

$$\Delta_{35} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{15}(Q_{21}Q_{31} - Q_{22}Q_{31}) - D_{25}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{D_{35}}{\eta_{11}}$$

$$\Delta_{36} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{16}(Q_{21}Q_{31} - Q_{22}Q_{31}) - D_{26}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{D_{36}}{\eta_{11}}$$

$$\Delta_{37} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{17}(Q_{21}Q_{31} - Q_{22}Q_{31}) - D_{27}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{D_{37}}{\eta_{11}}$$

$$\Delta_{38} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{18}(Q_{21}Q_{31} - Q_{22}Q_{31}) - D_{28}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{D_{38}}{\eta_{11}}$$

$$\Delta_{39} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[A_{17}(Q_{21}Q_{31} - Q_{22}Q_{31}) - A_{27}(Q_{11}Q_{32} - Q_{12}Q_{31})\right] + \frac{A_{37}}{\eta_{11}}$$

Then the optimal control is:

$$(A\text{-}8.4)\quad \begin{bmatrix} G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} -\theta_{11} & -\theta_{12} & 0 & -\theta_{14} & -\theta_{15} & 0 & 0 & -\theta_{16} \\ -\theta_{21} & -\theta_{22} & 0 & -\theta_{24} & -\theta_{25} & 0 & 0 & -\theta_{26} \\ -\theta_{31} & -\theta_{32} & 0 & -\theta_{34} & -\theta_{35} & 0 & 0 & -\theta_{36} \end{bmatrix} \begin{bmatrix} \bar{Y}^*_{1t-1} \\ \bar{Y}^*_{2t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix} + \begin{bmatrix} \frac{Q_{22}}{Q} & \frac{-Q_{12}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-Q_{21}}{Q} & \frac{Q_{11}}{Q} & \frac{Q_{11}}{Q} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\eta_{11}}\left(\frac{Q_{21}Q_{32}-Q_{22}Q_{31}}{Q}\right) & \frac{-1}{\eta_{11}}\left(\frac{Q_{11}Q_{32}-Q_{12}Q_{31}}{Q}\right) & \frac{1}{\eta_{11}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{Y}}_{1t} \\ \bar{\bar{Y}}_{2t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix}$$

$$+ \begin{bmatrix} -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & -\Delta_{15} & 0 & -\Delta_{19} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} \\ -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & -\Delta_{25} & 0 & -\Delta_{29} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} \\ -\Delta_{31} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & -\Delta_{35} & 1 & -\Delta_{39} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} \end{bmatrix} \begin{bmatrix} 1 \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix}$$

Since the targets $\bar{\bar{Y}}_{1t}$, $\bar{\bar{Y}}_{2t}$ and $\bar{B}_{1t}$ are defined as follows:

$$\bar{\bar{Y}}_{1t} = \delta_{10} + \delta_{12}\, L_{1t}$$

$$\bar{\bar{Y}}_{2t} = \delta_{20} + \delta_{22}\, L_{2t}$$

$$\bar{B}_{1t} = 0$$

after substitution and rearrangement of terms equation A-8.4 becomes:

$$\text{(A-8.5)}\quad \begin{bmatrix} G^*_{1t} \\ B^*_{2t} \\ r^*_{1t} \end{bmatrix} = \begin{bmatrix} -\theta_{11} & -\theta_{12} & 0 & -\theta_{14} & -\theta_{15} & 0 & 0 & -\theta_{16} \\ -\theta_{21} & -\theta_{22} & 0 & -\theta_{24} & -\theta_{25} & 0 & 0 & -\theta_{26} \\ -\theta_{31} & -\theta_{32} & 0 & -\theta_{34} & -\theta_{35} & 0 & 0 & -\theta_{36} \end{bmatrix} \begin{bmatrix} \tilde{Y}^*_{1t-1} \\ \tilde{Y}^*_{2t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix} + \begin{bmatrix} -\Delta_{11} + \frac{1}{Q}(Q_{22}\delta_{10} - Q_{12}\delta_{20}) & \frac{Q_{22}\delta_{12}}{Q} & \frac{-Q_{12}\delta_{22}}{Q} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & -\Delta_{15} & 0 & -\Delta_{19} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} \\ -\Delta_{21} + \frac{1}{Q}(-Q_{21}\delta_{10} + Q_{11}\delta_{20}) & \frac{-Q_{21}\delta_{12}}{Q} & \frac{Q_{11}\delta_{22}}{Q} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & -\Delta_{25} & 0 & -\Delta_{29} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} \\ -\Delta_{31} + \frac{1}{\eta_{11}Q}\left[\delta_{10}(Q_{21}Q_{32}-Q_{22}Q_{31}) - \delta_{20}(Q_{11}Q_{32}-Q_{12}Q_{31})\right] & \frac{\delta_{12}}{\eta_{11}Q}(Q_{21}Q_{32}-Q_{22}Q_{31}) & \frac{\delta_{22}}{\eta_{11}Q}(Q_{11}Q_{32}-Q_{12}Q_{31}) & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & -\Delta_{35} & 0 & -\Delta_{39} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} \end{bmatrix} \begin{bmatrix} 1 \\ L_{1t} \\ L_{2t} \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix}$$

APPENDIX A-9

EVALUATION OF THE WELFARE COST FOR THE TWO-COUNTRY CONTROL PROBLEM (CASE A)

The welfare cost is determined by:

$$\text{(A-5.1)} \quad \hat{J}^c = \frac{1}{2} \sum_{t=1}^{T} (x_t^{c*} - \overline{x}_t^{c*})' Q^c (x_t^{c*} - \overline{x}_t^{c*})$$

where

$$\text{(A-5.4)} \quad x_t^* = \left[I - B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]A^c\, x_{t-1}^{c*} + \left[B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]\overline{x}_t^c + \left[I - B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]C^c z_t^c$$

$$\text{(A-5.5)} \quad x_t^{c*} - \overline{x}_t^c = \left[I - B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]A^c x_{t-1}^{c*} + \left[B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c - I\right]\overline{x}_t^c + \left[I - B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]C^c z_t^c.$$

Let us compute for the two-country control problem the following matrices:

$$B^{c}(B^{c\prime}Q^{c}B^{c})^{-1}B^{c\prime}Q^{c} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & \eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{Q_{22}}{Q} & \frac{-Q_{12}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-Q_{21}}{Q} & \frac{Q_{11}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\eta_{11}Q}(Q_{21}Q_{32}-Q_{22}Q_{31}) & \frac{-1}{\eta_{11}Q}(Q_{11}Q_{32}-Q_{12}Q_{31}) & \frac{1}{\eta_{11}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{Q_{22}}{Q} & \frac{-Q_{12}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-Q_{21}}{Q} & \frac{Q_{11}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\eta_{11}Q}(Q_{21}Q_{32} - Q_{22}Q_{31}) & \frac{1}{\eta_{11}Q}(Q_{11}Q_{32} - Q_{12}Q_{31}) & \frac{1}{\eta_{11}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[I-B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{-Q_{22}}{Q} & \frac{Q_{12}}{Q} & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{Q_{21}}{Q} & \frac{-Q_{11}}{Q} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-1}{\eta_{11}Q}(Q_{21}Q_{32}-Q_{22}Q_{31}) & \frac{1}{\eta_{11}Q}(Q_{11}Q_{32}-Q_{12}Q_{31}) & \frac{-1}{\eta_{11}} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I-B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c]A^c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_{11} & 0 & 0 & a_{45} & 0 & 0 & 0 & -\gamma_{12} \\ 0 & \gamma_{21} & 0 & 0 & a_{55} & 0 & 0 & 0 \\ -\theta_{11} & -\theta_{12} & 0 & -\theta_{14} & -\theta_{15} & 0 & 0 & -\theta_{16} \\ -\theta_{21} & -\theta_{22} & 0 & -\theta_{24} & -\theta_{25} & 0 & 0 & -\theta_{26} \\ -\theta_{31} & -\theta_{32} & 0 & -\theta_{34} & -\theta_{35} & 0 & 0 & -\theta_{36} \end{bmatrix}$$

$$[I-B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c]C^c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{56} & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 \\ -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & -\Delta_{15} & 0 & -\Delta_{19} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} \\ -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & -\Delta_{25} & 0 & -\Delta_{29} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} \\ -\Delta_{31} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & -\Delta_{35} & 1 & -\Delta_{39} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} \end{bmatrix}$$

$$[B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c-I] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \frac{Q_{22}}{Q} & \frac{-Q_{12}}{Q} & 0 & 0 & 0 & -1 & 0 & 0 \\ \frac{-Q_{21}}{Q} & \frac{Q_{11}}{Q} & 0 & 0 & 0 & 0 & -1 & 0 \\ \frac{1}{\eta_{11}Q}(Q_{21}Q_{32}-Q_{22}Q_{31}) & \frac{-1}{\eta_{11}Q}(Q_{11}Q_{32}-Q_{12}Q_{31}) & \frac{1}{\eta_{11}} & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Then substituting the above matrices into equation A-5.5 we get:

$$
\text{(A-9.1)}\quad
\begin{bmatrix} Y^*_{1t} \\ Y^*_{2t} \\ B^*_{1t} \\ K^*_{1t} \\ K^*_{2t} \\ G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
-
\begin{bmatrix} \bar{\bar{Y}}_{1} \\ \bar{\bar{Y}}_{2t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{45} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 \\
-\theta_{11} & -\theta_{12} & 0 & -\theta_{14} & -\theta_{15} & 0 & 0 & -\theta_{16} \\
-\theta_{21} & -\theta_{22} & 0 & -\theta_{24} & -\theta_{25} & 0 & 0 & -\theta_{26} \\
-\theta_{31} & -\theta_{32} & 0 & -\theta_{34} & -\theta_{35} & 0 & 0 & -\theta_{36}
\end{bmatrix}
\begin{bmatrix} Y^*_{1t-1} \\ Y^*_{2t-1} \\ B^*_{1t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\frac{Q_{22}}{Q} & \frac{-Q_{12}}{Q} & 0 & 0 & 0 & -1 & 0 & 0 \\
\frac{-Q_{21}}{Q} & \frac{Q_{11}}{Q} & 0 & 0 & 0 & 0 & -1 & 0 \\
\frac{1}{\eta_{11}Q}(Q_{21}Q_{32}-Q_{22}Q_{31}) & \frac{1}{\eta_{11}Q}(Q_{11}Q_{32}-Q_{22}Q_{32}) & \frac{1}{\eta_{11}Q} & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix} \bar{\bar{Y}}_{1t} \\ \bar{\bar{Y}}_{2t} \\ \bar{B}_{1t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix}
$$

$$
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{56} & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 \\
-\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & -\Delta_{15} & 0 & -\Delta_{19} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} \\
-\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & -\Delta_{25} & 0 & -\Delta_{29} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} \\
-\Delta_{31} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & -\Delta_{35} & 0 & -\Delta_{39} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38}
\end{bmatrix}
\begin{bmatrix} 1 \\ IE_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ M_{3t} \end{bmatrix}
$$

Since

$$Q^c = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then it follows that:

$$(x_t^{c*} - \bar{x}_t^c)' Q \, (x_t^{c*} - \bar{x}_t^c) = 0$$

Therefore

$$\text{(A-9.2)} \quad \hat{J} = \frac{1}{2} \sum_{t=1}^{N} (x_t^{c*} - \bar{x}_t^c)' \, Q^c \, (x^{c*} - \bar{x}_t^c) = 0$$

APPENDIX A-10

THE REDUCED FORM OF THE TWO-COUNTRY MODEL (CASE B)

Under the assumption of linear dependent external balance, the variables of Votey's modified econometric model are classified as follows:

(1) endogenous variables:

B_{it}, C_{it}, I^G_{it}, I^n_{it}, K_{iy}, M_{it}, O_{1t}, Y_{it} $\qquad i = 1,2$

(2) controlled exogenous variables:

G_{1t}, G_{2t}, r_{1t}

(3) noncontrolled exogenous variables:

IE_{1t}, L_{it}, TT_{it}, r_{2t}, T_{it}, TR_{2t}, X^{III}_{it}, M^{III}_{it} $\qquad i = 1,2$

And the structural model is constituted of:

9 equations:

(E-1) $Y_{1t} = \delta_{10} + \delta_{11} K_{1t} + \delta_{12} L_{1t}$

(E-2) $C_{1t} = \alpha_{10} + \alpha_{11} Y_{1t} - \alpha_{11} T_{1t} - \alpha_{11} \delta^* K_{1t}$

(E-3) $M_{1t} = \beta_{10} + \beta_{11} Y_{1t} - \beta_{11} T_{1t} - \beta_{11} \delta^* K_{1t} + \beta_{12} TT_{1t};$

$TT_{1t} = \left(\frac{Px}{P_M}\right) 1t$

(E-4) $I^n_{1t} = \gamma_{10} + \gamma_{11} Y_{1t-1} - \gamma_{12} \delta^* - \gamma_{12} r_{1t-1} + \gamma_{13} K_{1t-1}$

(E-5) $0_{1t} = \eta_{10} + \eta_{11} r_{2t} - \eta_{11} r_{1t}$

(E-6) $Y_{2t} = \delta_{20} + \delta_{21} K_{1t} + \delta_{22} L_{1t}$

(E-7) $C_{2t} = \alpha_{20} + \alpha_{21} Y_{2t} - \alpha_{21} T_{2t} - \alpha_{21} TR_{2t} - \alpha_{21} \delta^* K_{2t}$

(E-8) $M_{2t} = \beta_{20} + \beta_{21} Y_{2t} - \beta_{21} T_{2t} - \beta_{21} TR_{2t} - \beta_{21} \delta^* K_{2t}$
$+ \beta_{22} TT_{2t}$; $TT_{2t} = \frac{P_x}{P_M} 2t$

(E-9) $I^n_{2t} = \gamma_{20} + \gamma_{21} Y_{2t-1} - \gamma_{22} \delta^* - \gamma_{22} r_{2t-1} + \gamma_{23} K_{2t-1}$

10 identities:

(I-1) $Y_{1t} = C_{1t} + I^G_{1t} + G_{1t} + X_{1t} - M_{1t}$

(I-2) $B_{1t} = X_{1t} - M_{1t} - 0_{1t}$

(I-3) $K_{1t} = I^n_{1t} + K_{1t-1}$

(I-4) $I^G_{1t} = I^n_{1t} + \delta^* K_{1t}$

(I-5) $X_{1t} = X^{III}_{1t} + (M_{2t} - M^{III}_{2t}) + IE_{1t}$

(I-6) $Y_{2t} = C_{2t} + I^G_{2t} + G_{2t} + X_{2t} - M_{2t}$

(I-7) $B_{2t} = X_{2t} - M_{2t} + 0_{1t}$

(I-8) $K_{2t} - I^{n}_{2t} + K_{2t-1}$

(I-9) $I^{G}_{2t} = I^{n}_{2t} + \delta^* K_{2t}$

(I-10) $X_{2t} = X^{III}_{2t} + (M_{1t} - M^{III}_{1t})$

Substituting (I-5) and (I-10) into (I-1), (I-2), (I-6) and (I-7):

(I-1a) $Y_{1t} = C_{1t} + I^{G}_{1t} + G_{1t} + X^{III}_{1t} + M_{2t} - M^{III}_{2t} + IE_{1t} - M_{1t}$

(I-2a) $B_{1t} = X^{III}_{1t} + M_{2t} - M^{III}_{2t} + IE_{1t} - M_{1t} + 0_{1t}$

(I-6a) $Y_{2t} = C_{2t} + I^{G}_{2t} + G_{2t} + X^{III}_{2t} + (M_{1t} - M^{III}_{1t}) - M_{2t}$

(I-7a) $B_{2t} = X^{III}_{2t} + M_{1t} - M^{III}_{1t} - M_{2t} + 0_{1t}$

$= (X^{III}_{1t} + X^{III}_{2t}) - (M^{III}_{1t} + M^{III}_{2t}) - B_{1t}$

Substituting {(I-4), (E-2), (E-3), (E-4)} into (I-1a); {(I-9), (E-7), (E-8), (E-9)} into (I-6a); {(E-3), (E-5)} into (I-2a); {(E-8, (E-5)} into (I-7a); (E-4) into (I-3) and (E-9) into (I-8) and regrouping the terms, we get the following reduced form system:

(I-1b) $(1-\alpha_{11} + \beta_{11})\, Y_{1t} - \beta_{21}\, Y_{2t} - \delta^*(1- \alpha_{11} + \beta_{11})\, K_{1t} + \beta_{21}\, \delta^*\, K_{2t}$

$= \gamma_{11}\, Y_{1t-1} + \gamma_{13}\, K_{1t-1} + G_{1t} - \gamma_{12}\, r_{1t-1}$

$+ (\alpha_{10} - \beta_{10} + \beta_{20} + \gamma_{10} - \gamma_{12}\, \delta^*) - \beta_{12}\, TT_{1t} + IE_{1t}$

$+ (\beta_{11} - \alpha_{11})\, T_{1t} + X^{III}_{1t} + \beta_{22}\, TT_{2t} - \beta_{21}\, T_{2t} - \beta_{21}\, TR_{2t} - M^{III}_{2t}$

(I-6b) $-\beta_{11} Y_{1t} + (1-\alpha_{21} + \beta_{21}) Y_{2t} - \delta^* (1-\alpha_{21} + \beta_{21}) K_{2t} + \beta_{11} \delta^* K_{1t}$

$= \gamma_{21} Y_{2t-1} + \gamma_{23} K_{2t-1} + G_{2t} - \gamma_{22} r_{2t-1}$

$+ (\alpha_{20} + \beta_{20} - \beta_{20} + \gamma_{20} - \gamma_{22} \delta^*) + \beta_{12} TT_{1t} - \beta_{11} T_{1t} - M_{1t}^{III}$

$- \beta_{22} TT_{2t} + (\beta_{21} - \alpha_{21}) T_{2t} + (\beta_{21} - \alpha_{21}) TR_{2t} + X_{2t}^{III}$

(I-2b) $\beta_{11} Y_{1t} - \beta_{21} Y_{2t} + B_{1t} - \beta_{11} \delta^* K_{1t} + \beta_{21} \delta^* K_{2t} = \eta_{11} r_{1t} - \eta_{11} r_{2t}$

$+ (\beta_{20} - \beta_{10} - \eta_{10}) - \beta_{12} TT_{1t} + \beta_{11} T_{1t} + IE_{1t} + X_{1t}^{III}$

$+ \beta_{22} TT_{2t} - \beta_{21} T_{2t} - \beta_{21} TR_{2t} - M_{2t}^{III}$

(I-7b) $B_{1t} + B_{2t} = X_{1t}^{III} - M_{1t}^{III} + X_{2t}^{III} - M_{2t}^{III}$

(I-3b) $K_{1t} = \gamma_{11} Y_{1t-1} + (1+\gamma_{13}) K_{1t-1} + (\gamma_{10} - \gamma_{12} \delta^*) - \gamma_{12} r_{1t-1}$

(I-8b) $K_{2t} = \gamma_{21} Y_{2t-1} + (1 + \gamma_{23}) K_{2t-1} + (\gamma_{20} - \gamma_{22} \delta^*) - \gamma_{22} r_{2t-1}$

To get rid of the problem of double-counting of K_{1t}, we make the following transformation of variables:

(I-11) $Y_{it} = \tilde{Y}_{it} + \delta_{i1} K_{it} = \delta_{i0} + \delta_{i2} L_{it}$ $\qquad i = 1,2$

(I-12) $Y_{it-1} = \tilde{Y}_{it-1} + \delta_{i1} K_{it-1} = \delta_{i0} + \delta_{i2} L_{it-1}$ $\qquad i = 1,2$

Substituting (I-11) and (I-12) for i = 1,2 into the above reduced form:

(I-1c) $a_{11} \tilde{Y}_{1t} - \beta_{21} \tilde{Y}_{2t} + a_{13} K_{1t} + a_{14} K_{2t} = \gamma_{11} \tilde{Y}_{1t-1} + a_{15} K_{1t-1}$

$+ G_{1t} - \gamma_{12} r_{1t-1} + a_{17} - \beta_{12} TT_{1t} + IE_{1t} + a_{18} T_{1t} + X_{1t}^{III}$

$+ \beta_{22} TT_{2t} - \beta_{21} T_{2t} - \beta_{21} TR_{2t} - M_{2t}^{III}$

(I-6c) $-\beta_{11}\tilde{Y}_{1t} + a_{22}\tilde{Y}_{2t} + a_{23}K_{1t} + a_{24}K_{2t} = \gamma_{21}\tilde{Y}_{2t-1} + a_{26}K_{2t-1}$

$+ G_{2t} - \gamma_{22}r_{2t-1} + a_{27} + \beta_{12}TT_{1t} - \beta_{11}T_{1t} - M^{III}_{1t} - \beta_{22}TT_{2t}$

$+ a_{28}T_{2t} + a_{28}TR_{2t} + X^{III}_{2t}$

(I-2c) $\beta_{11}\tilde{Y}_{1t} - \beta_{21}\tilde{Y}_{2t} + B_{1t} + a_{33}K_{1t} + a_{34}K_{2t} = \eta_{11}r_{1t} - \eta_{11}r_{2t}$

$+ a_{37} - \beta_{12}TT_{1t} + \beta_{11}T_{1t} + IE_{1t} + X^{III}_{1t} + \beta_{22}TT_{2t} - \beta_{21}T_{2t}$

$- \beta_{21}TR_{2t} - M^{III}_{2t}$

(I-7c) $B_{1t} + B_{2t} = X^{III}_{1t} - M^{III}_{1t} + X^{III}_{2t} - M^{III}_{2t}$

(I-3c) $K_{1t} = \gamma_{11}\tilde{Y}_{1t-1} + a_{55}K_{1t-1} - \gamma_{12}r_{1t-1} + a_{57}$

(I-8c) $K_{2t} = \gamma_{21}\tilde{Y}_{2t-1} + a_{66}K_{2t-1} - \gamma_{22}r_{2t-1} + a_{67}$

where:

$a_{11} = 1-\alpha_{11} + \beta_{11}$ $\qquad$ $a_{22} = 1-\alpha_{21} + \beta_{21}$

$a_{13} = (1-\alpha_{11} + \beta_{11})(\delta_{11} - \delta^*)$ $\qquad$ $a_{23} = \beta_{11}(\delta^* - \delta_{11})$

$a_{14} = \beta_{21}(\delta^* - \delta_{21})$ $\qquad$ $a_{24} = (a-\alpha_{21} + \beta_{21})(\delta_{21} - \delta^*)$

$a_{15} = (\gamma_{13} + \gamma_{11}\delta_{11})$ $\qquad$ $a_{26} = \gamma_{23} + \gamma_{21}\delta_{21}$

$a_{17} = (\alpha_{10} - \beta_{10} + \beta_{20} + \gamma_{10} - \gamma_{12}\delta^*)$

$a_{27} = \alpha_{20} + \beta_{10} - \beta_{20} + \gamma_{20} - \gamma_{22}\delta^*$

$a_{18} = \beta_{11} - \alpha_{11}$ $\qquad$ $a_{28} = \beta_{21}-\alpha_{21}$

$a_{33} = \beta_{11}(\delta_{11} - \delta^*)$

$a_{34} = \beta_{21}(\delta^* - \delta_{21})$

$a_{37} = \beta_{20} - \beta_{10} - \eta_{10}$

$a_{55} = 1 + \gamma_{13} + \gamma_{11}\delta_{11}$

$a_{57} = \gamma_{10} - \gamma_{12}\delta^*$

$a_{66} = 1 + \gamma_{23} + \gamma_{21}\delta_{21}$

$a_{67} = \gamma_{20} - \gamma_{22}\delta^*$

And in matrix notation it becomes:

$$(A\text{-}10.1)\quad \begin{bmatrix} a_{11} & -\beta_{21} & 0 & 0 & a_{13} & a_{14} \\ -\beta_{11} & a_{22} & 0 & 0 & a_{23} & a_{24} \\ \beta_{11} & -\beta_{21} & 1 & 0 & a_{33} & a_{34} \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \\ B_{1t} \\ B_{2t} \\ K_{1t} \\ K_{2t} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & 0 & 0 & 0 & a_{15} & 0 \\ 0 & \gamma_{21} & 0 & 0 & 0 & a_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_{11} & 0 & 0 & 0 & a_{55} & 0 \\ 0 & \gamma_{21} & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t-1} \\ \tilde{Y}_{2t-1} \\ B_{1t-1} \\ B_{2t-1} \\ Y_{1t-1} \\ K_{2t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G_{1t} \\ G_{2t} \\ r_{1t} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & -\gamma_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma_{12} \end{bmatrix} \begin{bmatrix} G_{1t-1} \\ G_{2t-1} \\ r_{1t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} a_{17} & 1 & 1 & 0 & -\beta_{12} & a_{18} & 0 & 0 & \beta_{22} & -\beta_{21} & -\beta_{21} & 0 & -1 \\ a_{27} & 0 & 0 & -1 & \beta_{12} & -\beta_{11} & 0 & -\gamma_{22} & -\beta_{22} & a_{28} & a_{28} & 1 & 0 \\ a_{37} & 1 & 1 & 0 & -\beta_{12} & \beta_{11} & -\eta_{11} & 0 & \beta_{22} & -\beta_{21} & -\beta_{21} & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ IE_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TE_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}$$

Multiplying both sides of the matrix-equation A-10.1 by:

$$\begin{bmatrix} a_{11} & -\beta_{21} & 0 & 0 & a_{13} & a_{14} \\ -\beta_{11} & a_{22} & 0 & 0 & a_{23} & a_{24} \\ \beta_{11} & -\beta_{21} & 1 & 0 & a_{33} & a_{34} \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & Q_{15} & Q_{16} \\ Q_{21} & Q_{22} & 0 & 0 & Q_{25} & Q_{26} \\ Q_{31} & -Q_{32} & 1 & 0 & -Q_{35} & -Q_{36} \\ -Q_{31} & Q_{32} & -1 & 1 & Q_{35} & Q_{36} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$|Q| = a_{11}\, a_{22} - \beta_{11}\, \beta_{21}$$

$$Q_{11} = \frac{a_{22}}{|Q|} \qquad Q_{12} = \frac{\beta_{21}}{|Q|}$$

$$Q_{21} = \frac{\beta_{11}}{|Q|} \qquad Q_{22} = \frac{a_{11}}{|Q|}$$

$$Q_{31} = \frac{\beta_{11}(\beta_{21} - a_{22})}{|Q|} \qquad Q_{32} = \frac{\beta_{21}(\beta_{11} - a_{11})}{|Q|}$$

$$Q_{15} = \frac{1}{|Q|}\,(a_{13}a_{22} + a_{23}\beta_{21})$$

$$Q_{25} = \frac{1}{|Q|}\,(a_{11}a_{23} + a_{13}\beta_{11})$$

$$Q_{35} = \frac{1}{|Q|}\,[a_{11}(a_{22}a_{33} + a_{23}\beta_{21}) - \beta_{11}\,\beta_{21}\,(a_{23} + a_{33}) + a_{13}\beta_{11}\,(\beta_{21} - a_{22})]$$

$$Q_{16} = \frac{1}{|Q|}\,(a_{24}\beta_{21} + a_{14}a_{22})$$

$$Q_{26} = \frac{-1}{|Q|}\,(a_{11}a_{24} + a_{14}\beta_{11})$$

$$Q_{36} = \frac{1}{|Q|} [a_{11}(a_{22}a_{34} + a_{24}\beta_{21}) - \beta_{11}\beta_{21} (a_{34} + a_{24}) + a_{14}\beta_{11} (\beta_{21} - a_{22})]$$

Then the reduced form is:

(A-10.2)

$$\begin{bmatrix} \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \\ B_{1t} \\ B_{2t} \\ K_{1t} \\ K_{2t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{13} & A_{14} \\ A_{21} & A_{22} & 0 & 0 & A_{23} & A_{24} \\ A_{31} & -A_{32} & 0 & 0 & A_{33} & -A_{34} \\ -A_{31} & A_{32} & 0 & 0 & -A_{33} & A_{34} \\ {}_{11} & 0 & 0 & 0 & a_{55} & 0 \\ 0 & {}_{21} & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t-1} \\ \tilde{Y}_{2t-1} \\ B_{1t-1} \\ B_{2t-1} \\ K_{1t-1} \\ K_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & -Q_{32} & {}_{11} \\ -Q_{31} & Q_{32} & -{}_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G_{1t} \\ G_{2t} \\ r_{1t} \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_{15} \\ 0 & 0 & a_{25} \\ 0 & 0 & -A_{35} \\ 0 & 0 & A_{35} \\ 0 & 0 & -\gamma_{12} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G_{1t-1} \\ G_{2t-1} \\ r_{1t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} D_{11} & Q_{11} & Q_{11} & -Q_{12} & D_{12} & D_{13} & 0 & A_{16} & D_{14} & D_{15} & D_{15} & Q_{12} & -Q_{11} \\ D_{21} & Q_{21} & Q_{21} & -Q_{22} & D_{22} & D_{23} & 0 & A_{26} & D_{24} & D_{25} & D_{25} & Q_{22} & -Q_{21} \\ D_{31} & 1+Q_{31} & 1+Q_{31} & Q_{32} & -D_{32} & D_{33} & -\eta_{11} & A_{36} & D_{34} & -D_{35} & -D_{35} & -Q_{32} & -(1+Q_{31}) \\ -D_{31} & -(1+Q_{31}) & -Q_{31} & -(1+Q_{32}) & D_{32} & -D_{33} & \eta_{11} & -A_{36} & -D_{34} & D_{35} & D_{35} & 1+Q_{32} & Q_{31} \\ a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ IE_{1t} \\ X_{1t}^{III} \\ M_{1t}^{III} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X_{2t}^{III} \\ M_{2t}^{III} \end{bmatrix}$$

where

$$A_{11} = \gamma_{11}(Q_{11} + Q_{15})$$

$$A_{21} = \gamma_{11}(Q_{21} + Q_{25})$$

$$A_{31} = \gamma_{11}(Q_{31} - Q_{35})$$

$$A_{12} = \gamma_{21}(Q_{12} + Q_{16})$$

$$A_{22} = \gamma_{21}(Q_{22} + Q_{26})$$

$$A_{32} = \gamma_{21}(Q_{32} + Q_{36})$$

$$A_{13} = a_{15}Q_{11} + a_{55}Q_{15}$$

$$A_{23} = a_{15}Q_{21} + a_{55}Q_{25}$$

$$A_{33} = a_{15}Q_{31} - a_{55}Q_{35}$$

$$A_{14} = a_{26}Q_{12} + a_{66}Q_{16}$$

$$A_{24} = a_{26}Q_{22} + a_{66}Q_{26}$$

$$A_{34} = a_{26}Q_{32} + a_{66}Q_{36}$$

$$A_{15} = -\gamma_{12}(Q_{11} + Q_{15})$$

$$A_{25} = -\gamma_{12}(Q_{21} + Q_{25})$$

$$A_{35} = \gamma_{12}(Q_{31} - Q_{35})$$

$$D_{11} = a_{17}Q_{11} + a_{27}Q_{12} + a_{57}Q_{15} + a_{67}Q_{16}$$

$$D_{21} = a_{17}Q_{21} + a_{27}Q_{22} + a_{57}Q_{25} + a_{67}Q_{26}$$

$$D_{31} = a_{17}Q_{31} - a_{27}Q_{32} + a_{37} - a_{57}Q_{35} - a_{67}Q_{36}$$

$$D_{12} = -\beta_{12}(Q_{11} - Q_{12})$$

$$D_{22} = -\beta_{12}(Q_{21} - Q_{22})$$

$$D_{32} = \beta_{12}(1 + Q_{31} + Q_{32})$$

$$D_{13} = a_{18}Q_{11} - \beta_{11}Q_{12}$$

$$D_{23} - a_{18}Q_{21} - \beta_{11}Q_{22}$$

$$D_{33} = a_{18}Q_{31} + \beta_{11}(1 + Q_{32})$$

$$D_{14} = \beta_{22}(Q_{11} - Q_{12})$$

$$D_{24} = \beta_{22}(Q_{21} - Q_{22})$$

$$D_{34} = \beta_{22}(1 + Q_{31} + Q_{32})$$

$$D_{15} = a_{28}Q_{12} - \beta_{21}Q_{11}$$

$$D_{25} = a_{28}Q_{22} - \beta_{21}Q_{21}$$

$$D_{35} = a_{38}Q_{32} + \beta_{21}(1 + Q_{31})$$

$A_{16} = -\gamma_{22}(Q_{12} + Q_{16})$

$A_{26} = -\gamma_{22}(Q_{22} + Q_{26})$

$A_{36} = \gamma_{22}(Q_{32} + Q_{36})$

In matrix notation, the structural form of the two-country econometric model under the assumption of linear dependent balance of payments is:

$$\text{(A-10.3)} \quad y_t = \overline{A}\, y_{t-1} + \overline{B}\, u_t + \overline{C}\, u_{t-1} + \overline{D}\, z_t$$

And Votey's estimated structural coefficients:

$\alpha_{10} = 52.57$	$\beta_{10} = 14.5131$	$\gamma_{10} = 31.707$	$\delta_{10} = 0$
$\alpha_{11} = -0.75$	$\beta_{11} = 0.1315$	$\gamma_{11} = -0.0027$	$\delta_{11} = 0.2051$
$\alpha_{12} =$ N.S.	$\beta_{12} = 12.53$	$\gamma_{12} = 461.7$	$\delta_{12} = 4.020$
$\alpha_{20} = 31.26$	$\beta_{21} = 0.098$	$\gamma_{13} = 0.0624$	$\delta^* = 0.0489$
$\alpha_{21} = 0.7345$	$\beta_{22} =$ N.S.	$\gamma_{20} = 1.5812$	$\delta_{20} = 0$
		$\gamma_{21} = 0.1668$	$\delta_{21} = 0.3634$
		$\gamma_{22} = 349.8519$	$\delta_{22} = 0.3552$
		$\gamma_{23} = 0.0429$	

$\eta_{10} = -0.6915$

$\eta_{11} = 46.279$

have been used to compute the numerical values for the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$:

$$\bar{A}=\begin{bmatrix} -0.0073 & 0.1298 & 0 & 0 & 0.0131 & 0.0806 \\ -0.0028 & 0.4538 & 0 & 0 & 0.0648 & -0.0329 \\ 0.0007 & +0.0324 & 0 & 0 & -0.0172 & 0.0201 \\ -0.0007 & -0.0324 & 0 & 0 & +0.0172 & -0.0201 \\ -0.0027 & 0 & 0 & 0 & 1.0618 & 0 \\ 0 & 0.1668 & 0 & 0 & 0 & 1.1033 \end{bmatrix}$$

$$;\bar{B}=\begin{bmatrix} 2.8932 & 0.7782 & 0 \\ 1.0475 & 3.0351 & 0 \\ -0.2785 & 0.1942 & 46.279 \\ 0.2785 & -0.1942 & -46.279 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ;\bar{C}=\begin{bmatrix} 0 & 0 & -1263.7107 \\ 0 & 0 & -\ 483.6539 \\ 0 & 0 & +\ 128.5835 \\ 0 & 0 & -\ 128.5835 \\ 0 & 0 & -\ 461.7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{D}=\begin{bmatrix}
129.6688 & 2.8932 & 2.8932 & -0.7782 & 59.9751 & -1.8918 & 0 & -272.2641 & NS & -0.7782 & -0.7782 & 0.7782 & -2.8932 \\
162.4172 & 1.0475 & 1.0475 & -3.0351 & -56.3610 & -1.0470 & 0 & -951.8105 & NS & -2.0351 & -2.0351 & 3.0351 & -1.0475 \\
-28.3120 & 0.7215 & 0.7215 & -0.1942 & 14.9535 & 0.2782 & -46.279 & -67.9307 & NS & -0.1942 & -0.1942 & 0.1942 & -0.7215 \\
28.3120 & -0.7215 & 0.2785 & -0.8058 & -14.9535 & -0.2782 & 46.279 & 67.9307 & NS & 0.1942 & 0.1942 & 0.80583 & -0.2785 \\
9.1299 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18.6889 & 0 & 0 & 0 & 0 & 0 & 0 & -349.8519 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

APPENDIX A-11

COMPUTATIONS OF THE OPTIMAL SOLUTION FOR THE TWO-COUNTRY CONTROL PROBLEM (CASE B)

The optimal result is given by equation 2-8

$$(2\text{-}8)\quad u^*_t = -(B^{c\prime}Q^cB^c)^{-1}\, B^cQ^cA^c\, x^{c*}_{t-1} + (B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^c\overline{x}^c_t - (B^{c\prime}Q^cB^c)^{-1}\, B^{c\prime}Q^cC^cz^c_t$$

where

$$x^{c*}_t = \left[\tilde{Y}^*_{1t}\,,\ \tilde{Y}^*_{2t}\,,\ B^*_{1t}\,,\ B^*_{2t},\ K^*_{1t},\ K^*_{2t},\ G^*_{1t},\ G^*_{2t},\ r^*_{1t}\right]'$$

$$u^{c*}_t = \left[G^*_{1t},\ G^*_{25},\ r^*_{1t}\right]'$$

$$z^c_t = \left[1,\ IE_{1t},\ X^{III}_{1t}\,,\ M^{III}_{1t}\,,\ TT_{1t},\ T_{1t},\ r_{2t},\ r_{2t-1},\ TT_{2t},\ T_{2t},\ TR_{2t},\ X^{III}_{2t}\,,\ M^{III}_{2t}\right]'$$

$$\overline{x}^c_t = \left[\tilde{Y}_{1t},\ \tilde{Y}_{2t},\ \overline{B}_{1t},\ \overline{B}_{2t},\ \overline{K}_{1t},\ \overline{K}_{2t},\ \overline{G}_{1t},\ \overline{G}_{2t},\ \overline{r}_{1t}\right]' \underline{\underline{d}} \left[\delta_{10} + \delta_{12}\, L_{1t},\ \delta_{20} + \delta_{22}\, L_{2t},\ 0,\ 0,\ \overline{K}_{1t},\ \overline{K}_{2t},\ \overline{G}_{1t},\ \overline{G}_{2t},\ \overline{r}_{1t}\right]'$$

$$A^c = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{13} & A_{14} & 0 & 0 & A_{15} \\ A_{21} & A_{22} & 0 & 0 & A_{23} & A_{24} & 0 & 0 & A_{25} \\ A_{31} & -A_{32} & 0 & 0 & A_{33} & -A_{34} & 0 & 0 & -A_{35} \\ -A_{31} & A_{32} & 0 & 0 & -A_{33} & A_{34} & 0 & 0 & A_{35} \\ \gamma_{11} & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & -\gamma_{12} \\ 0 & \gamma_{21} & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$;B^c = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & -Q_{32} & \eta_{11} \\ -Q_{31} & Q_{32} & -\eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ;Q^c = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C^c = \begin{bmatrix}
D_{11} & Q_{11} & Q_{11} & -Q_{12} & D_{12} & D_{13} & 0 & A_{16} & D_{14} & D_{15} & D_{15} & Q_{12} & -Q_{11} \\
D_{21} & Q_{21} & Q_{21} & -Q_{22} & D_{22} & D_{23} & 0 & A_{26} & D_{24} & D_{25} & D_{25} & Q_{22} & -Q_{21} \\
D_{31} & 1+Q_{31} & 1+Q_{31} & Q_{32} & -D_{32} & D_{33} & -\eta_{11} & A_{36} & D_{34} & -D_{35} & -D_{35} & -Q_{32} & -(1+Q_{31}) \\
-D_{31} & -(1+Q_{31}) & -Q_{31} & -(1+Q_{32}) & D_{32} & -D_{33} & \eta_{11} & -A_{36} & -D_{34} & D_{35} & D_{35} & 1+Q_{32} & Q_{31} \\
a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Let us compute

$$B^{c\prime}Q^{c} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} & -Q_{31} & 0 & 0 & 1 & 0 & 0 \\ Q_{12} & Q_{22} & -Q_{32} & Q_{32} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta_{11} & -\eta_{11} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} q_{11}Q_{11} & q_{22}Q_{21} & q_{33}Q_{31} & -q_{44}Q_{31} & 0 & 0 & 0 & 0 & 0 \\ q_{11}Q_{12} & q_{22}Q_{22} & -q_{33}Q_{32} & q_{44}Q_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33}\eta_{11} & -q_{44}\eta_{11} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(B^{c\prime}Q^cB^c) = \begin{bmatrix} q_{11}Q_{11}^2 + q_{22}Q_{21}^2 + (q_{33} + q_{44})Q_{31}^2 & q_{11}Q_{11}Q_{12} + q_{22}Q_{21}Q_{22} - (q_{33} + q_{44})\,Q_{31}Q_{32} & (q_{33} + q_{44})\eta_{11}Q_{31} \\ q_{11}Q_{11}Q_{12} + q_{22}\,Q_{21}Q_{22} - (q_{33}+q_{44})\,Q_{31}Q_{32} & q_{11}Q_{12}^2 + q_{22}Q_{22}^2 + (q_{33}+q_{44})Q_{32}^2 & -(q_{33} + q_{44})\eta_{11}Q_{32} \\ (q_{33} + q_{44})\,\eta_{11}Q_{31} & -(q_{33}+q_{44})\,\eta_{11}Q_{32} & (q_{33} + q_{44})\eta_{11}^2 \end{bmatrix}$$

As before, since $(B^{c\prime}Q^cB^c)$ is symmetric, the inverse of $(B^{c\prime}Q^cB^c)$ is obtained by the method of partitioning (Appendixes A-7 and A-8), which leads to:

$$(B^{c\prime}Q^cB^c)^{-1} = \frac{1}{(b_{11}b_{22}-b_{21}^2)b_{33}-b_{11}b_{32}^2-b_{22}b_{31}^2+2b_{21}b_{31}b_{32}} \begin{bmatrix} b_{22}b_{33} - b_{32}^2 & b_{31}b_{32} - b_{21}b_{33} & b_{22}b_{31} - b_{21}b_{32} \\ b_{31}b_{32} - b_{21}b_{33} & b_{11}b_{33} - b_{31}^2 & b_{11}b_{32} - b_{21}b_{31} \\ b_{22}b_{31} - b_{21}b_{32} & b_{11}b_{32} - b_{21}b_{31} & b_{11}b_{22} - b_{21}^2 \end{bmatrix}$$

where

$$b_{11} \stackrel{d}{=} q_{11}Q_{11}^2 + q_{22}Q_{21}^2 + (q_{33} + q_{44})Q_{31}^2$$

$$b_{22} \stackrel{d}{=} q_{11}Q_{12}^2 + q_{22}Q_{22}^2 + (q_{33} + q_{44})Q_{32}^2$$

$$b_{33} \stackrel{d}{=} (q_{33} + q_{44})\eta_{11}^2$$

$$b_{21} \stackrel{d}{=} q_{11}Q_{11}Q_{12} + q_{22}Q_{21}Q_{22} - (q_{33} + q_{44})Q_{31}Q_{32}$$

$$b_{31} \stackrel{d}{=} (q_{33} + q_{44})\eta_{11}Q_{31}$$

$$b_{32} \stackrel{d}{=} -(q_{33} + q_{44})\eta_{11}Q_{32}$$

Substituting the b_{ii}'s and b_{ij}'s ($i = 1,2,3$; $j = 1,2,3$) by their corresponding values, we get:

$$(b_{11}b_{22} - b_{21}^2)\, b_{33} - b_{11}b_{32}^2 - b_{22}b_{31}^2 + 2b_{21}b_{31}b_{32} \stackrel{d}{=} q_{11}q_{22}\,(q_{33} + q_{44})\, \eta_{11}^2\,(Q_{12}Q_{21} - Q_{11}Q_{22})^2$$

$$b_{22}b_{33} - b_{32}^2 \stackrel{d}{=} q_{11}\,(q_{33} + q_{44})\,\eta_{11}^2\,Q_{12}^2 + q_{22}(q_{33} + q_{44})\eta_{11}^2Q_{22}^2$$

$$b_{11}b_{33} - b_{31}^2 \stackrel{d}{=} q_{11}(q_{33} + q_{44})\eta_{11}^2Q_{11}^2 + q_{22}(q_{33} + q_{44})\,\eta_{11}^2\,Q_{21}^2$$

$$b_{11}b_{22} - b_{21}^2 \stackrel{d}{=} q_{11}q_{22}\,(Q_{12}Q_{21} - Q_{11}Q_{22})^2 + q_{11}(q_{33} + q_{44})(Q_{12}Q_{31} + Q_{11}Q_{32})^2 + q_{22}(q_{33} + q_{44})(Q_{22}Q_{31} + Q_{21}Q_{32})^2$$

$$b_{31}b_{32} - b_{21}b_{33} \stackrel{d}{=} -(q_{33} + q_{44})\,\eta_{11}^2\,(q_{11}Q_{11}Q_{12} + q_{22}Q_{21}Q_{22})$$

$$b_{22}b_{31} - b_{21}b_{32} \stackrel{d}{=} q_{11}(q_{33} + q_{44})\,\eta_{11}Q_{12}(Q_{12}Q_{31} + Q_{11}Q_{32}) + q_{22}\,(q_{33} + q_{44})\,\eta_{11}Q_{22}\,(Q_{22}Q_{31} + Q_{21}Q_{32})$$

$$b_{22}b_{31} - b_{21}b_{32} \stackrel{d}{=} q_{11}(q_{33} + q_{44})\, n_{11}Q_{11}(Q_{12}Q_{31} + Q_{11}Q_{32}) + q_{22}\,(q_{33} + q_{44})\, n_{11}Q_{21}\,(Q_{22}Q_{31} + Q_{21}Q_{32}$$

and

$$(B^{c\prime}Q^{c}B^{c})^{-1} = \begin{bmatrix} \dfrac{q_{11}Q_{12}^{2} + q_{22}Q_{22}^{2}}{q_{11}q_{22}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} & \dfrac{-(q_{11}Q_{11}Q_{12}+q_{22}Q_{21}Q_{22})}{q_{11}q_{22}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} & \dfrac{q_{11}Q_{12}(Q_{12}Q_{31}+Q_{11}Q_{32})+q_{22}Q_{22}(Q_{22}Q_{31}+Q_{21}Q_{32})}{q_{11}q_{22}n_{11}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} \\ \dfrac{-(q_{11}Q_{11}Q_{12}+q_{22}Q_{21}Q_{22})}{q_{11}q_{22}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} & \dfrac{q_{11}Q_{11}^{2} + q_{22}Q_{21}^{2}}{q_{11}q_{22}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} & \dfrac{-q_{11}Q_{11}(Q_{12}Q_{31}+Q_{11}Q_{32})-q_{22}Q_{21}(Q_{22}Q_{31}+Q_{21}Q_{32})}{q_{11}q_{22}n_{11}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} \\ \dfrac{q_{11}Q_{12}(Q_{12}Q_{31}+Q_{11}Q_{32})+q_{22}Q_{22}(Q_{22}Q_{31}+Q_{21}Q_{32})}{q_{11}q_{22}n_{11}(Q_{12}Q_{21} - Q_{11}Q_{22})^{2}} & \dfrac{-q_{11}Q_{11}(Q_{12}Q_{31}+Q_{11}Q_{32})\ q_{22}Q_{21}(Q_{22}Q_{31}+Q_{21}Q_{32})}{q_{11}q_{22}n_{11}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} & \dfrac{1}{(q_{33}+q_{44})n_{11}^{2}} + \dfrac{(Q_{12}Q_{31}+Q_{11}Q_{32})^{2}}{q_{22}n_{11}^{2}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} + \dfrac{(Q_{22}Q_{31}+Q_{21}Q_{32})}{q_{11}n_{11}^{2}(Q_{12}Q_{21}-Q_{11}Q_{22})^{2}} \end{bmatrix}$$

Now

(A-11.1)

$$(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c = \begin{bmatrix} \frac{-Q_{22}}{Q} & \frac{Q_{12}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{Q_{21}}{Q} & \frac{-Q_{11}}{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \frac{-(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{-q_{44}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$Q \stackrel{d}{=} Q_{12}Q_{21} - Q_{11}Q_{22}$$

(A-11.2)

$$(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^cA^c = \begin{bmatrix} \theta_{11} & \theta_{12} & 0 & 0 & \theta_{13} & \theta_{14} & 0 & 0 & \theta_{15} \\ \theta_{21} & \theta_{22} & 0 & 0 & \theta_{23} & \theta_{24} & 0 & 0 & \theta_{25} \\ \theta_{31} & \theta_{32} & 0 & 0 & \theta_{33} & \theta_{34} & 0 & 0 & \theta_{35} \end{bmatrix}$$

where

$$\theta_{11} \overset{d}{=} \frac{1}{Q}\left[A_{21}Q_{12} - A_{11}Q_{22}\right] \qquad \theta_{21} \overset{d}{=} \frac{1}{Q}\left[A_{11}Q_{21} - A_{21}Q_{11}\right]$$

$$\theta_{12} \overset{d}{=} \frac{1}{Q}\left[A_{22}Q_{12} - A_{12}Q_{22}\right] \qquad \theta_{22} \overset{d}{=} \frac{1}{Q}\left[A_{12}Q_{21} - A_{22}Q_{11}\right]$$

$$\theta_{13} \overset{d}{=} \frac{1}{Q}\left[A_{23}Q_{12} - A_{13}Q_{22}\right] \qquad \theta_{23} \overset{d}{=} \frac{1}{Q}\left[A_{13}Q_{21} - A_{23}Q_{11}\right]$$

$$\theta_{14} \overset{d}{=} \frac{1}{Q}\left[A_{24}Q_{12} - A_{14}Q_{22}\right] \qquad \theta_{24} \overset{d}{=} \frac{1}{Q}\left[A_{14}Q_{21} - A_{24}Q_{11}\right]$$

$$\theta_{15} \overset{d}{=} \frac{1}{Q}\left[A_{25}Q_{12} - A_{15}Q_{22}\right] \qquad \theta_{25} \overset{d}{=} \frac{1}{Q}\left[A_{15}Q_{21} - A_{25}Q_{11}\right]$$

$$\theta_{31} \overset{d}{=} \frac{1}{\eta_{11}}\left[A_{31} + \frac{A_{11}(Q_{22}Q_{31}+Q_{21}Q_{32})-A_{21}(Q_{12}Q_{31}+Q_{11}Q_{32})}{Q}\right]$$

$$\theta_{32} \overset{d}{=} \frac{1}{\eta_{11}}\left[A_{32} + \frac{A_{12}(Q_{22}Q_{31}+Q_{21}Q_{32})-A_{22}(Q_{12}Q_{31}+Q_{11}Q_{32})}{Q}\right]$$

$$\theta_{33} \overset{d}{=} \frac{1}{\eta_{11}}\left[A_{33} + \frac{A_{13}(Q_{22}Q_{31}+Q_{21}Q_{32})-A_{23}(Q_{12}Q_{31}+Q_{11}Q_{32})}{Q}\right]$$

$$\theta_{34} \overset{d}{=} \frac{1}{\eta_{11}}\left[A_{34} + \frac{A_{14}(Q_{22}Q_{31}+Q_{21}Q_{32})-A_{24}(Q_{12}Q_{31}+Q_{11}Q_{32})}{Q}\right]$$

$$\theta_{35} \overset{d}{=} \frac{1}{\eta_{11}}\left[-A_{35} + \frac{A_{15}(Q_{22}Q_{31}+Q_{21}Q_{32})-A_{25}(Q_{12}Q_{31}+Q_{11}Q_{32})}{Q}\right]$$

(A-11.3)

$$(B^{c\prime}Q^{c}B^{c})^{-1}B^{c\prime}Q^{c}C^{c} = \begin{bmatrix} \Delta_{10} & \Delta_{11} & \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} & 0 & \Delta_{15} & \Delta_{16} & \Delta_{17} & \Delta_{17} & \Delta_{18} & \Delta_{19} \\ \Delta_{20} & \Delta_{21} & \Delta_{21} & \Delta_{22} & \Delta_{23} & \Delta_{24} & 0 & \Delta_{25} & \Delta_{26} & \Delta_{27} & \Delta_{27} & \Delta_{28} & \Delta_{29} \\ \Delta_{30} & \Delta_{31} & \Delta_{41} & \Delta_{32} & \Delta_{33} & \Delta_{34} & 1 & \Delta_{35} & \Delta_{36} & \Delta_{37} & \Delta_{37} & \Delta_{38} & \Delta_{39} \end{bmatrix}$$

where

$$\Delta_{10} \stackrel{d}{=} \frac{1}{Q}(-D_{11}Q_{22} + D_{21}Q_{12}) \qquad \Delta_{20} \stackrel{d}{=} \frac{1}{Q}(D_{11}Q_{21} - D_{21}Q_{11})$$

$$\Delta_{11} \stackrel{d}{=} \frac{1}{Q}(-Q_{11}Q_{22} + Q_{12}Q_{21}) \qquad \Delta_{21} \stackrel{d}{=} \frac{1}{Q}(Q_{11}Q_{21} - Q_{11}Q_{21})$$

$$\Delta_{12} \stackrel{d}{=} \frac{1}{Q}(Q_{12}Q_{22} - Q_{12}Q_{22}) \qquad \Delta_{22} \stackrel{d}{=} \frac{1}{Q}(-Q_{12}Q_{21} + Q_{11}Q_{22})$$

$$\Delta_{13} \stackrel{d}{=} \frac{1}{Q}(-D_{12}Q_{22} + D_{22}Q_{12}) \qquad \Delta_{23} \stackrel{d}{=} \frac{1}{Q}(D_{12}Q_{21} - D_{22}Q_{11})$$

$$\Delta_{14} \stackrel{d}{=} \frac{1}{Q}(-D_{13}Q_{22} + D_{23}Q_{12}) \qquad \Delta_{24} \stackrel{d}{=} \frac{1}{Q}(D_{13}Q_{21} - D_{23}Q_{11})$$

$$\Delta_{15} \stackrel{d}{=} \frac{1}{Q}(-A_{16}Q_{22} + A_{26}Q_{12}) \qquad \Delta_{25} \stackrel{d}{=} \frac{1}{Q}(A_{16}Q_{21} - A_{26}Q_{11})$$

$$\Delta_{16} \stackrel{d}{=} \frac{1}{Q}(-D_{14}Q_{22} + D_{24}Q_{12}) \qquad \Delta_{26} \stackrel{d}{=} \frac{1}{Q}(D_{14}Q_{21} - D_{24}Q_{11})$$

$$\Delta_{17} \overset{d}{=} \frac{1}{Q}(-D_{15}Q_{22} + D_{25}Q_{12}) \qquad \Delta_{27} \overset{d}{=} \frac{1}{Q}(D_{15}Q_{21} - D_{25}Q_{11})$$

$$\Delta_{18} \overset{d}{=} \frac{1}{Q}(-Q_{12}Q_{22} + Q_{12}Q_{22}) \qquad \Delta_{28} \overset{d}{=} \frac{1}{Q}(Q_{12}Q_{21} - Q_{11}Q_{22})$$

$$\Delta_{19} \overset{d}{=} \frac{1}{Q}(Q_{11}Q_{22} - Q_{12}Q_{21}) \qquad \Delta_{29} \overset{d}{=} \frac{1}{Q}(-Q_{11}Q_{21} + Q_{11}Q_{21})$$

$$\Delta_{30} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{11}(Q_{22}Q_{31}+Q_{21}Q_{32})-D_{21}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] + \frac{D_{31}}{\eta_{11}}$$

$$\Delta_{31} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[Q_{11}(Q_{22}Q_{31}+Q_{21}Q_{32})-Q_{21}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] + \frac{1+Q_{31}}{\eta_{11}}$$

$$\Delta_{41} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[Q_{11}(Q_{22}Q_{31}+Q_{21}Q_{32})-Q_{21}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] + \frac{Q_{31}}{\eta_{11}} + \frac{q_{33}}{(q_{33}+q_{44})}$$

$$\Delta_{32} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[-Q_{12}(Q_{22}Q_{31}+Q_{21}Q_{32})+Q_{22}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] + \frac{Q_{32}}{\eta_{11}} + \frac{q_{44}}{(q_{33}+q_{44})}$$

$$\Delta_{33} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{12}(Q_{22}Q_{31}+Q_{21}Q_{32})-D_{22}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] - \frac{D_{32}}{\eta_{11}}$$

$$\Delta_{34} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{13}(Q_{22}Q_{31}+Q_{21}Q_{32})-D_{23}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] + \frac{D_{33}}{\eta_{11}}$$

$$\Delta_{35} \overset{d}{=} \frac{1}{\eta_{11}Q}\left|A_{16}(Q_{22}Q_{31}+Q_{21}Q_{32})-A_{26}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] + \frac{A_{36}}{\eta_{11}}$$

$$\Delta_{36} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{14}(Q_{22}Q_{31}+Q_{21}Q_{32})-D_{24}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] + \frac{D_{34}}{\eta_{11}}$$

$$\Delta_{37} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[D_{15}(Q_{22}Q_{31}+Q_{21}Q_{32})-D_{25}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] - \frac{D_{35}}{\eta_{11}}$$

$$\Delta_{38} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[Q_{12}(Q_{22}Q_{31}+Q_{21}Q_{32})-Q_{22}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] - \frac{Q_{32}}{\eta_{11}} - \frac{q_{44}}{(q_{33}+q_{44})}$$

$$\Delta_{39} \overset{d}{=} \frac{1}{\eta_{11}Q}\left[-Q_{11}(Q_{22}Q_{31}+Q_{21}Q_{32})+Q_{21}(Q_{12}Q_{31}+Q_{11}Q_{32})\right] - \frac{Q_{31}}{\eta_{11}} - \frac{q_{33}}{(q_{33}+q_{44})}$$

Then the optimal control is:

$$
(\text{A-11.4})\quad
\begin{bmatrix} G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
=
\begin{bmatrix}
-\theta_{11} & -\theta_{12} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15} \\
-\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25} \\
-\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35}
\end{bmatrix}
\begin{bmatrix} \bar{Y}^*_{1t-1} \\ \bar{Y}^*_{2t-1} \\ B^*_{1t-1} \\ B^*_{2t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
$$

$$
+\begin{bmatrix}
-Q_{22}/Q & +Q_{12}/Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{21}/Q & -Q_{11}/Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\dfrac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \dfrac{-(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \dfrac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \dfrac{-q_{44}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \bar{Y}_{1t} \\ \bar{Y}_{2t} \\ \bar{B}_{1t} \\ \bar{B}_{2t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix}
$$

$$
+\begin{bmatrix}
-\Delta_{10} & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\
-\Delta_{20} & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\
-\Delta_{30} & -\Delta_{31} & -\Delta_{41} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & 1 & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39}
\end{bmatrix}
\begin{bmatrix} 1 \\ IR_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

Since the targets $\bar{\tilde{Y}}_{1t}$, $\bar{\tilde{Y}}_{2t}$, $\bar{B}_{1t}$, $\bar{B}_{2t}$ are defined as follows

$$\bar{\tilde{Y}}_{1t} = \delta_{10} + \delta_{12} L_{1t}$$

$$\bar{\tilde{Y}}_{2t} = \delta_{20} + \delta_{22} L_{2t}$$

$$\bar{B}_{1t} = 0$$

$$\bar{B}_{2t} = 0$$

after substitution and rearrangement of terms, equation A-11.4 becomes:

$$
(A\text{-}11.5)\quad
\begin{bmatrix} G^{*}_{1t} \\ G^{*}_{2t} \\ r^{*}_{1t} \end{bmatrix}
=
\begin{bmatrix}
-\theta_{11} & -\theta_{12} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15} \\
-\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25} \\
-\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35}
\end{bmatrix}
\begin{bmatrix} \bar{Y}^{*}_{1t-1} \\ \bar{Y}^{*}_{2t-1} \\ B^{*}_{1t-1} \\ B^{*}_{2t-1} \\ K^{*}_{1t-1} \\ K^{*}_{2t-1} \\ G^{*}_{1t-1} \\ G^{*}_{2t-1} \\ r^{*}_{1t-1} \end{bmatrix}
+
\left[
\begin{matrix}
-\Delta_{10} + \frac{1}{Q}(Q_{22}\delta_{10} - Q_{12}\delta_{20}) & \frac{Q_{22}\delta_{12}}{Q} \\
-\Delta_{20} + \frac{1}{Q}(-Q_{21}\delta_{10} + Q_{11}\ \delta_{20}) & \frac{-Q_{21}\delta_{12}}{Q} \\
-\Delta_{30} + \frac{1}{\eta_{11}Q}(Q_{21}Q_{32}+Q_{22}Q_{31})\delta_{10}-\delta_{20}(Q_{11}Q_{32}-Q_{12}Q_{31}) & \frac{12}{\eta_{11}Q}(Q_{21}Q_{32}+Q_{22}Q_{31})
\end{matrix}
\right.
$$

$$
\left.
\begin{matrix}
\frac{-Q_{12}\delta_{22}}{Q} & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\
\frac{Q_{11}\delta_{22}}{Q} & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\
\frac{-\delta_{22}}{\eta_{11}Q}\, Q_{11}Q_{32}+Q_{12}Q_{31} & -\Delta_{31} & -\Delta_{41} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & 1 & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39}
\end{matrix}
\right]
\begin{bmatrix} 1 \\ L_{1t} \\ L_{2t} \\ IE_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

APPENDIX A-12

EVALUATION OF WELFARE COST FOR THE TWO-COUNTRY CONTROL PROBLEM (CASE B)

Under the assumption that the Certainty Equivalence Principle holds the welfare cost is determined by:

$$\text{(A-6.1)} \quad \hat{J}^c = \frac{1}{2} \sum_{t=1}^{N} (x_t^{c*} - \bar{x}_t^c)' Q \, (x_t^{c*} - \bar{x}_t^c)$$

where

$$\text{(A-6.2)} \quad x_t^{c*} - \bar{x}_t^c = \left[I - B^c (B^{c\prime} Q^c B^c)^{-1} B^{c\prime} Q^c\right] A^c \, x_{t-1}^{c*} + \left[B^c (B^{c\prime} Q^c B^c)^{-1} B^{c\prime} Q^c - I\right] \bar{x}_t^c + \left[I - B^c (B^{c\prime} Q^c B^c)^{-1} B^{c\prime} Q^c\right] C^c z_t^c$$

Let us compute for the two-country problem with the assumption of linear dependent external balance the following matrices:

$$B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & -Q_{32} & \eta_{11} \\ -Q_{31} & Q_{32} & -\eta_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -Q_{22}/Q & Q_{12}/Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{21}/Q & -Q_{11}/Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \frac{-(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{-q_{44}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{q_{33}}{(q_{33}+q_{44})} & \frac{-q_{44}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-q_{33}}{(q_{33}+q_{44})} & \frac{q_{44}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -Q_{22}/Q & Q_{12}/Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{21}/Q & -Q_{11}/Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \frac{-(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{-q_{44}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[I-B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{q_{44}}{(q_{33}+q_{44})} & \frac{q_{44}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{q_{33}}{(q_{33}+q_{44})} & \frac{q_{33}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
Q_{22}/Q & -Q_{12}/Q & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
-Q_{21}/Q & Q_{11}/Q & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
\frac{-(Q_{22}Q_{31}+Q_{21}Q_{32})}{\eta_{11}Q} & \frac{Q_{12}Q_{31}+Q_{11}Q_{32}}{\eta_{11}Q} & \frac{-q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{q_{44}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\left[I-B^c(B^{c\prime}Q^cB^c)^{-1}B^{c\prime}Q^c\right]A^c=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\gamma_{11} & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & -\gamma_{12}\\
0 & \gamma_{21} & 0 & 0 & 0 & a_{66} & 0 & 0 & 0\\
-\theta_{11} & -\theta_{12} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15}\\
-\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25}\\
-\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35}
\end{bmatrix}$$

$$\left[I-B^{c}(B^{c\prime}Q^{c}B^{c})^{-1}B^{c\prime}Q^{c}\right]C^{c} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q_{44}}{(q_{33}+q_{44})} & \frac{-q_{44}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{44}}{(q_{33}+q_{44})} & \frac{-q_{44}}{(q_{33}+q_{44})} \\
0 & 0 & \frac{q_{33}}{(q_{33}+q_{44})} & \frac{-q_{33}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{33}}{(q_{33}+q_{44})} & \frac{-q_{33}}{(q_{33}+q_{44})} \\
a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \\
-\Delta_{10} & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\
-\Delta_{20} & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\
-\Delta_{30} & -\Delta_{31} & -\Delta_{41} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & 1 & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39}
\end{bmatrix}$$

$$\left[B^{c}(B^{c\prime}Q^{c}B^{c})^{-1}B^{c\prime}Q^{c}-I^{c}\right] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-q_{44}}{(q_{33}+q_{44})} & \frac{-q_{44}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-q_{33}}{(q_{33}+q_{44})} & \frac{-q_{33}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-Q_{22}/Q & Q_{12}/Q & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
Q_{21}/Q & -Q_{11}/Q & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\frac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \frac{(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

Then substituting the above matrices into equation A-5.2 we get

$$
(\text{A-12.1}) \quad
\begin{bmatrix} Y^*_{1t} \\ Y^*_{2t} \\ B^*_{1t} \\ B^*_{2t} \\ K^*_{1t} \\ K^*_{2t} \\ G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
-
\begin{bmatrix} \bar{Y}_{1t} \\ \bar{Y}_{2t} \\ \bar{B}_{1t} \\ \bar{B}_{2t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_{11} & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & -\gamma_{12} \\
0 & \gamma_{21} & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 \\
-\theta_{11} & -\theta_{12} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15} \\
-\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25} \\
-\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35}
\end{bmatrix}
\begin{bmatrix} \bar{Y}^*_{1t-1} \\ \bar{Y}^*_{2t-1} \\ B^*_{1t-1} \\ B^*_{2t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
$$

$$
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-q_{44}}{(q_{33}+q_{44})} & \frac{-q_{44}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-q_{33}}{(q_{33}+q_{44})} & \frac{-q_{33}}{(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-Q_{22}/Q & Q_{12}/Q & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
Q_{21}/Q & -Q_{11}/Q & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\frac{Q_{22}Q_{31}+Q_{21}Q_{32}}{\eta_{11}Q} & \frac{-(Q_{12}Q_{31}+Q_{11}Q_{32})}{\eta_{11}Q} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & \frac{q_{33}}{\eta_{11}(q_{33}+q_{44})} & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix} \bar{Y}_{1t} \\ \bar{Y}_{2t} \\ \bar{B}_{1t} \\ \bar{B}_{2t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix},
$$

$$
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q_{44}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{44}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} \\
0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{33}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{33}}{q_{33}+q_{44}} \\
a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \\
-\Delta_{10} & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\
-\Delta_{20} & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\
-\Delta_{30} & -\Delta_{31} & -\Delta 41 & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & 1 & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39}
\end{bmatrix}
\begin{bmatrix} 1 \\ IE_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

Substituting $\bar{\bar{Y}}_{1t} = \delta_{10} + \delta_{12} L_{1t}$, $\bar{\bar{Y}}_{2t} = \delta_{20} - \delta_{22} L_{2t}$, $\bar{B}_{1t} = 0$, $\bar{B}_{2t} = 0$ into equation A-12.1 and rearranging terms:

$$
\text{(A-12.2)} \quad
\begin{bmatrix} \tilde{Y}^*_{1t} \\ \tilde{Y}^*_{2t} \\ B^*_{1t} \\ B^*_{2t} \\ K^*_{1t} \\ K^*_{2t} \\ G^*_{1t} \\ G^*_{2t} \\ r^*_{1t} \end{bmatrix}
-
\begin{bmatrix} \bar{\bar{Y}}_{1t} \\ \bar{\bar{Y}}_{2t} \\ \bar{B}_{1t} \\ \bar{B}_{2t} \\ \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix}
=
\left[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_{11} & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & -\gamma_{12} \\
0 & \gamma_{21} & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 \\
-\theta_{11} & -\theta_{12} & 0 & 0 & -\theta_{13} & -\theta_{14} & 0 & 0 & -\theta_{15} \\
-\theta_{21} & -\theta_{22} & 0 & 0 & -\theta_{23} & -\theta_{24} & 0 & 0 & -\theta_{25} \\
-\theta_{31} & -\theta_{32} & 0 & 0 & -\theta_{33} & -\theta_{34} & 0 & 0 & -\theta_{35}
\end{array}\right]
\begin{bmatrix} \tilde{Y}^*_{1t-1} \\ \tilde{Y}^*_{2t-1} \\ B^*_{1t-1} \\ B^*_{2t-1} \\ K^*_{1t-1} \\ K^*_{2t-1} \\ G^*_{1t-1} \\ G^*_{2t-1} \\ r^*_{1t-1} \end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix} \bar{K}_{1t} \\ \bar{K}_{2t} \\ \bar{G}_{1t} \\ \bar{G}_{2t} \\ \bar{r}_{1t} \end{bmatrix}
$$

$$
+ \left[\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{q_{44}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{44}}{q_{33}+q_{44}} & \frac{-q_{44}}{q_{33}+q_{44}} \\
0 & 0 & 0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{33}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{33}}{q_{33}+q_{44}} & \frac{-q_{33}}{q_{33}+q_{44}} \\
a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_{22} & 0 & 0 & 0 & 0 & 0 \\
-\Delta_{10} + \frac{1}{Q}(-Q_{22}\delta_{10} + Q_{12}\,\delta_{20}) & -(Q_{22}\ {}_{12})/Q & +(Q_{12}\ {}_{22})/Q & -\Delta_{11} & -\Delta_{11} & -\Delta_{12} & -\Delta_{13} & -\Delta_{14} & 0 & -\Delta_{15} & -\Delta_{16} & -\Delta_{17} & -\Delta_{17} & -\Delta_{18} & -\Delta_{19} \\
-\Delta_{20} + \frac{1}{Q}(Q_{21}\delta_{10} - Q_{11}\,\delta_{20}) & +(Q_{21}\ {}_{12})/Q & -(Q_{11}\ {}_{22})/Q & -\Delta_{21} & -\Delta_{21} & -\Delta_{22} & -\Delta_{23} & -\Delta_{24} & 0 & -\Delta_{25} & -\Delta_{26} & -\Delta_{27} & -\Delta_{27} & -\Delta_{28} & -\Delta_{29} \\
-\Delta_{30} + \frac{1}{\eta_{11}Q}\left[\delta_{10}(Q_{21}Q_{32}+Q_{22}Q_{31}) - \delta_{20}(Q_{11}Q_{32}-Q_{12}Q_{31})\right] & \frac{\delta_{12}}{\eta_{11}Q}(Q_{21}Q_{32}+Q_{22}Q_{31}) & \frac{-\delta_{22}}{\eta_{11}Q}(Q_{11}Q_{32}+Q_{12}Q_{31}) & -\Delta_{31} & -\Delta_{41} & -\Delta_{32} & -\Delta_{33} & -\Delta_{34} & & -\Delta_{35} & -\Delta_{36} & -\Delta_{37} & -\Delta_{37} & -\Delta_{38} & -\Delta_{39}
\end{array}\right]
\begin{bmatrix} 1 \\ L_{1t} \\ L_{2t} \\ IE_{1t} \\ X^{III}_{1t} \\ M^{III}_{1t} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X^{III}_{2t} \\ M^{III}_{2t} \end{bmatrix}
$$

Multiplying

$$Q^c = \begin{bmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

to equation A-12.2, we get:

(A-12.3)

$$Q^c(x_t^{c*} - \bar{x}_t^c) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q_{33}q_{44}}{q_{33}+q_{44}} & \frac{-q_{33}q_{44}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{33}q_{44}}{q_{33}+q_{44}} & \frac{-q_{33}q_{44}}{q_{33}+q_{44}} \\ 0 & 0 & 0 & 0 & \frac{q_{33}q_{44}}{q_{33}+q_{44}} & \frac{-q_{33}q_{44}}{q_{33}+q_{44}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_{33}q_{44}}{q_{33}+q_{44}} & \frac{-q_{33}q_{44}}{q_{33}+q_{44}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ L_{1t} \\ L_{2t} \\ IE_{1t} \\ X_{1t}^{III} \\ M_{1t}^{III} \\ TT_{1t} \\ T_{1t} \\ r_{2t} \\ r_{2t-1} \\ TT_{2t} \\ T_{2t} \\ TR_{2t} \\ X_{2t}^{III} \\ M_{2t}^{III} \end{bmatrix}$$

$$Q^c(x_t^{c*} - \bar{x}_t^c) = \begin{bmatrix} \frac{q_{33}q_{44}}{q_{33}+q_{44}} & \frac{-q_{33}q_{44}}{q_{33}+q_{44}} & \frac{q_{33}q_{44}}{q_{33}+q_{44}} & \frac{-q_{33}q_{44}}{q_{33}+q_{44}} \\ \frac{q_{33}q_{44}}{q_{33}+q_{44}} & \frac{-q_{33}q_{44}}{q_{33}+q_{44}} & \frac{q_{33}q_{44}}{q_{33}+q_{44}} & \frac{-q_{33}q_{44}}{q_{33}+q_{44}} \end{bmatrix} \begin{bmatrix} X_{1t}^{III} \\ M_{1t}^{III} \\ X_{2t}^{III} \\ M_{2t}^{III} \end{bmatrix}$$

And

$$(x_t^{c*} - \bar{x}_t^c)'Q\,(x_t^{c*} - \bar{x}_t^c) = \left[X_{1t}^{III},\ M_{1t}^{III},\ X_{2t}^{III},\ M_{2t}^{III}\right] \begin{bmatrix} \frac{q_{33}q_{44}^2+q_{44}q_{33}^2}{(q_{33}+q_{44})^2} & \frac{-(q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} & \frac{q_{33}q_{44}^2+q_{44}q_{33}^2}{(q_{33}+q_{44})^2} & \frac{-(q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} \\ \frac{-(q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} & \frac{q_{33}q_{44}^2+q_{44}q_{33}^2}{(q_{33}+q_{44})^2} & \frac{-(q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} & \frac{q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} \\ \frac{q_{33}q_{44}^2+q_{44}q_{33}^2}{(q_{33}+q_{44})^2} & \frac{-(q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} & \frac{q_{33}q_{44}^2+q_{44}q_{33}^2}{(q_{33}+q_{44})^2} & \frac{-(q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} \\ \frac{-q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} & \frac{q_{33}q_{44}^2+q_{44}q_{33}^2}{(q_{33}+q_{44})^2} & \frac{-(q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} & \frac{(q_{33}q_{44}^2+q_{44}q_{33}^2)}{(q_{33}+q_{44})^2} \end{bmatrix} \begin{bmatrix} X_{1t}^{III} \\ M_{1t}^{III} \\ X_{2t}^{III} \\ M_{2t}^{III} \end{bmatrix} \quad \text{(A-12.4a)}$$

$$\text{(A-12.4b)}\quad (x_t^{c*} - \bar{x}_t^c)'Q^c(x_t^{c*} - \bar{x}_t^c) = \frac{q_{33}q_{44}^2 + q_{44}q_{33}^2}{(q_{33}+q_{44})^2}\left[(X_{1t}^{III} - M_{1t}^{III}) + (X_{2t}^{III} - M_{2t}^{III})\right]^2$$

$$\text{(A-12.4c)}\quad (x_t^{c*} - \bar{x}_t^c)'Q^c(x_t^{c*} - \bar{x}_t^c) = \frac{(q_{33}q_{44})}{q_{33}+q_{44}}\left[(X_{1t}^{III} - M_{1t}^{III}) + (X_{2t}^{III} - M_{2t}^{III})\right]^2$$

Defining: $X_{1t}^{III} + X_{2t}^{III} = M_{3t}$

$M_{1t}^{III} + M_{2t}^{III} = X_{3t}$

then equation A-12.4c becomes:

$$\text{(A-12.5)}\quad (x_t^{c*} - \bar{x}_t^c)'Q^c(x_t^{c*} - \bar{x}_t^c) = \frac{(q_{33}q_{44})}{q_{33}+q_{44}}(M_{3t} - X_{3t})^2$$

Therefore

$$\text{(A-12.6)}\quad \hat{J}^c(u*) = \frac{1}{2}\,\frac{(q_{33}q_{44})}{q_{33}+q_{44}}\sum_{t=1}^{N}(M_{3t} - X_{3t})^2$$

APPENDIX A-13

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